Integration by Riemann Triangles

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In *Proofs Without Words II*, Roger B. Nelson collects a number of mathematical results with visual proofs that do not need any verbal explanations, including the following beautiful illustration of the formula

$$\int_0^A \frac{1}{(1+x^2)}\, dx = \arctan(A)$$

due to Aage Bondesen [4, p. 63].

Figure 1 The fan curve of $\frac{1}{2(1+x^2)}$.

Example 1 (The arctan function)**.** We consider the Riemann integral under the graph of the function

$$f(x) = \frac{1}{2(1 + x^2)}$$

as a limit of a sequence of rectangles. One of them is $CDBF$ in Figure 1. Here, the distance between C and D on the x-axis is just Δ. We connect the corners C and D with the point $R = (0, -1)$ and see that triangle RCD has area $\Delta/2$. We now reduce triangle RCD to triangle KJR, where K lies on a unit circle that is translated one unit in the negative y-direction. For the reduction factor from triangle CDR to KJR, we have

$$\frac{KR}{CR} = \frac{1}{\sqrt{1 + x^2}}.$$

Now we can calculate the area of the reduced triangle by squaring the reduction factor KR/CR. We get $\Delta/(2(1 + x^2))$ for the area of triangle KJR. This matches exactly the area of the Riemann rectangle below the graph of the function f. The factor $1/2$ was inserted in f to get this equivalence. Summing over the interval $[0, A]$, we get $\int_0^A \frac{1}{2(1+x^2)} \, dx$ on the one hand and the area of the sector of the circle corresponding to angle $\angle QRM$ on the other. This angle equals $\arctan(A)$. Thus, the area of sector QRM equals $\frac{1}{2} \arctan(A)$, and the proof is complete. The Riemann rectangles above are transferred to a fan-like figure consisting of Riemann triangles below. In this case, the fan curve approaches a sector of a circle whose area we know. This gives us the value of the integral above.

How could we know that we had to use a circle below the x-axis? Since triangles RMC and RLK are similar, we must have

$$\frac{t}{x} = \frac{1}{\sqrt{1 + x^2}}.$$

This gives us $t^2 = \frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$ or $\frac{1}{1+x^2} = 1 - t^2$, which again gives us

$$x = \frac{t}{\sqrt{1 - t^2}}.$$

For the fan curve g, we then have

$$g(t) + 1 = \frac{t}{x} = \frac{t}{t/\sqrt{1 - t^2}} = \sqrt{1 - t^2},$$

which shows that g is the unit circle translated one unit in the negative y-direction.

In this article, we try to generalize this idea. We want to establish a connection between two integrals, where one is the area under a function above the x-axis, and one is related to the fan curve below the x-axis. In Example 1, the limit of the fan curve was easy to compute. In some of our other examples, we will instead be able to express the limit of the fan curve in terms of other integrals. In all of our cases, converting Riemann rectangles to Riemann triangles will give us interesting relationships. We call this "the fan method."

Simple examples

We will start by presenting some simple examples, covered in some detail. In the next section, we introduce some more sophisticated examples, presented with less detail, and we assume more knowledge about integration.
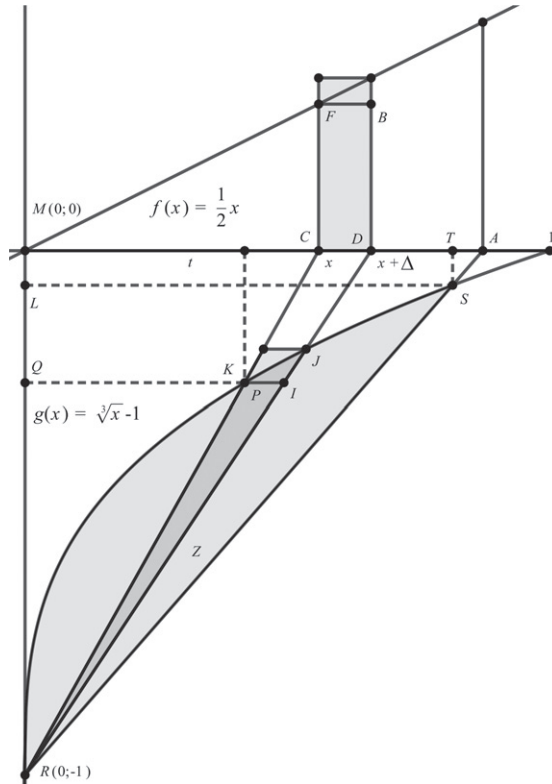

Figure 2 The fan curve of $\frac{1}{2}x$.

Example 2 (The linear function)**.** In our next example, we choose a linear function $y = x/2$ above the x-axis, see Figure 2. We now ask for the corresponding function g below the x-axis, where the Riemann rectangles have been replaced by Riemann triangles, as in Example 1. Since the triangles *CMR* and *PQR* are similar, we must have

$$\frac{x}{1} = \frac{t}{QR} = \frac{t}{g(t)+1}.\tag{1}$$

Since the Riemann rectangles and the Riemann triangles have the same area, and the reduction factor is t/x, we can determine g by comparing the areas. The area of *CBDF* is $\Delta \cdot x/2$, and the area of *CDR* is $\Delta/2$, so we must have

$$\Delta\frac{x}{2} = \frac{\Delta}{2}\left(\frac{t}{x}\right)^2.$$

Therefore, $x^3 = t^2$, which gives $g(t) + 1 = t/x = t^{1/3}$. From equation (1) we have

$$A = T/(g(T)+1).\tag{2}$$

Summing over the interval $[0, A]$, we see that the integral under the function f corresponds to the area Z of the curved figure between the graph of the third root function translated one unit in the negative y-direction and the line RS, which resembles a section through the wing of an airplane. Even though g is a complicated function, its inverse is the simple power function x^3. We can therefore relate Z to the area to the left of g, which is the integral of the inverse function translated one unit downward. This technique is also crucial in our further examples, and it is similar to the argument leading up to Young's inequality [**2**, Section 4.8]. Hence, for $0 < A < 1$, we get

$$\int_0^A \frac{x}{2} dx = Z = \frac{T \sqrt[3]{T}}{2} - \int_0^{\sqrt[3]{T}} x^3 \, dx.$$

By equation (2), we have

$$A = \frac{T}{g(T) + 1} = T/\sqrt[3]{T} = T^{\frac{2}{3}}.$$

Therefore, the area of triangle RLS is

$$\frac{T \sqrt[3]{T}}{2} = \frac{A^{\frac{3}{2}} \sqrt{A}}{2} = \frac{A^2}{2}.$$

The integral above is the area of a triangle, and we get $\int_0^A x/2 \, dx = A^2/4$. Altogether, we have

$$\frac{A^2}{4} = \frac{A^2}{2} - \int_0^{\sqrt{A}} x^3 \, dx,$$

that is,

$$\int_0^{\sqrt{A}} x^3 \, dx = \frac{A^2}{4}, \quad \text{or} \quad \int_0^V x^3 \, dx = \frac{V^4}{4}.$$

This confirms a known result from integral calculus.

Remark. This argument works for $A < 1$. If $A > 1$, the fan curve will cross the x-axis, but if we instead consider the function $g(x) = \sqrt[3]{x} - c$, where $c > A$, we can ensure that the graph of the fan curve, g, lies below the x-axis in order to keep the figures separated. This is not essential for our method, but is simply meant to make it easier to visualize. We can also use this extension of the domain for A in Examples 3 and 4 since in those cases f is increasing, which makes g also increase and eventually cross the x-axis.

We now generalize our observations from the above examples. If f is the function above the x-axis and g is the corresponding fan curve below the x-axis, where the Riemann rectangles are replaced by Riemann triangles, then we have the following two equations:

$$\frac{x}{1} = \frac{t}{g(t) + 1},$$

representing the similarity of triangles RCD and RPI, which we discussed above as equation (1), and

$$\Delta \cdot f(x) = \frac{\Delta}{2} \left(\frac{t}{x}\right)^2, \tag{3}$$

representing the equality of the areas of the Riemann rectangles and triangles. Altogether, this gives us the following theorem:

Theorem 1. *Given a function f, the corresponding fan curve can be described by*

$$f \left(\frac{t}{g(t) + 1} \right) = \frac{(g(t) + 1)^2}{2}. \tag{4}$$

Remark. This formula will be crucial in the following examples, and it settles the problem of finding the fan curve g below the x-axis, as soon as f is given. Since g is given implicitly by equation (4), we will not always be able to solve for g as a function, and we will instead just get a curve. We will see a number of examples of this phenomenon in Examples 6–10.
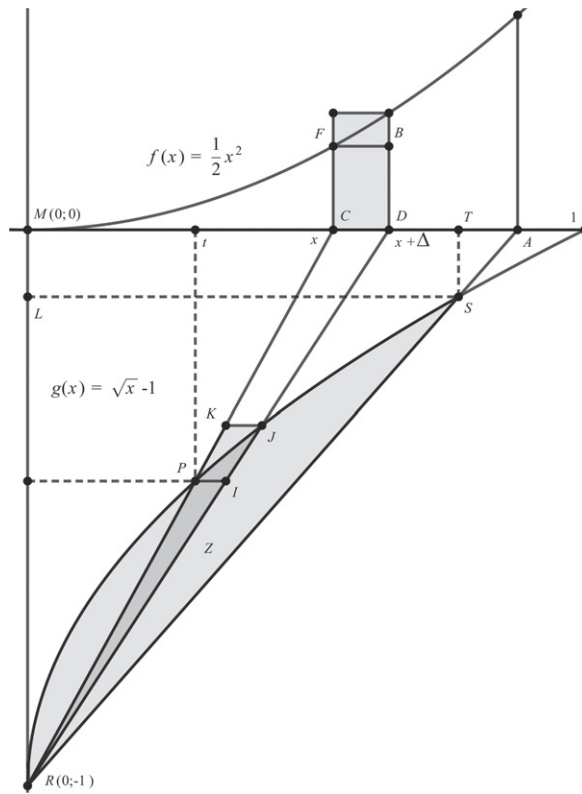

Figure 3   The fan curve of $\frac{1}{2}x^2$.

Example 3 (The quadratic function). In our next example, we choose a quadratic function $f(x) = x^2/2$ above the x-axis and look for the corresponding fan curve g below the x-axis, see Figure 3. According to equation (4), we have

$$f \left(\frac{t}{g(t) + 1} \right) = \frac{(g(t) + 1)^2}{2},$$

which gives us

$$g(t) + 1 = \frac{t}{x} = \sqrt{t}.$$

Summing over the interval $[0, A]$, we see that the integral under the function f corresponds to the area Z of the curved figure between the graph of the root function (translated one unit in the negative y-direction) and the line RS, which again resembles a section through the wing of an airplane. Now we apply the same technique as above. We observe that Z can be expressed as the difference between the area of the triangle RSL and the area to the left of the graph of g, but that is the integral of the inverse of g translated one unit downward, in other words, x^2. Geometrically speaking, both f and g represent parabolas. It is easy to see that $RL = g(T) + 1 = \sqrt{T}$, and by equation (2) we get

$$A = \frac{T}{g(T) + 1} = \frac{T}{\sqrt{T}},$$

and thus $A = RL = \sqrt{T}$. This gives us

$$\int_0^A \frac{x^2}{2} \, dx = Z = \frac{T\sqrt{T}}{2} - \int_0^A x^2 \, dx.$$

It follows that the area of triangle RLS is $T\sqrt{T}/2 = A^3/2$, and therefore

$$\frac{3}{2} \int_0^A \frac{x^2}{2} \, dx = \frac{T\sqrt{T}}{2} = \frac{A^3}{2}, \quad \text{i.e.,} \quad \int_0^A x^2 \, dx = \frac{A^3}{3}.$$

Again, we have been able to confirm a result from integral calculus, and we have been able to integrate the parabola.

Now we look at the rectangle $MAPW$ surrounding the graph of the function f over the interval $[0, A]$, see Figure 4. The area is $F = Af(A)$. Now, consider triangle RSL surrounding the corresponding fan shape. Here, the area is

$$G = T(g(T) + 1)/2.$$

Using equations (2) and (4), for $x = A$ and $t = T$ we have

$$F = Af(A) = \frac{T}{g(T) + 1} f\left(\frac{T}{g(T) + 1}\right)$$

$$= \frac{T(g(T) + 1)^2}{2(g(T) + 1)} = \frac{T(g(T) + 1)}{2} = G,$$

meaning that the areas of the rectangle and the triangle are the same. Thus, we know not only that the integral of f and the fan shape are equal in area, but also that the remaining gray areas in the rectangle and the triangle, respectively, are equal. These areas belong to the integrals of the inverse functions of f and g, respectively. This gives us new opportunities for finding further connections and formulae.

Example 4 (The power function). We can now apply our new findings. If we consider $f(x) = (1/2)x^k$, then we get

$$f\left(\frac{t}{g(t) + 1}\right) = \frac{(g(t) + 1)^2}{2}$$

by equation (4), and therefore,

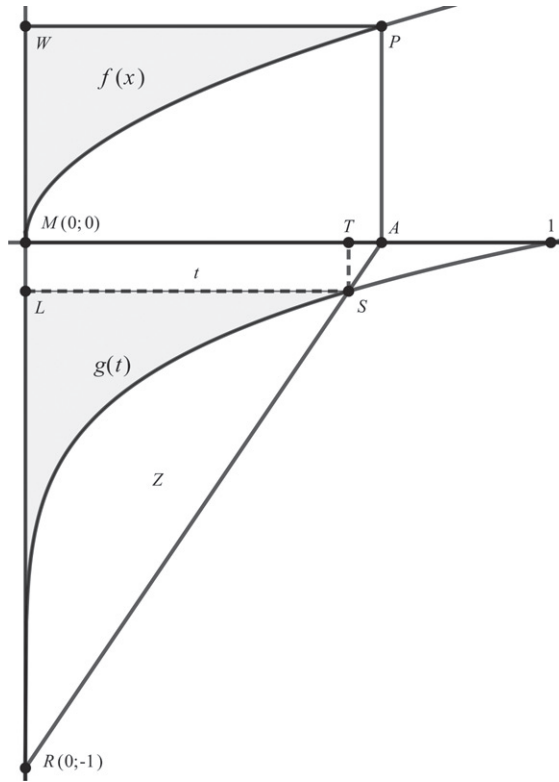$$t^k = (g(t) + 1)^{k+2} \quad \text{or} \quad g(t) + 1 = t^{k/(k+2)}.$$

Figure 4 Surrounding regions.

For $k = 1, 2$ the exponent in the fan curve becomes a unit fraction whose inverse becomes a power function. For $k \geq 3$, we do not get unit fractions, but we can write the exponent as $k/(k + 2) = 1/(1 + 2/k)$, and if we set $n = 1/k$, we get $1/(1 + 2/k) = 1/(2n + 1)$, which is again a unit fraction. In the sequel, we therefore choose $f(x) = \frac{1}{2}x^{1/n}$. Then equation (4) gives us

$$f\left(\frac{t}{g(t) + 1}\right) = (g(t) + 1)^2/2.$$

Therefore,

$$t = (g(t) + 1)^{2n+1} \quad \text{or} \quad g(t) = \sqrt[2n+1]{t} - 1.$$

The equality of the remaining areas corresponding to the integrals of the inverse functions then implies that

$$\int_0^{f(A)} (2x)^n \, dx = \int_0^{g(T)+1} t^{2n+1} \, dt.$$

Now, since

$$f(A) = f\left(\frac{T}{g(T) + 1}\right) = \frac{1}{2}(g(T) + 1)^2,$$

we can rewrite the last equation as

$$\int_0^{\frac{(g(T)+1)^2}{2}} (2x)^n\, dx = \int_0^{g(T)+1} t^{2n+1}\, dt \quad \text{or} \quad 2^n \int_0^{\frac{V^2}{2}} x^n\, dx = \int_0^V x^{2n+1}\, dx,$$

and we have established a result giving us many "new" integration formulae. Knowing a formula for $\int x\, dx$ gives us a formula for $\int x^3\, dx$. Knowing a formula for $\int x^2\, dx$ gives us a formula for $\int x^5\, dx$. Knowing a formula for $\int x^3\, dx$ gives us a formula for $\int x^7\, dx$, and so on.

To this point, we have assumed that the exponents n are positive numbers. But now we are tempted to look at negative exponents as well. As soon as $n \le 0$, we have to be prepared for the fact that the integrals starting at zero might be improper, and for $n \le -1$ even infinite. This problem can be circumvented by translating the left border for the integration from zero to one.
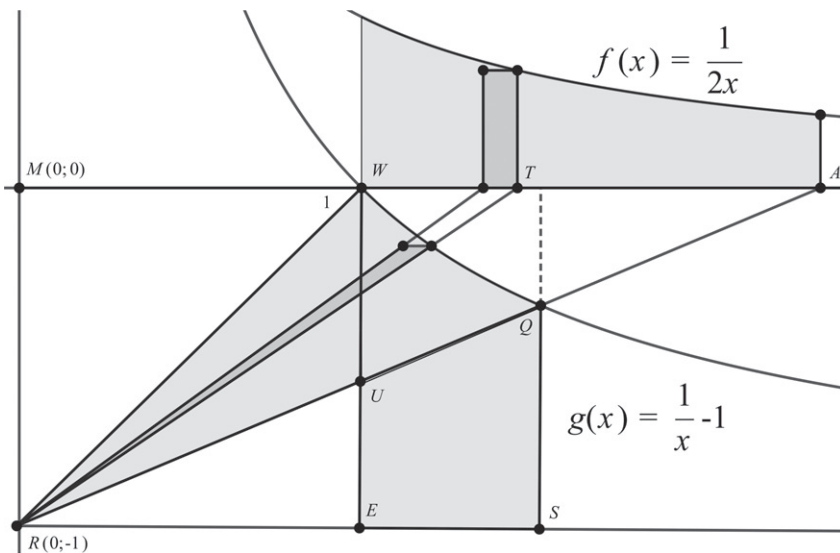

Figure 5   The fan curve of $\dfrac{1}{2x}$.

Example 5 (The hyperbola). The case $n = -1$ is especially interesting since the integral here is not trivial, and since the values of the exponents n and $2n + 1$ in the last formula coincide. This corresponds to $f(x) = 1/(2x)$ and $g(t) + 1 = 1/t$ in the beginning of Example 4. Since the coefficients on both sides are different, we can actually hope for a result on the integral of the hyperbola. We therefore study the integral of the function $f(x) = 1/(2x)$, see Figure 5. By the same arguments as above, we obtain that the area under the hyperbola $f(x) = 1/(2x)$ over the interval $[1, A]$ is equal to the area of the curved "triangle" RQW. Now, triangle REW has area $1/2$. The area of triangle RSQ is the same since $\frac{1}{2}T\frac{1}{T} = \frac{1}{2}$, implying that triangle RUW and the trapezoid $ESQU$ have the same area. Hence,

$$\int_1^A \frac{dx}{2x} = \int_1^T \frac{dx}{x},$$

and since

$$A = \frac{T}{g(T) + 1} = \frac{T}{1/T} = T^2,$$

this gives us

$$\int_1^{T^2} \frac{dx}{x} = 2 \int_1^T \frac{dx}{x}.$$

So, if we define $L(t) = \int_1^t (1/x)\, dx$, we get $L(T^2) = 2L(T)$, which is the logarithmic property of the integral of the hyperbola.
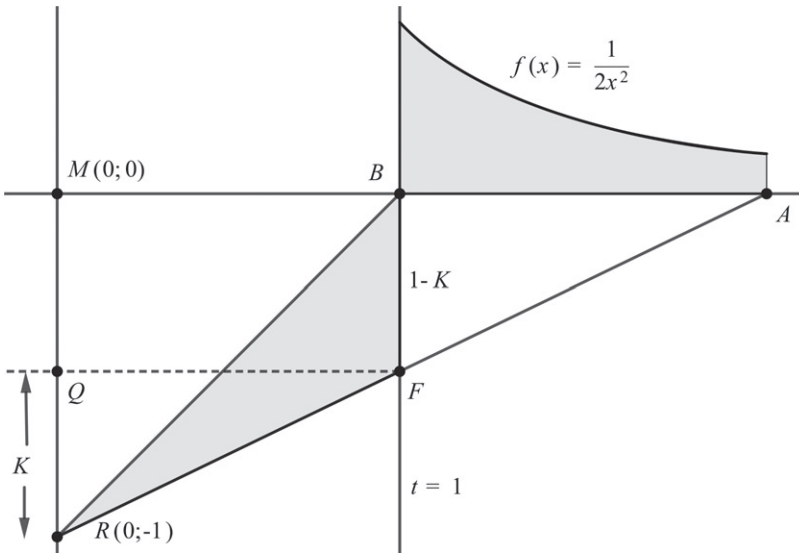

Figure 6   The fan curve of $1/2x^2$.

Example 6 (The function $f(x) = 1/2x^2$)**.** We now consider the integral of $1/2x^2$, see Figure 6. From equation (4) we have

$$f\left(\frac{t}{g(t) + 1}\right) = \frac{1}{2}(g(t) + 1)^2,$$

leading to

$$\frac{(g(t) + 1)^2}{2t^2} = \frac{(g(t) + 1)^2}{2}$$

or $t^2 = 1$, which gives us $t = \pm 1$. We choose $t = 1$ and consider triangles RAM and RFQ. Similarity then gives us $A/1 = 1/K$. Therefore, the area of triangle RFB is

$$\frac{1 - K}{2} = \frac{1}{2} - \frac{1}{2A}.$$

Thus, we have

$$\int_1^A \frac{dx}{2x^2} = \frac{1}{2} - \frac{1}{2A}.$$

Because of the vertical line, the situation here is especially simple. Notice that the fan curve g is here given by $t = 1$, which is no longer the graph of a function.
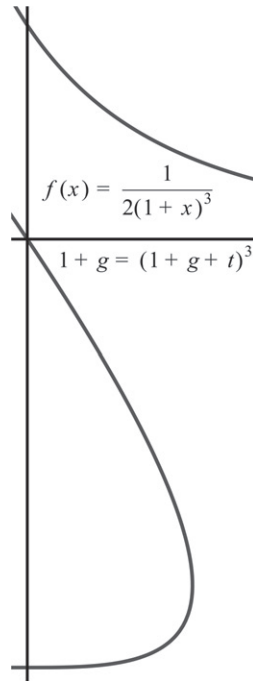
Advanced examples

Figure 7   The nose curve for $n = 3$.

Example 7 (The nose curve). Our next example is the class of functions

$$f(x) = \frac{1}{2(1 + x)^n}.$$

Equation (4) gives us

$$f\left(\frac{t}{g(t) + 1}\right) = \frac{1}{2}(g(t) + 1)^2.$$

In our case, this reads

$$\frac{1}{2(1 + \frac{t}{1+g})^n} = \frac{1}{2}(g(t) + 1)^2,$$

giving us $(1 + g)^{n-2} = (1 + g + t)^n$.

  We call this a "nose curve." Observe that for $n = 3$, it is simply a cubic curve that has been rotated. Figure 7 shows the nose curve for $n = 3$. Here, the improper integral $\int_0^\infty 1/2(1 + x)^3 \, dx = 1/4$, and thus the area of the "nose" is $1/4$. This is another example of the case where the fan curve g is no longer a function.

Remark. The case $n = 2$ is especially simple since we get $g(t) = -t$, see Figure 8. For linear functions, we know the integral by geometric arguments, so this case can be used to prove

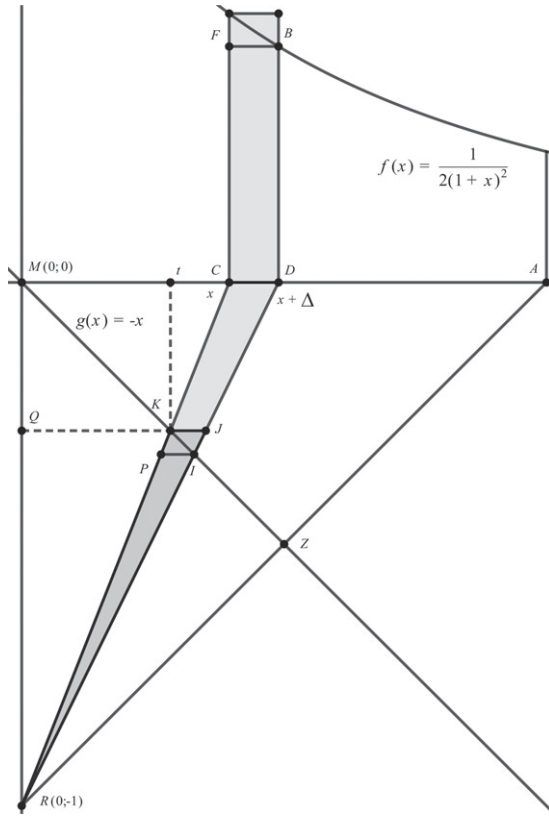$$\int_0^A \frac{dx}{(1 + x)^2} = 1 - \frac{1}{1 + A}.$$

Figure 8   The fan curve of $\dfrac{1}{2(1+x)^2}$.

Thus, we have shown the corresponding result from calculus by simple geometrical means. We also see directly that the integral $\int_0^\infty dx/2(1+x)^2$ has to be finite because the intersection point Z never moves below the line $y=-1$. Therefore, the integral is always less than or equal to $1/2$. A similar argument could be used in Example 1 to show that $\int_0^\infty dx/2(1+x^2)=\pi/4$.

Remark. The case $n=1$ also gives us some interesting insight. Choosing $f(x)=1/2(1+x)$, equation (4) gives us

$$\frac{1}{2\left(\frac{t}{g+1}+1\right)}=\frac{1}{2}(g+1)^2,$$

which is equivalent to

$$g+1=\frac{-t\pm\sqrt{4+t^2}}{2}.$$

Thus $\int dx/2(1+x)$, which we know is a logarithmic function, can now be coupled to $\int(-t\pm\sqrt{4+t^2})/2\,dt$, which helps explain why the integral $\int\sqrt{a^2+t^2}\,dt$ will involve a logarithmic function. The details are somewhat complicated and are omitted.
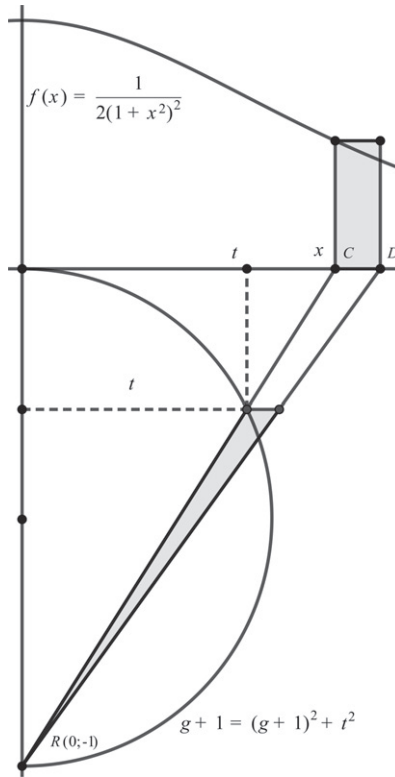
Figure 9   The fan curve of $\dfrac{1}{2(1+x^2)^2}$.

Example 8 (The bell-shaped functions)**.** The following example is due to Stephan Berendonk at the University of Cologne (private communication). We consider an alteration of the initial example by Aage Bondesen, where we replace the function above the x-axis with $f(x) = 1/2(1+x^2)^2$, see Figure 9. By equation (4) we get

$$\frac{1}{\left(1+\left(\frac{t}{g(t)+1}\right)^2\right)^2} = (g(t)+1)^2 \quad \text{or} \quad g(t)+1 = (g(t)+1)^2 + t^2,$$

representing a circle with center$(0, -1/2)$ and radius $1/2$, another example in which the fan curve g is no longer a function. Strictly speaking, the equation above gives us a pair of circles, but we ignore the circle that lies below $y = -1$. Since the area of the semicircle is $(1/2)\pi(1/2)^2 = \pi/8$, we can see that

$$\int_0^\infty \frac{dx}{2(1+x^2)^2} = \frac{\pi}{8}, \quad \text{or} \quad \int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi}{4},$$

a result you can find in many integration tables.

Remark. In fact, the whole class of bell-shaped functions

$$f(x) = \frac{1}{2(1+x^2)^n},$$

containing both Bondesen's quarter of a circle ($n = 1$) and Berendonk's semicircle ($n = 2$), can be treated by our method. For higher values of n, we use equation (4) to get

$$\frac{1}{(1 + (\frac{t}{g+1})^2)^n} = (g + 1)^2 \quad \text{or} \quad (g + 1)^{2(n-1)} = ((g + 1)^2 + t^2)^n,$$

which represents a pair of ovals. Notice that, as in the above example, we have terms of the form $(g + 1)^2$ on both sides. For $n = 1$, we get a constant on the left-hand side in the equation above, and therefore a single circle. For $n > 1$, we have to take a square root and therefore get pairs of ovals for the curve g, see Figure 10 for the case $n = 3$. However, for our purposes, only the upper branch above $y = -1$ is of interest.
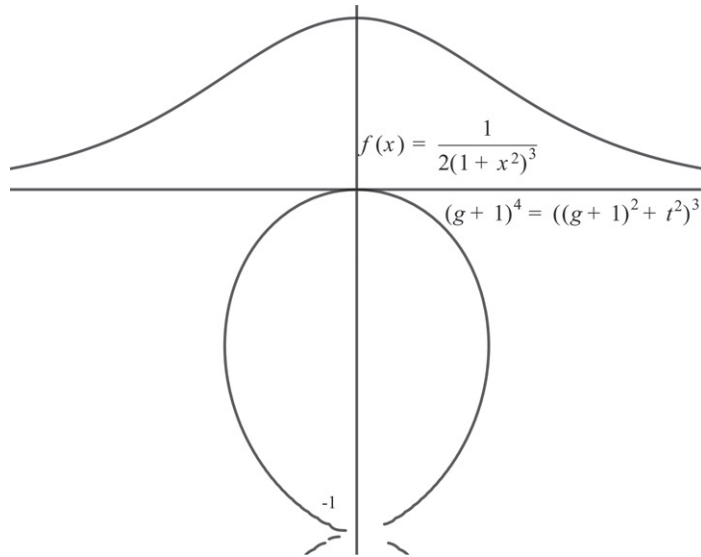

Figure 10   The fan curve of $\dfrac{1}{2(1 + x^2)^3}$.

Since we know the integral of f from zero to infinity, we can find the area of the ovals, too. We set

$$I_n = \int_0^\infty \frac{dx}{(1 + x^2)^n},$$

and for $n > 1$, we have

$$I_n = \int_0^\infty \frac{1 + x^2 - x^2}{(1 + x^2)^n} \, dx = I_{n-1} - \int_0^\infty \frac{x^2 \, dx}{(1 + x^2)^n} = I_{n-1} - \int_0^\infty uv' \, dx,$$

where we have chosen $u = x$ and

$$v = \frac{-1}{2(n - 1)} \frac{1}{(1 + x^2)^{n-1}}.$$

Integrating by parts, we get

$$\int_0^\infty uv' \, dx = uv \, \Big|_0^\infty - \int_0^\infty u'v \, dx = 0 - \left(\frac{-1}{2(n - 1)} \right) \int_0^\infty \frac{dx}{(1 + x^2)^{n-1}}.$$

Altogether, this leads to

$$I_n = I_{n-1} - \left(\frac{1}{2(n-1)}\right) I_{n-1} = \left(\frac{2n-3}{2n-2}\right) I_{n-1},$$

giving us

$$I_n = \left(\frac{2n-3}{2n-2}\right) \left(\frac{2n-5}{2n-4}\right) \cdots \frac{3}{4} \frac{1}{2} \frac{\pi}{2},$$

since $I_1 = \frac{\pi}{2}$ by Aage Bondesen's result. Multiplying by 2 gives us the area of the ovals.

Remark. In the following examples, we no longer use the factor 2 in the denominator to be canceled in equation (3). The effect is that $g(0)$ might be greater than 0, as we can see in Figures 11–14. However, this is no problem since we only kept the graph of g below the x-axis to make it easier to visualize the method.
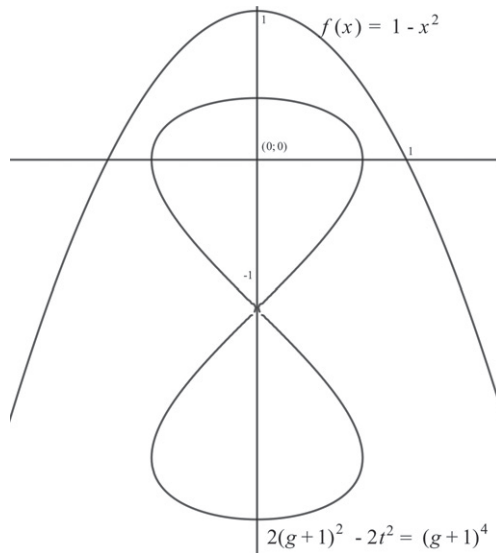

Figure 11   The fan curve of $f(x) = 1 - x^2$.

Example 9 (Upside-down parabola)**.** What about the parabola $f(x) = 1 - x^2$? Here, equation (4) gives us $1 - (t/(g+1))^2 = \frac{1}{2}(g+1)^2$ or

$$2(g+1)^2 - 2t^2 = (g+1)^4.$$

This is the equation of the lemniscate of Gerono [**3**, p. 124], [**1**, p. 117]. Since we know that the area under the parabola in the first quadrant is $\int_0^1 (1 - x^2)\,dx = 2/3$, we also know that the area of the lemniscate is $4\frac{2}{3} = 8/3$, see Figure 11.

Remark. This lemniscate of Gerono is defined as a midpoint curve [**1**]. Start with two parabolas with the same vertical symmetry line, one opening upwards, the other opening downwards. Now we intersect the parabolas with varying horizontal lines and ask for the midpoints between the intersection points. In this way, we obtain the lemniscate of Gerono.

Remark. Face-down parabolas with zeros apart from the origin always give us lemniscates as fan curves. Face-up parabolas give other curves, the root function of Example 3 being a special case.

Remark. The whole class of functions $f(x) = 1 - x^{2n}$ could be treated in the same way. Here new lemniscate-like shapes arise. Since $\int_0^1 (1 - x^{2n})\, dx = 2n/(2n+1)$, the area of the corresponding lemniscate-like curve is $8n/(2n+1)$.
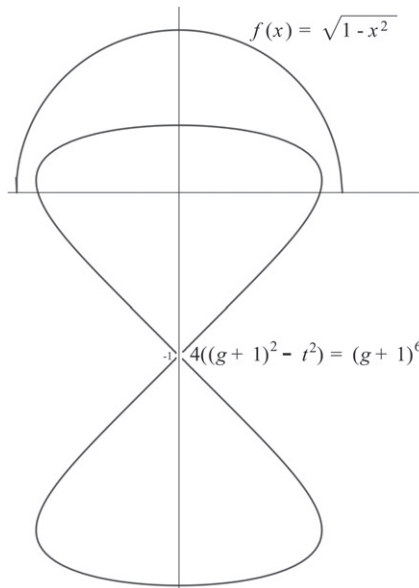

Figure 12   The fan curve of the circle $f(x) = \sqrt{1 - x^2}$.

Example 10 (The circle). We choose a circle $f(x) = \sqrt{1 - x^2}$. Then by equation (4) we have

$$4((g + 1)^2 - t^2) = (g + 1)^6,$$

which also corresponds to a lemniscate-like curve. Since this curve comes from a quarter circle, the area contained in the whole lemniscate-like curve is $4\frac{\pi}{4} = \pi$, see Figure 12.

Up to now we have only considered graphs of functions f, but we can generalize our results to parametric curves, which actually will simplify our formulae. In fact the transition equation (4) from f to g becomes much simpler when we use parametric representations for curves instead of functions.

Given a curve above the x-axis with parametric representation $x = x(s)$, $y = y(s)$, we now look for the corresponding fan curve $x = t(s)$, $y = g(s)$. The area of the Riemann rectangle is $A = y(s)\Delta$, while the corresponding triangle has area

$$B = \frac{\Delta}{2} \left(\frac{t(s)}{x(s)} \right)^2.$$

Since we want them to be equal, we have

$$y(s) = \frac{1}{2} \left(\frac{t(s)}{x(s)} \right)^2, \quad \text{i.e.,} \quad t(s) = x(s)\sqrt{2y(s)}.$$

In addition, we have the similarity equation

$$\frac{x(s)}{1} = \frac{t(s)}{g(s)+1} \quad \text{or} \quad g(s)+1 = \frac{t(s)}{x(s)} = \frac{x(s)\sqrt{2y(s)}}{x(s)} = \sqrt{2y(s)}.$$

Thus, we have found a very simple connection between the parametric representation of the curve above the x-axis and the parametric representation of the corresponding fan curve. We formulate this result as a theorem.

Theorem 2. *Given a parametric curve $x = x(s)$, $y = y(s)$, the corresponding fan curve can be described by*

$$x = t(s) = x(s)\sqrt{2y(s)} \quad and \quad y+1 = g(s)+1 = \sqrt{2y(s)}.$$

Remark. By the same formulae, it is also possible to calculate the original curve $(x(s), y(s))$ given the fan curve $(t(s), g(s))$. We get

$$y(s) = \frac{(1+g(s))^2}{2} \quad \text{and} \quad x(s) = \frac{t(s)}{g(s)+1},$$

but note that we do not necessarily get an expression for y as a function of x.

Remark. If the curve above the x-axis is a function, we have $x = x$ and $y = f(x)$, and we find $g+1 = \sqrt{2f(x)}$, giving us

$$\frac{(g+1)^2}{2} = f(x) = f(\frac{t}{g+1}).$$

Thus, we have proven Theorem 1 again.

Remark. If we use a parametric representation $x = \cos(s)$ and $y = \sin(s)$ for the circle from Example 10, then we get

$$x = t(s) = \cos(s)\sqrt{2\sin(s)}$$

and

$$y+1 = g(s)+1 = \sqrt{2\sin(s)}.$$

This gives us the same lemniscate-like curve

$$4x^2 = 4(y+1)^2 - (y+1)^6$$

that we already found in Example 10, see Figure 12.

Remark. If the original curve above the x-axis is a function, then we are now able to distinguish between the cases where the fan curve is a function or a curve. If $x = t(s)$ is increasing, then the fan curve is a function itself. Thus, if $t'(s) > 0$, then we can be sure that the fan curve is a function. Now, we have $t(x) = x\sqrt{2f(x)}$, and our condition reads

$$t'(x) = \sqrt{2}\left(\sqrt{f(x)} + \frac{x}{2\sqrt{f(x)}}f'(x)\right) > 0$$

or $2f(x) + xf'(x) > 0$. Since we only consider $x \geq 0$ and $f(x) \geq 0$, the condition is automatically satisfied if f' is positive. For decreasing f, this is equivalent to $|f(x)/f'(x)| > x/2$. Now, the expression $|f(x)/f'(x)| = \tau$ is the so-called *subtangent*, the length of the tangent's "shadow" between the tangent's intersection with the

x-axis and the point on the function graph. Thus, our condition reads $2\tau > x$, meaning that the original function must not decrease "too quickly." Alternatively, we can divide by f to get $2 + x(f'/f) > 0$, which gives us $f'/f > -(2/x)$. It follows that for $0 < a < x$, we get

$$\int_a^x \frac{f'(t)}{f(t)}\, dt > \int_a^x \frac{-2dt}{t},$$

which gives us

$$\log f(x) - \log(a) > -2\log x + 2\log(a) \quad \text{or } f(x) > f(a)\frac{a^2}{x^2}.$$

This explains why the nose curves gave us functions for $n \leq 2$, and the bell-shaped curves gave us functions for $n = 1$.
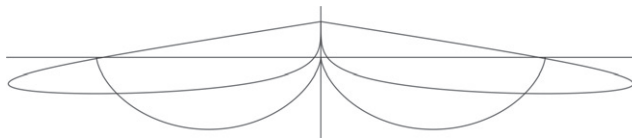

Figure 13   The fan curve of the cycloid.

Example 11 (The cycloid)**.** Let us now apply Theorem 2. For the cycloid we have $x(s) = s - \sin(s)$ and $y(s) = 1 - \cos(s)$. Thus,

$$t(s) = (s - \sin(s))\sqrt{2(1 - \cos(s)}$$

and

$$g(s) + 1 = \sqrt{2(1 - \cos(s))} = \sin(s/2).$$

So, the fan curve corresponding to a cycloid looks like the wing of a dragonfly. Since the area under a single arc of a cycloid is 3π, we can conclude that the pair of wings of the dragonfly covers an area of 6π, see Figure 13.

Example 12 (Surfaces in space)**.** In this final example, we try to mimic Aage Bondesen's idea in three dimensions. We start with the surface function

$$f(x, y) = \frac{1}{3\sqrt{1 + x^2 + y^2}^3}.$$

The graph is shown in Figure 14.

Here we consider a Riemann column at the position (x, y) in the xy-plane, the base of the column being Δ^2, see Figure 15. On the same base we construct a pyramid having its vertex at $R = (0, 0, -1)$. This pyramid has volume $\Delta^2/3$ since its height is 1. Now, we reduce this pyramid until the reduced base reaches a unit sphere which is translated one unit in the negative z-direction, the reduction factor being $1/\sqrt{1 + x^2 + y^2}$. The volume of the reduced wedge is $\Delta^2/\left((3\sqrt{1 + x^2 + y^2})^3\right)$ since we have to use the cube of the reduction factor in order to find the volume of the
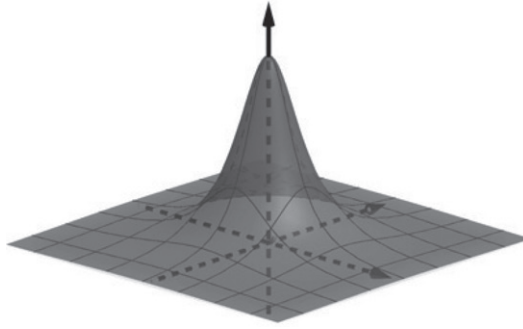
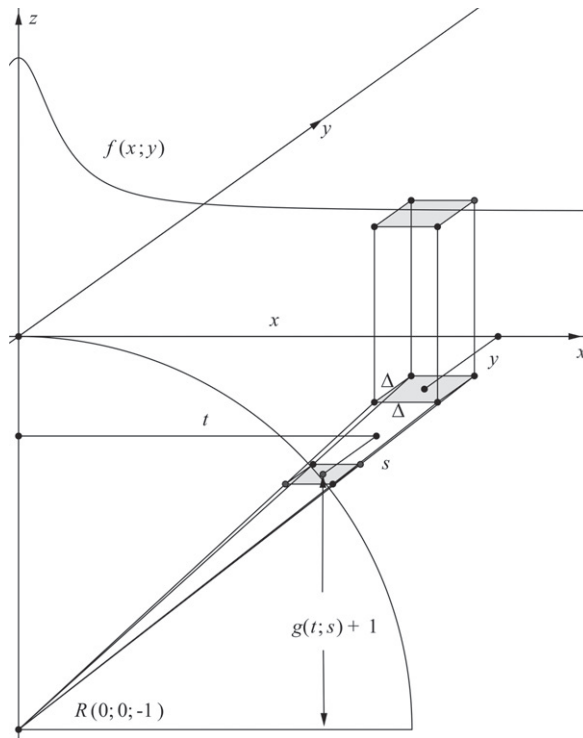**Figure 14**　The surface $\dfrac{1}{3\sqrt{1+x^2+y^2}^{\,3}}$.

Figure 15　Riemann columns.

wedge. Thus, the reduced wedge has the same volume as the corresponding Riemann column $\Delta^2 f(x, y)$. Summing over the whole plane we get

$$\iint_{\substack{\text{over the whole} \\ xy-\text{plane}}} \frac{dx\,dy}{3\sqrt{1+x^2+y^2}^{\,3}} = \frac{2\pi}{3},$$

which is the volume of the hemisphere.

Remark. For a general surface function $f(x, y)$, we must have

$$\Delta^2 f(x, y) = \frac{\Delta^2}{3}\left(\frac{t}{x}\right)^3,$$

where $t/x = s/y$ is the reduction factor. Similarity gives us $x/1 = t/(g(t, s) + 1)$ and also $y/1 = s/(g(t, s) + 1)$. Thus, we can find g by the following formula corresponding to equation (4):

$$f\left(\frac{t}{g(t, s) + 1}, \frac{s}{g(t, s) + 1}\right) = \frac{(g(t, s) + 1)^3}{3}.$$

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Summary. In this article we present a method of integration inspired by an example from Proofs Without Words II by Nelsen. Riemann rectangles under a curve are transformed into triangles. The limit of the collection of triangles gives us a new shape. The area under the original curve and the area of the new shape are the same. In this way, interesting connections between areas under different curves are established and integration formulae are obtained. Only elementary concepts like similarity and Riemann rectangles are used. A variety of examples is worked out to show the strength of the method. The method may be used for first year calculus students to deepen their understanding of the Riemann integral.

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