Rigorous derivation of some asymptotic models for free water surfaces and interfaces

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Thesis for the degree of Philosophiae Doctor (PhD)
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Abstract

The origins of contemporary mathematical hydrodynamics can be traced back to the 18th century. In 1757, Euler published a paper where he introduced equations that could describe the motion of fluids [31], known today as the Euler equations. Other notable figures such as Lagrange, Laplace, Poisson, and Cauchy around this time were also drawn to the study of waves in fluids, which continues to be an active field of research to this day (see [21] for a historical review). However, the Euler equations contain several difficulties related to the complexity of the system, both analytically and in applications. To overcome some of these issues, one typically considers simplified models characterized by dimensionless parameters that describe the main mechanisms involved.

In this work, we rigorously derive asymptotic models from the irrotational Euler equations with a free surface. Specifically, we derive several new models and prove that their solutions converge to the solution of its reference model with respect to the scaling parameters. We say that an asymptotic model is a *fully justified* if we can answer the following points in the affirmative:

1. The solutions of the reference model exist on the relevant time scale.
2. The solutions of the asymptotic model exist (at least) on the same time scale.
3. We must establish the *consistency* between the asymptotic model and the reference model. This means the solutions of the reference model solve the asymptotic model up to a certain precision. Then show that the error is “small” when comparing the two solutions.

This thesis consists of two main parts. The first part consists of three papers and concerns the study of asymptotic models in the case of a single fluid in shallow water. In papers 1 and 2, we study Whitham-type systems that were previously derived in the sense of consistency by Emerald [27]. The reference model, in this case, is the *water waves equations*, where the first point is proved in the seminal paper by Alvarez-Samaniego and Lannes [7]. The remaining point in their full justification is to prove the well-posedness of these systems on the relevant time scale. In paper 3, we derive new models, in the sense of consistency, with an improved description of the variation of the bottom toporgaphy. Here, we derive models with improved frequency dispersion, where the goal is to describe waves passing over an obstacle that is studied experimentally in the classical paper by Dingemans [23].

For the second part of the thesis, we prove the full justification of the Benjamin-Ono equation as an asymptotic model for the unidirectional propagation of long internal water waves in a two-layer fluid, where one layer is of great depth. In this case, the second point is well-known, while the first and third point is proved in paper 4. The proof of the consistency is based on the paper by Bona, Lannes, and Saut [14]. While the existence result for the general two-layer fluid model on the relevant time scale is a nontrivial extension of the work of Lannes [40], where both fluids are of finite depth.

Sammendrag

Opprinnelsen til moderne matematisk hydrodynamikk kan spores tilbake til 1700-tallet. I 1757 publiserte Euler artikkelen der han introduserte ligninger, i dag kjent som Euler likningene, som kan beskrive væskedynamikk [31]. Hvordan beskrive bølger i væsker ble også studert av Lagrange, Laplace, Poisson, Cauchy og det er fortsatt et aktivt forskningsfelt i dag (for en historisk gjennomgang se [21]). Men fra et praktisk og analytisk perspektiv, er Euler ligningene veldig kompliserte. Det er derfor vanlig å introdusere forenklede modeller, som er karakterisert av dimensjonsløse parametre og som gir en god beskrivelse av den opprinnelige modellen.

I dette arbeidet, utleder vi asymptotiske modeller fra Euler likningene med en fri overflate og for irroterende flyt. Mer spesifikt, så utleder vi flere modeller, hvor vi kvantifiserer feilen med den opprinnelige modellen. Altså, vi beviser at løsningene fra modellen konvergerer til løsningen av referansemodellen med hensyn på noen dimensjonsløse parametre. Vi sier at en modell *fullstendig rettferdiggjort* dersom vi kan svare på følgende punkter:

1. Løsningen av referanse modellen eksisterer på den relevante tidsskalaen.
2. Løsningen av den asymptotiske modellen eksisterer (minst) på den samme tidsskalaen.
3. Sist må vi vise at modellene er *konsistente*. Altså at løsningen av referansemodellen er også en løsning av den asymptotiske modellen opp til gitt toleranse. Deretter må vi vise at differansen mellom disse løsningene er "liten."

Denne oppgaven består hovedsakelig i to deler. Den første delen består av tre artikler som angår utledningen av asymptotiske modeller for å beskrive en enkel væske i grunt vann. I artikkel 1 og 2, studerer vi såkalte Whitham-type modeller som ble tidligere utledet av Emerald [27] hvor han viste at ligningene var konsistente. I dette tilfellet er referansemodellen kjent som *vannbølge ligningene*, hvor punkt en er bevisst i den viktige artikkelen av Alvarez-Samaniego og Lannes [7]. Dermed, for å fullstendig rettferdiggjøre Whitham-modellene, gjenstår det bare å vise at de eksisterer på den relevante tidsskalaen. I artikkel 3, utleder vi nye modeller, der vi viser at de er konsistente med vannbølge likningene, og hvor presisjonen gir en bedre beskrivelse av endringen av bunnen. Modellene har også en forbedret beskrivelse av dispersjonsforholdet til referansemodellen, og målet er å beskrive bølger som beveger seg over en bunn med bråe endringer. Dette er motivert av de eksperimentelle resultatene i den klassiske artikkelen til Dingemans [23].

I den andre delen av avhandlingen gir vi et bevis for den fullstendige rettferdiggjørelsen av Benjamin-Ono likningen. Dette er en asymptotisk modell som beskriver lange bølger som beveger seg i en retning mellom to fluider. Her er dybden for til den ene væsken mye dypere enn den andre. I dette tilfellet er punkt nummer to velkjent, mens det første og tredje punktet er bevist i artikkel 4 av denne avhandlingen. Beviset for at modellene er konsistente er basert på artikkelen til Bona, Lannes, og Saut [14]. Mens eksistens resultatet for det generelle systemet for to væsker på den relevante tidsskalaen er en ikke triviell forlengelse av resultatet til Lannes [40] hvor begge fluidene har en endelig dybde.

List of publications

1. M. O. Paulsen. *Long time well-posedness of Whitham-Boussinesq systems.* Nonlinearity, 35 (12) : 6284-6348, 2022.

2. L. Emerald and M. O. Paulsen. *Long time well-posedness and full justification of a Whitham-Green-Naghdi system.* arXiv preprint arXiv:2306.00711, 2023.

3. L. Emerald and M. O. Paulsen. *Rigorous derivation of weakly dispersive shallow water models with large amplitude topography variations.* arXiv preprint arXiv:2306.02186, 2023.

4. M. O. Paulsen. *Justification of the Benjamin-Ono equation as an internal water waves model.* arXiv preprint arXiv:2311.10058, 2023.

Index of notations

- For the defintion of $\varepsilon, \mu, \beta, \gamma, \mathrm{bo}$ see equation (1.1.16).
- We let c denote a positive constant independent of the small parameters μ, ε, β, and bo that may change from line to line. Also, as a shorthand, we use the notation $a \lesssim b$ to mean $a \leqslant c\, b$.
- Let $k \in \mathbb{N}, l \in \mathbb{N}$ and $m \in \mathbb{N}$. A function R is said to be of order $\mathcal{O}(\mu^k \varepsilon^l)$, denoted $R = \mathcal{O}(\mu^k \varepsilon^l)$, if divided by $\mu^k \varepsilon^l$ this function is uniformly bounded with respect to $\mu, \varepsilon \in (0,1)$ in the Sobolev norms $|\cdot|_{H^s}$.
- We define the gradient by $\nabla_{x,z} = (\partial_x, \partial_z)^T$ and the Laplace operator by $\Delta_{x,z} = \partial_x^2 + \partial_z^2$. We also and introduce their scaled versions

$$\nabla^\mu_{x,z} = (\sqrt{\mu}\partial_x, \partial_z)^T \quad \text{and} \quad \Delta^\mu_{x,z} = \nabla^\mu_{x,z} \cdot \nabla^\mu_{x,z} = \mu\partial_x^2 + \partial_z^2.$$

- Let the normal derivative be given by $\partial_{\mathbf{n}_f} = \mathbf{n}_f \cdot \nabla_{x,z}$ where $\mathbf{n}_f = \frac{1}{\sqrt{1+(\partial_x f)^2}} \begin{pmatrix} -\partial_x f \\ 1 \end{pmatrix}$.
- Let $L^2(\mathbb{R})$ be the usual space of square integrable functions with norm $|f|_{L^2} = \sqrt{\int_{\mathbb{R}} |f(x)|^2 \, dx}$. Also, for any $f, g \in L^2(\mathbb{R})$ we denote the scalar product by $\big(f, g\big)_{L^2} = \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx$.
- Let $f : \mathbb{R} \to \mathbb{R}$ be a tempered distribution, let $\widehat{f}$ or $\mathcal{F}f$ be its Fourier transform. Let $F : \mathbb{R} \to \mathbb{R}$ be a bounded function. Then the Fourier multiplier associated with $F(\xi)$ is denoted F and defined by the formula:

$$\mathcal{F}\big(\mathrm{F}(\mathrm{D})f(x)\big)(\xi) = F(\xi)\widehat{f}(\xi).$$

- For any $s \in \mathbb{R}$ we call the multiplier $\widehat{|\mathrm{D}|^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$ the Riesz potential of order $-s$.
- For any $s \in \mathbb{R}$ we call the multiplier $\Lambda^s = (1 + \mathrm{D}^2)^{\frac{s}{2}} = \langle \mathrm{D} \rangle^s$ the Bessel potential of order $-s$.
- The Sobolev space $H^s(\mathbb{R})$ is equivalent to the weighted L^2-space with $|f|_{H^s} = |\Lambda^s f|_{L^2}$.
- Let $H^{s+1}_{\gamma,\mathrm{bo}}(\mathbb{R}) = H^{s+1}(\mathbb{R})$ with norm

$$|u|^2_{H^{s+1}_{\gamma,\mathrm{bo}}} = (1-\gamma)|u|^2_{H^s} + \mathrm{bo}^{-1}|\partial_x u|^2_{H^s}.$$

- Let $H^\infty(U)$ be given in terms of $\partial^\alpha f \in L^2(U)$ for all $\alpha \in \mathbb{N}$ where $\partial = \partial_x$ if $U = \mathbb{R}$. In the case $U \subset \mathbb{R}^2$, then we let ∂ be either ∂_x or ∂_z.
- For any $s \geqslant 0$ we will denote $\mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$ the homogeneous Sobolev space with $|f|_{\mathring{H}^{s+\frac{1}{2}}} = |\mathrm{D}^{\frac{1}{2}} f|_{H^s}$. One should note that $|\mathrm{D}| = \mathcal{H}\partial_x$, where $\widehat{\mathcal{H}f}(\xi) = -i \,\mathrm{sgn}(\xi)\widehat{f}(\xi)$ is the Hilbert transform.
- For any $s \geqslant 0$ we will denote $\dot{H}^{s+1}(\mathbb{R})$ the Beppo-Levi space with $|f|_{\dot{H}^{s+1}} = |\Lambda^s \partial_x f|_{L^2}$.

- For any $s \geqslant 0$ we will denote $\dot{H}_{\mu}^{s+\frac{1}{2}}(\mathbb{R}) = \dot{H}^{s+\frac{1}{2}}(\mathbb{R})$ with $|f|_{\dot{H}_{\mu}^{s+\frac{1}{2}}} = |\mathfrak{B} f|_{L^2}$ and where $\mathfrak{B}$ is a Fourier multiplier defined in frequency by:

$$\mathcal{F}(\mathfrak{B} f)(\xi) = \frac{|\xi|}{(1+\sqrt{\mu}|\xi|)^{\frac{1}{2}}} \widehat{f}(\xi).$$

- We say f is a Schwartz function $\mathscr{S}(\mathbb{R})$, if $f \in C^{\infty}(\mathbb{R})$ and satisfies for all $j, k \in \mathbb{N}$,

$$\sup_x |x^j \partial_x^k f| < \infty.$$

- Let $a < b$ be real numbers and consider the domain $\mathcal{S} = (a,b) \times \mathbb{R}$. Then the space $\dot{H}^{s+1}(\mathcal{S})$ is endowed with the seminorm

$$\|f\|_{\dot{H}^{s+1}(\mathcal{S})}^2 = \int_a^b |\nabla_{x,z} f(\cdot, z)|_{H^s}^2 \, \mathrm{d}z.$$

Contents

Chapter 1

Introduction

1.1 The governing equations

We start this chapter with a brief description of the governing equations that will serve as a basis for the thesis. The presentation is inspired by the book of David Lannes [41], and we also refer the reader to [3,24,36] for their excellent presentation of the subject. Moreover, we will angle the presentation towards the main results of the thesis. In particular, the focus will be on specific results in water wave theory to motivate the scientific result of the thesis, [29,30,44,45], presented in Chapter 2. For simplicity, we restrict the presentation to having one horizontal dimension and we shall only make formal computations in this section where we suppose the functions are regular and decay sufficiently fast at infinity[1].

The reference model is given in terms of two free surface incompressible Euler equations given on the upper-fluid domain:

$$\Omega^- = \{(x,z) \in \mathbb{R}^2 \; : \; z > \zeta\},$$

and the lower fluid domain:

$$\Omega^+ = \{(x,z) \in \mathbb{R}^2 \; : \; -H + b < z < \zeta\}.$$

Here $\zeta = \zeta(x,t) \in \mathbb{R}$ denotes the free surface, the given function $b = b(x) \in \mathbb{R}$ is the variation of the bottom and $H > 0$ is the still water depth when $b = 0$, see Figure 1.

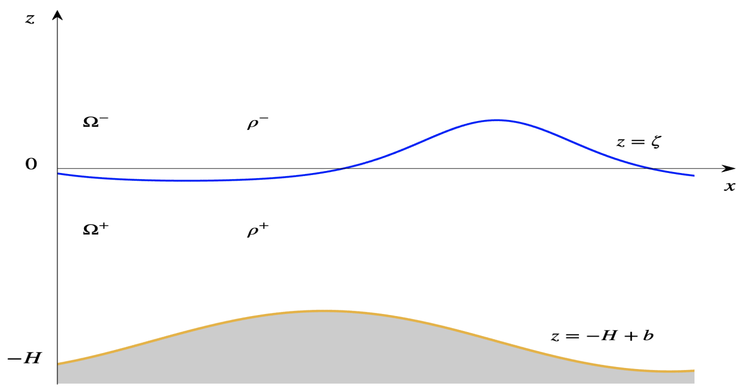


Figure 1: The blue line denotes the surface elevation $z = \zeta$ where $\zeta = \zeta(x,t)$ is a free variable. The free surface separates two fluids with density $0 \leqslant \rho^- < \rho^+$. The brown line $z = -H + b$ is the topography variation where $b = b(x)$ is a given function.

[1] To be precise, we will assume the functions in this section belong to $H^\infty(U)$ where $U = \Omega^\pm$ or $\mathbb{R}$.

Remark 1.1.1. Here, we let the lower fluid domain be of finite depth, while the upper fluid domain is unbounded in the vertical direction. This is done out of convenience since we will later consider shallow water models in papers $1-3$ where we simply suppose the top layer is air with density $\rho^- = 0$. On the other hand, in paper 4, we are concerned with the derivation of the Benjamin-Ono equation where one of the layers is of infinite depth and $0 < \rho^- < \rho^+$.

Let the velocity of a fluid particle at a point (x,z,t) written as $\mathbf{U}^{\pm}(x,z,t) \in \mathbb{R}^2$, where the notation $\{-,+\}$ stands for the upper or lower fluid respectively. Moreover, let $P^{\pm}(x,z,t) \in \mathbb{R}$ be the pressure, $-g\mathbf{e}_z$ is the acceleration of gravity, where $g > 0$ and $\mathbf{e}_z$ is the unit upward vector in the vertical direction, and $\rho^{\pm} \geqslant 0$ are the densities of the two homogeneous and incompressible fluids. Then the momentum balance and mass conservation in each fluid are given by

$$\begin{cases} \rho^{\pm}(\partial_t \mathbf{U}^{\pm} + (\mathbf{U}^{\pm} \cdot \nabla_{x,z})\mathbf{U}^{\pm}) = -\nabla_{x,z}P^{\pm} - g\mathbf{e}_z \\ \mathrm{div}_{x,z}\mathbf{U}^{\pm} = 0, \end{cases} \tag{1.1.1}$$

in $\Omega^{\pm}$. We will further suppose that the fluid is irrotational, meaning we have that

$$\omega^{\pm} := \mathrm{curl}_{x,z}\mathbf{U}^{\pm} = 0 \quad \text{in} \quad \Omega^{\pm}. \tag{1.1.2}$$

Moreover, we note for smooth solutions of the Euler equations, it is sufficient to impose the condition at time $t = 0$ [3]. Also, it is worth noting that assumption (1.1.2) is fundamental to reduce the Euler equations (1.1.1) to a system of equations that is defined at the interface. We will see this in the next section.

Continuing, we have that for a simply connected domain and curl-free vector field that there exist a potential $\Phi^{\pm}(x,z) \in \mathbb{R}$ such that

$$\mathbf{U}^{\pm} = \nabla_{x,z}\Phi^{\pm}.$$

As a result, rewriting the momentum equation, $(1.1.1)_1$, we have that $\Phi^{\pm}$ satisfies the Bernoulli equation:

$$\rho^{\pm}\big(\partial_t\Phi^{\pm} + \frac{1}{2}|\nabla_{x,z}\Phi^{\pm}|^2 + gz\big) = -P^{\pm}, \tag{1.1.3}$$

while the mass conservation equation, $(1.1.1)_2$, implies that

$$\Delta_{x,z}\Phi^{\pm} = 0. \tag{1.1.4}$$

So far, we have reduced the equations of motion within the fluid to a set of equations described by the auxiliary function $\Phi^{\pm}$. To have a closed system of equations, we need to impose boundary conditions. At the interface, we impose the *kinematic boundary condition*:

$$\partial_t\zeta = \sqrt{1 + (\partial_x\zeta)^2}(\partial_{\mathbf{n}_\zeta}\Phi^{\pm})|_{z=\zeta}, \tag{1.1.5}$$

and which tells you that the flow of the fluid propagates along the surface, or in other words, a particle at the surface will remain at the surface [3]. Lastly, we will assume that the bottom is fixed and impermeable, meaning we impose that

$$\partial_{\mathbf{n}_b}\Phi^{+}|_{z=-H+b} = 0. \tag{1.1.6}$$

Remark 1.1.2. The same condition also holds for $\nabla_{x,z}\Phi^-$ as z tends to infinity. However, this will be a property of the solution of the Laplace problem formulated for Φ^- (under given restrictions on the Dirichlet data). We will give some details on this point in Observation 1.1.8 below.

The internal water waves equations

We may now gather all the equations above to formulate a coupled system of PDEs with boundary conditions in terms of the unknowns $(\zeta, \Phi^{\pm})$. However, following the *Zakharov-Craig-Sulem* formulation [19, 20, 62], we choose to formulate the system in terms of

$$(\zeta, \psi^{\pm}), \quad \text{where} \quad \psi^{\pm}(t,x) = \Phi^{\pm}(t,x,\zeta(x,t)).$$

Then we can use the chain rule, the kinematic boundary condition (1.1.5), and the Bernoulli equation (1.1.3) to obtain the celebrated *water waves equations* in the Zakharov-Craig-Sulem formulation:

$$\begin{cases} \partial_t \zeta - \mathcal{G}^{\pm}[\zeta, b]\psi^{\pm} = 0 \\ \rho^{\pm}\Big(\partial_t \psi^{\pm} + g\zeta + \frac{1}{2}(\partial_x \psi^{\pm})^2 - \frac{1}{2}\frac{(\mathcal{G}^{\pm}[\zeta,b]\psi^{\pm} + \partial_x\zeta\partial_x\psi^{\pm})^2}{1+(\partial_x\zeta)^2}\Big) = -P^{\pm}|_{z=\zeta}. \end{cases} \tag{1.1.7}$$

Here $\mathcal{G}^{\pm}[\zeta, b]\psi^{\pm}$ are known as the Dirichlet-Neumann operators and is given by

$$\mathcal{G}^{\pm}[\zeta, b]\psi^{\pm} = \sqrt{1 + (\partial_x\zeta)^2}(\partial_{\mathbf{n}_\zeta}\Phi^{\pm})|_{z=\zeta},$$

where $\Phi^{\pm}$ is a solution of the Laplace problems

$$\begin{cases} \Delta_{x,z}\Phi^{\pm} = 0 & \text{in} \quad \Omega^{\pm} \\ \Phi^{\pm}|_{z=\zeta} = \psi^{\pm} & \partial_{\mathbf{n}_b}\Phi^{+}|_{z=-H+b} = 0. \end{cases} \tag{1.1.8}$$

The last system is deduced from the mass conservation (1.1.4) and the impermeability of the bottom (1.1.6).

Next, we need to comment on the pressure force at the free surface. If there is a difference in pressure at the interface then it is proportional to the mean curvature of the interface:

$$(P^{+} - P^{-})|_{z=\zeta} = \sigma\kappa(\zeta),$$

where $\sigma \in (0, 1)$ is the surface tension parameter and $\kappa(\zeta)$ is defined by

$$\kappa(\zeta) = -\partial_x\Big(\frac{\partial_x\zeta}{\sqrt{1 + (\partial_x\zeta)^2}}\Big).$$

Here we choose to stay consistent with the notation used in [41], where it was noted in the case of a single fluid that the effect of surface tension is only relevant for small characteristic scales. In fact, the capillary effects are relevant for ripples with wavelengths of the size of a couple of centimeters (see Example 9.1 in [41] for a formal argument).

Remark 1.1.3. Since we are concerned with long wave dynamics, it is safe to neglect the effects of surface tension from a practical perspective. However, as we will see later, from a mathematical perspective, even a small amount of surface tension is, in some cases, necessary for models to be well-posed.

Remark 1.1.4. The irrotationality condition and mass conservation reduce the Euler equations (1.1.1) into two scalar evolution equations that are defined on the boundary and independent from the vertical variable. Moreover, the water waves equations are Hamiltonian [62]. For simplicity we let $\rho^{-} = 0$, then we have that

$$H = \frac{g}{2}\int_{\mathbb{R}} \zeta^2 \,\mathrm{d}x + \sigma\int_{\mathbb{R}} \Big(\sqrt{1 + (\partial_x\zeta)^2} - 1\Big)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \psi^{+}\mathcal{G}^{+}[\zeta, b]\psi^{+}\,\mathrm{d}x,$$

with H satisfying the system

$$\partial_t\zeta = \delta_{\psi^+}H, \quad \text{and} \quad \partial_t\psi^{+} = -\delta_\zeta H,$$

where δ_{ψ^+} and δ_ζ are functional derivatives.

Finally, we turn to the presentation of the *internal water waves equations*. First, we ease the notation, by making the following simplifications

$$\gamma = \frac{\rho^-}{\rho^+}, \quad \rho^- < \rho^+ = 1, \quad g = 1.$$

Next, we follow the approach by Lannes [40], where we reduce the number of variables by using the first equation in (1.1.7) to see that

$$\mathcal{G}^{-}[\zeta, b]\psi^{-} = \mathcal{G}^{+}[\zeta, b]\psi^{+}.$$

Then we write ψ^- as a function of ψ^+ through the inverse relation

$$\psi^- = (\mathcal{G}^-[\zeta, b])^{-1}\mathcal{G}^+[\zeta, b]\psi^+,$$

and we define (formally) a new variable ψ by the formula

$$\begin{aligned}\psi &= \psi^+ - \gamma\psi^- \\ &= \big(1 - \gamma(\mathcal{G}^-[\zeta, b])^{-1}\mathcal{G}^+[\zeta, b]\big)\psi^+ \\ &= \mathcal{J}[\zeta, b]\psi^+.\end{aligned}$$

The unknowns ζ and ψ will form the primary variables of the internal water waves system. Moreover, we use these relations to (formally) define a new operator

$$\mathcal{G}[\zeta, b] = \mathcal{G}^+[\zeta, b](\mathcal{J}[\zeta, b])^{-1}. \tag{1.1.9}$$

Remark 1.1.5. In the case of the two-fluid problem, we only will consider the case of a flat bottom. For a precise definition of $\mathcal{G}[\zeta, 0]$, we refer the reader to Section 2 of [45].

From the above expressions, we obtain the internal water waves equations in dimensional form:

$$\begin{cases}\partial_t\zeta - \mathcal{G}[\zeta, b]\psi = 0 \\ \partial_t\psi + (1-\gamma)\zeta + \frac{1}{2}\big((\partial_x\psi^+)^2 - \gamma(\partial_x\psi^-)^2\big) + \mathcal{N}[\zeta, b, \psi^\pm] = -\sigma\kappa(\zeta),\end{cases} \tag{1.1.10}$$

where

$$\mathcal{N}[\zeta, b, \psi^\pm] = \frac{\gamma(\mathcal{G}^-[\zeta, b]\psi^- + \partial_x\zeta\partial_x\psi^-)^2 - (\mathcal{G}^+[\zeta, b]\psi^+ + \partial_x\zeta\partial_x\psi^+)^2}{2(1 + (\partial_x\zeta)^2)}.$$

Before we turn to some comments on the structure of (1.1.10), we make a simple observation on how $\mathcal{G}[0, 0]$ depends on the geometry of the problem.

Observation 1.1.6. In the formula (1.1.9) we note that we choose to invert $\mathcal{G}^-[\zeta, b]$ with domain $\mathcal{G}^+[\zeta, b]$. To see why the order of composition is important, we consider their linearized operators with a flat bottom. In particular, we have that the operators are defined by

$$\mathcal{G}^\pm[0, 0]\psi^\pm = \partial_z\Phi^\pm|_{z=0},$$

where $\Phi^\pm$ is the solutions of

$$\begin{cases}\Delta_{x,z}\Phi^\pm = 0 & \text{in} \quad \Omega^\pm \\ \Phi^\pm|_{z=0} = \psi^\pm & \partial_z\Phi^+|_{z=-H} = 0.\end{cases}$$

Then for regular Dirichlet data, we simply apply the Fourier transform on the horizontal variable and solve the corresponding ODEs:

$$\partial_z^2\widehat{\Phi}^\pm - \xi^2\widehat{\Phi}^\pm = 0.$$

Then use the Fourier multiplier notation to find that

$$\Phi^+ = \frac{\cosh((z+H)|\mathrm{D}|)}{\cosh(H|\mathrm{D}|)}\psi^+ \quad \text{and} \quad \Phi^- = \mathrm{e}^{-z|\mathrm{D}|}\psi^-. \tag{1.1.11}$$

As a result, we obtain the operators

$$\mathcal{G}^+[0, 0]\psi^+ = |\mathrm{D}|\tanh(H|\mathrm{D}|)\psi^+ \quad \text{and} \quad \mathcal{G}^-[0, 0]\psi^- = |\mathrm{D}|\psi^-. \tag{1.1.12}$$

By looking at the operator at the Fourier side, we see that $\mathcal{G}^+[0, 0]$ has a double root in zero, while in the infinite depth case, $\mathcal{G}^-[0, 0]$ has a simple root. This is the formal reason why we choose to compose the inverse of $\mathcal{G}^-$ with $\mathcal{G}^+$. See Remark 2.7 and the proof of Proposition 2.4 in [45] for details on this point.

The definition of $\mathcal{G}^{\pm}$ in (1.1.12) also serves as a hint that having one fluid of infinite depth and the other of finite depth alters the functional setting for $\psi^{\pm}$:

Observation 1.1.7. To understand how to define the elements that should be in the definition of the energy space, it is instructive to look at the linearized Hamiltonian H_0 of the system. In the case of the water waves equation in finite depth (i.e., $\rho^- = 0$), we find that

$$H_0 = \frac{g}{2}\int_{\mathbb{R}} \zeta^2 \,\mathrm{d}x + \sigma \int_{\mathbb{R}} (\partial_x \zeta)^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \psi^+ |\mathrm{D}| \tanh(H|\mathrm{D}|)\psi^+ \,\mathrm{d}x.$$

Using Plancherel's identity and the equivalence:

$$\frac{H|\xi|^2}{(1+H|\xi|)} \lesssim |\xi| \tanh(H|\xi|) \lesssim \frac{H|\xi|^2}{(1+H|\xi|)},$$

allows us to identify the energy associated with the system. It should provide (at least) a control on:

$$\int_{\mathbb{R}} \Big(g\zeta^2 + \sigma(\partial_x\zeta)^2 \Big) \,\mathrm{d}x + H \int_{\mathbb{R}} \Big| \frac{|\mathrm{D}|}{(1+H|\mathrm{D}|)^{\frac{1}{2}}} \psi^+ \Big|^2 \,\mathrm{d}x.$$

On the other hand, in the case of $H \to -\infty$, we see that the energy space must provide a control on the trace of the velocity potential in $\mathring{H}^{\frac{1}{2}}(\mathbb{R})$.

Lastly, for the linearized solution, we can observe that the decay estimate in Remark 1.1.2 holds and that there is a smoothing effect from the Poisson kernel:

Observation 1.1.8. Suppose $\psi^- \in \mathring{H}^{\frac{1}{2}}(\mathbb{R})$ and let Φ^- be given in terms of the Poisson kernel (1.1.11). We first observe that $\Phi^- \in \dot{H}^1(\mathcal{S}^-)$. Indeed, we obtain directly by Plancherel's identity, Fubini, and integration in z that

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} |\partial_z \Phi^-(x,z)|^2 \,\mathrm{d}x\mathrm{d}z &= -\frac{1}{2}\int_{\mathbb{R}} ||\xi|^{\frac{1}{2}} \widehat{\psi}^-(\xi)|^2 \int_0^\infty \partial_z \Big(\mathrm{e}^{-2z|\xi|} \Big) \,\mathrm{d}z\mathrm{d}\xi \\ &= \frac{1}{2} ||\mathrm{D}|^{\frac{1}{2}} \psi^-|_{L^2}^2. \end{aligned}$$

The same computation can be done for $\partial_x \Phi^-$. Next, we verify the decay at infinity. For $s > \frac{1}{2}$ and $z > 0$, we observe by virtue of the Sobolev embedding $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and Plancherel's identity that

$$|\partial_z \Phi^-(\cdot, z)|_{L^\infty} \lesssim ||\mathrm{D}| \langle \mathrm{D} \rangle^s \mathrm{e}^{-z|\mathrm{D}|} \psi^-(\cdot)|_{L^2} \lesssim \frac{1}{\sqrt{z}} ||\mathrm{D}|^{\frac{1}{2}} \psi^-|_{L^2}.$$

Taking the limit in z, we obtain the result:

$$\lim_{z\to\infty} \sup_{x\in\mathbb{R}} |\partial_z \Phi^-(x,z)| = 0.$$

The same estimate holds for $\partial_x \Phi^-$.

On the structure of the internal water waves equations

Regarding the structure of the equations, we see that $\mathcal{G}^{\pm}[\zeta]\psi^{\pm}$ is determined by the solution of an elliptic problem that depends on the fluid domain. However, by applying a simple change of variable, problem (1.1.8) can be transformed to a more general elliptic problem with non-constant coefficients (see, for instance, Chapter 2 in [41] for a detailed account). On the other hand, the water waves equations (1.1.7) seem to have a more complex structure. In fact, if we let $\underline{\mathbf{U}}^{\pm} = \mathbf{U}^{\pm}|_{z=\zeta} = (\underline{V}^{\pm}, \underline{w}^{\pm})^T$ be the velocity field at the free surface, then under a seemingly technical condition:

$$\underline{\mathfrak{a}}^{\pm} = g + (\partial_t + \underline{V}^{\pm}\partial_x)\underline{w}^{\pm} > 0, \tag{1.1.13}$$

the system has a hyperbolic character [41]. We will give some details on this in the next observation, where we follow the book of Lannes [41], Section 4.3.5.

Observation 1.1.9. Briefly put, the mathematical relevance of the criterion stems from the "quasilinearization" of the water waves equations. In particular, since we want to estimate the solutions in higher Sobolev norms we apply α derivatives on the equation and reformulate it in terms of the "good unknowns"[2]:

$$\zeta_{(\alpha)} = \partial_x^\alpha \zeta, \qquad \psi^\pm_{(\alpha)} = \partial_x \psi^\pm - \underline{w}\zeta_{(\alpha)},$$

where the equations take the form

$$\begin{cases} \partial_t \zeta_{(\alpha)} - \mathcal{G}^\pm[\zeta, b]\psi^\pm_{(\alpha)} + \partial_x(\underline{V}\psi_{(\alpha)}) = 0 \\ \partial_t \psi^\pm_{(\alpha)} + \underline{\mathfrak{a}}^\pm \zeta_{(\alpha)} + \underline{V}^\pm \partial_x \psi^\pm_{(\alpha)} = 0, \end{cases}$$

up to a rest consisting of lower order derivative terms, see Proposition 4.10 in [41]. At this point, it is natural to define an energy that cancels the linear terms in the energy estimates. However, to do so, we need the coefficient $\underline{\mathfrak{a}}^\pm$ to have a positive lower bound.

With this brief explanation in mind, let us now consider the physical relevance of criteria (1.1.13). We will relate the definition in (1.1.13) with a condition on the pressure force at the free surface. To do so, to reformulate the Euler equations (1.1.1) and relate it to the trace $\underline{\mathbf{U}}^\pm$:

$$\begin{cases} (\partial_t V^\pm)|_{z=\zeta} + \underline{V}^\pm(\partial_x V^\pm)|_{z=\zeta} + \underline{w}^\pm(\partial_z V^\pm)_{z=\zeta} = -(\partial_x P^\pm)|_{z=\zeta} \\ (\partial_t w^\pm)|_{z=\zeta} + \underline{V}^\pm(\partial_x w)|_{z=\zeta} + \underline{w}^\pm(\partial_z w^\pm)_{z=\zeta} = -(\partial_z P^\pm + g)|_{z=\zeta}. \end{cases}$$

This is simply done by applying the chain rule. In particular, we observe: ($\partial = \partial_t$ or ∂_x)

$$\partial \underline{\mathbf{U}}^\pm = (\partial \mathbf{U}^\pm)|_{z=\zeta} + \partial\zeta(\partial_z \mathbf{U}^\pm)|_{z=\zeta},$$

and since the pressure is constant at the surface:

$$0 = (\partial_x P^\pm)|_{z=\zeta} + \partial_x \zeta (\partial_z P).$$

The last equation on the pressure will provide a link between the two equations for $\underline{V}^\pm$ and $\underline{w}^\pm$. Indeed, together with the kinematic boundary condition (1.1.5) which is directly related to $\underline{\mathbf{U}}^\pm$ through:

$$\begin{aligned} \partial_t \zeta &= \sqrt{1 + (\partial_x \zeta)^2}\mathbf{n}_\zeta \cdot \underline{\mathbf{U}}^\pm \\ &= \underline{w}^\pm - \underline{V}^\pm \partial_x \zeta, \end{aligned}$$

we find after some algebra that

$$\begin{cases} \partial_t \underline{V}^\pm + \underline{V}^\pm \partial_x \underline{V}^\pm = \partial_x \zeta (\partial_z P)|_{z=\zeta} \\ \partial_t \underline{w}^\pm + \underline{V}^\pm \partial_x \underline{w}^\pm = -(\partial_z P + g)|_{z=\zeta}. \end{cases} \tag{1.1.14}$$

Here the first equation (1.1.14) can be given by

$$\partial_t \underline{V}^\pm + \underline{\mathfrak{a}}^\pm \partial_x \zeta + \underline{V}^\pm \partial_x \underline{V}^\pm = 0,$$

by substituting the pressure with the second equation which is related to (1.1.13):

$$\underline{\mathfrak{a}}^\pm = -(\partial_z P^\pm)|_{z=\zeta}.$$

In other words, (1.1.13) is equivalent to imposing positivity of the vertical pressure force at the surface. The condition is known as the Rayleigh-Taylor stability criterion and it gives a condition to exclude instabilities at the surface [41, 55].

[2] The definition of $\psi_{(\alpha)}$ is deduced from the shape derivative formula of the Dirichlet-Neumann operator, see [41].

In the case of having one fluid, with $\rho^- = 0$, the Rayleigh-Taylor stability criterion (1.1.13) is actually a necessary condition for the system to be well-posed [25]. It was later proved that under this condition, the water waves system was locally well-posed; we refer the reader to the pioneering work of Wu [57, 58] in the case of infinite depth and the work of Lannes in finite depth [39]. In fact, as we will discuss in the next section, the solutions of the water waves equations exist on a long time, which was proved by Alvarez-Samaniego and Lannes [7] (see also the more recent work on extended life span and improved regularity results [1, 2, 4–6, 59–61]).

In the case of the two fluid problems, with $0 < \rho^- < \rho^+$, then we also need to impose criteria on the data in order to obtain a stable configuration of the free surface. However, in this case, the situation is more subtle. In fact, the problem becomes ill-posed unless there is surface tension $\sigma > 0$ [26, 33, 34]. There are several results that utilize this fact to obtain well-posedness results in different configurations of the fluid domain where the time of existence $T = T(\sigma)$ tends to zero as $\sigma \to 0$ [9, 10, 16, 52, 53]. From a modeling point of view, this is a paradox, since surface tension is not relevant in the dynamics of long waves but is necessary for there to be solutions. In particular, the asymptotic models that are derived from the two-fluid systems neglect surface tension. A solution to this problem was provided by Lannes [40] in the case of two fluids with finite depth. He was able to generalize the Rayleigh-Taylor criterion for a two-fluid system to include surface tension and is given by

$$\underline{\mathfrak{a}}^+ - \underline{\mathfrak{a}}^- > \frac{1}{4\sigma} \frac{(\rho^+\rho^-)^2}{(\rho^+ + \rho^-)^2} \mathfrak{c}(\zeta) |\underline{V}^+ - \underline{V}^-|_{L^\infty}. \tag{1.1.15}$$

Here $\mathfrak{c}(\zeta) > 0$ is some constant that depends on the geometry of the problem, which in the case considered in [40] are two fluids of finite depth. We will adapt this criterion in the case where one fluid is of infinite depth. The main point is that this criterion allowed for an existence time that does not shrink to zero as $\sigma \to 0$. In fact, by introducing the small parameters into the equations, Lannes obtained a long-time existence and unique result.

Remark 1.1.10. The criterion is, of course, only assumed for the initial data and then propagated for $t > 0$ by using the equation and the Fundamental Theorem of Calculus. Clearly, the existence time would then depend on σ. However, by introducing the small parameters, we can bypass this difficulty and still obtain a long time existence and uniqueness result. See Remark 1.5 in [45] for more on this point in the case with one layer of infinite depth.

Nondimensionalization of the internal water waves equations and comments

From a practical point of view, it is often better to consider simplified models rather than full water wave equations. The idea dates back to Lagrange in 1781 [38], where we instead consider asymptotic models that are derived by "zooming" in specific regimes that capture the physics that you set out to describe. In particular, to describe long waves in shallow water with bathymetry effects it is instructive to introduce the quantities H, λ, a_{surf} and a_{bott}, the characteristic water depth, the characteristic wavelength in the longitudinal direction, the characteristic surface amplitude, and the characteristic amplitude of the bathymetry of the system. From these characteristic quantities, we define the following non-dimensional parameters

$$\varepsilon = \frac{a_{\text{surf}}}{H}, \qquad \mu = \frac{H^2}{\lambda^2}, \qquad \beta = \frac{a_{\text{bott}}}{H}, \qquad \gamma = \frac{\rho^-}{\rho^+}, \qquad \text{bo} = \frac{\rho^+ g \lambda^2}{\sigma}, \tag{1.1.16}$$

where the last number is related to surface tension and is known as the Bond number. Then the natural scaling for long waves, with one fluid of shallow depth is given by

$$x = \lambda x', \quad z = Hz', \quad t = \frac{\lambda}{c_{\text{ref}}} t', \quad \zeta = a_{\text{surf}} \zeta', \quad b = a_{\text{bott}} b',$$

where the prime notation denotes a nondimensional quantity and c_{ref} is the reference speed. We are yet to identify the dimensions of the auxiliary variable ψ and the reference speed in terms of the characteristic quantities. To do so, it is instructive to consider the linearized system (with $\beta = \sigma = 0$):

$$\begin{cases} \partial_t \zeta - \mathcal{G}[0,0]\psi = 0 \\ \partial_t \psi + (1-\gamma)\zeta = 0, \end{cases} \tag{1.1.17}$$

where $\mathcal{G}[0,0]$ is a Fourier multiplier given by

$$\mathcal{G}[0,0] = \mathcal{G}^+[0,0]\Big(1 - \gamma(\mathcal{G}^-[0,0])^{-1}\mathcal{G}^+[0,0]\Big)^{-1},$$

and using the relations (1.1.12) we find that

$$\mathcal{G}[0,0]\psi(x) = \mathcal{F}^{-1}\Big(|\xi|\frac{\tanh(H|\xi|)}{1+\gamma\tanh(H|\xi|)}\widehat{\psi}(\xi)\Big)(x).$$

We can simplify this expression under the shallow water assumption in the lower fluid, i.e. $\mu \ll 1$. In particular, for a wave with wavelength λ, the frequencies are concentrated around $|\xi| = \frac{2\pi}{\lambda}$, so that a Taylor expansion of the hyperbolic tangent implies

$$|\xi|\frac{\tanh(H|\xi|)}{1+\gamma\tanh(H|\xi|)} = H\xi^2,$$

up to an order $\mathcal{O}(\mu)$. Thus, we find

$$\mathcal{G}[0,0]\psi = -H\partial_x^2\psi.$$

From this simplification we can reduce (1.1.17) to a wave equation:

$$\partial_t^2\zeta - c_{\mathrm{ref}}^2\partial_x^2\zeta = 0,$$

where we identify the reference speed $c_{\mathrm{ref}}^2 = H(1-\gamma)$. Moreover, and from the second equation of (1.1.17) we can now find the dimensions of ψ:

$$\psi = \frac{a_{\mathrm{surf}}\lambda}{\sqrt{H}}\psi'.$$

Thus, performing the change of variables above to the internal water waves system (1.1.10), and omitting the prime notation, yields:

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi = 0 \\ \partial_t\psi + (1-\gamma)\zeta + \frac{1}{2}\big(\varepsilon(\partial_x\psi^+)^2 - \gamma\varepsilon(\partial_x\psi^-)^2\big) + \varepsilon\mathcal{N}[\varepsilon\zeta,\beta b,\psi^\pm] = -\frac{1}{\mathrm{bo}}\frac{1}{\varepsilon\sqrt{\mu}}\kappa(\varepsilon\sqrt{\mu}\zeta), \end{cases} \tag{1.1.18}$$

where

$$\mathcal{N}[\varepsilon\zeta,\beta b,\psi^\pm] = \frac{1}{2\mu}\frac{\gamma(\mathcal{G}_\mu^-[\varepsilon\zeta,\beta b]\psi^- + \varepsilon\mu\partial_x\zeta\partial_x\psi^-)^2 - (\mathcal{G}_\mu^+[\varepsilon\zeta,\beta b]\psi^+ + \varepsilon\mu\partial_x\zeta\partial_x\psi^+)^2}{(1+\varepsilon^2\mu(\partial_x\zeta)^2)}.$$

and

$$\mathcal{G}_\mu[\varepsilon\zeta,\beta b] = \mathcal{G}_\mu^+[\varepsilon\zeta,\beta b](\mathcal{J}_\mu[\varepsilon\zeta,\beta b])^{-1}. \tag{1.1.19}$$

The operators $\mathcal{G}_\mu^\pm[\varepsilon\zeta]$ are defined by

$$\mathcal{G}_\mu^\pm[\varepsilon\zeta]\psi^\pm = (\partial_z\Phi^\pm - \varepsilon\mu\partial_x\zeta\partial_x\Phi^\pm)|_{z=\varepsilon\zeta},$$

through the solutions of the scaled Laplace equations:

$$\begin{cases} (\mu\partial_x^2 + \partial_z^2)\Phi^\pm = 0 \quad \text{for} \quad \Omega^\pm \\ \Phi^\pm|_{z=\varepsilon\zeta} = \psi^\pm \quad \partial_z\Phi^+|_{z=-1} = 0, \end{cases} \tag{1.1.20}$$

where

$$\Omega^+ = \{(x,z) \,:\, -1+\beta b < z < \varepsilon\zeta\} \quad \text{and} \quad \Omega^- = \{(x,z) \,:\, z > \varepsilon\zeta\}.$$

We will now list long time existence and uniqueness results related to (1.1.18) for different cases in $\rho^\pm$, σ, and fluid depth.

Remark 1.1.11. We only consider the results that will later be used, and so the list is not exhaustive. In any of the cases, we will impose the standard non-cavitation condition that ensures that the fluid depth:

$$h = 1 + \varepsilon\zeta - \beta b$$

does not touch the bottom. This means there is a minimum depth $h_{\min} \in (0,1)$ such that $h \geqslant h_{\min}$ for all time the solution exists. Of course the, condition is imposed at time $t = 0$ and then verified for $t = \mathcal{O}(\frac{1}{\max\{\varepsilon,\beta\}})$. See Lemma 5.2 in [44] for a detailed demonstration of this point. Now, under this assumption, we have the following results:

1. In the case of a single fluid in finite depth with no surface tension, the long time well-posedness is given in [7] (see also [41], Chapter 4). The proof relies on the energy method where the authors define an energy functional $\mathcal{E}(t) = \mathcal{E}(\zeta(t), \psi^+(t))$ that depends on the norm of $(\zeta, \psi^+) \in C([0,T]; H^s(\mathbb{R}) \times \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R}))$ (see Observation 1.1.7). Then under the non-cavitation condition and a non-dimensional version of (1.1.13) they prove an estimate on the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \lesssim \max\{\varepsilon,\beta\}\big(\mathcal{E}(t)\big)^{\frac{3}{2}},$$

   which implies an existence time $t = \mathcal{O}(\frac{1}{\max\{\varepsilon,\beta\}})$. The main difficulty is handling higher order derivatives on the Dirichlet-Neumann operator and deriving precise estimates with respect to the small parameters ε, μ, β.

2. In the case of a single fluid in finite depth with surface tension, the same result holds. We refer the reader to [41], Chapter 9 for the proof. The main difference in the proof is in the definition of the energy that now needs to include ζ in $H^{s+1}_{0,\mathrm{bo}}(\mathbb{R})$. This gives additional difficulties when compared to the case $\sigma = 0$. In particular, having to control one more derivative in ζ means that it will induce subprincipal terms in ψ^+ that need to be accounted for in the energy estimates. The solution is to define energy that controls both space and time derivatives and is first introduced in [46]. See also Remark 9.9 in [41] on this technical point.

3. In the case of the two fluid systems where both layers are of finite depth, with $\beta = 0$, $\sigma > 0$, and $\rho^- < \rho^+$, the long time existence and uniqueness are proved in [40]. Here, the energy estimate is deduced under a non-dimensional version of (1.1.15), which allows for an existence time that is independent of σ. We should also note that due to the complex nature of the operator $\mathcal{G}_\mu$, there are several additional difficulties that arise. In particular, the proof relies on several symbolic expressions of the operators involved in $\mathcal{G}_\mu$ using pseudodifferential methods and is used to have precise estimates in terms of the small parameters and to deduce the skew-adjointness in some cases.

4. In the case of a single fluid in infinite depth without surface tension, a similar result as the first point holds and is proved in [41], Chapter 4. The main difference is the scaling of the equation and the definition of the energy, which is now modified to include the norm of the trace of the velocity potential in $\mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$.

5. In the case of the two fluid systems where one layer is of infinite depth with $\beta = 0$ and $\rho^- < \rho^+$, the long time existence and uniqueness are proved in [45], and is one of the main results of this thesis. Here we need to combine the methods used in the previous points where the domain alters the functional setting, the surface tension induces subprincipal terms in the energy estimates, and we need new pseudodifferential estimates in this context.

We end this section with comments on the dispersive nature of internal water waves. To understand the dispersive properties of the internal water waves equation we follow the presentation of [24] in the case of a single fluid. In particular, it is instructive to consider plane waves solutions of the linearized equations in non-dimensional variables for a flat bottom[3]:

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu}\mathcal{G}_\mu[0,0]\psi = 0 \\ \partial_t \psi + \big((1-\gamma) - \mathrm{bo}^{-1}\partial_x^2\big)\zeta = 0. \end{cases} \tag{1.1.21}$$

Then in order to find the plane wave solutions

$$\zeta = \zeta_0 e^{i(\xi x - \omega_{\mathrm{iww}}(\xi)t)} \quad \text{and} \quad \psi = \psi_0 e^{i(\xi x - \omega_{\mathrm{iww}}(\xi)t)},$$

form the initial data (ζ_0, ψ_0), one can verify from the equation (1.1.21) that we need $i\omega_{\mathrm{iww}}\psi_0 = (1-\gamma)\zeta_0$ and that

$$\omega_{\mathrm{iww}}(\xi)^2 = \big((1-\gamma) + \mathrm{bo}^{-1}\xi^2\big)\frac{\xi}{\sqrt{\mu}}\frac{\tanh(\sqrt{\mu}\xi)}{1+\gamma\tanh(\sqrt{\mu}|\xi|)}. \tag{1.1.22}$$

The quantity $\omega_{\mathrm{iww}}(\xi)$ is known as the *dispersion relation* of the linearized internal water waves. One should also note that (1.1.22) is specific to the fact that we have one layer of infinite depth. Moreover, we define the *phase function* $\theta(x,t) = \xi(x - \frac{\omega_{\mathrm{iww}}(\xi)}{\xi}t)$ and the *phase speed* by

$$c_{\mathrm{p,iww}}^2 = \frac{\omega_{\mathrm{iww}}(\xi)^2}{\xi^2},$$

and it reveals the dispersive character of the waves (see Figure 2). Indeed, if we suppose bo^{-1} is small then since the speed depends on $|\xi| \sim \frac{1}{\lambda}$ we see that waves travel faster for long waves. Meaning that if we take a wave packet comprised of different frequencies, the short frequency part (long wave) will travel faster than the large frequencies (short waves), therefore causing the wave to spread out.

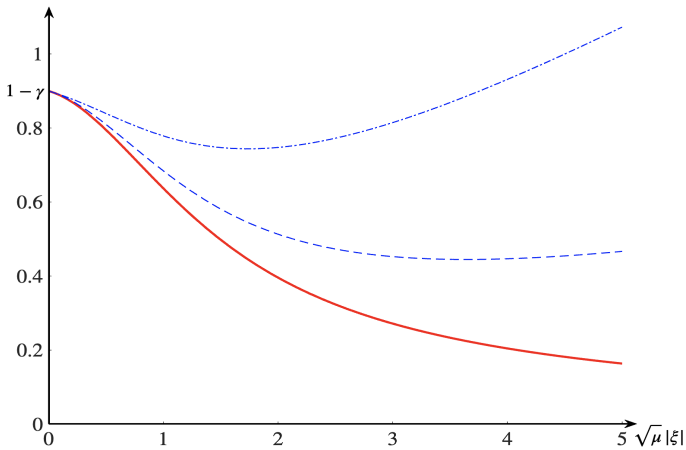


Figure 2: A plot of $c_{\mathrm{p,iww}}^2$ in the case $\mathrm{bo}^{-1} = 0$ (red solid line), $\mathrm{bo}^{-1} \ll 1$ (blue dased line), and $\mathrm{bo}^{-1} \sim 1$ (blue dashed-dot line).

Note that the effect of surface tension is only relevant for large frequencies (short waves). But as noted above, it is necessary for the internal water waves to be well-posed. This is consistent with results in points 3 and 5 mentioned in Remark 1.1.11, where adding a small amount of surface tension allowed for long time existence results without affecting the dynamics of long waves.

Remark 1.1.12. There are several asymptotic models that capture the dispersive nature of the internal water wave equations. We comment briefly on some important results. We first consider classical examples in the

[3] For a nonflat bottom the linearized Dirichlet-Neumann operators are much more complicated.

case of a single fluid and flat bottom that will help to put the main results of this thesis in perspective. In this case, we know that the solutions of the water waves equations exist on a long time scale (see previous point 1. in Remark 1.1.11). Therefore, to fully justify the asymptotic model under consideration, its solutions should exist on the same time scale and satisfy a convergence estimate.

1. One of the most classical example is the KdV equation [15, 37]:
$$\partial_t\zeta + (1+\frac{\mu}{6}\partial_x^2)\partial_x\zeta + \frac{3\varepsilon}{2}\zeta\partial_x\zeta = 0,$$
which describes the evolution of waves propagating in one direction. The first rigorous derivation of this model was given by Craig [17] where the precision[4] of the model with respect to the water waves equation is $\mathcal{O}(\mu^2+\varepsilon\mu)$.

2. For waves propagating in two directions, Bona-Chen-Saut [13] derived a family of Boussinesq systems:
$$\begin{cases}(1-b\mu\partial_x^2)\partial_t\zeta + (1+a\mu\partial_x^2)\partial_x v + \varepsilon\partial_x(\zeta v) = 0\\ (1-d\mu\partial_x^2)\partial_t v + (1+c\mu\partial_x^2)\partial_x\zeta + \varepsilon v\partial_x v = 0,\end{cases} \tag{1.1.23}$$
where a, b, c, and d are real parameters satisfying $a+b+c+d=\frac{1}{3}$ and $v(x,t)\in\mathbb{R}$ approximates the fluid velocity at some height in the fluid domain. The precision of the system is the same as for the KdV and was proved later by [12] in some cases of $a+b+c+d=\frac{1}{3}$. It is worth noting that the long-time well-posedness of (1.1.23) is far from trivial and has undergone extensive studies: [43, 47–50].

3. The higher-order extension of the Boussinesq models is the Green-Naghdi equation. For simplicity we consider the following version [51, 54]:
$$\begin{cases}\partial_t\zeta + \partial_x(hv) = 0\\ (1+\mu\mathcal{T}[h,\beta b])\big(\partial_t v + \varepsilon v\partial_x v\big) + \partial_x\zeta + \mu\varepsilon\mathcal{Q}[h,v] = 0,\end{cases} \tag{1.1.24}$$
where $h = 1+\varepsilon\zeta$ is the fluid depth in the case of a flat bottom and with
$$\mathcal{T}[h]v = -\frac{1}{3h}\partial_x\big(h^3\partial_x v\big), \qquad \mathcal{Q}[h,v] = \frac{2}{3h}\partial_x\big(h^3(\partial_x v)^2\big).$$
Comparing its solutions with the ones of the water waves equations evolving from the same data, one obtains a precision of order $\mathcal{O}(\mu^2)$ [8, 42]. In other words, it is an improvement from the Boussinesq equations in the case where nonlinear effects are dominant (i.e., when $\mu \ll \varepsilon$).

The next results are some generalizations of the models mentioned above and that preserve the dispersive properties of the linearized water wave equations.

4. An important example for modeling unidirectional waves in shallow water is the Whitham equation [56]:
$$\partial_t\zeta + \sqrt{\mathrm{F}_1}(\mathrm{D})\partial_x\zeta + \frac{3\varepsilon}{2}\zeta\partial_x\zeta = 0, \tag{1.1.25}$$
where $\sqrt{\mathrm{F}_1}(\mathrm{D})$ is a Fourier multiplier defined as the square root of the symbol of $\mathrm{F}_1(\mathrm{D})$:
$$\mathrm{F}_1(\mathrm{D})f(x) = \mathcal{F}^{-1}\Big(\frac{\tanh(\sqrt{\mu}\xi)}{\sqrt{\mu}\xi}\widehat{f}(\xi)\Big)(x).$$
The first rigorous result, justifying the model, was given by Klein, Linares, Pilod, and Saut [35], where they compared its solution rigorously with those of the KdV equation to obtain the same precision. More recently, this result was improved by Emerald [28] where the precision[5] of (1.1.25) is of order $\mathcal{O}(\varepsilon\mu)$ when compared to the water waves equation. In other words, it is exact at the linear level and we therefore call it a *full dispersion* model.

[4] Actually, Craig restricted the parameters to be $\varepsilon=\mu$. The general precision was remarked in [28].
[5] This is for well-prepared initial data, see [28] for the precise statement.

5. There are several full dispersion versions of the Boussinesq systems. As an example, consider the generalization of (1.1.23) in the case $(a,b,c,d) = (0,0,\frac{1}{3},0)$ [32]:

$$\begin{cases} \partial_t \zeta + \partial_x (hv) = 0 \\ \partial_t v + \mathrm{F}_1(\mathrm{D})\partial_x \zeta + \varepsilon v \partial_x v = 0. \end{cases} \tag{1.1.26}$$

The system was rigorously derived in [27] in the sense of consistency. Moreover, combining this result with the long time well-posedness result in [44] implies the full justification of the model. The precision of (1.1.26) is of order $\mathcal{O}(\varepsilon\mu)$ when compared to the water waves equations.

6. A full dispersion version of the Green-Naghdi equations (1.1.24) is derived in [27] in the sense of consistency, where we need to replace $\mathcal{T}$ and $\mathcal{Q}$ by:

$$\mathcal{T}_{\mathrm{F}_2}[h]v = -\frac{1}{3h}\partial_x \sqrt{\mathrm{F}_2}(\mathrm{D})\big(h^3\sqrt{\mathrm{F}_2}(\mathrm{D})\partial_x v\big), \quad \mathcal{Q}_{\mathrm{F}_2}[h,v] = \frac{2}{3h}\partial_x \sqrt{\mathrm{F}_2}(\mathrm{D})\big(h^3(\sqrt{\mathrm{F}_2}(\mathrm{D})\partial_x v)^2\big),$$

with

$$\mathrm{F}_2(\mathrm{D}) = \frac{3}{\mu|\mathrm{D}|^2}(1 - \mathrm{F}_1(\mathrm{D})).$$

The precision of the model of order $\mathcal{O}(\varepsilon\mu^2)$, and the full justification as a shallow water model was established in [29]. If we compare the precision with the classical Green-Naghdi, we observe that there is a qualitative gain in the parameters.

Before we turn to the case of the two-fluid system, we can make a formal argument for the difference in the precision of full dispersion models and their classical versions. To do so, we compare the dispersion relation of the models full dispersion models in point 5 and 6, which is equal to the dispersion relation of the water wave system[6], the Green-Naghdi system, and the $(0,0,\frac{1}{3},0)-$Boussinesq system:

$$\frac{\omega_{\mathrm{ww}}(\xi)^2}{\xi^2} = \frac{\tanh(\sqrt{\mu}\xi)}{\sqrt{\mu}\xi}, \qquad \frac{\omega_{\mathrm{gn}}(\xi)^2}{\xi^2} = \frac{1}{1+\frac{\mu}{3}\xi^2}, \qquad \frac{\omega_{\mathrm{b}}(\xi)^2}{\xi^2} = (1 - \frac{\mu}{3}\xi^2).$$

Then we see that the phase speed of the asymptotic models as Taylor expansion that is equal to the phase speed of its full dispersion version up to an order of $\mathcal{O}(\mu^2)$ (see also Figure 3). Meaning in cases where high-frequency interactions are dominant, they would not offer a good description of the behavior of the waves.

Finally, we end this remark by commenting on asymptotic models derived from the two-layer system. Here, there are several possibilities depending on the depths of the two fluids.

5. An application of the long time existence result by Lannes [40], is the derivation of the shallow water equations [18]:

$$\begin{cases} \partial_t \zeta + \partial_x\Big(\frac{h_1 h_2}{\delta + \gamma h_2} v\Big) = 0 \\ \partial_t v + (1-\gamma)\partial_x \zeta + \frac{\varepsilon}{2}\partial_x\Big(\frac{(\delta h_2)^2 - \gamma h_2^2}{(\delta h_1 + \gamma h_2)^2} v^2\Big) = 0, \end{cases}$$

where the depth of the upper fluid is H_1, the depth of the lower fluid H_2, the ratio $\delta = \frac{H_1}{H_2}$, with $h_1 = 1 - \varepsilon\zeta$, $h_1 = 1 + \varepsilon\delta\zeta$, and $v = \partial_x \psi$. The system was derived rigourously by Bona, Lannes, and Saut [14] in the sense of consistency with the precision of order $\mathcal{O}(\mu)$.

6. There are several important consistency results given in [14] (see also [18]). However, the main point to keep in mind, is that the result in [40] is not uniform with respect to the fluid depth. Meaning that if one considers models outside the shallow water regime, then one also has to revisit this result to have uniform estimates, and of course, accompanied by the long time well-posedness of the asymptotic model.

[6] ω_{ww} is given by formula (1.1.22) in the case $\gamma = \mathrm{bo}^{-1} = 0$.

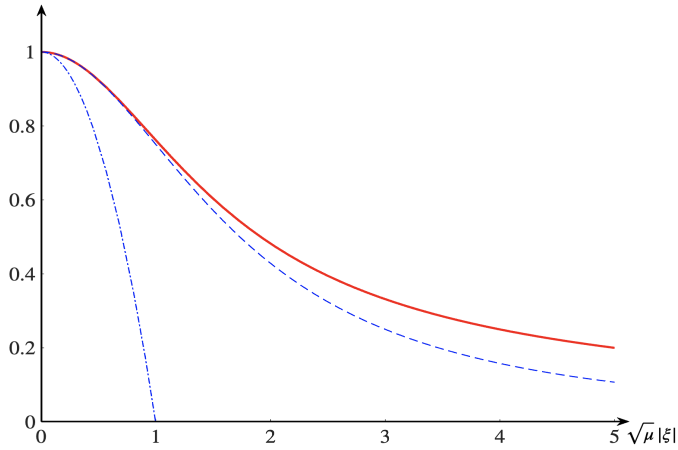


Figure 3: A plot of the phase velocities of the linearized water waves equation (red line), a Green-Naghdi system (blue dashed line), and a Boussinesq system (blue dashed-dot line).

6. In the case of one fluid of infinite depth, given by (1.1.18) one can derive the Benjamin-Ono equation (BO) [11, 22]:
$$\partial_t\zeta + c(1-\frac{\gamma}{2}\sqrt{\mu}|\mathrm{D}|)\partial_x\zeta + c\frac{3\varepsilon}{2}\zeta\partial_x\zeta = 0,$$
where $c^2 = (1-\gamma)$. The rigorous justification of the BO equation is proved in [45]. In particular, the long-time existence of the internal water waves equations (1.1.18) with a small amount of surface tension was established, and the precision of the BO equation is proved to be of order $\mathcal{O}(\mu)$ when compared to the internal water waves equation.

1.2 Main results

1.2.1 Paper 1: Long time well-posedness of Whitham-Boussinesq systems

Published in Nonlinearity [44].

In paper 1, we study the long time well-posedness of three importaint full dispersion systems in one and two horizontal dimensions. These systems are called Whitham-Boussinesq systems which were obtained by improving the dispersion of the surface waves Boussinesq systems. To clarify the result, we consider again (1.1.26) as an example:
$$\begin{cases} \partial_t\zeta + \partial_x(hv) = 0 \\ \partial_t v + \mathrm{F}_1(\mathrm{D})\partial_x\zeta + \varepsilon v\partial_x v = 0, \end{cases}$$
with $h = 1+\varepsilon\zeta$, and as noted in Remark 1.1.12 (point 5). There was previously no local well-posedness result for this system, even on a short time, and the system corresponds to a full dispersion version of the $(0, 0, \frac{1}{3}, 0)-$Boussinesq system, which is believed to be ill-posed [35]. However, in the weakly dispersive case, we proved that system (1.1.26) is well-posedness on the long time scale of order $\mathcal{O}(\frac{1}{\varepsilon})$, for initial data that satisfies the non-cavitation condition. The proof is based on theory for hyperbolic system, where one has to find a suitable energy to handle the dispersive terms. In the case of system (1.1.26), we construct a modified energy functional[7] that are adapted to the dispersive system, that reads:
$$\mathcal{E}_{\mathrm{wh}}(\zeta, v) := \int_{\mathbb{R}} \Big((\sqrt{\mathrm{F}_1}(\mathrm{D})\Lambda_\mu^{\frac{1}{2}}\Lambda^s\zeta)^2 + h(\Lambda_\mu^{\frac{1}{2}}\Lambda^s v)^2 \Big)\,\mathrm{d}x,$$

[7] In the paper [44], the energy was actually given in scaled variables $\varepsilon\zeta$ and εv.

where $\Lambda_\mu^{\frac{1}{2}}$ is the scaled Bessel potential defined by the symbol $\xi \mapsto (1+\mu\xi^2)^{\frac{1}{4}}$ in frequency. With this energy, we were able to prove through a careful analysis of commutators and product estimates involving the multipliers above, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\mathrm{wh}}(\zeta,v) \lesssim \varepsilon\big(\mathcal{E}_{\mathrm{wh}}(\zeta,v)\big)^{\frac{3}{2}}. \tag{1.2.1}$$

Then with the equivalence

$$\mathcal{E}_{\mathrm{wh}}(\zeta,v) \sim |\zeta|^2_{H^s} + |\Lambda_\mu^{\frac{1}{2}} v|^2_{H^s},$$

and an estimate on the difference of two solutions, a well-posedness result was proved in the sense of Hadamar. See Theorem 1.11 in this paper for the precise statement and for the novelties related to the other systems that were considered.

We end this brief summary with two remarks on the result.

Remark 1.2.1. A consequence of the energy estimates of the paper and the consistency result provided by Emerald in [27] is the full justification of (1.1.26) as a water waves model with precision $\mathcal{O}(\varepsilon\mu)$.

Remark 1.2.2. If we add the effect of the bottom, then we simply need to change the defintion of fluid depth:

$$h = 1 + \varepsilon\zeta - \beta b.$$

Meaning, by replacing h in the equation (1.1.26) would yield a system with precision $\mathcal{O}(\mu(\varepsilon+\beta))$ [24]. Moreover, the contribution of the bottom in the energy estimates is not principal terms since b is a given function. So, it seems straightforward to deduce a long time existence result using the same method. However, in this case, the time of existence is of order $\mathcal{O}(\frac{1}{\max\{\varepsilon,\beta\}})$.

1.2.2 Paper 2: Long time well-posedness of a Whitham-Green-Naghdi system

Joint work with Louis Emerald.
Submitted for publication.

In paper 2, is an extension of the previous work where we consider a weakly dispersive[8] version of the Green-Nagdi system with bathymetry. In particular, we again use the energy method to prove the long time well-posedness of a system on the form

$$\begin{cases} \partial_t\zeta + \partial_x(hv) = 0 \\ (1+\mu\mathcal{T}_{\mathrm{F}_2}[h,\beta b])\big(\partial_t v + \varepsilon v\partial_x v\big) + \partial_x\zeta + \mu\varepsilon(\mathcal{Q}_{\mathrm{F}_2}[h,v] + \mathcal{Q}_{b,\mathrm{F}_2}[h,b,v]) = 0. \end{cases} \tag{1.2.2}$$

Here $h = 1+\varepsilon\zeta - \beta b$ is the height of the fluid and

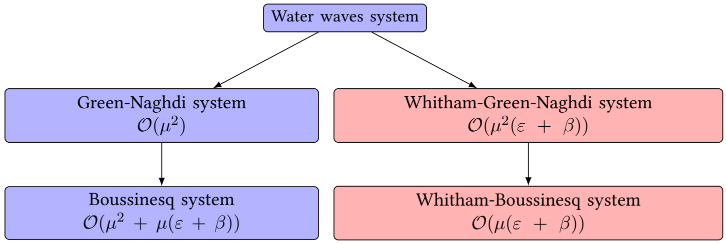
$$\mathcal{T}_{\mathrm{F}_2}[h,\beta b]v, \quad \mathcal{Q}_{\mathrm{F}_2}[h,v], \quad \mathcal{Q}_{b,\mathrm{F}_2}[h,\beta b,v], \tag{1.2.3}$$

are complicated expressions that depend on the nonlocal operators

$$\mathrm{F}_1(\mathrm{D}) = \frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{\sqrt{\mu}|\mathrm{D}|}, \qquad \mathrm{F}_2(\mathrm{D})f = \frac{3}{\mu|\mathrm{D}|^2}(1-\mathrm{F}_1(\mathrm{D})).$$

The system is a similar version to a system derived by Duchêne in [24] in the sense of consistency. The precision of the weakly dispersive Green-Naghdi system with bottom effect was proved to be of order $\mathcal{O}(\mu^2(\varepsilon+\beta))$ when compared to the water waves equations. To put the result in context, we have the following comparisons in the case of bathymetry with the models mentioned so far:

[8] It is not fully dispersive since ω_{ww} was computed in the case $\beta = 0$. When $\beta > 0$, this is highly nontrivial.



To derive system (1.2.2), there are many possibilities that give the same precision. On the other hand, it is not clear which one is well-posed. In fact, the novelty of the proof was to determine the quantities (1.2.3), in the weakly dispersive case, such that we could prove an energy estimate of the type (1.2.1). From this estimate, we deduce the well-posedness of the system with an existence time of order $t = \mathcal{O}(\frac{1}{\max\{\varepsilon,\beta\}})$. Then we use this result, to compare the solutions with the ones of the water waves equations and prove that the two solutions are close up to an error of order $\mathcal{O}(\mu^2(\varepsilon + \beta))$ on the same time interval.

Remark 1.2.3. The main motivation behind studying full dispersion models is that it allows us to decouple the parameters in the precision (see discussion in Remark 1.1.12). This, of course, implies a larger regime for which the models are applicable.

1.2.3 Paper 3: Rigorous derivation of weakly dispersive shallow water models with large amplitude topography variations

Joint work with Louis Emerald.
Submitted for publication.

In this paper, we derive new shallow water models in one and two horizontal dimensions with improved precision with respect to μ and the bathymetry parameter β. The motivation stems from Remark 1.2.3, where we want to derive a model that describes waves propagating over an obstacle. This is an important modeling problem (see, for instance, Chapter 5, Section 2.3 in [41]) and is based on the experimental results by Dingemans [23] (see illustration of the set-up in Figure 4). In these experiments, it was observed that waves long waves tend to steepen as they would pass an obstacle, where high frequency waves are generated behind the obstacle.

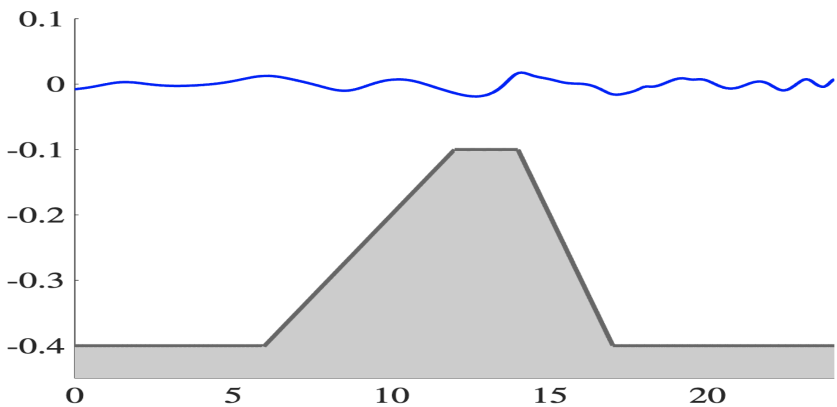


Figure 4: A long wave propagating from left to right towards an obstacle.

In other words, on one side of the obstacle we need a good description of long waves ($\mu, \varepsilon \ll 1$), on top of the obstacle we enter the shallow water regime ($\mu \ll 1$). While after the waves pass, we want a good description of the weakly nonlinear regime ($\varepsilon \ll 1$).

The first result of this paper is the rigorous derivation of models with uneven bathymetries at the order of precision $\mathcal{O}(\mu\varepsilon)$. In the one-dimensional case, this model reads

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu}\mathcal{G}_b \psi = 0 \\ \partial_t \psi + \zeta + \dfrac{\varepsilon}{2}(\partial_x \psi)^2 = 0, \end{cases} \tag{1.2.4}$$

where $\mathcal{G}_b$ is an operator given by

$$\frac{1}{\mu}\mathcal{G}_b \psi = -\partial_x \Big(\frac{1 + \varepsilon\zeta - \beta b}{1 - \beta b} \int_{-1+\beta b}^{0} \partial_x \Phi \, \mathrm{d}z \Big),$$

and is defined by the solutions of the elliptic problem

$$\begin{cases} \Delta^{\mu}_{x,z}\Phi = 0 \ \text{ in } \ \mathbb{R} \times [-1 + \beta b, 0], \\ \Phi|_{z=0} = \psi, \ \ \big[\partial_z \Phi - \mu\beta \partial_x b \partial_x \Phi\big]\big|_{z=-1+\beta b} = 0. \end{cases} \tag{1.2.5}$$

System (1.2.4) can be viewed as an extension of the full dispersion models with uneven bathymetry, where the elliptic problem (1.2.5) corresponds to the $(1.1.20)^{+}$ in the case $\varepsilon = 0$. A drawback of this model is that it depends on the solutions of an elliptic problem. This is costly from a numerical perspective.

To simplify system (1.2.4), we instead find approximate solutions of (1.2.5) given in terms of pseudodifferential operators. Then we use them to rigorously derive new models in the sense of consistency with precision:

$$\mathcal{O}(\mu\varepsilon + \mu^2\beta^2) \quad \text{and} \quad \mathcal{O}(\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2).$$

1.2.4 Paper 4: Justification of the Benjamin-Ono equation as an internal water waves model

Submitted for publication.

This paper gives the first rigorous justification of the Benjamin-Ono equation:

$$\partial_t \zeta + c(1 - \frac{\gamma}{2}\sqrt{\mu}|\mathrm{D}|)\partial_x \zeta + c\frac{3\varepsilon}{2}\zeta\partial_x\zeta = 0, \tag{1.2.6}$$

as an internal water wave model on the physical time scale. The BO equation was derived formally in the 60s as an asymptotic model from a two-fluid system where one fluid is of great depth [11,22]. The equation has generated much interest since its inception. However, it is still an open question whether its solutions are close to the ones of the original physical system.

The primary step is to prove the long-time existence of the internal water wave equations with one fluid of infinite depth and the second fluid layer of finite depth and flat bottom. Then, we show that the difference between two regular solutions of the internal water waves equations and the BO equation, which evolves from the same initial datum, provides a good and stable approximation to the system on the natural time scale. In particular, there are regular solutions[9] (ζ, ψ) solving (1.1.18) on a positive time interval with $t = \mathcal{O}(\frac{1}{\varepsilon})$. Moreover, from the same data, we have a unique solution of (1.2.6) denoted by ζ^{BO}, and we prove that

$$|\zeta - \zeta^{\mathrm{bo}}|_{L^\infty} \lesssim \mu t,$$

on a long time where the implicit constant depends on the norm of the initial data in the energy space (see Theorem 1.18 in [45] for the precise statement).

[9]Of course, with $\beta = 0$ and under certain assumption on the data. See for instance Remark 1.1.10 on this point.

The novelties of the proof are related to the geometry of the problem, where having one fluid of finite depth and one of infinite depth alters the functional setting for the Dirichlet-Neumann operators involved. A simple argument was put forward in Observation 1.1.7 and Remark 1.1.11 on this point. As a result, we study the various compositions of the operators involved in the expression of (1.1.19). This requires a refined symbolic analysis of the Dirichlet-Neumann operator on infinite depth and deriving new pseudo-differential estimates, where we extend the results by Lannes [40] in the case of two fluids in the shallow water regime.

Remark 1.2.4. As noted in point 6. in Remark 1.1.12, there are several asymptotic regimes that are not captured by the result of Lannes [40]. In particular, the case of having one layer of great depth. An interesting example would be the intermediate long wave regime where one layer is allowed to be larger than the other (see [14, 36]).

Chapter 2

Scientific Results

Paper I

2.1 Long time well-posedness of Whitham-Boussinesq systems

M. O. Paulsen.

Published in Nonlinearity [44].

LONG TIME WELL-POSEDNESS OF WHITHAM-BOUSSINESQ SYSTEMS

MARTIN OEN PAULSEN

Abstract. Consideration is given to three different full dispersion Boussinesq systems arising as asymptotic models in the bi-directional propagation of weakly nonlinear surface waves in shallow water. We prove that, under a non-cavitation condition on the initial data, these three systems are well-posed on a time scale of order $\mathcal{O}(\frac{1}{\varepsilon})$, where ε is a small parameter measuring the weak non-linearity of the waves. For one of the systems, this result is new even for short time. The two other systems involve surface tension, and for one of them, the non-cavitation condition has to be sharpened when the surface tension is small. The proof relies on suitable symmetrizers and the classical theory of hyperbolic systems. However, we have to track the small parameters carefully in the commutator estimates to get the long time well-posedness.

Finally, combining our results with the recent ones of Emerald provide a full justification of these systems as water wave models in a larger range of regimes than the classical (a, b, c, d)-Boussinesq systems.

1. Introduction

1.1. **Full dispersion models.** The Korteweg-de Vries (KdV) equation is an asymptotic model for the unidirectional propagation of small amplitude, long waves on the surface of an ideal fluid of constant depth. It was introduced in [8, 32] to model the propagation of solitary waves in shallow water with a wide range of applications both mathematically and physically. However, its dispersion is too strong in high frequencies when compared to the full water wave system. In particular, the KdV equation does not feature wave breaking or peaking waves. To overcome these shortcomings, Whitham introduced in [52] an equation with an improved dispersion relation. He replaced the KdV dispersion with the exact dispersion of the linearized water wave system obtaining the equation

$$\partial_t \zeta + \sqrt{\mathcal{K}_\mu}(\mathrm{D})\partial_x \zeta + \varepsilon \zeta \partial_x \zeta = 0, \tag{1.1}$$

for $(x, t) \in \mathbb{R} \times \mathbb{R}^+$, where the function $\zeta(x, t) \in \mathbb{R}$ denotes the surface elevation and the operator $\sqrt{\mathcal{K}_\mu}(\mathrm{D})$ is the square root of the Fourier multiplier $\mathcal{K}_\mu(\mathrm{D})$ defined in frequency by

$$K_\mu(\xi) = \frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}\left(1 + \sigma\mu|\xi|^2\right). \tag{1.2}$$

Moreover, μ and ε are small parameters related to the level of dispersion and nonlinearity, and σ is a nonnegative parameter related to the surface tension[1,2].

Whitham conjectured in [52] that equation (1.1) would allow, in addition to the KdV traveling-wave regime, the occurrence of waves of greatest height with a sharp crest as well as the formation of shocks. However, it was not until recently that these phenomena

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Key words and phrases. Whitham-Boussinesq; Long time well-posedness; Symmetrizers.

[1] Actually, Whitham introduced the equation formally without the parameters μ and ε.

[2] He also did not include surface tension, *i.e.* $\sigma = 0$.

were rigorously proved. We mention among others the existence of periodic waves [18], the existence and stability of traveling waves [17, 4, 48, 27], the formation of shocks [24, 45], Benjamin-Feir instabilities [47, 25], the existence of periodic waves of greatest height [20] and solitary waves of greatest height [50]. Note that in the case of surface tension ($\sigma > 0$), the dynamics appear to be rather different (see e.g. [31] and the references therein).

These results illustrate some mathematical properties uniquely related to an improved dispersion relation, though there are some phenomena that the Whitham equation does not feature due to its unidirectionality. For instance, the Euler equations admit non-modulational instabilities of small-amplitude periodic traveling waves [36], but the unidirectional nature of the Whitham equation is believed to prohibit such instabilities [10].

Regarding the two-way propagation of waves at the surface of a fluid and in the long wave regime, Bona, Chen, and Saut derived a three-parameter family of Boussinesq systems [5]

$$\begin{cases}(1-b\mu\partial_x^2)\partial_t\zeta + (1+a\mu\partial_x^2)\partial_x v + \varepsilon\partial_x(\zeta v) = 0\\ (1-d\mu\partial_x^2)\partial_t v + (1+c\mu\partial_x^2)\partial_x\zeta + \varepsilon v\partial_x v = 0,\end{cases} \tag{1.3}$$

where a, b, c and d are real parameters satisfying $a+b+c+d=\frac{1}{3}$, $\zeta(x,t)\in\mathbb{R}$ is the deviation of the free surface with respect to its rest state, and $v(x,t)\in\mathbb{R}$ approximates the fluid velocity at some height in the fluid domain. Like the KdV equation, the Boussinesq systems are celebrated models for surface waves in coastal oceanography. Analogously to the unidirectional case, one could replace the dispersion with the linearized dispersion of the water wave equations in (1.3). These improved dispersion versions are expected to lead to a more "accurate" description of the full water wave system. Those systems are commonly referred to as the Whitham-Boussinesq systems or full dispersion Boussinesq systems.

Actually, there are different possibilities of full dispersion Boussinesq systems. This paper will focus on three important ones, linking them to some specific cases of the Boussinesq systems without BBM terms ($b=d=0$). To be precise, we introduce the operator $\mathcal{T}_\mu(\mathrm{D})$ corresponding to $\mathcal{K}_\mu(\mathrm{D})$ for $\sigma=0$, and whose Fourier symbol is defined by

$$T_\mu(\xi) = \frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}. \tag{1.4}$$

First, we consider the system

$$\begin{cases}\partial_t\zeta + \mathcal{K}_\mu(\mathrm{D})\partial_x v + \varepsilon\partial_x(\zeta v) = 0\\ \partial_t v + \partial_x\zeta + \varepsilon v\partial_x v = 0,\end{cases} \tag{1.5}$$

introduced in [33, 1, 38] without surface tension and in [31] with surface tension. Here, as above, ζ denotes the elevation of the surface around its equilibrium position, while v approximates the fluid velocity at the free surface. We also consider its two-dimensional counterpart

$$\begin{cases}\partial_t\zeta + \mathcal{K}_\mu(\mathrm{D})\nabla\cdot\mathbf{v} + \varepsilon\nabla\cdot(\zeta\mathbf{v}) = 0\\ \partial_t\mathbf{v} + \nabla\zeta + \frac{\varepsilon}{2}\nabla|\mathbf{v}|^2 = \mathbf{0},\end{cases} \tag{1.6}$$

where $x\in\mathbb{R}^2$ and $\mathbf{v}(x,t)\in\mathbb{R}^2$ approximates the fluid velocity at the surface in two space dimensions. In the case zero surface tension, it is proved that (1.5) models solitary waves [39] and admit high-frequency (non-modulational) instabilities of small-amplitude periodic traveling waves [19]. We also observe that (1.5) is related to (1.3) by expanding (1.2) in low frequencies. Indeed, since $K_\mu(\xi)\simeq 1+\mu(\sigma-\frac{1}{3})\xi^2$ by a Taylor expansion we see that (1.5) reduce to (1.3) with $(a,b,c,d)=(\frac{1}{3}-\sigma,0,0,0)$.

A second system is obtained by applying the operator (1.4) to $\partial_x\zeta$, which gives

$$\begin{cases}\partial_t\zeta + \partial_x v + \varepsilon\partial_x(\zeta v) = 0\\ \partial_t v + \mathcal{T}_\mu(\mathrm{D})\partial_x\zeta + \varepsilon v\partial_x v = 0,\end{cases} \tag{1.7}$$

and in two dimensions reads

$$\begin{cases}\partial_t\zeta + \nabla\cdot\mathbf{v} + \varepsilon\nabla\cdot(\zeta\mathbf{v}) = 0\\ \partial_t\mathbf{v} + \mathcal{T}_\mu(\mathrm{D})\nabla\zeta + \frac{\varepsilon}{2}\nabla|\mathbf{v}|^2 = \mathbf{0}.\end{cases} \tag{1.8}$$

This system was first introduced in [26], where it is proved that (1.7) features Benjamin-Feir (modulational) instabilities. Note that while ζ plays the same role as for system (1.5), it is $\mathcal{T}_\mu^{-1}(\mathrm{D})v$ which approximates the velocity potential at the free surface in this case (and $\mathcal{T}_\mu^{-1}(\mathrm{D})\mathbf{v}$ in two dimensions). We also observe that (1.7) reduces in the formal limit $\sqrt{\mu}|\xi| \to 0$ to the Boussinesq system (1.3) with $(a,b,c,d) = (0,0,\frac{1}{3},0)$ in low frequencies.

Finally, we will also consider a full dispersion version of (1.3) when $\mathcal{T}_\mu(\mathrm{D})$ is applied to the nonlinear terms, while $\mathcal{K}_\mu(\mathrm{D})$ is applied on the $\partial_x\zeta$. This system reads

$$\begin{cases}\partial_t\zeta + \partial_x v + \varepsilon\mathcal{T}_\mu(\mathrm{D})\partial_x(\zeta v) = 0\\ \partial_t v + \mathcal{K}_\mu(\mathrm{D})\partial_x\zeta + \varepsilon\mathcal{T}_\mu(\mathrm{D})(v\partial_x v) = 0,\end{cases} \tag{1.9}$$

and in two dimensions is given by

$$\begin{cases}\partial_t\zeta + \nabla\cdot\mathbf{v} + \varepsilon\mathcal{T}_\mu(\mathrm{D})\nabla\cdot(\zeta\mathbf{v}) = 0\\ \partial_t v + \mathcal{K}_\mu(\mathrm{D})\nabla\zeta + \frac{\varepsilon}{2}\mathcal{T}_\mu(\mathrm{D})\nabla|\mathbf{v}|^2 = \mathbf{0}.\end{cases} \tag{1.10}$$

Here ζ and v play the same roles as for system (1.7) (similarly, $\mathbf{v}$ has the same role as in (1.8)). It was introduced in [13] and has the advantage of being Hamiltonian. Moreover, the existence of solitary waves is proved in [14].

1.2. **Full justification.** A fundamental question in the derivation of an asymptotic model is whether its solution converges to the solution of the original physical system. In particular, we say that an asymptotic model is a valid approximation of the Euler equations with a free surface if we can answer the following points in the affirmative [33]:

1. The solutions of the water wave equations exist on the relevant scale $\mathcal{O}(\frac{1}{\varepsilon})$.
2. The solutions of the asymptotic model exist (at least) on the scale $\mathcal{O}(\frac{1}{\varepsilon})$.
3. Lastly, we must establish the *consistency* between the asymptotic model and the water wave equations, and then show that the error is of order $\mathcal{O}(\mu\varepsilon t)$ when comparing the two solutions.

The first point was proved by Alvarez-Samaniego and Lannes [2] for surface gravity waves and Ming, Zhang and Zhang [37] for gravity-capillary waves in the weakly transverse regime, while points 2. and 3. are specific to the asymptotic model under consideration. For instance, in the case of the Whitham equation, Klein *et al.* [31] compared its solution rigorously with those of the KdV equation. In particular, they proved that the difference between two solutions evolving from the same initial datum is bounded by $\mathcal{O}(\varepsilon^2 t)$ for all $0 \leq t \lesssim \varepsilon^{-1}$ with ε, μ in the KdV-regime:

$$\mathcal{R}_{KdV} = \{(\varepsilon,\mu) \,:\, 0 \leq \mu \leq 1, \quad \mu = \varepsilon\},$$

which justified the Whitham equation as a water wave model in this regime by relying on the justification of the KdV equation [9, 33].

On the other hand, due to the improved dispersion relation of (1.1), Emerald [22] was able to decouple the parameters (ε, μ) and prove an error estimate between the Whitham equation and the water wave system with a precision $\mathcal{O}(\mu\varepsilon t)$ for $0 \leq t \lesssim \varepsilon^{-1}$ in the shallow water regime:

$$\mathcal{R}_{SW} = \{(\varepsilon, \mu) \,:\, 0 \leq \mu \leq 1, \quad 0 \leq \varepsilon \leq 1\}. \tag{1.11}$$

Moreover, Emerald decoupled the small parameters for the KdV equation and proved its precision to be $\mathcal{O}(\mu^2 + \mu\varepsilon)t$ for $0 \leq t \lesssim \varepsilon^{-1}$. Consequently, the Witham equation is valid for a larger set of small parameters when compared to the KdV equation. Specifically, when $\varepsilon \ll \mu$, these estimates imply that (1.1) equation is a better approximation of the water wave equations.

In the case of the Boussinesq systems (1.3), consistency was first proved in [6] for $(\varepsilon, \mu) \in \mathcal{R}_{KdV}$ by relying on intermediate symmetric systems for which the long time well-posedness follows by classical arguments. However the long time well-posedness for the (a, b, c, d) Boussinesq is far from trivial. This result was proved[3] later by Saut, Xu and Wang [42, 46]. The proof relies on suitable symmetrizers and hyperbolic theory.

The natural next step is to consider the Whitham-Boussinesq systems for $(\varepsilon, \mu) \in \mathcal{R}_{SW}$. *In particular, the goal of this paper is to establish the well-posedness of* (1.5)-(1.10)*, with uniform bounds, on time intervals of size* $\mathcal{O}(\frac{1}{\varepsilon})$. Since point 1. of the justification is already established, the long-time existence and consistency remain. Using the method of Emerald, one can prove the consistency of any Whitham-Boussinesq system with the water wave system (see also [21] for other full dispersion shallow water models). Therefore, having the long time well-posedness theory for (1.5)-(1.10) will provide the final step for the full justification of these systems.

1.3. **Former well-posedness results.** Regarding system (1.5) and (1.6), we know from previous studies that surface tension plays a fundamental role in the well-posedness theory. In fact, when $\sigma = 0$ the initial value problem associated to system (1.5) is probably ill-posed unless $\zeta > 0$ (see the formal argument in Section 4 in [31]). We refer to [40] for a well-posedness under the non-physical condition $\zeta \geq c_0 > 0$. When surface tension is taken into account, system (1.5) was proved to be locally well-posed by Kalisch and Pilod [28] for $(\zeta, v) \in H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R})$, $s > \frac{5}{2}$ (and $s > 3$ in two dimensions), by using a modified energy method. We also refer to the work by Wang [51] for an alternative proof using a nonlocal symmetrizer. However, it is worth noting that all these well-posedness resutls were proved on a short time without considering the small parameters ε and μ. Finally, in the formal limit $\sqrt{\mu}|\xi| \to 0$, one recovers the Boussinesq system corresponding to $(a, b, c, d) = (\frac{1}{3} - \sigma, 0, 0, 0)$. This system has been proved in [46] to be well-posed on large time for $\sigma > \frac{1}{3}$, while it is known to be ill-posed for $\sigma < \frac{1}{3}$ [3]. This is a formal indication that the threshold $\sigma = \frac{1}{3}$ will play an important role for the long well-posedness of (1.5) and (1.6). We will come back to this issue in the next section (see Figure 1).

As far as we know, there are no well-posedness results for system (1.7) and (1.8) even on short time. In the formal limit $\sqrt{\mu}|\xi| \to 0$, system (1.7) reduces to the Boussinesq system corresponding to $(a, b, c, d) = (0, 0, \frac{1}{3}, 0)$, which is believed to be ill-posed [31].

Next, attention is turned to (1.9) and (1.10). There are several results when $\sigma = 0$. In this case, Dinvay [12] proved short time local well-posedness for $(\zeta, v) \in H^{s+\frac{1}{2}}(\mathbb{R}) \times H^{s+1}(\mathbb{R})$, $s \geq 0$ in the one-dimensional case. The proof is based on standard hyperbolic theory that involves a modified energy similar to [28]. This result was then extended in [15] by exploiting the smoothing effect of the linear flow using dispersive techniques improving the regularity

[3]In the most dispersive case $(a, b, c, d) = (\frac{1}{6}, \frac{1}{6}, 0, 0)$, the relevant time scale $\mathcal{O}(\varepsilon^{-1})$ is still missing; the best results being on a time scale $\mathcal{O}(\varepsilon^{-\frac{2}{3}})$ [43, 44], (see also [35] on a time scale $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$ by using dispersive techniques).

to $H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R})$, $s > -\frac{1}{10}$. Furthermore, when considering small data, the system is globally well-posed due to the control of the Hamiltonian. The estimates derived in the aforementioned papers are not uniform in μ. However, a recent study by Tesfahun [49] proved that the corresponding 2-dimensional system (1.10) without surface tension is well-posed on a time interval of order $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$ in the KdV−regime. Indeed, dispersive techniques are tailored-made for short waves and therefore seem not to be well suited to capture the long wave regime (see for instance [35] for similar results for the Boussinesq system in the KdV-KdV case). Finally, in the case of surface tension $\sigma > 0$, Dinvay proved in [11] the short time local well-posedness of (1.9) and (1.10) by using modified energy techniques. This result also implies the small data global well-posedness in this case.

Lastly,[4] we would like to comment on a recent work by Emerald [23]. Here he considered a class of non-local quasi-linear systems in one and two dimensions that include the following family of Whitham-Boussinesq systems,

$$\begin{cases} \partial_t\zeta + \mathcal{T}_\mu(\mathrm{D})\nabla\cdot\mathbf{v} + \varepsilon(\mathcal{T}_\mu)^\alpha(\mathrm{D})\nabla\cdot(\zeta(\mathcal{T}_\mu)^\alpha(\mathrm{D})\mathbf{v}) = 0 \\ \partial_t\mathbf{v} + \nabla\zeta + \varepsilon((\mathcal{T}_\mu)^\alpha(\mathrm{D})\mathbf{v}\cdot\nabla)((\mathcal{T}_\mu)^\alpha(\mathrm{D})\mathbf{v}) = \mathbf{0}, \end{cases} \tag{1.12}$$

with $\alpha \geq \frac{1}{2}$. In the paper, the author proves the long time well-posedness of (1.12), and deomonstrate that the error between the water wave system is of order $\mathcal{O}(\mu\varepsilon t)$. Also, note that in the case $\alpha = 0$, then (1.12) corresponds to system (1.6) in the case $\sigma = 0$. This case is still an open problem. However, combining the results of [23] with the ones in this paper, accounts for many of the possible Whitham-Boussinesq systems, and thus complete each other well.

1.4. **Main results.** In the current paper, we take into account the small parameters (ε, μ) and prove the well-posedness of (1.5), (1.7), (1.9), and their two-dimensional versions, on a time scale $\mathcal{O}(\frac{1}{\varepsilon})$.

In the case of systems (1.7)-(1.8) and (1.9)-(1.10), we will work under the standard non-cavitation condition.

Definition 1.1 (Non-cavitation condition)**.** *Let $d = 1$ or 2 with $s > \frac{d}{2}$ and $\varepsilon \in (0,1)$. We say the initial surface elevation $\zeta_0 \in H^s(\mathbb{R}^d)$ satisfies the "non-cavitation condition" if there exist $h_0 \in (0,1)$ such that*

$$1 + \varepsilon\zeta_0(x) \geq h_0, \quad \text{for all } x \in \mathbb{R}^d. \tag{1.13}$$

In the case of system (1.5) and (1.6), we will distinguish between the cases $\sigma \geq \frac{1}{3}$ and $0 < \sigma < \frac{1}{3}$. More precisely, for $\sigma \geq \frac{1}{3}$, we will also assume the non-cavitation condition in Definition 1.1, while for $0 < \sigma < \frac{1}{3}$, we have to impose the following $\sigma-$dependent surface condition.

Definition 1.2 ($\sigma-$dependent surface condition)**.** *Let $d = 1$ or 2 with $s > \frac{d}{2}$, $\varepsilon \in (0,1)$ and $\sigma \in (0, \frac{1}{3})$. We say the initial surface elevation $\zeta_0 \in H^s(\mathbb{R}^d)$ satisfy the "$\sigma-$dependent surface condition" if*

$$1 + \varepsilon\zeta_0(x) \geq h_\sigma, \quad \text{for all } x \in \mathbb{R}^d, \tag{1.14}$$

where $h_\sigma = 1 - \frac{\sigma}{2}$.

Remark 1.3. *For $0 < \sigma < \frac{1}{3}$, $K_\mu(\xi)$ is not a monotone function for positive frequencies, as we can be seen in the figure below. This is why we choose to impose condition (1.14) in this case.*

[4] See also [16] for a survey on resent developments in the field.

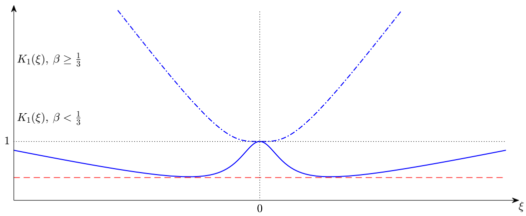

FIGURE 1. The multipiler $K_1(\xi)$ in the case when $\sigma \geq \frac{1}{3}$ (dash-dot) and $\sigma < \frac{1}{3}$ (line). The horizontal line (dashed) specifies the minimum.

Remark 1.4. *One can see the $\sigma-$dependent surface condition as a constraint on the initial data that is related to the minimum of the function $K_\mu(\xi)$. For instance, if we consider the multiplier in Figure 1, then an admissible initial datum must satisfy the constraint in the figure below.*

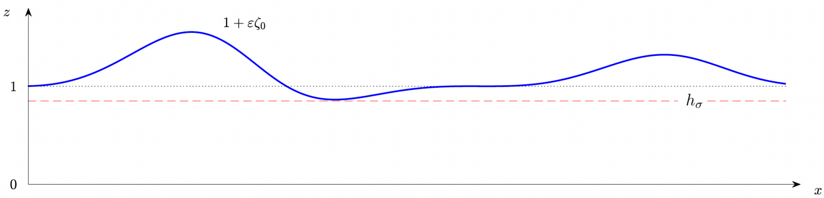

FIGURE 2. The blue line denotes the initial surface elevation $1 + \varepsilon\zeta_0$, and is restricted by h_σ when $0 < \sigma < \frac{1}{3}$.

Before we state the main results, we define a natural solution space for systems (1.5)-(1.6) and (1.7)-(1.8).

Definition 1.5. *We define the norm on the function space $V^s_\mu(\mathbb{R}^d)$ to be*

$$|(\zeta, \mathbf{v})|^2_{V^s_\mu} := |\zeta|^2_{H^s} + |\mathbf{v}|^2_{H^s} + \sqrt{\mu}|\mathrm{D}^{\frac{1}{2}}\mathbf{v}|^2_{H^s}.$$

Theorem 1.6. *Let $d = 1$ or 2 with $s > \frac{d}{2} + \frac{3}{2}$, $\sigma > 0$ and $\varepsilon, \mu \in (0, 1)$. Assume that $(\zeta_0, \mathbf{v}_0) \in V^s_\mu(\mathbb{R}^d)$ satisfies either the non-cavitation condition (1.13) in the case $\sigma \geq 1/3$ or the $\sigma-$dependent surface condition (1.14) in the case $0 < \sigma < \frac{1}{3}$, where curl $\mathbf{v}_0 = \mathbf{0}$ if $d = 2$. Moreover, we assume that*

$$0 < \varepsilon \leq \frac{1}{k^2_\sigma |(\zeta_0, \mathbf{v}_0)|_{V^s_\mu}} \quad \textit{for} \quad k^2_\sigma = \begin{cases} \frac{c}{\sigma} & \textit{for} \ \ 0 < \sigma < \frac{1}{3} \\ c\sigma & \textit{for} \ \ \sigma \geq \frac{1}{3} \end{cases} \tag{1.15}$$

for some $c > 0$. Then there exists a positive T given by

$$T = \frac{1}{k^1_\sigma |(\zeta_0, \mathbf{v}_0)|_{V^s_\mu}} \quad \textit{with} \quad k^1_\sigma = \begin{cases} \frac{c}{\sigma} & \textit{for} \ 0 < \sigma < \frac{1}{3} \\ c\sigma^2 & \textit{for} \ \sigma \geq \frac{1}{3} \end{cases} \tag{1.16}$$

such that (1.5) and (1.6) admits a unique solution

$$(\zeta, \mathbf{v}) \in C([0, T/\varepsilon] : V^s_\mu(\mathbb{R}^d)) \cap C^1([0, T/\varepsilon] : V^{s-\frac{3}{2}}_\mu(\mathbb{R}^d)),$$

that satisfies

$$\sup_{t\in[0,T/\varepsilon]} |(\zeta,\mathbf{v})|_{V^s_\mu} \lesssim |(\zeta_0,\mathbf{v}_0)|_{V^s_\mu}. \tag{1.17}$$

Furthermore, there exists a neighborhood $\mathcal{U}$ of $(\zeta_0,\mathbf{v}_0)$ such that the flow map

$$F^s_{T,\varepsilon,\mu} : V^s_\mu(\mathbb{R}^d) \ \to \ C([0,\tfrac{T}{2\varepsilon}]; V^s_\mu(\mathbb{R}^d)), \quad (\zeta_0,\mathbf{v}_0) \ \mapsto \ (\zeta,\mathbf{v}),$$

is continuous.

Remark 1.7. *The proof of the continuous dependence on long time of order $\mathcal{O}(\frac{1}{\varepsilon})$ seems to be new for Boussinesq type systems. It relies on the Bona-Smith argument* [7] *and could be easily adapted for the (a,b,c,d)-Boussinesq systems.*

Remark 1.8. *A heuristic argument can be made to argue that the physical solutions appear when the initial data is of order one in terms of ε* [41]. *To illustrate this point, take the Burgers equation*

$$u_t - \varepsilon u u_x = 0,$$

a simple model that can describe an inviscid fluid in shallow water theory. Then by the energy method, it is easy to deduce that the time of existence is of order $T \sim \frac{1}{\varepsilon|u_0|_{H^s}}$ for $s > \frac{3}{2}$. As a consequence, we have that $T \sim \frac{1}{\varepsilon}$ if the initial data is of size $\mathcal{O}_\varepsilon(1)$.

Remark 1.9. *If $\sigma \sim 1$ then $\varepsilon \lesssim 1$ by* (1.15)*, and so* (1.16) *implies that $T/\varepsilon \sim 1/\varepsilon$. On the other hand, in the case of having $\sigma \ll 1$,* (1.15) *would impose $\varepsilon \lesssim \sigma$, and by* (1.16) *we have the existence on the timescale $T/\varepsilon \sim \sigma/\varepsilon$.*

Remark 1.10. *Regarding the $\sigma-$dependent surface condition, we demonstrate that the solution will persist for a long time and satisfy $\varepsilon\zeta(x,t) \geq -c\sigma$ for some constant $c > 0$. One should also note that this is coherent since $0 < \varepsilon \lesssim \sigma$ as explained in the previous remark. For a related discussion on this physical condition see Subsection 1.3.*

Next, we state a well-posedness result for (1.7) and (1.8). These systems does not feature any surface tension and is well-posed for a long time under the standard non-cavitation condition.

Theorem 1.11. *Let $d = 1$ or 2 with $s > \frac{d}{2}+1$ and $\mu \in (0,1)$. Assume that $(\zeta_0, v_0) \in V^s_\mu(\mathbb{R})$ satisfies the non-cavitation condition* (1.13)*, where $\operatorname{curl}\mathbf{v}_0 = \mathbf{0}$ if $d = 2$. Also assume that for some $c > 0$ that $0 < \varepsilon \leq c\big(|(\zeta_0,\mathbf{v}_0)|_{V^s_\mu}\big)^{-1}$. Then there exists $T = c\big(|(\zeta_0,\mathbf{v}_0)|_{V^s_\mu}\big)^{-1}$ such that* (1.7) *and* (1.8) *admits a unique solution*

$$(\zeta,\mathbf{v}) \in C([0,T/\varepsilon] : V^s_\mu(\mathbb{R}^d)) \cap C^1([0,T/\varepsilon] : V^{s-1}_\mu(\mathbb{R}^d)),$$

that satisfies

$$\sup_{t\in[0,T/\varepsilon]} |(\zeta,\mathbf{v})|_{V^s_\mu} \lesssim |(\zeta_0,\mathbf{v}_0)|_{V^s_\mu}.$$

In addition, the flow map is continuous with respect to the initial data.

Remark 1.12. *As far as we know, Theorem 1.11 is the first well-posedness result for systems* (1.7)-(1.8).

Similarly, we can combine the techniques used to prove Theorem 1.6 and Theorem 1.11 to establish the long time well-posedness of (1.9)-(1.10) in the space:

Definition 1.13. *Define the norm on the function space $X^s_{\sigma,\mu}(\mathbb{R}^d)$ to be*

$$|(\zeta,\mathbf{v})|^2_{X^s_{\sigma,\mu}} := |\zeta|^2_{H^s} + \sigma\mu|\mathrm{D}^1\zeta|^2_{H^s} + |\mathbf{v}|^2_{H^s} + \sqrt{\mu}|\mathrm{D}^{\frac{1}{2}}\mathbf{v}|^2_{H^s}.$$

Theorem 1.14. *Let $d = 1$ or 2 with $s > \frac{d}{2} + 1$, $\sigma \geq 0$ and $\mu \in (0,1)$. Assume that $(\zeta_0, \mathbf{v}_0) \in X^s_{\sigma,\mu}(\mathbb{R})$ satisfies the non-cavitation condition* (1.13)*, where curl $\mathbf{v}_0 = \mathbf{0}$ if $d = 2$. Also assume that for some $c > 0$ that $0 < \varepsilon \leq c\big(|(\zeta_0, \mathbf{v}_0)|_{X^s_{\sigma,\mu}}\big)^{-1}$. Then there exists $T = c\big(|(\zeta_0, \mathbf{v}_0)|_{X^s_{\sigma,\mu}}\big)^{-1}$ such that* (1.9) *and* (1.10) *admits a unique solution*

$$(\zeta, \mathbf{v}) \in C([0, T/\varepsilon] : X^s_{\sigma,\mu}(\mathbb{R}^d) \cap C^1([0, T/\varepsilon] : X^{s-1}_{\sigma,\mu}(\mathbb{R}^d)),$$

that satisfies

$$\sup_{t\in[0,T/\varepsilon]} |(\zeta, \mathbf{v})|_{X^s_{\sigma,\mu}} \lesssim |(\zeta_0, \mathbf{v}_0)|_{X^s_{\sigma,\mu}}.$$

In addition, the flow map is continuous with respect to the initial data.

Remark 1.15. *Including $\sigma > 0$ in the norm in the definition of $X^s_{\sigma,\mu}(\mathbb{R}^d)$ will allow us to obtain a long time well-posedness result under the non-cavitation condition. Additionally, when $0 < \sigma < \frac{1}{3}$ then ε is independent from the surface tension parameter, and in the case $\sigma = 0$ we have that $X^s_{0,\mu}(\mathbb{R}^d)$ is equal to $V^s_\mu(\mathbb{R}^d)$.*

Remark 1.16. *For the sake of clarity, we will mainly focus on the one-dimensional case. Theorems 1.6, 1.11 and 1.14 can be easily extended to the 2-dimensional case by following the same methods since the symbols $\mathcal{K}_\mu(\mathrm{D})$ and $\mathcal{T}_\mu(\mathrm{D})$ are radial. We give a brief outline of what would be the main changes in Section 6.*

1.5. **Strategy and outline.** The proof of Theorem 1.6 relies mainly on energy estimates similar to the ones provided in [28] on a fixed time. Though, we use the idea of Wang [51], who included the nonlocal operator $\mathcal{K}_\mu(\mathrm{D})$ in the definition of the energy[5]:

Definition 1.17. *Let $(\eta, u) = \varepsilon(\zeta, v)$ and Λ^s be the bessel potential of order $-s$. Then we define the energy associated to* (1.5) *in the one-dimensional case to be:*

$$E_s(\eta, u) := \int_{\mathbb{R}} \left((\Lambda^s \eta)^2 + \eta(\Lambda^s u)^2 + (\sqrt{\mathcal{K}_\mu}(\mathrm{D})\Lambda^s u)^2 \right) dx.$$

This energy formulation will free us to cancel out specific nonlinear terms that appear naturally in the computations yielding the estimate

$$\frac{d}{dt} E_s(\eta, u) \lesssim_\sigma \big(E_s(\eta, u)\big)^{\frac{3}{2}}. \tag{1.18}$$

Combined with the coercivity of the energy, then by a standard bootstrap argument, one deduces a solution with the lifespan of $T_0 = \mathcal{O}(\frac{1}{\varepsilon})$. We refer the reader to Proposition 3.1 and Lemma 5.3 for these results. The proof of the energy estimate is similar to the one presented in [51], but we keep track of the small parameters. We should also note that estimate (1.18) is applied to a regularized version of (1.5), where we recover the original system using a Bona-Smith argument.

To run the Bona-Smith argument for $s > 2$, one classically needs to estimate the difference between two solutions at the $V^0_\mu(\mathbb{R})-$level. These estimates will be the most technical point of the paper and are specific to the dependence of the small parameters. In short, the technical difficulty is related to the apparent need for 'generalized' Kato-Ponce type commutator estimates on $\mathcal{K}_\mu(\mathrm{D})$ (see Lemma 2.9 and the generalization for $\mathcal{K}_\mu(\mathrm{D})$ in Lemma 2.11). Whereas for the case $\mu = 1$, one can use Calderón type estimates to simplify $\mathcal{K}_\mu(\mathrm{D})$ directly (see [28] and the reformulated system (2.1)). The main idea will be to split $\mathcal{K}_\mu(\mathrm{D})$

[5] Wang actually used this multiplier in the case $\mu = 1$.

in high and low frequencies, and then derive new commutator estimates that allow us to obtain the necessary order of μ in the estimates related to the energy.

For the proof of Theorem 1.11, we follow the same strategy, but in this case, the dispersion operator (1.4) is regularizing. The trick will be to introduce a scaled Bessel potential in the energy, allowing us to mimic the properties of (1.2). The energy is given by:

Definition 1.18. *Let $(\eta, u) = \varepsilon(\zeta, v)$ and $\Lambda_\mu^{\frac{1}{2}}$ be the scaled Bessel potential defined by the symbol $\xi \mapsto (1+\mu\xi^2)^{\frac{1}{4}}$ in frequency. Then the energy associated to (1.7) in the one-dimensional case reads:*

$$\mathcal{E}_s(\eta,u) := \int_{\mathbb{R}} \Big((\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac{1}{2}}\Lambda^s\eta)^2 + (1+\eta)(\Lambda_\mu^{\frac{1}{2}}\Lambda^s u)^2 \Big)\, dx.$$

The energy formulated in Definition 1.18 is new and will require commutator estimates specific to the equation. This will, in turn, allow us to decouple the parameters μ and ε in the estimates and, by extension, provide an estimate in the form of (1.18).

In the same spirit, we define a modified energy for system (1.9):

Definition 1.19. *Let $(\eta, u) = \varepsilon(\zeta, v)$ and $\sigma > 0$. Then the energy associated to (1.9) in the one-dimensional case reads:*

$$\mathscr{E}_s(\eta,u) := \int_{\mathbb{R}} \Big((\Lambda^s\eta)^2 + \sigma\mu(\mathrm{D}^1\Lambda^s\eta)^2 + \eta(\Lambda^s u)^2 + (\sqrt{\mathcal{T}}_\mu^{-1}(\mathrm{D})\Lambda^s u)^2 \Big)\, dx.$$

Note also that the energy includes the surface tension parameter σ and will allow us to deduce an estimate on the form (1.18), where the coercivity estimate will be uniform in σ. In turn, this will provide the long time well-posedness for $\sigma \ll 1$ and $T/\varepsilon \sim 1/\varepsilon$ as pointed out in Remark 1.15.

The paper is organized as follows. In Section 2, we introduce some important technical results whose proofs will be postponed to the appendix. In the same section, we also present new commutator estimates needed to treat the nonlinear terms when estimating the energy in Sections 3 and 4. Then we conclude in Section 5 by combining the results obtained in the former sections to prove Theorem 1.6 in full detail in the one-dimensional case. Lastly, we comment briefly on the changes to adapt the proof in the two-dimensional setting, while the proof of Theorem 1.11 and Theorem 1.14 will follow by the same arguments.

1.6. **Notation.**

- We let c denote a positive constant independent of μ, ε that may change from line to line. Also, as a shorthand, we use the notation $a \lesssim b$ to mean $a \leq c\, b$. Similarly, if the constant depends on σ, we write $a \lesssim_\sigma b$. In particular, we define the constants depending on σ,

$$c_\sigma^1 = \begin{cases} c\sigma & \text{for } 0<\sigma<\frac{1}{3} \\ c & \text{for } \sigma \geq \frac{1}{3} \end{cases} \quad \text{and} \quad c_\sigma^2 = \begin{cases} c & \text{for } 0<\sigma<\frac{1}{3} \\ c\sigma & \text{for } \sigma \geq \frac{1}{3} \end{cases} \tag{1.19}$$

- Let $(V, |\cdot|_V)$ be a vector space. Then for $\alpha \geq 0$, $\lambda > 0$ and $f_\lambda \in V$ be a function depending on λ, we define the "big$-\mathcal{O}$" notation to be

$$|f_\lambda|_V = \mathcal{O}(\lambda^\alpha) \iff \lim_{\lambda\to 0} \lambda^{-\alpha}|f_\lambda|_V < \infty.$$

Similarly, we define the "small$-o$" notation to be

$$|f_\lambda|_V = o(\lambda^\alpha) \iff \lim_{\lambda\to 0} \lambda^{-\alpha}|f_\lambda|_V = 0.$$

- Let $L^2(\mathbb{R})$ be the usual space of square integrable functions with norm $|f|_{L^2} = \sqrt{\int_{\mathbb{R}} |f(x)|^2\, dx}$. Also, for any $f, g \in L^2(\mathbb{R})$ we denote the scalar product by $\big(f,g\big)_{L^2} = \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx$.
- For any tempered distribution f, the operator $\mathcal{F}$ denoting the Fourier transform, applied to f, will be written as $\hat{f}(\xi)$ or $\mathcal{F}f(\xi)$.
- Let $m : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then we will use the notation $m(\mathrm{D})$ for a multiplier defined in frequency by $\widehat{m(\mathrm{D})f}(\xi) = m(\xi)\hat{f}(\xi)$.
- For any $s \in \mathbb{R}$ we call the multiplier $\widehat{\mathrm{D}^s f}(\xi) = |\xi|^s \hat{f}(\xi)$ the Riesz potential of order $-s$. One should note that $\mathrm{D}^1 = \mathcal{H}\partial_x$, where $\widehat{\mathcal{H}f}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$ is the Hilbert transform.
- For any $s \in \mathbb{R}$ we call the multiplier $\Lambda^s = (1 + \mathrm{D}^2)^{\frac{s}{2}} = \langle \mathrm{D}\rangle^s$ the Bessel potential of order $-s$. Moreover, the Sobolev space $H^s(\mathbb{R})$ is equivalent to the weighted L^2-space; $|f|_{H^s} = |\Lambda^s f|_{L^2}$. We also find it convenient to define $\Lambda_\mu^{\frac{1}{2}}$ which is a multiplier assosiated to the symbol:
$$\mathcal{F}(\Lambda_\mu^{\frac{1}{2}} f)(\xi) = (1 + \mu\xi^2)^{\frac{1}{4}}\hat{f}(\xi). \tag{1.20}$$
- We say f is a Schwartz function $\mathscr{S}(\mathbb{R})$, if $f \in C^\infty(\mathbb{R})$ and satisfies for all $\alpha, \beta \in \mathbb{N}$,
$$\sup_x |x^\alpha \partial_x^\beta f| < \infty.$$
- If A and B are two operators, then we denote the commutator between them to be $[A, B] = AB - BA$.

2. Preliminary results

2.1. Pointwise estimates. The first result concerns the properties of the dispersive part of the equation. Namely, we deduce pointwise estimates for the multipliers (1.2) and (1.4) that are needed to obtain the coercivity of the energy (see, for instance, equation (3.7) below). Moreover, these estimates will prove essential when dealing with the nonlinear parts of the equation that appear in the energy estimates.

Lemma 2.1. *Let $\mu \in (0,1)$. Then we have the following pointwise estimates on the kernel $K_\mu(\xi)$:*

- *For $\sigma \geq 0$, we have the upper bound*
$$K_\mu(\xi) \lesssim 1 + \sigma(1 + \sigma\sqrt{\mu}|\xi|). \tag{2.1}$$
- *If $\sigma \geq \frac{1}{3}$, then for all $h_0 \in (0,1)$ we have the lower bound*
$$K_\mu(\xi) \geq (1 - \frac{h_0}{2}) + c\sqrt{\mu}|\xi|, \tag{2.2}$$
whereas, if $0 < \sigma < \frac{1}{3}$, we have the lower bound
$$K_\mu(\xi) \geq \sigma + c\sigma\sqrt{\mu}|\xi|. \tag{2.3}$$
- *The derivative of the symbol $K_\mu(\xi)$ satisfies*
$$\left|\frac{d}{d\xi}\sqrt{K_\mu(\xi)}\right| \lesssim \langle\xi\rangle^{-1} + \sqrt{\sigma}\mu^{\frac{1}{4}}\langle\xi\rangle^{-\frac{1}{2}}. \tag{2.4}$$
- *We have the following comparison of $\sqrt{\mathcal{K}_\mu(\xi)}$ by*
$$\big|\sqrt{K_\mu(\xi)} - \sqrt{\sigma}\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\big| \lesssim \sqrt{\sigma} + \sigma. \tag{2.5}$$

- *There holds*

$$\sqrt{K_\mu(\xi)}\langle\xi\rangle^{s-1}|\xi| \lesssim (\sqrt{\sigma}+\sigma)\langle\xi\rangle^{s} + \sqrt{\sigma}\mu^{\frac{1}{4}}\langle\xi\rangle^{s}|\xi|^{\frac{1}{2}}. \tag{2.6}$$

Remark 2.2. *For inequality* (2.3)*, it is crucial to specify the dependence in* σ *as it will provide the coercivity of the energy when* $0<\sigma<\frac{1}{3}$*. The same is true for* (2.2)*, whose importance will be revealed in the proof of Proposition 3.1 below. Though, we note that* (2.3) *does not agree with* (2.2) *when* $\sigma=\frac{1}{3}$*. This is because the lower bound in* (2.3) *is not optimal, but it does not play a role in the overall result.*

Remark 2.3. *We also trace the dependence in* σ *for the first pointwise estimate* (2.1)*, and it will sometimes be replaced with* c_σ^2 *given by* (1.19)*. This constant will again appear when we prove the energy estimates which will provide the size of the time of existence (see Lemma 5.3 in the proof Theorem 1.6).*

The proof of Lemma 2.1 is technical and postponed to the Appendix in Section A.2. A corollary of Proposition 2.1 may now be stated.

Corollary 2.4. *Take* $f\in\mathscr{S}(\mathbb{R})$*,* $\mu\in(0,1)$ *and* $s\in\mathbb{R}$*. Then in the case* $\sigma\geq\frac{1}{3}$ *and for all* $h_0\in(0,1)$ *we have*

$$(1-\frac{h_0}{2})|f|_{H^s}^2 + c\sqrt{\mu}|\mathrm{D}^{\frac{1}{2}}f|_{H^s}^2 \leq |\sqrt{\mathcal{K}_\mu}(\mathrm{D})f|_{H^s}^2 \leq c_\sigma^2|f|_{H^s}^2 + c\sigma\sqrt{\mu}|\mathrm{D}^{\frac{1}{2}}f|_{H^s}^2. \tag{2.7}$$

Similarly, in the case $0<\sigma<\frac{1}{3}$ *there holds*

$$\sigma|f|_{H^s}^2 + c\sigma\sqrt{\mu}|\mathrm{D}^{\frac{1}{2}}f|_{H^s}^2 \leq |\sqrt{\mathcal{K}_\mu}(\mathrm{D})f|_{H^s}^2 \leq c_\sigma^2|f|_{H^s}^2 + c\sqrt{\mu}|\mathrm{D}^{\frac{1}{2}}f|_{H^s}^2. \tag{2.8}$$

Proof. The upper bound in (2.7) follows by Plancherel's identity and the pointwise estimate (2.1), while the lower bound is a consequence of (2.2).

In the same way, for $0<\sigma<\frac{1}{3}$, then (2.8) is deduced from (2.3). □

Similarly, we state some useful pointwise estimates on $T_\mu(\xi)$ and the scaled Bessel potential $\Lambda_\mu^{\frac{1}{2}}$, where the proof is presented in Appendix A.2.

Lemma 2.5. *Let* $\mu\in(0,1)$*. Then we have the following pointwise estimates on the kernel* $T_\mu(\xi)$ *:*

- *For all* $h_0\in(0,1)$ *there holds*

$$(1-\frac{h_0}{2}) + c\sqrt{\mu}|\xi| \leq (T_\mu(\xi))^{-1} \lesssim 1+\sqrt{\mu}|\xi|. \tag{2.9}$$

- *There holds*

$$1\lesssim T_\mu(\xi)\langle\sqrt{\mu}\xi\rangle \lesssim 1. \tag{2.10}$$

- *For* $s\in\mathbb{R}$ *there holds*

$$\left|\frac{d}{d\xi}\langle\xi\rangle^s\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}}\right| \lesssim \langle\xi\rangle^{s-1}\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}}. \tag{2.11}$$

- *For* $s\in\mathbb{R}$ *there holds*

$$\left|\frac{d}{d\xi}\sqrt{T_\mu(\xi)}\langle\xi\rangle^s\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}}\right| \lesssim \langle\xi\rangle^{s-1}. \tag{2.12}$$

- *There holds*

$$\left|\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}} - \mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\right| \lesssim 1. \tag{2.13}$$

A direct consequence of the above estimates can now be given.

Corollary 2.6. *Let $f \in \mathscr{S}(\mathbb{R})$, $\mu \in (0,1)$, $s \in \mathbb{R}$ and $c > 0$. Then for all $h_0 \in (0,1)$ there holds*

$$|\sqrt{\mathcal{T}_\mu}(\mathrm{D})f|_{L^2} \leq |f|_{L^2}. \tag{2.14}$$

$$(1 - \frac{h_0}{2})|f|_{H^s}^2 + c\sqrt{\mu}|\mathrm{D}^{\frac{1}{2}}f|_{H^s}^2 \leq |\sqrt{\mathcal{T}_\mu}^{-1}(\mathrm{D})f|_{H^s}^2 \lesssim |f|_{H^s}^2 + c\sqrt{\mu}|\mathrm{D}^{\frac{1}{2}}f|_{H^s}^2. \tag{2.15}$$

$$|f|_{H^s} \lesssim |\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac{1}{2}}f|_{H^s} \lesssim |f|_{H^s}. \tag{2.16}$$

$$|f|_{H^s}^2 + \sqrt{\mu}|\mathrm{D}^{\frac{1}{2}}f|_{H^s}^2 \lesssim |\Lambda_\mu^{\frac{1}{2}}f|_{H^s}^2 \lesssim |f|_{H^s}^2 + \sqrt{\mu}|\mathrm{D}^{\frac{1}{2}}f|_{H^s}^2. \tag{2.17}$$

2.2. **Commutator estimates.** To handle derivatives in the nonlinear parts of the equations, we need commutator estimates on $\mathcal{K}_\mu(\mathrm{D})$ and $\mathcal{T}_\mu(\mathrm{D})$.

Lemma 2.7. *Let $f, g \in \mathscr{S}(\mathbb{R})$, $\mu \in (0,1)$, $s \geq 1$, and $t_0 > \frac{1}{2}$. Then we have the following commutator estimate*

$$\begin{aligned}|[\sqrt{\mathcal{K}_\mu}(\mathrm{D})\Lambda^s, f]\partial_x g|_{L^2} \lesssim (c_\sigma^2|f|_{H^s} + \sqrt{\sigma}\mu^{\frac{1}{4}}|\mathrm{D}^{\frac{1}{2}}f|_{H^s})|\partial_x g|_{H^{t_0}} \\ + (c_\sigma^2|g|_{H^s} + \sqrt{\sigma}\mu^{\frac{1}{4}}|\mathrm{D}^{\frac{1}{2}}g|_{H^s})|\partial_x f|_{H^{t_0}}.\end{aligned} \tag{2.18}$$

In the high regularity setting, the proof will follow the same lines as in [51], but we track the dependence in μ and σ using the pointwise estimates above.

Proof. First, write the commutator as a bilinear form:

$$\big|[\sqrt{\mathcal{K}_\mu}(\mathrm{D})\Lambda^s, f]\partial_x g\big|_{L^2} = \Big|\int_{\mathbb{R}} \Big(\sqrt{K_\mu(\xi)}\langle\xi\rangle^s - \sqrt{K_\mu(\rho)}\langle\rho\rangle^s\Big)\hat{f}(\xi-\rho)\widehat{\partial_x g}(\rho)\,d\rho\Big|_{L^2_\xi}.$$

Then if $a = \min\{\xi,\rho\}$ and $b = \max\{\xi,\rho\}$, we can use the mean value theorem, leaving us to estimate the following terms

$$\Big|\sqrt{K_\mu(\xi)}\langle\xi\rangle^s - \sqrt{K_\mu(\rho)}\langle\rho\rangle^s\Big| \lesssim \sup_{\omega\in(a,b)}|m(\omega)|\,|\xi-\rho|,$$

where

$$m(\omega) = m_1(\omega) + m_2(\omega) = \langle\omega\rangle^s\frac{d}{d\omega}\sqrt{K_\mu(\omega)} + \langle\omega\rangle^{s-1}\sqrt{K_\mu(\omega)}.$$

But using (2.5) to estimate $m_1(\omega)$ and (2.4) to treat $m_2(\omega)$, we deduce

$$m(\omega) \lesssim c_\sigma^2\langle\omega\rangle^{s-1} + \sqrt{\sigma}\mu^{\frac{1}{4}}\langle\omega\rangle^{s-1}|\omega|^{\frac{1}{2}}, \tag{2.19}$$

where the upper bound is increasing for $s \geq 1$. Therefore an upper bound is attained at $|\rho|$ or $|\xi| \leq |\xi-\rho| + |\rho|$. In particular, if $\omega = |\xi-\rho|$ then we may conclude by Minkowski integral inequality, the Cauchy-Schwarz inequality and (2.19) that

$$\begin{aligned}\big|[\sqrt{\mathcal{K}_\mu}(\mathrm{D})\Lambda^s, f]\partial_x g\big|_{L^2} &\lesssim c_\sigma^2\Big|\int_{\mathbb{R}}\langle\xi-\rho\rangle^{s-1}|\xi-\rho|\,|\hat{f}(\xi-\rho)|\,|\widehat{\partial_x g}(\rho)|\,d\rho\Big|_{L^2_\xi} \\ &\quad + \sqrt{\sigma}\mu^{\frac{1}{4}}\Big|\int_{\mathbb{R}}\langle\xi-\rho\rangle^{s-1}|\xi-\rho|^{\frac{1}{2}}|\xi-\rho|\,|\hat{f}(\xi-\rho)|\,|\widehat{\partial_x g}(\rho)|\,d\rho\Big|_{L^2_\xi} \\ &\lesssim (c_\sigma^2|f|_{H^s} + \sqrt{\sigma}\mu^{\frac{1}{4}}|\mathrm{D}^{\frac{1}{2}}f|_{H^s})|\partial_x g|_{H^{t_0}},\end{aligned}$$

for $t_0 > \frac{1}{2}$. On the other hand, if $\omega = |\rho|$, then we make a change of coordinates and argue similarly to deduce,

$$\begin{aligned}\big|[\sqrt{\mathcal{K}_\mu}(\mathrm{D})\Lambda^s, f]\partial_x g\big|_{L^2} &\lesssim c_\sigma^2 \Big| \int_{\mathbb{R}} \langle \xi-\nu\rangle^{s-1} |\widehat{\partial_x g}(\xi-\nu)|\,|\nu|\,|\hat{f}(\nu)|\,d\nu\Big|_{L^2_\xi} \\ &\quad + \sqrt{\sigma}\mu^{\frac{1}{4}} \Big| \int_{\mathbb{R}} \langle \nu\rangle^{s-1} |\xi-\nu|^{\frac{1}{2}} |\widehat{\partial_x g}(\xi-\nu)|\,|\nu|\,|\hat{f}(\nu)|\,d\nu\Big|_{L^2_\xi} \\ &\lesssim (c_\sigma^2|g|_{H^s} + \sqrt{\sigma}\mu^{\frac{1}{4}}|\mathrm{D}^{\frac{1}{2}}g|_{H^s})|\partial_x f|_{H^{t_0}}.\end{aligned}$$

Adding the two scenarios, we may conclude that (2.18) holds.

□

We will also need a commutator estimates on $\mathcal{T}_\mu(\mathrm{D})$ and $\Lambda_\mu^{\frac{1}{2}}$.

Lemma 2.8. *Let $f, g \in \mathscr{S}(\mathbb{R})$, $s \geq 1$, $t_0 > \frac{1}{2}$, $\mu \in (0,1)$ and $\Lambda_\mu^{\frac{1}{2}}$ as defined in (1.20).*

- *Then we have a Kato-Ponce type estimate*

$$\begin{aligned}|[\Lambda^s\Lambda_\mu^{\frac{1}{2}}, f]\partial_x g|_{L^2} &\lesssim (|f|_{H^s} + \mu^{\frac{1}{4}}|\mathrm{D}^{\frac{1}{2}}f|_{H^s})|\partial_x g|_{H^{t_0}} \\ &\quad + (|g|_{H^s} + \mu^{\frac{1}{4}}|\mathrm{D}^{\frac{1}{2}}g|_{H^s})|\partial_x f|_{H^{t_0}}.\end{aligned} \tag{2.20}$$

- *There holds*

$$|[\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda^s\Lambda_\mu^{\frac{1}{2}}, f]\partial_x g|_{L^2} \lesssim |f|_{H^s}|g|_{H^{t_0+1}} + |f|_{H^{t_0+1}}|g|_{H^s}. \tag{2.21}$$

Proof. The proof is similar to the one of Lemma 2.7 and relies on the pointwise estimates established in Lemma 2.5. Indeed, for (2.20) we define $a_1(\mathrm{D})(f,g) := [\Lambda^s\Lambda_\mu^{\frac{1}{2}}, f]\partial_x g$ and use the mean value theorem combined with (2.11) to deduce

$$\begin{aligned}|\hat{a}_1(\xi)(f,g)| &\leq \int_{\mathbb{R}} \Big|\langle\xi\rangle^s\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}} - \langle\rho\rangle^s\langle\sqrt{\mu}\rho\rangle^{\frac{1}{2}}\Big| |\hat{f}(\xi-\rho)|\,|\widehat{\partial_x g}(\rho)|\,d\rho \\ &\lesssim \int_{\mathbb{R}} \langle\xi-\rho\rangle^{s-1}\langle\sqrt{\mu}(\xi-\rho)\rangle^{\frac{1}{2}}|\xi-\rho|\,|\hat{f}(\xi-\rho)|\,|\widehat{\partial_x g}(\eta)|\,d\rho \\ &\quad + \int_{\mathbb{R}} \langle\rho\rangle^{s-1}\langle\sqrt{\mu}\rho\rangle^{\frac{1}{2}}|\xi-\rho|\,|\hat{f}(\xi-\rho)|\,|\widehat{\partial_x g}(\rho)|\,d\rho.\end{aligned}$$

Then if we apply the $L^2(\mathbb{R})-$norm with respect to ξ, we can argue as in Lemma 2.7 that

$$|\hat{a}_1(\xi)(f,g)|_{L^2_\xi} \lesssim |\Lambda_\mu^{\frac{1}{2}}f|_{H^s}\int_{\mathbb{R}}|\widehat{\partial_x g}(\rho)|\,d\rho + |\Lambda_\mu^{\frac{1}{2}}g|_{H^s}\int_{\mathbb{R}}|\rho|\,|\hat{f}(\rho)|\,d\rho.$$

Then use the definiton of $a_1(\mathrm{D})(f,g)$ and (2.17) to conclude.

The proof of (2.21) is the same, with $a_2(\mathrm{D})(f,g) := [\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda^s\Lambda_\mu^{\frac{1}{2}}, f]\partial_x g$. We use (2.12) to find that

$$\begin{aligned}|\hat{a}_2(\xi)(f,g)| &\leq \int_{\mathbb{R}} \Big|\sqrt{T_\mu(\xi)}\langle\xi\rangle^s\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}} - \sqrt{T_\mu(\rho)}\langle\rho\rangle^s\langle\sqrt{\mu}\rho\rangle^{\frac{1}{2}}\Big| |\hat{f}(\xi-\rho)|\,|\widehat{\partial_x g}(\rho)|\,d\rho \\ &\lesssim \int_{\mathbb{R}} \langle\xi-\rho\rangle^{s-1}|\xi-\rho|\,|\hat{f}(\xi-\rho)|\,|\widehat{\partial_x g}(\rho)|\,d\rho \\ &\quad + \int_{\mathbb{R}} \langle\rho\rangle^{s-1}|\xi-\rho|\,|\hat{f}(\xi-\rho)|\,|\widehat{\partial_x g}(\rho)|\,d\rho,\end{aligned}$$

and the result follows.

□

Next, we state the classical Kato-Ponce commutator estimate. We will use it repeatedly to commute the Bessel potential with functions to obtain the desired energy estimates in the coming sections.

Lemma 2.9 (Kato - Ponce commutator estimates [29]). *Let $s \geq 0$, $p, p_2, p_3 \in (1,\infty)$ and $p_1, p_4 \in (1,\infty]$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. Then*

$$|\Lambda^s(fg)|_{L^p} \lesssim |f|_{L^{p_1}}|\Lambda^s g|_{L^{p_2}} + |\Lambda^s f|_{L^{p_3}}|g|_{L^{p_4}} \tag{2.22}$$

and

$$|[\Lambda^s, f]g|_{L^p} \lesssim |\partial_x f|_{L^{p_1}}|\Lambda^{s-1}g|_{L^{p_2}} + |\Lambda^s f|_{L^{p_3}}|g|_{L^{p_4}}. \tag{2.23}$$

Similar commutator estimates also hold for more general multipliers. In fact, by splitting the frequency domain into two parts using smooth cut-off functions defined in frequency, we can obtain sharper commutator estimates specific to equation (1.5).

Definition 2.10. *We define the smooth cut-off functions $\chi^{(i)} \in \mathscr{S}(\mathbb{R})$ as Fourier multipliers*

$$\mathcal{F}(\chi^{(i)}(\mathrm{D})f)(\xi) = \chi^{(i)}(|\xi|)\hat{f}(\xi),$$

for any $f \in \mathscr{S}(\mathbb{R})$ with the following properties:

$$0 \leq \chi^{(i)}(\xi) \leq 1, \qquad (\chi^{(1)}(\xi))^2 + (\chi^{(2)}(\xi))^2 = 1 \quad on \quad \mathbb{R},$$

and

$$supp\, \chi^{(1)} \subset [-1,1], \quad supp\, \chi^{(2)} \subset \mathbb{R}\backslash\big[-\frac{1}{2},\frac{1}{2}\big].$$

Moreover, we denote the scaled version in μ by $\chi^{(i)}_\mu(\xi) = \chi^{(i)}(\sqrt{\mu}\xi)$.

We have the results:

Lemma 2.11. *Let $s > \frac{3}{2}$, $\mu \in (0,1)$ and $f, g \in \mathscr{S}(\mathbb{R})$.*

- *Let $(\chi^{(1)}_\mu\sqrt{\mathcal{K}_\mu})(\mathrm{D})$ be the multiplier of the symbol $(\chi^{(1)}_\mu\sqrt{K_\mu})(\xi)$. Then*

$$|(\chi^{(1)}_\mu\sqrt{\mathcal{K}_\mu})(\mathrm{D})f|_{L^2} \lesssim_\sigma |f|_{L^2}, \tag{2.24}$$

  and

$$|[(\chi^{(1)}_\mu\sqrt{\mathcal{K}_\mu})(\mathrm{D}), f]\partial_x g|_{L^2} \lesssim_\sigma |f|_{H^s}|g|_{L^2}. \tag{2.25}$$

- *We define the symbol*

$$\mathrm{F}_{\mu,\frac{1}{2}}(\mathrm{D}) := \Big(\frac{1}{\sqrt{\mu}|\mathrm{D}|} + \sigma\sqrt{\mu}|\mathrm{D}|\Big)^{\frac{1}{2}}. \tag{2.26}$$

  Then

$$|(\chi^{(2)}_\mu\mathrm{F}_{\mu,\frac{1}{2}})(\mathrm{D})f|_{L^2} \lesssim_\sigma |f|_{L^2} + \mu^{\frac{1}{4}}|\mathrm{D}^{\frac{1}{2}}f|_{L^2} \tag{2.27}$$

  and

$$|[(\chi^{(2)}_\mu\mathrm{F}_{\mu,\frac{1}{2}})(\mathrm{D}), f]\partial_x g|_{L^2} \lesssim_\sigma \mu^{\frac{1}{4}}|f|_{H^s}|g|_{H^{\frac{1}{2}}}. \tag{2.28}$$

- *Lastly, we define the symbol $\mathrm{F}_{\mu,0}(\mathrm{D})$ to be*

$$\mathrm{F}_{\mu,0}(\mathrm{D}) := \Big(\frac{1}{\sqrt{\mu}|\mathrm{D}|} + \sigma\sqrt{\mu}|\mathrm{D}| - \mathcal{K}_\mu(\mathrm{D})\Big)^{\frac{1}{2}}. \tag{2.29}$$

  Then

$$|(\chi^{(2)}_\mu\mathrm{F}_{\mu,0})(\mathrm{D})f|_{L^2} \lesssim_\sigma |f|_{L^2} \tag{2.30}$$

  and

$$|[(\chi^{(2)}_\mu\mathrm{F}_{\mu,0})(\mathrm{D}), f]\partial_x g|_{L^2} \lesssim_\sigma |f|_{H^s}|g|_{L^2}. \tag{2.31}$$

The proof is postponed to Appendix A.3, where we also will prove the following commutator estimates at the $L^2(\mathbb{R})-$level:

Lemma 2.12. *Let $s > \frac{3}{2}$, $\mu \in (0,1)$ and $f, g \in \mathscr{S}(\mathbb{R})$.*

- *For the composition of $\sqrt{\mathcal{T}_\mu}(\mathrm{D})$ and $\Lambda_\mu^{\frac{1}{2}}$ there holds,*
$$|[\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac{1}{2}}, f]\partial_x g|_{L^2} \lesssim |f|_{H^s}|g|_{L^2}. \tag{2.32}$$
- *While for the usual Bessel potential there holds,*
$$|[\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda^s, f]\partial_x g|_{L^2} \lesssim |f|_{H^s}|\Lambda^s g|_{L^2}. \tag{2.33}$$
- *Similarly, when the operator Λ^s is the identity, we have*
$$|[\sqrt{\mathcal{T}_\mu}(\mathrm{D}), f]\partial_x g|_{L^2} \lesssim |f|_{H^s}|g|_{L^2}. \tag{2.34}$$
- *The derivative of the following commutator satisfies*
$$|\partial_x[\sqrt{\mathcal{T}_\mu}(\mathrm{D}), f]g|_{L^2} \lesssim |f|_{H^s}|g|_{L^2}. \tag{2.35}$$
- *Lastly, we can commute $\Lambda_\mu^{\frac{1}{2}}$ by*
$$|[\Lambda_\mu^{\frac{1}{2}}, f]\partial_x g|_{L^2} \lesssim |f|_{H^s}|\Lambda_\mu^{\frac{1}{2}} g|_{L^2}. \tag{2.36}$$

2.3. **Classical estimates.** Before turning to the proof of the energy estimates, we state some necessary results that will also be used throughout the paper. First, recall the embeddings (see, for example [34]).

Lemma 2.13 (Sobolev embeddings)**.** *Let $f \in \mathscr{S}(\mathbb{R})$ and $s \in (0, \frac{1}{2})$. Then $H^s(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ with $p = \frac{2}{1-2s}$, and there holds*
$$|f|_{L^p} \lesssim |\mathrm{D}^s f|_{L^2}. \tag{2.37}$$
Moreover, In the case $s > \frac{1}{2}$, then $H^s(\mathbb{R})$ is continuously embedded in $L^\infty(\mathbb{R})$.

We also will use the Leibniz rule for the Riesz potential on multiple occasions.

Lemma 2.14 (Fractional Leibniz rule [30])**.** *Let $\sigma = \sigma_1 + \sigma_2 \in (0,1)$ with $\sigma_i \in [0, \sigma]$ and $p, p_1, p_2 \in (1, \infty)$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, for $f, g \in \mathscr{S}(\mathbb{R})$*
$$|\mathrm{D}^\sigma(fg) - f\mathrm{D}^\sigma g - g\mathrm{D}^\sigma f|_{L^p} \lesssim |\mathrm{D}^{\sigma_1} f|_{L^{p_1}}|\mathrm{D}^{\sigma_2} g|_{L^{p_2}}. \tag{2.38}$$
Moreover, the case $\sigma_2 = 0$, $p_2 = \infty$ is also allowed.

Finally, we recall the following results for the Bona-Smith argument (provided in the classical paper [7]) on the multiplier $\varphi_\delta(\mathrm{D})$ defined by:

Definition 2.15. *Let $\varphi \in \mathscr{S}(\mathbb{R})$ such that $\int \varphi = 1$ and for $\delta > 0$ define the regularization operators $\varphi_\delta(\mathrm{D})$ in frequency by*
$$\forall f \in L^2(\mathbb{R}), \quad \forall \xi \in \mathbb{R}, \quad \widehat{\varphi_\delta f}(\xi) := \varphi(\delta\xi)\hat{f}(\xi),$$
where φ is a real valued and $\varphi(0) = 1$.

We give the version of the regularization estimates as presented in [34] (Proposition 9.1).

Proposition 2.16. *Let $s > 0$, $\delta > 0$ and $f \in \mathscr{S}(\mathbb{R})$. Then*
$$|\varphi_\delta(\mathrm{D})f|_{H^{s+\alpha}} \lesssim \delta^{-\alpha}|f|_{H^s}, \quad \forall \alpha > 0, \tag{2.39}$$
and
$$|\varphi_\delta(\mathrm{D})f - f|_{H^{s-\beta}} \lesssim \delta^\beta |f|_{H^s}, \quad \forall \beta \in [0, s]. \tag{2.40}$$

Moreover, there holds

$$|\varphi_\delta(\mathrm{D})f - f|_{H^{s-\beta}} \underset{\delta\to 0}{=} o(\delta^\beta), \qquad \forall \beta \in [0,s]. \tag{2.41}$$

3. A priori estimates

In this section, we give *a priori* estimates for solutions of the three systems (1.5), (1.7), and (1.9).

3.1. Estimates for system (1.5).

As noted in the introduction, we revisit the energy estimate in [51] to keep track of the parameters σ, ε and μ. For simplicity, we adopt the notation $\mathbf{U} = (\eta, u)^T = \varepsilon(\zeta, v)^T$, where we write (1.5) on the compact form:

$$\partial_t \mathbf{U} + M(\mathbf{U}, \mathrm{D})\mathbf{U} = \mathbf{0}, \tag{3.1}$$

with

$$M(\mathbf{U}, \mathrm{D}) = \begin{pmatrix} u\partial_x & (\mathcal{K}_\mu(\mathrm{D}) + \eta)\partial_x \\ \partial_x & u\partial_x \end{pmatrix}. \tag{3.2}$$

Also, we simplify the notation for the energy given in Definition 1.17 by introducing the symmetrizer

$$Q(\mathbf{U}, \mathrm{D}) = Q^{(1)}(\mathbf{U}, \mathrm{D}) + Q^{(2)}(\mathbf{U}, \mathrm{D}) = \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{K}_\mu(\mathrm{D}) \end{pmatrix}. \tag{3.3}$$

Then the energy given in Definition 1.17 can be rewritten as

$$E_s(\mathbf{U}) = \big(\Lambda^s \mathbf{U}, Q(\mathbf{U}, D)\Lambda^s \mathbf{U}\big)_{L^2}.$$

Proposition 3.1. *Let $s > 2$, $\varepsilon, \mu \in (0,1)$ and $(\eta, u) = \varepsilon(\zeta, v) \in C([0,T_0]; V^s_\mu(\mathbb{R}))$ be a solution to (3.1) on a time interval $[0,T_0]$ for some $T_0 > 0$. Moreover, assume there exist $h_0 \in (0,1)$ and $h_1 > 0$ such that*

$$h_0 - 1 \leq \eta(x,t), \quad \forall (x,t) \in \mathbb{R}\times[0,T_0] \quad \textit{and} \quad \sup_{t\in[0,T_0]} |(\eta,u)|_{H^s\times H^s} \leq h_1, \tag{3.4}$$

when $\sigma \geq \frac{1}{3}$, and that

$$-\frac{\sigma}{2} \leq \eta(x,t), \quad \forall (x,t) \in \mathbb{R}\times[0,T_0] \quad \textit{and} \quad \sup_{t\in[0,T_0]} |(\eta,u)|_{H^s\times H^s} \leq h_1, \tag{3.5}$$

when $0 < \sigma < \frac{1}{3}$.

Then, for the energy given in Definition 1.17 and c^i_σ defined by (1.19),

$$\frac{d}{dt}E_s(\mathbf{U}) \leq c^2_\sigma \left(E_s(\mathbf{U})\right)^{\frac{3}{2}}, \tag{3.6}$$

for all $0 < t < T_0$, and

$$c^1_\sigma |(\eta,u)|^2_{V^s_\mu} \leq E_s(\mathbf{U}) \leq c^2_\sigma |(\eta,u)|^2_{V^s_\mu}, \tag{3.7}$$

for all $0 < t < T_0$.

Remark 3.2. *Note that we aim to prove (3.6) with power $\frac{3}{2}$ on the right-hand side. This result will prove essential in getting the time of existence $T \sim \frac{1}{\varepsilon}$ in the proof of Theorem 1.6. One should also note that if we have (3.7), then it is enough to show*

$$\frac{d}{dt}E_s(\mathbf{U}) \lesssim_\sigma |(\eta,u)|^3_{V^s_\mu},$$

to obtain (3.6). With this in mind, in the proof of the proposition, we will repeatedly use assumption (3.4)−(3.5) to discard higher powers in the norm of the solution than 3. Meaning *the terms of form* $|(\eta,u)|^{3+n}_{V^s_\mu}$ *for* $n\in\mathbb{N}$ *will be bounded by* $|(\eta,u)|^3_{V^s_\mu}$ *since this seems to be the best we can hope for when using the current method.*

Proof of Proposition 3.1. We first prove estimate (3.7) in the case $\sigma \geq 1/3$. By definition, we have that

$$E_s(\mathbf{U}) = |\Lambda^s\eta|^2_{L^2} + \big(\Lambda^s u, (\mathcal{K}_\mu(\mathrm{D})+\eta)\Lambda^s u\big)_{L^2}.$$

Thus, as a result of the non-cavitation condition (3.4) and the estimate (2.7), there holds

$$\big(\Lambda^s u, (\mathcal{K}_\mu(\mathrm{D})+\eta)\Lambda^s u\big)_{L^2} \geq \frac{h_0}{2}|u|^2_{H^s} + c\sqrt{\mu}|\mathrm{D}^{\frac12}u|^2_{H^s}.$$

The reverse inequality holds for any $\sigma>0$ and is a consequence of (2.7), Hölder's inequality, the Sobolev embedding with $s>\frac32$, and conditions (3.4)$-$(3.5). Indeed, we observe that

$$E_s(\mathbf{U}) \leq |\eta|^2_{H^s} + |\sqrt{\mathcal{K}_\mu}(\mathrm{D})u|^2_{H^s} + |\eta|_{L^\infty}|u|^2_{H^s} \leq c_\sigma|(\eta,u)|^2_{V^s_\mu}.$$

In the case $0<\sigma<\frac13$, we impose the $\sigma-$dependent surface condition (3.5), leaving less to be absorbed for the coercivity and in conjunction with (2.8). This implies

$$\big(\Lambda^s u, (\mathcal{K}_\mu(\mathrm{D})+\eta)\Lambda^s u\big)_{L^2} \geq \frac{\sigma}{2}|u|^2_{H^s} + c\sqrt{\mu}|\mathrm{D}^{\frac12}u|^2_{H^s}.$$

As a consequence, we have that (3.7) is established for all $\sigma>0$.

Next, we prove (3.6). By using (3.1) and the fact that $Q(\mathbf{U},\mathrm{D})$ is self-adjoint, we compute

$$\begin{aligned}\frac12\frac{d}{dt}E_s(\mathbf{U}) &= \big(\Lambda^s\partial_t\mathbf{U}, Q(\mathbf{U},\mathrm{D})\Lambda^s\mathbf{U}\big)_{L^2} + \frac12\big(\Lambda^s\mathbf{U}, (\partial_t Q(\mathbf{U},\mathrm{D}))\Lambda^s\mathbf{U}\big)_{L^2}\\ &= -\big(\Lambda^s M(\mathbf{U},\mathrm{D})\mathbf{U}, Q(\mathbf{U},\mathrm{D})\Lambda^s\mathbf{U}\big)_{L^2} + \frac12\big(\Lambda^s\mathbf{U}, (\partial_t Q(\mathbf{U},\mathrm{D}))\Lambda^s\mathbf{U}\big)_{L^2}\\ &=: -I + II.\end{aligned}$$

<u>Control of I</u>. We may write

$$\begin{aligned}I &= \big([\Lambda^s, M(\mathbf{U},\mathrm{D})]\mathbf{U}, Q^{(1)}(\mathbf{U},\mathrm{D})\Lambda^s\mathbf{U}\big)_{L^2} + \big(Q^{(1)}(\mathbf{U},\mathrm{D})M(\mathbf{U},\mathrm{D})\Lambda^s\mathbf{U},\Lambda^s\mathbf{U}\big)_{L^2}\\ &\quad+ \big(\Lambda^s M(\mathbf{U},\mathrm{D})\mathbf{U}, Q^{(2)}(\mathbf{U},\mathrm{D})\Lambda^s\mathbf{U}\big)_{L^2}\\ &=: I_1+I_2+I_3.\end{aligned}$$

Control of I_1. It follows from the Cauchy-Schwarz inequality that

$$|I_1| \leq |[\Lambda^s, M(\mathbf{U},\mathrm{D})]\mathbf{U}|_{L^2}|Q^{(1)}(\mathbf{U},\mathrm{D})\Lambda^s\mathbf{U}|_{L^2}.$$

The second term is easily treated,

$$|Q^{(1)}(\mathbf{U},\mathrm{D})\Lambda^s\mathbf{U}|_{L^2} \lesssim |\Lambda^s\eta|_{L^2} + |\eta|_{L^\infty}|\Lambda^s u|_{L^2} \lesssim |(\eta,u)|_{V^s_\mu},$$

by Hölder's inequality, the Sobolev embedding with $s>\frac12$, and assumption (3.4). Furthermore, using the Kato-Ponce commutator estimate (2.23) yields

$$\begin{aligned}|[\Lambda^s, M(\mathbf{U},\mathrm{D})]\mathbf{U}|_{L^2} &\leq |[\Lambda^s,u]\partial_x\eta|_{L^2} + |[\Lambda^s,\eta]\partial_x u|_{L^2} + |[\Lambda^s,u]\partial_x u|_{L^2}\\ &\leq |\eta|_{H^s}|u|_{H^s} + |u|^2_{H^s}\\ &\leq |(\eta,u)|^2_{V^s_\mu}.\end{aligned}$$

The desired bound on I_1 follows:

$$|I_1| \lesssim |(\eta,u)|^3_{V^s_\mu}.$$

Control of $I_2 + I_3$. First note that $(a_{ij}) = Q^{(1)}(\mathbf{U},\mathrm{D})M(\mathbf{U},\mathrm{D})$ is given by,

$$(a_{ij}) = \begin{pmatrix} u\partial_x & (\mathcal{K}_\mu(\mathrm{D})+\eta)\partial_x \\ \eta\partial_x & \eta u\partial_x \end{pmatrix}.$$

We must estimate each piece below,

$$\begin{aligned}&\big(Q^{(1)}(\mathbf{U},\mathrm{D})M(\mathbf{U},\mathrm{D})\Lambda^s\mathbf{U},\Lambda^s\mathbf{U}\big)_{L^2}\\ &= \big(a_{11}\Lambda^s\eta,\Lambda^s\eta\big)_{L^2} + \big(a_{12}\Lambda^s u,\Lambda^s\eta\big)_{L^2} + \big(a_{21}\Lambda^s\eta,\Lambda^s u\big)_{L^2} + \big(a_{22}\Lambda^s u,\Lambda^s u\big)_{L^2}\\ &=: A_{11}+A_{12}+A_{21}+A_{22}.\end{aligned}$$

As we will shortly see, $A_{12}+A_{21}$ needs to be compensated by B_{21}, that is defined by the remaining part:

$$\begin{aligned}\big(\Lambda^s M(\mathbf{U},\mathrm{D})\mathbf{U},Q^{(2)}(\mathbf{U},\mathrm{D})\Lambda^s\mathbf{U}\big)_{L^2} &= \big(\partial_x\Lambda^s\eta,\mathcal{K}_\mu(\mathrm{D})\Lambda^s u\big)_{L^2} + \big(\Lambda^s(u\partial_x u),\mathcal{K}_\mu(\mathrm{D})\Lambda^s u\big)_{L^2}\\ &=: B_{21}+B_{22},\end{aligned}$$

while B_{22} is the price we pay for symmetry.

Control of A_{11}. Integration by part and the Sobolev embedding yields

$$|A_{11}| \le \frac{1}{2}\big|\big(\partial_x u\Lambda^s\eta,\Lambda^s\eta\big)_{L^2}\big| \le \frac{1}{2}|\partial_x u|_{L^\infty}|\eta|^2_{H^s} \lesssim |(\eta,u)|^3_{V^s_\mu}.$$

Control of $A_{12}+A_{21}+B_{21}$. By definition, consideration is given to the expression

$$A_{12}+A_{21}+B_{21} = \big((\mathcal{K}_\mu(\mathrm{D})+\eta)\partial_x\Lambda^s u,\Lambda^s\eta\big)_{L^2} + \big((\mathcal{K}_\mu(\mathrm{D})+\eta)\partial_x\Lambda^s\eta,\Lambda^s u\big)_{L^2}.$$

Observe, after integration by parts that

$$A_{12} = -\big(\Lambda^s u,(\mathcal{K}_\mu(\mathrm{D})+\eta)\partial_x\Lambda^s\eta\big)_{L^2} - \big(\Lambda^s u,\partial_x\eta\Lambda^s\eta\big)_{L^2}.$$

The first term cancels with $(A_{21}+B_{21})$, while the Sobolev embedding easily controls the remaining part,

$$|\big(\Lambda^s u,\partial_x\eta\Lambda^s\eta\big)_{L^2}| \le |\partial_x\eta|_{L^\infty}|\eta|_{H^s}|u|_{H^s} \lesssim |(\eta,u)|^3_{V^s_\mu}.$$

Control of A_{22}. We simply use integration by parts as above together with (3.4)−(3.5) to deduce

$$|A_{22}| \le |\big(\eta u\partial_x\Lambda^s u,\Lambda^s u\big)_{L^2}| \le c_\sigma^2|(\eta,u)|^3_{V^s_\mu}.$$

Control of B_{22}. We observe, after integrating by parts that

$$\begin{aligned}B_{22} &= \big(\Lambda^s(u\partial_x u),\mathcal{K}_\mu(\mathrm{D})\Lambda^s u\big)_{L^2}\\ &= \big([\sqrt{\mathcal{K}_\mu}(\mathrm{D})\Lambda^s,u]\partial_x u,\sqrt{\mathcal{K}_\mu}(\mathrm{D})\Lambda^s u\big)_{L^2} - \frac{1}{2}\big((\partial_x u)\sqrt{\mathcal{K}_\mu}(\mathrm{D})\Lambda^s u,\sqrt{\mathcal{K}_\mu}(\mathrm{D})\Lambda^s u\big)_{L^2}.\end{aligned}$$

Thus, we deduce by using Hölder's inequality, estimates (2.18) and (2.7) that

$$|B_{22}| \le c_\sigma^2|(\eta,u)|^3_{V^s_\mu}.$$

Control of II. First we claim that $|\mathcal{K}_\mu(\mathrm{D})\partial_x u|_{L^\infty} \lesssim_\sigma |(\eta,u)|_{V^s_\mu}$ for $s>2$. Indeed, it follows from (2.1) and the Sobolev embedding $H^{\frac{1}{2}^+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ that

$$\begin{aligned}|\mathcal{K}_\mu(\mathrm{D})\partial_x u|_{L^\infty} &\le |\partial_x u|_{H^{s-\frac{3}{2}}} + \sigma\sqrt{\mu}|\mathrm{D}^1\partial_x u|_{H^{s-\frac{3}{2}}}\\ &\le c_\sigma^2(|u|_{H^s}+\sqrt{\mu}|\mathrm{D}^{\frac{1}{2}}u|_{H^s}).\end{aligned} \tag{3.8}$$

Then we observe by using equation (3.1) yields,

$$II = \big(\Lambda^s u,(\partial_t\eta)\Lambda^s u\big)_{L^2} = -\big(\Lambda^s u,(\mathcal{K}_\mu(\mathrm{D})\partial_x u)\Lambda^s u\big)_{L^2} - \big(\Lambda^s u,(\partial_x(\eta u))\Lambda^s u\big)_{L^2}.$$

Consequently, the desired estimate follows from Hölder's inequality, the Sobolev embedding, and the above claim that,

$$|II| \lesssim |\mathcal{K}_\mu(\mathrm{D})\partial_x u|_{L^\infty}|u|_{H^s}^2 + |\partial_x(\eta u)|_{L^\infty}|u|_{H^s}^2 \lesssim_\sigma |(\eta,u)|_{V_\mu^s}^3. \tag{3.9}$$

Adding together all the estimates, combined with (3.7) yields,

$$\frac{d}{dt}E_s(\mathbf{U}) \leq c_\sigma^2 \left(E_s(\mathbf{U})\right)^{\frac{3}{2}},$$

and completes the proof of Proposition 3.1.

□

3.2. **Estimates for system** (1.7)**.** As in the former subsection we define $\mathbf{U} = (\eta,u)^T = \varepsilon(\zeta,v)^T$ and we write the system on a compact form:

$$\partial_t \mathbf{U} + \mathcal{M}(\mathbf{U},\mathrm{D})\mathbf{U} = \mathbf{0}, \tag{3.10}$$

with

$$\mathcal{M}(\mathbf{U},\mathrm{D}) = \begin{pmatrix} u\partial_x & (1+\eta)\partial_x \\ \mathcal{T}_\mu(\mathrm{D})\partial_x & u\partial_x \end{pmatrix}. \tag{3.11}$$

We define the symmetrizer associated to (3.10) to be

$$\mathcal{Q}(\mathbf{U},\mathrm{D}) = \begin{pmatrix} \mathcal{T}_\mu(\mathrm{D}) & 0 \\ 0 & 1+\eta \end{pmatrix}. \tag{3.12}$$

Then the energy given in Definition 1.18 can be written as

$$\mathcal{E}_s(\mathbf{U}) = \left(\Lambda^s\Lambda_\mu^{\frac{1}{2}}\mathbf{U}, \mathcal{Q}(\mathbf{U},D)\Lambda^s\Lambda_\mu^{\frac{1}{2}}\mathbf{U}\right)_{L^2}, \tag{3.13}$$

and the *a priori estimate* for (1.7) is stated in the following proposition.

Proposition 3.3. *Let* $s > \frac{3}{2}$*,* $\varepsilon,\mu \in (0,1)$ *and* $(\eta,u) = \varepsilon(\zeta,v) \in C([0,T_0];V_\mu^s(\mathbb{R}))$ *be a solution to* (1.7) *on a time interval* $[0,T_0]$ *for some* $T_0 > 0$*. Moreover, assume there exist* $h_0 \in (0,1)$ *and* $h_1 > 0$ *such that*

$$h_0 - 1 \leq \eta(x,t), \quad \forall (x,t) \in \mathbb{R}\times[0,T_0] \quad \textit{and} \quad \sup_{t\in[0,T_0]} |(\eta,u)|_{H^s\times H^s} \leq h_1. \tag{3.14}$$

Then, for the energy given in Definition 1.18, there holds

$$\frac{d}{dt}\mathcal{E}_s(\mathbf{U}) \lesssim \left(\mathcal{E}_s(\mathbf{U})\right)^{\frac{3}{2}}, \tag{3.15}$$

for all $0 < t < T_0$*, and*

$$|(\eta,u)|_{V_\mu^s}^2 \lesssim \mathcal{E}_s(\mathbf{U}) \lesssim |(\eta,u)|_{V_\mu^s}^2, \tag{3.16}$$

for all $0 < t < T_0$*.*

Proof of Proposition 3.3. We begin by proving (3.16). By Definition (1.18) of the energy, the non-cavitation condition (3.14), (2.17), and (2.16) we obtain the lower bound

$$\begin{aligned}\mathcal{E}_s(\mathbf{U}) &= |\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac{1}{2}}\eta|_{H^s}^2 + \left(\Lambda_\mu^{\frac{1}{2}}\Lambda^s u, (1+\eta)\Lambda_\mu^{\frac{1}{2}}\Lambda^s u\right)_{L^2} \\ &\geq c|\eta|_{H^s}^2 + h_0|\Lambda_\mu^{\frac{1}{2}}u|_{H^s}^2 \\ &\geq c|\eta|_{H^s}^2 + c\cdot h_0(|u|_{H^s}^2 + \sqrt{\mu}|\mathrm{D}^{\frac{1}{2}}u|_{H^s}^2),\end{aligned}$$

for some $c > 0$. The reverse inequality follows by the estimates (2.16), (2.17), Hölder's inequality, the Sobolev embedding, and (3.14):

$$\mathcal{E}_s(\mathbf{U}) \leq |\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac{1}{2}}\eta|_{H^s}^2 + |\Lambda_\mu^{\frac{1}{2}}u|_{H^s}^2 + |\eta|_{L^\infty}|\Lambda_\mu^{\frac{1}{2}}u|_{H^s}^2 \lesssim |(\eta,u)|_{V_\mu^s}^2.$$

Next, we prove (3.15). There follows by using (3.10) and the self-adjointness of $\mathcal{Q}(U,\mathrm{D})$ that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathcal{E}_s(\mathbf{U}) &= -\big(\Lambda^s\Lambda_\mu^{\frac12}\mathcal{M}(\mathbf{U},\mathrm{D})\mathbf{U},\mathcal{Q}(\mathbf{U},\mathrm{D})\Lambda^s\Lambda_\mu^{\frac12}\mathbf{U}\big)_{L^2}\\ &\quad+\frac12\big(\Lambda^s\Lambda_\mu^{\frac12}\mathbf{U},(\partial_t\mathcal{Q}(\mathbf{U},\mathrm{D}))\Lambda^s\Lambda_\mu^{\frac12}\mathbf{U}\big)_{L^2}\\ &=: -\mathcal{I}+\mathcal{II}.\end{aligned}$$

Control of $\mathcal{I}$. By definition of (3.13) we decompose $\mathcal{I}$ in four pieces,

$$\begin{aligned}\mathcal{I} &= \big(\Lambda^s\Lambda_\mu^{\frac12}(u\partial_x\eta),\Lambda^s\Lambda_\mu^{\frac12}\mathcal{T}_\mu(\mathrm{D})\eta\big)_{L^2}+\big(\Lambda^s\Lambda_\mu^{\frac12}\big((1+\eta)\partial_x u\big),\Lambda^s\Lambda_\mu^{\frac12}\mathcal{T}_\mu(\mathrm{D})\eta\big)_{L^2}\\ &\quad+\big(\Lambda^s\Lambda_\mu^{\frac12}\mathcal{T}_\mu(\mathrm{D})\partial_x\eta,(1+\eta)\Lambda^s\Lambda_\mu^{\frac12}u\big)_{L^2}+\big(\Lambda^s\Lambda_\mu^{\frac12}u\partial_x u,(1+\eta)\Lambda^s\Lambda_\mu^{\frac12}u\big)_{L^2}\\ &=: \mathcal{A}_{11}+\mathcal{A}_{12}+\mathcal{A}_{21}+\mathcal{A}_{22}.\end{aligned}$$

Control of $\mathcal{A}_{11}$. We aim to exploit symmetries, and we first write $\mathcal{A}_{11}$ as

$$\begin{aligned}\mathcal{A}_{11} &= \big([\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D}),u]\partial_x\eta,\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\eta\big)_{L^2}\\ &\quad+\big(u\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\partial_x\eta,\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\eta\big)_{L^2}\\ &=: \mathcal{A}_{11}^1+\mathcal{A}_{11}^2.\end{aligned}$$

The first term is treated by the commutator estimate (2.21) with $s>\frac32$, the Cauchy-Schwarz inequality and (2.16). Thus, there holds

$$|\mathcal{A}_{11}^1|\leq|[\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D}),u]\partial_x\eta|_{L^2}|\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\eta|_{L^2}\lesssim|u|_{H^s}|\eta|_{H^s}^2.$$

Similar to previous estimates, we use integration by parts and exploit the symmetries of $\mathcal{A}_{11}^2$, then conclude by (2.16), and the Sobolev embedding $H^{\frac12^+}(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$ that

$$|\mathcal{A}_{11}|\leq\frac12\big|\big((\partial_x u)\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\eta,\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\eta\big)_{L^2}\big|\lesssim|(\eta,u)|_{V_\mu^s}^3.$$

Control of $\mathcal{A}_{12}+\mathcal{A}_{21}$. We first decompose $\mathcal{A}_{12}$ in two parts

$$\begin{aligned}\mathcal{A}_{12} &= \big([\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D}),\eta]\partial_x u,\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\eta\big)_{L^2}\\ &\quad+\big((1+\eta)\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\partial_x u,J^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\eta\big)_{L^2}\\ &= \mathcal{A}_{12}^1+\mathcal{A}_{12}^2.\end{aligned}$$

We estimate $\mathcal{A}_{12}^1$ the same way we did for $\mathcal{A}_{11}^1$ and obtain

$$|\mathcal{A}_{12}^1|\lesssim|u|_{H^s}|\eta|_{H^s}^2.$$

For the second term, after integration by parts, we find

$$\begin{aligned}\mathcal{A}_{12}^2 &= -\big((\partial_x\eta)\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})u,\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\eta\big)_{L^2}\\ &\quad-\big((1+\eta)\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})u,\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\partial_x\eta\big)_{L^2}\\ &= \mathcal{A}_{12}^{2,1}+\mathcal{A}_{12}^{2,2}.\end{aligned}$$

By using the Sobolev embedding and (2.16), we find that

$$|\mathcal{A}_{12}^{2,1}|\leq|\partial_x\eta|_{L^\infty}|\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}u|_{H^s}|\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}\eta|_{H^s}\lesssim|u|_{H^s}|\eta|_{H^s}^2.$$

On the other hand, we cannot estimate $\mathcal{A}_{12}^{2,2}$ on its own. We must therefore cancel it with $\mathcal{A}_{21}$. Observe

$$\begin{aligned}\mathcal{A}_{21} &= \big(\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\partial_x\eta, [\sqrt{\mathcal{T}_\mu}(\mathrm{D}),\eta]\Lambda^s\Lambda_\mu^{\frac12}u\big)_{L^2}\\ &\quad + \big(\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\partial_x\eta, (1+\eta)\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})u\big)_{L^2}\\ &= \mathcal{A}_{21}^1 + \mathcal{A}_{21}^2.\end{aligned}$$

First, by using integration by parts, the Cauchy-Schwarz inequality, (2.16), (2.35) and (2.17) we find that

$$|\mathcal{A}_{21}^1| = |\Lambda^s\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\eta|_{L^2}|\partial_x[\sqrt{\mathcal{T}_\mu}(\mathrm{D}),\eta]\Lambda^s\Lambda_\mu^{\frac12}u|_{L^2} \lesssim |(\eta,u)|_{V_\mu^s}^3. \tag{3.17}$$

On the other hand, we observe that $\mathcal{A}_{21}^2 = -\mathcal{A}_{21}^{2,2}$. We may therefore conclude that the sum satisfies:

$$|\mathcal{A}_{12}+\mathcal{A}_{21}| \lesssim |(\eta,u)|_{V_\mu^s}^3.$$

Control of $\mathcal{A}_{22}$*.* Similar to $\mathcal{A}_{11}$ we write the expression with the good commutator:

$$\begin{aligned}\mathcal{A}_{22} &= \big([\Lambda^s\Lambda_\mu^{\frac12},u]\partial_x u, (1+\eta)\Lambda^s\Lambda_\mu^{\frac12}u\big)_{L^2} + \big(u\Lambda^s\Lambda_\mu^{\frac12}\partial_x u, (1+\eta)\Lambda^s\Lambda_\mu^{\frac12}u\big)_{L^2}\\ &= \mathcal{A}_{22}^1 + \mathcal{A}_{22}^2.\end{aligned}$$

Then use the Cauchy-Schwarz inequality, (3.14), (2.20) with $s > \frac32$, and the Sobolev embedding to get

$$|\mathcal{A}_{22}^1| \lesssim |[\Lambda^s\Lambda_\mu^{\frac12},u]\partial_x u|_{L^2}(1+|\eta|_{L^\infty})|\Lambda^s\Lambda_\mu^{\frac12}u|_{L^2} \lesssim |(\eta,u)|_{V_\mu^s}^3.$$

While for $\mathcal{A}_{22}^2$ we integrate by parts, apply the Sobolev embedding, and again bound each term by the V_μ^s-norm of (η,u) to obtain that

$$|\mathcal{A}_{22}^2| \lesssim (|\partial_x u|_{L^\infty} + |\partial_x\eta|_{L^\infty})|\Lambda_\mu^{\frac12}u|_{H^s}^2 \lesssim |(\eta,u)|_{V_\mu^s}^3.$$

Gathering all these estimates, we conclude that

$$|\mathcal{I}| \lesssim |(\eta,u)|_{V_\mu^s}^3. \tag{3.18}$$

<u>Control of $\mathcal{II}$.</u> By defintion of (3.12) and (3.10) we get that,

$$\begin{aligned}\mathcal{II} &= \big(\Lambda^s\Lambda_\mu^{\frac12}u, (\partial_t\eta)\Lambda^s\Lambda_\mu^{\frac12}u\big)_{L^2}\\ &= -\big(\Lambda^s\Lambda_\mu^{\frac12}u, (\partial_x u)\Lambda^s\Lambda_\mu^{\frac12}u\big)_{L^2} - \big(\Lambda^s\Lambda_\mu^{\frac12}u, (\partial_x(\eta u))\Lambda^s\Lambda_\mu^{\frac12}u\big)_{L^2}\end{aligned}$$

Then, by using Hölder's inequality, the Sobolev embedding, (3.14) and (2.17), we deduce that

$$|\mathcal{II}| \lesssim (|\partial_x u|_{L^\infty} + |\partial_x(\eta u)|_{L^\infty})|\Lambda_\mu^{\frac12}u|_{H^s}^2 \lesssim |(\eta,u)|_{V_\mu^s}^3. \tag{3.19}$$

Consequently, we may add (3.18) and (3.19), then apply (3.16) to conclude the proof of estimate (3.15).

□

3.3. **Estimates for system** (1.9)**.** As in the former subsections we let $\mathbf{U} = (\eta, u)^T = \varepsilon(\zeta, v)^T$ and write the system on the form

$$\partial_t \mathbf{U} + \mathscr{M}(\mathbf{U}, \mathrm{D})\mathbf{U} = \mathbf{0}, \tag{3.20}$$

with

$$\mathscr{M}(\mathbf{U}, \mathrm{D}) = \begin{pmatrix} \mathcal{T}_\mu(\mathrm{D})(u\partial_x \cdot) & \partial_x + \mathcal{T}_\mu(\mathrm{D})(\eta\partial_x \cdot) \\ \mathcal{K}_\mu(\mathrm{D})\partial_x & \mathcal{T}_\mu(\mathrm{D})(u\partial_x \cdot) \end{pmatrix}. \tag{3.21}$$

The symmetrizer is defined by

$$\mathscr{Q}(\mathbf{U}, D) = \begin{pmatrix} \mathcal{T}_\mu^{-1}(\mathrm{D})\mathcal{K}_\mu(\mathrm{D}) & 0 \\ 0 & \mathcal{T}_\mu^{-1}(\mathrm{D}) + \eta \end{pmatrix}. \tag{3.22}$$

Then the energy given in Definition 1.19 can be written as

$$\mathscr{E}_s(\mathbf{U}) = \big(\Lambda^s \mathbf{U}, \mathscr{Q}(\mathbf{U}, D)\Lambda^s \mathbf{U}\big)_{L^2}, \tag{3.23}$$

and the *a priori estimate* for (1.9) is stated in the following proposition.

Proposition 3.4. *Let $s > \frac{3}{2}$, $\varepsilon, \mu \in (0,1)$, $\sigma \geq 0$, and let $(\eta, u) = \varepsilon(\zeta, v) \in C([0, T_0]; X^s_{\sigma,\mu}(\mathbb{R}))$ be a solution to* (1.9) *on a time interval $[0, T_0]$ for some $T_0 > 0$. Moreover, assume that there exist $h_0 \in (0,1)$ and $h_1 > 0$ such that*

$$h_0 - 1 \leq \eta(x,t), \quad \forall (x,t) \in \mathbb{R} \times [0, T_0] \quad \textit{and} \quad \sup_{t \in [0,T_0]} |(\eta, u)|_{H^s \times H^s} \leq h_1. \tag{3.24}$$

Then, for the energy given in Definition 1.19, there holds,

$$\frac{d}{dt}\mathscr{E}_s(\mathbf{U}) \lesssim c_\sigma^2 \big(\mathscr{E}_s(\mathbf{U})\big)^{\frac{3}{2}}, \tag{3.25}$$

and the energy is coercive:

$$|(\eta, u)|^2_{X^s_{\sigma,\mu}} \lesssim \mathscr{E}_s(\mathbf{U}) \lesssim |(\eta, u)|^2_{X^s_{\sigma,\mu}}. \tag{3.26}$$

Proof of Proposition 3.4. We will first provide the coercivity estimate (3.26). By Definition 1.19 for the energy, the non-cavitation condition (3.24) and (2.15) we obtain the lower bound

$$\begin{aligned} \mathscr{E}_s(\mathbf{U}) &= |\langle \sqrt{\sigma\mu}\mathrm{D}^1\rangle\eta|^2_{H^s} + \big(\Lambda^s u, (\mathcal{T}_\mu^{-1}(\mathrm{D}) + \eta)\Lambda^s u\big)_{L^2} \\ &\geq |\eta|^2_{H^s} + \sigma\mu|\mathrm{D}^1\eta|^2_{H^s} + \frac{h_0}{2}|u|^2_{H^s} + c\sqrt{\mu}|\mathrm{D}^{\frac{1}{2}}u|^2_{H^s}, \end{aligned}$$

for some $c > 0$ and $\sigma \geq 0$. The reverse inequality follows by the upper bound in (2.15), the Sobolev embedding and (3.24):

$$\mathscr{E}_s(\mathbf{U}) \leq |\langle \sqrt{\sigma\mu}\mathrm{D}^1\rangle\eta|^2_{H^s} + |\sqrt{\mathcal{T}_\mu}^{-1}(\mathrm{D})\Lambda^s u|^2 + |\eta|_{L^\infty}|\Lambda^s u|^2 \lesssim |(\eta, u)|^2_{X^s_{\sigma,\mu}}.$$

We may now prove (3.25). To do so, we use (3.20) and the self-adjointness of $\mathscr{Q}(\mathbf{U}, \mathrm{D})$ to write

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\mathscr{E}_s(\mathbf{U}) &= -\big(\Lambda^s \mathscr{M}(\mathbf{U}, \mathrm{D})\mathbf{U}, \mathscr{Q}(\mathbf{U}, \mathrm{D})\Lambda^s \mathbf{U}\big)_{L^2} + \frac{1}{2}\big(\Lambda^s \mathbf{U}, (\partial_t \mathscr{Q}(\mathbf{U}, \mathrm{D}))\Lambda^s \mathbf{U}\big)_{L^2} \\ &=: \mathscr{I} + \mathscr{I}\mathscr{I}. \end{aligned}$$

Control of $\mathscr{I}$. By definition of (3.23) we must estimate the following terms:

$$\begin{aligned} \mathscr{I} &= \big(\Lambda^s(u\partial_x\eta), \mathcal{K}_\mu(\mathrm{D})\Lambda^s\eta\big)_{L^2} + \big(\Lambda^s\partial_x u + \Lambda^s \mathcal{T}_\mu(\mathrm{D})(\eta\partial_x u), \mathcal{T}_\mu^{-1}(\mathrm{D})\mathcal{K}_\mu(\mathrm{D})\Lambda^s\eta\big)_{L^2} \\ &\quad + \big(\Lambda^s \mathcal{K}_\mu(\mathrm{D})\partial_x\eta, (\mathcal{T}_\mu^{-1}(\mathrm{D}) + \eta)\Lambda^s u\big)_{L^2} + \big(\Lambda^s \mathcal{T}_\mu(\mathrm{D})(u\partial_x u), (\mathcal{T}_\mu^{-1}(\mathrm{D}) + \eta)\Lambda^s u\big)_{L^2} \\ &=: \mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}. \end{aligned}$$

Control of $\mathscr{A}_{11}$. We rewrite $\mathscr{A}_{11}$ as

$$\begin{aligned}\mathscr{A}_{11} &= \big([\Lambda^s\sqrt{\mathcal{K}_\mu}(\mathrm{D}),u]\partial_x\eta,\Lambda^s\sqrt{\mathcal{K}_\mu}(\mathrm{D})\eta\big)_{L^2} + \big(u\Lambda^s\sqrt{\mathcal{K}_\mu}(\mathrm{D})\partial_x\eta,\Lambda^s\sqrt{\mathcal{K}_\mu}(\mathrm{D})\eta\big)_{L^2}\\ &=: \mathscr{A}_{11}^1 + \mathscr{A}_{11}^2.\end{aligned}$$

Then in the case $\sigma > 0$, we first observe by interpolation and Young's inequality that

$$\sqrt{\sigma}\mu^{\frac14}|\eta|_{H^{s+\frac12}} \leq |\eta|_{H^s}^{\frac12}\big(\sigma\sqrt{\mu}|\eta|_{H^{s+1}}\big)^{\frac12} \lesssim_\sigma |\eta|_{H^s} + \sigma\sqrt{\mu}|\mathrm{D}^1\eta|_{H^s}, \tag{3.27}$$

and thus $\mathscr{A}_{11}^1$ is treated by the Cauchy-Schwarz inequality, the commutator estimate (2.18) with $s > \frac32$, (2.7), and (3.27):

$$|\mathscr{A}_{11}^1| \lesssim c_\sigma^2(|u|_{H^s} + \mu^{\frac12}|u|_{H^{s+\frac12}})(|\eta|_{H^s} + \sqrt{\sigma}\mu^{\frac14}|\eta|_{H^{s+\frac12}})^2 \lesssim c_\sigma^2|(\eta,u)|^3_{X^s_{\sigma,\mu}}.$$

On the other hand, for $\mathscr{A}_{11}^2$ we conclude by integration by parts, (2.7), (3.27), and the Sobolev embedding with $s > \frac32$ that

$$|\mathscr{A}_{11}| \lesssim |\mathscr{A}_{11}^1| + |\partial_x u|_{L^\infty}|\sqrt{\mathcal{K}_\mu}(\mathrm{D})\eta|_{H^s}|\sqrt{\mathcal{K}_\mu}(\mathrm{D})\eta|_{H^s} \lesssim c_\sigma^2|(\eta,u)|^3_{X^s_{\sigma,\mu}},$$

for $\sigma > 0$. Moreover, in the case $\sigma = 0$, then $\mathcal{K}_\mu(\mathrm{D})$ is equal to $\mathcal{T}_\mu(\mathrm{D})$ and we simply use Hölder's inequality, (2.33), (2.14), the Sobolev embedding, and integration by parts to deduce the estimate

$$\begin{aligned}|\mathscr{A}_{11}| &\leq |\big([\Lambda^s\sqrt{\mathcal{T}_\mu}(\mathrm{D}),u]\partial_x\eta,\Lambda^s\sqrt{\mathcal{T}_\mu}(\mathrm{D})\eta\big)_{L^2}| + |\big(u\Lambda^s\sqrt{\mathcal{T}_\mu}(\mathrm{D})\partial_x\eta,\Lambda^s\sqrt{\mathcal{T}_\mu}(\mathrm{D})\eta\big)_{L^2}|\\ &\lesssim |u|_{H^s}|\eta|^2_{H^s}.\end{aligned}$$

Control of $\mathscr{A}_{12}+\mathscr{A}_{21}$. By using integration by parts we write,

$$\begin{aligned}\mathscr{A}_{12} &= \big(\Lambda^s(\eta\partial_x u),\mathcal{K}_\mu(\mathrm{D})\Lambda^s\eta\big)_{L^2} - \big(\Lambda^s u,\mathcal{T}_\mu^{-1}(\mathrm{D})\mathcal{K}_\mu(\mathrm{D})\Lambda^s\partial_x\eta\big)_{L^2}\\ &= \mathscr{A}_{12}^1 + \mathscr{A}_{12}^2.\end{aligned}$$

For $\mathscr{A}_{12}^1$, observe

$$\begin{aligned}\mathscr{A}_{12}^1 &= \big([\Lambda^s,\eta]\partial_x u,\mathcal{K}_\mu(\mathrm{D})\Lambda^s\eta\big)_{L^2} - \big((\partial_x\eta)\Lambda^s u,\mathcal{K}_\mu(\mathrm{D})\Lambda^s\eta\big)_{L^2} - \big(\eta\Lambda^s u,\mathcal{K}_\mu(\mathrm{D})\Lambda^s\partial_x\eta\big)_{L^2}\\ &= \mathscr{A}_{12}^{1,1} + \mathscr{A}_{12}^{1,2} + \mathscr{A}_{12}^{1,3}.\end{aligned}$$

Then in the case $\sigma > 0$ we use the Kato-Ponce commutator estimate (2.9), the Sobolev embedding, and the pointwise estimate (2.1) combined with Plancherel imply that

$$|\mathscr{A}_{12}^{1,1} + \mathscr{A}_{12}^{1,2}| \lesssim c_\sigma^2|u|_{H^s}|\eta|_{H^s}(|\eta|_{H^s} + \sigma\sqrt{\mu}|\mathrm{D}^1\eta|_{H^s}) \lesssim c_\sigma^2|(\eta,u)|^3_{X^s_{\sigma,\mu}}.$$

While for the case $\sigma = 0$, we simply use the boundedness of $\mathcal{T}_\mu(\mathrm{D})$ on $L^2(\mathbb{R})$ to deduce,

$$|\mathscr{A}_{12}^{1,1} + \mathscr{A}_{12}^{1,2}| \lesssim |u|_{H^s}|\eta|^2_{H^s}.$$

However, in either case the contribution of remaining terms, $\mathscr{A}_{12}^{1,3} + \mathscr{A}_{12}^2$, will be canceled by $\mathscr{A}_{21}$. Indeed, we observe that

$$\mathscr{A}_{21} = \big(\Lambda^s\mathcal{K}_\mu(\mathrm{D})\partial_x\eta,\eta\Lambda^s u\big)_{L^2} + \big(\Lambda^s\mathcal{K}_\mu(\mathrm{D})\partial_x\eta,\mathcal{T}_\mu^{-1}(\mathrm{D})\Lambda^s u\big)_{L^2} = -\mathscr{A}_{12}^{1,3} - \mathscr{A}_{12}^2.$$

Hence, combining these identities and estimates gives the bound

$$|\mathscr{A}_{12} + \mathscr{A}_{21}| \lesssim c_\sigma^2|(\eta,u)|^3_{X^s_{\sigma,\mu}}.$$

Control of $\mathscr{A}_{22}$. Consider the two terms:

$$\mathscr{A}_{22} = \big(\Lambda^s(u\partial_x u),\Lambda^s u\big)_{L^2} + \big(\Lambda^s\mathcal{T}_\mu(\mathrm{D})(u\partial_x u),\eta\Lambda^s u\big)_{L^2} = \mathscr{A}_{22}^1 + \mathscr{A}_{22}^2.$$

The control of $\mathscr{A}_{22}^1$ is a direct consequence of the Kato-Ponce commutator estimate (2.9) and integration by parts. Since $s > \frac{3}{2}$, we have that

$$|\mathscr{A}_{22}^1| \leq \big|\big([\Lambda^s, u]\partial_x u, \Lambda^s u\big)_{L^2}\big| + \frac{1}{2}\big|\big((\partial_x u)\Lambda^s u, \Lambda^s u\big)_{L^2}\big| \lesssim |u|_{H^s}^3.$$

To deal with $\mathscr{A}_{22}^2$, we make the decomposition

$$\begin{aligned}\mathscr{A}_{22}^2 &= \big([\Lambda^s\sqrt{\mathcal{T}_\mu}(\mathrm{D}), u]\partial_x u, \sqrt{\mathcal{T}_\mu}(\mathrm{D})\eta\Lambda^s u\big)_{L^2} + \big(u\Lambda^s\sqrt{\mathcal{T}_\mu}(\mathrm{D})\partial_x u, [\sqrt{\mathcal{T}_\mu}(\mathrm{D}), \eta]\Lambda^s u\big)_{L^2} \\ &\quad + \big(u\Lambda^s\sqrt{\mathcal{T}_\mu}(\mathrm{D})\partial_x u, \eta\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda^s u\big)_{L^2} \\ &= \mathscr{A}_{22}^{2,1} + \mathscr{A}_{22}^{2,2} + \mathscr{A}_{22}^{2,3}.\end{aligned}$$

Then for $\mathscr{A}_{22}^{2,1}$ we employ the Cauchy-Schwarz inequality, (2.33), (3.24), (2.14), and the Sobolev embedding to deduce

$$|\mathscr{A}_{22}^{2,1}| \leq |[\Lambda^s\sqrt{\mathcal{T}_\mu}(\mathrm{D}), u]\partial_x u|_{L^2}|\sqrt{\mathcal{T}_\mu}(\mathrm{D})(\eta\Lambda^s u)|_{L^2} \lesssim |(\eta, u)|_{X_{\sigma,\mu}^s}^3.$$

Before we treat $\mathscr{A}_{22}^{2,2}$, we note that $|[\sqrt{\mathcal{T}_\mu}(\mathrm{D}), \eta]\Lambda^s u|_{L^2} \lesssim |\eta|_{H^s}|u|_{H^s}$. Indeed, using (2.14) and the Sobolev embedding we find that

$$|[\sqrt{\mathcal{T}_\mu}(\mathrm{D}), \eta]\Lambda^s u|_{L^2} \lesssim |\eta|_{L^\infty}|\Lambda^s u|_{L^2} \lesssim |\eta|_{H^s}|u|_{H^s}. \tag{3.28}$$

Consequently, using integration by parts, the Cauchy-Schwarz inequality, (3.28) and (2.35) we get

$$\begin{aligned}|\mathscr{A}_{22}^{2,2}| &= \big|\big((\partial_x u)\Lambda^s\sqrt{\mathcal{T}_\mu}(\mathrm{D})u, [\sqrt{\mathcal{T}_\mu}(\mathrm{D}), \eta]\Lambda^s u\big)_{L^2}\big| \\ &\quad + \big|\big(u\Lambda^s\sqrt{\mathcal{T}_\mu}(\mathrm{D})u, \partial_x[\sqrt{\mathcal{T}_\mu}(\mathrm{D}), \eta]\Lambda^s u\big)_{L^2}\big| \\ &\lesssim |\eta|_{H^s}|u|_{H^s}^3,\end{aligned}$$

then use (3.24) on one term. Similarly, for $\mathscr{A}_{22}^{2,3}$ we use integration by parts, the Sobolev embedding, and (2.14) to get the bound

$$|\mathscr{A}_{22}^{2,3}| \lesssim |\partial_x(\eta u)|_{L^\infty}|u|_{H^s}^2.$$

Therefore, we conclude by (3.24) and gathering all these estimates that

$$|\mathscr{A}_{22}| \lesssim |(\eta, u)|_{X_{\sigma,\mu}^s}^3,$$

and by extension, we have the bound

$$|\mathscr{I}| \lesssim_\sigma |(\eta, u)|_{X_{\sigma,\mu}^s}^3.$$

Control of $\mathscr{I}\mathscr{I}$. By defintion of (3.22) and (3.20) we get that,

$$\begin{aligned}\mathscr{I}\mathscr{I} &= \big(\Lambda^s u, (\partial_t\eta)\Lambda^s u\big)_{L^2} \\ &= -\big(\Lambda^s u, (\partial_x u)\Lambda^s u\big)_{L^2} - \big(\Lambda^s u, (\mathcal{T}_\mu(\mathrm{D})\partial_x(\eta u))\Lambda^s u\big)_{L^2},\end{aligned}$$

Then the final estimate follows by the Cauchy-Schwarz inequality, (3.24) and the fact that $\mathcal{T}_\mu(\mathrm{D})$ is bounded on $L^2(\mathbb{R})$, then apply Hölder's inequality, and the Sobolev embedding to deduce

$$|\mathscr{I}\mathscr{I}| \leq |\partial_x u|_{L^\infty}|u|_{H^s}^2 + |\mathcal{T}_\mu(\mathrm{D})\partial_x(\eta u)|_{L^\infty}|u|_{H^s}^2 \lesssim |(\eta, u)|_{X_{\sigma,\mu}^s}^3.$$

□

4. Estimates for the difference of two solutions

4.1. **Estimates for system** (1.5)**.** We will now estimate the difference between two solutions of (1.5) given by $\mathbf{U}_1 = (\eta_1, u_1)^T = \varepsilon(\zeta_1, v_1)^T$ and $\mathbf{U}_2 = (\eta_2, u_2)^T = \varepsilon(\zeta_2, v_2)^T$. For convenience, we define $(\psi, w) = (\eta_1 - \eta_2, u_1 - u_2)$. Then $\mathbf{W} = (\psi, w)^T$ solves

$$\partial_t \mathbf{W} + M(\mathbf{U}_1, \mathrm{D})\mathbf{W} = \mathbf{F}, \tag{4.1}$$

with M defined as in (3.2) and $\mathbf{F} = -\big(M(\mathbf{U}_1, \mathrm{D}) - M(\mathbf{U}_2, \mathrm{D})\big)\mathbf{U}_2$. Specifically, the source term is given by

$$\mathbf{F} = -\begin{pmatrix} w\partial_x \eta_2 + \psi \partial_x u_2 \\ w\partial_x u_2 \end{pmatrix}. \tag{4.2}$$

The energy associated to (4.1) is given in terms of the symmetrizer $Q(\mathbf{U}_1, \mathrm{D})$ defined in (3.3) and reads

$$\tilde{E}_s(\mathbf{W}) := \big(\Lambda^s \mathbf{W}, Q(\mathbf{U}_1, \mathrm{D})\Lambda^s \mathbf{W}\big)_{L^2}. \tag{4.3}$$

The main result of this section reads:

Proposition 4.1. *Take $s > 2$ and $\varepsilon, \mu \in (0,1)$. Let $(\eta_1, u_1), (\eta_2, u_2) \in C([0, T_0] : V^s_\mu(\mathbb{R}))$ be two solutions of* (1.5) *on a time interval $[0, T_0]$ for some $T_0 > 0$. Moreover, assume there exist $h_0 \in (0,1)$ and $h_1 > 0$ such that*

$$h_0 - 1 \leq \eta_1(x,t), \quad \forall (x,t) \in \mathbb{R} \times [0, T_0] \quad \textit{and} \quad \sup_{t \in [0,T_0]} |(\eta_1, u_1)|_{H^s \times H^s} \leq h_1, \tag{4.4}$$

when $\sigma \geq \frac{1}{3}$, and that

$$-\frac{\sigma}{2} \leq \eta_1(x,t), \quad \forall (x,t) \in \mathbb{R} \times [0, T_0] \quad \textit{and} \quad \sup_{t \in [0,T_0]} |(\eta_1, u_1)|_{H^s \times H^s} \leq h_1, \tag{4.5}$$

when $0 < \sigma < \frac{1}{3}$.

Define the difference to be $\mathbf{W} = (\psi, w) = (\eta_1 - \eta_2, u_1 - u_2)$. Then, for the energy defined by (4.3)*, there holds*

$$\frac{d}{dt}\tilde{E}_0(\mathbf{W}) \lesssim_\sigma \max_{i=1,2} |(\eta_i, u_i)|_{V^s_\mu} |(\psi, w)|^2_{V^0_\mu}, \tag{4.6}$$

and

$$|(\psi, w)|^2_{V^0_\mu} \lesssim_\sigma \tilde{E}_0(\mathbf{W}) \lesssim_\sigma |(\psi, w)|^2_{V^0_\mu}. \tag{4.7}$$

Furthermore, we have the following estimate at the $V^s_\mu-$ level:

$$\frac{d}{dt}\tilde{E}_s(\mathbf{W}) \lesssim_\sigma |\big(\Lambda^s \mathbf{F}, Q(\mathbf{U}_1, \mathrm{D})\Lambda^s \mathbf{W}\big)_{L^2}| + \max_{i=1,2} |(\eta_i, u_i)|_{V^s_\mu} |(\psi, w)|^2_{V^s_\mu}, \tag{4.8}$$

and

$$|(\psi, w)|^2_{V^s_\mu} \lesssim_\sigma \tilde{E}_s(\mathbf{W}) \lesssim_\sigma |(\psi, w)|^2_{V^s_\mu}. \tag{4.9}$$

Remark 4.2. *The source term corresponding to $\mathbf{F}$ given by* (4.8) *will be treated in the proof of Theorem 1.6 by using regularization estimates and a classical Bona-Smith argument* [7]*.*

Proof of Proposition 4.1. First, the proofs of (4.7) and (4.9) are similar to the one of (3.7).

Next, we only prove (4.6), where (4.8) is more straightforward and follows the same line, utilizing similar estimates to those applied for the proof of Proposition 3.1.

To prove (4.6), we use (4.1) and the self-adjointness of $Q(\mathbf{U}_1, \mathrm{D})$ to write

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\tilde{E}_0(\mathbf{W}) &= \frac{1}{2}\big(\mathbf{W}, (\partial_t Q(\mathbf{U}_1, \mathrm{D}))\mathbf{W}\big)_{L^2} + \big(\mathbf{F}, Q(\mathbf{U}_1, \mathrm{D})\mathbf{W}\big)_{L^2} \\ &\quad - \big(M(\mathbf{U}_1, \mathrm{D})\mathbf{W}, Q(\mathbf{U}_1, \mathrm{D})\mathbf{W}\big)_{L^2} \\ &=: I - II - III. \end{aligned}$$

Control of I. We estimate the first term for $s>2$ by arguing similarly to estimate (3.9). Indeed, we have that

$$I=(w,(\partial_t\eta_1)w)\lesssim|\partial_t\eta_1|_{L^\infty}|w|_{L^2}^2\lesssim_\sigma|(u_1,\eta_1)|_{V_\mu^s}|(\psi,w)|_{V_\mu^0}^2.$$

Control of II. For II, we write

$$\begin{aligned}II&=\big(w\partial_x\eta_2,\psi\big)_{L^2}+\big(\psi\partial_x\eta_2,\psi\big)_{L^2}+\big(w\partial_x u_2,\eta_1 w\big)_{L^2}+\big(w\partial_x u_2,\mathcal{K}_\mu(\mathrm{D})w\big)_{L^2}\\&=:II_1+II_2+II_3+II_4.\end{aligned}$$

The first three terms are treated by the Cauchy-Schwarz inequality and the Sobolev embedding. Take, for instance, II_1:

$$|II_1|\lesssim|w\partial_x\eta_2|_{L^2}|\psi|_{L^2}\lesssim|\eta_2|_{H^s}|(\psi,w)|_{V_\mu^0}^2,$$

for $s>\frac{3}{2}$. Then estimating II_2+II_3 similarly gives

$$|II_1+II_2+II_3|\lesssim\max_{i=1,2}|(\eta_i,u_i)|_{V_\mu^s}|(\psi,w)|_{V_\mu^0}^2.$$

Regarding the term containing the multiplier $\mathcal{K}_\mu(\mathrm{D})$, we write

$$II_4\leq|\sqrt{\mathcal{K}_\mu}(\mathrm{D})(w\partial_x u_2)|_{L^2}|\sqrt{\mathcal{K}_\mu}(\mathrm{D})w|_{L^2}=:II_4^1\cdot II_4^2,$$

and make the observation

$$\begin{aligned}II_4^1&\leq|(\sqrt{\mathcal{K}_\mu}(\mathrm{D})-\sqrt{\sigma}\mu^{\frac14}\mathrm{D}^{\frac12})(w\partial_x u_2)|_{L^2}+\sqrt{\sigma}\mu^{\frac14}|\mathrm{D}^{\frac12}(w\partial_x u_2)|_{L^2}\\&=:II_4^{1,1}+\sqrt{\sigma}II_4^{1,2}.\end{aligned}$$

For the first term, we note that $(\sqrt{\mathcal{K}_\mu}(\mathrm{D})-\sqrt{\sigma}\mu^{\frac14}\mathrm{D}^{\frac12})$ is bounded on $L^2(\mathbb{R})$ by (2.5), and we can conclude by the Sobolev embedding that

$$II_4^{1,1}\lesssim_\sigma|w\partial_x\eta_2|_{L^2}\lesssim_\sigma|\eta_2|_{H^s}|(\psi,w)|_{V_\mu^0}.$$

For the remaining term, $II_4^{1,2}$, we first make an observation. Let $\nu=\frac{1}{2}^-$ and $(p_1,p_2)=(\frac{1}{\nu},\frac{2}{1-2\nu})$ then by (2.37) there holds

$$|\mathrm{D}^{\frac12}\partial_x u_2|_{L^{p_1}}\mu^{\frac14}|w|_{L^{p_2}}\lesssim|u_2|_{H^{2-\nu}}\mu^{\frac14}|\mathrm{D}^{\frac12}w|_{L^2}.\tag{4.10}$$

Moreover, by the fractional Leibniz rule (2.38), the triangle inequality and Hölder's inequality yields the bound

$$\begin{aligned}II_4^{1,2}&\lesssim\mu^{\frac14}|\mathrm{D}^{\frac12}(w\partial_x u_2)-w\mathrm{D}^{\frac12}\partial_x u_2-(\partial_x u_2)\mathrm{D}^{\frac12}w|_{L^2}+\mu^{\frac14}|w\mathrm{D}^{\frac12}\partial_x u_2|_{L^2}\\&\quad+\mu^{\frac14}|(\partial_x u_2)\mathrm{D}^{\frac12}w|_{L^2}\\&\lesssim|\mathrm{D}^{\frac12}\partial_x u_2|_{L^{p_1}}\mu^{\frac14}|w|_{L^{p_2}}+|w|_{L^2}\mu^{\frac14}|\mathrm{D}^{\frac12}\partial_x u_2|_{L^\infty}+|\partial_x u_2|_{L^\infty}\mu^{\frac14}|\mathrm{D}^{\frac12}w|_{L^2}.\end{aligned}$$

Now, since $\frac{1}{p_1}+\frac{1}{p_2}=\nu+\frac{1-2\nu}{2}=\frac{1}{2}$, we may apply (4.10) to deal with the first term, and combined with the Sobolev embedding $H^{\frac{1}{2}^+}(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$ we deduce that

$$II_4^{1,2}\lesssim|u_2|_{H^s}\mu^{\frac14}|\mathrm{D}^{\frac12}w|_{L^2}+|w|_{L^2}\mu^{\frac14}|\mathrm{D}^{\frac12}u_2|_{H^s},$$

with $s>\frac{3}{2}$. Consequently, the bound on II_4 is given by

$$II_4\lesssim_\sigma\max_{i=1,2}|(\eta_i,u_i)|_{V_\mu^s}|(\psi,w)|_{V_\mu^0}^2,$$

which allows us to conclude that

$$II\lesssim_\sigma\max_{i=1,2}|(\eta_i,u_i)|_{V_\mu^s}|(\psi,w)|_{V_\mu^0}^2.$$

Control of III. By definition, we must estimate:

$$\begin{aligned} III &= \big(u_1\partial_x\psi,\psi\big)_{L^2} + \big((\mathcal{K}_\mu(\mathrm{D})+\eta_1)\partial_x w,\psi\big)_{L^2} \\ &\quad + \big(\partial_x\psi,(\mathcal{K}_\mu(\mathrm{D})+\eta_1)w\big)_{L^2} + \big(u_1\partial_x w,(\mathcal{K}_\mu(\mathrm{D})+\eta_1)w\big)_{L^2} \\ &= A_1 + A_2 + A_3 + A_4. \end{aligned}$$

The first term is handled by integration by parts and the Sobolev embedding

$$|A_1| \lesssim |\partial_x u_1|_{L^\infty}|\psi|_{L^2}^2 \lesssim |u_1|_{H^s}|\psi|_{L^2}^2.$$

Next, we observe a cancelation in the off-diagonal terms due to the symmetry. Indeed, we see after integrating by parts that

$$A_2 = -\big((\partial_x\eta_1)w,\psi\big)_{L^2} - A_3.$$

Consequently, we observe after using Hölder's inequality and the Sobolev embedding that

$$|A_2 + A_3| \lesssim |\partial_x\eta_1|_{L^\infty}|w|_{L^2}|\psi|_{L^2}.$$

The only term remaining is A_4, which contains the multiplier that will need some more care. In particular, we write

$$\begin{aligned} A_4 &= \big(u_1\partial_x w,\eta_1 w\big)_{L^2} + \big(u_1\partial_x w,\mathcal{K}_\mu(\mathrm{D})w\big)_{L^2} \\ &=: A_4^1 + A_4^2. \end{aligned}$$

The first term is again treated by integration by parts, and we obtain the bound

$$|A_4^1| \lesssim |u_1|_{H^s}|\eta_1|_{H^s}|w|_{L^2}^2.$$

Lastly, to estimate A_4^2, we split the kernel $\mathcal{K}_\mu(\mathrm{D})$ into several pieces that are localized in low and high frequencies:

$$\mathcal{K}_\mu(\mathrm{D}) = ((\chi_\mu^{(1)})^2\mathcal{K}_\mu)(\mathrm{D}) + ((\chi_\mu^{(2)})^2(\mathrm{F}_{\mu,\frac{1}{2}})^2)(\mathrm{D}) - ((\chi_\mu^{(2)})^2(\mathrm{F}_{\mu,0})^2)(\mathrm{D}), \tag{4.11}$$

where $\mathrm{F}_{\mu,\frac{1}{2}}(\mathrm{D})$ is defined in (2.26), $\mathrm{F}_{\mu,0}(\mathrm{D})$ is defined in (2.29) and $\chi_\mu^{(i)}(\mathrm{D})$ with its porperties given by Definition 2.10. Then, we get that

$$\begin{aligned} \big(u_1\partial_x w,\mathcal{K}_\mu(\mathrm{D})w\big)_{L^2} &= \big((\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(\mathrm{D})(u_1\partial_x w),(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(\mathrm{D})w\big)_{L^2} \\ &\quad + \big((\chi_\mu^{(2)}\mathrm{F}_{\mu,\frac{1}{2}})(\mathrm{D})(u_1\partial_x w),(\chi_\mu^{(2)}\mathrm{F}_{\mu,\frac{1}{2}})(\mathrm{D})w\big)_{L^2} \\ &\quad - \big((\chi_\mu^{(2)}\mathrm{F}_{\mu,0})(\mathrm{D})(u_1\partial_x w),(\chi_\mu^{(2)}\mathrm{F}_{\mu,0})(\mathrm{D})w\big)_{L^2} \\ &=: A_4^{2,1} + A_4^{2,2} - A_4^{2,3}. \end{aligned}$$

We treat each term individually using the commutator estimates in Lemma 2.11, where the remaining part is symmetric and is treated by using integration by parts and the Sobolev embedding in the usual way.

Control of $A_4^{2,1}$. Proceeding as explained above, we have that

$$\begin{aligned} A_4^{2,1} &= \big([(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(\mathrm{D}),u_1]\partial_x w,(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(\mathrm{D})w\big)_{L^2} \\ &\quad + \big(u_1(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(\mathrm{D})\partial_x w,(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(\mathrm{D})w\big)_{L^2} \\ &= A_4^{2,1,1} + A_4^{2,1,2}. \end{aligned}$$

For $A_4^{2,1,1}$ we use the Cauchy-Schwarz inequality, (2.24), and (2.25) to obtain the bound

$$\begin{aligned} |A_4^{2,1,1}| &\lesssim |[(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(\mathrm{D}),u_1]\partial_x w|_{L^2}|(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(\mathrm{D})w|_{L^2} \\ &\lesssim |u_1|_{H^s}|w|_{L^2}^2. \end{aligned}$$

For the remaining term, we deduce from (2.24) that

$$|A_4^{2,1,2}| = \frac{1}{2}|\big((\partial_x u_1)(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(\mathrm{D})w, (\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(\mathrm{D})w\big)_{L^2}|$$
$$\lesssim |\partial_x u_1|_{L^\infty}|w|_{L^2}^2.$$

Control of $A_4^{2,2}$. Similarly we get from the estimates (2.27), (2.28) and the Sobolev embedding that

$$|A_4^{2,2}| \lesssim |\big([(\chi_\mu^{(2)}\mathrm{F}_{\mu,\frac{1}{2}})(\mathrm{D}), u_1]\partial_x w, (\chi_\mu^{(2)}\mathrm{F}_{\mu,\frac{1}{2}})(\mathrm{D})w\big)_{L^2}|$$
$$+ \frac{1}{2}|\big((\partial_x u_1)(\chi_\mu^{(2)}\mathrm{F}_{\mu,\frac{1}{2}})(\mathrm{D})w, (\chi_\mu^{(2)}\mathrm{F}_{\mu,\frac{1}{2}})(\mathrm{D})w\big)_{L^2}|$$
$$\lesssim |u_1|_{H^s}(|w|_{L^2} + \mu^{\frac{1}{4}}|\mathrm{D}^{\frac{1}{2}}w|_{L^2})^2.$$

Control of $A_4^{2,3}$. By the same approach as above, combined with estimates (2.30) and (2.31) leaves us with the bound

$$|A_4^{2,3}| \lesssim |\big([(\chi_\mu^{(2)}\mathrm{F}_{\mu,0})(\mathrm{D}), u_1]\partial_x w, (\chi_\mu^{(2)}\mathrm{F}_{\mu,0})(\mathrm{D})w\big)_{L^2}|$$
$$+ \frac{1}{2}|\big((\partial_x u_1)(\chi_\mu^{(2)}\mathrm{F}_{\mu,0})(\mathrm{D})w, (\chi_\mu^{(2)}\mathrm{F}_{\mu,0})(\mathrm{D})w\big)_{L^2}|$$
$$\lesssim |u_1|_{H^s}|w|_{L^2}^2 + |\partial_x u_1|_{L^\infty}|w|_{L^2}^2.$$

Gathering all these estimates, we obtain the result

$$|A_4| = |A_4^1 + A_4^2 + A_4^3| \lesssim_\sigma (|u_1|_{H^s} + |\eta_1|_{H^s})|(\psi, w)|_{V_\mu^0}^2.$$

Adding $I + II + III$ concludes the proof. □

4.2. **Estimates for system** (1.7)**.** As in the prvious subsection, we let $\mathbf{U}_1 = (\eta_1, u_1)^T = \varepsilon(\zeta_1, v_1)^T$ and $\mathbf{U}_2 = (\eta_2, u_2)^T = \varepsilon(\zeta_2, v_2)^T$ be two solutions of (1.7) and define the difference $(\psi, w) = (\eta_1 - \eta_2, u_1 - u_2)$. Then $\mathbf{W} = (\psi, w)^T$ solves

$$\partial_t \mathbf{W} + \mathcal{M}(\mathbf{U}_1, \mathrm{D})\mathbf{W} = \mathbf{F}, \tag{4.12}$$

with $\mathcal{M}$ defined as in (3.11) and $\mathbf{F}$ will remain the same as previously defined by (4.2). Then the energy associated to (4.12) is given in terms of the symmetrizer (3.12):

$$\tilde{\mathcal{E}}_s(\mathbf{W}) := \big(\Lambda_\mu^{\frac{1}{2}}\Lambda^s \mathbf{W}, \mathcal{Q}(\mathbf{U}_1, \mathrm{D})\Lambda_\mu^{\frac{1}{2}}\Lambda^s \mathbf{W}\big)_{L^2}. \tag{4.13}$$

Proposition 4.3. *Take* $s > \frac{3}{2}$ *and* $\varepsilon, \mu \in (0,1)$*. Let* $(\eta_1, u_1), (\eta_2, u_2) \in C([0, T_0] : V_\mu^s(\mathbb{R}))$ *be two solutions of* (1.7) *on a time interval* $[0, T_0]$ *for some* $T_0 > 0$*. Moreover, assume there exists* $h_0 \in (0,1)$ *and* $h_1 > 0$ *such that*

$$h_0 - 1 \leq \eta_1(x,t), \quad \forall (x,t) \in \mathbb{R} \times [0, T_0] \quad \textit{and} \quad \sup_{t\in[0,T_0]} |(\eta_1, u_1)|_{H^s \times H^s} \leq h_1. \tag{4.14}$$

Define the difference to be $\mathbf{W} = (\psi, w) = (\eta_1 - \eta_2, u_1 - u_2)$*. Then, for the energy defined by* (4.13)*, there holds*

$$\frac{d}{dt}\tilde{\mathcal{E}}_0(\mathbf{W}) \lesssim \max_{i=1,2} |(\eta_i, u_i)|_{V_\mu^s}|(\psi, w)|_{V_\mu^0}^2, \tag{4.15}$$

and

$$|(\psi, w)|_{V_\mu^0}^2 \lesssim \tilde{\mathcal{E}}_0(\mathbf{W}) \lesssim |(\psi, w)|_{V_\mu^0}^2. \tag{4.16}$$

Furthermore, we have the following estimate at the V_μ^s- *level:*

$$\frac{d}{dt}\tilde{\mathcal{E}}_s(\mathbf{W}) \lesssim |\big(\Lambda^s \mathbf{F}, \mathcal{Q}(\mathbf{U}_1, \mathrm{D})\Lambda^s \mathbf{W}\big)_{L^2}| + \max_{i=1,2} |(\eta_i, u_i)|_{V_\mu^s}|(\psi, w)|_{V_\mu^s}^2, \tag{4.17}$$

and

$$|(\psi,w)|_{V^s_\mu}^2 \lesssim \tilde{\mathcal{E}}_s(\mathbf{W}) \lesssim |(\psi,w)|_{V^s_\mu}^2. \tag{4.18}$$

Proof. The proofs of (4.16) and (4.18) are similar to the proof of (3.16).

Also, we only prove (4.15) since the control of (4.17) follows by the proof of Proposition 3.3.

To prove (4.15), we use (4.12) and the self-adjointness of $\mathcal{Q}(\mathbf{U}_1,\mathrm{D})$ to write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\tilde{\mathcal{E}}_0(\mathbf{W}) &= \frac{1}{2}\big(\Lambda_\mu^{\frac12}\mathbf{W},(\partial_t\mathcal{Q}(\mathbf{U}_1,\mathrm{D}))\Lambda_\mu^{\frac12}\mathbf{W}\big)_{L^2} + \big(\Lambda_\mu^{\frac12}\mathbf{F},\mathcal{Q}(\mathbf{U}_1,\mathrm{D})\Lambda_\mu^{\frac12}\mathbf{W}\big)_{L^2}\\ &\quad - \big(\Lambda_\mu^{\frac12}\mathcal{M}(\mathbf{U}_1,\mathrm{D})\mathbf{W},\mathcal{Q}(\mathbf{U}_1,\mathrm{D})\Lambda_\mu^{\frac12}\mathbf{W}\big)_{L^2}\\ &=: \mathcal{I} - \mathcal{II} - \mathcal{III}.\end{aligned}$$

Control of $\mathcal{I}$. Using (4.12), (2.17), the Sobolev embedding and (4.14) yields

$$|\mathcal{I}| = \frac{1}{2}|\big(\Lambda_\mu^{\frac12}w,(\partial_t\eta_1)\Lambda_\mu^{\frac12}w\big)_{L^2}| \lesssim |(\eta_1,u_1)|_{V^s_\mu}|\Lambda_\mu^{\frac12}w|_{L^2}^2,$$

since $s>\frac{3}{2}$.

Control of $\mathcal{II}$. The contribution of the source term is given by

$$\begin{aligned}\mathcal{II} &= \big(\Lambda_\mu^{\frac12}(w\partial_x\eta_2),\mathcal{T}_\mu(\mathrm{D})\Lambda_\mu^{\frac12}\psi\big)_{L^2} + \big(\Lambda_\mu^{\frac12}(\psi\partial_x u_2),\mathcal{T}_\mu(\mathrm{D})\Lambda_\mu^{\frac12}\psi\big)_{L^2}\\ &\quad + \big(\Lambda_\mu^{\frac12}(w\partial_x u_2),\Lambda_\mu^{\frac12}w\big)_{L^2} + \big(\Lambda_\mu^{\frac12}(w\partial_x u_2),\eta_1\Lambda_\mu^{\frac12}w\big)_{L^2}\\ &=: \mathcal{II}_1+\mathcal{II}_2+\mathcal{II}_3+\mathcal{II}_4.\end{aligned}$$

Control of $\mathcal{II}_1+\mathcal{II}_2$. The estimate of $\mathcal{II}_1$ is a direct consequence of the Cauchy-Schwarz inequality, (2.16) and the Sobolev embedding. Indeed, since $s>\frac{3}{2}$, we get

$$|\mathcal{II}_1| \leq |\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}(w\partial_x\eta_2)|_{L^2}|\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}\psi|_{L^2} \lesssim |\eta_2|_{H^s}|w|_{L^2}|\psi|_{L^2}.$$

Next, the control of $\mathcal{II}_2$ follows by the same estimates and gives

$$|\mathcal{II}_2| \lesssim |\eta_2|_{H^s}|\psi|_{H^1}^2.$$

Control of $\mathcal{II}_3+\mathcal{II}_4$. We first deduce from (2.13) that

$$|\Lambda_\mu^{\frac12}(w\partial_x u_2)|_{L^2} \leq |w\partial_x u_2|_{L^2} + \mu^{\frac14}|\mathrm{D}^{\frac12}(w\partial_x u_2)|_{L^2}.$$

The first term is estimated by the Sobolev embedding, while the second term is equal to the term $II_4^{1,2}$ in the proof of Proposition 4.1. Since the terms w and u_2 in $II_4^{1,2}$ belong to the same function space, we can apply the same estimates. Thus, there holds for $s>\frac{3}{2}$ that

$$|\Lambda_\mu^{\frac12}(w\partial_x u_2)|_{L^2} \lesssim \max_{i=1,2}|(\eta_i,u_i)|_{V^s_\mu}|(\psi,w)|_{V^0_\mu}^2. \tag{4.19}$$

Therefore, by using the Cauchy-Schwarz inequality, (4.19), (2.17), (4.14), and the Sobolev embedding implies

$$|\mathcal{II}_3|+|\mathcal{II}_4| \lesssim (1+|\eta_1|_{L^\infty})|\Lambda_\mu^{\frac12}(w\partial_x u_2)|_{L^2}|\Lambda_\mu^{\frac12}w|_{L^2} \lesssim \max_{i=1,2}|(\eta_i,u_i)|_{V^s_\mu}|(\psi,w)|_{V^0_\mu}^2.$$

Control of $\mathcal{III}$. Lastly, the symmetrized term reads:

$$\begin{aligned}\mathcal{III} &= \big(\Lambda_\mu^{\frac12}(u_1\partial_x\psi), \mathcal{T}_\mu(\mathrm{D})\Lambda_\mu^{\frac12}\psi\big)_{L^2} + \big(\Lambda_\mu^{\frac12}\big((1+\eta_1)\partial_x w\big), \mathcal{T}_\mu(\mathrm{D})\Lambda_\mu^{\frac12}\psi\big)_{L^2} \\ &\quad + \big(\mathcal{T}_\mu(\mathrm{D})\Lambda_\mu^{\frac12}\partial_x\psi, (1+\eta_1)\Lambda_\mu^{\frac12}w\big)_{L^2} + \big(\Lambda_\mu^{\frac12}(u_1\partial_x w), (1+\eta_1)\Lambda_\mu^{\frac12}w\big)_{L^2} \\ &= \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}.\end{aligned}$$

Each term is treated by using integration by parts and suitable commutator estimates.

Control of $\mathcal{A}_{11}$. For $\mathcal{A}_{11}$, we use integration by parts to find that

$$\mathcal{A}_{11} = \big([\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}, u_1]\partial_x\psi, \sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}\psi\big)_{L^2} - \frac12\big((\partial_x u_1)\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}\psi, \sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}\psi\big)_{L^2}.$$

Thus, it follows from the commutator estimate (2.32) with $s > \frac32$ and estimate (2.16) that

$$|\mathcal{A}_{11}| \lesssim \max_{i=1,2}|(\eta_i, u_i)|_{V_\mu^s}|\psi|_{L^2}^2.$$

Control of $\mathcal{A}_{12} + \mathcal{A}_{21}$. Treating the off-diagonal terms we first observe,

$$\begin{aligned}\mathcal{A}_{12} &= \big([\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}, \eta_1]\partial_x w, \sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}\psi\big)_{L^2} \\ &\quad + \big((1+\eta_1)\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}\partial_x w, \sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}\psi\big)_{L^2} \\ &= \mathcal{A}_{12}^1 + \mathcal{A}_{12}^2.\end{aligned}$$

The commutator estimate (2.32) and estimate (2.16) deals with the first term. Indeed, we get the bound

$$|\mathcal{A}_{12}^1| \leq |[\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}, \eta_1]\partial_x w|_{L^2}|\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}\psi|_{L^2} \lesssim \max_{i=1,2}|(\eta_i, u_i)|_{V_\mu^s}|(\psi, w)|_{V_\mu^0}^2.$$

Next, we integrate $\mathcal{A}_{12}^2$ by parts to obtain two new terms

$$\begin{aligned}\mathcal{A}_{12}^2 &= -\big((\partial_x\eta_1)\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})w, \Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\psi\big)_{L^2} \\ &\quad - \big((1+\eta_1)\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})w, \Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\partial_x\psi\big)_{L^2} \\ &= \mathcal{A}_{12}^{2,1} + \mathcal{A}_{12}^{2,2}.\end{aligned}$$

Arguing as above, we find that

$$|\mathcal{A}_{12}^{2,1}| \leq |\partial_x\eta_1|_{L^\infty}|\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})w|_{L^2}|\Lambda_\mu^{\frac12}\sqrt{\mathcal{T}_\mu}(\mathrm{D})\psi|_{L^2} \lesssim |\eta_1|_{H^s}|(\psi, w)|_{V_\mu^0}^2,$$

for $s > \frac32$. On the other hand, the term $\mathcal{A}_{12}^{2,2}$, is absorbed by $\mathcal{A}_{21}$. Indeed,

$$\begin{aligned}\mathcal{A}_{21} &= -\big(\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}\psi, \partial_x[\sqrt{\mathcal{T}_\mu}(\mathrm{D}), \eta_1]\Lambda_\mu^{\frac12}w\big)_{L^2} \\ &\quad + \big(\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}\partial_x\psi, (1+\eta_1)\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}w\big)_{L^2} \\ &= \mathcal{A}_{21}^1 + \mathcal{A}_{21}^2,\end{aligned}$$

with $\mathcal{A}_{21}^2 = -\mathcal{A}_{12}^{2,2}$. We estimate $\mathcal{A}_{21}^1$ by using the Cauchy-Schwarz inequality, (2.16), (2.35), and (2.17) to get

$$|\mathcal{A}_{21}^1| \leq |\sqrt{\mathcal{T}_\mu}(\mathrm{D})\Lambda_\mu^{\frac12}\psi|_{L^2}|\partial_x[\sqrt{\mathcal{T}_\mu}(\mathrm{D}), \eta_1]\Lambda_\mu^{\frac12}w|_{L^2} \lesssim |\eta_1|_{H^s}|(\psi, w)|_{V_\mu^0}^2.$$

Thus, we deduce by gathering all these estimates that

$$|\mathcal{A}_{12} + \mathcal{A}_{21}| \lesssim \max_{i=1,2}|(\eta_i, u_i)|_{V_\mu^s}|(\psi, w)|_{V_\mu^0}^2.$$

Control of $\mathcal{A}_{22}$. Lastly, the term $\mathcal{A}_{22}$ is estimated by (2.36) for $s > \frac{3}{2}$, (4.14), and integration by parts

$$\begin{aligned}|\mathcal{A}_{22}^2| &\leq \left|\left([\Lambda_\mu^{\frac{1}{2}}, u_1]\partial_x w, (1+\eta_1)\Lambda_\mu^{\frac{1}{2}} w\right)_{L^2}\right| + \left|\left(u_1 \Lambda_\mu^{\frac{1}{2}}\partial_x w, (1+\eta_1)\Lambda_\mu^{\frac{1}{2}} w\right)_{L^2}\right| \\ &\lesssim \max_{i=1,2} |(\eta_i, u_i)|_{V_\mu^s} |(\psi, w)|_{V_\mu^0}^2.\end{aligned}$$

Therefore, we deduce that

$$\frac{d}{dt}\tilde{\mathcal{E}}_0(\mathbf{W}) \lesssim |\mathcal{I}| + |\mathcal{II}| + |\mathcal{III}| \lesssim \max_{i=1,2} |(\eta_i, u_i)|_{V_\mu^s} |(\psi, w)|_{V_\mu^0}^2,$$

which concludes the proof of Proposition 4.3.

□

4.3. **Estimates for system** (1.9)**.** Again, we let $\mathbf{U}_1 = (\eta_1, u_1)^T = \varepsilon(\zeta_1, v_1)^T$ and $\mathbf{U}_2 = (\eta_2, u_2)^T = \varepsilon(\zeta_2, v_2)^T$ be two solutions of (1.9) and define the difference $(\psi, w) = (\eta_1 - \eta_2, u_1 - u_2)$. Then $\mathbf{W} = (\psi, w)^T$ solves

$$\partial_t \mathbf{W} + \mathscr{M}(\mathbf{U}_1, \mathrm{D})\mathbf{W} = \mathbf{F}, \tag{4.20}$$

with $\mathscr{M}$ defined as in (3.21) and $\mathbf{F}$ is defined by

$$\mathbf{F} = -\begin{pmatrix} \mathcal{T}_\mu(\mathrm{D})(w\partial_x \eta_2) + \mathcal{T}_\mu(\mathrm{D})(\psi \partial_x u_2) \\ \mathcal{T}_\mu(\mathrm{D})(w\partial_x u_2)\end{pmatrix}. \tag{4.21}$$

The energy associated with (4.20) is given in terms of the symmetrizer (3.22) by

$$\tilde{\mathscr{E}}_s(\mathbf{W}) := \left(\Lambda^s \mathbf{W}, \mathscr{Q}(\mathbf{U}_1, \mathrm{D})\Lambda^s \mathbf{W}\right)_{L^2}. \tag{4.22}$$

Proposition 4.4. *Take $s > \frac{3}{2}$, $\varepsilon, \mu \in (0,1)$ and $\sigma \geq 0$. Let $(\eta_1, u_1), (\eta_2, u_2) \in C([0,T_0] : X_{\sigma,\mu}^s(\mathbb{R}))$ be two solutions of* (1.9) *on a time interval $[0, T_0]$ for some $T_0 > 0$. Moreover, assume there exist $h_0 \in (0,1)$ and $h_1 > 0$ such that*

$$h_0 - 1 \leq \eta_1(x,t), \quad \forall (x,t) \in \mathbb{R}\times[0,T_0] \quad \textit{and} \quad \sup_{t\in[0,T_0]} |(\eta_1, u_1)|_{H^s\times H^s} \leq h_1. \tag{4.23}$$

Define the difference to be $\mathbf{W} = (\psi, w) = (\eta_1 - \eta_2, u_1 - u_2)$. Then, for the energy defined by (4.22)*, there holds*

$$\frac{d}{dt}\tilde{\mathscr{E}}_0(\mathbf{W}) \lesssim_\sigma \max_{i=1,2} |(\eta_i, u_i)|_{H^s} |(\psi, w)|_{X_{\sigma,\mu}^0}^2, \tag{4.24}$$

and

$$|(\psi, w)|_{X_{\sigma,\mu}^0}^2 \lesssim \tilde{\mathscr{E}}_0(\mathbf{W}) \lesssim |(\psi, w)|_{X_{\sigma,\mu}^0}^2. \tag{4.25}$$

Furthermore, we have the following estimate at the $X_{\sigma,\mu}^s-$ level:

$$\frac{d}{dt}\tilde{\mathscr{E}}_s(\mathbf{W}) \lesssim_\sigma \left|\left(\Lambda^s \mathbf{F}, \mathscr{Q}(\mathbf{U}_1, \mathrm{D})\Lambda^s \mathbf{W}\right)_{L^2}\right| + \max_{i=1,2} |(\eta_i, u_i)|_{X_{\sigma,\mu}^s} |(\psi, w)|_{X_{\sigma,\mu}^s}^2, \tag{4.26}$$

and

$$|(\psi, w)|_{X_{\sigma,\mu}^s}^2 \lesssim \tilde{\mathscr{E}}_s(\mathbf{W}) \lesssim |(\psi, w)|_{X_{\sigma,\mu}^s}^2. \tag{4.27}$$

Proof. By previous arguments, we note that the proofs of (4.25) and (4.27) are similar to the proof of (3.26).

Moreover, we will only prove (4.24) since the control of (4.26) follows by the proof of Proposition 3.4.

We will now prove (4.24). Then we first use (4.20) and the self-adjointness of $\mathscr{Q}(\mathbf{U}_1, \mathrm{D})$ to write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\tilde{\mathscr{E}}_0(\mathbf{W}) &= \frac{1}{2}\big(\mathbf{W}, (\partial_t \mathscr{Q}(\mathbf{U}_1, \mathrm{D}))\mathbf{W}\big)_{L^2} + \big(\mathbf{F}, \mathscr{Q}(\mathbf{U}_1, \mathrm{D})\mathbf{W}\big)_{L^2} \\ &\quad - \big(\mathscr{M}(\mathbf{U}_1, \mathrm{D})\mathbf{W}, \mathscr{Q}(\mathbf{U}_1, \mathrm{D})\mathbf{W}\big)_{L^2} \\ &=: \mathscr{I} - \mathscr{I}\mathscr{I} - \mathscr{I}\mathscr{I}\mathscr{I}.\end{aligned}$$

<u>Control of $\mathscr{I}$.</u> By (4.20), Hölder's inequality, the Sobolev embedding and (4.23) we deduce

$$|\mathscr{I}| = \frac{1}{2}\big|\big(w, (\partial_t \eta_1)w\big)_{L^2}\big| \lesssim |u_1|_{H^s}(1 + |\eta_1|_{H^s})|w|_{L^2}^2 \lesssim \max_{i=1,2} |(\eta_i, u_i)|_{H^s}|(\psi, w)|^2_{X^0_{\sigma,\mu}}.$$

<u>Control of $\mathscr{I}\mathscr{I}$.</u> The contribution from the source term is given by,

$$\begin{aligned}\mathscr{I}\mathscr{I} &= \big(w\partial_x \eta_2, \mathcal{K}_\mu(\mathrm{D})\psi\big)_{L^2} + \big(\psi\partial_x \eta_2, \mathcal{K}_\mu(\mathrm{D})\psi\big)_{L^2} \\ &\quad + \big(w\partial_x u_2, w\big)_{L^2} + \big(\mathcal{T}_\mu(\mathrm{D})(w\partial_x u_2), \eta_1 w\big)_{L^2} \\ &=: \mathscr{I}\mathscr{I}_1 + \mathscr{I}\mathscr{I}_2 + \mathscr{I}\mathscr{I}_3 + \mathscr{I}\mathscr{I}_4.\end{aligned}$$

Control of $\mathscr{I}\mathscr{I}_1 + \mathscr{I}\mathscr{I}_2$. For $\sigma > 0$, we first apply the Cauchy-Schwarz inequality, (2.1), and the Sobolev embedding to deduce that for $s > \frac{3}{2}$

$$\begin{aligned}|\mathscr{I}\mathscr{I}_1| + |\mathscr{I}\mathscr{I}_2| &\lesssim (|w|_{L^2} + |\psi|_{L^2})|\eta_2|_{H^s}|\mathcal{K}_\mu(\mathrm{D})\psi|_{L^2} \\ &\lesssim |\eta_2|_{H^s}|(\psi, w)|^2_{X^0_{\sigma,\mu}}.\end{aligned}$$

The case $\sigma = 0$, is similar where we instead use the boundedness of $\mathcal{T}_\mu(\mathrm{D})$ on $L^2(\mathbb{R})$ to obtain

$$|\mathscr{I}\mathscr{I}_1| + |\mathscr{I}\mathscr{I}_2| \lesssim |\eta_2|_{H^s}|(\psi, w)|^2_{L^2}.$$

Control of $\mathscr{I}\mathscr{I}_3 + \mathscr{I}\mathscr{I}_4$. Both terms are treated with the Cauchy-Schwatz inequality, (2.14) and the Sobolev embedding. Consequently, for $s > \frac{3}{2}$ and $\sigma \geq 0$ there holds

$$|\mathscr{I}\mathscr{I}_3| + |\mathscr{I}\mathscr{I}_4| \lesssim (1 + |\eta_1|_{H^s})|u_2|_{H^s}|w|^2_{L^2} \lesssim \max_{i=1,2} |(\eta_i, u_i)|_{H^s}|(\psi, w)|^2_{X^0_{\sigma,\mu}}.$$

Gathering all these estimates yields

$$|\mathscr{I}\mathscr{I}| \lesssim \max_{i=1,2} |(\eta_i, u_i)|_{H^s}|(\psi, w)|^2_{X^0_{\sigma,\mu}}.$$

<u>Control of $\mathscr{I}\mathscr{I}\mathscr{I}$.</u> The symmetrized term $\mathscr{I}\mathscr{I}\mathscr{I}$ reads:

$$\begin{aligned}\mathscr{I}\mathscr{I}\mathscr{I} =& \big(u_1\partial_x\psi, \mathcal{K}_\mu(\mathrm{D})\psi\big)_{L^2} + \big((1 + \mathcal{T}_\mu(\mathrm{D})\eta_1)\partial_x w, \mathcal{T}_\mu^{-1}(\mathrm{D})\mathcal{K}_\mu(\mathrm{D})\psi\big)_{L^2} \\ &+ \big(\mathcal{K}_\mu(\mathrm{D})\partial_x\psi, (\mathcal{T}_\mu^{-1}(\mathrm{D}) + \eta_1)w\big)_{L^2} + \big(\mathcal{T}_\mu(\mathrm{D})(u_1\partial_x w), (\mathcal{T}_\mu^{-1}(\mathrm{D}) + \eta_1)w\big)_{L^2} \\ =& \mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}.\end{aligned}$$

Control of $\mathscr{A}_{11}$. In the case $\sigma > 0$, we decompose $\mathscr{A}_{11}$ as

$$\begin{aligned}\mathscr{A}_{11} &= \big(u_1\partial_x\psi, ((\chi_\mu^{(1)})^2\mathcal{K}_\mu)(\mathrm{D})\psi\big)_{L^2} + \big(u_1\partial_x\psi, ((\chi_\mu^{(2)})^2(\mathrm{F}_{\mu,\frac{1}{2}})^2)(\mathrm{D})\psi\big)_{L^2} \\ &\quad - \big(u_1\partial_x\psi, ((\chi_\mu^{(2)})^2(\mathrm{F}_{\mu,0})^2)(\mathrm{D})\psi\big)_{L^2},\end{aligned}$$

where we have divided the multiplier $\mathcal{K}_\mu(\mathrm{D})$ into three pieces in the same way as we did in (4.11). We may therefore apply the same estimates as for A_4^3 in the proof of Proposition 4.1, where we change the role of ψ and w to obtain

$$|\mathscr{A}_{11}| = \big|\big(u_1\partial_x\psi, \mathcal{K}_\mu(\mathrm{D})\psi\big)_{L^2}\big| \lesssim_\sigma |u_1|_{H^s}(|\psi|^2_{L^2} + \sqrt{\sigma}\mu^{\frac{1}{4}}|\psi|^2_{H^{\frac{1}{2}}}).$$

Then use inequality (3.27) to conclude that

$$|\mathscr{A}_{11}| \lesssim |u_1|_{H^s}|(\psi, w)|^2_{X^0_{\sigma,\mu}}.$$

In the case $\sigma = 0$, we simply use Hölder's inequality, (2.14), (2.34), to obtain

$$\begin{aligned}|\mathscr{A}_{11}| &\lesssim |\big((\partial_x u_1)\sqrt{\mathcal{T}_\mu}(\mathrm{D})\psi, \sqrt{\mathcal{T}_\mu}(\mathrm{D})\psi\big)_{L^2}| + |\big([\sqrt{\mathcal{T}_\mu}(\mathrm{D}), u_1]\partial_x\psi, \sqrt{\mathcal{T}_\mu}(\mathrm{D})\psi\big)_{L^2}| \\ &\lesssim |u_1|_{H^s}|\psi|^2_{L^2}.\end{aligned}$$

Control of $\mathscr{A}_{12} + \mathscr{A}_{21}$. Treating the off-diagonal terms we first observe by integrating by parts that

$$\mathscr{A}_{12} = -\big((\partial_x\eta_1)w, \mathcal{K}_\mu(\mathrm{D})\psi\big)_{L^2} - \mathscr{A}_{21}.$$

Therefore, we may apply Hölder's inequality, the Sobolev embedding, and (2.1) for $\sigma \geq 0$, to deduce

$$|\mathscr{A}_{12} + \mathscr{A}_{21}| \lesssim |\eta_1|_{H^s}|(\psi, w)|_{X^0_{\sigma,\mu}}.$$

Control of $\mathscr{A}_{22}$. We decompose $\mathscr{A}_{22}$ into two terms

$$\begin{aligned}\mathscr{A}_{22} &= \big(u_1\partial_x w, w\big)_{L^2} + \big(\mathcal{T}_\mu(\mathrm{D})(u_1\partial_x w), \eta_1 w\big)_{L^2} \\ &= \mathscr{A}^1_{22} + \mathscr{A}^2_{22}.\end{aligned}$$

We see that $\mathscr{A}^1_{22}$ is easily treated by the Cauchy-Schwarz inequality, integration by parts, the Sobolev embedding, and (4.23). Indeed, there holds

$$|\mathscr{A}^1_{22}| \lesssim |u_1|_{H^s}|w|^2_{L^2}.$$

Next, we decompose $\mathscr{A}^2_{22}$ into three parts

$$\begin{aligned}\mathscr{A}^2_{22} &= \big([\sqrt{\mathcal{T}_\mu}(\mathrm{D}), u_1]\partial_x w, \sqrt{\mathcal{T}_\mu}(\mathrm{D})(\eta_1 w)\big)_{L^2} + \big(u_1\sqrt{\mathcal{T}_\mu}(\mathrm{D})\partial_x w, [\sqrt{\mathcal{T}_\mu}(\mathrm{D}), \eta_1]w\big)_{L^2} \\ &\quad + \big(u_1\sqrt{\mathcal{T}_\mu}(\mathrm{D})\partial_x w, \eta_1\sqrt{\mathcal{T}_\mu}(\mathrm{D})w\big)_{L^2}. \\ &= \mathscr{A}^{2,1}_{22} + \mathscr{A}^{2,2}_{22} + \mathscr{A}^{2,3}_{22}.\end{aligned}$$

For $\mathscr{A}^{2,1}_{22}$, we simply apply Hölder's inequality, (2.34), (2.14), the Sobolev embedding to find that

$$|\mathscr{A}^{2,1}_{22}| \leq |[\sqrt{\mathcal{T}_\mu}(\mathrm{D}), u_1]\partial_x w|_{L^2}|\sqrt{\mathcal{T}_\mu}(\mathrm{D})(\eta_1 w)|_{L^2} \lesssim |u_1|_{H^s}|\eta_1|_{H^s}|w|^2_{L^2}.$$

For $\mathscr{A}^{2,2}_{22}$, we first remark that

$$|[\sqrt{\mathcal{T}_\mu}(\mathrm{D}), \eta_1]w|_{L^2} \lesssim |\eta_1|_{L^\infty}|w|_{L^2}, \tag{4.28}$$

simply by using Hölder's inequality and (2.14). Then after integrating by parts, we use Hölder's inequality, the Sobolev embedding, (2.35), (4.23), and (4.28) to deduce that

$$\begin{aligned}|\mathscr{A}^{2,2}_{22}| &\leq |\partial_x u_1|_{L^\infty}|\sqrt{\mathcal{T}_\mu}(\mathrm{D})w|_{L^2}|[\sqrt{\mathcal{T}_\mu}(\mathrm{D}), \eta_1]w|_{L^2} \\ &\quad + |u_1|_{L^\infty}|\sqrt{\mathcal{T}_\mu}(\mathrm{D})w|_{L^2}|\partial_x[\sqrt{\mathcal{T}_\mu}(\mathrm{D}), \eta_1]w|_{L^2} \\ &\lesssim \max_{i=1,2}|(\eta_i, u_i)|_{H^s}|w|^2_{L^2}.\end{aligned}$$

Lastly, we use integration by parts, then apply Hölder's inequality, (2.14), the Sobolev embedding, and (4.23) to obtain that

$$|\mathscr{A}^{2,3}_{22}| \leq \frac{1}{2}|\partial_x(u_1\eta_1)|_{L^\infty}|\sqrt{\mathcal{T}_\mu}(\mathrm{D})w|^2_{L^2} \lesssim \max_{i=1,2}|(\eta_i, u_i)|_{H^s}|w|^2_{L^2}.$$

We may now gather these estimates to conclude that

$$|\mathscr{A}_{22}| \lesssim \max_{i=1,2}|(\eta_i, u_i)|_{H^s}|(\psi, w)|^2_{X^0_{\sigma,\mu}},$$

and as a result the proof of Proposition 4.4 is now complete. □

5. Proof of Theorem 1.6 in the one-dimensional case

Proof. The proof is divided into eight steps, utilizing the results above.

Step 1: *Existence of solutions for a regularized system.* Let $s > \frac{1}{2}$, $0 < \nu < 1$ and $\alpha = \frac{3}{2}^{+}$. Then, for any initial data $\mathbf{U}_0 := (\eta_0, u_0) \in V^s_\mu(\mathbb{R})$, we claim that there exist $c_\sigma > 0$, and a time

$$0 < T_\nu := T_\nu\big(|(\eta_0, u_0)|_{V^s_\mu}\big) = \left(\frac{c_\sigma \nu^{\frac{2}{3\alpha}}}{1 + |(\eta_0, u_0)|_{V^s_\mu}}\right)^{\frac{1}{1-\frac{2}{3\alpha}}} \tag{5.1}$$

such that $\mathbf{U}^\nu := (\eta^\nu, u^\nu)^T \in C([0, T_\nu]; V^s_\mu(\mathbb{R}))$ is a unique solution of the regularized Cauchy problem:

$$\begin{cases} \partial_t \eta^\nu + u^\nu \partial_x \eta^\nu + (\mathcal{K}_\mu(\mathrm{D}) + \eta^\nu)\partial_x u^\nu = -\nu \langle D\rangle^\alpha \eta^\nu \\ \partial_t u^\nu + \partial_x \eta^\nu + u^\nu \partial_x u^\nu = -\nu \langle D \rangle^\alpha u^\nu. \end{cases} \tag{5.2}$$

The proof of the existence of a unique solution is a consequence of the contraction mapping principle. First, we find the diagonalisation of the linear part, $S^\nu(t)$, of (5.2) to be

$$S^\nu(t) = \frac{1}{2}\begin{pmatrix} -\sqrt{\mathcal{K}_\mu}(\mathrm{D}) & \sqrt{\mathcal{K}_\mu}(\mathrm{D}) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \exp(-t\mathcal{L}^\nu_-(\mathrm{D})) & 0 \\ 0 & \exp(-t\mathcal{L}^\nu_+(\mathrm{D})) \end{pmatrix} \begin{pmatrix} -\mathcal{K}_\mu^{-\frac{1}{2}}(\mathrm{D}) & 1 \\ \mathcal{K}_\mu^{-\frac{1}{2}}(\mathrm{D}) & 1 \end{pmatrix}$$

where $\mathcal{L}^\nu_\pm(\mathrm{D}) = \pm i D \sqrt{\mathcal{K}_\mu}(\mathrm{D}) + \nu \langle D \rangle^\alpha$. Then we shall show that

$$\Phi_{\mathbf{U}_0}(\mathbf{U}^\nu)(t) := S^\nu(t)\mathbf{U}_0 - \int_0^t S^\nu(t-s)\partial_x \begin{pmatrix} \eta^\nu u^\nu \\ \frac{(u^\nu)^2}{2} \end{pmatrix}(s)\, ds, \tag{5.3}$$

defines a contraction on the closed subspace $B(a)$ of $C([0, T]; V^s_\mu(\mathbb{R}))$, whose norm is bounded by a, and is centered at the point $S^\nu(t)\mathbf{U}_0$. However, we note by Plancherel that for $|\xi| > 1$ there holds,

$$|S^\nu(t)\partial_x \mathbf{U}|_{L^2} \lesssim_\sigma \| |\xi|^{\frac{3}{2}} e^{-\nu|\xi|^\alpha t} \hat{\mathbf{U}}|_{L^2} \lesssim_\sigma \frac{1}{(\nu t)^{\frac{2}{3\alpha}}} |\mathbf{U}|_{L^2}. \tag{5.4}$$

The same is trivially true for $|\xi| \leq 1$. Now, combining (5.4) with the fact that $\frac{2}{3\alpha} < 1$ and the algebra property of $H^{\frac{1}{2}^+}(\mathbb{R})$ we deduce that

$$|\Phi_{\mathbf{U}_0}(\mathbf{U}^\nu) - S^\nu(t)\mathbf{U}_0|_{H^s} \lesssim_\sigma T^{1-\frac{2}{3\alpha}} \nu^{-\frac{2}{3\alpha}} |\mathbf{U}^\nu|^2_{H^s}$$

and

$$|\Phi_{\mathbf{U}_0}(\mathbf{U}^\nu_1) - \Phi_{\mathbf{U}_0}(\mathbf{U}^\nu_2)|_{H^s} \lesssim_\sigma T^{1-\frac{2}{3\alpha}} \nu^{-\frac{2}{3\alpha}} |\mathbf{U}^\nu_1 - \mathbf{U}^\nu_2|_{H^s} (|\mathbf{U}^\nu_1|_{H^s} + |\mathbf{U}^\nu_2|_{H^s})$$

Therefore, by choosing $a = |\mathbf{U}_0|_{L^\infty_T V^s_\mu}$ and T as in (5.1) we can use the above estimates to conclude by the Fixed Point Theorem that there exist a unique solution of (5.2) in $C([0, T_\nu]; V^s_\mu(\mathbb{R}))$.

Remark 5.1. *A consequence of Step* 1*, is the continuity of the flow map associated with* (5.2)*. But this is only for the 'short' time T_ν given by* (5.1)*, and is therefore not useful for the limit equation.*

Step 2: *The blow-up alternative.* We define the maximal time of existence to be

$$T^\star_\nu = \sup\Big\{T_\nu > 0 \ : \ \exists!\, \mathbf{U}^\nu = (\eta^\nu, u^\nu)^T \text{ solution of (5.2) in } C([0, T_\nu]; V^s_\mu(\mathbb{R}))\Big\}.$$

Then we claim that the solution of (5.2) satisfies the blow-up alternative:

$$\text{If}\quad T_\nu^\star < \infty, \quad \text{then} \quad \lim_{t\nearrow T_\nu^\star} |(\eta^\nu, u^\nu)(t)|_{V_\mu^s} = \infty. \tag{5.5}$$

First, we argue by contradiction that $T_\nu^\star < \infty$ and there exist $A \in \mathbb{R}^+$ such that

$$\sup_{t\in[0,T_\nu^\star)} |(\eta^\nu, u^\nu)(t)|_{V_\mu^s} = A. \tag{5.6}$$

We use (5.1) to define $\tau_{\nu,A} = T_\nu^\star - \frac{T_\nu(A)}{2}$. Then we have that

$$a := |(\eta^\nu, u^\nu)(\tau_{\nu,A})|_{V_\mu^s} \leq A.$$

Therefore, if we let $\mathbf{V}_0^\nu = (\eta^\nu, u^\nu)^T(\tau_{\nu,A})$ serve as initial data, then (5.2) has a unique solution given by

$$\mathbf{V}^\nu(t) = S^\nu(t)\mathbf{V}_0^\nu - \int_0^t S^\nu(t-s)\partial_x \begin{pmatrix} v_1^\nu v_2^\nu \\ \frac{(v_2^\nu)^2}{2} \end{pmatrix}(s)\, ds \tag{5.7}$$

with $\mathbf{V}^\nu = (v_1^\nu, v_2^\nu) \in C([0, T_\nu(a)]; V_\mu^s(\mathbb{R}))$. Here, $T_\nu(a)$ is given by (5.1) due to Step 1. Moreover, we observe that $T_\nu(a) \geq T_\nu(A)$ by definition (5.1), and implies $\tau_{\nu,A} + T_\nu(a) \geq T_\nu^\star + \frac{T_\nu(A)}{2}$. Thus, we define the extension of $\mathbf{U}^\nu = (\eta^\nu, u^\nu)^T$ by the function

$$\mathbf{Z}^\nu(t) = \begin{cases} \mathbf{U}^\nu(t), & \text{if} \quad 0 \leq t < \tau_{\nu,A} \\ \mathbf{V}^\nu(t - \tau_{\nu,A}), & \text{if} \quad \tau_{\nu,A} \leq t \leq \tau_{\nu,A} + T_\nu(a), \end{cases}$$

and one can verify that it is a solution of (5.2) for all $t \in [0, T_\nu^\star + \frac{T_\nu(A)}{2}] \subset [0, \tau_{\nu,A} + T_\nu(a)]$. This contradicts the definition of $T_\nu^\star$. Thus, we conclude that if $T_\nu^\star < \infty$, then necessarily $A = \infty$ in (5.6), and implies

$$\limsup_{t\nearrow T_\nu^\star} |(\eta^\nu, u^\nu(t))|_{V_\mu^s} = \infty. \tag{5.8}$$

To conclude the proof of the claim, we use (5.8) to verify that for all $R > 0$ there exists an open interval $(t_R, T_\nu^\star)$ such that $|(\eta^\nu, u^\nu(t))|_{V_\mu^s} > R$, for all $t \in (t_R, T_\nu^\star)$. Indeed, we argue by contradiction that there exists $R \in \mathbb{R}^+$ such that for all $0 < t_R < T_\nu^\star$, we have

$$|(\eta^\nu, u^\nu)(t)|_{V_\mu^s} \leq R, \tag{5.9}$$

for some $t \in (t_R, T_\nu^\star)$. By (5.8) there is a time such that $\tau_{R,0} > T_\nu^\star - \frac{T_\nu(R)}{2}$ and satisfying

$$|(\eta^\nu, u^\nu)(\tau_{R,0})|_{V_\mu^s} > R.$$

On the other hand, by assumption (5.9) we can take $t_R = \tau_{R,0}$ and use the fact that there is a time $\tau_{R,1} \in (t_R, T_\nu^\star)$ such that

$$|(\eta^\nu, u^\nu)(\tau_{R,1})|_{V_\mu^s} \leq R.$$

Thus, by the same argument as above we can take $(\eta^\nu, u^\nu)(\tau_{R,1})$ as initial data of (5.2) to find an extended solution defined on $[0, T_\nu^\star + \frac{T_\nu(R)}{2}] \subset [0, \tau_{R,1} + T_\nu(R)]$. This contradicts the definition of $T_\nu^\star$. As a result, we conclude that (5.5) holds true.

Step 3: *The existence time is independent of* $\nu > 0$. We claim that there exists

$$T = \frac{1}{k_\sigma^1 |(\zeta_0, v_0)|_{V_\mu^s}},$$

as in (1.16), such that the regularized solution $\varepsilon(\zeta^\nu, v^\nu) = (\eta^\nu, u^\nu)$ exists on the interval $[0, \frac{T}{\varepsilon}]$.

The proof relies on a bootstrap argument similar to the proof of Lemma 5.1 in [28]. In fact, the long time existence is a direct consequence of the following remark and lemma.

Remark 5.2. *We will invoke the estimates in Proposition 3.1 for system* (5.2)*. However, due to the parabolic regularisation, we must also control the additional terms given by*

$$\frac{d}{dt}E_s(\mathbf{U}^\nu) \lesssim_\sigma \big(E_s(\mathbf{U}^\nu)\big)^{\frac{3}{2}} - \nu\big(\Lambda^{s+\alpha}\mathbf{U}^\nu, Q(\mathbf{U}^\nu, \mathrm{D})\Lambda^s\mathbf{U}^\nu\big)_{L^2}.$$

But decomposing the last term, we note that

$$\begin{aligned}\big(\Lambda^{s+\alpha}\mathbf{U}^\nu, Q(\mathbf{U}^\nu,\mathrm{D})\Lambda^s\mathbf{U}^\nu\big)_{L^2} &= \big(\Lambda^{s+\frac{\alpha}{2}}\eta^\nu, \Lambda^{s+\frac{\alpha}{2}}\eta^\nu\big)_{L^2} + \big(\Lambda^{s+\frac{\alpha}{2}}u^\nu, (\mathcal{K}_\mu(\mathrm{D})+\eta^\nu)\Lambda^{s+\frac{\alpha}{2}}u^\nu\big)_{L^2} \\ &\quad + \big(\Lambda^{s+\frac{\alpha}{2}}u^\nu, [\Lambda^{\frac{\alpha}{2}}, \eta^\nu]\Lambda^s u^\nu\big)_{L^2} \\ &= I + II + III.\end{aligned}$$

The first two terms has a positive sign, while the third term, III, can be absorbed by the second term by using Cauchy-Schwarz, (A.9) *and Young's inequality:*

$$|III| \leq 2c_1|u^\nu|_{H^{s+\frac{\alpha}{2}}}|\eta^\nu|_{H^s}|u^\nu|_{H^s} \leq c_2|u^\nu|^2_{H^{s+\frac{\alpha}{2}}} + \frac{c_1}{c_2}|\eta^\nu|^2_{H^s}|u^\nu|^2_{H^s},$$

by choosing $0 < c_2 < \min\{\frac{h_0}{2}, \frac{\sigma}{2}\}$. Indeed, by (2.7)*,* (2.8)*,* (3.4)*,* (3.5) *and* (3.7) *we get the bound*

$$\begin{aligned}-\nu(I+II+III) \leq -\nu|\eta^\nu|^2_{H^{s+\frac{\alpha}{2}}} &+ \nu(c_2 - \min\{\frac{h_0}{2}, \frac{\sigma}{2}\})|u^\nu|^2_{H^{s+\frac{\alpha}{2}}} \\ &+ \frac{c_1}{c_2}c_\sigma\big(E_s(\mathbf{U}^\nu)\big)^{\frac{3}{2}}.\end{aligned}$$

Therefore, we have that Proposition 3.1 holds for the regularised system.

Lemma 5.3. *Let $s > 2$ and ε be as in* (1.15)*. Let $(\eta^\nu, u^\nu) = \varepsilon(\zeta^\nu, v^\nu) \in C([0, T^\star_\nu); V^s_\mu(\mathbb{R}))$ be a solution of* (5.2) *with initial data $\varepsilon(\zeta_0, v_0) = (\eta_0, u_0) \in V^s_\mu(\mathbb{R})$, defined on its maximal time of existence and satisfying the blow-up alternative* (5.5)*. Moreover, let $\eta_0 = \varepsilon\zeta_0$ satisfy either the non-cavitation condition* (1.13) *or the $\sigma-$dependent surface condition* (1.14)*, depending on whether $\sigma \geq \frac{1}{3}$ or $0 < \sigma < \frac{1}{3}$, respectively. Then there exists a time*

$$T_0 = \frac{1}{k^1_\sigma|(\eta_0, u_0)|_{V^s_\mu}}, \tag{5.10}$$

such that $T^\star_\nu > T_0$ and

$$\sup_{t\in[0,T_0]}|(\eta^\nu, u^\nu)(t)|_{V^s_\mu} \leq 4k^2_\sigma|(\eta_0, u_0)|_{V^s_\mu}. \tag{5.11}$$

The constants are on the form

$$k^2_\sigma = \frac{c^2_\sigma}{c^1_\sigma} \quad \textit{and} \quad k^1_\sigma = \begin{cases}\frac{C_1}{\sigma} & \textit{for } 0 < \sigma < \frac{1}{3} \\ C_2\sigma^2 & \textit{for } \sigma \geq \frac{1}{3}\end{cases}$$

where C_1 and C_2 are two positive constants to be fixed in the proof.

Proof. We define the set

$$\tilde{T}_\nu = \sup\Big\{T_\nu \in (0, T^\star_\nu) \; : \; \sup_{t\in[0,T_\nu]}|(\eta^\nu, u^\nu)(t)|_{V^s_\mu} \leq 4k^2_\sigma|(\eta_0, u_0)|_{V^s_\mu}\Big\}. \tag{5.12}$$

Then we first note that $\tilde{T}_\nu < T^\star_\nu$, or else it would contradict the blow-up alternative (5.5). For the proof we argue by contradiction that $\tilde{T}_\nu \leq T_0$.

The main idea is to improve the estimate given in (5.12). First, we verify that the solution (η^ν, u^ν) satisfy (3.4). Indeed, recalling assumption (1.15):

$$0 < \varepsilon \leq \frac{1}{k_\sigma^2 |(\zeta_0, v_0)|_{V_\mu^s}},$$

implies

$$|(\eta^\nu, u^\nu)|_{H^s} \leq k_\sigma^2 |(\eta_0^\nu, u_0^\nu)|_{V_\mu^s} = 4\varepsilon k_\sigma^2 |(\zeta_0, v_0)|_{V_\mu^s} \leq 4, \tag{5.13}$$

for all $t \in [0, \tilde{T}_\nu]$. Next, the solution (η^ν, u^ν) satisfy the non-cavitation condition. We will prove this as a consequence of the bound

$$\sup_{\tau \in [0, \tilde{T}_\nu]} |\partial_t \eta^\nu(\tau)| \leq k_\sigma^2 \sigma^2.$$

Indeed, by similar argument as for (3.8), we use (5.2), (5.13), and $H^{\frac{1}{2}^+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ to find

$$|\partial_t \eta^\nu|_{L^\infty} \leq c_\sigma^2 |(\eta^\nu, u^\nu)|_{V_\mu^s} + |\eta^\nu|_{H^s} |u^\nu|_{H^s} \leq 4 k_\sigma^2 c_\sigma^2 |(\eta_0, u_0)|_{V_\mu^s}.$$

Then, by the Fundamental Theorem of Calculus we obtain

$$1 + \eta^\nu(x,t) = 1 + \eta_0 + \int_0^t \partial_t \eta^\nu(x,s)\, ds \geq h_0 - k_\sigma^2 \sigma^2 \tilde{T}_\nu, \tag{5.14}$$

for all $t \in [0, \tilde{T}_\nu]$. On the one hand, if $\sigma \geq \frac{1}{3}$, then

$$k_\sigma^2 = \frac{c_\sigma^2}{c_\sigma^1} = c\sigma.$$

Thus, for $C_2 > 0$ large enough, we get that $k_\sigma^1 \geq C_2 \sigma^2 \geq \frac{c\sigma^2}{h_0}$. Moreover, by the assumption $\tilde{T}_\nu \leq T_0$, we conclude from (5.14) that

$$1 + \eta^\nu(x,t) \geq h_0 - k_\sigma^2 \sigma^2 T_0 \geq \frac{h_0}{2},$$

for all $t \in [0, \tilde{T}_\nu]$. On the other hand, in the case when $\sigma \in (0, \frac{1}{3})$ we need to verify (3.5). But this can be done the same way by choosing $k_\sigma^1 \geq \frac{C_1}{\sigma} \geq \frac{c}{\sigma h_\sigma}$ for $C_1 > 0$ large enough.

Having remark 5.2 in mind, the hypotheses of Proposition 3.1 are now verified, leaving us (3.6) and (3.7) at our disposal. With this at hand, we observe that

$$E_s(\mathbf{U}^\nu)(t) \leq E_s(\mathbf{U}^\nu)(0) + c_\sigma^2 \int_0^t \big(E_s(\mathbf{U}^\nu)(s')\big)^{\frac{3}{2}}\, ds' =: \tilde{\psi}(t).$$

By the above inequality, we then have $\tilde{\psi}'(t) \leq c_\sigma^2 \big(E_s(\mathbf{U}^\nu)(t)\big)^{\frac{3}{2}} \leq c_\sigma^2\, (\tilde{\psi}(t))^{\frac{3}{2}}$. We solve the differential inequality and use (3.7) to relate the energy with the V_μ^s-norm of the solution and deduce that

$$c_\sigma^1 |(\eta^\nu, u^\nu)(t)|_{V_\mu^s} \leq \frac{c_\sigma^2 |(\eta_0, u_0)|_{V_\mu^s}}{1 - \frac{(c_\sigma^2)^2}{2} t |(\eta_0, u_0)|_{V_\mu^s}}, \tag{5.15}$$

for all $t \in [0, \tilde{T}_\nu]$. Finally, if $C_1, C_2 > 0$ is large enough then since $\tilde{T}_\nu \leq T_0$ we have that

$$|(\eta^\nu, u^\nu)(t)|_{V_\mu^s} \leq 2k_\sigma^2 |(\eta_0^\nu, u_0^\nu)|_{V_\mu^s}.$$

Though, by continuity of the solution in time $t \in [0, T_\nu^\star)$, there exists $\tau > 0$ such that $|(\eta^\nu, u^\nu)(\tau)|_{V_\mu^s} \leq 3k_\sigma^2 |(\eta_0, u_0)|_{V_\mu^s}$ for $\tilde{T}_\nu < \tau < T_\nu^\star$. This contradicts the definition of $\tilde{T}_\nu$. Thus, we may conclude $T_0 < \tilde{T}_\nu$ for all $\nu > 0$ and that T_0 is independent from ν by its definition in (5.10).

□

Remark 5.4. *For $0<\sigma<\frac{1}{3}$ we observe that $k_\sigma^1 \sim k_\sigma^2 \sim \frac{1}{\sigma}$ and is due to the appearance of c_σ^1 in the coercivity estimate* (3.7)*. This will impact the size of the time interval when σ is small (see Remark 1.9). On the other hand, for system* (1.9) *the coercivity estimate* (3.26) *is independent of σ and therefore gives a longer time of existence, as noted in Remark 1.15.*

Step 4: *Uniqueness.* Given a solution of (1.5), then we claim that it must be unique.

We consider two solutions

$$\varepsilon(\zeta_1, v_1) = (\eta_1, u_1) \text{ and } \varepsilon(\zeta_1, v_1) = (\eta_1, u_1) \text{ in } C([0,T_0]; V_\mu^s(\mathbb{R})),$$

with the same initial data. Then define $\mathbf{W} = (\eta_1 - \eta_2, u_1 - u_2)^T$, which is associated to the initial datum $\mathbf{W}(0) = \mathbf{0}$. Since $(\eta_1, u_1) \in H^s(\mathbb{R})$, there exist a number $h_1 > 0$ such that $|(\eta_1, u_1)|_{H^s \times H^s} \leq h_1$. Moreover, η_1 satisfies the non-cavitation condition by the Fundamental Theorem of Calculus and the argument made in the proof of Lemma 5.3. Thus, we may use Proposition 4.1 to deduce

$$\frac{d}{dt}\tilde{E}_0(\mathbf{W}) \lesssim_\sigma \max_{i=1,2} |(\eta_i, u_i)|_{V_\mu^s} \tilde{E}_0(\mathbf{W}).$$

Then Grönwall's lemma and (4.7) implies that $|(\eta_1 - \eta_2, u_1 - u_2)(t)|_{V_\mu^s} = 0$ for all $t \in [0, T_0]$. We therefore conclude the proof of the uniqueness.

Step 5: *Existence of solutions.* We claim that for all $0 \leq s' < s$ there exists a solution $(\zeta, v) = \varepsilon^{-1}(\eta, u) \in C([0,T_0]; V_\mu^{s'}(\mathbb{R})) \cap L^\infty([0,T_0]; V_\mu^s(\mathbb{R}))$ of (1.5) with $T_0 = \mathcal{O}(\frac{1}{\varepsilon})$ defined by (5.10).

Using the change of variable $(\zeta, v) = \varepsilon^{-1}(\eta, u)$, we see that the claim in Step 5 is equivalent to proving that (η^ν, u^ν) solving (5.2) will satisfy system (3.1) in the limit $\nu \searrow 0$ on $[0, T_0]$. In fact, the main idea is to prove the convergence of $\{(\eta^\nu, u^\nu)\}$ as $\nu \searrow 0$ by considering the difference between two solutions

$$\mathbf{W} = (\psi, w) := (\eta^{\nu'} - \eta^\nu, u^{\nu'} - u^\nu).$$

with $0 < \nu' < \nu < \mu$ and where $(\eta^{\nu'}, u^{\nu'}), (\eta^\nu, u^\nu)$ are two sets of solutions to system (5.2), obtained in Step 1. Then for $\alpha = \frac{3}{2}^+$ we have that (ψ, w) satisfies a regularized version of (4.1):

$$\partial_t \mathbf{W} + M(\mathbf{U}^{\nu'}, \mathrm{D})\mathbf{W} = \mathbf{F}^\nu - \nu' \Lambda^\alpha \mathbf{W} + (\nu - \nu')\Lambda^\alpha \mathbf{U}^\nu,$$

with

$$\mathbf{F}^\nu = -\begin{pmatrix} w\partial_x \eta^\nu + \psi \partial_x u^\nu \\ w \partial_x u^\nu \end{pmatrix}, \tag{5.16}$$

and the same initial data.

The system also satisfies the estimates of Proposition 4.1 by simply noting that

$$\frac{d}{dt}\tilde{E}_0(\mathbf{W}) \lesssim_\sigma \tilde{E}_0(\mathbf{W}) - \nu'\big(\Lambda^\alpha \mathbf{W}, Q(\mathbf{U}^{\nu'}, \mathrm{D})\mathbf{W}\big)_{L^2} + (\nu - \nu')\big(\Lambda^\alpha \mathbf{U}^\nu, Q(\mathbf{U}^{\nu'}, \mathrm{D})\mathbf{W}\big)_{L^2}$$

with

$$\begin{aligned}\big(\Lambda^\alpha \mathbf{W}, Q(\mathbf{U}^{\nu'}, \mathrm{D})\mathbf{W}\big)_{L^2} &= \big(\Lambda^{\frac{\alpha}{2}}\psi, \Lambda^{\frac{\alpha}{2}}\psi\big)_{L^2} + \big(\Lambda^{\frac{\alpha}{2}} w, (\mathcal{K}_\mu(\mathrm{D}) + \eta^{\nu'})\Lambda^{\frac{\alpha}{2}} w\big)_{L^2} \\ &\quad + \big(\Lambda^{\frac{\alpha}{2}} w, [\Lambda^{\frac{\alpha}{2}}, \eta^{\nu'}] w\big)_{L^2}.\end{aligned}$$

The two first term has a positive sign, while the last term can be absorbed arguing exactly as in remark 5.2. On the other hand, we have directly that for $s > \frac{3}{2}$

$$\begin{aligned}|\big(\Lambda^\alpha \mathbf{U}^\nu, Q(\mathbf{U}^{\nu'}, \mathrm{D})\mathbf{W}\big)_{L^2}| \lesssim |\eta^\nu|_{H^s}|\psi|_{L^2} + |\eta^{\nu'}|_{H^s}|u^\nu|_{H^s}|w|_{L^2} \\ + |\sqrt{\mathcal{K}_\mu}(\mathrm{D})u^\nu|_{H^s}|\sqrt{\mathcal{K}_\mu}(\mathrm{D})w|_{L^2}.\end{aligned}$$

Thus, gathering these estimates with (1.17), (2.7) and (4.7) we find that

$$\frac{d}{dt}\tilde{E}_0(\mathbf{W}) \lesssim_\sigma |(\eta_0, u_0)|_{V^s_\mu}(\tilde{E}_0(\mathbf{W}) + (\nu - \nu')(\tilde{E}_0(\mathbf{W}))^{\frac{1}{2}}). \tag{5.17}$$

Step 5.1: *Convergence in* $C([0,T_0]; V^0_\mu(\mathbb{R}))$. Define the difference (ψ, w) as above, then use (5.17) and (4.7), combined with Grönwall's inequality and (5.11) to find the estimate

$$\sup_{t\in[0,T_0]} |(\psi, w)(t)|_{V^0_\mu} \lesssim_\sigma |(\eta_0, u_0)|_{V^s_\mu}(\nu - \nu').$$

Consequently, $\{(\eta^\nu, u^\nu)\}_{0<\nu\leq 1}$ defines a Cauchy sequence in $C([0,T_0]; V^0_\mu(\mathbb{R}))$ and we conclude that there exists a limit $(\eta, u) \in C([0,T_0]; V^0_\mu(\mathbb{R}))$ by completeness.

Step 5.2: *Solution in* $C([0,T_0]; V^{s'}_\mu(\mathbb{R})) \cap L^\infty([0,T_0]; V^s_\mu(\mathbb{R}))$ *for* $s' \in [0, s)$. As a direct consequence of (5.11) and the previous step, we deduce by interpolation

$$\begin{aligned}|(\psi, w)|_{L^\infty_{T_0}V^{s'}_\mu} &\lesssim_\sigma |(\psi, w)|^{\frac{s'}{s}}_{L^\infty_{T_0}V^s_\mu}|(\psi, w)|^{1-\frac{s'}{s}}_{L^\infty_{T_0}V^0_\mu} \\ &\lesssim_\sigma (\nu - \nu')^{1-\frac{s'}{s}}|(\eta_0, u_0)|_{V^s_\mu} \underset{\nu\to 0}{\longrightarrow} 0.\end{aligned} \tag{5.18}$$

Step 6: *The solution is bounded by the initial data.* We claim that the solution obtained in Step 5 satisfies (1.17).

Indeed, using the notation from the previous step, we deduce by (5.11) that

$$\{(\eta^\nu, u^\nu)\}_{0<\nu\leq 1} \subset C([0,T_0]; V^s_\mu(\mathbb{R}))$$

is a bounded sequence in a reflexive Banach space. As a result, we have by Eberlein-Šmulian's Theorem that $(\eta^\nu, u^\nu) \underset{\nu\to 0}{\rightharpoonup} (\eta, u)$ weakly in $V^s_\mu(\mathbb{R})$ for all $t \in [0, T_0]$ and implies

$$\sup_{t\in[0,T_0]} |(\eta, u)|_{V^s_\mu} \lesssim_\sigma |(\eta_0, u_0)|_{V^s_\mu}. \tag{5.19}$$

Remark 5.5. *For smooth data* $(\eta_0, u_0) \in H^\infty(\mathbb{R})$ *of* (3.1) *we could reapply the arguments above to deduce the existence of a smooth solution* $(\eta, u) \in C([0,T_0]; H^\infty(\mathbb{R}))$, *who satisfy the bound* (5.19) *for any* $s > 2$ *and with* T_0 *as defined in* (5.10).

Step 7: *Presistence of the solution.* We claim that there exists a unique solution $(\zeta, v) = \varepsilon^{-1}(\eta, u) \in C([0,T_0]; V^s_\mu(\mathbb{R}))$ of (1.5).

We consider (η^δ, u^δ), solving (3.1) with regularised initial data: $(\eta^\delta_0, u^\delta_0) = (\varphi_\delta(\mathrm{D})\eta_0, \varphi_\delta(\mathrm{D})u_0)$ and with $\varphi_\delta(\mathrm{D})$ as in definition 2.15. Then for any $\delta > 0$ we have by remark 5.5 that the solution is smooth and satisfy

$$|(\eta^\delta, u^\delta)|_{L^\infty_{T_0}V^s_\mu} \lesssim |(\eta^\delta_0, u^\delta_0)|_{V^s_\mu}, \tag{5.20}$$

for $t \in [0, T_0]$. To conclude the proof, we let $0 < \delta' < \delta < 1$ and again consider the difference

$$\mathbf{W} = (\psi, w) := (\eta^{\delta'} - \eta^\delta, u^{\delta'} - u^\delta),$$

which also satisfy

$$\partial_t \mathbf{W} + M(\mathbf{U}^{\delta'}, \mathrm{D})\mathbf{W} = \mathbf{F}^\delta$$

with

$$\mathbf{F}^\delta = -\begin{pmatrix} w\partial_x\eta^\delta + \psi\partial_x u^\delta \\ w\partial_x u^\delta \end{pmatrix}, \tag{5.21}$$

and with initial data

$$(\psi(0), w(0)) = ((\varphi_{\delta'}(\mathrm{D}) - \varphi_\delta(\mathrm{D}))\eta_0, (\varphi_{\delta'}(\mathrm{D}) - \varphi_\delta(\mathrm{D}))u_0). \tag{5.22}$$

The system satisfies the estimates of Proposition 4.1 and we use (4.6) and (4.7), combined with Grönwall's inequality and (5.20) to first find the estimate

$$\sup_{t\in[0,T_0]} |(\psi, w)(t)|_{V^0_\mu} \leq e^{c_\sigma^2 |(\eta_0,u_0)|_{V^s_\mu} T_0} |(\psi(0), w(0))|_{V^0_\mu},$$

with $|(\eta_0, u_0)|_{V^s_\mu} T_0 \lesssim_\sigma 1$ by definition (5.10). As a result, we use (5.22), the triangle inequality and (2.40) to deduce that

$$\sup_{t\in[0,T_0]} |(\psi, w)(t)|_{V^0_\mu} \lesssim_\sigma \delta^s |(\eta_0, u_0)|_{V^s_\mu} \underset{\delta\to 0}{\longrightarrow} 0. \tag{5.23}$$

Moreover, as a direct consequence of (5.11) and (5.23), we deduce by interpolation

$$\begin{aligned} |(\psi, w)|_{L^\infty_{T_0} V^{s'}_\mu} &\lesssim_\sigma |(\psi, w)|^{\frac{s'}{s}}_{L^\infty_{T_0} V^s_\mu} |(\psi, w^\delta)|^{1-\frac{s'}{s}}_{L^\infty_{T_0} V^0_\mu} \\ &\lesssim_\sigma \delta^{s-s'} |(\eta_0, u_0)|_{V^s_\mu} \underset{\delta\to 0}{\longrightarrow} 0. \end{aligned} \tag{5.24}$$

To conclude, we keep these estimates in mind where we aim to apply estimate (4.8), following the Bona-Smith argument [7]. But first we must control $\mathbf{F}^\delta$ in $V^s_\mu(\mathbb{R})$. The elements of $\mathbf{F}^\delta$ are given in (4.2), and we must therefore control the terms given by:

$$\begin{aligned} &\big(\Lambda^s \mathbf{F}^\delta, Q(\mathbf{U}^{\delta'}, D)\Lambda^s \mathbf{W}\big)_{L^2} \\ &= \big(\Lambda^s(w\partial_x\eta^\delta), \Lambda^s\psi\big)_{L^2} + \big(\Lambda^s(\psi\partial_x\eta^\delta), \Lambda^s\psi\big)_{L^2} \\ &\quad + \big(\Lambda^s(w\partial_x u^\delta), \eta^{\delta'}\Lambda^s w\big)_{L^2} + \big(\Lambda^s(w\partial_x u^\delta), \mathcal{K}_\mu(\mathrm{D})\Lambda^s w\big)_{L^2} \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned}$$

The terms A_1, A_2 and A_3 are treated similarly. For instance, take A_1. Then we observe that

$$A_1 \leq |\Lambda^s(w\partial_x\eta^\delta)|_{L^2} |\Lambda^s\psi|_{L^2}.$$

Furthermore, using (2.22) and the Sobolev embedding, we obtain that

$$|\Lambda^s(w\partial_x\eta^\delta)|_{L^2} \lesssim |w|_{L^\infty} |\Lambda^s\partial_x\eta^\delta|_{L^2} + |\Lambda^s w|_{L^2} |\eta^\delta|_{H^s}. \tag{5.25}$$

Using the triangle inequality, (2.39), and (5.20), we observe that

$$|\Lambda^s\partial_x\eta^\delta|_{L^2} \leq \delta^{-1} |\eta_0|_{H^s}, \tag{5.26}$$

which needs to be compensated to close the estimate. With this in mind, we use the Sobolev embedding $H^{\frac{1}{2}^+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and (5.24) to deduce

$$|w|_{L^\infty_{T_0} L^\infty} \lesssim |(\psi, w)|_{L^\infty_{T_0} V^{\frac{1}{2}^+}_\mu} \lesssim \delta^{s-\frac{1}{2}^+} |(\eta_0, u_0)|_{V^s_\mu}. \tag{5.27}$$

Thus, combining (5.25) with (5.26) and (5.27) we get that

$$|A_1| \lesssim \sup_{t\in[0,T_0]} \big(|w|_{H^s} |\eta^\delta|_{H^s} + \delta^{s-\frac{3}{2}^+} |(\eta_0, u_0)|_{V^s_\mu}\big) |\psi|_{H^s},$$

as $\delta \searrow 0$. Arguing similarly, and using estimate (5.20), we deduce that

$$|A_1| + |A_2| + |A_3| \lesssim \sup_{t\in[0,T_0]} |(\eta_0, u_0)|_{V^s_\mu}\big(|(\psi, w)|^2_{V^s_\mu} + \delta^{s-\frac{3}{2}^+}|(\psi, w)|_{V^s_\mu}\big).$$

For A_4, we write

$$\begin{aligned} A_4 = &\big([\Lambda^s\sqrt{\mathcal{K}_\mu}(\mathrm{D}), w]\partial_x u^\delta, \Lambda^s\sqrt{\mathcal{K}_\mu}(\mathrm{D})w\big)_{L^2} \\ &+ \big(w\Lambda^s\sqrt{\mathcal{K}_\mu}(\mathrm{D})\partial_x u^\delta, \Lambda^s\sqrt{\mathcal{K}_\mu}(\mathrm{D})w\big)_{L^2}. \end{aligned}$$

The commutator is treated by (2.18). While in the second term, we use (2.7) and argue as above, giving the estimate

$$|A_4| \lesssim_\sigma \sup_{t\in[0,T_0]} |(\eta_0, u_0)|_{V^s_\mu}\big(|(\psi, w)|^2_{V^s_\mu} + \delta^{s-\frac{3}{2}^+}|(\psi, w)|_{V^s_\mu}\big).$$

We may therefore conclude by (4.8):

$$\frac{d}{dt}\tilde{E}_s(\mathbf{W}) \lesssim_\sigma |(\eta_0, u_0)|_{V^s_\mu}\big(\tilde{E}_s(\mathbf{W}) + \delta^{s-\frac{3}{2}^+}\tilde{E}_s(\mathbf{W})^{\frac{1}{2}}\big).$$

Then Grönwall's inequality and (4.9) implies

$$|(\psi, w)|_{L^\infty_{T_0}V^s_\mu} \lesssim_\sigma \delta^{s-\frac{3}{2}^+}|(\eta_0, u_0)|_{V^s_\mu} \underset{\delta\to 0}{\longrightarrow} 0.$$

Thus, (η^δ, u^δ) is a Cauchy sequence in $C([0,T_0]; V^s_\mu(\mathbb{R}))$ and we conclude by the uniqueness of the limit that the solution $(\eta, u) \in C([0,T_0]; V^s_\mu(\mathbb{R}))$.

Step 8: *Continuous dependence of the flow map data solution.* Consider two sets of initial data $(\zeta_1, v_1)(0), (\zeta_2, v_2)(0) \in V^s_\mu(\mathbb{R})$. Then we claim that for all $\lambda > 0$, there exists $\gamma > 0$ such that having

$$|(\zeta_1 - \zeta_2, v_1 - v_2)(0)|_{V^s_\mu} < \gamma,$$

implies

$$|(\zeta_1 - \zeta_2, v_1 - v_2)|_{L^\infty_{\frac{T_0}{2}}V^s_\mu} < \lambda.$$

Equivalently, we will prove that for $\varepsilon(\zeta_1, \zeta_2, v_1, v_2) = (\eta_1, \eta_2, u_1, u_2)$ such that

$$|(\eta_1 - \eta_2, u_1 - u_2)(0)|_{V^s_\mu} < \varepsilon\gamma,$$

implies

$$|(\eta_1 - \eta_2, u_1 - u_2)|_{L^\infty_{\frac{T_0}{2}}V^s_\mu} < \varepsilon\lambda.$$

Using the notation in Step 7, we let $0 < \delta < 1$ to be fixed, and $(\eta_1^\delta, u_1^\delta), (\eta_2^\delta, u_2^\delta) \in C([0, \frac{T_0}{2}]; V^s_\mu(\mathbb{R}))$ be two solutions of (5.2) on large time with corresponding initial data $(\varphi_\delta(\mathrm{D})\eta_1, \varphi_\delta(\mathrm{D})u_1)(0)$ and $(\varphi_\delta(\mathrm{D})\eta_2, \varphi_\delta(\mathrm{D})u_2)(0)$. Then observe

$$\begin{aligned} |(\eta_1 - \eta_2, u_1 - u_2)|_{V^s_\mu} &\leq |(\eta_1 - \eta_1^\delta, u_1 - u_1^\delta)|_{V^s_\mu} + |(\eta_2 - \eta_2^\delta, u_2 - u_2^\delta)|_{V^s_\mu} \\ &\quad + |(\eta_1^\delta - \eta_2^\delta, u_1^\delta - u_2^\delta)|_{V^s_\mu} \\ &=: B_1 + B_2 + B_3. \end{aligned} \tag{5.28}$$

For the first two terms we use that $\varepsilon^{-1}(\eta^\delta, u^\delta) = (\zeta^\delta, v^\delta) \to (\zeta, v) = \varepsilon^{-1}(\eta, u)$ as $\delta \searrow 0$ by Step 6. Therefore it follows that B_1 and B_2 must at least satisfy the estimate,

$$\sup_{t\in[0,\frac{T_0}{2}]} (B_1 + B_2)(t) \lesssim_\sigma \varepsilon o_\delta(1). \tag{5.29}$$

While for B_3, we need the continuity of the flow map of the regularized system (5.2) on a long time (see Remark 5.1).

We let $\tilde{\mathbf{W}} = (\tilde{\psi}, \tilde{w}) = (\eta_1^\delta - \eta_2^\delta, u_1^\delta - u_2^\delta)$. Then staying consistent with previous notation, we have that the difference between two regularized solutions will satisfy the equation:

$$\partial_t \tilde{\mathbf{W}} + M(\mathbf{U}_1^\delta, \mathrm{D})\tilde{\mathbf{W}} = \tilde{\mathbf{F}}^\delta, \tag{5.30}$$

with

$$\tilde{\mathbf{F}}^\delta = -\begin{pmatrix} \tilde{w}\partial_x \ \eta_2^\delta + \tilde{\psi}\partial_x u_2^\delta \\ \tilde{w}\partial_x u_2^\delta \end{pmatrix},$$

and initial data

$$(\tilde{\psi}, \tilde{w})(0) = (\varphi_\delta(\mathrm{D})\eta_1 - \varphi_\delta(\mathrm{D})\eta_2, \varphi_\delta(\mathrm{D})u_1 - \varphi_\delta(\mathrm{D})u_2)(0).$$

We will use this information to estimate B_3 by suitable energy estimates at $V^0_\mu(\mathbb{R})$ and $V^s_\mu(\mathbb{R})-$level.

Similar to Step 6, we first obtain the estimate in $V^0_\mu(\mathbb{R})$ by using (4.7) and (4.9). Indeed, there holds

$$\frac{d}{dt}\tilde{E}_0(\tilde{\mathbf{W}}) \lesssim_\sigma \max_{i=1,2} |(\eta_i^\delta, u_i^\delta)|_{V^s_\mu} \tilde{E}_0(\tilde{\mathbf{W}}). \tag{5.31}$$

For simplicity we let $|(\eta_1, u_1)(0)|_{V^s_\mu} = \varepsilon K$. Moreover, we observe that if $\varepsilon\gamma < \frac{1}{2}|(\eta_1, u_1)(0)|_{V^s_\mu}$, then we have by (5.20) that

$$\begin{aligned} |(\eta_1^\delta, u_1^\delta)|_{L^\infty_{\frac{T_0}{2}} V^s_\mu} + |(\eta_2^\delta, u_2^\delta)||_{L^\infty_{\frac{T_0}{2}} V^s_\mu} &\lesssim_\sigma |(\eta_1, u_1)(0)|_{V^s_\mu} + |(\eta_2, u_2)(0)|_{V^s_\mu} \\ &\lesssim_\sigma \varepsilon K. \end{aligned} \tag{5.32}$$

As a result, we have an estimate of the difference in $V^0_\mu(\mathbb{R})$. Indeed, by Grönwall's inequality, (5.31), (5.32), the triangle inequality, and (2.40) implies

$$|(\tilde{\psi}, \tilde{w})|_{V^0_\mu} \lesssim_\sigma |(\tilde{\psi}, \tilde{w})(0)|_{V^0_\mu} \lesssim_\sigma \varepsilon K(\delta^s + K^{-1}\gamma). \tag{5.33}$$

We will now use this decay estimate to deal with (4.8), which is at the $V^s_\mu(\mathbb{R})-$level. Similar to Step 6, we decompose the source term (4.2) in four pieces

$$\begin{aligned} \tilde{A} &:= \big(\Lambda^s \tilde{\mathbf{F}}^\delta, Q(\mathbf{U}_1^\delta, \mathrm{D})\Lambda^s \tilde{\mathbf{W}}^\delta\big)_{L^2} \\ &= \big(\Lambda^s(\tilde{w}\partial_x \eta_2^\delta), \Lambda^s\tilde{\psi}\big)_{L^2} + \big(\Lambda^s(\tilde{\psi}\partial_x\eta_2^\delta), \Lambda^s\tilde{\psi}\big)_{L^2} \\ &\quad + \big(\Lambda^s(\tilde{w}\partial_x u_2^\delta), \eta_2^\delta \Lambda^s\tilde{w}\big)_{L^2} + \big(\Lambda^s(\tilde{w}\partial_x u_2^\delta), \mathcal{K}_\mu(\mathrm{D})\Lambda^s\tilde{w}\big)_{L^2} \\ &=: \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3 + \tilde{A}_4. \end{aligned}$$

To estimate $\tilde{A}_1$, we first obtain a bound similar to (5.24). Indeed, using the Sobolev emebedding, interpolation, (5.32), and (5.33) yields

$$\begin{aligned} \sup_{t\in[0,T_0]} |\tilde{w}|_{L^\infty} &\lesssim \sup_{t\in[0,T_0]} |(\tilde{\psi}, \tilde{w})|_{V_\mu^{\frac{1}{2}^+}} \\ &\lesssim |(\tilde{\psi}, \tilde{w})|^{\frac{1}{2s}^+}_{L^\infty_{T_0} V^s_\mu} |(\tilde{\psi}, \tilde{w})|^{1-\frac{1}{2s}^+}_{L^\infty_{T_0} V^0_\mu} \\ &\lesssim \varepsilon K(\delta^{s-\frac{1}{2}^+} + (K^{-1}\gamma)^{1-\frac{1}{2s}^+}) \end{aligned}$$

where $1 - \frac{1}{2s}^+ > 0$ for $s > \frac{1}{2}^+$. Then arguing as we did for A_1 in Step 7, we obtain that

$$\begin{aligned}|\tilde{A}_1| &\lesssim \sup_{t\in[0,T_0]} \big(|\tilde{w}|_{H^s}|\eta_2^\delta|_{H^s} + |\tilde{w}|_{L^\infty}\delta^{-1}|\eta_2^\delta|_{H^s}\big)|\tilde{\psi}|_{H^s} \\ &\lesssim \varepsilon K \sup_{t\in[0,T_0]} \big(|\tilde{w}|_{H^s} + \delta^{s-\frac{3}{2}^+} + \delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+}\big)|\tilde{\psi}|_{H^s}.\end{aligned}$$

Moreover, for the remaining terms, we can use similar estimates, recalling that for $\tilde{A}_4$ we also need to deal with the non-local operator $\mathcal{K}_\mu(\mathrm{D})$ (see step 7 for details). Indeed,

$$\tilde{A} \lesssim_\sigma \varepsilon K \sup_{t\in[0,T_0]} \big(|(\tilde{\psi},\tilde{w})|_{V^s_\mu} + \delta^{s-\frac{3}{2}^+} + \delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+}\big)|(\tilde{\psi},\tilde{w})|_{V^s_\mu}. \tag{5.34}$$

Consequently, combining estimates (4.8) and (4.9) with (5.34) yields

$$\frac{d}{dt}\tilde{E}_s(\tilde{\mathbf{W}}) \lesssim_\sigma \varepsilon K\big(\tilde{E}_s(\tilde{\mathbf{W}}) + (\delta^{s-\frac{3}{2}^+} + \delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+})\tilde{E}_s(\tilde{\mathbf{W}})^{\frac{1}{2}}\big).$$

Thus, we have an estimate on B_3 by the energy estimate (4.9), Grönwall's inequality and (2.41). Indeed, there holds

$$\begin{aligned}B_3 = |(\tilde{\psi},\tilde{w})|_{V^s_\mu} &\lesssim_\sigma |(\tilde{\psi},\tilde{w})(0)|_{V^s_\mu} + \varepsilon K(\delta^{s-\frac{3}{2}^+} + \delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+}) \\ &\lesssim_\sigma \varepsilon o_\delta(1) + \varepsilon\gamma + \varepsilon K(\delta^{s-\frac{3}{2}^+} + \delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+}).\end{aligned} \tag{5.35}$$

Returning to (5.28), we may conclude the proof of the continuous dependence. We first fix $0 < \delta < 1$ to be small enough and satisfying

$$o_\delta(1) + K\delta^{s-\frac{3}{2}^+} < \frac{\lambda}{2c_\sigma},$$

for some constant c_σ depending on σ. Then let γ verify the restriction:

$$\varepsilon\gamma < \frac{1}{2}|(\eta_1,u_1)(0)|_{V^s_\mu},$$

such that $\gamma + K\delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+} < \frac{\lambda}{2c_\sigma}$. Consequently, we have by (5.28), (5.29) and (5.35) that

$$\begin{aligned}\sup_{t\in[0,\frac{T_0}{2}]} |(\eta_1-\eta_2,u_1-u_2)(t)|_{V^s_\mu} &\le \varepsilon c_\sigma(o_\delta(1) + \gamma + K(\delta^{s-\frac{3}{2}^+} + \delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+})) \\ &< \varepsilon\lambda.\end{aligned}$$

As a result, we have demonstrated that the solution of (1.5) depends continuously on the initial data and thus completes the proof of Theorem 1.6.

□

6. The two-dimensional case

Let $x = (x_1,x_2) \in \mathbb{R}^2$ and $\mathbf{v} = (v_1,v_2)^T$. Then we observe that under the curl-free condition on the initial datum in Theorem 1.6 that (1.6) enjoys a similar structure to (1.5) (see also Lemma 4.2 in [35]). Indeed, since $\operatorname{curl}\mathbf{v}_0 = \mathbf{0}$ we can take the curl of the second equation in (1.6) and find that $\operatorname{curl}\mathbf{v} = \mathbf{0}$, courtesy of the Fundamental Theorem of Calculus. We therefore have the relation

$$\partial_{x_1}v_2 = \partial_{x_2}v_1. \tag{6.1}$$

Now, let $\mathbf{u} = \varepsilon\mathbf{v}$ and define $\mathbf{U} = (\eta,\mathbf{u})^T = \varepsilon(\zeta,\mathbf{v})^T$. Then use (6.1) to rewrite system (1.6) as

$$\partial_t\mathbf{U} + M(\mathbf{U},D)\mathbf{U} = \mathbf{0}, \tag{6.2}$$

with

$$M(\mathbf{U},D) = \begin{pmatrix} \mathbf{u}\cdot\nabla & (\mathcal{K}_\mu(\mathrm{D})+\eta)\partial_{x_1} & (\mathcal{K}_\mu(\mathrm{D})+\eta)\partial_{x_2} \\ \partial_{x_1} & (\mathbf{u}\cdot\nabla)\,\bullet & 0 \\ \partial_{x_2} & 0 & (\mathbf{u}\cdot\nabla)\,\bullet \end{pmatrix}.$$

Then the natural symmetrizer is given by

$$Q(\mathbf{U},D) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\mathcal{K}_\mu(\mathrm{D})+\eta) & 0 \\ 0 & 0 & (\mathcal{K}_\mu(\mathrm{D})+\eta) \end{pmatrix},$$

and by extension, an energy associated to (6.2) reads

$$E_s(\mathbf{U}) = \big(\Lambda^s\mathbf{U}, Q(\mathbf{U},\mathrm{D})\Lambda^s\mathbf{U}\big).$$

The energy estimates are similar to the one-dimensional case. Indeed, for $(\eta,\mathbf{u})\in V^s_\mu(\mathbb{R}^2)$ and $s>\frac{5}{2}$, we observe that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_s(\mathbf{U}) = &-\big(\Lambda^s(\mathbf{u}\cdot\nabla\eta),\Lambda^s\eta\big)_{L^2} - \big(\Lambda^s((\mathcal{K}_\mu(\mathrm{D})+\eta)\nabla\cdot\mathbf{u}),\Lambda^s\eta\big)_{L^2} \\ &-\big(\Lambda^s\nabla\eta,(\mathcal{K}_\mu(\mathrm{D})+\eta)\Lambda^s\mathbf{u}\big)_{L^2} - \big(\Lambda^s((\mathbf{u}\cdot\nabla)\mathbf{u}),(\mathcal{K}_\mu(\mathrm{D})+\eta)\Lambda^s\mathbf{u}\big)_{L^2} \\ &+\frac{1}{2}\big(\Lambda^s\mathbf{u},(\partial_t\eta)\Lambda^s\mathbf{u}\big)_{L^2}.\end{aligned}$$

An estimate analogous to the ones of Proposition 3.1 is a consequence of two-dimensional versions of estimates in Section 2. However, these are easily extended to 2-d by noting that $\mathcal{K}_\mu(\mathrm{D})$ and $\mathcal{T}_\mu(\mathrm{D})$ is radial.

The estimate of the difference between two solutions is similar to the proof of Proposition 4.1.

Appendix A

A.1. **Pointwise estimates for $\sqrt{K_\mu(\xi)}$ and $\sqrt{T_\mu(\xi)}$.** Before turning to the proof of the pointwise estimates in Lemma 2.1 and Lemma 2.5, we make an important observation. Let $\sqrt{\mathcal{T}_\mu}(\mathrm{D})$ be the Fourier multiplier associated with the symbol

$$\sqrt{T_\mu(\xi)} = \sqrt{\frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}}.$$

Then the operator is regularizing for $\mu>0$ on $L^2(\mathbb{R})$, and acts similar to the scaled Bessel potential $\Lambda_\mu^{-\frac{1}{2}}$ defined by the symbol $\xi\mapsto(1+\mu\xi^2)^{-\frac{1}{4}}$. While $\sqrt{K_\mu(\xi)}$ has a similar behaviour in low frequency for $\sigma<\frac{1}{3}$, but acts like $\Lambda_\mu^{\frac{1}{2}}$ in high frequencies.

Lemma A.1. *Let $\mu>0$ and take any $n\in\mathbb{N}$.*

- *Then $T_\mu(\xi)$ satisfies*

$$\Big|\frac{d^n}{d\xi^n}\sqrt{T_\mu(\xi)}\Big| \lesssim \mu^{\frac{n}{2}}\langle\sqrt{\mu}\xi\rangle^{-\frac{1}{2}-n}. \tag{A.1}$$

- *Similarly, $K_\mu(\xi)$ satisfies*

$$\Big|\frac{d^n}{d\xi^n}\sqrt{K_\mu(\xi)}\Big| \lesssim_\sigma \mu^{\frac{n}{2}}\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}-n}. \tag{A.2}$$

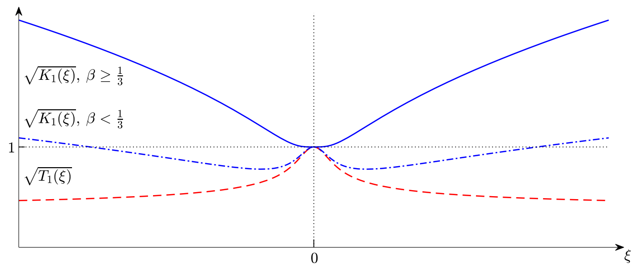

FIGURE 3. The multiplier $\sqrt{K_1(\xi)}$ in the cases $\sigma \geq \frac{1}{3}$ (line) and $\sigma < \frac{1}{3}$ (dash-dots). The red curve is a plot of $\sqrt{T_1(\xi)}$ (dash).

Proof. The proof is a generalization of Lemma 8 in [15]. Following their arguments, we observe that since $T_1(\sqrt{\mu}r) = T_\mu(r)$ for $r > 0$, it is sufficient to show that

$$\left|\frac{d^n}{dr^n}\sqrt{T_1(r)}\right| \lesssim \langle r\rangle^{-\frac{1}{2}-n}.$$

We divide the proof into two steps.

First, let $0 < r < \frac{1}{2}$ and prove that any derivative of $\sqrt{T_1(r)}$ is bounded. We have that $\sqrt{T_1(r)}$ is bounded in this region. By direct computation, we observe

$$\frac{d}{dr}\sqrt{T_1(r)} = \frac{\operatorname{sech}^2(r)}{(2r)^2\sqrt{T_1(r)}}\left(2r + \frac{e^{-2r} - e^{2r}}{2}\right) =: \frac{\operatorname{sech}^2(r)}{\sqrt{T_1(r)}}G(r), \tag{A.3}$$

where $G(r)$ can be written as a series by expanding the exponentials. Indeed, we have that

$$G(r) = \frac{1}{(2r)^2}\left(2r + \frac{e^{-2r} - e^{2r}}{2}\right) = -\sum_{k=0}^{\infty}\frac{(2r)^{2k+1}}{(2k+3)!}.$$

The series is uniformly convergent for $r \geq 0$. Moreover, $G(r)$ and its derivatives are bounded for $0 < r < \frac{1}{2}$. By extension, since for all $n \geq 0$ there holds $\frac{d^n}{dr^n} sech^2(r) \lesssim e^{-2r}$ we have that

$$\left|\frac{d^n}{dr^n}\sqrt{T_1(r)}\right| \lesssim 1.$$

Now, we let $r \geq \frac{1}{2}$ and prove the necessary decay estimate. We use the identity

$$\tanh(r) = 1 - \frac{2}{e^{2r} + 1}, \tag{A.4}$$

and deduce by the chain rule that

$$\left|\frac{d^n}{dr^n}\sqrt{T_1(r)}\right| \lesssim \sum_{k=0}^{n}\left(\frac{1}{r}\left(1 - \frac{2}{e^{2r}+1}\right)\right)^{\frac{1}{2}-k} r^{-k-n} \lesssim \langle r\rangle^{-\frac{1}{2}-n}.$$

Lastly, we have that (A.2) follows by the Leibniz rule. Indeed, we observe

$$\begin{aligned}\Big|\frac{d^n}{dr^n}\sqrt{K_1(r)}\Big| &= \Big|\frac{d^n}{dr^n}\sqrt{T_1(r)(1+\sigma r^2)}\Big| \\ &\lesssim \sum_{k=0}^{n}\Big|\frac{d^{n-k}}{dr^{n-k}}\sqrt{T_1(r)}\Big|\Big|\frac{d^k}{dr^k}\sqrt{1+\sigma r^2}\Big| \\ &\lesssim_\sigma \langle r\rangle^{\frac{1}{2}-n},\end{aligned}$$

which concludes the proof of Lemma A.1. □

A.2. **Proof of Lemmas 2.1 and 2.5.**

Proof of Lemma 2.1. First, we again make the observation that $K_1(\sqrt{\mu}\xi) = K_\mu(\xi)$. Therefore, we simply let $r > 0$ and consider $K_1(r)$. To establish the upper bound given in (2.1), we note that for $r < 1$ we have

$$K_1(r) \leq 1 + \sigma.$$

This is because $\frac{\tanh(r)}{r} \leq 1$. On the other hand, when $r \geq 1$ then $\tanh(r) < 1$ and it follows that

$$K_1(r) \leq 1 + \sigma r.$$

Consequently, for all $r > 0$ there holds $K_1(r) \leq c_\sigma^2(1 + r)$ with c_σ^2 as defined in (1.19).

Next, we prove the lower bound given by (2.2) with $\sigma \geq \frac{1}{3}$. We will again split r into two intervals, where we aim to prove

$$K_1(r) = \frac{\tanh(r)}{r}(1 + \sigma r^2) \geq (1 - \frac{h_0}{2}) + cr, \tag{A.5}$$

for some positive constant $c > 0$ and any $h_0 \in (0, 1)$. We prove (A.5) by considering two cases for r. When $0 \leq r \leq \frac{h_0}{4}$ we use that

$$\tanh(r) = \int_0^r (1 - \tanh^2(x))\, dx \geq r - \frac{r^3}{3}, \tag{A.6}$$

since $\tanh^2(x) \leq x^2$ by the mean value theorem. Therefore, we have that

$$K_1(r) \geq \Big(1 - \frac{r^2}{3}\Big)\Big(1 + \frac{r^2}{3}\Big) \geq \Big(1 - \frac{h_0}{2}\Big) + \Big(\frac{h_0}{4} - \frac{r^4}{9}\Big) + r,$$

which implies (A.5) since $0 \leq r \leq \frac{h_0}{4}$.

For the remaining part, we use the identity (A.4) and show that (A.5) holds for $r \geq \frac{h_0}{4}$ if:

$$\begin{aligned} &\tanh(r)\bigg(1 + \frac{r^2}{3}\bigg) - r(1 + cr) > 0 \\ \iff &\bigg(1 + \frac{r^2}{3} - r(1 + cr)\bigg) - \frac{2}{e^{2r} + 1}\bigg(1 + \frac{r^2}{3}\bigg) > 0 \\ \iff &\bigg(1 + \frac{r^2}{3} - r(1 + cr)\bigg)\big(e^{2r} + 1\big) - 2\bigg(1 + \frac{r^2}{3}\bigg) := G(r) > 0.\end{aligned}$$

But this holds since

$$\begin{aligned}G'''(r) &= \frac{4}{3}e^{2r}\big(2r^2 - 3c(2r^2+6r+3)\big)\\ &\geq \frac{4}{3}e^{2r}\Big(r^2(1-6c) + r(\frac{h_0}{8} - 18c) + (\frac{h_0^2}{32} - 9c)\Big),\end{aligned}$$

and is positive for $0 < c < 10^{-3}h_0^2$ with $r \geq \frac{h_0}{4}$. Indeed, as a consequence we have the following chain of implications

$$\begin{aligned}&0 < G''\Big(\frac{h_0}{2}\Big) \leq G''(r) = \frac{2}{3}\big(e^{2r}(1-2r+2r^2-3c(2r^2+4r+1)) - 3c - 1\big)\\ \Longrightarrow\ &0 < G'\Big(\frac{h_0}{2}\Big) \leq G'(r) = \frac{1}{3}\big(e^{2r}(3-4r+2r^2-6cr(r+1)) - 2r - 3 - 6cr\big)\\ \Longrightarrow\ &0 < G\Big(\frac{h_0}{2}\Big) \leq G(r).\end{aligned}$$

We have therefore verified (A.5) for all $r \geq 0$ and we conclude that (2.2) holds true.

Similarly, for $0 < \sigma < \frac{1}{3}$, we have that (2.3) is a consequence of the inequality

$$\frac{\tanh(r)}{r}(1+\sigma r^2) \geq \sigma + cr.$$

One should note that we do not require sharp estimates. In fact, we simply need to obtain the estimate

$$\Big(1+\sigma r^2 - r(\sigma + cr)\Big)\big(e^{2r}+1\big) - 2\Big(1+\sigma r^2\Big) =: H(r) \geq 0,$$

for $r \geq 0$. On the other hand, we observe that

$$H'''(r) = 4e^{2r}\Big(2 + 2\sigma r(2+r) - 3c + 2cr(3+r)\Big) > 0$$

for all $r \geq 0$ if

$$(2-3c) + 2r(2\sigma - 3c) + 2r^2(\sigma - c) > 0$$

and is ensured for $0 < c \leq 10^{-3}\sigma$. Consequently,

$$\begin{aligned}&0 < H''(0) \leq H''(r) = 2e^{2r}\Big(2 + \sigma(2r^2+2r-1) - c(2r^2+4r+1)\Big) - 2(\sigma + c)\\ &0 < H'(0) \leq H'(r) = e^{2r}\Big(2 + \sigma(2r^2-1) - 2c(r^2+2r)\Big) - \sigma(2r+1) - 2cr\\ &0 < H(0) \leq H(r),\end{aligned}$$

and we argue as above to conclude.

The proof of estimate (2.4) is a direct consequence of Lemma A.1 and (A.2) with $n = 1$ if we trace the dependence in σ:

$$\Big|\frac{d}{dr}\sqrt{K_1(r)}\Big| \lesssim \langle r\rangle^{-\frac{1}{2}-1}(1+\sigma r^2)^{\frac{1}{2}} + \langle r\rangle^{-\frac{1}{2}}\frac{\sigma r}{(1+\sigma r^2)^{\frac{1}{2}}} \lesssim \langle r\rangle^{-1} + \sqrt{\sigma}\langle r\rangle^{-\frac{1}{2}}.$$

Estimate (2.5) concerns the following bound on the difference:

$$\Big|\sqrt{K_\mu(\xi)} - \sqrt{\sigma}\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\Big| = \Big|\Big(\frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}(1+\sigma\mu\xi^2)\Big)^{\frac{1}{2}} - \sqrt{\sigma}\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\Big|.$$

For $\sigma\sqrt{\mu}|\xi| \le 1$ there holds trivially by using the triangle inequality that

$$\Big|\sqrt{K_\mu(\xi)} - \sqrt{\sigma}\mu^{\frac14}|\xi|^{\frac12}\Big| \lesssim 1.$$

While for $\sigma\sqrt{\mu}|\xi| > 1$ we observe by direct calculations that

$$\begin{aligned}\Big|\sqrt{K_\mu(\xi)} - \sqrt{\sigma}\mu^{\frac14}|\xi|^{\frac12}\Big| &= \sqrt{\sigma}\mu^{\frac14}|\xi|^{\frac12}\Big|\Big(\frac{\tanh(\sqrt{\mu}|\xi|)}{\sigma\mu\xi^2} + \tanh(\sqrt{\mu}|\xi|)\Big)^{\frac12} - 1\Big| \\ &\lesssim \sqrt{\sigma}\mu^{\frac14}|\xi|^{\frac12}\Big|\frac{\tanh(\sqrt{\mu}|\xi|)}{\sigma\mu\xi^2} + (\tanh(\sqrt{\mu}|\xi|) - 1)\Big| \\ &\lesssim \frac{1}{(\sigma\sqrt{\mu}|\xi|)^{\frac12}}\frac{1}{\sqrt{\mu}|\xi|} + \sqrt{\sigma}(\sqrt{\mu}|\xi|)^{\frac12}e^{-2\sqrt{\mu}|\xi|} \\ &\lesssim \sigma + \sqrt{\sigma},\end{aligned}$$

where we used the triangle inequality and that $\sigma\sqrt{\mu}|\xi| > 1$.

Lastly, we prove (2.6) by using (2.5):

$$\begin{aligned}\sqrt{K_\mu(\xi)}\langle\xi\rangle^{s-1}|\xi| &= \Big(\sqrt{K_\mu(\xi)} - \sqrt{\sigma}\mu^{\frac14}|\xi|^{\frac12}\Big)\langle\xi\rangle^{s-1}|\xi| + \sqrt{\sigma}\mu^{\frac14}\langle\xi\rangle^{s-1}|\xi|^{\frac32} \\ &\lesssim (\sigma + \sqrt{\sigma})\langle\xi\rangle^{s} + \sqrt{\sigma}\mu^{\frac14}\langle\xi\rangle^{s}|\xi|^{\frac12}.\end{aligned}$$

□

Proof of Lemma 2.5. To prove (2.9), since $T_1(\sqrt{\mu}\xi) = T_\mu(\xi)$, we only need to establish the following inequality:

$$1 - \frac{h_0}{2} + cr \le \frac{r}{\tanh(r)} \lesssim 1 + r, \tag{A.7}$$

for all $r > 0$ and some $c > 0$. We also note that the upper bound is trivial, so we only prove the lower bound. Let $h_0 \in (0,1)$. By the mean value theorem we find that $tanh(r) \le r$ and observe

$$\frac{r}{\tanh(r)} = (1 - \frac{h_0}{2})\frac{r}{\tanh(r)} + \frac{h_0}{2}\frac{r}{\tanh(r)} \ge 1 - \frac{h_0}{2} + \frac{h_0}{2}r.$$

Next, we consider (2.10). For $\sqrt{\mu}|\xi| \le 1$ we have that $T_\mu(\xi) \sim 1$ and $\langle\sqrt{\mu}\xi\rangle \sim 1$. On the other hand, when $\sqrt{\mu}|\xi| \ge 1$ then $T_\mu(\xi) \sim \frac{1}{\sqrt{\mu}|\xi|}$ and $\langle\sqrt{\mu}\xi\rangle \sim \sqrt{\mu}|\xi|$. Multiplying the two functions, we obtain the desired result.

We estimate the derivative (2.11) directly and using that $\mu \in (0,1)$:

$$\Big|\frac{d}{d\xi}\langle\xi\rangle^{s}\langle\sqrt{\mu}\xi\rangle^{\frac12}\Big| \lesssim \langle\xi\rangle^{s-1}\langle\sqrt{\mu}\xi\rangle^{\frac12} + \langle\xi\rangle^{s}\langle\sqrt{\mu}\xi\rangle^{\frac12}\sqrt{\mu}\langle\sqrt{\mu}\xi\rangle^{-1} \lesssim \langle\xi\rangle^{s-1}\langle\sqrt{\mu}\xi\rangle^{\frac12},$$

since $\sqrt{\mu}\langle\sqrt{\mu}\xi\rangle^{-1} \le \langle\xi\rangle^{-1}$.

Similarly, we have that (2.12) follows by the same argument after using (A.1) and (2.11):

$$\begin{aligned}\Big|\frac{d}{d\xi}\sqrt{T_\mu(\xi)}\langle\xi\rangle^{s}\langle\sqrt{\mu}\xi\rangle^{\frac12}\Big| &\lesssim \sqrt{\mu}\langle\xi\rangle^{-\frac32}\langle\xi\rangle^{s}\langle\sqrt{\mu}\xi\rangle^{\frac12} + \langle\sqrt{\mu}\xi\rangle^{-\frac12}\langle\xi\rangle^{s-1}\langle\sqrt{\mu}\xi\rangle^{\frac12} \\ &\lesssim \langle\xi\rangle^{s-1}.\end{aligned}$$

For estimate (2.13), we observe that

$$\langle\sqrt{\mu}\xi\rangle^{\frac12} - \mu^{\frac14}|\xi|^{\frac12} = \mu^{\frac14}|\xi|^{\frac12}\Big((\frac{1}{\mu\xi^2} + 1)^{\frac14} - 1\Big) \lesssim 1.$$

□

A.3. **Proof of Lemmas 2.11 and 2.12.** For the proof of Lemma 2.11 and Lemma 2.12, we need a "generalized" version of the Kato-Ponce commutator estimate which holds for symbols defined by:

Definition A.2 (Symbol class [33] Def. *B.7*)**.** *We say that a symbol* $\mathrm{F}(\mathrm{D})$ *is a member of the symbol class* $\mathcal{S}^s$ *with* $s \in \mathbb{R}$*, if* $\xi \mapsto F(\xi) \in \mathbb{C}$ *is smooth and satisfies*

$$\forall \alpha \in \mathbb{N}, \quad \sup_{\xi\in\mathbb{R}} \langle \xi \rangle^{\alpha - s} \Big| \frac{d^\alpha}{d\xi^\alpha} F(\xi) \Big| < \infty.$$

One also associates the following seminorm on $\mathcal{S}^s$ *:*

$$\mathcal{N}^s(\mathrm{F}) = \sup_{\alpha\in\mathbb{N},\ \alpha\leq 4} \ \sup_{\xi\in\mathbb{R}} \langle \xi \rangle^{\alpha - s} \Big| \frac{d^\alpha}{d\xi^\alpha} F(\xi) \Big|. \tag{A.8}$$

The following results is found in Appendix B of [33].

Lemma A.3. *Let* $t_0 > 1/2$*,* $s \geq 0$ *and* $\sigma \in \mathcal{S}^s$*. If* $f \in H^s \cap H^{t_0+1}(\mathbb{R})$*, then for all* $g \in H^{s-1}(\mathbb{R})$*,*

$$|[\mathrm{F}(\mathrm{D}), f]g|_{L^2} \lesssim \mathcal{N}^s(\mathrm{F}) |f|_{H^{\max\{t_0+1, s\}}} |g|_{H^{s-1}}. \tag{A.9}$$

With this at hand, we may give the proof.

Proof of Lemma 2.11. To prove (2.24) and (2.25), it suffices to verify for all $n \in \mathbb{N}$ that

$$\sup_{\xi\in\mathbb{R}} \langle \xi \rangle^n \left| \frac{d^n}{d\xi^n} (\chi_\mu^{(1)} \sqrt{\mathcal{K}_\mu})(\xi) \right| \lesssim_\sigma 1, \tag{A.10}$$

for any $0 < \mu < 1$. Indeed, in agreement with Definition A.2, then $(\chi_\mu^{(1)} \sqrt{\mathcal{K}_\mu}) \in \mathcal{S}^0$ and (2.25) holds true due to Lemma A.3. Moreover, using Plancherel and (A.10) with $n = 0$ we have

$$|(\chi_\mu^{(1)} \sqrt{\mathcal{K}_\mu})(\mathrm{D}) f|_{L^2} \lesssim_\sigma |f|_{L^2},$$

proving (2.24). Now, let us prove (A.10). We observe that

$$\mu^{\frac{k}{2}} \langle \xi \rangle^k \Big| (\frac{d^k}{d\xi^k} \chi^{(1)})(\sqrt{\mu}\xi) \Big| \lesssim 1, \quad k \geq 0, \tag{A.11}$$

since $\sqrt{\mu}|\xi| \gtrsim 1$ on the support of $\chi_\mu^{(1)}(\xi)$. Moreover, we observe by Lemma A.1 and $\mu \in (0, 1)$ that

$$\chi_\mu^{(1)}(\xi) \left| \frac{d^k}{d\xi^k} \sqrt{K_\mu(\xi)} \right| \lesssim_\sigma \chi_\mu^{(1)}(\xi) \langle \sqrt{\mu}\xi \rangle^{\frac{1}{2}} \mu^{\frac{k}{2}} \langle \sqrt{\mu}\xi \rangle^{-k} \lesssim_\sigma \langle \xi \rangle^{-k}.$$

Combining these estimates with the Leibniz rule yields

$$\begin{aligned}
\langle \xi \rangle^n \Big| \frac{d^n}{d\xi^n} \big(\chi_\mu^{(1)}(\xi) \sqrt{K_\mu(\xi)} \big) \Big| &\lesssim \langle \xi \rangle^n \sum_{k=0}^n \Big| \frac{d^{n-k}}{d\xi^{n-k}} \big(\chi^{(1)}(\sqrt{\mu}\xi) \big) \ \frac{d^k}{d\xi^k} \big(\sqrt{K_\mu(\xi)} \big) \Big| \\
&\lesssim_\sigma \langle \xi \rangle^n \sum_{k=0}^n \mu^{\frac{n-k}{2}} \big| (\frac{d^{n-k}}{d\xi^{n-k}} \chi^{(1)})(\sqrt{\mu}\xi) \big| \langle \xi \rangle^{-k} \\
&\lesssim_\sigma \sum_{k=0}^n \mu^{\frac{n-k}{2}} \langle \xi \rangle^{n-k} \big| (\frac{d^{n-k}}{d\xi^{n-k}} \chi^{(1)})(\sqrt{\mu}\xi) \big| \lesssim_\sigma 1.
\end{aligned}$$

Hence, $(\chi_\mu^{(1)} \sqrt{\mathcal{K}_\mu}) \in \mathcal{S}^0$ and $\mathcal{N}^0(\chi_\mu^{(1)} \sqrt{\mathcal{K}_\mu}) \lesssim_\sigma 1$ independently from μ, proves (A.10).

Next, we consider estimates (2.27) and (2.28). Recalling (2.26) we define

$$\tilde{F}_{\mu,\frac{1}{2}}(\xi) = \mu^{-\frac{1}{4}} F_{\mu,\frac{1}{2}}(\xi) = \frac{1}{\mu^{\frac{1}{4}}} \frac{1}{\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}} (1+\mu\sigma\xi^2)^{\frac{1}{2}}. \tag{A.12}$$

Then, it suffices to prove that

$$\sup_{\xi\in\mathbb{R}} \langle\xi\rangle^{n-\frac{1}{2}} \Big| \frac{d^n}{d\xi^n} (\chi_\mu^{(2)} \tilde{F}_{\mu,\frac{1}{2}})(\xi) \Big| \lesssim_\sigma 1, \tag{A.13}$$

for all $n \in \mathbb{N}$ and any $\mu \in (0,1)$. Indeed, if we assume (A.13) and take $n = 0$ we deduce from Plancherel's identity that

$$|(\chi_\mu^{(2)} F_{\mu,\frac{1}{2}})(\mathrm{D}) f|_{L^2} \lesssim_\sigma \mu^{\frac{1}{4}} |f|_{H^{\frac{1}{2}}} \lesssim_\sigma |f|_{L^2} + \mu^{\frac{1}{4}} |\mathrm{D}^{\frac{1}{2}} f|_{L^2}.$$

which proves (2.27). Moreover, (A.13) also implies that $(\chi_\mu^{(2)} \tilde{F}_{\mu,\frac{1}{2}}) \in \mathcal{S}^{\frac{1}{2}}$ with $\mathcal{N}^{\frac{1}{2}}(\chi_\mu^{(2)} \tilde{\mathrm{F}}_{\mu,\frac{1}{2}}) \lesssim_\sigma 1$ so that

$$|[(\chi_\mu^{(2)} \mathrm{F}_{\mu,\frac{1}{2}})(\mathrm{D}), f]\partial_x g|_{L^2} \lesssim_\sigma \mu^{\frac{1}{4}} |f|_{H^s} |g|_{H^{\frac{1}{2}}},$$

by Lemma A.3. Now we prove (A.13). First, we consider the functions,

$$a_\mu(\xi) = \frac{1}{\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}} \quad \text{and} \quad b_\mu(\xi) = (1+\mu\sigma\xi^2)^{\frac{1}{2}}.$$

Then, since $|\xi| > \sqrt{\mu}|\xi| > 1$ on the support of $\chi_\mu^{(2)}$, we observe that

$$\chi_\mu^{(2)}(\xi) \Big| \frac{d^k}{d\xi^k} a_\mu(\xi) \Big| \lesssim \mu^{\frac{1}{4}} \chi_\mu^{(2)}(\xi) \frac{1}{\sqrt{\mu}|\xi|} \frac{1}{|\xi|^{k-\frac{1}{2}}} \lesssim \mu^{\frac{1}{4}} \langle\xi\rangle^{\frac{1}{2}-k} \langle\sqrt{\mu}\xi\rangle^{-1}. \tag{A.14}$$

While $b_\mu(\xi) \lesssim_\sigma \langle\sqrt{\mu}\xi\rangle$ and its deriatives satisfy the bound,

$$\chi_\mu^{(2)}(\xi) \Big| \frac{d^k}{d\xi^k} b_\mu(\xi) \Big| \lesssim_\sigma \chi_\mu^{(2)}(\xi) \mu^{\frac{k}{2}} \langle\sqrt{\mu}\xi\rangle^{1-k} \lesssim_\sigma \langle\sqrt{\mu}\xi\rangle \langle\xi\rangle^{-k}. \tag{A.15}$$

Thus, if all derivatives falls on $\tilde{\mathrm{F}}_{\mu,\frac{1}{2}}$, the Leibniz rule, (A.14) and (A.15) imply

$$\chi_\mu^{(2)}(\xi) \Big| \frac{d^k}{d\xi^k} \tilde{F}_{\mu,\frac{1}{2}}(\xi) \Big| \lesssim \mu^{-\frac{1}{4}} \chi_\mu^{(2)}(\xi) \sum_{j=0}^{k} \Big| \frac{d^{k-j}}{d\xi^{k-j}} a_\mu(\xi) \Big| \Big| \frac{d^j}{d\xi^j} b_\mu(\xi) \Big| \lesssim_\sigma \langle\xi\rangle^{\frac{1}{2}-k}.$$

On the other hand, when derivatives fall the cut-off function, we observe

$$\mu^{\frac{k}{2}} \langle\xi\rangle^k \Big| (\frac{d^k}{d\xi^k} \chi^{(2)})(\sqrt{\mu}\xi) \Big| \lesssim 1, \quad k \geq 1, \tag{A.16}$$

since the support of $\frac{d^k}{d\xi^k}\chi_\mu^{(2)}$ is contained in the support of $\chi_\mu^{(1)}$. As a result, there holds

$$\begin{aligned}
\langle\xi\rangle^{n-\frac{1}{2}} \Big| \frac{d^n}{d\xi^n} \big(\chi_\mu^{(2)}(\xi) \tilde{F}_{\mu,\frac{1}{2}}(\xi)\big) \Big| &\lesssim \langle\xi\rangle^{n-\frac{1}{2}} \sum_{k=0}^{n} \Big| \frac{d^{n-k}}{d\xi^{n-k}} \big(\chi^{(2)}(\sqrt{\mu}\xi)\big) \Big| \Big| \frac{d^k}{d\xi^k} \big(\tilde{F}_{\mu,\frac{1}{2}}(\xi)\big) \Big| \\
&\lesssim_\sigma \langle\xi\rangle^{n-\frac{1}{2}} \sum_{k=0}^{n} \mu^{\frac{n-k}{2}} \big| (\frac{d^{n-k}}{d\xi^{n-k}} \chi^{(2)})(\sqrt{\mu}\xi) \big| \langle\xi\rangle^{\frac{1}{2}-k} \\
&\lesssim_\sigma \sum_{k=0}^{n} \mu^{\frac{n-k}{2}} \langle\xi\rangle^{n-k} \big| (\frac{d^{n-k}}{d\xi^{n-k}} \chi^{(2)})(\sqrt{\mu}\xi) \big| \\
&\lesssim_\sigma 1.
\end{aligned}$$

The estimate is uniform in $\mu \in (0,1)$, and (A.13) is proved, which provides the desired result.

Lastly, we prove (2.30) and (2.31) arguing in the same vein. First, we write:

$$\begin{aligned}\chi_\mu^{(2)}(\xi)F_{\mu,0}(\xi) &= \chi_\mu^{(2)}(\xi)\cdot \frac{1}{\mu^{\frac14}|\xi|^{\frac12}}\cdot\Big(1+\mu\sigma\xi^2\Big)^{\frac12}\cdot\Big(\frac{2}{e^{2\sqrt{\mu}|\xi|}+1}\Big)^{\frac12}\\ &=: \chi_\mu^{(2)}(\xi)\, a_\mu(\xi)\, b_\mu(\xi)\, c_\mu(\xi),\end{aligned}$$

making use of the identity (A.4). Then we observe for all $N\in\mathbb{N}$ that

$$\langle\xi\rangle^k\langle\sqrt{\mu}\xi\rangle^N\Big|\frac{d^k}{d\xi^k}c_\mu(\xi)\Big| \lesssim \mu^{\frac{k}{2}}\langle\xi\rangle^k\langle\sqrt{\mu}\xi\rangle^N e^{-\sqrt{\mu}|\xi|}\lesssim 1. \tag{A.17}$$

As a result, we deduce by (A.15) and (A.17) with $N=1$ that

$$\begin{aligned}\chi_\mu^{(2)}(\xi)\Big|\frac{d^k}{d\xi^k}\big(b_\mu(\xi)\,c_\mu(\xi)\big)\Big| &\lesssim \chi_\mu^{(2)}(\xi)\sum_{j=0}^k\big|\frac{d^{k-j}}{d\xi^{k-j}}\big(b_\mu(\xi)\big)\big|\,\big|\frac{d^j}{d\xi^j}\big(c_\mu(\xi)\big)\big|\\ &\lesssim_\sigma \sum_{j=0}^k\langle\xi\rangle^{-(k-j)}\langle\sqrt{\mu}\xi\rangle\,\langle\xi\rangle^{-j}\langle\sqrt{\mu}\xi\rangle^{-1}\\ &\lesssim_\sigma \langle\xi\rangle^{-k}.\end{aligned}$$

Moreover, we use (A.14) to deduce

$$\chi_\mu^{(2)}(\xi)\Big|\frac{d^k}{d\xi^k}F_{\mu,0}(\xi)\Big| \lesssim \chi_\mu^{(2)}(\xi)\sum_{j=0}^k\Big|\frac{d^{k-j}}{d\xi^{k-j}}\big(a_\mu(\xi)\big)\Big|\,\Big|\frac{d^j}{d\xi^j}\big(b_{\mu,\sigma}(\xi)\,c_\mu(\xi)\big)\Big|\lesssim_\sigma\langle\xi\rangle^{-k},$$

from which we find

$$\langle\xi\rangle^n\Big|\frac{d^n}{d\xi^n}\big(\chi_\mu^{(2)}(\xi)F_{\mu,0}(\xi)\big)\Big|\lesssim\langle\xi\rangle^n\sum_{k=0}^n\Big|\frac{d^{n-k}}{d\xi^{n-k}}\big(\chi_\mu^{(2)}(\xi)\big)\Big|\,\Big|\frac{d^k}{d\xi^k}\big(\mathrm{F}_{\mu,0}(\xi)\big)\Big|\lesssim_\sigma 1,$$

by (A.16). Arguing as above, we may conclude that the estimates (2.30) and (2.31) hold. □

Proof of Lemma 2.12. In order to prove (2.32), we simply verify that $\sqrt{\mathcal{T}_\mu}J_\mu^{\frac12}\in\mathcal{S}^0$ and $\mathcal{N}^0(\sqrt{\mathcal{T}_\mu}\Lambda_\mu^{\frac12})\lesssim 1$ uniformly in $\mu\in(0,1)$. But this is a direct consequence of Lemma A.1 and the Leibniz rule:

$$\begin{aligned}\langle\xi\rangle^n\Big|\frac{d^n}{d\xi^n}\sqrt{T_\mu(\xi)}\langle\sqrt{\mu}\xi\rangle^{\frac12}\Big| &\lesssim\langle\xi\rangle^n\sum_{k=0}^n\Big|\frac{d^{n-k}}{d\xi^{n-k}}\sqrt{T_\mu(\xi)}\Big|\,\Big|\frac{d^k}{d\xi^k}\langle\sqrt{\mu}\xi\rangle^{\frac12}\Big|\\ &\lesssim\langle\xi\rangle^n\sum_{k=0}^n\mu^{\frac{n-k}{2}}\langle\sqrt{\mu}\xi\rangle^{-\frac12-(n-k)}\mu^{\frac{k}{2}}\langle\sqrt{\mu}\xi\rangle^{\frac12-k}\\ &\lesssim\langle\xi\rangle^n\mu^{\frac{n}{2}}\langle\sqrt{\mu}\xi\rangle^{-n},\end{aligned}$$

and is bounded by a constant independent from $\mu\in(0,1)$. Hence, we may conclude by Lemma A.3 that (2.32) holds true.

A similar approach is used for the proof of (2.33). Indeed, we observe that

$$\langle\xi\rangle^{n-s}\Big|\frac{d^n}{d\xi^n}\sqrt{T_\mu(\xi)}\langle\xi\rangle^s\Big|\lesssim\langle\xi\rangle^{n-s}\sum_{k=0}^n\mu^{\frac{n-k}{2}}\langle\sqrt{\mu}\xi\rangle^{-\frac12-(n-k)}\langle\xi\rangle^{s-k}\lesssim 1.$$

Hence, $\sqrt{\mathcal{T}_\mu}\Lambda^s \in \mathcal{S}^s$ and $\mathcal{N}^s(\sqrt{\mathcal{T}_\mu}\Lambda^s) \lesssim 1$ uniformly in $\mu \in (0,1)$, allowing us to conclude by Lemma A.3.

The proof of (2.34) is the same, by a direct application of (A.1) we deduce that $\sqrt{\mathcal{T}_\mu} \in \mathcal{S}^0$ uniformly in $\mu \in (0,1)$.

Next, we consider (2.35). Define the bilinear form: $a_1(\mathrm{D})(f,g) = \partial_x[\sqrt{\mathcal{T}_\mu}(D), f]g$. Then we may use Plancherel to write

$$|\hat{a}_1(\xi)(f,g)| \leq \int_{\mathbb{R}} |\xi|\Big|\sqrt{T_\mu(\xi)} - \sqrt{T_\mu(\rho)}\Big||\hat{f}(\xi-\rho)|\,|\hat{g}(\rho)|\,d\rho.$$

Clearly, if we can prove that

$$b_1(\xi,\rho) := |\xi|\Big|\sqrt{T_\mu(\xi)} - \sqrt{T_\mu(\rho)}\Big| \lesssim 1 + |\xi-\rho|, \tag{A.18}$$

then we can conclude as we did for the proof of Lemma 2.7. Indeed, assuming the claim (A.18), then there holds

$$|\partial_x[\sqrt{\mathcal{T}_\mu}(D), f]g|_{L^2} = |\hat{a}_1(\xi)(f,g)|_{L^2} \lesssim (|f|_{H^{t_0}} + |\partial_x f|_{H^{t_0}})|g|_{L^2}.$$

Now, in order to estimate $b_1(\xi,\rho)$ we consider three cases. First, if $|\rho| \leq 1$, then we have by the triangle inequality,

$$b_1(\xi,\rho) \leq (1+|\xi-\rho|)\Big(\sqrt{T_\mu(\xi)} + \sqrt{T_\mu(\rho)}\Big) \lesssim 1 + |\xi-\rho|,$$

since $\xi \mapsto \sqrt{T_\mu(\xi)}$ is bounded by one. Secondly, consider the region where $|\rho| > 1$ and $|\xi| \geq |\rho|$. Then since $\xi \mapsto \tanh(\sqrt{\mu}|\xi|)$ is increasing and $\xi \mapsto T_\mu(\xi)$ is decreasing, we have that

$$\frac{|\rho|}{|\xi|} \leq \Big(\frac{|\rho|}{|\xi|}\Big)^{\frac{1}{2}} \leq \Big(\frac{T_\mu(\xi)}{T_\mu(\rho)}\Big)^{\frac{1}{2}} \leq 1.$$

Thus, there holds

$$b_1(\xi,\rho) = |\xi|\Big(1 - \Big(\frac{T_\mu(\xi)}{T_\mu(\rho)}\Big)^{\frac{1}{2}}\Big)\sqrt{T_\mu(\rho)} \leq |\xi| - |\rho| \leq |\xi-\rho|.$$

For $|\rho| > 1$ and $|\xi| < |\rho|$ we use a similar argument to find,

$$b_1(\xi,\rho) = |\xi|\Big(1 - \Big(\frac{T_\mu(\rho)}{T_\mu(\xi)}\Big)^{\frac{1}{2}}\Big)\sqrt{T_\mu(\xi)} \leq \frac{|\xi|}{|\rho|}(|\rho| - |\xi|) \leq |\xi-\rho|.$$

Finally, we estimate (2.36) using a similar approach. We define the bilinear form $a_2(D)(f,g) = [\Lambda_\mu^{\frac{1}{2}}, f]\partial_x g$ and look in frequency:

$$|\hat{a}_2(\xi)(f,g)| \leq \int_{\mathbb{R}} \Big|\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}} - \langle\sqrt{\mu}\rho\rangle^{\frac{1}{2}}\Big||\hat{f}(\xi-\rho)|\,|\widehat{\partial_x g}(\rho)|\,d\rho.$$

Then by the same argument as above, we only need to prove that

$$b_2(\xi,\rho) = \Big|\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}} - \langle\sqrt{\mu}\rho\rangle^{\frac{1}{2}}\Big|\frac{|\rho|}{\langle\sqrt{\mu}\rho\rangle^{\frac{1}{2}}} \lesssim 1 + |\xi-\rho|. \tag{A.19}$$

We consider three cases. If $|\rho| \leq 1$ then since $\mu \in (0,1)$, there holds by the triangle inequality:

$$b_2(\xi,\rho) \lesssim 1 + \langle\xi-\rho\rangle^{\frac{1}{2}}.$$

In the case $|\xi| \geq |\rho| > 1$, observe that

$$\frac{1+\mu\rho^2}{1+\mu\xi^2} \leq \frac{(1+\mu\rho^2)^{\frac{1}{4}}}{(1+\mu\xi^2)^{\frac{1}{4}}} \leq 1, \tag{A.20}$$

and we have

$$\frac{\xi^2-\rho^2}{|\xi-\rho|\,|\xi|} \leq \frac{|\xi|+|\rho|}{|\xi|} \lesssim 1. \tag{A.21}$$

As a consequence, recalling $\mu \in (0,1)$, we have that

$$\begin{aligned} b_2(\xi,\rho) &\leq \Big(1-\frac{1+\mu\rho^2}{1+\mu\xi^2}\Big)\frac{\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}}}{\langle\sqrt{\mu}\rho\rangle^{\frac{1}{2}}}|\rho| \\ &\leq \frac{\mu(\xi^2-\rho^2)}{\mu^{\frac{1}{4}}(1+\mu\xi^2)^{\frac{3}{4}+\frac{1}{4}}}\frac{(1+\mu\xi^2)^{\frac{1}{4}}}{\langle\rho\rangle^{\frac{1}{2}}}|\rho| \\ &\leq \frac{\xi^2-\rho^2}{|\xi|}\frac{|\rho|}{|\xi|^{\frac{1}{2}}\langle\rho\rangle^{\frac{1}{2}}} \\ &\lesssim |\xi-\rho|. \end{aligned}$$

Lastly, in the case $|\rho| > 1$ and $|\xi| < |\rho|$, we can simply change the role of ξ and ρ in (A.20) and (A.21). As result, we get

$$b_2(\xi,\rho) \leq \Big(1-\frac{1+\mu\xi^2}{1+\mu\rho^2}\Big)|\rho| \leq \frac{\rho^2-\xi^2}{|\rho|} \lesssim |\xi-\rho|.$$

We may therefore conclude that (A.19) holds and the estimate (2.36) follows. □

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Paper II

2.2 Long time well-posedness and full justification of a Whitham-Green-Naghdi system

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LONG TIME WELL-POSEDNESS AND FULL JUSTIFICATION OF A WHITHAM-GREEN-NAGHDI SYSTEM

LOUIS EMERALD AND MARTIN OEN PAULSEN

ABSTRACT. We establish the full justification of a "Whitham-Green-Naghdi" system modeling the propagation of surface gravity waves with bathymetry in the shallow water regime. It is an asymptotic model of the water waves equations with the same dispersion relation. The model under study is a nonlocal quasilinear symmetrizable hyperbolic system without surface tension. We prove the consistency of the general water waves equations with our system at the order of precision $\mathcal{O}(\mu^2(\varepsilon+\beta))$, where μ is the shallow water parameter, ε the nonlinearity parameter, and β the topography parameter. Then we prove the long time well-posedness on a time scale $\mathcal{O}(\frac{1}{\max\{\varepsilon,\beta\}})$. Lastly, we show the convergence of the solutions of the Whitham-Green-Naghdi system to the ones of the water waves equations on the later time scale.

1. INTRODUCTION

In this article, we study a full dispersion Green-Naghdi system that describes strongly dispersive surface waves over a variable bottom. The system under consideration is described in terms of the unknowns ζ, v, and b. Here $\zeta(t,x)\in\mathbb{R}$ denotes the surface elevation, $v(t,x)\in\mathbb{R}$ is related to the velocity field described by the full Euler equations, and b is the elevation of the bathymetry. The system reads,

$$\begin{cases}\partial_t\zeta+\partial_x(hv)=0\\ (h+\mu h\mathcal{T}[h,\beta b])\big(\partial_t v+\varepsilon v\partial_x v\big)+h\partial_x\zeta+\mu\varepsilon h\big(\mathcal{Q}[h,v]+\mathcal{Q}_b[h,b,v]\big)=0,\end{cases}\tag{1.1}$$

where $h=1+\varepsilon\zeta-\beta b$ and

$$\begin{aligned}\mathcal{T}[h,\beta b]v=&-\frac{1}{3h}\partial_x\mathrm{F}^{\frac12}\big(h^3\mathrm{F}^{\frac12}\partial_x v\big)+\frac{1}{2h}\big(\partial_x\mathrm{F}^{\frac12}(h^2(\beta\partial_x b)v)-h^2(\beta\partial_x b)\mathrm{F}^{\frac12}\partial_x v\big)\\&+(\beta\partial_x b)^2 v,\end{aligned}\tag{1.2}$$

and

$$\mathcal{Q}[h,v]=\frac{2}{3h}\partial_x\mathrm{F}^{\frac12}\big(h^3(\mathrm{F}^{\frac12}\partial_x v)^2\big)\tag{1.3}$$

$$\mathcal{Q}_b[h,\beta b,v]=h(\mathrm{F}^{\frac12}\partial_x v)^2(\beta\partial_x b)+\frac{1}{2h}\partial_x\mathrm{F}^{\frac12}(h^2v^2\beta\partial_x^2 b)+v^2(\beta\partial_x^2 b)(\beta\partial_x b),\tag{1.4}$$

with $\mathrm{F}^{\frac12}$ being a Fourier multiplier associated with the dispersion relation of the water waves system. Specifically, if we let $\hat{f}(\xi)$ be the Fourier transform of f, then the symbol is defined in frequency by

$$\widehat{\mathrm{F}^{\frac12}f}(\xi)=\sqrt{\frac{3}{\mu\xi^2}\Big(\frac{\sqrt{\mu}\xi}{\tanh(\sqrt{\mu}\xi)}-1\Big)}\hat{f}(\xi).\tag{1.5}$$

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The parameters μ, ε, and β are defined by the comparison between characteristic quantities of the system under study. Among those are the characteristic water depth H_0, the characteristic wave amplitude a_s, the characteristic bathymetry amplitude a_b, and the characteristic wavelength L. From these comparisons appear three adimensional parameters of main importance:

- $\mu := \frac{H_0^2}{L^2}$ is the shallow water parameter,
- $\varepsilon := \frac{a_s}{H_0}$ is the nonlinearity parameter,
- $\beta := \frac{a_b}{H_0}$ is the bathymetry parameter.

Replacing the Fourier multiplier $\mathrm{F}^{\frac{1}{2}}$ by identity in system (1.1) we retrieve the classical Green-Naghdi system. The later system is proved to be consistent with the water waves equations, in the sense of Definition 5.1 in [31], at the order of precision $\mathcal{O}(\mu^2)$ for parameters $(\mu, \varepsilon, \beta)$ in the shallow water regime:

Definition 1.1. *Let $\mu_{\max} > 0$, then we define the shallow water regime to be*

$$\mathcal{A}_{\mathrm{SW}} := \{(\mu, \varepsilon, \beta) : \mu \in (0, \mu_{\max}], \varepsilon \in [0,1], \beta \in [0,1]\}.$$

Taking ε to be zero in (1.1), we get the linearized water waves equations around the rest state with the following dispersion relation

$$\omega_{WW}(\xi)^2 = \xi^2 \frac{\tanh(\sqrt{\mu}\xi)}{\sqrt{\mu}\xi}. \tag{1.6}$$

This is why we say that system (1.1) is a full dispersion Green-Naghdi model. Moreover, it is proved in the present paper that the water waves equations are consistent, in the sense of Proposition 3.2, with system (1.1) at the order of precision $\mathcal{O}(\mu^2(\varepsilon + \beta))$. The improved precision compared to the classical Green-Naghdi system allows for a change in the propagation of the waves. Such occurrences have been studied in the Dingemans experiments [7]. In these experiments, they investigated a long wave passing over a submerged obstacle. They observed that waves tend to steepen due to a compression effect from the bottom, where high harmonics generated by topography-induced nonlinear interactions are freely released behind the obstacle. This last phenomenon makes it natural that one wants to improve the frequency dispersion of the classical shallow water models. Deriving such models has been the subject of active research. Here are some references in the case of the Boussinesq model [24, 33, 5]. In the case of the Green-Naghdi model, one can consult [42] and [6], where the authors compared the classical Green-Naghdi model with one-parameter and three-parameters Green-Naghdi models in one case of the Dingemans experiments for which the propagation and interaction of highly dispersive waves are under study. By tuning the parameters, they are able to describe the dispersion relation of the water waves equations for a larger set of frequencies. As an example, the dispersion relation of the three-parameter model is

$$\omega_{GN}(\xi)^2 = \xi^2 \frac{(1 + \mu\frac{\theta+\gamma}{3}\xi^2)(1 + \mu\frac{\alpha-1}{3}\xi^2)}{(1 + \mu\frac{\gamma}{3}\xi^2)(1 + \mu\frac{\alpha+\theta}{3}\xi^2)}, \tag{1.7}$$

where the parameters α, γ and θ are chosen such that (1.7) approximates well the dispersion relation of the water waves equations, (1.6), for higher frequencies. In particular, for $(\theta, \alpha, \gamma) = (-1, 1, 1)$ we obtain the original Green-Naghdi system. Moreover, in the case $(\theta, \alpha, \gamma) = (0.207, 1, 0.071)$ it was demonstrated in [6], that (1.7) is a better approximation of (1.6) (see Figure 1). This improvement allowed the authors to describe strongly dispersive waves with uneven bathymetry accurately. In fact, in the case where high frequencies are dominant, the improved Green-Naghdi models tend to describe the propagation of the

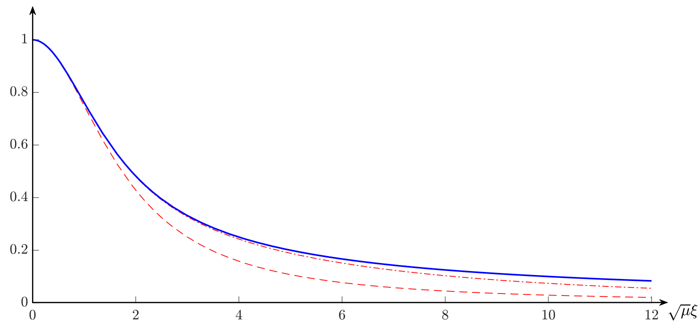

FIGURE 1. The blue curve is a plot of $\omega_{WW}(\xi)/\xi^2$ (line). The red curves plots $\omega_{GN}(\xi)/\xi^2$ in the case $(\theta, \alpha, \gamma) = (-1, 1, 1)$ (dash) and $(\theta, \alpha, \gamma) = (0.207, 1, 0.071)$ (dash-dots).

waves more correctly. However, in general, one can expect to have even higher frequency interactions for which one needs to keep the full dispersion relation of the water waves equations.

The first full dispersion model, called the Whitham equations, was introduced by Whitham in [43] to study breaking waves and Stokes waves of maximal amplitude. The existence of these phenomena for this model has been proved in the recent papers [20, 25, 38, 40]. The Whitham is a classical model in oceanography and can be seen as a modified version of the Kordeweg-de Vries equations with lower frequency dispersion. In addition, the existence of periodic waves was proved in [19], and the existence of Benjamin-Feir instabilities was demonstrated in [26, 37]. See also the series of papers on the stability of traveling waves [2, 17, 29, 39],

The study of bidirectional full dispersion models for a flat bottom has also been the subject of active research. One class of such systems is the Whitham-Boussinesq ones. They are the full dispersion versions of the Boussinesq system, meaning they have the same dispersion relation as the water waves equations (1.6). Like the Whitham equation, these type of systems features solitary waves [8, 34], Benjamin-Feir instabilities [27, 35, 41], high-frequency instabilities of small-amplitude periodic traveling waves [18]. See also some comparative studies between the Boussinesq and the Whitham-Boussinesq models [4, 11].

The full dispersion Whitham-Green-Naghdi models are next order approximations of the water waves equations when compared to the Whitham-Boussinesq systems. These systems were recently derived in [21] for a flat bottom and extended to include bathymetry in [14]. See also [13] where the authors derived a two-layers Whitham-Green-Naghdi system. There is still a lot of research left to be done on the study of qualitative properties of these systems, but we mention the work of Duchene et. al [16], which proved the existence of solitary waves where they consider both surface gravity waves and internal waves.

An important part of the study of the full dispersion systems is the full justification as an asymptotic model of the water waves equations in the shallow water regime. To be more precise, we say a model is fully justified if the following points are proven:

- The solutions of the water waves equations exist on the scale $\mathcal{O}(\frac{1}{\max\{\varepsilon,\beta\}})$.
- The solutions of the asymptotic model exist on the scale $\mathcal{O}(\frac{1}{\max\{\varepsilon,\beta\}})$.

- Solutions of the water waves equations solve the asymptotic model up to remainder terms of a specified order of precision in terms of the adimensional parameters μ, ε, and β. This last point is called the consistency of the water waves equations with respect to the asymptotic model.
- By virtue of the previous points, one has to show that the difference between the solutions of the water waves equations and the asymptotic model satisfies an error estimate depending polynomially on μ, ε and β.

If we can verify these four points, then we can compare solutions of the water waves equations with solutions of the asymptotic models up to times of order $\mathcal{O}(\frac{1}{\max\{\varepsilon,\beta\}})$. The first point is proved by Alvarez-Samaniego and Lannes in [1].

The three remaining points are specific to the asymptotic model. For instance, in the case of the Whitham equation, the local well-posedness in the relevant time scale follow by classical arguments on hyperbolic systems. The consistency of the water waves equations with this model has been recently proved in [22] at the order of precision $\mathcal{O}(\mu\varepsilon)$ in the unidirectional case, but the method supposes well-prepared initial conditions. In the bidirectional case, the author proved an order of precision $\mathcal{O}(\mu\varepsilon + \varepsilon^2)$ and doesn't suppose well-prepared initial conditions. In conclusion, we have the full justification of the Whitham equation at the order of precision $\mathcal{O}(\mu\varepsilon)$ in the unidirectional case under the restriction of well-prepared initial conditions. In the bidirectional case, the order of precision is $\mathcal{O}(\mu\varepsilon + \varepsilon^2)$.

Regarding the Whitham-Boussinesq systems for flat bottoms, the consistency of the water waves equations with the later models has been proved in [21] with an order of precision $\mathcal{O}(\mu\varepsilon)$ in the shallow water regime. When nonflat bottoms are considered, it has been proved in [14] to be consistent with the water waves with a precision $\mathcal{O}(\mu(\varepsilon+\beta))$. With respect to the second point of the justification, it has been proved for a large class of Whitham-Boussinesq systems with flat bottoms [36, 23], to be well-posed on the time scale $\mathcal{O}(\frac{1}{\varepsilon})$. Lastly, we also mention earlier results on the local-well posedness on a fixed time scale given in [9, 10, 12].

For the Whitham-Green-Naghdi systems, it is proved in [21] that for a flat bottom, the water waves equations are consistent with the later systems at the order of precision $\mathcal{O}(\mu^2\varepsilon)$ in the shallow water regime. Moreover, in the case of uneven bathymetry, it has been proved in [14] that the precision order is $\mathcal{O}(\mu^2(\varepsilon+\beta))$. In [13], the authors proved the local well-posedness with a relevant time scale for a two-layer full dispersion Green-Naghdi model with surface tension. This system can be seen as a generalization of (1.1). However, their method relies on adding surface tension, where the time of existence tends to zero as the surface tension parameter goes to zero. Moreover, this system has only been proved to be consistent with the water waves equations at the order of precision $\mathcal{O}(\mu^2)$ even if, based on numerical experiments, the expected seems to be $\mathcal{O}(\mu^2\varepsilon)$.

In the present paper, we prove the full justification of the Whitham-Green-Naghdi system without surface tension (1.1) as an asymptotic model of the water waves equations at the order of precision $\mathcal{O}(\mu^2(\varepsilon+\beta))$.

1.1. **Definition and notations.**

- We let c denote a positive constant independent of μ, ε, β that may change from line to line. Also, as a shorthand, we use the notation $a \lesssim b$ to mean $a \leq c\, b$.
- Let $s \in \mathbb{R}$ then the function $\lceil s \rceil$ returns the smallest integer greater than or equal to s.

- Let $L^2(\mathbb{R})$ be the usual space of square integrable functions with norm $|f|_{L^2} = \sqrt{\int_{\mathbb{R}} |f(x)|^2 \, dx}$. Also, for any $f, g \in L^2(\mathbb{R})$ we denote the scalar product by $\big(f, g\big)_{L^2} = \int_{\mathbb{R}} f(x)\overline{g(x)} \, dx$.
- Let $f : \mathbb{R} \to \mathbb{R}$ be a tempered distribution, let $\hat{f}$ or $\mathcal{F}f$ be its Fourier transform. Let $G : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then the Fourier multiplier associated with $G(\xi)$ is denoted G and defined by the formula:

$$\widehat{\mathrm{G}f}(\xi) = G(\xi)\hat{f}(\xi).$$

- For any $s \in \mathbb{R}$ we call the multiplier $\widehat{\mathrm{D}^s f}(\xi) = |\xi|^s \hat{f}(\xi)$ the Riesz potential of order $-s$.
- For any $s \in \mathbb{R}$ we call the multiplier $\Lambda^s = (1 + \mathrm{D}^2)^{\frac{s}{2}} = \langle \mathrm{D} \rangle^s$ the Bessel potential of order $-s$.
- The Sobolev space $H^s(\mathbb{R})$ is equivalent to the weighted L^2-space with $|f|_{H^s} = |\Lambda^s f|_{L^2}$.
- For any $s \geq 1$ we will denote $\dot{H}^s(\mathbb{R})$ the Beppo Levi space with $|f|_{\dot{H}^s} = |\Lambda^{s-1}\partial_x f|_{L^2}$.
- Let $k \in \mathbb{N}, l \in \mathbb{N}$ and $m \in \mathbb{N}$. A function R is said to be of order $\mathcal{O}(\mu^k(\varepsilon^l + \beta^m))$, denoted $R = \mathcal{O}(\mu^k(\varepsilon^l + \beta^m))$, if divided by $\mu^k(\varepsilon^l + \beta^m)$ this function is uniformly bounded with respect to $(\mu, \varepsilon, \beta) \in \mathcal{A}_{\mathrm{SW}}$ in the Sobolev norms $|\cdot|_{H^s}$, $s \geq 0$.
- We say f is a Schwartz function $\mathscr{S}(\mathbb{R})$, if $f \in C^\infty(\mathbb{R})$ and satisfies for all $j, k \in \mathbb{N}$,

$$\sup_x |x^j \partial_x^k f| < \infty.$$

- If A and B are two operators, then we denote the commutator between them to be $[A, B] = AB - BA$.

1.2. **Main results.** Throughout this paper, we will always make the following fundamental assumption.

Definition 1.2 (Non-cavitation assumption)**.** *Let $s > \frac{1}{2}$, $\varepsilon \in (0, 1)$ and $\beta \geq 0$. We say the initial surface elevation $\zeta_0 \in H^s(\mathbb{R})$ and the bottom profile $b \in L^\infty(\mathbb{R})$ satisfies the "non-cavitation assumption" if there exist $h_0 \in (0, 1)$ such that*

$$1 + \varepsilon\zeta_0(x) - \beta b(x) \geq h_0, \quad \textit{for all } \; x \in \mathbb{R}. \tag{1.8}$$

Next, before we state the main results, we define the energy space associated to (1.1).

Definition 1.3. *We define the complete function space $Y^s_\mu(\mathbb{R}^d) = H^s(\mathbb{R}) \times X^s_\mu(\mathbb{R})$, where $X^s_\mu(\mathbb{R})$ is a subspace of $H^{s+\frac{1}{2}}(\mathbb{R})$ equipped with the norm*

$$|v|^2_{X^s_\mu} := |v|^2_{H^s} + \sqrt{\mu}|\mathrm{D}^{\frac{1}{2}} v|^2_{H^s},$$

and we make the definition

$$|(\zeta, v)|^2_{Y^s_\mu} := |\zeta|^2_{H^s} + |v|^2_{X^s_\mu}.$$

The following Theorem is one of the main results of the paper and concerns the local well-posedness of (1.1) on the relevant time scale $\mathcal{O}(\frac{1}{\max\{\varepsilon,\beta\}})$ in the energy space.

Theorem 1.4 (Well-posedness)**.** *Let $s > \frac{3}{2}$ and $(\mu, \varepsilon, \beta) \in \mathcal{A}_{\mathrm{SW}}$. Assume that $(\zeta_0, v_0) \in Y^s_\mu(\mathbb{R})$ satisfies the non-cavitation condition (1.8) and $b \in H^{s+2}(\mathbb{R})$. Then there exists $T = c\big(|(\zeta_0, v_0)|_{Y^s_\mu}\big)^{-1}$ such that (1.1) admits a unique solution*

$$(\zeta, v) \in C([0, \frac{T}{\max\{\varepsilon, \beta\}}] : Y^s_\mu(\mathbb{R})) \cap C^1([0, \frac{T}{\max\{\varepsilon, \beta\}}] : Y^{s-1}_\mu(\mathbb{R})),$$

that satisfies

$$\sup_{t\in[0,\frac{T}{\max\{\varepsilon,\beta\}}]} |(\zeta,v)|_{Y^s_\mu} \lesssim |(\zeta_0,v_0)|_{Y^s_\mu}. \tag{1.9}$$

Furthermore, there exists a neighborhood of (ζ_0, v_0) *such that the flow map*

$$: Y^s_\mu(\mathbb{R}) \;\to\; C([0,\tfrac{T}{2\max\{\varepsilon,\beta\}}];Y^s_\mu(\mathbb{R})), \quad (\zeta_0,v_0) \;\mapsto\; (\zeta,v),$$

is continuous.

Remark 1.5. *For the sake of simplicity, we restrict our study to the one-dimensional setting. We comment on the possible extension to two dimensions at the end of Section 3.*

For the next Theorem, we will state the full justification of (1.1) as a water waves model. To give the result, we first state the water waves equations:

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta]\psi = 0 \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}(\partial_x\psi)^2 - \frac{\mu\varepsilon}{2}\frac{(\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta]\psi + \varepsilon\partial_x\zeta\cdot\partial_x\psi)^2}{1+\varepsilon^2\mu(\partial_x\zeta)^2} = 0, \end{cases} \tag{1.10}$$

where $\mathcal{G}^\mu[\varepsilon\zeta]$ stands for the Dirichlet-Neumann operator and ψ is the trace at the surface of the velocity potential Φ, see [31] for more information. To compare solutions between the water waves equations and system (1.1), we define the vertical average of the horizontal component of the velocity field through the formula

$$\overline{V} = \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta} \partial_x\Phi \,\mathrm{d}z, \tag{1.11}$$

where Φ stands for the velocity potential in the water domain $\Omega_t := \{(x,z)\in\mathbb{R}^2, -1+\beta b \leq z \leq \varepsilon\zeta\}$. It is the solution of the following elliptic problem

$$\begin{cases} \partial_z^2\Phi + \mu\partial_x^2\Phi = 0, \quad \text{in} \;\; \Omega_t \\ \Phi|_{z=\varepsilon\zeta} = \psi, \quad \partial_n\Phi|_{z=-1+\beta b} = 0, \end{cases} \tag{1.12}$$

where $\partial_n\Phi|_{z=-1+\beta b} = \partial_z\Phi - \mu\beta\partial_x b\partial_x\Phi$. The last ingredient in justifying the full dispersion Green-Naghdi system is a long time existence result for the water waves equations. As explaind above, this was proved in [1] and is also detailed in [31] (see Theorem 4.16). The result holds for regular data satisfying the non-cavitation condition and the classical Rayleigh-Taylor stability condition. Since this is a technical condition related to the hyperbolicity of the system, we will simply refer to [31] for the precise statement.

We may now state the final result of this paper.

Theorem 1.6 (Full justification)**.** *Let* $s\in\mathbb{N}$ *such that* $s\geq 4$ *and* $(\mu,\varepsilon,\beta)\in\mathcal{A}_{\mathrm{SW}}$*. Then for* $b\in H^{s+2}(\mathbb{R})$ *and any initial data* $(\zeta_0,\psi_0)\in H^s(\mathbb{R})\times\dot{H}^s(\mathbb{R})$ *satisfying the non-cavitation assumption* (1.8) *and the Rayleigh-Taylor stability condition* (4.29) *given in* [31]*, there exist a time* $\tilde{T}>0$ *and unique classical solution of the water waves equations* (1.10) *given by*

$$(\zeta,\psi)\in C([0,\frac{\tilde{T}}{\max\{\varepsilon,\beta\}}] : H^s(\mathbb{R})\times\dot{H}^s(\mathbb{R})),$$

from which we define $\overline{V}\in C([0,\frac{\tilde{T}}{\max\{\varepsilon,\beta\}}] : H^s(\mathbb{R}))$ *through* (1.11) *and let* $\mathbf{U} = (\zeta,\overline{V})$*.*

Moreover, if we let $v_0^{WGN} = \overline{V}|_{t=0}$*. Then* $v_0^{WGN}\in X^s_\mu(\mathbb{R})$ *and there exist a unique classical solution, denoted by*

$$\mathbf{U}^{WGN} = (\zeta^{WGN}, v^{WGN}) \in C([0,\frac{T}{\max\{\varepsilon,\beta\}}] : Y^s_\mu(\mathbb{R})),$$

of the Whitham-Green-Naghdi system (1.1) *sharing the same initial data*

$$\mathbf{U}^{WGN}|_{t=0} = (\zeta, \overline{V})|_{t=0}.$$

Comparing the two solutions, we have that for $s \in \mathbb{N}$ *large enough such that for all* $0 \leq \max\{\varepsilon, \beta\}t \leq \min\{\tilde{T}, T\}$ *there holds*

$$|\mathbf{U} - \mathbf{U}^{WGN}|_{L^\infty([0,t]\times\mathbb{R})} \leq C(|\zeta_0|_{H^s}, |\overline{V}|_{t=0}|_{H^s}, |b|_{H^{s+2}})\mu^2(\varepsilon + \beta)t,$$

with $\tilde{T}, T, C$ *positive constants uniform with respect to* $(\mu, \varepsilon, \beta) \in \mathcal{A}_{SW}$.

Remark 1.7. *In the statement of the theorem, we simply let* s *be large enough. The reason is due to the consistency result given by Theorem* 10.5 *in* [14], *which links the water waves equations with a similar Whitham-Green-Naghdi system. However, it is possible to have a precise range of* s *if one reproves this theorem and carefully tracks the "loss of derivatives". See Section 3 for more on this point.*

1.3. **Outline.** In Section 2, we state the technical estimates that will be used throughout the paper. In Subsection 2.1, we state some classical estimates. In Subsection 2.2 we study the properties of the Fourier multiplier $\mathrm{F}^{\frac{1}{2}}$. Lastly, in Subsection (2.3) we establish the properties related to the operator $\mathcal{T}[h, \beta b]$ defined by (1.2).

In Section 3 we prove the consistency of the water waves equations with system (1.1) at the order of precision $\mathcal{O}(\mu^2(\varepsilon + \beta))$ in the shallow water regime $\mathcal{A}_{\mathrm{SW}}$. The starting point of this proof is the full dispersion Green-Naghdi system derived [14] where the precision with respect to the water waves equations (1.10) is proved to be $\mathcal{O}(\mu^2(\varepsilon + \beta))$.

Sections, 4 and 5 are about establishing the energy estimates with uniform bounds on the solutions. Then as a result of the energy estimates provided in the aforementioned sections, we are in the position to prove Theorem 1.4 in Section 6. The proof relies on classical hyperbolic theory for quasilinear systems.

In Section 7, we prove the full justification result of system (1.1) resulting from all previous sections.

2. Preliminary results

2.1. **Classical estimates.** In this section, we state some classical results that will be used throughout the paper. First, recall the embedding results (see, for example, [32]).

Proposition 2.1 (Sobolev embedding)**.** *Let* $f \in \mathscr{S}(\mathbb{R})$ *and* $s \in (0, \frac{1}{2})$. *Then* $H^s(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ *with* $p = \frac{2}{1-2s}$, *and there holds*

$$|f|_{L^p} \lesssim |\mathrm{D}^s f|_{L^2}. \tag{2.1}$$

Moreover, In the case $s > \frac{1}{2}$, *then* $H^s(\mathbb{R})$ *is continuously embedded in* $L^\infty(\mathbb{R})$.

Next, we state the Leibniz rule for the Riesz potential.

Proposition 2.2 (Fractional Leibniz rule [30])**.** *Let* $\sigma = \sigma_1 + \sigma_2 \in (0, 1)$ *with* $\sigma_i \in [0, \sigma]$ *and* $p, p_1, p_2 \in (1, \infty)$ *satisfy* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Then, for* $f, g \in \mathscr{S}(\mathbb{R})$

$$|\mathrm{D}^\sigma(fg) - f\mathrm{D}^\sigma g - g\mathrm{D}^\sigma f|_{L^p} \lesssim |\mathrm{D}^{\sigma_1} f|_{L^{p_1}} |\mathrm{D}^{\sigma_2} g|_{L^{p_2}}. \tag{2.2}$$

Moreover, the case $\sigma_2 = 0$, $p_2 = \infty$ *is also allowed.*

Corollary 2.3. *Let* $r \in (\frac{1}{2}, 1)$, $f \in H^r(\mathbb{R})$ *and* $g \in H^{\frac{1}{2}}(\mathbb{R})$. *Then*

$$|fg|_{L^2} \lesssim |\mathrm{D}^{r-\frac{1}{2}} f|_{L^2} |g|_{H^{\frac{1}{2}}} \tag{2.3}$$

and

$$|\mathrm{D}^{\frac{1}{2}}(fg)|_{L^2} \lesssim |f|_{H^r} |g|_{H^{\frac{1}{2}}}. \tag{2.4}$$

Proof. To prove (2.3), we first let $\nu \in (0, \frac{1}{2})$ to be fixed later. Then combine Hölder's inequality with the conjugate pair $\frac{1}{p_1} + \frac{1}{p_2} = \nu + \frac{1-2\nu}{2} = \frac{1}{2}$ and (2.1) to get that

$$|fg|_{L^2} \lesssim |f|_{L^{\frac{1}{\nu}}} |g|_{L^{\frac{2}{1-2\nu}}} \lesssim |\mathrm{D}^{\frac{1-2\nu}{2}} f|_{L^2} |\mathrm{D}^{\nu} g|_{L^2}.$$

However, for any $r \in (\frac{1}{2}, 1)$ we observe that we may choose ν such that $\frac{1-2\nu}{2} = r - \frac{1}{2}$, and the proof follows.

Next, we prove (2.4). We will use Hölder's inequality, (2.2) with $(\sigma_1, \sigma_2) = (\frac{1}{2}, 0)$, and $\frac{1}{2} = \nu + \frac{1-2\nu}{2}$ with $\nu \in (0, \frac{1}{2})$ as above to deduce

$$\begin{aligned} |\mathrm{D}^{\frac{1}{2}}(fg)|_{L^2} &\leq |\mathrm{D}^{\frac{1}{2}}(fg) - f\mathrm{D}^{\frac{1}{2}} g - g\mathrm{D}^{\frac{1}{2}} f|_{L^2} + |f\mathrm{D}^{\frac{1}{2}} g|_{L^2} + |g\mathrm{D}^{\frac{1}{2}} f|_{L^2} \\ &\lesssim |\mathrm{D}^{\frac{1}{2}} f|_{L^{\frac{1}{\nu}}} |g|_{L^{\frac{2}{1-2\nu}}} + |f|_{L^\infty} |\mathrm{D}^{\frac{1}{2}} g|_{L^2} \\ &\lesssim |f|_{H^r} |g|_{H^{\frac{1}{2}}}, \end{aligned}$$

where we used (2.1) in the last line with $\frac{1-2\nu}{2} = r - \frac{1}{2}$, and the Sobolev embedding $H^r(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

□

Definition 2.4. *Let $d = 1, 2$ We say that a Fourier multiplier* $\mathrm{G}(\mathrm{D})$ *is of order s ($s \in \mathbb{R}$) and write* $\mathrm{G} \in \mathcal{S}^s$ *if $\xi \in \mathbb{R}^d \mapsto G(\xi) \in \mathbb{C}$ is smooth and satisfies*

$$\forall \xi \in \mathbb{R}^d, \forall \beta \in \mathbb{N}^d, \quad \sup_{\xi \in \mathbb{R}^d} \langle \xi \rangle^{|\beta| - s} |\partial^\beta G(\xi)| < \infty.$$

We also introduce the seminorm

$$\mathcal{N}^s(\mathrm{G}) = \sup_{\beta \in \mathbb{N}^d, |\beta| \leq 2 + d + \lceil \frac{d}{2} \rceil} \sup_{\xi \in \mathbb{R}^d} \langle \xi \rangle^{|\beta| - s} |\partial^\beta G(\xi)|.$$

Proposition 2.5. *Let $d = 1, 2$, $t_0 > d/2$, $s \geq 0$ and $\mathrm{G} \in \mathcal{S}^s$. If $f \in H^s \cap H^{t_0+1}(\mathbb{R}^d)$ then, for all $g \in H^{s-1}(\mathbb{R}^d)$,*

$$|[\mathrm{G}, f]g|_2 \leq \mathcal{N}^s(\mathrm{G}) |f|_{H^{\max\{t_0+1, s\}}} |g|_{H^{s-1}}. \tag{2.5}$$

See Appendix B.2 in [31] for the proof of this proposition. Next, we will need the following results to run the Bona-Smith argument (provided in the classical paper [3]) on the multiplier $\chi_\delta(\mathrm{D})$ defined by:

Definition 2.6. *Let $\chi \in \mathscr{S}(\mathbb{R})$ be a real valued function such that $\chi(0) = 1$ and $\int_{\mathbb{R}} \chi(\xi) \mathrm{d}\xi = 1$. Then for $\delta > 0$ we define the regularisation operators $\chi_\delta(\mathrm{D})$ in frequency by*

$$\forall f \in L^2(\mathbb{R}), \quad \forall \xi \in \mathbb{R}, \quad \widehat{\chi_\delta f}(\xi) := \chi(\delta \xi) \hat{f}(\xi).$$

We give the version of the regularisation estimates as presented in [32] (Proposition 9.1).

Proposition 2.7. *Let $s > 0$, $\delta > 0$ and $f \in \mathscr{S}(\mathbb{R})$. Then*

$$|\chi_\delta(\mathrm{D}) f|_{H^{s+\alpha}} \lesssim \delta^{-\alpha} |f|_{H^s}, \quad \forall \alpha \geq 0, \tag{2.6}$$

and

$$|\chi_\delta(\mathrm{D}) f - f|_{H^{s-\beta}} \lesssim \delta^\beta |f|_{H^s}, \quad \forall \beta \in [0, s]. \tag{2.7}$$

Moreover, there holds

$$|\chi_\delta(\mathrm{D}) f - f|_{H^{s-\beta}} \underset{\delta \to 0}{=} \mathcal{O}(\delta^\beta), \qquad \forall \beta \in [0, s]. \tag{2.8}$$

Lastly, we need an interpolation inequality. In particular, for any $s, r, t \in \mathbb{R}$ such that $r < t$ and $s \in (r, t)$, then for $\theta \in (0,1)$ given by $\theta + (1-\theta) = \frac{t-s}{t-r} + \frac{s-r}{t-r}$, we have by Plancherel's identity and Hölder's inequality that

$$|f|_{H^s} \leq |f|_{H^r}^{\theta} |f|_{H^t}^{(1-\theta)}. \tag{2.9}$$

2.2. **Properties of** F. In this section, we prove estimates concerning the dispersive properties of the equation.

Proposition 2.8. *Let $s \in \mathbb{R}$ and $f \in \mathscr{S}(\mathbb{R})$, then there exist $c > 0$ such that*

$$c^{-1}|f|_{X_\mu^s}^2 \leq |f|_{H^s}^2 + \mu|\mathrm{F}^{\frac{1}{2}}\partial_x f|_{H^s}^2 \leq c|f|_{X_\mu^s}^2, \tag{2.10}$$

$$|\mathrm{F}^{-\frac{1}{2}} f|_{H^s} \leq c|f|_{X_\mu^s}, \tag{2.11}$$

$$\sqrt{\mu}|\mathrm{F}^1 f|_{H^s} \leq c|f|_{H^{s-1}}, \tag{2.12}$$

$$\mu^{\frac{1}{4}}|\mathrm{F}^{\frac{1}{2}} f|_{H^s} \leq c|f|_{H^{s-\frac{1}{2}}}. \tag{2.13}$$

Proof. The behaviour at low frequency of the three Fourier multipliers $\mathrm{F}^{\frac{1}{2}}, \mathrm{F}^{-\frac{1}{2}}$ and F^1 at low frequency is

$$F^{\frac{1}{2}}(\xi), F^{-\frac{1}{2}}(\xi), F^1(\xi) \underset{0}{\sim} 1.$$

At high frequency, their respective behavior is

$$F^{\frac{1}{2}}(\xi) \underset{\infty}{\sim} \frac{1}{\mu^{\frac{1}{4}}\sqrt{|\xi|}}, \quad F^{-\frac{1}{2}}(\xi) \underset{\infty}{\sim} \mu^{\frac{1}{4}}\sqrt{|\xi|}, \quad F^1(\xi) \underset{\infty}{\sim} \frac{1}{\sqrt{\mu}|\xi|}.$$

This gives us (2.11), (2.12), (2.13), and the right-hand side inequality of (2.10). It only remains to prove the left-hand side inequality of (2.10):

$$|f|_{X_\mu^s}^2 = |f|_{H^s}^2 + \sqrt{\mu}|\mathrm{D}^{\frac{1}{2}} f|_{H^s}^2.$$

Now, let $\mathcal{F}(\mathbb{1}_{\{\sqrt{\mu}|\mathrm{D}|\leq 1\}} f)(\xi) = \mathbb{1}_{\{\sqrt{\mu}|\xi|\leq 1\}}\hat{f}(\xi)$ where $\mathbb{1}_{\{\sqrt{\mu}|\xi|\leq 1\}}$ is the usual indicator function supported on the frequencies $\sqrt{\mu}|\xi| \leq 1$. Then we get that

$$\begin{aligned}
\sqrt{\mu}|\mathrm{D}^{\frac{1}{2}} f|_{H^s}^2 &= \sqrt{\mu}|\mathbb{1}_{\{\sqrt{\mu}|\mathrm{D}|\leq 1\}}\mathrm{D}^{\frac{1}{2}}\Lambda^s f|_{L^2}^2 + \sqrt{\mu}|\mathbb{1}_{\{\sqrt{\mu}|\mathrm{D}|> 1\}}\mathrm{D}^{\frac{1}{2}}\Lambda^s f|_{L^2}^2 \\
&\lesssim \sqrt{\mu}|\mathbb{1}_{\{\sqrt{\mu}|\mathrm{D}|\leq 1\}}\mathrm{F}^{\frac{1}{2}}\mathrm{D}^{\frac{1}{2}}\Lambda^s f|_{L^2}^2 + \mu^{\frac{3}{2}}|\mathbb{1}_{\{\sqrt{\mu}|\mathrm{D}|> 1\}}\mathrm{F}^1\mathrm{D}^{\frac{1}{2}}\Lambda^s f|_{L^2}^2 \\
&\lesssim |f|_{H^s}^2 + \mu|\mathrm{F}^{\frac{1}{2}} f|_{H^s}^2.
\end{aligned}$$

□

Proposition 2.9. *Let $f, g \in \mathscr{S}(\mathbb{R})$ and $t_0 > \frac{1}{2}$. Then for $s \geq \frac{1}{2}$ there holds,*

$$|[\Lambda^s \mathrm{F}^{\frac{1}{2}}, f]g|_{L^2} \lesssim |f|_{H^{\max\{t_0+1, s-\frac{1}{2}\}}}|\mathrm{F}^{\frac{1}{2}} g|_{H^{s-1}}. \tag{2.14}$$

In the case $s \geq 1$, there holds

$$|[\Lambda^s, \mathrm{F}^{\frac{1}{2}}\partial_x(f\cdot)]\partial_x g|_{L^2} \lesssim |\mathrm{F}^{\frac{1}{2}}\partial_x f|_{H^s}|\partial_x g|_{H^{t_0}} + |\mathrm{F}^{\frac{1}{2}}\partial_x g|_{H^s}|\partial_x f|_{H^{t_0}}. \tag{2.15}$$

Moreover, in the case $s = 0$ we have that

$$|[\mathrm{F}^{\frac{1}{2}}, f]g|_{L^2} \lesssim |f|_{X_\mu^{t_0+1}}|\mathrm{F}^{\frac{1}{2}} g|_{H^{-1}}. \tag{2.16}$$

Proof. To prove (2.14) we note that the Fourier multiplier $\Lambda^s \mathrm{F}^{\frac{1}{2}}$ is of order $s-\frac{1}{2}$ in the sense of Definition 2.4. Moreover, we observe that

$$\mathcal{N}^{s-\frac{1}{2}}(\Lambda^s \mathrm{F}^{\frac{1}{2}}) \lesssim \mu^{-\frac{1}{4}}.$$

Thanks to the commutator estimates of Proposition 2.5 and estimate (2.10), we have

$$|[\Lambda^s \mathrm{F}^{\frac{1}{2}}, f]g|_{L^2} \lesssim \mu^{-\frac{1}{4}} |f|_{H^{\max\{t_0+1, s-\frac{1}{2}\}}} |g|_{H^{s-\frac{3}{2}}} \lesssim |f|_{H^{\max\{t_0+1, s-\frac{1}{2}\}}} |\mathrm{F}^{\frac{1}{2}} g|_{H^{s-1}},$$

and proves estimate (2.14).

For the proof of (2.15), we start by estimating the bilinear form:

$$\mathrm{a(D)}(f,g) = [\Lambda^s, \mathrm{F}^{\frac{1}{2}} \partial_x (f \cdot)] \partial_x g,$$

given by

$$|\hat{a}(\xi)(f,g)| \leq \int_{\mathbb{R}} F^{\frac{1}{2}}(\xi)|\xi| \big| \langle \xi \rangle^s - \langle \rho \rangle^s \big| |\hat{f}(\xi - \rho)| \, |\widehat{\partial_x g}(\rho)| \, d\rho.$$

First, if $|\xi| \leq |\rho|$ we can use the mean value theorem to deduce that

$$|\hat{a}(\xi)(f,g)| \lesssim \int_{\mathbb{R}} F^{\frac{1}{2}}(\rho)|\rho| \langle \rho \rangle^{s-1} |\xi - \rho| |\hat{f}(\xi - \rho)| \, |\widehat{\partial_x g}(\rho)| \, d\rho,$$

since both $\omega \mapsto \langle \omega \rangle^{s-1}$ and $\omega \mapsto F^{\frac{1}{2}}(\omega)|\omega|$ are increasing functions. By extension, we make a change of variable $\gamma = \xi - \rho$, apply Minkowski integral inequality and Cauchy-Schwarz to find the estimate

$$\begin{aligned}
|[\Lambda^s, \mathrm{F}^{\frac{1}{2}} \partial_x (f \cdot)] \partial_x g|_{L^2} &\lesssim \Big| \int_{\mathbb{R}} F^{\frac{1}{2}}(\cdot - \gamma)| \cdot - \gamma| \langle \cdot - \gamma \rangle^{s-1} |\gamma| |\hat{f}(\gamma)| \, |\widehat{\partial_x g}(\cdot - \gamma)| \, d\gamma \Big|_{L^2_\xi} \\
&\lesssim |\mathrm{F}^{\frac{1}{2}} \partial_x g|_{H^s} \int_{\mathbb{R}} |\gamma| |\hat{f}(\gamma)| \, d\gamma \\
&\lesssim |\mathrm{F}^{\frac{1}{2}} \partial_x g|_{H^s} |\partial_x f|_{H^{t_0}}.
\end{aligned}$$

On the other hand, when $|\rho| \leq |\xi|$ then we can argue similarly to find that

$$\begin{aligned}
|\hat{a}(\xi)(f,g)| \lesssim & \int_{\mathbb{R}} F^{\frac{1}{2}}(\xi - \rho)|\xi - \rho| \langle \xi - \rho \rangle^{s-1} |\xi - \rho| |\hat{f}(\xi - \rho)| \, |\widehat{\partial_x g}(\rho)| \, d\rho \\
& + \int_{\mathbb{R}} F^{\frac{1}{2}}(\rho)|\rho| \langle \rho \rangle^{s-1} |\xi - \rho| |\hat{f}(\xi - \rho)| \, |\widehat{\partial_x g}(\rho)| \, d\rho,
\end{aligned}$$

and as using the estimate above we conclude in this case that

$$\begin{aligned}
|[\Lambda^s, \mathrm{F}^{\frac{1}{2}} \partial_x (f \cdot)] \partial_x g|_{L^2} &\lesssim \Big| \int_{\mathbb{R}} F^{\frac{1}{2}}(\cdot - \rho)| \cdot - \rho| \langle \cdot - \rho \rangle^{s-1} | \cdot - \rho| |\hat{f}(\cdot - \rho)| \, |\widehat{\partial_x g}(\rho)| \, d\rho \Big|_{L^2_\xi} \\
&\quad + |\mathrm{F}^{\frac{1}{2}} \partial_x g|_{H^s} |\partial_x f|_{H^{t_0}} \\
&\lesssim |\mathrm{F}^{\frac{1}{2}} \partial_x f|_{H^s} |\partial_x g|_{H^{t_0}}. + |\mathrm{F}^{\frac{1}{2}} \partial_x g|_{H^s} |\partial_x f|_{H^{t_0}}.
\end{aligned}$$

Adding the two cases completes the proof.

Next, we prove (2.16) by estimating the bilinear form:

$$|\hat{a}(\xi)(f,g)| \lesssim \int_{\mathbb{R}} \big| F^{\frac{1}{2}}(\xi) - F^{\frac{1}{2}}(\rho) \big| |\hat{f}(\xi - \rho)| |\hat{g}(\rho)| \, d\rho.$$

Clearly, it is enough to prove that

$$k(\xi,\rho) := \big|F^{\frac{1}{2}}(\xi) - F^{\frac{1}{2}}(\rho)\big|F^{-\frac{1}{2}}(\rho)\langle\rho\rangle \lesssim 1 + |\xi-\rho|F^{-\frac{1}{2}}(\xi-\rho). \tag{2.17}$$

Indeed, assuming the claim (2.17) and using Plancherel, Minkowski integral inequality, the Cauchy-Schwarz inequality and (2.11) we obtain the desired estimate

$$\begin{aligned}
|[\mathrm{F}^{\frac{1}{2}}, f]g|_{L^2} &\leq \Big|\int_{\mathbb{R}} k(\xi,\rho)\,|\hat{f}(\xi-\rho)|\,F^{\frac{1}{2}}(\rho)\langle\rho\rangle^{-1}|\hat{g}(\rho)|\,d\rho\Big|_{L^2_\xi} \\
&\lesssim \Big|\int_{\mathbb{R}} \big(1+|\gamma|F^{-\frac{1}{2}}(\gamma)\big)\,|\hat{f}(\gamma)|\,F^{\frac{1}{2}}(\xi-\gamma)\langle\xi-\gamma\rangle^{-1}|\hat{g}(\xi-\gamma)|\,d\gamma\Big|_{L^2_\xi} \\
&\lesssim |f|_{X^{t_0+1}_\mu}|\mathrm{F}^{\frac{1}{2}}g|_{H^{-1}}.
\end{aligned}$$

Now, to prove the claim (2.17), we consider three cases. First, in the case $|\rho|\leq 1$ it follows directly that

$$k(\xi,\rho)\lesssim 1,$$

since $\xi\mapsto F^{\frac{1}{2}}(\xi)$ and $\rho\mapsto F^{-\frac{1}{2}}(\rho)\langle\rho\rangle$ is bounded. Next, consider the case $|\rho|>1$ and $|\xi|\leq|\rho|$. Then we note that since $\xi\mapsto F^{\frac{1}{2}}(\xi)$ is decreasing for $\xi>0$, we have the estimate

$$\frac{F(\rho)}{F(\xi)}\leq 1,$$

and moreover since $\xi\mapsto\big(\frac{\sqrt{\mu}\xi}{\tanh(\sqrt{\mu}\xi)}-1\big)$ is increasing for $\xi>0$, we get that

$$\frac{|\xi|}{|\rho|}\leq\Big(\frac{|\xi|}{|\rho|}\Big)^{\frac{1}{2}}\leq\Big(\frac{F(\rho)}{F(\xi)}\Big)^{\frac{1}{2}}\leq 1. \tag{2.18}$$

Thus, we obtain the bound

$$\begin{aligned}
k(\xi,\rho) &= \Big(1-\Big(\frac{F(\rho)}{F(\xi)}\Big)^{\frac{1}{2}}\Big)\Big(\frac{F(\xi)}{F(\rho)}\Big)^{\frac{1}{2}}\langle\rho\rangle \\
&\lesssim (|\rho|-|\xi|)\Big(\frac{F(\xi)}{F(\rho)}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Finally, to conclude this case, we make the observation that if $|\xi|\sim|\rho|$, then

$$\Big(\frac{F(\xi)}{F(\rho)}\Big)^{\frac{1}{2}}\lesssim 1.$$

Otherwise, we obtain

$$\Big(\frac{F(\xi)}{F(\rho)}\Big)^{\frac{1}{2}}\lesssim 1+\mu^{\frac{1}{4}}|\rho|^{\frac{1}{2}}\lesssim 1+\mu^{\frac{1}{4}}|\rho-\xi|^{\frac{1}{2}}.$$

Gathering these estimates allows us to conclude that

$$k(\xi,\rho)\lesssim(|\rho|-|\xi|)\Big(\frac{F(\xi)}{F(\rho)}\Big)^{\frac{1}{2}}\lesssim 1+|\xi-\rho|F^{-\frac{1}{2}}(\xi-\rho).$$

On the other hand, the case $|\xi| > |\rho| > 1$ follows directly by changing the role of ξ and ρ in (2.18). Indeed, we obtain that

$$\begin{aligned} k(\xi,\rho) &= \left(1 - \left(\frac{F(\xi)}{F(\rho)}\right)^{\frac{1}{2}}\right)\langle\rho\rangle \\ &\lesssim (|\xi| - |\rho|), \end{aligned}$$

and the proof of (2.16) is complete.

□

Proposition 2.10. *Let $s \geq 0$, and let $f \in H^{s+2}(\mathbb{R})$, then we have the following estimation on the Fourier multiplier* $\mathrm{F}^{\frac{1}{2}}$

$$|(\mathrm{F}^{\frac{1}{2}} - 1)f|_{H^s} \lesssim \mu |f|_{H^{s+2}}.$$

Proof. First, remark that it is enough to prove the result only when $s = 0$. The function defining the Fourier multiplier $\mathrm{F}^{\frac{1}{2}}$ is a smooth function on $(0, +\infty)$, continuous in 0 with $F^{\frac{1}{2}}(0) = 1$ and its first derivative is zero. Moreover, its second derivative is bounded in $[0, +\infty)$, so that from Plancherel identity and the Taylor-Lagrange formula, we get

$$|(F^{\frac{1}{2}}(\sqrt{\mu}|\xi|) - 1)\hat{f}|_2 \leq \mu||\xi|^2\hat{f}|_{L^2}.$$

In the end, we have the estimate

$$|(\mathrm{F}^{\frac{1}{2}} - 1)f|_{L^2} \leq \mu |f|_{H^2}.$$

□

2.3. **Properties of $\mathscr{T}[h,\beta b]$.** In this section, we study an elliptic operator associated with $\mathcal{T}[h,\beta b]$ given by (1.2). The main result is given in the following proposition where the main reference is [28].

Proposition 2.11. *Let $(\mu,\varepsilon,\beta) \in \mathcal{A}_{\mathrm{SW}}$, $s \geq 0$, $\zeta \in H^{\max\{1,s\}}(\mathbb{R})$, $b \in H^{s+2}(\mathbb{R})$ and let $h = 1 + \varepsilon\zeta - \beta b$ satisfy the non-cavitation condition (1.8). Define the application*

$$\mathscr{T}[h,\beta b] \ : \ \begin{cases} H^1(\mathbb{R}) & \to L^2(\mathbb{R}) \\ v & \mapsto hv + \mu h \mathcal{T}[h,\beta b]v \end{cases} \tag{2.19}$$

Then we have the following properties:

1. *The operator (2.19) is well-defined and for $v \in H^1(\mathbb{R})$ there holds,*
$$|\mathscr{T}[h,\beta b]v|_{L^2} \lesssim |v|_{X_\mu^{\frac{1}{2}}}. \tag{2.20}$$
2. *The operator (2.19) is one-to-one and onto.*
3. *For $s \geq 0$ and $f \in H^s(\mathbb{R})$ there holds,*
$$|\mathscr{T}^{-1}[h,\beta b]f|_{X_\mu^s} \lesssim |f|_{H^s}. \tag{2.21}$$
4. *For $s \geq 1$ and $f \in H^{s-1}(\mathbb{R})$ there holds,*
$$\sqrt{\mu}|\mathrm{F}^{\frac{1}{2}}\mathscr{T}^{-1}[h,\beta b]f|_{H^s} \lesssim |f|_{H^{s-1}}. \tag{2.22}$$

Proof. We give the proof in four steps.

Step 1: The application (2.19) is well-defined. Indeed, by assumption and Sobolev embedding $H^{\frac{1}{2}^+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ we have that $h \in L^\infty(\mathbb{R})$. Therefore, by (2.10) we get that

$$\begin{aligned}
|\mathscr{T}[h,\beta b]v|_{L^2} &\lesssim |hv|_{L^2} + \mu|\partial_x \mathrm{F}^{\frac{1}{2}}(h^3\mathrm{F}^{\frac{1}{2}}\partial_x v)|_{L^2} + \mu|\partial_x \mathrm{F}^{\frac{1}{2}}(h^2(\beta\partial_x b)v)|_{L^2} \\
&\quad + \mu|h^2(\beta\partial_x b)\mathrm{F}^{\frac{1}{2}}\partial_x v|_{L^2} + \mu|h(\beta\partial_x b)^2 v|_{L^2} \\
&\lesssim |h|_{L^\infty}|v|_{L^2} + \mu^{\frac{3}{4}}|\mathrm{D}^{\frac{1}{2}}(h^3\mathrm{F}^{\frac{1}{2}}\partial_x v)|_{L^2} + \mu^{\frac{3}{4}}|\mathrm{D}^{\frac{1}{2}}(h^2(\beta\partial_x b)v)|_{L^2} \\
&\quad + \mu|h^2(\beta\partial_x b)\mathrm{F}^{\frac{1}{2}}\partial_x v|_{L^2} + \mu|h|_{L^\infty}|(\beta\partial_x b)^2 v|_{L^2} \\
&=: A_1 + A_2 + A_3 + A_4 + A_5.
\end{aligned}$$

To conclude, we note that $(h-1) \in H^1(\mathbb{R})$ and together with Hölder's inequality, the Sobolev embedding, and (2.10) we estimate $A_1 + A_4 + A_5$:

$$A_1 + A_4 + A_5 \leq c(|h-1|_{H^1}, |h^2-1|_{H^1}, \beta|\partial_x b|_{L^\infty})|v|_{X_\mu^{\frac{1}{2}}}.$$

The remaining terms are treated similarly, after an application of (2.4), and yield the desired estimate

$$|\mathscr{T}[h,\beta b]v|_{L^2} \lesssim |v|_{X_\mu^{\frac{1}{2}}}.$$

Step 2. The application (2.19) is one-to-one and onto. Equivalently, we prove that there exist a unique solution $v \in H^1(\mathbb{R})$ to the equation

$$\mathscr{T}[h,\beta b]v = f, \tag{2.23}$$

for $f \in L^2(\mathbb{R})$. To construct a solution, we first consider the variational formulation of (2.23) that is given by

$$a(v,\varphi) = L(\varphi), \tag{2.24}$$

for any $\varphi \in C_c^\infty(\mathbb{R})$ and with

$$\begin{cases} a(v,\varphi) := \big(v, h\varphi\big)_{L^2} + \big(v, \mu h\mathcal{T}[h,\beta b]\varphi\big)_{L^2} \\ L(\varphi) := \big(f,\varphi\big)_{L^2}. \end{cases}$$

Then, through a direct application of the Lax-Milgram lemma, we prove there exists a unique variational solution $v \in \overline{C_c^\infty(\mathbb{R})}^{|\cdot|_{H^{\frac{1}{2}}}} = H^{\frac{1}{2}}(\mathbb{R})$. Indeed, we observe that the application $(u,v) \mapsto a(u,v)$ is continuous on $H^{\frac{1}{2}}(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$:

$$|a(v,\varphi)| \leq c(|h-1|_{H^1}, \beta|\partial_x b|_{L^\infty})|v|_{X_\mu^0}|\varphi|_{X_\mu^0},$$

by integration by parts, Hölder's inequality and (2.10). Moreover, the coercivity estimate is deduced by first making the observation:

$$\begin{aligned}
a(v,v) &= \big(v, hv\big)_{L^2} + \mu\Big(h\big(\frac{h}{\sqrt{3}}\mathrm{F}^{\frac{1}{2}}\partial_x v - \frac{\sqrt{3}}{2}(\beta\partial_x b)v\big), \frac{h}{\sqrt{3}}\mathrm{F}^{\frac{1}{2}}\partial_x v - \frac{\sqrt{3}}{2}(\beta\partial_x b)v\Big)_{L^2} \\
&\quad + \frac{\mu}{4}\big(h(\beta\partial_x b)v, (\beta\partial_x b)v\big)_{L^2} \\
&\geq h_0|v|_{L^2}^2 + \mu h_0\Big|\frac{h}{\sqrt{3}}\mathrm{F}^{\frac{1}{2}}\partial_x v - \frac{\sqrt{3}}{2}(\beta\partial_x b)v\Big|_{L^2}^2 + \frac{\mu\beta^2}{4}|\sqrt{h}(\partial_x b)v|_{L^2}^2 \\
&=: I.
\end{aligned}$$

Now, let $\nu > 0$ be chosen later and make the decomposition $I = (1-\nu)I + \nu I$. Then the first term can be bounded below by

$$(1-\nu)I \geq (1-\nu)h_0|v|_{L^2}^2 + \frac{(1-\nu)\mu\beta^2}{4}|\sqrt{h}(\partial_x b)v|_{L^2}^2.$$

On the other hand, the remaining part is estimated by Cauchy-Schwarz and Young's inequality:

$$\begin{aligned}\nu I &\geq \frac{\nu\mu h_0^3}{3}|\mathrm{F}^{\frac{1}{2}}\partial_x v|_{L^2}^2 - \mu\nu h_0\beta|\sqrt{h}|_{L^\infty}|\sqrt{h}(\partial_x b)v|_{L^2}|\mathrm{F}^{\frac{1}{2}}\partial_x v|_{L^2} \\ &\geq \frac{\nu\mu h_0^3}{3}|\mathrm{F}^{\frac{1}{2}}\partial_x v|_{L^2}^2 - \mu\nu h_0\Big(\frac{h_0^2}{6}|\mathrm{F}^{\frac{1}{2}}\partial_x v|_{L^2}^2 + \frac{3}{2h_0^2}|\sqrt{h}|_{L^\infty}^2\beta^2|\sqrt{h}(\partial_x b)v|_{L^2}\Big).\end{aligned}$$

So that

$$I \geq (1-\nu)h_0|v|_{L^2}^2 + \frac{\nu\mu h_0^3}{6}|\mathrm{F}^{\frac{1}{2}}\partial_x v|_{L^2}^2 + \mu\beta^2\Big(\frac{1}{4} - \nu\Big(\frac{1}{4} + \frac{3|\sqrt{h}|_{L^\infty}^2}{2h_0}\Big)\Big)|\sqrt{h}(\partial_x b)v|_{L^2}.$$

Thus, to conclude, simply choose ν small enough, from which we deduce the desired estimate

$$a(v,v) \geq c|v|_{X_\mu^0}^2. \tag{2.25}$$

Lastly, the application $\varphi \mapsto L(\varphi)$ is continuous on $\varphi \in H^{\frac{1}{2}}(\mathbb{R})$ by Cauchy-Schwarz. Consequently, we have a unique variational solution $v \in H^{\frac{1}{2}}(\mathbb{R})$ satisfying (2.24) for any $\varphi \in H^{\frac{1}{2}}(\mathbb{R})$. Let us show that this solution is in $H^1(\mathbb{R})$, so that it also satisfies (2.23).

Let $0 < \delta \leq 1$ and take $\chi_\delta(D)$ as in Definition 2.6 and define a sequence of smooth functions given by $v_\delta := \chi_\delta v \in \cap_{s>0}H^s(\mathbb{R})$. Then using $\Lambda^1(\chi_\delta)^2 v \in H^{\frac{1}{2}}(\mathbb{R})$ as a test function, we get

$$\begin{aligned}a(\Lambda^{\frac{1}{2}}v_\delta, \Lambda^{\frac{1}{2}}v_\delta) = a\Big(v, \Lambda^1(\chi_\delta)^2 v\Big) &- \big([\Lambda^{\frac{1}{2}}\chi_\delta, h]v, \Lambda^{\frac{1}{2}}v_\delta\big)_{L^2} - \frac{\mu}{3}\big([\Lambda^{\frac{1}{2}}\chi_\delta, h^3]\mathrm{F}^{\frac{1}{2}}\partial_x v, \Lambda^{\frac{1}{2}}\mathrm{F}^{\frac{1}{2}}\partial_x v_\delta\big)_{L^2} \\ &+ \frac{\mu}{2}\big([\Lambda^{\frac{1}{2}}\chi_\delta, h^2(\beta\partial_x b)]\mathrm{F}^{\frac{1}{2}}\partial_x v, \Lambda^{\frac{1}{2}}v_\delta\big)_{L^2} + \frac{\mu}{2}\big([\Lambda^{\frac{1}{2}}\chi_\delta, h^2(\beta\partial_x b)]v, \Lambda^{\frac{1}{2}}\mathrm{F}^{\frac{1}{2}}\partial_x v_\delta\big)_{L^2} \\ &- \mu\big([\Lambda^{\frac{1}{2}}\chi_\delta, h(\beta\partial_x b)^2]v, \Lambda^{\frac{1}{2}}v_\delta\big)_{L^2}.\end{aligned}$$

Then using (2.24) and (2.25), we get

$$\begin{aligned}c|v_\delta|_{X_\mu^{\frac{1}{2}}}^2 &\leq a(\Lambda^{\frac{1}{2}}v_\delta, \Lambda^{\frac{1}{2}}v_\delta) \\ &= |\big(f, \Lambda^1(\chi_\delta)^2 v\big)_{L^2} - \big([\Lambda^{\frac{1}{2}}\chi_\delta, h]v, \Lambda^{\frac{1}{2}}v_\delta\big)_{L^2} - \frac{\mu}{3}\big([\Lambda^{\frac{1}{2}}\chi_\delta, h^3]\mathrm{F}^{\frac{1}{2}}\partial_x v, \Lambda^{\frac{1}{2}}\mathrm{F}^{\frac{1}{2}}\partial_x v_\delta\big)_{L^2} \\ &\quad + \frac{\mu}{2}\big([\Lambda^{\frac{1}{2}}\chi_\delta, h^2(\beta\partial_x b)]\mathrm{F}^{\frac{1}{2}}\partial_x v, \Lambda^{\frac{1}{2}}v_\delta\big)_{L^2} + \frac{\mu}{2}\big([\Lambda^{\frac{1}{2}}\chi_\delta, h^2(\beta\partial_x b)]v, \Lambda^{\frac{1}{2}}\mathrm{F}^{\frac{1}{2}}\partial_x v_\delta\big)_{L^2} \\ &\quad - \mu\big([\Lambda^{\frac{1}{2}}\chi_\delta, h(\beta\partial_x b)^2]v, \Lambda^{\frac{1}{2}}v_\delta\big)_{L^2}|.\end{aligned}$$

Now remark that $\Lambda^{\frac{1}{2}}\chi_\delta(D)$ is a Fourier multiplier of order $\frac{1}{2}$ in the sense of Definition 2.4, and that $\mathcal{N}^{\frac{1}{2}}(\Lambda^{\frac{1}{2}}\chi_\delta(D)) \lesssim 1$ uniformly in δ. Hence, from Cauchy-Schwarz inequality, (2.10) and the commutator estimates of Proposition 2.5, we get

$$c|v_\delta|_{X_\mu^{\frac{1}{2}}}^2 \leq c(|f|_{L^2} + |v|_{L^2})|v_\delta|_{H^1}.$$

We, therefore, deduce the estimate

$$\sqrt{\mu}c|v_\delta|_{H^1} \lesssim |f|_{L^2} + |v|_{L^2}. \tag{2.26}$$

The family $\{v_\delta\}_{0<\delta\leq 1}$ is uniformly bounded in $H^1(\mathbb{R})$. Hence, since $H^1(\mathbb{R})$ is a reflexive Banach space, there exists $V \in H^1(\mathbb{R})$ and a subsequence $\{v_{\delta_n}\}_{0<\delta_n\leq 1}$ with $\delta_n \to 0$ such that $v_{\delta_n} \rightharpoonup V$. By uniqueness of the limit in $L^2(\mathbb{R})$, we deduce that $v = V \in H^1(\mathbb{R})$.
To conclude, we may now use (2.24) and integration by parts to find that

$$\big(\mathscr{T}[h,\beta b]v,\varphi\big)_{L^2} = \big(f,\varphi\big)_{L^2},$$

for any $\varphi \in C_c^\infty(\mathbb{R})$. Hence, we conclude that the variational solution also provides a unique solution of (2.23).

<u>Step 3.</u> The estimate (2.21) holds. To prove the claim, we first consider v, the solution of (2.23). From the coercivity estimate (2.25) and Cauchy-Schwarz inequality, we have

$$|v|_{X^0_\mu}^2 \lesssim a(v,v) = L(v) \leq |f|_{L^2}|v|_{L^2} \leq |f|_{L^2}|v|_{X^0_\mu},$$

so that

$$|v|_{X^0_\mu} \lesssim c|f|_{L^2}. \tag{2.27}$$

Next, we apply Λ^s to (2.23) and observe that $\Lambda^s v$ is a distributional solution of the equation

$$\mathscr{T}[h,\beta b]v_s = \Lambda^s f - [\Lambda^s,h]v + \frac{\mu}{3}\partial_x \mathrm{F}^{\frac12}([\Lambda^s,h^3]\mathrm{F}^{\frac12}\partial_x v) - \frac{\mu}{2}\partial_x\mathrm{F}^{\frac12}([\Lambda^s,h^2(\beta\partial_x b)]v) \tag{2.28}$$

$$+ \frac{\mu}{2}[\Lambda^s,h^2(\beta\partial_x b)]\mathrm{F}^{\frac12}\partial_x v - [\Lambda^s,h(\beta\partial_x b)^2]v \tag{2.29}$$

Moreover, from the coercivity estimate (2.25), the variational solution of (2.28), v_s, satisfies

$$\begin{aligned} c|v_s|_{X^0_\mu}^2 \leq a(v_s,v_s) &= \Big(\Lambda^s f - [\Lambda^s,h]v + \frac{\mu}{3}\partial_x\mathrm{F}^{\frac12}([\Lambda^s,h^3]\mathrm{F}^{\frac12}\partial_x v) - \frac{\mu}{2}\partial_x\mathrm{F}^{\frac12}([\Lambda^s,h^2(\beta\partial_x b)]v) \\ &\quad + \frac{\mu}{2}[\Lambda^s,h^2(\beta\partial_x b)]\mathrm{F}^{\frac12}\partial_x v - [\Lambda^s,h(\beta\partial_x b)^2]v, v_s\Big)_{L^2} \\ &= \big(\Lambda^s f, v_s\big)_{L^2} - \big([\Lambda^s,h]v,v_s\big)_{L^2} - \frac{\mu}{3}\big([\Lambda^s,h^3]\mathrm{F}^{\frac12}\partial_x v, \mathrm{F}^{\frac12}\partial_x v_s\big)_{L^2} \\ &\quad + \frac{\mu}{2}\big([\Lambda^s,h^2(\beta\partial_x b)]v,\mathrm{F}^{\frac12}\partial_x v_s\big)_{L^2} + \frac{\mu}{2}\big([\Lambda^s,h^2(\beta\partial_x b)]\mathrm{F}^{\frac12}\partial_x v, v_s\big)_{L^2} \\ &\quad - \big([\Lambda^s,h(\beta\partial_x b)^2]v,v_s\big)_{L^2}. \end{aligned}$$

Then using Cauchy-Schwarz inequality and the commutator estimates of Proposition 2.5, we get

$$|v_s|_{X^0_\mu}^2 \lesssim c(|f|_{H^s} + |v|_{X^{s-1}_\mu})|v_s|_{X^s_\mu}.$$

To conclude, we first consider $s\in\mathbb{N}$ and simply argue by induction using (2.27) as a base case noting that the distributional and variational solutions must coincide, i.e. $v_s = \Lambda^s v$. Then use the interpolation inequality (2.9) to obtain (2.21) for any s real number ≥ 0.

<u>Step 4.</u> The estimate (2.22) holds. Arguing as above, we apply $\mathrm{F}^{\frac12}\Lambda^s$ to (2.23) and get

$$\begin{aligned} |\mathrm{F}^{\frac12}v|_{X^s_\mu} &\lesssim a(\mathrm{F}^{\frac12}\Lambda^s v, \mathrm{F}^{\frac12}\Lambda^s v) \\ &= \big(\mathrm{F}^1\Lambda^s f,\Lambda^s v\big)_{L^2} - \big([\Lambda^s\mathrm{F}^{\frac12},h]v,\mathrm{F}^{\frac12}\Lambda^s v\big)_{L^2} - \frac{\mu}{3}\big([\Lambda^s\mathrm{F}^{\frac12},h^3]\mathrm{F}^{\frac12}\partial_x v, \mathrm{F}^1\partial_x\Lambda^s v\big)_{L^2} \\ &\quad + \frac{\mu}{2}\big([\Lambda^s\mathrm{F}^{\frac12},h^2(\beta\partial_x b)]v,\mathrm{F}^1\partial_x\Lambda^s v\big)_{L^2} + \frac{\mu}{2}\big([\Lambda^s\mathrm{F}^{\frac12},h^2(\beta\partial_x b)]F^{\frac12}\partial_x v,\mathrm{F}^{\frac12}\Lambda^s v\big)_{L^2} \\ &\quad - \big([\Lambda^s\mathrm{F}^{\frac12},h(\beta\partial_x b)^2]v,\mathrm{F}^{\frac12}\Lambda^s v\big)_{L^2}. \end{aligned}$$

Now using Cauchy-Schwarz inequality, the commutator estimates (2.14) and (2.12), we get

$$|\mathrm{F}^{\frac{1}{2}}v|^2_{X^s_\mu} \lesssim (\frac{1}{\sqrt{\mu}}|f|_{H^{s-1}} + |v|_{H^{s-1}})|v|_{H^s}.$$

Moreover, for all $s \in \mathbb{R}$ there holds,

$$\begin{aligned}
|v|_{H^s} &\lesssim |\mathbb{1}_{\{\sqrt{\mu}|D|\leq 1\}}v|_{H^s} + |\mathbb{1}_{\{\sqrt{\mu}|D|>1\}}v|_{H^s} \\
&\lesssim |\mathbb{1}_{\{\sqrt{\mu}|D|\leq 1\}}\mathrm{F}^{\frac{1}{2}}v|_{H^s} + \sqrt{\mu}|\mathbb{1}_{\{\sqrt{\mu}|D|>1\}}\mathrm{F}^1\partial_x v|_{H^s} \\
&\lesssim |\mathrm{F}^{\frac{1}{2}}v|_{H^s} + \sqrt{\mu}|\mathrm{F}^1\partial_x v|_{H^s} \\
&\lesssim |\mathrm{F}^{\frac{1}{2}}v|_{X^s_\mu}.
\end{aligned}$$

Thus, by gathering these estimates we get

$$\sqrt{\mu}|\mathrm{F}^{\frac{1}{2}}v|_{X^s_\mu} \lesssim c\big(|f|_{H^{s-1}} + \sqrt{\mu}|\mathrm{F}^{\frac{1}{2}}v|_{X^{s-1}_\mu}\big),$$

and allows us to argue by induction for $s \in \mathbb{N}\backslash\{0\}$, where the base case reads

$$|\mathrm{F}^{\frac{1}{2}}v|_{X^0_\mu} \lesssim |v|_{X^0_\mu} \lesssim |f|_{L^2}$$

Then use (2.9) to conclude the proof. In the end, we have the estimate

$$\sqrt{\mu}|\mathrm{F}^{\frac{1}{2}}v|_{H^s} \leq \sqrt{\mu}|\mathrm{F}^{\frac{1}{2}}v|_{X^s_\mu} \leq c|f|_{H^{s-1}}.$$

□

3. Consistency between (1.1) and (1.10)

To derive system (1.1), we start from the full dispersion Green-Naghdi model derived in [14] for which we know the order of precision with respect to the water waves equations (1.10).

Proposition 3.1 (Theorem 10.5 in [14]). *There exists $n \in \mathbb{N}$ and $T > 0$ such that for all $s \geq 0$ and $(\mu, \varepsilon, \beta) \in \mathcal{A}_{\mathrm{SW}}$, with $b \in H^{s+n}(\mathbb{R})$ and for every solution $(\zeta, \psi) \in C([0, \frac{T}{\varepsilon}]; H^{s+n}(\mathbb{R}) \times \dot{H}^{s+n}(\mathbb{R}))$ to the water waves equations* (1.10) *one has*

$$\begin{cases} \partial_t\zeta + \partial_x(h\overline{V}) = 0 \\ \partial_t(\overline{V} + \mu\mathcal{T}[h, \beta\partial_x b]\overline{V}) + \partial_x\zeta + \varepsilon\overline{V}\partial_x\overline{V} + \mu\varepsilon\partial_x\mathcal{R}[h, \beta\partial_x b, \overline{V}] = \mu^2(\varepsilon + \beta)R, \end{cases} \tag{3.1}$$

where $\overline{V}$ is defined through (1.12) *and* (1.11), *and*

$$\begin{aligned}
\mathcal{T}[h, \beta b]\overline{V} = -\frac{1}{3h}\partial_x\mathrm{F}^{\frac{1}{2}}(h^3\mathrm{F}^{\frac{1}{2}}\partial_x\overline{V}) + \frac{1}{2h}\Big(\partial_x\mathrm{F}^{\frac{1}{2}}(h^2(\beta\partial_x b)\overline{V}) - h^2(\beta\partial_x b)\mathrm{F}^{\frac{1}{2}}\partial_x\overline{V}\Big) \\
+ (\beta\partial_x b)^2\overline{V},
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}[h, \beta b, \overline{V}] = -\frac{\overline{V}}{3h}\partial_x\mathrm{F}^{\frac{1}{2}}\big(h^3\mathrm{F}^{\frac{1}{2}}\partial_x\overline{V}\big) - \frac{1}{2}h^2\big(\mathrm{F}^{\frac{1}{2}}\partial_x\overline{V}\big)^2 \\
+ \frac{1}{2}\Big(\frac{\overline{V}}{h}\partial_x\mathrm{F}^{\frac{1}{2}}\big(h^2(\beta\partial_x b)\overline{V}\big) + h(\beta\partial_x b)\overline{V}\mathrm{F}^{\frac{1}{2}}\partial_x\overline{V} + (\beta\partial_x b)^2\overline{V}^2\Big),
\end{aligned}$$

and where $|R|_{H^s} \leq C(\frac{1}{h_0}, \mu_{\max}, |\zeta|_{H^{s+n})}, |\partial_x\psi|_{H^{s+n}}, |b|_{H^{s+n}})$.

Furthermore, we say that the water waves equations are consistent with the system (3.1) *at the order of precision $\mathcal{O}(\mu^2(\varepsilon + \beta))$ in the shallow water regime.*

Proposition 3.2. *The water waves equations are consistent with the system*

$$\begin{cases} \partial_t\zeta + \partial_x(h\overline{V}) = 0 \\ (h + \mu h\mathcal{T}[h, \beta\partial_x b])\big(\partial_t\overline{V} + \varepsilon\overline{V}\partial_x\overline{V}\big) + h\partial_x\zeta + \mu\varepsilon h(\mathcal{Q}[h, \overline{V}] + \mathcal{Q}_b[h, b, \overline{V}]) = 0, \end{cases}$$

at the order of precision $\mathcal{O}(\mu^2(\varepsilon + \beta))$, where

$$\mathcal{Q}[h, \overline{V}] = \frac{2}{3h}\partial_x \mathrm{F}^{\frac{1}{2}}\big(h^3(\mathrm{F}^{\frac{1}{2}}\partial_x\overline{V})^2\big) \tag{3.2}$$

$$\mathcal{Q}_b[h, \beta b, \overline{V}] = h(\mathrm{F}^{\frac{1}{2}}\partial_x\overline{V})^2(\beta\partial_x b) + \frac{1}{2h}\partial_x\mathrm{F}^{\frac{1}{2}}(h^2\overline{V}^2\beta\partial_x^2 b) + \overline{V}^2(\beta\partial_x^2 b)(\beta\partial_x b). \tag{3.3}$$

Proof. Let us first remark that we only have to work on the second equation of system (3.1) and that the first equation can also be written

$$\partial_t h = -\varepsilon\partial_x(h\overline{V}). \tag{3.4}$$

Then multiplying the second equation of (3.1) by h we can write

$$h\partial_t\big(\overline{V} + \mu\mathcal{T}[h, \beta b]\overline{V}\big) = (h + \mu h\mathcal{T}[h, \beta b])\partial_t\overline{V} + h[\partial_t, \mu\mathcal{T}[h, \beta b]]\overline{V}.$$

Now, using (3.4) we observe that the following terms are of order $\mu\varepsilon$:

$$h[\partial_t, \mu\mathcal{T}[h, \beta b]]\overline{V} + \mu\varepsilon h\mathcal{R}[h, v],$$

and so we can use Proposition 2.10 to trade the multiplier $\mathrm{F}^{\frac{1}{2}}$ with identity and terms of order $\mu^2\varepsilon$. Thus, following the derivation presented in [28] we obtain that

$$(h + \mu h\mathcal{T}[h, \beta\partial_x b])\big(\partial_t\overline{V} + \varepsilon\overline{V}\partial_x\overline{V}\big) + h\partial_x\zeta + \mu\varepsilon h(\tilde{\mathcal{Q}}[h, \overline{V}] + \tilde{\mathcal{Q}}_b[h, b, \overline{V}]) = \mathcal{O}(\mu^2\varepsilon)$$

where

$$\tilde{\mathcal{Q}}[h, \overline{V}] = \frac{2}{3h}\partial_x\big(h^3(\partial_x\overline{V})^2\big)$$

$$\tilde{\mathcal{Q}}_b[h, \beta b, \overline{V}] = h(\partial_x\overline{V})^2(\beta\partial_x b) + \frac{1}{2h}\partial_x(h^2\overline{V}^2\beta\partial_x^2 b) + \overline{V}^2(\beta\partial_x^2 b)(\beta\partial_x b).$$

To conclude, we simply apply Proposition 2.10 once more to see that

$$\tilde{\mathcal{Q}}[h, \overline{V}] = \mathcal{Q}[h, \overline{V}] + \mathcal{O}(\mu)$$

and

$$\tilde{\mathcal{Q}}_b[h, \beta b, \overline{V}] = \mathcal{Q}_b[h, \beta b, \overline{V}] + \mathcal{O}(\mu).$$

□

Remark 3.3. *If we consider the two-dimensional case where we let $X = (x_1, x_2)$ and $\overline{V}, R \in \mathbb{R}^2$, then system (3.1) reads*

$$\begin{cases} \partial_t\zeta + \nabla_X\cdot(h\overline{V}) = 0 \\ \partial_t(\overline{V} + \mu\mathcal{T}[h, \beta b]\overline{V}) + \nabla_X\zeta + \frac{\varepsilon}{2}\nabla_X|\overline{V}|^2 + \mu\varepsilon\nabla_X\mathcal{R}[h, \beta\partial_x b, \overline{V}] = \mu^2(\varepsilon + \beta)R. \end{cases} \tag{3.5}$$

In this case, one can exploit the observation that the quantity

$$U = \overline{V} + \mu\mathcal{T}[h, \beta b]\overline{V} \tag{3.6}$$

approximates the gradient of the velocity potential at the free surface. Consequently, for regular solutions, one can impose the condition $\mathrm{curl}\,U|_{t=0} = 0$ *and using the second equation in* (3.5)*, we can deduce that* $\mathrm{curl}\,U = 0$ *whenever the solution is defined. However, this observation does not carry over to* (1.1) *since the two systems are not equivalent. On the other hand, if* $\mathrm{F} = \mathrm{Id}$*, then the two systems are equivalent, and one may exploit this insight to deal with the two-dimensional case.*

Remark 3.4. *The estimates in Section* 2 *can be extended to two dimensions where we note that* $F^{\frac{1}{2}}(\xi)$ *is a radial function. Also, in light of the previous remark, it could be possible to work on system* (3.5) *directly where we estimate the variables* ζ, U *and with* $\overline{V} = \overline{V}[h, \beta b, U]$ *uniquely defined by* (3.6) *(see* [15] *for similar observations). However, doing this change of unknowns would change the mathematical structure of the equations. So that it is not obvious that we can close the energy method in that case.*

4. A priori estimates

In this section, we establish *a priori* bounds on the solutions of (1.1). To this end, we let $\mathbf{U} = (\zeta, v)$ and for simplicity we introduce the notation

$$\mathscr{T} = \mathscr{T}[h, \beta b], \quad \mathcal{Q} = \mathcal{Q}[h, v], \quad \mathcal{Q}_b = \mathcal{Q}_b[h, \beta b, v],$$

allowing us to write (1.1) on the more compact form:

$$S(\mathbf{U})(\partial_t \mathbf{U} + M_1(\mathbf{U})\partial_x \mathbf{U}) + M_2(\mathbf{U})\partial_x \mathbf{U} + Q(\mathbf{U}) + Q_b(\mathbf{U}) = \mathbf{0}, \tag{4.1}$$

with

$$S(\mathbf{U}) := \begin{pmatrix} 1 & 0 \\ 0 & \mathscr{T}(\cdot) \end{pmatrix}, \quad M_1(\mathbf{U}) := \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon v \end{pmatrix}, \quad M_2(\mathbf{U}) := \begin{pmatrix} \varepsilon v & h \\ h & 0 \end{pmatrix},$$

and where the quadratic terms are

$$Q(\mathbf{U}) = \begin{pmatrix} 0 \\ \mu\varepsilon h \mathcal{Q} \end{pmatrix}, \quad Q_b(\mathbf{U}) = \begin{pmatrix} -(\beta \partial_x b) v \\ \mu \varepsilon h \mathcal{Q}_b \end{pmatrix}, \tag{4.2}$$

with $\mathcal{Q}$ as defined by (1.3) and $\mathcal{Q}_b$ defined by (1.4). We may now give the energy and the energy estimate of (4.1). In particular, we make the definition:

$$E_s(\mathbf{U}) = \big(\Lambda^s \mathbf{U}, S(\mathbf{U})\Lambda^s \mathbf{U}\big)_{L^2}, \tag{4.3}$$

allowing us to state the following result.

Proposition 4.1. *Let* $s > \frac{3}{2}$, $(\mu, \varepsilon, \beta) \in \mathcal{A}_{SW}$, *and* $(\zeta, v) \in C([0,T]; Y^s_\mu(\mathbb{R}))$ *be a solution to* (4.1) *on a time interval* $[0, T]$ *for some* $T > 0$. *Moreover, assume* $b \in H^{s+2}(\mathbb{R})$ *and there exist* $h_0 \in (0,1)$ *such that*

$$h_0 - 1 + \beta b \leq \varepsilon \zeta(x,t), \quad \forall (x,t) \in \mathbb{R} \times [0,T], \tag{4.4}$$

and suppose that

$$N(s) := \varepsilon \sup_{t \in [0,T]} |(\zeta(t,\cdot), v(t,\cdot))|_{Y^s_\mu} + \beta |b|_{H^{s+2}} \leq N^\star, \tag{4.5}$$

for some $N^\star \in \mathbb{R}^+$. *Then, for the energy given by* (4.3)*, there holds,*

$$\frac{d}{dt} E_s(\mathbf{U}) \lesssim N(s) E_s(\mathbf{U}), \tag{4.6}$$

and

$$|(\zeta, v)|^2_{Y^s_\mu} \lesssim E_s(\mathbf{U}) \lesssim |(\zeta, v)|^2_{Y^s_\mu}, \tag{4.7}$$

for all $0 < t < T$.

Proof. We first prove (4.7). We note that the energy is similar to the bilinear form defined in (2.24). Thus, the estimate is a direct consequence of Step 2. in the proof of Proposition 2.11 and (4.4).

Next, we prove (4.6). Using (4.1), the self-adjointness of $S(\mathbf{U})$ and the invertibility provided by Proposition 2.11 under assumption (4.4), we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_s(\mathbf{U}) &= \frac{1}{2}\big(\Lambda^s\mathbf{U},(\partial_t S(\mathbf{U}))\Lambda^s\mathbf{U}\big)_{L^2} + \big(\Lambda^s\partial_t\mathbf{U}, S(\mathbf{U})\Lambda^s\mathbf{U}\big)_{L^2}\\ &= \frac{1}{2}\big(\Lambda^s\mathbf{U},(\partial_t S(\mathbf{U}))\Lambda^s\mathbf{U}\big)_{L^2} - \big(\Lambda^s M_1(\mathbf{U})\partial_x\mathbf{U}, S(\mathbf{U})\Lambda^s\mathbf{U}\big)_{L^2}\\ &\quad - \big(M_2(\mathbf{U})\partial_x\Lambda^s\mathbf{U},\Lambda^s\mathbf{U}\big)_{L^2} - \big([\Lambda^s,(S^{-1}M_2)(\mathbf{U})]\partial_x\mathbf{U}, S(\mathbf{U})\Lambda^s\mathbf{U}\big)_{L^2}\\ &\quad - \big(\Lambda^s(S^{-1}Q)(\mathbf{U}), S(\mathbf{U})\Lambda^s\mathbf{U}\big)_{L^2} - \big(\Lambda^s(S^{-1}Q_b)(\mathbf{U}), S(\mathbf{U})\Lambda^s\mathbf{U}\big)_{L^2}\\ &=: I + II + III + IV + V + VI.\end{aligned}$$

<u>Control of I</u>. We first use the equation for $\partial_t\zeta$ in (4.1), together with the Sobolev embedding $H^{s-1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ with $s-1>\frac{1}{2}$ and the algebra property to deduce the estimate:

$$\begin{aligned}|\partial_t h|_{L^\infty} &\leq \varepsilon|\partial_x(hv)|_{L^\infty}\\ &\lesssim \varepsilon(1+\varepsilon|\zeta|_{H^s}+\beta|b|_{H^s})|v|_{H^s}.\end{aligned} \tag{4.8}$$

Therefore, by definition (2.19) of $\mathscr{T}[h,\beta b]$, using integration by parts, Hölder's inequality, (4.5), Sobolev embedding, and (4.7) we obtain the bound

$$\begin{aligned}|I| &\leq \frac{1}{2}|\big(\Lambda^s v,(\partial_t h)\Lambda^s v\big)_{L^2}| + \frac{\mu}{6}|\big(\mathrm{F}^{\frac{1}{2}}\partial_x\Lambda^s v,(\partial_t(h^3))\mathrm{F}^{\frac{1}{2}}\partial_x\Lambda^s v\big)_{L^2}|\\ &\quad + \frac{\mu}{2}|\big(\mathrm{F}^{\frac{1}{2}}\partial_x\Lambda^s v,(\partial_t(h^2))\beta(\partial_x b)\Lambda^s v\big)_{L^2}| + \mu|\big(\Lambda^s v,(\partial_t h)(\beta\partial_x b)^2\Lambda^s v\big)_{L^2}|\\ &\lesssim N(s)E_s(\mathbf{U}),\end{aligned}$$

for $s>\frac{3}{2}$.

<u>Control of II.</u> By definition of $\mathscr{T}[h,\beta b]$ we must deal with the terms:

$$\begin{aligned}II &= -\varepsilon\big(\Lambda^s(v\partial_x v), h\Lambda^s v\big)_{L^2} + \frac{\mu\varepsilon}{3}\big(\Lambda^s(v\partial_x v),\partial_x\mathrm{F}^{\frac{1}{2}}(h^3\partial_x\mathrm{F}^{\frac{1}{2}}\Lambda^s v)\big)_{L^2}\\ &\quad - \frac{\mu\varepsilon\beta}{2}\big(\Lambda^s(v\partial_x v),\partial_x\mathrm{F}^{\frac{1}{2}}(h^2(\partial_x b)\Lambda^s v)\big)_{L^2} + \frac{\mu\varepsilon\beta}{2}\big(\Lambda^s(v\partial_x v), h^2(\partial_x b)\partial_x\mathrm{F}^{\frac{1}{2}}\Lambda^s v\big)_{L^2}\\ &\quad - \frac{\mu\varepsilon\beta}{2}\big(\Lambda^s(v\partial_x v), h(\beta\partial_x b)^2\Lambda^s v\big)_{L^2}\\ &=: II_1 + II_2 + II_3 + II_4 + II_5.\end{aligned}$$

Using integration by parts, we may decompose II_1 into two pieces

$$\begin{aligned}II_1 &= -\varepsilon\big(hv\Lambda^s\partial_x v,\Lambda^s v\big)_{L^2} - \varepsilon\big([\Lambda^s,v]\partial_x v, h\Lambda^s v\big)_{L^2}\\ &= \frac{\varepsilon}{2}\big((\partial_x(hv))\Lambda^s v,\Lambda^s v\big)_{L^2} - \varepsilon\big([\Lambda^s,v]\partial_x v, h\Lambda^s v\big)_{L^2}.\end{aligned}$$

Then by Hölder's inequality, Sobolev embedding, and the commutator estimate (2.5), we obtain the estimate:

$$|II_1| \lesssim \varepsilon(1+|h-1|_{H^s})|v|_{H^s}^3.$$

We also note that II_5 can be estimated in the same way, and we obtain easily that

$$|II_5| \lesssim \varepsilon(1+|h-1|_{H^s})|b|_{H^{s+1}}^2|v|_{H^s}^3.$$

For II_2, we also use integration by parts to make the observation:

$$\begin{aligned}II_2 &= -\frac{\mu\varepsilon}{3}\big([\Lambda^s,\mathrm{F}^{\frac{1}{2}}\partial_x(v\cdot)]\partial_x v, h^3\partial_x\mathrm{F}^{\frac{1}{2}}\Lambda^s v\big)_{L^2} - \frac{\mu\varepsilon}{3}\big(\mathrm{F}^{\frac{1}{2}}\partial_x(v\Lambda^s\partial_x v), h^3\partial_x\mathrm{F}^{\frac{1}{2}}\Lambda^s v\big)_{L^2}\\ &=: II_2^1 + II_2^2.\end{aligned}$$

Then we treat II_2^1 with Hölder's inequality, Sobolev embedding, and (2.15) to get

$$|II_2^1| \lesssim (1+|h^3-1|_{H^s})|v|_{H^s}|v|_{X_\mu^s}^2.$$

On the other hand, we need to decompose II_2^2 further and carefully distribute the μ:

$$\begin{aligned} II_2^2 &= -\frac{\mu\varepsilon}{3}\big(v\mathrm{F}^{\frac12}\Lambda^s\partial_x^2 v, h^3\partial_x\mathrm{F}^{\frac12}\Lambda^s v\big)_{L^2} - \frac{\mu\varepsilon}{3}\big([\mathrm{F}^{\frac12}, v]\Lambda^s\partial_x^2 v, h^3\partial_x\mathrm{F}^{\frac12}\Lambda^s v\big)_{L^2} \\ &\quad - \frac{\mu\varepsilon}{3}\big(\mathrm{F}^{\frac12}\big((\partial_x v)\Lambda^s\partial_x v\big), h^3\partial_x\mathrm{F}^{\frac12}\Lambda^s v\big)_{L^2} \\ &=: II_2^{2,1} + II_2^{2,2} + II_2^{2,3}. \end{aligned}$$

For $II_2^{2,1}$, we simply integrate by parts and argue as we did for II_1 to obtain

$$\begin{aligned} |II_2^{2,1}| &\lesssim \mu\varepsilon|\partial_x(h^3 v)|_{H^{s-1}}|\mathrm{F}^{\frac12}\Lambda^s\partial_x v|_{H^s}^2 \\ &\lesssim \varepsilon(1+|h^3-1|_{H^s})|v|_{H^s}|v|_{X_\mu^s}^2. \end{aligned}$$

For $II_2^{2,2}$, we use Hölder's inequality, Sobolev embedding, and (2.16) to directly obtain that

$$\begin{aligned} |II_2^{2,2}| &\lesssim \mu\varepsilon(1+|h^3-1|_{H^s})|v|_{X_\mu^s}|\mathrm{F}^{\frac12}\partial_x^2 v|_{H^{s-1}}|\mathrm{F}^{\frac12}\partial_x v|_{H^s} \\ &\lesssim \varepsilon(1+|h^3-1|_{H^s})|v|_{X_\mu^s}^3. \end{aligned}$$

For $II_2^{2,3}$, we also need to be careful in the distribution of μ. In fact, we need to use Plancherel, then Cauchy-Schwarz and (2.11) to get

$$\begin{aligned} |II_2^{2,3}| &= \frac{\mu\varepsilon}{3}\big|\big(\mathrm{F}^{\frac12}\Lambda^s\partial_x v, \mathrm{F}^{-\frac12}\Big((\partial_x v)\mathrm{F}^{\frac12}\big(h^3\partial_x\mathrm{F}^{\frac12}\Lambda^s v\big)\Big)\big)_{L^2}\big| \\ &\lesssim \varepsilon|v|_{X_\mu^s}\Big(\sqrt{\mu}|(\partial_x v)\mathrm{F}^{\frac12}\big(h^3\partial_x\mathrm{F}^{\frac12}\Lambda^s v\big)|_{L^2} + \mu^{\frac34}|\mathrm{D}^{\frac12}\Big((\partial_x v)\mathrm{F}^{\frac12}\big(h^3\partial_x\mathrm{F}^{\frac12}\Lambda^s v\big)\Big)|_{L^2}\Big) \\ &=: \varepsilon|v|_{X_\mu^s}(A+B). \end{aligned}$$

Then estimate A by Hölder's inequality, the Sobolev embedding, and the boundedness of $\mathrm{F}^{\frac12}$ on $L^2(\mathbb{R})$ to get

$$A \lesssim |v|_{H^s}(1+|h^3-1|_{H^s})|v|_{X_\mu^s},$$

while for B, we also use (2.4) and (2.13) to get

$$\begin{aligned} B &\lesssim \mu^{\frac34}|v|_{H^s}|\mathrm{F}^{\frac12}\big(h^3\partial_x\mathrm{F}^{\frac12}\Lambda^s v\big)|_{H^{\frac12}} \\ &\lesssim \sqrt{\mu}|v|_{H^s}|h^3\partial_x\mathrm{F}^{\frac12}\Lambda^s v|_{L^2} \\ &\lesssim |v|_{H^s}(1+|h^3-1|_{H^s})|v|_{X_\mu^s}. \end{aligned}$$

Next, we use integration by parts to decompose II_3 into several pieces:

$$\begin{aligned} II_3 &= -\frac{\mu\varepsilon\beta}{2}\big([\Lambda^s, v]\partial_x v, \partial_x\mathrm{F}^{\frac12}(h^2(\partial_x b)\Lambda^s v)\big)_{L^2} + \frac{\mu\varepsilon\beta}{2}\big(\mathrm{F}^{\frac12}((\partial_x v)\Lambda^s\partial_x v), h^2(\partial_x b)\Lambda^s v\big)_{L^2} \\ &\quad + \frac{\mu\varepsilon\beta}{2}\big(\mathrm{F}^{\frac12}(v\Lambda^s\partial_x^2 v), h^2(\partial_x b)\Lambda^s v\big)_{L^2} \\ &=: II_3^1 + II_3^2 + II_3^3. \end{aligned}$$

Then for II_3^1, we apply (2.5), (2.13), and (2.4) to obtain that

$$\begin{aligned} |II_3^1| &\lesssim \varepsilon|v|_{H^s}^2\mu^{\frac34}|\mathrm{D}^{\frac12}(h^2(\partial_x b)\Lambda^s v)|_{L^2} \\ &\lesssim \varepsilon(1+|h^2-1|_{H^s})|b|_{H^{s+1}}|v|_{X_\mu^s}^3. \end{aligned}$$

For II_3^2, we argue as for $II_2^{2,3}$ to get that

$$\begin{aligned}|II_3^2| &\lesssim \varepsilon\mu\big|\big(F^{\frac12}\Lambda^s\partial_x v, F^{-\frac12}((\partial_x v)\mathrm{F}^{\frac12}(h^2(\partial_x b)\Lambda^s v))\big)_{L^2}\big| \\ &\lesssim \varepsilon|v|_{X^s_\mu}\sqrt{\mu}(|(\partial_x v)\mathrm{F}^{\frac12}(h^2(\partial_x b)\Lambda^s v)|_{L^2} + \mu^{\frac14}|\mathrm{D}^{\frac12}((\partial_x v)\mathrm{F}^{\frac12}(h^2(\partial_x b)\Lambda^s v))|_{L^2}) \\ &\lesssim \varepsilon|v|_{X^s_\mu}(1+|h^2-1|_{H^s})|b|_{H^{s+1}}|v|^2_{H^s}.\end{aligned}$$

For II_3^3, we first make the decomposition

$$\begin{aligned}II_3^2 &= \frac{\mu\varepsilon\beta}{2}\big([\mathrm{F}^{\frac12}, v]\Lambda^s\partial_x^2 v, h^2(\partial_x b)\Lambda^s v\big)_{L^2} + \frac{\mu\varepsilon\beta}{2}\big(v\mathrm{F}^{\frac12}\Lambda^s\partial_x^2 v, h^2(\partial_x b)\Lambda^s v\big)_{L^2} \\ &=: II_3^{2,1} + II_3^{2,2}.\end{aligned}$$

For $II_3^{2,1}$, we employ Hölder's inequality, Sobolev embedding, and (2.16) to get that

$$|II_3^{2,1}| \lesssim \mu\varepsilon|v|_{X^s_\mu}|\mathrm{F}^{\frac12}\partial_x v|_{H^s}|b|_{H^{s+1}}(1+|h^2-1|_{H^s})|v|_{H^s}.$$

Lastly, for $II_3^{2,2}$, we use integration by parts to make the observation that

$$\begin{aligned}II_3^{2,2} &= -\frac{\mu\varepsilon\beta}{2}\big(\mathrm{F}^{\frac12}\Lambda^s\partial_x v, (vh^2(\partial_x b))v\Lambda^s\partial_x v\big)_{L^2} - \frac{\mu\varepsilon\beta}{2}\big(\mathrm{F}^{\frac12}\Lambda^s\partial_x v, (\partial_x(vh^2(\partial_x b)))\Lambda^s v\big)_{L^2} \\ &= \frac{\mu\varepsilon\beta}{2}\big(\mathrm{F}^{\frac12}\Lambda^s\partial_x v, vh^2(\partial_x b)[\Lambda^s, v]\partial_x v\big)_{L^2} - II_4 - \frac{\mu\varepsilon\beta}{2}\big(\mathrm{F}^{\frac12}\Lambda^s\partial_x v, (\partial_x(vh^2(\partial_x b)))\Lambda^s v\big)_{L^2}.\end{aligned}$$

Then we may use Hölder's inequality, (2.5), (4.5), (4.7), and Sobolev embedding to get that

$$|II_3^{2,2} + II_4| \lesssim N(s)E_s(\mathbf{U}).$$

Gathering all these estimates, using (4.5) and (4.7), allows us to conclude that

$$|II| \lesssim N(s)E_s(\mathbf{U}).$$

<u>Control of III.</u> Then by definition, we must estimate the terms:

$$III = -\varepsilon\big(v\partial_x\Lambda^s\zeta, \Lambda^s\zeta\big)_{L^2} - \big(h\partial_x\Lambda^s v, \Lambda^s\zeta\big)_{L^2} - \big(h\partial_x\Lambda^s\zeta, \Lambda^s v\big)_{L^2}$$

For the estimate on these terms, we integrate by parts and apply Hölder's inequality, Sobolev embedding, and (4.7) to deduce

$$\begin{aligned}|III| &\le \frac{\varepsilon}{2}|\big((\partial_x v)\Lambda^s\zeta, \Lambda^s\zeta\big)_{L^2}| + |\big((\partial_x h)\Lambda^s v, \Lambda^s\zeta\big)_{L^2}| \\ &\lesssim N(s)E_s(\mathbf{U}).\end{aligned}$$

<u>Control of IV.</u> We decompose each term in IV and estimate them separately. In particular, we must estimate the following terms,

$$\begin{aligned}IV &= -\varepsilon\big([\Lambda^s, v]\partial_x\zeta, \Lambda^s\zeta\big)_{L^2} - \big([\Lambda^s, h]\partial_x v, \Lambda^s\zeta\big)_{L^2} - \big([\Lambda^s, \mathscr{T}^{-1}(h\bullet)]\partial_x\zeta, \mathscr{T}\Lambda^s v\big)_{L^2} \\ &=: IV_1 + IV_2 + IV_3.\end{aligned}$$

The first two terms are easily controlled by Cauchy-Schwarz and (2.5):

$$|IV_1| + |IV_2| \lesssim \varepsilon|v|_{L^\infty}|\zeta|^2_{H^s} + (\varepsilon|\zeta|_{L^\infty} + \beta|b|_{L^\infty})|\zeta|_{H^s}|v|_{H^s}.$$

Then use Sobolev embedding and (4.7) to conclude. However, need to decompose the remaining term further. To do so, we make the observation that

$$\begin{aligned}\mathscr{T}[\Lambda^s, \mathscr{T}^{-1}(h\cdot)]f = &-[\Lambda^s, h]\mathscr{T}^{-1}(hf) + \frac{\mu}{3}\partial_x \mathrm{F}^{\frac{1}{2}}[\Lambda^s, h^3]\partial_x \mathrm{F}^{\frac{1}{2}}\big(\mathscr{T}^{-1}(hf)\big) \\ &- \frac{\mu}{2}\partial_x \mathrm{F}^{\frac{1}{2}}[\Lambda^s, h^2\beta(\partial_x b)]\mathscr{T}^{-1}(hf) + \frac{\mu}{2}[\Lambda^s, h^2\beta(\partial_x b)]\partial_x \mathrm{F}^{\frac{1}{2}}\mathscr{T}^{-1}(hf) \\ &+ \mu[\Lambda^s, h(\beta\partial_x b)^2]\mathscr{T}^{-1}(hf) + [\Lambda^s, h]f.\end{aligned} \tag{4.9}$$

Then by this identity, the self-adjointness of $\mathscr{T}[h, \beta b]$, and integration by parts, we may decompose IV_3 into six pieces:

$$\begin{aligned}IV_3 = &\big([\Lambda^s, h]\mathscr{T}^{-1}(h\partial_x\zeta), \Lambda^s v\big)_{L^2} + \frac{\mu}{3}\big([\Lambda^s, h^3]\partial_x \mathrm{F}^{\frac{1}{2}}\big(\mathscr{T}^{-1}(h\partial_x\zeta)\big), \partial_x \mathrm{F}^{\frac{1}{2}}\Lambda^s v\big)_{L^2} \\ &- \frac{\mu}{2}\big([\Lambda^s, h^2\beta(\partial_x b)]\mathscr{T}^{-1}(h\partial_x\zeta), \partial_x \mathrm{F}^{\frac{1}{2}}\Lambda^s v\big)_{L^2} - \frac{\mu}{2}\big([\Lambda^s, h^2\beta(\partial_x b)]\partial_x \mathrm{F}^{\frac{1}{2}}\mathscr{T}^{-1}(h\partial_x\zeta), \Lambda^s v\big)_{L^2} \\ &- \mu\big([\Lambda^s, h(\beta\partial_x b)^2]\mathscr{T}^{-1}(h\partial_x\zeta), \Lambda^s v\big)_{L^2} - \big([\Lambda^s, h]\partial_x\zeta, \Lambda^s v\big)_{L^2} \\ =:\ & IV_3^1 + IV_3^2 + IV_3^3 + IV_3^4 + IV_3^5 + IV_3^6.\end{aligned}$$

For IV_3^1, use Cauchy-Schwarz inequality, (2.5), Sobolev embedding, (2.21), (4.5), and the algebra property of $H^{s-1}(\mathbb{R})$ for $s - 1 > \frac{1}{2}$ to get the bound

$$\begin{aligned}|IV_3^1| &\lesssim (\varepsilon|\partial_x\zeta|_{L^\infty} + \beta|\partial_x b|_{L^\infty})|h\partial_x\zeta|_{H^{s-1}}|v|_{H^s} \\ &\lesssim \varepsilon|\zeta|_{H^s}^2|v|_{H^s} + \beta|\partial_x b|_{L^\infty}|\zeta|_{H^s}|v|_{H^s}.\end{aligned}$$

Similarly, when estimating IV_3^2 we also use (2.10) and the inverse estimate (2.22) to deduce

$$\begin{aligned}|IV_3^2| &\lesssim \varepsilon\mu|\zeta|_{H^s}|\mathrm{F}^{\frac{1}{2}}\mathscr{T}^{-1}(h\partial_x\zeta)|_{H^s}|\partial_x \mathrm{F}^{\frac{1}{2}}\Lambda^s v|_{H^s} \\ &\lesssim \varepsilon|\zeta|_{H^s}^2|v|_{X_\mu^s} + \beta|\partial_x b|_{L^\infty}|\zeta|_{H^s}|v|_{X_\mu^s}.\end{aligned}$$

Next, we see that $IV_3^3 + IV_3^4 + IV_3^5$ offers no other difficulties. In fact, applying the same estimates as above, with (4.5), yields

$$|IV_3^3| + |IV_3^4| + |IV_3^5| \lesssim (1 + \varepsilon|\zeta|_{H^s})\beta|\partial_x b|_{L^\infty}|\zeta|_{H^s}|v|_{X_\mu^s}.$$

Lastly, IV_3^6 is controlled by Cauchy-Schwarz inequality, (2.5) and Sobolev emebedding:

$$|IV_3^6| \lesssim \varepsilon|\zeta|_{H^s}^2|v|_{H^s}.$$

<u>Control of V.</u> We need to make a careful decomposition of the following term

$$\mathscr{T}\big(\Lambda^s\mathscr{T}^{-1}(h\mathcal{Q})\big) = \frac{2}{3}\mathscr{T}\Big(\Lambda^s\mathscr{T}^{-1}\big(\partial_x \mathrm{F}^{\frac{1}{2}}(h^3(\mathrm{F}^{\frac{1}{2}}\partial_x v)^2)\big)\Big).$$

To do so, we use the identity

$$\begin{aligned}\mathscr{T}\Big(\Lambda^s\mathscr{T}^{-1}\big(\mathrm{F}^{\frac{1}{2}}\partial_x(fg)\big)\Big) = &-[\Lambda^s, \mathscr{T}]\mathscr{T}^{-1}(\mathrm{F}^{\frac{1}{2}}\partial_x(fg)) + [\Lambda^s, \mathrm{F}^{\frac{1}{2}}\partial_x(f\cdot)]g \\ &+ \mathrm{F}^{\frac{1}{2}}\partial_x(f\Lambda^s g),\end{aligned}$$

then use integration by parts to make the decomposition

$$\begin{aligned}V = &\frac{2\mu\varepsilon}{3}\Big[\big([\Lambda^s, \mathscr{T}]\mathscr{T}^{-1}(h\mathcal{Q}), \Lambda^s v\big)_{L^2} + \big([\Lambda^s, h^3]\big((\mathrm{F}^{\frac{1}{2}}\partial_x v)^2\big), \mathrm{F}^{\frac{1}{2}}\partial_x\Lambda^s v\big)_{L^2} \\ &+ \big(h^3\Lambda^s\big((\mathrm{F}^{\frac{1}{2}}\partial_x v)^2\big), \mathrm{F}^{\frac{1}{2}}\partial_x\Lambda^s v\big)_{L^2}\Big] \\ = &\ V_1 + V_2 + V_3.\end{aligned}$$

We treat V_1 first, where we must control the following terms:

$$\begin{aligned}V_1 = &\mu\varepsilon\big([\Lambda^s,h]\mathscr{T}^{-1}(h\mathcal{Q}),\Lambda^s v\big)_{L^2} + \frac{\mu^2\varepsilon}{3}\big([\Lambda^s,h^3]\partial_x \mathrm{F}^{\frac12}\big(\mathscr{T}^{-1}(h\mathcal{Q})\big),\partial_x\mathrm{F}^{\frac12}\Lambda^s v\big)_{L^2}\\ &-\frac{\mu^2\varepsilon}{2}\big([\Lambda^s,h^2\beta(\partial_x b)]\mathscr{T}^{-1}(h\mathcal{Q}),\partial_x\mathrm{F}^{\frac12}\Lambda^s v\big)_{L^2} - \frac{\mu^2\varepsilon}{2}\big([\Lambda^s,h^2\beta(\partial_x b)]\partial_x\mathrm{F}^{\frac12}\mathscr{T}^{-1}(h\mathcal{Q}),\Lambda^s v\big)_{L^2}\\ &-\mu^2\varepsilon\big([\Lambda^s,h(\beta\partial_x b)^2]\mathscr{T}^{-1}(h\partial_x\mathcal{Q}),\Lambda^s v\big)_{L^2}\\ =:& V_1^1+V_1^2+V_1^3+V_1^4+V_1^5.\end{aligned}$$

To estimate the first term, V_1, we simply argue as above. Indeed, by (2.5), the Sobolev embedding, and using that $X^{s-1}(\mathbb{R})\subset H^{s-1}(\mathbb{R})$ with (2.21) yields

$$\begin{aligned}|V_1^1| &\lesssim \mu\varepsilon|[\Lambda^s,h]\mathscr{T}^{-1}(h\mathcal{Q})|_{L^2}|v|_{H^s}\\ &\lesssim \mu\varepsilon|h\mathcal{Q}|_{H^{s-1}}|\zeta|_{H^s}|v|_{H^s}.\end{aligned}$$

Then to estimate $|h\mathcal{Q}|_{H^{s-1}}$, we first observe by the interpolation inequality (2.9) and Young's inequality that

$$\begin{aligned}\sqrt{\mu}|\mathrm{F}^{\frac12}\partial_x v|^2_{H^{s-\frac12}} &\lesssim |\mathrm{F}^{\frac12}\partial_x v|_{H^{s-1}}\sqrt{\mu}|\mathrm{F}^{\frac12}\partial_x v|_{H^s}\\ &\lesssim |v|^2_{H^s}+\mu|\mathrm{F}^{\frac12}\partial_x v|^2_{H^s}.\end{aligned}$$

Thus, we may estimate $|h\mathcal{Q}|_{H^{s-1}}$ by using (2.21), the algebra property of $H^{s-\frac12}(\mathbb{R})$ for $s-\frac12>1$ and combined with (2.14) and (2.13):

$$\begin{aligned}\mu|h\mathcal{Q}|_{H^{s-1}} &= \frac{\mu}{3}|\partial_x\mathrm{F}^{\frac12}\big(h^3((\mathrm{F}^{\frac12}\partial_x v)^2)\big)|_{H^{s-1}}\\ &\lesssim \mu|\partial_x\mathrm{F}^{\frac12}((\mathrm{F}^{\frac12}\partial_x v)^2)|_{H^{s-1}} + \mu|\partial_x\mathrm{F}^{\frac12}\big((h^3-1)\mathrm{F}^{\frac12}((\partial_x v)^2)\big)|_{H^{s-1}}\\ &\lesssim \sqrt{\mu}|\mathrm{F}^{\frac12}\partial_x v|^2_{H^{s-\frac12}} + \mu|[\Lambda^s\mathrm{F}^{\frac12},h^3]((\mathrm{F}^{\frac12}\partial_x v)^2)|_{L^2} + \mu|(h^3-1)\mathrm{F}^{\frac12}((\mathrm{F}^{\frac12}\partial_x v)^2)|_{H^s}\\ &\lesssim N(s)|v|^2_{H^s},\end{aligned}$$

and using (4.5), we deduce that

$$|V_1^1|\lesssim N(s)|v|^2_{H^s}.$$

Next, we consider V_1^2 and observe that we can have a similar bound. Indeed, using (4.5), (2.5), and (2.14) we observe that

$$\begin{aligned}|V_1^2| &\lesssim \mu^2\varepsilon|[\Lambda^s,h^3]\partial_x\mathrm{F}^{\frac12}\mathscr{T}^{-1}(h\mathcal{Q})|_{L^2}|\mathrm{F}^{\frac12}\partial_x v|_{H^s}\\ &\lesssim \mu^{\frac32}\varepsilon|\mathrm{F}^{\frac12}\mathscr{T}^{-1}(h\mathcal{Q})|_{H^s}|v|_{X^s_\mu}\\ &\lesssim \mu\varepsilon|h\mathcal{Q}|_{H^{s-1}}|v|_{X^s_\mu},\end{aligned}$$

and we use the previous estimates to obtain that

$$|V_1^2|\lesssim N(s)|v|^2_{X^s_\mu}.$$

Moreover, we note that it is straightforward to estimate $|V_1^3|+...+|V_1^5|$ arguing as we did for V_1^1 and V_1^2. Thus, gathering all these estimates and using (4.7) yields,

$$|V_1|\lesssim N(s)E_s(\mathbf{U}).$$

Next, we estimate V_2 using Hölder's inequality, (4.5) and (2.5) to obtain

$$\begin{aligned}|V_2| &\lesssim \mu\varepsilon(1+|h^3-1|_{H^s})|\mathrm{F}^{\frac12}v|^2_{H^s}|\partial_x\mathrm{F}^{\frac12}v|_{H^s}\\ &\lesssim N(s)|v|^2_{X^s_\mu}.\end{aligned}$$

Lastly, for V_3, we inject a commutator and use Hölder's inequality, Sobolev embedding, (4.5) and (2.5) to get

$$\begin{aligned}|V_3| &\lesssim \mu\varepsilon\big|\big(h^3[\Lambda^s,(\mathrm{F}^{\frac12}\partial_x v)](\mathrm{F}^{\frac12}\partial_x v),\mathrm{F}^{\frac12}\partial_x\Lambda^s v\big)_{L^2}\big| + \mu\varepsilon\big|\big(h^3(\mathrm{F}^{\frac12}\partial_x v)\Lambda^s\mathrm{F}^{\frac12}\partial_x v,\mathrm{F}^{\frac12}\partial_x\Lambda^s v\big)_{L^2}\big|\\ &\lesssim \mu\varepsilon|\mathrm{F}^{\frac12}\partial_x v|_{H^s}|\mathrm{F}^{\frac12}\partial_x v|_{H^{s-1}}|\mathrm{F}^{\frac12}\partial_x\Lambda^s v|_{L^2}\\ &\lesssim N(s)|v|^2_{X^s_\mu}.\end{aligned}$$

Control of VI. To complete the proof we need to estimate the remaining part:

$$\begin{aligned}VI &= -\varepsilon\mu\big([\Lambda^s,\mathscr{T}]\mathscr{T}^{-1}\mathcal{Q}_b,\Lambda^s v\big)_{L^2} - \varepsilon\mu\big(\Lambda^s\mathcal{Q}_b,\Lambda^s v\big)_{L^2}\\ &\quad - \big(\Lambda^s((\beta\partial_x b)v),\Lambda^s\zeta\big)_{L^2}\\ &=: VI_1 + VI_2 + VI_3.\end{aligned}$$

The estimate in VI is similar to the one of V, where we now have to deal with the following terms

$$\begin{aligned}VI_1 &= -\mu\varepsilon\big([\Lambda^s,h]\mathscr{T}^{-1}\mathcal{Q}_b,\Lambda^s v\big)_{L^2} + \frac{\mu^2\varepsilon}{3}\big(\mathrm{F}^{\frac12}\partial_x\big([\Lambda^s,h^3]\mathrm{F}^{\frac12}\partial_x(\mathscr{T}^{-1}\mathcal{Q}_b)\big),\Lambda^s v\big)_{L^2}\\ &\quad - \frac{\mu^2\varepsilon}{2}\big(\mathrm{F}^{\frac12}\partial_x\big([\Lambda^s,h^2\beta(\partial_x b)]\mathscr{T}^{-1}\mathcal{Q}_b\big),\Lambda^s v\big)_{L^2}\\ &\quad + \frac{\mu^2\varepsilon}{2}\big([\Lambda^s,h^2\beta(\partial_x b)]\mathrm{F}^{\frac12}\partial_x(\mathscr{T}^{-1}\mathcal{Q}_b),\Lambda^s v\big)_{L^2}\\ &\quad - \mu^2\varepsilon\big([\Lambda^s,h(\beta\partial_x b)^2]\mathscr{T}^{-1}\mathcal{Q}_b,\Lambda^s v\big)_{L^2}\\ &=: VI_1^1 + VI_1^2 + VI_1^3 + VI_1^4 + VI_1^5.\end{aligned}$$

Each term is treated similarly. For instance, take VI_1^2, which is the term with the least margin. Arguing as above, we use Cauchy-Schwarz inequality, (2.5), (2.22) and (4.5) to deduce that

$$|VI_1^2| \lesssim \varepsilon\mu|\mathcal{Q}_b|_{H^{s-1}}|v|_{X^s_\mu},$$

where use the algebra property of $H^{s-1}(\mathbb{R})$ for $s > \frac32$ to get:

$$\begin{aligned}\mu|\mathcal{Q}_b|_{H^{s-1}} &\lesssim \mu|h^2(\partial_x\mathrm{F}^{\frac12}v)^2(\beta\partial_x b)|_{H^{s-1}} + \mu|\partial_x\mathrm{F}^{\frac12}(h^2v^2\beta\partial_x^2 b)|_{H^{s-1}} + \mu|hv^2(\beta\partial_x^2 b)(\beta\partial_x b)|_{H^{s-1}}\\ &\lesssim |v|^2_{H^s}.\end{aligned}$$

Using similar estimates for the remaining terms, it is easy to deduce that

$$|VI_1| \lesssim \varepsilon|v|^3_{X^s_\mu}.$$

For IV_2, we use integration by parts to make the decomposition:

$$\begin{aligned}VI_2 &= -\varepsilon\mu\big(\Lambda^s(h^2(\partial_x\mathrm{F}^{\frac12}v)^2(\beta\partial_x b)),\Lambda^s v\big)_{L^2} + \frac{\varepsilon\mu}{2}\big(\Lambda^s(h^2v^2\beta\partial_x^2 b),\partial_x\mathrm{F}^{\frac12}\Lambda^s v\big)_{L^2}\\ &\quad - \varepsilon\mu\big(\Lambda^s(hv^2(\beta\partial_x^2 b)(\beta\partial_x b)),\Lambda^s v\big)_{L^2}.\end{aligned}$$

Each term is estimated by Hölder's inequality, Sobolev embedding, the algebra property of $H^s(\mathbb{R})$, and (4.7), leaving us with the estimate

$$|VI_2| \lesssim N(s)E_s(\mathbf{U}).$$

Lastly, VI_3 is estimated using the same estimates and gives

$$|VI_3| \lesssim \beta|\partial_x b|_{H^s}|v|_{H^s}|\zeta|_{H^s}.$$

Consequently, we have the estimate

$$|VI| \lesssim N(s)E_s(\mathbf{U}),$$

and thus completes the proof of Proposition 4.1.

□

Remark 4.2. *Under the provision of Proposition 4.1, using the algebra property of $H^{s-1}(\mathbb{R})$ for $s > \frac{3}{2}$, (2.21), suitable commutator estimates one can easily obtain that*

$$|(M_1 + S^{-1}M_2)(\mathbf{U})\partial_x \mathbf{U}|_{H^{s-1}} \lesssim |\mathbf{U}|_{H^s}, \tag{4.10}$$

and

$$|(S^{-1}(Q + Q_b)(\mathbf{U})|_{H^{s-1}} \lesssim |\mathbf{U}|_{H^s}. \tag{4.11}$$

5. Estimates on the difference of two solutions

We will now estimate the difference between two solutions of (1.1) given by $\mathbf{U}_1 = (\zeta_1, v_1)^T$ and $\mathbf{U}_2 = \varepsilon(\zeta_2, v_2)^T$. For convenience, we define $(\eta, w) = (\zeta_1 - \zeta_2, v_1 - v_2)$. Then $\mathbf{W} = (\eta, w)^T$ solves

$$\partial_t \mathbf{W} + (M_1 + S^{-1}M_2)(\mathbf{U}_1)\partial_x \mathbf{W} = \mathbf{F}, \tag{5.1}$$

with S, M_1, M_2, Q, Q_b defined as in (4.1) and

$$\begin{aligned}\mathbf{F} = &-\Big[(M_1 + (S^{-1}M_2))(\mathbf{U}_1) - (M_1 + S^{-1}M_2)(\mathbf{U}_2)\Big]\partial_x \mathbf{U}_2 \\ &- \Big[(S^{-1}(Q + Q_b))(\mathbf{U}_1) - (S^{-1}(Q + Q_b))(\mathbf{U}_2)\Big] \\ =:& \ \mathbf{F}_1 + \mathbf{F}_2.\end{aligned}$$

The energy associated to (7.1) is given in terms of the symmetrizer $S(\mathbf{U}_1)$ and reads

$$\tilde{E}_s(\mathbf{W}) := \big(\Lambda^s \mathbf{W}, S(\mathbf{U}_1)\Lambda^s \mathbf{W}\big)_{L^2}. \tag{5.2}$$

The main result of this section reads:

Proposition 5.1. *Let $s > \frac{3}{2}$, $(\mu, \varepsilon, \beta) \in \mathcal{A}_{SW}$, and $(\zeta_1, v_1), (\zeta_2, v_2) \in C([0,T]; Y^s_\mu(\mathbb{R}))$ be a solution to (4.1) on a time interval $[0,T]$ for some $T > 0$. Moreover, assume $b \in H^{s+2}(\mathbb{R})$ and there exist $h_0 \in (0,1)$ such that*

$$h_0 - 1 + \beta b \leq \varepsilon \zeta_i(x,t), \quad \forall (x,t) \in \mathbb{R} \times [0,T],$$

for $i = 1, 2$, and suppose also that

$$N(s) := \varepsilon \sup_{t \in [0,T]} |(\zeta_i(t,\cdot), v_i(t,\cdot))|_{Y^s_\mu} + \beta |b|_{H^{s+2}} \leq N^\star, \tag{5.3}$$

for some $N^\star \in \mathbb{R}^+$. Define the difference to be $\mathbf{W} = (\eta, w) = (\zeta_1 - \zeta_2, v_1 - v_2)$. Then, for the energy defined by (5.2), there holds

$$\frac{d}{dt}\tilde{E}_0(\mathbf{W}) \lesssim N(s)|(\eta, w)|^2_{Y^0_\mu}, \tag{5.4}$$

and

$$|(\eta, w)|^2_{Y^0_\mu} \lesssim \tilde{E}_0(\mathbf{W}) \lesssim |(\eta, w)|^2_{Y^0_\mu}. \tag{5.5}$$

Furthermore, we have the following estimate at the $Y^s_\mu-$ level:

$$\frac{d}{dt}\tilde{E}_s(\mathbf{W}) \lesssim |\big(\Lambda^s \mathbf{F}, S(\mathbf{U}_1)\Lambda^s \mathbf{W}\big)_{L^2}| + N(s)|(\eta, w)|^2_{Y^s_\mu}, \tag{5.6}$$

and

$$|(\eta, w)|^2_{Y^s_\mu} \lesssim \tilde{E}_s(\mathbf{W}) \lesssim |(\eta, w)|^2_{Y^s_\mu}. \tag{5.7}$$

Proof. We note that (5.5), (5.6) and (5.7) follow by the same arguments as in the proof of Proposition 4.1 and is therefore omitted.

To prove (5.4), we use (7.1), the self-adjointness $S(\mathbf{U})$, and Proposition 2.11 to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\tilde{E}_0(\mathbf{W}) &= \frac{1}{2}\big(\mathbf{W},(\partial_t S(\mathbf{U}_1))\mathbf{W}\big)_{L^2} - \big(M_1(\mathbf{U}_1)\partial_x\mathbf{W}, S(\mathbf{U}_1)\mathbf{W}\big)_{L^2}\\ &\quad - \big(M_2(\mathbf{U}_1)\partial_x\mathbf{W},\mathbf{W}\big)_{L^2} + \big(\mathbf{F}_1, S(\mathbf{U}_1)\mathbf{W}\big)_{L^2} + \big(\mathbf{F}_2, S(\mathbf{U}_1)\mathbf{W}\big)_{L^2}\\ &=: \mathcal{I} + \mathcal{II} + \mathcal{III} + \mathcal{IV} + \mathcal{V}.\end{aligned}$$

Control of $\mathcal{I}$. The estimate of $\mathcal{I}$ is a direct consequence of Hölder's inequality, (5.3), (4.8), and (5.3):

$$|\mathcal{I}| \lesssim N(s)|w|^2_{X^0_\mu},$$

for $s > \frac{3}{2}$.

Control of $\mathcal{II}$. By definition of $\mathcal{II}$, after performing an integration by parts, yields

$$\begin{aligned}\mathcal{II} &= \frac{\varepsilon}{2}\big((\partial_x(v_1h_1))w, w\big)_{L^2} + \frac{\mu\varepsilon}{3}\big(\partial_x \mathrm{F}^{\frac{1}{2}}(v_1\partial_x w), h_1^3\partial_x\mathrm{F}^{\frac{1}{2}}w\big)_{L^2}\\ &\quad - \frac{\mu\varepsilon\beta}{2}\big(v_1\partial_x w, \partial_x\mathrm{F}^{\frac{1}{2}}(h_1^2(\partial_x b)w)\big)_{L^2} + \frac{\mu\varepsilon\beta}{2}\big(v_1\partial_x w, h_1^2(\partial_x b)\partial_x\mathrm{F}^{\frac{1}{2}}w\big)_{L^2}\\ &\quad + \frac{\mu\varepsilon\beta}{2}\big(v_1\partial_x w, h_1(\beta\partial_x b)^2 w\big)_{L^2}\\ &=: \mathcal{II}_1 + \mathcal{II}_2 + \mathcal{II}_3 + \mathcal{II}_4 + \mathcal{II}_5.\end{aligned}$$

For $\mathcal{II}_1$ and $\mathcal{II}_5$, we simply use Hölders inequality and Sobolev embedding to obtain

$$|\mathcal{II}_1| + |\mathcal{II}_5| \lesssim \varepsilon(1 + |h_1 - 1|_{H^s} + (1 + |h_1 - 1|_{H^s})|b|^2_{H^{s+1}})|v_1|_{H^s}|w|^2_{L^2}.$$

For $\mathcal{II}_2$, we observe that is similar to II_2^2 in the proof of Proposition 4.1 where w plays the role of $\Lambda^s v$. Then reapplying the same estimates yields:

$$|\mathcal{II}_2| \lesssim \varepsilon(1 + |h_1^3 - 1|_{H^s})|w|^3_{X^0_\mu}.$$

For $\mathcal{II}_3$, we integrate by parts to make the decomposition

$$\begin{aligned}\mathcal{II}_3 &= \frac{\mu\varepsilon\beta}{2}\big(\mathrm{F}^{\frac{1}{2}}((\partial_x v_1)\partial_x w), h_1^2(\partial_x b)w\big)_{L^2} + \frac{\mu\varepsilon\beta}{2}\big(\mathrm{F}^{\frac{1}{2}}(v_1\partial_x^2 w), h_1^2(\partial_x b)w\big)_{L^2}\\ &=: \mathcal{II}_3^1 + \mathcal{II}_3^2.\end{aligned}$$

Here $\mathcal{II}_3^1$ is similar to II_3^2 in the proof of Proposition 4.1 and applying the estimates yields,

$$|\mathcal{II}_3^1| \lesssim \varepsilon(1 + |h_1^2 - 1|_{H^s})|b|_{H^{s+1}}|v_1|_{H^s}|w|_{L^2}|w|_{X^0_\mu}.$$

On the other hand, $\mathcal{II}_3^2$ is similar to II_3^3 and we observe that

$$\begin{aligned}\mathcal{II}_3^2 &= \frac{\mu\varepsilon\beta}{2}\big([\mathrm{F}^{\frac{1}{2}}, v_1]\partial_x^2 w, h_1^2(\partial_x b)w\big)_{L^2} - \frac{\mu\varepsilon\beta}{2}\big(v_1\mathrm{F}^{\frac{1}{2}}\partial_x w, h_1^2(\partial_x b)\partial_x w\big)_{L^2}\\ &\quad - \frac{\mu\varepsilon\beta}{2}\big(v_1\mathrm{F}^{\frac{1}{2}}\partial_x w, \big(\partial_x(h_1^2(\partial_x b))\big)w\big)_{L^2}\\ &= \mathcal{II}_3^{2,1} + \mathcal{II}_3^{2,2} + \mathcal{II}_3^{2,3}.\end{aligned}$$

Then we observe that $\mathcal{II}_3^{2,2} = -\mathcal{II}_4$, while for $\mathcal{II}_3^{2,1}$ and $\mathcal{II}_3^{2,3}$ we apply Hölder's inequality, (2.16), and Sobolev embedding to obtain the bound

$$|\mathcal{II}_3^{2,1}| + |\mathcal{II}_3^{2,3}| \lesssim \varepsilon|v_1|_{H^s}(1 + |h_1^2 - 1|_{H^s})|b|_{H^{s+1}}|w|_{L^2}|w|_{X^0_\mu}.$$

Gathering these estimates and using (5.3) yields

$$|\mathcal{II}| \lesssim N(s)|w|^2_{X^0_\mu}.$$

Control of $\mathcal{III}$. By definition of $\mathcal{III}$ we must estimate the terms:

$$\begin{aligned}\mathcal{III} &= -\varepsilon\big(v_1\partial_x\eta,\eta\big)_{L^2} - \big(h_1\partial_x w,\eta\big)_{L^2} - \big(h_1\partial_x\eta,w\big)_{L^2}\\ &=: \mathcal{III}_1 + \mathcal{III}_2 + \mathcal{III}_3.\end{aligned}$$

Starting with $\mathcal{III}_1$, we simply integrate by parts and use Hölder's inequality and Sobolev embedding to deduce

$$|\mathcal{III}_1| \lesssim \varepsilon|v_1|_{H^s}|\eta|^2_{L^2}.$$

Similarly, for $\mathcal{III}_2 + \mathcal{III}_3$ we use integration by parts, the Sobolev embedding, and (5.3) to get that

$$\begin{aligned}|\mathcal{III}_2 + \mathcal{III}_3| &\lesssim |\partial_x h_1|_{L^\infty}|w|_{L^2}|\eta|_{L^2}\\ &\lesssim N(s)|w|_{L^2}|\eta|_{L^2}.\end{aligned}$$

In conclusion, we obtain the bound

$$|\mathcal{III}| \lesssim N(s)|(\eta,w)|^2_{Y^0_\mu}.$$

Control of $\mathcal{IV}$. First define the notation

$$\mathscr{T}_i = \mathscr{T}[h_i,\beta b],$$

for $i = 1,2$ and consider the terms

$$\begin{aligned}\mathcal{IV} &= -\varepsilon\big(w\partial_x\zeta_2,\eta\big)_{L^2} - \varepsilon\big(\eta\partial_x v_2,\eta\big)_{L^2} - \big((\mathscr{T}_1^{-1}(h_1\cdot) - \mathscr{T}_2^{-1}(h_2\cdot))\partial_x\zeta_2, \mathscr{T}_1 w\big)_{L^2}\\ &\quad - \varepsilon\big(w\partial_x v_2, \mathscr{T}_1 w\big)_{L^2}\\ &=: \mathcal{IV}_1 + \mathcal{IV}_2 + \mathcal{IV}_3 + \mathcal{IV}_4.\end{aligned}$$

For the first two terms, we use Hölder's inequality and the Sobolev embedding to deduce the bound:

$$|\mathcal{IV}_1| + |\mathcal{IV}_2| \leq \varepsilon|\zeta_2|_{H^s}|w|_{L^2}|\eta|_{L^2} + \varepsilon|v_2|_{H^s}|\eta|^2_{L^2},$$

for $s > \frac{3}{2}$. Next, we make the observation

$$\mathscr{T}_1(\mathscr{T}_1^{-1}f_1 - \mathscr{T}_2^{-1}f_2) = (f_1 - f_2) - (\mathscr{T}_1 - \mathscr{T}_2)\mathscr{T}_2^{-1}f_2. \tag{5.8}$$

Using (5.8) and invertability of $\mathscr{T}_i$ we observe that

$$\begin{aligned}\mathcal{IV}_3 &= -\varepsilon\big(\eta\partial_x\zeta_2, w\big)_{L^2} + \frac{\mu\varepsilon}{3}\big(\mathrm{F}^{\frac{1}{2}}\partial_x(\eta(h_1^2 + h_1h_2 + h_2^2)\partial_x\mathrm{F}^{\frac{1}{2}}\mathscr{T}_2^{-1}(h_2\partial_x\zeta_2)), w\big)_{L^2}\\ &\quad - \frac{\mu\varepsilon}{2}\big(\partial_x\mathrm{F}^{\frac{1}{2}}(\eta(h_1+h_2)(\beta\partial_x b)\mathscr{T}_2^{-1}(h_2\partial_x\zeta_2)), w\big)_{L^2}\\ &\quad + \frac{\mu\varepsilon}{2}\big(\eta(h_1+h_2)(\beta\partial_x b)\partial_x\mathrm{F}^{\frac{1}{2}}\mathscr{T}_2^{-1}(h_2\partial_x\zeta_2), w\big)_{L^2}\\ &\quad - \mu\varepsilon\big(\eta(\beta\partial_x b)^2\mathscr{T}_2^{-1}(h_2\partial_x\zeta_2), w\big)_{L^2}\\ &=: \mathcal{IV}_3^1 + \mathcal{IV}_3^2 + \mathcal{IV}_3^3 + \mathcal{IV}_3^4 + \mathcal{IV}_3^5,\end{aligned}$$

where $\mathcal{IV}_3^1 = \mathcal{IV}_1$ which is already treated. While for the second term, we use integration by parts, Hölder's inequality, Sobolev embedding, (5.3), and (2.22) to obtain

$$\begin{aligned}|\mathcal{IV}_3^2| &\leq \varepsilon|\eta|_{L^2}|(h_1^2 + h_1h_2 + h_2^2)|_{L^\infty}|\partial_x \mathrm{F}^{\frac{1}{2}}\mathscr{T}_2^{-1}(h_2\partial_x\zeta_2))|_{L^\infty}|w|_{X^0_\mu}\\ &\lesssim N(s)|\eta|_{L^2}|w|_{X^0_\mu}\end{aligned}$$

for $s > \frac{3}{2}$. For II_3^3 we apply the same estimates together with (2.21) to deduce

$$\begin{aligned}|\mathcal{IV}_3^3| &\lesssim \varepsilon|(h_1 + h_2)|_{L^\infty}|\beta\partial_x b|_{L^\infty}|h_2|_{L^\infty}|\partial_x\zeta_2|_{L^\infty}|\eta|_{L^2}|w|_{X^0_\mu}\\ &\lesssim N(s)|\eta|_{L^2}|w|_{X^0_\mu}.\end{aligned}$$

Next, we see that $\mathcal{IV}_3^4$ is estimated similarly to $\mathcal{IV}_3^2$ and we get that

$$|\mathcal{IV}_3^4| \lesssim N(s)|\eta|_{L^2}|w|_{X^0_\mu}.$$

The part $\mathcal{IV}_3^5$ is easily treated with Hölder's inequality and Sobolev embedding. Thus, gathering these estimates and applying (5.5) yields,

$$|\mathcal{IV}_3| \lesssim N(s)|(\eta, w)|^2_{Y^0_\mu}.$$

Lastly, we deal with $\mathcal{IV}_4$:

$$\begin{aligned}\mathcal{IV}_4 = &-\varepsilon\big(w\partial_x v_2, h_1 w\big)_{L^2} + \frac{\mu\varepsilon}{3}\big(w\partial_x v_2, \partial_x \mathrm{F}^{\frac{1}{2}}(h_1^3\mathrm{F}^{\frac{1}{2}}\partial_x w)\big)_{L^2}\\ &- \frac{\mu\varepsilon}{2}\big(w\partial_x v_2, \partial_x\mathrm{F}^{\frac{1}{2}}(h_1^2(\beta\partial_x b)w)\big)_{L^2} + \frac{\mu\varepsilon}{2}\big(w\partial_x v_2, h_1^2(\beta\partial_x b)\partial_x\mathrm{F}^{\frac{1}{2}}w\big)_{L^2}\\ &- \mu\varepsilon\big(w\partial_x v_2, h_1(\beta\partial_x b)^2 w\big)_{L^2}\\ =:&\ \mathcal{IV}_4^1 + \mathcal{IV}_4^2 + \mathcal{IV}_4^3 + \mathcal{IV}_4^4.\end{aligned}$$

Each term is treated similarly, and we only give the details for $\mathcal{IV}_4^2$ since it is the term with the least margin. In particular, using integration by parts, Hölder's inequality, Sobolev embedding, (5.3),

$$\begin{aligned}|\mathcal{IV}_4^2| &\lesssim \varepsilon\sqrt{\mu}(1 + |h_1^3 - 1|_{H^s})|\partial_x\mathrm{F}^{\frac{1}{2}}(w\partial_x v_2)|_{L^2}|w|_{X^0_\mu}\\ &\lesssim N(s)\mu^{\frac{1}{4}}|w\partial_x v_2|_{H^{\frac{1}{2}}}|w|_{X^0_\mu}.\end{aligned}$$

Then we use Hölder's inequality, Sobolev embedding, and (2.4) to deduce that

$$\begin{aligned}\mu^{\frac{1}{4}}|w\partial_x v_2|_{H^{\frac{1}{2}}} &\lesssim \mu^{\frac{1}{4}}(|\partial_x v_2|_{L^\infty}|w|_{L^2} + |D^{\frac{1}{2}}(w\partial_x v_2)|_{L^2})\\ &\lesssim |v_2|_{H^s}|w|_{L^2} + \mu^{\frac{1}{4}}|v_2|_{H^{r+1}}|w|_{H^{\frac{1}{2}}},\end{aligned}$$

for any $r > \frac{1}{2}$. Now choose r such that $s > r + 1 > \frac{3}{2}$ allowing us to conclude that

$$\mu^{\frac{1}{4}}|w\partial_x v_2|_{H^{\frac{1}{2}}} \lesssim |v_2|_{H^s}|w|_{X^0_\mu},$$

and from which we obtain:

$$|\mathcal{IV}_4^2| \lesssim N(s)|w|^2_{X^0_\mu}.$$

To summarize this part, we can use (5.3) to obtain the estimate

$$|\mathcal{IV}| \lesssim N(s)|(\eta, w)|^2_{Y^0_\mu}.$$

Control of $\mathcal{V}$. Define the notation

$$\mathcal{Q}_i = \mathcal{Q}[h_i, v_i], \quad \mathcal{Q}_{b,i} = \mathcal{Q}_b[h_i, \beta b, v_i],$$

with $i = 1, 2$, and using the identity (5.8), then we obtain the following terms:

$$\begin{aligned}\mathcal{V} &= \beta\big(\partial_x b w, \eta\big)_{L^2} - \mu\varepsilon\big(h_1\mathcal{Q}_1 - h_2\mathcal{Q}_2, w\big)_{L^2} + \mu\varepsilon\big((\mathscr{T}_1 - \mathscr{T}_2)\mathscr{T}_2^{-1}(h_2\mathcal{Q}_2), w\big)_{L^2}\\ &\quad - \mu\varepsilon\big(h_1\mathcal{Q}_{b,1} - h_2\mathcal{Q}_{b,2}, w\big)_{L^2} + \mu\varepsilon\big((\mathscr{T}_1 - \mathscr{T}_2)\mathscr{T}_2^{-1}(h_2\mathcal{Q}_{b,2}), w\big)_{L^2}\\ &=: \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 + \mathcal{V}_4 + \mathcal{V}_5.\end{aligned}$$

The estimate of $\mathcal{V}_1$ follows directly by Hölder's inequality:

$$|\mathcal{V}_1| \lesssim \beta|\partial_x b|_{L^\infty}|w|_{L^2}|\eta|_{L^2}.$$

For $\mathcal{V}_2$, we use the definition of Q_i and then integration by parts to make the following decomposition

$$\begin{aligned}\mathcal{V}_2 &= \frac{2\mu\varepsilon}{3}\big(\big(h_1^3(\partial_x \mathrm{F}^{\frac12} v_1)^2 - h_2^3(\partial_x\mathrm{F}^{\frac12}v_2)^2\big), \partial_x\mathrm{F}^{\frac12}w\big)_{L^2}\\ &= \frac{2\mu\varepsilon}{3}\big(\big(\eta(h_1+h_2)\big)(\partial_x\mathrm{F}^{\frac12}v_1)^2, \partial_x\mathrm{F}^{\frac12}w\big)_{L^2}\\ &\quad + \frac{2\mu\varepsilon}{3}\big(h_2^3\big(\partial_x\mathrm{F}^{\frac12}(v_1+v_2)\big)(\partial_x\mathrm{F}^{\frac12}w,), \partial_x\mathrm{F}^{\frac12}w\big)_{L^2}.\end{aligned}$$

Now, estimate each term by Hölder's inequality and Sobolev embedding to obtain that

$$\begin{aligned}|\mathcal{V}_2| &\lesssim \mu\varepsilon\big(|h_1+h_2|_{L^\infty}|\partial_x\mathrm{F}^{\frac12}v_1|^2_{L^\infty}|\eta|_{L^2}|\partial_x\mathrm{F}^{\frac12}w|_{L^2} + |h_2^3|_{L^\infty}|\partial_x\mathrm{F}^{\frac12}(v_1+v_2)|_{L^\infty}|\partial_x\mathrm{F}^{\frac12}w|^2_{L^2}\big)\\ &\lesssim \varepsilon\max_{i=1,2}\big((1+|h_i-1|_{H^s})|v_1|^2_{H^s}|\eta|_{L^2}|w|_{X^0_\mu} + (1+|h_2^3-1|_{H^s})|v_i|_{H^s}|w|^2_{X^0_\mu}\big).\end{aligned}$$

Then conclude this estimate by applying (5.3):

$$|\mathcal{V}_2| \lesssim N(s)|(\eta,w)|^2_{Y^0_\mu}.$$

For $\mathcal{V}_3$, we use the same decomposition as for $\mathcal{IV}_3$ and find that

$$\begin{aligned}\mathcal{V}_3 &= \frac{\mu^2\varepsilon}{3}\big(\mathrm{F}^{\frac12}\partial_x\big(\eta(h_1^2+h_1h_2+h_2^2)\partial_x\mathrm{F}^{\frac12}\mathscr{T}_2^{-1}(h_2\mathcal{Q}_2)\big), w\big)_{L^2}\\ &\quad - \frac{\mu^2\varepsilon}{2}\big(\partial_x\mathrm{F}^{\frac12}\big(\eta(h_1+h_2)(\beta\partial_x b)\mathscr{T}_2^{-1}(h_2\mathcal{Q}_2)\big), w\big)_{L^2}\\ &\quad + \frac{\mu^2\varepsilon}{2}\big(\eta(h_1+h_2)(\beta\partial_x b)\partial_x\mathrm{F}^{\frac12}\mathscr{T}_2^{-1}(h_2\mathcal{Q}_2), w\big)_{L^2} - \mu^2\varepsilon\big(\eta(\beta\partial_x b)^2\mathscr{T}_2^{-1}(h_2\mathcal{Q}_2), w\big)_{L^2}\\ &=: \mathcal{V}_3^1 + \mathcal{V}_3^2 + \mathcal{V}_3^3 + \mathcal{V}_3^4.\end{aligned}$$

Each term is treated similarly, but the term with the least margin is $\mathcal{V}_3^1$. In fact, we use integration by parts, Hölder's inequality, the Sobolev embedding $H^{s-1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, (5.3), and (2.22) to get the following estimate

$$\begin{aligned}|\mathcal{V}_3^1| &\lesssim \mu^2\varepsilon|\eta|_{L^2}|h_1^2+h_1h_2+h_2^2|_{L^\infty}|\mathrm{F}^{\frac12}\mathscr{T}_2^{-1}(h_2\mathcal{Q}_2)|_{H^s}|\mathrm{F}^{\frac12}\partial_x w|_{L^2}\\ &\lesssim \mu\varepsilon|h_2\mathcal{Q}_2|_{H^{s-1}}|\eta|_{L^2}|w|_{X^0_\mu}.\end{aligned}$$

Then using (2.13) and the algebra property of $H^s(\mathbb{R})$ and the boundedness of $\mathrm{F}^{\frac12}$, we observe that

$$|h_2\mathcal{Q}_2|_{H^{s-1}} \lesssim |\mathrm{F}^{\frac12}\partial_x(h_2^3(\mathrm{F}^{\frac12}\partial_x v_2)^2)|_{H^{s-1}} \lesssim (1+|h_2^3-1|_{H^s})|\mathrm{F}^{\frac12}\partial_x v_2|^2_{H^s}.$$

Consequently, we may gather these estimates to deduce the bound

$$|\mathcal{V}_3^1| + ... + |\mathcal{V}_3^4| \lesssim N(s)|(\eta,w)|^2_{X^0_\mu},$$

where $|\mathcal{V}_3^2| + ... + |\mathcal{V}_3^4|$ are easier versions of $\mathcal{V}_3^2$.

To conclude we must estimate $\mathcal{V}_4$ and $\mathcal{V}_5$. However, since $\mathcal{Q}_b$ contains fewer derivatives than $\mathcal{Q}$, these terms could be considered to be of lower order. In fact, $\mathcal{V}_4$ is estimated by a similar decomposition to the one of $\mathcal{V}_2$, while $\mathcal{V}_5$ is a just a simpler version of $\mathcal{V}_3$. We may therefore conclude that

$$|\mathcal{V}| \lesssim N(s)|(\eta, w)|^2_{Y^0_\mu}.$$

Gathering all these estimates, we obtain (5.4), and the proof of Proposition 5.1 is complete. □

Remark 5.2. *From the proof of the proposition, it is easy to make the rough estimate of the source term in* (5.6)*:*

$$|\big(\Lambda^s \mathbf{F}, S(\mathbf{U}_1)\Lambda^s \mathbf{W}\big)_{L^2}| \lesssim N(s+1)\,|(\eta, w)|_{Y^{s-1}_\mu}\Big(\tilde{E}_s(\mathbf{W}))^{\frac{1}{2}} + N(s)\tilde{E}_s(\mathbf{W})\Big),$$

combining the estimates used below (see control of II) and using the product estimate for $H^s(\mathbb{R})$. The estimate (5.6) *serves two purposes. One is to prove the full justification of* (1.1) *as a water waves model, where we allow for a loss of derivatives (see Section 7).*

On the other hand, to get the continuity of the flow, one needs to compensate the norms on the right of (5.6)*:*

$$\max_{i=1,2} |(\zeta_i, v_i)|_{Y^{s+1}_\mu} |(\eta, w)|_{Y^{s-1}_\mu},$$

and is done by regularising the initial data and a Bona-Smith argument [3].

6. Long time Well-posedness of (1.1)

For the proof of Theorem 1.4 we will use the parabolic regularisation method for the existence of solutions and a Bona–Smith regularisation argument [3] to prove the continuous dependence of the solutions with respect to the initial data. This method is classical in the case of quasilinear equations and we will only outline the steps that are unique to system (1.1) and needed to run the argument. In particular, one can read [14] for a similar argument in the case of the classical Green-Naghdi system. Lastly, the reader might also find it useful to read the detailed proof, using these methods, in the case of the Benjamin-Ono equation in [32], and likewise in the case of Whitham-Boussinesq systems demonstrated in [36].

Proof. Step 1: *Existence of solutions for a regularised system.* Let $s > \frac{3}{2}$, $\alpha \in (1, \frac{3}{2}]$ and take $\nu > 0$ small. Moreover let $\mathbf{U}_0 = (\zeta_0, v_0)^T \in Y^s_\mu(\mathbb{R})$, $b \in H^{s+2}(\mathbb{R})$ satisfying (1.2) and define $T_\nu > 0$ such that

$$T_\nu \searrow 0 \quad \text{as} \quad \nu \searrow 0, \quad \text{and} \quad T_\nu = T_\nu(|(\zeta_0, v_0)|_{Y^s_\mu}), \tag{6.1}$$

with the property that

$$\text{if} \quad a < b \quad \text{then} \quad T_\nu(a) > T_\nu(b). \tag{6.2}$$

Then we claim there is a unique solution $\mathbf{U}^\nu = (\zeta^\nu, v^\nu)^T \in C([0, T_\nu]; Y^s_\mu(\mathbb{R}))$ associated to $\mathbf{U}_0$ that satisfy the regularised version of (4.1) given by,

$$S(\mathbf{U}^\nu)(\partial_t \mathbf{U}^\nu + M_1(\mathbf{U}^\nu)\partial_x \mathbf{U}^\nu) + M_2(\mathbf{U}^\nu)\partial_x \mathbf{U}^\nu + Q(\mathbf{U}^\nu) + Q_b(\mathbf{U}^\nu) = -\nu S(\mathbf{U}^\nu)\Lambda^\alpha \mathbf{U}^\nu. \tag{6.3}$$

To prove the claim, we first suppose the non-cavitation condition for $\mathbf{U}^\nu$ and use Proposition 2.11 to apply the inverse of $\mathscr{T}[h,\beta b]$ on the second equation in (6.3). Then we study the Duhamel formulation:

$$\mathbf{U}^\nu(t) = \mathrm{e}^{-\nu\langle \mathrm{D}\rangle^\alpha t}\mathbf{U}_0 + \int_0^t \mathrm{e}^{-\nu\langle \mathrm{D}\rangle^\alpha (t-s)}\mathcal{N}(\mathbf{U}^\nu)(s)\,\mathrm{d}s,$$

where $\mathrm{e}^{-\nu\langle \mathrm{D}\rangle^\alpha t}$ is the Fourier multiplier defined by

$$\mathcal{F}(\mathrm{e}^{-\nu\langle \mathrm{D}\rangle^\alpha t}f)(\xi) = e^{-\nu\langle \xi\rangle^\alpha t}\hat{f}(\xi),$$

and with

$$\mathcal{N}(\mathbf{U}^\nu) = (M_1 + S^{-1}M_2)(\mathbf{U}^\nu)\partial_x\mathbf{U}^\nu + (S^{-1}(Q+Q_b))(\mathbf{U}^\nu).$$

In particular, we prove that the application

$$\Phi : \mathbf{U}^\nu \mapsto \mathrm{e}^{-\nu\langle \mathrm{D}\rangle^\alpha t}\mathbf{U}_0 + \int_0^t \mathrm{e}^{-\nu\langle \mathrm{D}\rangle^\alpha (t-s)}\mathcal{N}(\mathbf{U}^\nu)(s)\,\mathrm{d}s, \tag{6.4}$$

is a contraction map on the subspace

$$B(R,h_0) = \Big\{\mathbf{U} = (\zeta,v) \in C([0,T];Y^s_\mu(\mathbb{R})) \ : \ |(\zeta,v)|_{Y^s_\mu} < R, \ \inf_{t\in(0,T)}(1+\varepsilon\zeta^\nu - \beta b) \geq h_0\Big\},$$

with $R>0$ to be determined. First, observe by Plancherel's identity and then splitting in high and low frequencies that

$$\begin{aligned} |\mathrm{e}^{-\nu\langle \mathrm{D}\rangle^\alpha t}\mathbf{U}|_{H^s} &\lesssim |\mathbf{U}|_{L^2} + (\nu t)^{-\frac{1}{\alpha}}|(\nu t)^{\frac{1}{\alpha}}|\xi|e^{-((\nu t)^{\frac{1}{\alpha}}|\xi|)^\alpha}\hat{\mathbf{U}}|_{H^{s-1}} \\ &\lesssim (1+(\nu t)^{-\frac{1}{\alpha}})|\mathbf{U}|_{H^{s-1}}, \end{aligned}$$

and trivially that

$$|\mathrm{e}^{-\nu\langle \mathrm{D}\rangle^\alpha t}\mathbf{U}|_{H^s} \leq |\mathbf{U}|_{H^s}.$$

Thus, as a consequence of these estimates and Remark 4.2 we obtain that

$$\sup_{t\in[0,T]}|\Phi(\mathbf{U}^\nu)(t)|_{H^s} \leq c|\mathbf{U}_0|_{H^s} + cT^{1-\frac{1}{\alpha}}\nu^{-\frac{1}{\alpha}}|\mathbf{U}|_{H^s}.$$

Now, choose R to be

$$R = 2c|(\zeta_0,v_0)|_{Y^s_\mu}.$$

Additionally, since $1-\frac{1}{\alpha}>0$ we may take T positive depending on ν and R on the form

$$T^{1-\frac{1}{\alpha}} \sim \frac{\nu^{\frac{1}{\alpha}}}{R},$$

small enough, and such that

$$1+\varepsilon\zeta^\nu(x,t) - \beta b(x) = h_0 + \int_0^t \partial_t\zeta^\nu(x,s)\,\mathrm{d}s \geq h_0 - cT(R+R^2) \geq \frac{h_0}{2},$$

using the Fundamental theorem of calculus and (4.8). Then the map (6.4) is well-defined on $B(R,\frac{h_0}{2})$, and the contraction estimate is obtained similarly after some straightforward algebraic manipulations. We may therefore conclude this step by the Banach fixed point Theorem.

Remark 6.1 (The blow-up alternative)**.** *If we define the maximal time of existence T^{ν}_{Max} to be*

$$T^{\nu}_{Max} = \sup\big\{T_{\nu} > 0 \ : \ \exists! \quad \mathbf{U}^{\nu} \quad \textit{solution of} \quad (6.3) \quad \textit{in} \quad C([0,T_{\nu}];Y^s_{\mu}(\mathbb{R}))\big\},$$

then by a standard contradiction argument, one can deduce that

$$\textit{if} \quad T^{\nu}_{Max} < \infty, \quad \textit{then} \quad \lim_{t\nearrow T^{\nu}_{Max}} |(\zeta^{\nu},v^{\nu})|_{Y^s_{\mu}} = \infty \quad \textit{or} \quad \lim_{t\nearrow T^{\nu}_{Max}} \inf_{x\in\mathbb{R}} 1+\varepsilon\zeta^{\nu}+\beta b = 0. \quad (6.5)$$

This is due to the fact that if (6.5) *does not hold, one can use Step* 1. *and the properties of T^{ν} given by* (6.1) *and* (6.2) *to extend the solution beyond the maximal time.*

Step 2: *The existence time is independent of $\nu > 0$.* Let $s > \frac{3}{2}$ and $(\zeta^{\nu},v^{\nu}) \in C([0,T^{\nu}_{\text{Max}});Y^s_{\mu}(\mathbb{R}))$ be a solution of (6.3) with initial data $(\zeta_0,v_0) \in Y^s_{\mu}(\mathbb{R})$, defined on its maximal time of existence and satisfying the blow-up alternative (6.5). Moreover, let ζ_0 satisfy (1.2). Then for $\tilde{N} = |(\zeta_0,v_0)|_{Y^s_{\mu}} + |b|_{H^{s+2}}$, there exist a time

$$T = \frac{1}{\tilde{N}}, \tag{6.6}$$

such that $T < T^{\nu}_{\text{Max}}$ and

$$\sup_{t\in[0,\frac{T}{\max\{\varepsilon,\beta\}}]} |(\zeta^{\nu},v^{\nu})(t)|_{Y^s_{\mu}} \lesssim |(\zeta_0,v_0)|_{Y^s_{\mu}}. \tag{6.7}$$

Indeed, if the solution of (6.3) also satisfies estimate (4.6), then one could combine this estimate with (6.5) and a bootstrap argument to get the result. However, to obtain the same estimate for (6.3), one has to take into account an additional term:

$$\frac{d}{dt}E_s(\mathbf{U}^{\nu}) \lesssim N(s)E_s(\mathbf{U}^{\nu}) - \nu\big(\Lambda^{s+\alpha}\mathbf{U}^{\nu}, S(\mathbf{U}^{\nu})\Lambda^s\mathbf{U}^{\nu}\big)_{L^2},$$

appearing due to the regularisation. To control this additional term, we make the decomposition

$$\begin{aligned}\big(\Lambda^{s+\alpha}\mathbf{U}^{\nu}, S(\mathbf{U}^{\nu})\Lambda^s\mathbf{U}^{\nu}\big)_{L^2} &= |\zeta^{\nu}|^2_{H^{s+\frac{\alpha}{2}}} + \big(\mathscr{T}\Lambda^{s+\frac{\alpha}{2}}v^{\nu},\Lambda^{s+\frac{\alpha}{2}}v^{\nu}\big)_{L^2} + \big([\Lambda^{\frac{\alpha}{2}},\mathscr{T}]\Lambda^s v^{\nu},\Lambda^{s+\frac{\alpha}{2}}v^{\nu}\big)_{L^2}\\ &= I_1 + I_2 + I_3.\end{aligned}$$

Then the two first terms will have a positive sign, where

$$I_2 \geq c(h_0)|v^{\nu}|^2_{X^{s+\frac{\alpha}{2}}},$$

arguing as we did in the proof of Proposition 2.11, step 2. On the other hand, I_3 is further decomposed by using integration by parts:

$$\begin{aligned}I_3 = &-\big([\Lambda^{\frac{\alpha}{2}},h^{\nu}]\Lambda^s v^{\nu},\Lambda^{s+\frac{\alpha}{2}}v^{\nu}\big)_{L^2} - \frac{\mu}{3}\big([\Lambda^{\frac{\alpha}{2}},(h^{\nu})^3]\Lambda^s\mathrm{F}^{\frac12}\partial_x v^{\nu},\Lambda^{s+\frac{\alpha}{2}}\mathrm{F}^{\frac12}\partial_x v^{\nu}\big)_{L^2}\\ &-\frac{\mu}{2}\big([\Lambda^{\frac{\alpha}{2}},(h^{\nu})^2(\beta\partial_x b)]\Lambda^s v^{\nu},\Lambda^{s+\frac{\alpha}{2}}\mathrm{F}^{\frac12}\partial_x v^{\nu}\big)_{L^2} + \frac{\mu}{2}\big([\Lambda^{\frac{\alpha}{2}},(h^{\nu})^2(\beta\partial_x b)]\Lambda^s v^{\nu},\Lambda^{s+\frac{\alpha}{2}}v^{\nu}\big)_{L^2}\\ &+\mu\big([\Lambda^{\frac{\alpha}{2}},h^{\nu}(\beta\partial_x b)^2]\Lambda^s\mathrm{F}^{\frac12}\partial_x v^{\nu},\Lambda^{s+\frac{\alpha}{2}}v^{\nu}\big)_{L^2}.\end{aligned}$$

We recall that $\alpha \in (1,\frac{3}{2}]$. We may therefore estimate each term by Hölder's inequality, (2.5), Sobolev embedding, and then use Young's inequality to deduce that

$$\begin{aligned}|I_3| &\leq N(s)|v^{\nu}|_{X^s_{\mu}}|v^{\nu}|_{X^{s+\frac{\alpha}{2}}_{\mu}}\\ &\leq \frac{N(s)}{c_1}|v^{\nu}|^2_{X^s_{\mu}} + c_1N(s)|v^{\nu}|^2_{X^{s+\frac{\alpha}{2}}_{\mu}},\end{aligned}$$

for $c_1 > 0$ small enough such that

$$-\nu\big(\Lambda^{s+\alpha}\mathbf{U}^\nu, \Lambda^s\mathbf{U}^\nu\big)_{L^2} = -\nu(I_1 + I_2 + I_3) \lesssim N(s)|v^\nu|^2_{X^s_\mu},$$

and by extension, we obtain that

$$\frac{d}{dt}E_s(\mathbf{U}^\nu) \lesssim N(s)E_s(\mathbf{U}^\nu),$$

allowing us to conclude this step.

Remark 6.2. *Since $\frac{\alpha}{2} \in (\frac{1}{2}, \frac{3}{4})$, one can obtain a similar estimate on $|I_3|$ in the case $\Lambda^s = \mathrm{Id}$. Indeed, there holds*

$$|I_3| \leq N(r)|v^\nu|_{X^0_\mu}|v^\nu|_{X^{\frac{\alpha}{2}}_\mu},$$

for $r > \frac{3}{2}$.

Step 3: *Existence of solutions.* We claim that for all $0 \leq s' < s$ there exists a solution $(\zeta, v) \in C([0, \frac{T}{\max\{\varepsilon,\beta\}}]; Y^{s'}_\mu(\mathbb{R})) \cap L^\infty([0, \frac{T}{\max\{\varepsilon,\beta\}}]; Y^s_\mu(\mathbb{R}))$ of (1.1) with T defined by (6.6).

To prove the claim, we let $0 < \nu' < \nu < 1$ where we take $(\zeta^{\nu'}, v^{\nu'}), (\zeta^\nu, v^\nu)$ to be two sets of solutions to system (6.3), obtained in Step 1, and with the same initial data. Then define the difference to be

$$\mathbf{W} = (\eta, w) := (\zeta^{\nu'} - \zeta^\nu, v^{\nu'} - v^\nu),$$

with $\alpha \in (1, \frac{3}{2}]$. Observe that (η, w) satisfies a regularised version of (7.1):

$$\partial_t\mathbf{W} + (S^{-1}M)(\mathbf{U}^{\nu'})\mathbf{W} = \mathbf{F} - \nu'\Lambda^\alpha\mathbf{W} + (\nu - \nu')\Lambda^\alpha\mathbf{U}^\nu,$$

where $\mathbf{F}$ is defined by

$$\begin{aligned}\mathbf{F} = &-\Big[(M_1 + (S^{-1}M_2))(\mathbf{U}^{\nu'}) - (M_1 + S^{-1}M_2)(\mathbf{U}^\nu)\Big]\partial_x\mathbf{U}^\nu \\ &- \Big[(S^{-1}(Q + Q_b))(\mathbf{U}^{\nu'}) - (S^{-1}(Q + Q_b))(\mathbf{U}^\nu)\Big].\end{aligned}$$

Now, we can easily extend the estimates in Proposition 5.1 and use Remark 6.2 to deduce the estimate

$$\frac{d}{dt}\tilde{E}_0(\mathbf{W}) \lesssim N(s)(\tilde{E}_0(\mathbf{W}) + (\nu - \nu')\big(\Lambda^\alpha\mathbf{U}^\nu, S(\mathbf{U}^\nu)\mathbf{W}\big)_{L^2},$$

where the last term can be bounded using the definition of $\mathscr{T}$ and the fact that $\alpha \in (1, \frac{3}{2}]$. In particular, we obtain that

$$\frac{d}{dt}\tilde{E}_0(\mathbf{W}) \lesssim N(s)(\tilde{E}_0(\mathbf{W}) + (\nu - \nu')(\tilde{E}_0(\mathbf{W}))^{\frac{1}{2}}). \tag{6.8}$$

By (6.7) and definition of $N(s)$, we have that $N(s) \lesssim 1$. Moreover, using Grönwall's inequality on (6.8) and (5.5) yields,

$$\sup_{t\in[0,\frac{T}{\max\{\varepsilon,\beta\}}]} |(\eta, w)(t)|_{Y^0_\mu} \lesssim \nu - \nu'.$$

Then using this estimate combined with interpolation we get that

$$\sup_{t\in[0,\frac{T}{\max\{\varepsilon,\beta\}}]} |(\eta, w)|_{Y^{s'}_\mu} \lesssim (\nu - \nu')^{1-\frac{s'}{s}} \underset{\nu\to 0}{\longrightarrow} 0, \tag{6.9}$$

from which we deduce that $\{(\zeta^\nu, v^\nu)\}_{0<\nu\leq 1}$ defines a Cauchy sequence in $C([0,\frac{T}{\max\{\varepsilon,\beta\}}];Y^{s'}_\mu(\mathbb{R}))\cap L^\infty([0,\frac{T}{\max\{\varepsilon,\beta\}}];Y^s_\mu(\mathbb{R}))$ for $s'\in[0,s)$. Thus, we conclude that there exists a limit by completeness.

Step 4: *The solution is bounded by the initial data.* We claim that the solution obtained in Step 3 satisfies (1.9).

Indeed, using the notation from the previous step, we deduce by (6.7) that

$$\{(\zeta^\nu, u^\nu)\}_{0<\nu\leq 1}\subset C([0,\frac{T}{\max\{\varepsilon,\beta\}}];Y^s_\mu(\mathbb{R})),$$

is a bounded sequence in a reflexive Banach space. As a result, we have by Eberlein-Šmulian's Theorem that $(\zeta^\nu, v^\nu)\underset{\nu\to 0}{\rightharpoonup}(\zeta,v)$ weakly in $Y^s_\mu(\mathbb{R})$ for a.e. $t\in[0,\frac{T}{\max\{\varepsilon,\beta\}}]$. In particular, we have that

$$\sup_{t\in[0,\frac{T}{\max\{\varepsilon,\beta\}}]}|(\zeta,v)|_{V^s_\mu}\leq \liminf_{\nu\searrow 0}\sup_{t\in[0,\frac{T}{\max\{\varepsilon,\beta\}}]}|(\zeta^\nu,v^\nu)|_{Y^s_\mu}\lesssim|(\zeta_0,v_0)|_{Y^s_\mu}. \tag{6.10}$$

Step 5: *Persistence and continuity of the flow.* There is a solution $(\zeta,v)\in C([0,\frac{T}{\max\{\varepsilon,\beta\}}];V^s_\mu(\mathbb{R}))$ of (1.1) that depends continuously on the initial data.

For the proof of this step, we define a new sequence of functions (ζ^δ,v^δ) solving (1.1), with mollified initial data, i.e.

$$(\zeta^\delta_0,v^\delta_0)=(\chi_\delta(\mathrm{D})\zeta_0,\chi_\delta(\mathrm{D})v_0)\in H^\infty(\mathbb{R}):=\cap_{s>0}H^s(\mathbb{R}).$$

Reapplying the arguments of Step 1 and Step 2, combined with Proposition 2.7, one can deduce that

$$(\zeta^\delta,v^\delta)\in C([0,\frac{T}{\max\{\varepsilon,\beta\}}];H^\infty(\mathbb{R})),$$

satisfying (6.10). Now that the sequence is well-defined one can again define the difference between two solutions and use Proposition 2.7, together with Proposition 5.1 and Remark 5.2 to deduce the result. As mentioned above, at this stage in the proof, the argument is classical and the details can be found in e.g. [3, 32, 36].

□

7. Justification of (1.1) as a water waves model

We now give the proof of Theorem 1.6.

Proof. First, we let $s\geq 4$ and take initial data $(\zeta_0,\psi_0)\in H^s(\mathbb{R})\times\dot{H}^s(\mathbb{R})$ and $b\in H^{s+2}(\mathbb{R})$. Then the solutions of the water waves equations (1.10):

$$(\zeta,\psi)\in C([0,\frac{\tilde{T}}{\max\{\varepsilon,\beta\}}];H^s(\mathbb{R})\times\dot{H}^s(\mathbb{R})),$$

are given by Theorem 4.16 in [1]. Moreover, we can define $\overline{V}\in C([0,\frac{\tilde{T}}{\max\{\varepsilon,\beta\}}];X^s_\mu(\mathbb{R}))$. Now, use Proposition 3.2 and formulation (4.1) to say that for some $\tilde{T}>0$ the functions $\mathbf{U}=(\zeta,\overline{V})^T$ solves

$$\partial_t\mathbf{U}+(M_1+(S^{-1}M_2))(\mathbf{U})\partial_x\mathbf{U}+(S^{-1}Q)(\mathbf{U})+(S^{-1}Q_b)(\mathbf{U})=\mu^2(\varepsilon+\beta)\mathbf{R},$$

for any $t\in[0,\frac{\tilde{T}}{\max\{\varepsilon,\beta\}}]$ and with S,M_1,M_2,Q,Q_b defined as in (4.1) and $\mathbf{R}=(0,R)\in L^\infty([0,\frac{\tilde{T}}{\max\{\varepsilon,\beta\}}];X^r_\mu(\mathbb{R}))$ for some $r\in\mathbb{N}$.

The next step is to let $v_0^{\mathrm{WGN}} = \overline{V}|_{t=0} \in X^s_\mu(\mathbb{R})$ and then use Theorem 1.4 deduce the existence of $T > 0$ such that

$$\mathbf{U}^{\mathrm{WGN}} = (\zeta^{\mathrm{WGN}}, v^{\mathrm{WGN}}) \in C([0, \frac{T}{\max\{\varepsilon,\beta\}}]; Y^s_\mu(\mathbb{R})),$$

solves system (4.1):

$$\partial_t \mathbf{U}^{\mathrm{WGN}} + (M_1 + (S^{-1}M_2))(\mathbf{U}^{\mathrm{WGN}})\partial_x \mathbf{U}^{\mathrm{WGN}} + (S^{-1}Q)(\mathbf{U}^{\mathrm{WGN}}) + (S^{-1}Q_b)(\mathbf{U}^{\mathrm{WGN}}) = \mathbf{0},$$

for any $t \in [0, \frac{T}{\max\{\varepsilon,\beta\}}]$. Consequently, taking the difference between the two solutions

$$\mathbf{W} = (\eta, w)^T = \mathbf{U} - \mathbf{U}^{\mathrm{WGN}},$$

we obtain the following system

$$\partial_t \mathbf{W} + (M_1 + S^{-1}M_2)(\mathbf{U})\partial_x \mathbf{W} = \tilde{\mathbf{F}}, \tag{7.1}$$

similar to (7.1) and with

$$\begin{aligned}\tilde{\mathbf{F}} = &-\Big[(M_1 + (S^{-1}M_2))(\mathbf{U}) - (M_1 + S^{-1}M_2)(\mathbf{U}^{\mathrm{WGN}})\Big]\partial_x \mathbf{U}^{\mathrm{WGN}} \\ &- \Big[(S^{-1}(Q+Q_b))(\mathbf{U}) - (S^{-1}(Q+Q_b))(\mathbf{U}^{\mathrm{WGN}})\Big] + \mu^2(\varepsilon+\beta)\mathbf{R} \\ = &\ \mathbf{F} + \mu^2(\varepsilon+\beta)\mathbf{R},\end{aligned}$$

for any $t \in [0, \frac{\min\{\tilde{T},T\}}{\max\{\varepsilon,\beta\}}]$. Then using the estimates (5.6),(5.7), and Remark 5.2 we deduce for $r > \frac{3}{2}$ that

$$\begin{aligned}\frac{d}{dt}\tilde{E}_r(\mathbf{W}) &\lesssim |\big(\Lambda^r \tilde{\mathbf{F}}, S(\mathbf{U})\Lambda^r \mathbf{W}\big)_{L^2}| + N(r)\tilde{E}_r(\mathbf{W}) \\ &\lesssim \mu^2(\varepsilon+\beta)|\big(\Lambda^r R, \mathscr{T}[h,\beta b]\Lambda^r w\big)_{L^2}| + N(r+1)\tilde{E}_r(\mathbf{W}).\end{aligned}$$

However, by definition of $\mathscr{T}[h,\beta b]$ and using integration by parts, Hölder's inequality and the Sobolev embedding we easily obtain the estimate

$$|\big(\Lambda^r R, \mathscr{T}[h,\beta b]\Lambda^r w\big)_{L^2}| \lesssim N(r)|R|_{X^r_\mu}|w|_{X^r_\mu}.$$

Gathering these estimates, together with (5.7), we observe

$$\frac{d}{dt}\tilde{E}_r(\mathbf{W}) \lesssim \mu^2(\varepsilon+\beta)|R|_{X^r_\mu}(\tilde{E}_r(\mathbf{W}))^{\frac{1}{2}} + N(r+1)\tilde{E}_r(\mathbf{W}).$$

Now, a simple application of Grönwall's inequality and (5.7) yields

$$|(\eta, w)|_{Y^r_\mu} \lesssim \mu^2(\varepsilon+\beta)t\,|R|_{X^r_\mu}e^{N(r+1)t}. \tag{7.2}$$

Finally, to conclude we use that $Y^r_\mu(\mathbb{R}) \subset H^r(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for $r > \frac{3}{2}$, and (7.2) to get

$$\begin{aligned}|\mathbf{U} - \mathbf{U}^{\mathrm{WGN}}|_{L^\infty([0,t];\mathbb{R})} &\lesssim |(\eta,w)|_{L^\infty([0,t];Y^r_\mu(\mathbb{R}))} \\ &\lesssim \mu^2(\varepsilon+\beta)t\,|R|_{X^r_\mu}e^{N(r+1)t}.\end{aligned}$$

To conclude, we let s be large enough such that $r+1 < s$ to get that

$$|\mathbf{U} - \mathbf{U}^{\mathrm{WGN}}|_{L^\infty([0,t];\mathbb{R})} \lesssim \mu^2(\varepsilon+\beta)t,$$

for all $t \in [0, \frac{\min\{\tilde{T},T\}}{\max\{\varepsilon,\beta\}}]$.

□

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Paper III

2.3 Rigorous derivation of weakly dispersive shallow water models with large amplitude topography variations

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RIGOROUS DERIVATION OF WEAKLY DISPERSIVE SHALLOW WATER MODELS WITH LARGE AMPLITUDE TOPOGRAPHY VARIATIONS

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Abstract. We derive rigorously from the water waves equations new irrotational shallow water models for the propagation of surface waves in the case of uneven topography in horizontal dimensions one and two. The systems are made to capture the possible change in the waves' propagation, which can occur in the case of large amplitude topography. The main contribution of this work is the construction of new multi-scale shallow water approximations of the Dirichlet-Neumann operator. We prove that the precision of these approximations is given at the order $\mathcal{O}(\mu\varepsilon)$, $\mathcal{O}(\mu\varepsilon+\mu^2\beta^2)$ and $\mathcal{O}(\mu^2\varepsilon+\mu\varepsilon\beta+\mu^2\beta^2)$. Here μ, ε, and β denote respectively the shallow water parameter, the nonlinear parameter, and the bathymetry parameter. From these approximations, we derive models with the same precision as the ones above. The model with precision $\mathcal{O}(\mu\varepsilon)$ is coupled with an elliptic problem, while the other models do not present this inconvenience.

1. Introduction

1.1. **Motivations.** The general model of surface waves in coastal oceanography is often considered too complex to be used in practical situations. As a result, the simplification of the water waves equations in specific asymptotic regimes has been a subject of active research. In the derivation of asymptotic models, one considers characteristic quantities of the system under study. In our paper, we will denote by H_0, L, a_{surf} and a_{bott}, the characteristic water depth, the characteristic wavelength in the longitudinal direction, the characteristic surface amplitude and the characteristic amplitude of the bathymetry of the system. From these characteristic quantities, we define the following non-dimensional parameters

$$\mu := \frac{H_0^2}{L^2}, \quad \varepsilon := \frac{a_{\text{surf}}}{H_0}, \quad \beta := \frac{a_{\text{bott}}}{H_0}.$$

We will focus on the shallow water regime defined by $\mu \ll 1$ and the weakly nonlinear regime $\varepsilon \ll 1$, in the case of uneven topography.

Numerous shallow water models were derived in the literature, giving approximations of the solutions of the water waves system in the shallow water regime or long wave regime $\mu \sim \varepsilon \ll 1$, at the order of precision $\mathcal{O}(\mu^k)$ with $k = 1, 2$ or 3. However, it is not clear that these classical models capture the change in the propagation of the waves which can happen in the case of large amplitude topographies. Such occurrences have been studied in the Dingemans experiments [9]. In these experiments, the authors investigate a long wave passing over a submerged obstacle. They observed that waves tend to steepen due to a compression effect from the bottom, where high harmonics generated by topography-induced nonlinear interactions are freely released behind the obstacle. This last phenomenon makes it natural to improve the frequency dispersion of the classical shallow water models.

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The ideal precision of the resulting model would be of the form $\mathcal{O}(\mu^k\varepsilon^l)$, with $k, l \geq 1$, in order to capture both the shallow water and the weakly non-linear regimes.

One way to improve the frequency dispersion is to consider multi-parameters Boussinesq or Green-Naghdi models; see [14, 17] for a comparison between the classical Boussinesq and Green-Naghdi models with their multi-parameters versions in the case of the Dingemans experiments. The improved frequency dispersion allowed the authors to describe strongly dispersive waves with uneven bathymetry accurately. However, the order of precision of these multi-parameters systems is the same as the classical ones.

In the flat bottom case, another way to improve the frequency dispersion of the classical shallow water models is to consider full dispersion models, for which the dispersion relation is the same as the one of the water waves equations:

$$\omega_{\mathrm{WW}}(\xi)^2 = \frac{\tanh\left(\sqrt{\mu}|\xi|\right)}{\sqrt{\mu}|\xi|}|\xi|^2.$$

In [11], the author rigorously derived these models at the order of precision $\mathcal{O}(\mu\varepsilon)$ and $\mathcal{O}(\mu^2\varepsilon)$. To obtain this non-trivial order of precision, it is fundamental to keep the exact dispersion relation. In comparison, for the classical Boussinesq and Green-Naghdi model, the dispersion relation is

$$\omega_{\mathrm{B}}(\xi)^2 = \big(1 - \frac{\mu}{3}|\xi|^2\big)|\xi|^2, \quad \omega_{\mathrm{GN}}(\xi)^2 = \frac{1}{1+\frac{\mu}{3}|\xi|^2}|\xi|^2,$$

so that by a Taylor expansion, one makes errors of order $\mathcal{O}(\mu^2)$ from the approximation of the dispersion relation of the water waves equations.

In [10], the author extended the work in [11] in the case of variable bottom. He derived models with a precision of order $\mathcal{O}(\mu\varepsilon + \mu\beta)$ when compared to the water waves equations for a class of weakly dispersive Boussinesq system, and a precision order $\mathcal{O}(\mu^2\varepsilon + \mu^2\beta)$ with respect to the water waves equations for a class of weakly dispersive Green-Naghdi systems.

The first result of this paper is the rigorous derivation of an extension of the full dispersion models in the case of uneven bathymetries at the order of precision $\mathcal{O}(\mu\varepsilon)$. This model reads

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}\mathcal{G}_b\psi = 0 \\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}|\nabla_X\psi|^2 = 0, \end{cases} \tag{1.1}$$

where $\zeta(t,X) \in \mathbb{R}$ represents the water surface elevation, $b(X) \in \mathbb{R}$ represents the bottom elevation, $\psi(t,X) \in \mathbb{R}$ is the trace at $z = 0$, where z is the variable, of the potential ϕ, solving

$$\begin{cases} \Delta^{\mu}_{X,z}\phi = 0 \quad \text{in} \quad \mathbb{R}^d \times [-1+\beta b, 0], \\ \phi|_{z=0} = \psi, \quad \big[\partial_z\phi - \mu\beta\nabla_X b \cdot \nabla_X\phi\big]\big|_{z=-1+\beta b} = 0, \end{cases} \tag{1.2}$$

and where $\mathcal{G}_b$ is an operator given by

$$\frac{1}{\mu}\mathcal{G}_b\psi = -\nabla_X \cdot \Big(\frac{1+\varepsilon\zeta-\beta b}{1-\beta b}\int_{-1+\beta b}^{0} \nabla_X\phi \,\mathrm{d}z\Big). \tag{1.3}$$

The model (1.1) can be viewed as a simplified version of the water waves model, where the elliptic problem is given on a fixed domain independent of time. The precision of the model is $\mathcal{O}(\mu\varepsilon)$ and makes it an ideal extension of the full dispersion models in the case of a variable bottom with which one can capture the change of behavior in the propagation of the wave during the aforementioned Dingemans experiments. A drawback from a numerical point of view would be that one would need to solve an elliptic problem at each time step when computing the solutions of the model.

To simplify the model even further, one can construct explicit approximations of the solutions of the elliptic problem (1.2). From these approximations, one deduce expansions of the Dirichlet-Neumman operator at the same order of precision. In [8], the authors derived such approximations of the Dirichlet-Neumann operator in the long wave regime/ small amplitude waves regime using an explicit formula for the solution of the elliptic problem (1.2), where the formula depends on the inversion of a pseudo-differential operator. They make use of these approximations to derive a Boussinesq type model. Their result is formal and holds only in horizontal dimension one. The extension in the variable bottom case of full dispersion models was considered in [6] in the case of horizontal dimension one. The authors used the result in [8] to formally derive three models in the shallow water regime with order of precision $\mathcal{O}(\mu\varepsilon + \varepsilon^2)$. A drawback is that their models depend on the inversion of a pseudo-differential operator and consequently seem to create instabilities in the simulations. Moreover, if one inverts a pseudo-differential operator, it is not clear how one could quantify the error of approximation in the Sobolev spaces uniformly in the parameters μ, ε and β and then make the derivation rigorous with the correct order of precision.

In [7], the author derived rigorously, from the water waves equations, a classical type Boussinesq system in the long wave regime with an order of precision $\mathcal{O}(\mu^2)$ when $\beta = \mathcal{O}(1)$. Translated in the shallow water regime, the precision is $\mathcal{O}(\mu^2 + \mu\varepsilon + \mu^2\beta^2)$. One should also note that the bathymetry related terms in the aforementioned system are of higher order when compared to the linear terms. Therefore, the well-posedness of such a system is not clear.

In the present work, we construct new shallow water approximations of the Dirichlet-Neumann operator at the order of precision $\mathcal{O}(\mu\varepsilon)$, $\mathcal{O}(\mu\varepsilon + \mu^2\beta^2)$ and $\mathcal{O}(\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)$. We also quantify the error in the Sobolev spaces uniformly in μ, ε and β. With these approximations, we prove that system (1.1) is consistent with the water waves equations at order $\mathcal{O}(\mu\varepsilon)$. Then we derive new weakly-dispersive Boussinesq type systems with the order of precision $\mathcal{O}(\mu\varepsilon + \mu^2\beta^2)$, with respect to the water waves equations. In addition, we derive new weakly-dispersive Green-Naghdi type systems with the order of precision $\mathcal{O}(\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)$. We emphasize the fact that the orders of precision are non-trivial in terms of the bathymetry parameter. Contrary to the models presented in[7], the contribution of the bathymetry terms does not contain higher order derivatives when compared to the linear terms. Moreover, they have a similar quasi-linear hyperbolic structure as the full dispersion models in the flat bottom case. We expect then, in light of the recent works of [5, 4], to be able to prove a long time well-posedness result for these models. This will be an objective for future work. Lastly, we discuss the derivation of extensions that have a Hamiltonian structure.

Notations 1.1.

- *Let* Id *be the* $d \times d$ *identity matrix, and take* $\mathbf{0} = (0,0)^T$ *if* $d = 2$, $\mathbf{0} = 0$ *if* $d = 1$. *Then we define the* $(d+1) \times (d+1)$ *matrix* I^μ *by*

$$I^\mu = \begin{pmatrix} \sqrt{\mu}\mathrm{Id} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}.$$

- *We define the* d*-dimensional Laplace operator by*

$$\Delta_X = \begin{cases} \partial_x^2 & \text{when} \quad d = 1 \\ \partial_x^2 + \partial_y^2 & \text{when} \quad d = 2. \end{cases}$$

- *We define the $(d+1)$-dimensional scaled gradient by*
$$\nabla^{\mu}_{X,z} = I^{\mu}\nabla_{X,z} = \begin{cases} (\sqrt{\mu}\partial_x, \partial_z)^T & \text{when} \quad d=1 \\ (\sqrt{\mu}\partial_x, \sqrt{\mu}\partial_y, \partial_z)^T & \text{when} \quad d=2, \end{cases}$$
and we introduce the scaled Laplace operator
$$\Delta^{\mu}_{X,z} = \nabla^{\mu}_{X,z}\cdot\nabla^{\mu}_{X,z} = \mu\Delta_X + \partial_z^2.$$
- *Let $f:\mathbb{R}^d\to\mathbb{R}$ be a tempered distribution, let $\hat{f}$ or $\mathcal{F}f$ be its Fourier transform and $\mathcal{F}^{-1}f$ be its inverse Fourier transform.*
- *For any $s\in\mathbb{R}$ we call the multiplier $\Lambda^s = (1+|\mathrm{D}|^2)^{\frac{s}{2}} = \langle \mathrm{D}\rangle^s$ the Bessel potential of order $-s$.*
- *The Sobolev space $H^s(\mathbb{R}^d)$ is equivalent to the weighted L^2-space with $|f|_{H^s} = |\Lambda^s f|_{L^2}$.*
- *For any $s\geq 1$ we will denote $\dot{H}^s(\mathbb{R}^d)$ the Beppo-Levi space with $|f|_{\dot{H}^s} = |\Lambda^{s-1}\nabla_X f|_{L^2}$.*
- *Let $\Omega\subset\mathbb{R}^{d+1}$. For any $k\in\mathbb{N}$, we define the space $H^{k,0}(\Omega)$ with norm*
$$\|f\|^2_{H^{k,0}(\Omega)} = \sum_{|\gamma|\leq k}\int_{\Omega}|\partial_X^{\gamma}f(X,z)|^2\,\mathrm{d}z\mathrm{d}X,$$
and similarly, for $l\in\mathbb{N}$ such that $l\leq k$, we define the space $H^{k,l}(\Omega)$ with norm $\|f\|_{H^{k,l}(\Omega)} = \sum\limits_{j=0}^{l}\|\partial_z^j f\|_{H^{k-j,0}(\Omega)}$.
- *We say that f is a Schwartz function $\mathscr{S}(\mathbb{R}^d)$, if $f\in C^{\infty}(\mathbb{R}^d)$ and satisfies for all $\alpha,\beta\in\mathbb{N}^d$,*
$$\sup_{X\in\mathbb{R}^d}|X^{\alpha}\partial_X^{\beta}f| < \infty.$$
- *If A and B are two operators, then we denote the commutator between them to be $[A,B] = AB - BA$.*
- *We let c denote a positive constant independent of μ,ε,β that may change from line to line. Also, as a shorthand, we use the notation $a\lesssim b$ to mean $a\leq c\,b$.*
- *Let $t_0 > \frac{d}{2}$, $s\geq 0$, $h_{\min}, h_{b,\min}\in(0,1)$. Then for $\zeta, b, \nabla_X\psi$ sufficiently regular and $C(\cdot)$ a positive, non-decreasing function of its argument, we define the constants*
$$\begin{aligned} M_0 &= C(\frac{1}{h_{\min}}, \frac{1}{h_{b,\min}}, |\zeta|_{H^{t_0}}, |b|_{H^{t_0}}) \\ M(s) &= C(M_0, |\zeta|_{H^{\max\{t_0+2,s\}}}, |b|_{H^{\max\{t_0+2,s\}}}) \\ N(s) &= C(M(s), |\nabla_X\psi|_{H^s}). \end{aligned}$$

1.2. **The consistency problem and main results.** Throughout this paper, d will be the dimension of the horizontal variable, denoted $X\in\mathbb{R}^d$. The reference model of our study is the water waves equations, written under the Zakharov-Craig-Sulem formulation:
$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}\mathcal{G}^{\mu}[\varepsilon\zeta,\beta b]\psi = 0 \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla_X\psi|^2 - \frac{\mu\varepsilon}{2}\frac{(\frac{1}{\mu}\mathcal{G}^{\mu}[\varepsilon\zeta,\beta b]\psi + \varepsilon\nabla_X\zeta\cdot\nabla_X\psi)^2}{1+\varepsilon^2\mu|\nabla_X\zeta|^2} = 0. \end{cases} \tag{1.4}$$
Here the free surface elevation is the graph of $\zeta(t,X)$, which is a function of time t and horizontal space $X\in\mathbb{R}^d$. The bottom elevation is the graph of $b(X)$, which is a time-independent function. The function $\psi(t,X)$ is the trace at the surface of the velocity potential, and $\mathcal{G}^{\mu}$ is the Dirichlet-to-Neumann operator defined later in Definition 1.3.

Moreover, every variable and function in (1.4) is compared with physical characteristic parameters of the same dimension $H_0, a_{\mathrm{surf}}, a_{\mathrm{bott}}$ or L.

Throughout the paper, we will always make the following fundamental assumption:

Definition 1.2 (Non-cavitation condition)**.** *Let $\varepsilon \in [0,1]$, $\beta \in [0,1]$ and $s > \frac{d}{2}$. Let also $b \in C_c^\infty(\mathbb{R}^d)$ be a smooth function with compact support, and take $\zeta \in H^s(\mathbb{R}^d)$. We say ζ and b satisfies the "non-cavitation condition" if there exists $h_{min} \in (0,1)$ such that*

$$h := 1 + \varepsilon\zeta(X) - \beta b(X) \geq h_{min}, \quad \textit{for all} \;\; X \in \mathbb{R}^d. \tag{1.5}$$

Under the non-cavitation condition, we may define the Dirichlet-Neumann operator by [14]:

Definition 1.3. *Let $t_0 > \frac{d}{2}$, $\psi \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$, $b \in C_c^\infty(\mathbb{R}^d)$, and $\zeta \in H^{t_0+1}(\mathbb{R}^d)$ be such that (1.5) is satisfied. Let Φ be the unique solution in $\dot{H}^2(\Omega_t)$ of the boundary value problem*

$$\begin{cases} \Delta^\mu_{X,z}\Phi = 0 & \textit{in} \quad \Omega_t := \{(X,z) \in \mathbb{R}^{d+1}, -1+\beta b(X) < z < \varepsilon\zeta(X)\} \\ \partial_{n_b}\Phi = 0 & \textit{on} \quad z = -1 + \beta b(X) \\ \Phi = \psi & \textit{on} \quad z = \varepsilon\zeta(t,X), \end{cases} \tag{1.6}$$

where

$$\partial_{n_b} = \mathbf{n}_b \cdot I^\mu \nabla^\mu_{X,z}, \qquad \mathbf{n}_b = \frac{1}{\sqrt{1+\beta^2|\nabla_X b|^2}} \begin{pmatrix} -\beta\nabla_X b \\ 1 \end{pmatrix},$$

then $\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi \in H^{\frac{1}{2}}(\mathbb{R}^d)$ is defined by

$$\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi = (\partial_z\Phi - \mu\varepsilon\nabla_X\zeta \cdot \nabla_X\Phi)_{|z=\varepsilon\zeta}. \tag{1.7}$$

For convenience, it is easier to work with the vertical average of the horizontal component of the velocity. We make the following definition using Proposition 3.35 in [14].

Definition 1.4. *Let $t_0 > \frac{d}{2}$, $\psi \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$, $b \in C_c^\infty(\mathbb{R}^d)$, and $\zeta \in H^{t_0+1}(\mathbb{R}^d)$ such that (1.5) is satisfied. Let $\Phi \in \dot{H}^2(\Omega_t)$ be the solution of (1.6), then we define the operator:*

$$\overline{V}^\mu[\varepsilon\zeta, \beta b]\psi = \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta} \nabla_X\Phi \,\mathrm{d}z, \tag{1.8}$$

and the following relation holds,

$$\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi = -\mu\nabla_X \cdot (h\overline{V}^\mu[\varepsilon\zeta, \beta b]\psi). \tag{1.9}$$

Throughout this paper, we will denote $\overline{V}^\mu[\varepsilon\zeta, \beta b]\psi$ by $\overline{V}$ when no confusion is possible.

In order to write the main results of this paper, we need to define two types of differential operators. The first type is the Fourier multipliers.

Definition 1.5. *Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be a tempered distribution, and let $\widehat{u}$ be its Fourier transform. Let $F : \mathbb{R}^d \to \mathbb{R}$ be a smooth function with polynomial decay. Then the Fourier multiplier associated with $F(\xi)$ is denoted* $\mathrm{F(D)}$ *(denoted* F *when no confusion is possible) and defined by the formula:*

$$\widehat{\mathrm{F(D)}u}(\xi) = F(\xi)\widehat{u}(\xi).$$

Definition 1.6. *Let F_0 be a Fourier multiplier depending on the transverse variable:*

$$\mathrm{F}_0 u(X) = \mathcal{F}^{-1}\Big(\frac{\cosh((z+1)\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\hat{u}(\xi)\Big)(X),$$

for $z \in [-2,0]$. *We also define the four Fourier multipliers* F_1, F_2, F_3 *and* F_4 *by the expressions:*

$$\mathrm{F}_1 = \frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{\sqrt{\mu}|\mathrm{D}|}, \quad \mathrm{F}_2 = \frac{3}{\mu|\mathrm{D}|^2}(1-\mathrm{F}_1), \quad \mathrm{F}_3 = \mathrm{sech}(\sqrt{\mu}|D|), \quad \mathrm{F}_4 = \frac{2}{\mu|\mathrm{D}|^2}(1-\mathrm{F}_3).$$

Next, we would like to define operators of the form

$$\mathcal{L}[X,D]u(X) := \mathcal{F}^{-1}\big(L(X,\xi)\hat{u}(\xi)\big)(X), \tag{1.10}$$

where L is a smooth function in a particular symbol class given in the next definition.

Definition 1.7. *Let* $d = 1,2$ *and* $m \in \mathbb{R}$. *We say* $L \in S^m$ *is a symbol of order* m *if* $L(X,\xi)$ *is* $C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ *and satisfies*

$$\forall \alpha \in \mathbb{N}^d, \quad \forall \gamma \in \mathbb{N}^d, \quad \langle \xi \rangle^{-(m-|\gamma|)} |\partial_X^\alpha \partial_\xi^\gamma L(X,\xi)| < \infty.$$

We also introduce the seminorm

$$\mathcal{M}_m(L) = \sup_{|\alpha| \leq \lceil \frac{d}{2} \rceil + 1} \sup_{|\gamma| \leq \lceil \frac{d}{2} \rceil + 1} \sup_{(X,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} \Big\{ \langle \xi \rangle^{-(m-|\gamma|)} |\partial_X^\alpha \partial_\xi^\gamma L(X,\xi)| \Big\}. \tag{1.11}$$

The next result allows us to justify the formula (1.10) for functions u in Sobolev spaces.

Theorem 1.8. *Let* $d = 1,2$, $s \geq 0$, *and* $L \in S^m$. *Then formula* (1.10) *defines a bounded pseudo-differential operator of order* m *from* $H^{s+m}(\mathbb{R}^d)$ *to* $H^s(\mathbb{R})$ *and satisfies*

$$|\mathcal{L}[X,D]u|_{H^s} \leq \mathcal{M}_m(L)|u|_{H^{s+m}}. \tag{1.12}$$

We refer to [2] for this result, where the constant is given implicitly in the proof (see also [15, 1]). We will define operators of interest under the assumption:

Assumption/Definition 1.9. *Let* $d = 1,2$ *and* $\beta \in [0,1]$. *Throughout this paper, we will always assume that the bathymetry* $\beta b \in C_c^\infty(\mathbb{R}^d)$ *satisfies the following: There exists* $b_{\max} \in (0,1)$ *such that*

$$\beta|b(X)| \leq b_{\max} < 1, \quad \textit{for all } \ X \in \mathbb{R}^d. \tag{1.13}$$

We also define the water depth at the rest state $h_b := 1 - \beta b(X)$. *As a consequence of* (1.13), *there exists a constant* $h_{b,\min} \in (0,1)$ *such that*

$$0 < h_{b,\min} \leq h_b. \tag{1.14}$$

Through (1.14), we suppose the bottom topography is submerged under the still water level. We may now define the pseudo-differential operators that will play an important role in deriving new models that allow for large amplitude topography variations.

Definition/Proposition 1.10. *Let* $\mu, \beta \in [0,1]$, $d = 1,2$, $s \geq 0$ *and* $b \in C_c^\infty(\mathbb{R}^d)$ *such that* (1.13) *is satisfied. We define the following pseudo-differential operators of order zero, bounded uniformly with respect to* μ *and* β *in* $H^s(\mathbb{R}^d)$:

$$\mathcal{L}_1^\mu[\beta b] = -\frac{1}{\beta}\sinh(\beta b(X)\sqrt{\mu}|\mathrm{D}|)\mathrm{sech}(\sqrt{\mu}|\mathrm{D}|)\frac{1}{\sqrt{\mu}|\mathrm{D}|}$$

$$\mathcal{L}_2^\mu[\beta b] = -(\mathcal{L}_1^\mu[\beta b] + b)\frac{1}{\mu|\mathrm{D}|^2}$$

$$\mathcal{L}_3^\mu[\beta b] = -\big(\cosh(\beta b(X)\sqrt{\mu}|\mathrm{D}|)\mathrm{sech}(\sqrt{\mu}|\mathrm{D}|) - 1\big)\frac{1}{\mu|\mathrm{D}|^2}.$$

Moreover, for $u \in \mathscr{S}(\mathbb{R}^d)$ we have the following estimates

$$|\mathcal{L}_1^\mu[\beta b]u|_{H^s} \leq M(s)|u|_{H^s}. \tag{1.15}$$

$$|\mathcal{L}_2^\mu[\beta b]u|_{H^s} \leq M(s)|u|_{H^s} \tag{1.16}$$

$$|\mathcal{L}_3^\mu[\beta b]u|_{H^s} \leq M(s)|u|_{H^s} \tag{1.17}$$

$$|\mathcal{L}_1^\mu[\beta b]u + bu|_{H^s} \leq \mu M(s)|u|_{H^{s+2}} \tag{1.18}$$

$$|\mathcal{L}_1^\mu[\beta b]u - (-b - \frac{\mu\beta^2}{6}b^3|\mathrm{D}|^2)\mathrm{F}_3 u|_{H^s} \leq \mu^2\beta^4 M(s)|u|_{H^{s+4}} \tag{1.19}$$

$$|\mathcal{L}_2^\mu[\beta b]u - (-\frac{1}{2}b\mathrm{F}_4 + \frac{\beta^2}{6}b^3\mathrm{F}_3)u|_{H^s} \leq \mu\beta^4 M(s)|u|_{H^{s+2}}. \tag{1.20}$$

Remark 1.11. *Under assumption (1.13) the operators $\mathcal{L}_1^\mu$, $\mathcal{L}_2^\mu$, and $\mathcal{L}_3^\mu$ are "classical pseudo-differential operators of order zero". We will share the details of the proof in Appendix A, Subsection A.1.*

Proposition 1.12. *Let $t_0 > \frac{d}{2}$, $\psi \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$, $b \in C_c^\infty(\mathbb{R}^d)$, and $\zeta \in H^{t_0+1}(\mathbb{R}^d)$ such that (1.5) is satisfied. Let $\phi \in \dot{H}^2(\mathbb{R}^d \times [-1+\beta b, 0])$ be the solution of*

$$\begin{cases} \Delta_{X,z}^\mu \phi = 0 \quad \text{in} \quad \mathcal{S}_b, \\ \phi|_{z=0} = \psi, \quad \big[\partial_z\phi - \mu\beta\nabla_X b \cdot \nabla_X\phi\big]\big|_{z=-1+\beta b} = 0. \end{cases} \tag{1.21}$$

Then we can define

$$\frac{1}{\mu}\mathcal{G}_b\psi = -\nabla_X \cdot \Big(\frac{h}{h_b}\int_{-h_b}^0 \nabla_X\phi \,\mathrm{d}z\Big). \tag{1.22}$$

Moreover, for $\psi \in \dot{H}^{s+5}(\mathbb{R}^d)$ and $\zeta \in H^{\max\{t_0+2,s+3\}}(\mathbb{R}^d)$ we have the estimate

$$\frac{1}{\mu}|\mathcal{G}^\mu\psi - \mathcal{G}_b\psi|_{H^s} \leq \mu\varepsilon M(s+3)|\nabla_X\psi|_{H^{s+4}}.$$

Remark 1.13. *The operator $\mathcal{G}_b$ contains terms of order $\varepsilon\zeta$ and is different from $\mathcal{G}^\mu[0,\beta b]$ defined by (1.9). To be precise, we can relate the two operators by expanding $\frac{h}{h_b}$:*

$$\frac{1}{\mu}\mathcal{G}_b\psi = \frac{1}{\mu}\mathcal{G}^\mu[0,\beta b]\psi - \varepsilon\nabla_X \cdot \Big(\frac{\zeta}{h_b}\int_{-h_b}^0 \nabla_X\phi\,\mathrm{d}z\Big).$$

Proposition 1.14. *Let $d = 1,2$, $t_0 > \frac{d}{2}$ and $s \geq 0$. Also let $b \in C_c^\infty(\mathbb{R}^d)$ and $\zeta \in H^{\max\{t_0+2,s+3\}}(\mathbb{R}^d)$ such that (1.5) and (1.13) are satisfied. From the previously defined operators, we have the following approximations of the Dirichlet-Neumann operator:*

$$\begin{aligned}\frac{1}{\mu}\mathcal{G}_0\psi = &-\mathrm{F}_1\Delta_X\psi - \beta(1 + \frac{\mu}{2}\mathrm{F}_4\Delta_X)\nabla_X \cdot \big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) - \varepsilon\nabla_X \cdot \big(\zeta\mathrm{F}_1\nabla_X\psi\big) \\ &+ \frac{\mu\beta^2}{2}\nabla_X \cdot \big(\mathcal{B}[\beta b]\nabla_X\psi\big),\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\mu}\mathcal{G}_1\psi = &-\nabla_X \cdot (h\nabla_X\psi) - \frac{\mu}{3}\Delta_X\Big(\frac{h^3}{h_b^3}\mathrm{F}_2\Delta_X\psi\Big) - \mu\beta\Delta_X\big(\mathcal{L}_2^\mu[\beta b]\Delta_X\psi\big) \\ &- \frac{\mu\beta}{2}\mathrm{F}_4\Delta_X\nabla_X \cdot \big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) + \frac{\mu\beta^2}{2}\nabla_X \cdot \big(\mathcal{B}[\beta b]\nabla_X\psi\big),\end{aligned}$$

where

$$\mathcal{B}[\beta b]\nabla_X\psi = b\mathrm{F}_4\nabla_X(\nabla_X\cdot(b\nabla_X\psi)) \tag{1.23}$$
$$+ h_b\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\nabla_X\psi)\big) + 2h_b(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\nabla_X\psi).$$

Moreover, we have the following estimates on the Dirichlet-Neumann operator

$$\frac{1}{\mu}|\mathcal{G}^\mu\psi - \mathcal{G}_0\psi|_{H^s} \leq (\mu\varepsilon + \mu^2\beta^2)M(s+3)|\nabla_X\psi|_{H^{s+5}} \tag{1.24}$$

$$\frac{1}{\mu}|\mathcal{G}^\mu\psi - \mathcal{G}_1\psi|_{H^s} \leq (\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)M(s+3)|\nabla_X\psi|_{H^{s+5}}. \tag{1.25}$$

Proposition 1.14 is the key result from which we will derive our new models. However, before presenting these models, we need to define the notion of consistency of the water waves equations (1.4) with a given asymptotic model.

Definition 1.15 (Consistency)**.** *Let $\mu,\varepsilon,\beta \in [0,1]$. We denote by* (A) *an asymptotic model of the following form:*

$$\text{(A)}\quad \begin{cases} \partial_t\zeta + \mathcal{N}_1(\zeta,b,\psi) = 0 \\ \partial_t(\mathcal{T}[\zeta,b]\psi) + \mathcal{N}_2(\zeta,b,\psi) = 0, \end{cases}$$

where $\mathcal{T}$ is a linear operator with respect to ψ and possibly nonlinear with respect to ζ and b. While $\mathcal{N}_1$ and $\mathcal{N}_2$ are possibly nonlinear operators.

We say that the water waves equations are consistent at order $\mathcal{O}(\sum \mu^k\varepsilon^l\beta^m)$ with (A) *if there exists $n \in \mathbb{N}$ and a universal constant $T > 0$ such that for any $s \geq 0$ and every solution $(\zeta,\psi) \in C([0,\frac{T}{\varepsilon}]; H^{s+n}(\mathbb{R}^d) \times \dot{H}^{s+n}(\mathbb{R}^d))$ to the water waves equations* (1.4)*, one has for all $t \in [0,\frac{T}{\varepsilon}]$,*

$$\begin{cases} \partial_t\zeta + \mathcal{N}_1(\zeta,b,\psi) = \big(\sum \mu^k\varepsilon^l\beta^m\big)R_1 \\ \partial_t(\mathcal{T}[\zeta,b]\psi) + \mathcal{N}_2(\zeta,b,\psi) = \big(\sum \mu^k\varepsilon^l\beta^m\big)R_2, \end{cases}$$

where $|R_i|_{H^s} \leq N(s+n)$ for all $t \in [0,\frac{T}{\varepsilon}]$ with $i = 1,2$.

We should note that the existence time for solutions of the water waves equations is proved to be on the scale $\mathcal{O}(\frac{1}{\max\{\varepsilon,\beta\}})$ and uniformly with respect to μ (see [3]). However, it was proved that when one includes surface tension with a strength of the same order as the shallow water parameter μ, then the time existence is improved and becomes of order $\mathcal{O}(\frac{1}{\varepsilon})$ [5]. For the sake of clarity, we will omit the surface tension in this paper. But one could easily add it to every model of this work without changing the results. With this in mind, we may now state our consistency results.

Theorem 1.16. *Let $\mathcal{G}_b$ be defined by* (2.22)*. Then for any $\mu \in (0,1]$, $\varepsilon \in [0,1]$, and $\beta \in [0,1]$ the water waves equations* (1.4) *are consistent, in the sense of Definition 1.15 with $n = 5$, at order $\mathcal{O}(\mu\varepsilon)$ with the Boussinesq-type system:*

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}\mathcal{G}_b\psi = 0 \\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}|\nabla_X\psi|^2 = 0, \end{cases} \tag{1.26}$$

Remark 1.17. *The system is a simplified version of the water waves equations where $\mathcal{G}_b$ is defined implicitly in terms of the solution of* (1.21)*. The elliptic problem is defined on the fixed domain $\mathcal{S}_b = \mathbb{R}^d \times [-1+\beta b, 0]$, but depends on the Dirichlet data ψ which in turn depend on ζ through* (3.1)*.*

Theorem 1.18. *Let* F_1 *and* F_4 *be the two Fourier multipliers given in Definition 1.6, and let* $\mathcal{L}_1^\mu$ *be given in Definition 1.10. Then for any* $\mu \in (0,1]$, $\varepsilon \in [0,1]$, *and* $\beta \in [0,1]$ *the water waves equations* (1.4) *are consistent, in the sense of Definition 1.15 with* $n = 6$, *at order* $\mathcal{O}(\mu\varepsilon + \mu^2\beta^2)$ *with the Boussinesq type system:*

$$\begin{cases} \partial_t\zeta + \mathrm{F}_1\Delta_X\psi + \beta(1+\frac{\mu}{2}\mathrm{F}_4\Delta_X)\nabla_X\cdot(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi) \\ \qquad\qquad +\varepsilon\mathrm{G}_1\nabla_X\cdot(\zeta\mathrm{G}_2\nabla_X\psi) - \frac{\mu\beta^2}{2}\nabla_X\cdot\big(\mathcal{B}[\beta b]\nabla_X\psi\big) = 0 \\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}(\mathrm{G}_1\nabla_X\psi)\cdot(\mathrm{G}_2\nabla_X\psi) = 0, \end{cases} \tag{1.27}$$

where

$$\mathcal{B}[\beta b]\nabla_X\psi = b\mathrm{F}_4\nabla_X(\nabla_X\cdot(b\nabla_X\psi)) + h_b\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\nabla_X\psi)\big) + 2h_b(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\nabla_X\psi),$$

and $\mathrm{G}_1, \mathrm{G}_2$ *are any Fourier multipliers such that for any* $s \geq 0$ *and* $u \in H^{s+2}(\mathbb{R}^d)$, *we have*

$$|(\mathrm{G}_j - 1)u|_{H^s} \lesssim \mu|u|_{H^{s+2}}. \tag{1.28}$$

Remark 1.19.

- *Taking* $\beta = 0$ *in* (3.4), *we get the class of full dispersion Boussinesq systems derived rigorously in* [11] *with a precision* $\mathcal{O}(\mu\varepsilon)$. *These systems were rigorously justified on a time scale of order* $\mathcal{O}(\frac{1}{\varepsilon})$ *under additional decrease constraint on the Fourier multipliers* G_1 *and* G_2 *(see* [12] *for more information).*
- *In the case* $\mathrm{G}_1 = \mathrm{G}_2 = \mathrm{Id}$, (3.4) *is believed to be ill-posed* [13] *in the case* $\beta = 0$, *unless one includes surface tension* [16]. *Alternatively, one can exploit the regularizing effect provided by the multipliers* G_j [12].
- *One could also add Fourier multipliers* G_j, *defined by* (1.28), *in the term of order* $\mathcal{O}(\mu\beta^2)$ *without changing the precision of the model.*
- *Neglecting terms of order* $\mathcal{O}(\mu\varepsilon+\mu\beta)$ *and approximating* $\mathcal{L}_1^\mu[\beta b]$ *with estimate* (1.18), *we arrive at the same models derived in* [10].

One can replace the pseudo-differential operator in (3.4) with estimate (1.19). Indeed, we have the following result:

Corollary 1.20. *Under the same assumptions as in Theorem 1.18, we can take*

$$\mathcal{L}_1^\mu[\beta b]\bullet = -(b + \frac{\mu\beta^2}{6}b^3|\mathrm{D}|^2)\mathrm{F}_3\bullet,$$

in system (3.4) *and keep the precision* $\mathcal{O}(\mu\varepsilon + \mu^2\beta^2)$.

We also derive a Boussinesq-type system in the variables $(\zeta, \overline{V})$:

Theorem 1.21. *Let* F_1 *and* F_4 *be the two Fourier multipliers given in Definition 1.6, and let* $\mathcal{L}_1^\mu$ *be given in Definition 1.10. Then for any* $\mu \in (0,1]$, $\varepsilon \in [0,1]$, *and* $\beta \in [0,1]$ *the water waves equations* (1.4) *are consistent, in the sense of Definition 1.15 with* $n = 7$, *at order* $\mathcal{O}(\mu\varepsilon + \mu^2\beta^2)$ *with the Boussinesq type system:*

$$\begin{cases} \partial_t\zeta + \nabla_X\cdot(h\overline{V}) = 0 \\ \partial_t\overline{V} + \mathcal{T}_0^\mu[\beta b, \varepsilon\zeta]\nabla_X\zeta + \frac{\varepsilon}{2}\nabla_X|\overline{V}|^2 = \mathbf{0}, \end{cases} \tag{1.29}$$

where

$$\mathcal{T}_0^\mu[\beta b,\varepsilon\zeta]\bullet = \frac{1}{h}\Big(\mathrm{F}_1\bullet + \beta\mathcal{L}_1^\mu[\beta b]\bullet + \varepsilon\zeta\mathrm{F}_1\bullet\Big) + \frac{\mu\beta}{2}\nabla_X\mathrm{F}_4\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\bullet\big)$$
$$- \frac{\mu\beta^2}{2}\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\bullet)\big) - \mu\beta^2(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\bullet).$$

Remark 1.22.

- *The first equation in* (1.29) *is exact and is a formulation of the conservation of mass.*
- *Taking* $\beta = 0$ *in* (1.29)*, we get the class of full dispersion Boussinesq systems derived rigorously in* [16] *with a precision* $\mathcal{O}(\mu\varepsilon)$*.*
- *One could also add Fourier multipliers* G_j*, defined by* (1.28)*, in the term of order* $\mathcal{O}(\mu\beta^2)$ *without changing the precision of the model.*

Corollary 1.23. *Under the same assumptions as in Theorem 1.21, we can take*

$$\mathcal{T}_0^{\mu}[\beta b, \varepsilon\zeta]\bullet = \mathrm{F}_1 \bullet + \frac{\beta b}{h}(\mathrm{F}_1 - \mathrm{F}_3) \bullet + \frac{\mu\beta^3}{6h} b^3 |\mathrm{D}|^2 \mathrm{F}_3 \bullet - \frac{\mu\beta}{2} \nabla_X \mathrm{F}_4 \nabla_X \cdot (b\bullet) \\ - \frac{\mu\beta^2}{2} \nabla_X \big(b\mathrm{F}_4 \nabla_X \cdot (b\bullet)\big) - \mu\beta^2 (\nabla_X b) \mathrm{F}_1 \nabla_X \cdot (b\bullet),$$

in system (1.29) *and keep the precision* $\mathcal{O}(\mu\varepsilon + \mu^2\beta^2)$*.*

The next two results concern full dispersion Green-Naghdi systems.

Theorem 1.24. *Let* F_2 *and* F_4 *be the two Fourier multipliers given in Definition 1.6, and let* $\mathcal{L}_2^{\mu}$ *be given in Definition 1.10. Then for any* $\mu \in (0,1]$*,* $\varepsilon \in [0,1]$*, and* $\beta \in [0,1]$ *the water waves equations* (1.4) *are consistent, in the sense of Definition 1.15 with* $n = 6$*, at order* $\mathcal{O}(\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)$ *with the Green-Naghdi type system:*

$$\begin{cases} \partial_t \zeta + \nabla_X \cdot (h\mathcal{T}_1^{\mu}[\beta b, \varepsilon\zeta]\nabla_X\psi) - \frac{\mu\beta^2}{2}\nabla_X \cdot \big(\mathcal{B}[\beta b]\nabla_X\psi\big) = 0 \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla_X\psi|^2 - \frac{\mu\varepsilon}{2} h^2 (\sqrt{\mathrm{F}_2}\Delta_X\psi)^2 = 0, \end{cases} \tag{1.30}$$

where

$$\mathcal{B}[\beta b]\bullet = b\mathrm{F}_4 \nabla_X(\nabla_X \cdot (b\bullet)) + h_b \nabla_X \big(b\mathrm{F}_4 \nabla_X \cdot (b\bullet)\big) + 2h_b (\nabla_X b)\mathrm{F}_1 \nabla_X \cdot (b\bullet),$$

and

$$\mathcal{T}_1^{\mu}[\beta b, \varepsilon\zeta]\bullet = \mathrm{Id} + \frac{\mu}{3h}\nabla_X \sqrt{\mathrm{F}_2}\Big(\frac{h^3}{h_b^3}\sqrt{\mathrm{F}_2}\nabla_X \cdot \bullet\Big) + \frac{\mu\beta}{h}\nabla_X\Big(\mathcal{L}_2^{\mu}[\beta b]\nabla_X \cdot \bullet\Big) \\ + \frac{\mu\beta}{2h}\mathrm{F}_4 \nabla_X \nabla_X \cdot \big(\mathcal{L}_1^{\mu}[\beta b]\bullet\big),$$

and $\sqrt{\mathrm{F}_2}$ *is the square root of* F_2*.*

Remark 1.25.

- *System* (1.30) *was first derived in* [11] *in the case* $\beta = 0$*.*
- *In* [10]*, the author derived a weakly dispersive Green-Naghdi type system with an order of precision given by* $\mathcal{O}(\mu^2\varepsilon + \mu^2\beta)$*.*
- *One could also add Fourier multipliers* G_j*, defined by* (1.28)*, in the term of order* $\mathcal{O}(\mu\beta^2)$ *without changing the precision of the model.*

Again, we can simplify the system using Proposition 1.10 to obtain a system only depending on Fourier multipliers.

Corollary 1.26. *Under the same assumptions as in Theorem 1.24, we can take*

$$\mathcal{L}_1^{\mu}[\beta b]\bullet = -b\mathrm{F}_3\bullet,$$

and

$$\mathcal{L}_2^{\mu}[\beta b]\bullet = -\frac{1}{2}b\mathrm{F}_4 \bullet + \frac{\beta^2}{6}b^3\mathrm{F}_3\bullet,$$

in system (1.30) *keeping the precision* $\mathcal{O}(\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)$*.*

Several generalizations can now be made, where the next system is chosen to mimic some of the properties of the classical Green-Naghdi systems:

Theorem 1.27. *Let* F_2 *and* F_4 *be the two the Fourier multipliers given in Definition 1.6, let* $\mathcal{L}_1^\mu$ *and* $\mathcal{L}_2^\mu$ *be given in Definition 1.10. Then for any* $\mu \in (0,1]$, $\varepsilon \in [0,1]$, *and* $\beta \in [0,1]$ *the water waves equations* (1.4) *are consistent, in the sense of Definition 1.15 with* $n = 7$, *at order* $\mathcal{O}(\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)$ *with the Green-Naghdi type system:*

$$\begin{cases} \partial_t\zeta + \nabla_X \cdot (h\overline{V}) = 0, \\ \partial_t(\mathcal{I}^\mu[h]\overline{V}) + \mathcal{I}^\mu[h]\mathcal{T}_2^\mu[\beta b, h]\nabla_X\zeta + \frac{\varepsilon}{2}\nabla_X\big(|\overline{V}|^2\big) + \mu\varepsilon\nabla_X\mathcal{R}_1^\mu[\beta b, h, \overline{V}] = \mathbf{0}, \end{cases} \tag{1.31}$$

where $\overline{V}$ *defined by* (1.8),

$$\mathcal{I}^\mu[h]\bullet = \mathrm{Id} - \frac{\mu}{3h}\sqrt{\mathrm{F}_2}\nabla_X\Big(h^3\sqrt{\mathrm{F}_2}\nabla_X\cdot\bullet\Big),$$

$$\begin{aligned}\mathcal{T}_2^\mu[\beta b, \varepsilon\zeta]\bullet = \mathrm{Id} &+ \frac{\mu}{3h}\sqrt{\mathrm{F}_2}\nabla_X\Big(\frac{h^3}{h_b^3}\sqrt{\mathrm{F}_2}\nabla_X\cdot\bullet\Big) + \frac{\mu\beta}{h}\nabla_X\Big(\mathcal{L}_2^\mu[\beta b]\nabla_X\cdot\bullet\Big) \\ &+ \frac{\mu\beta h_b}{2h}\nabla_X\mathrm{F}_4\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\bullet\big) - \frac{\mu\beta^2 h_b}{2h}\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\bullet)\big) \\ &- \frac{\mu\beta^2 h_b}{h}(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\bullet),\end{aligned}$$

and

$$\mathcal{R}_1^\mu[\beta b, h, \overline{V}] = -\frac{h^2}{2}(\nabla_X\cdot\overline{V})^2 - \frac{1}{3h}\big(\nabla_X(h^3\nabla_X\cdot\overline{V})\big)\cdot\overline{V} - \frac{1}{2}h^3\Delta_X(|\overline{V}|^2) + \frac{1}{6h}h^3\Delta_X(|\overline{V}|^2).$$

Remark 1.28.

- *As for the classical Green-Naghdi system, we observe that the first equation is a formulation of mass conservation.*
- *The system depends on the elliptic operator* $h\mathcal{I}^\mu[h]$ *and is similar to the systems derived in* [14, 11, 10] *in that sense.*
- *The presence of the term* $\mathcal{I}^\mu[h]\mathcal{T}_2^\mu[\beta b, h]\nabla_X\zeta$ *in the second equation makes it quite unique. Note that one may simplify it, but we chose to keep it under this form because in the study of the local well-posedness theory, one would apply the inverse of the elliptic operator* $h\mathcal{I}^\mu[h]$ *to the equation.*

Corollary 1.29. *Under the same assumptions as in Theorem 1.27, we can take*

$$\mathcal{L}_1^\mu[\beta b]\bullet = -b\mathrm{F}_3\bullet,$$

and

$$\mathcal{L}_2^\mu[\beta b]\bullet = -\frac{1}{2}b\mathrm{F}_4\bullet + \frac{\beta^2}{6}b^3\mathrm{F}_3\bullet,$$

in system (4.5) *keeping the precision* $\mathcal{O}(\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)$.

1.3. **Outline.** The paper is organized as follows. In Section 2, we set out to prove Proposition 1.14. First, we start Subsection 2.1 by transforming the elliptic problem (1.6) so that its domain is time-independent. Then we use this new formulation to perform multi-scale expansions. In particular, in Subsection 2.2 and 2.3, we make several expansions of the velocity potential in terms of μ, ε and β. From these expansions, we approximate the vertically averaged velocity potential $\overline{V}$ in Subsection 2.4, from which the proof of Proposition 1.14 is deduced in Subsection 2.5. Section 3 is dedicated to the proofs of Theorem 1.18 and Theorem 1.21. We also formally derive a Hamiltonian Boussinesq type system. In Section

4 we prove Theorem 1.24 and Theorem 1.27. Lastly, the appendix is composed of three subsections. The Subsection A.1 is dedicated to the proof of Proposition 1.10. In the last two Subsections A.2 and A.3, we state and prove technical tools.

2. Asymptotic expansions of the Dirichlet-Neumann operator

In this section, we perform expansions of the Dirichlet-Neumann operator with an error of order $\mathcal{O}(\mu\varepsilon + \mu^2\beta^2)$ and $\mathcal{O}(\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)$. The standard approach to deriving asymptotic models is by approximating the velocity potential Φ, which in turn will give an approximation of (1.7). Classically, one straightens the fluid domain to work on the flat strip, where we can easily make approximations. However, if we straighten the bottom, there will be an appearance of β that will give approximations on the form $\mathcal{O}(\mu\varepsilon + \mu\beta)$ in the case Boussinesq type systems and $\mathcal{O}(\mu^2\varepsilon + \mu^2\beta)$ in the case Green-Naghdi type systems (see [10] for the derivation of such models).

2.1. **The transformed Laplace equation.**

Motivated by the previous discussion, we make a change of variable that only straightens the top of the fluid domain.

Definition 2.1. *Let $s > \frac{d}{2} + 1$, $b \in C_c^\infty(\mathbb{R}^d)$, and $\zeta \in H^s(\mathbb{R}^d)$ such that the non-cavitation assumptions* (1.5) *and* (1.13) *are satisfied. We define the time-dependent diffeomorphism mapping the domain*

$$\mathcal{S}_b := \{(X,z) \in \mathbb{R}^{d+1} : -1 + \beta b \leq z \leq 0\},$$

onto the water domain Ω_t through

$$\Sigma_b : \begin{cases} \mathcal{S}_b & \longrightarrow \quad \Omega_t \\ (X,z) & \mapsto \quad (X, z + \sigma(X,z)) \end{cases}$$

with

$$\sigma(X,z) = \frac{\varepsilon\zeta(X)}{1 - \beta b(X)} z + \varepsilon\zeta(X). \tag{2.1}$$

Remark 2.2. *The map given in Definition 2.1 is a diffeomorphism. Indeed, by computing the Jacobi matrix, we find that*

$$J_{\Sigma_b} = \begin{pmatrix} \mathrm{Id} & \mathbf{0} \\ (\nabla_X \sigma)^T & 1 + \partial_z \sigma \end{pmatrix},$$

where

$$1 + \partial_z \sigma = \frac{h}{h_b}.$$

Therefore, under the non-cavitation condition as stated in Definition 1.2, we have a non-zero determinant:

$$|J_{\Sigma_b}| \geq \frac{h_{\min}}{1 + \beta |b|_{L^\infty}}.$$

The next result shows that the properties of solutions of the boundary problem (1.6) can be obtained from the study of an equivalent elliptic boundary value problem defined on $\mathcal{S}_b$.

Proposition 2.3. *Let $\phi_b = \Phi \circ \Sigma_b$ where the map Σ_b is given in Definition 2.1. Then under the provisions of Definition 1.3 we have that Φ is a (variational, classical) solution of* (1.6) *if and only if ϕ_b is a (variational, classical) solution of*

$$\begin{cases} \nabla^\mu_{X,z} \cdot P(\Sigma_b) \nabla^\mu_{X,z} \phi_b = 0 \quad in \quad S_b \\ \phi_b|_{z=0} = \psi, \quad \partial^{P_b}_{n_b} \phi_b|_{z=-h_b} = 0, \end{cases} \tag{2.2}$$

where the matrix $P(\Sigma_b)$ is given by

$$P(\Sigma_b) = |J_{\Sigma_b}|(I^\mu)^{-1} J_{\Sigma_b}^{-1}(I^\mu)^2 (J_{\Sigma_b}^{-1})^T (I^\mu)^{-1}, \tag{2.3}$$

and the Neumann condition reads

$$\partial_{n_b}^{P_b} \phi_b|_{z=-h_b} = \mathbf{n}_b \cdot I^\mu P(\Sigma_b)\nabla_{X,z}^\mu \phi_b|_{z=-h_b}. \tag{2.4}$$

Moreover, the matrix $P(\Sigma_b)$ is coercive, i.e. there exists $c > 0$ such that for all $Y \in \mathbb{R}^{d+1}$ and any $(X,z) \in \mathcal{S}_b$ there holds,

$$P(\Sigma_b)Y \cdot Y \geq c|Y|^2. \tag{2.5}$$

Remark 2.4. *We may compute the inverse Jacobi-matrix $J_{\Sigma_b}^{-1}$ so that using the expression for $P(\Sigma_b)$ (2.3), we find*

$$P(\Sigma_b) = \begin{pmatrix} (1+\partial_z\sigma)\mathrm{Id} & -\sqrt{\mu}\nabla_X\sigma \\ -\sqrt{\mu}(\nabla_X\sigma)^T & \dfrac{1+\mu h_b|\nabla_X\sigma|^2}{1+\partial_z\sigma} \end{pmatrix}.$$

Now, since σ is given by (2.1) we find that

$$1+\partial_z\sigma = 1 + \frac{\varepsilon\zeta}{h_b} = \frac{h}{h_b},$$

and

$$P(\Sigma_b) = \begin{pmatrix} \dfrac{h}{h_b}\mathrm{Id} & -\sqrt{\mu}\nabla_X\sigma \\ -\sqrt{\mu}(\nabla_X\sigma)^T & \dfrac{h_b+\mu h_b|\nabla_X\sigma|^2}{h} \end{pmatrix},$$

where

$$\nabla_X\sigma = \varepsilon\nabla_X\Big(\frac{\zeta}{h_b}\Big)z + \varepsilon\nabla_X\zeta. \tag{2.6}$$

Proof. The fact that $\nabla_{X,z}^\mu \cdot P(\Sigma_b)\nabla_{X,z}^\mu\phi_b = 0$ in $\mathcal{S}_b$ and that $P(\Sigma_b)$ satisfies (2.5) is classical and we simply refer to [14], Proposition 2.25 and Lemma 2.26.

To verify the Neumann condition, we first use the chain rule to make the observation

$$\nabla_{X,z}^\mu\phi_b = I^\mu(J_{\Sigma_b})^T(I^\mu)^{-1}(\nabla_{X,z}^\mu\Phi)\circ\Sigma_b. \tag{2.7}$$

Then by (2.4), we get that

$$\begin{aligned}
\partial_{n_b}^{P_b}\phi_b &= |J_{\Sigma_b}|\mathbf{n}_b \cdot (J_{\Sigma_b})^{-1} I^\mu(\nabla_{X,z}^\mu\Phi)\circ\Sigma_b \\
&= \mathbf{n}_b \cdot I^\mu \begin{pmatrix} (1+\partial_z\sigma)\mathrm{Id} & \mathbf{0} \\ -(\nabla_X\sigma)^T & 1 \end{pmatrix} (\nabla_{X,z}^\mu\Phi)\circ\Sigma_b \\
&= \mathbf{n}_b \cdot I^\mu(\nabla_{X,z}^\mu\Phi)\circ\Sigma_b + \mathbf{n}_b \cdot I^\mu \begin{pmatrix} \partial_z\sigma\mathrm{Id} & \mathbf{0} \\ -(\nabla_X\sigma)^T & 0 \end{pmatrix} (\nabla_{X,z}^\mu\Phi)\circ\Sigma_b.
\end{aligned}$$

Now, use (1.6) with the fact that for $z = -h_b$ then

$$0 = \partial_{n_b}\Phi|_{-h_b} = \mathbf{n}_b \cdot I^\mu(\nabla_{X,z}^\mu\Phi)\circ\Sigma_b.$$

Therefore, we are left with the expression

$$\partial_{n_b}^{P_b}\phi_b|_{z=-h_b} = \frac{1}{|\mathbf{n}_b|}\begin{pmatrix}-\beta\nabla_X b\\ 1\end{pmatrix}\cdot\begin{pmatrix}\mu\partial_z\sigma(\nabla_X\Phi)|_{z=-h_b}\\ -\mu\nabla_X\sigma\cdot(\nabla_X\Phi)|_{z=-h_b}\end{pmatrix}$$
$$= -\frac{\mu}{|\mathbf{n}_b|}(\beta\partial_z\sigma\nabla_X b + \nabla_X\sigma)\cdot(\nabla_X\Phi)\big|_{z=-h_b}.$$

But $\partial_z\sigma|_{z=-h_b} = \dfrac{\varepsilon\zeta}{h_b}$, $\nabla_X\sigma|_{z=-h_b} = -\beta\dfrac{\varepsilon\zeta}{h_b}\nabla_X b$, and thus the proof is complete. □

In the next section, we will make expansions of ϕ_b and then use the expression of (1.8) to approximate the Dirichelet-Naumann operator. But first, we must relate the definition of $\overline{V}^{\mu}[\varepsilon\zeta,\beta b]\psi$ with the new velocity potential on $\mathcal{S}_b$.

Proposition 2.5. *Let Σ_b be given in Definition 2.1. Then under the provisions of Definition 1.4, the operator* (1.8) *is equivalent to the following formulation*

$$\overline{V}^{\mu}[\varepsilon\zeta,\beta b]\psi = \frac{1}{h}\int_{-1+\beta b}^{0}\big[\frac{h}{h_b}\nabla_X\phi_b - (\varepsilon\nabla_X\Big(\frac{\zeta}{h_b}\Big)z + \varepsilon\nabla_X\zeta)\partial_z\phi_b\big]\,\mathrm{d}z, \tag{2.8}$$

where $\phi_b = \Phi\circ\Sigma_b$.

Proof. We use the new variables defined by the mapping Σ_b and the chain rule to get

$$\overline{V}^{\mu}[\varepsilon\zeta,\beta b]\psi = \frac{1}{h}\int_{-1+\beta b}^{0}(\nabla_X\Phi)\circ\Sigma_b\,|J_{\Sigma_b}|\,\mathrm{d}z$$
$$= \frac{1}{h}\int_{-1+\beta b}^{0}\big[\frac{h}{h_b}\nabla_X\phi_b - \nabla_X\sigma\partial_z\phi_b\big]\,\mathrm{d}z.$$

Then using (2.1), we obtain the result. □

2.2. **Multi-scale expansions.** In order to make expansions of ϕ_b we first make several observations on how to decompose system (2.2).

Observation 2.6. *We can decompose the elliptic operator given in Remark 2.4 into:*

$$\frac{h}{h_b}\nabla^{\mu}_{X,z}\cdot P(\Sigma_b)\nabla^{\mu}_{X,z}\phi_b = \Delta^{\mu}_{X,z}\phi_b + \mu\varepsilon A[\nabla_X,\partial_z]\phi_b,$$

where

$$A[\nabla_X,\partial_z]\phi_b = \frac{\zeta}{h_b}\Delta_X\phi_b + \frac{h}{h_b}\nabla_X\cdot\big(\frac{\zeta}{h_b}\nabla_X\phi_b\big) - \frac{h}{h_b}\nabla_X\cdot\big(\frac{1}{\varepsilon}\nabla_X\sigma\partial_z\phi_b\big)$$
$$- \frac{h}{h_b}\partial_z\big(\frac{1}{\varepsilon}\nabla_X\sigma\cdot\nabla_X\phi_b\big) + \partial_z\big(\frac{1}{\varepsilon}|\nabla_X\sigma|^2\partial_z\phi_b\big).$$

We may simplify this expression by using formula (2.6) *for $\nabla_X\sigma$ to get that*

$$A[\nabla_X,\partial_z]\phi_b = \frac{\zeta}{h_b}(1+\frac{h}{h_b})\Delta_X\phi_b - \frac{h}{h_b}\nabla_X\zeta\cdot\nabla_X\partial_z\phi_b \tag{2.9}$$
$$- \frac{h}{h_b}\nabla_X\cdot\big(\frac{1}{\varepsilon}\nabla_X\sigma\partial_z\phi_b\big) + \partial_z\big(\frac{1}{\varepsilon}|\nabla_X\sigma|^2\partial_z\phi_b\big).$$

In this formula, we emphasize the terms that do not contain $\partial_z\phi_b$. This is because these are the leading terms in the approximations that are performed below.

Observation 2.7. *Similarly, we can also decompose the Neumann condition into*

$$\begin{aligned}\frac{h}{h_b}|\mathbf{n}_b|\partial^{P_b}_{n_b}\phi_b|_{z=-h_b} &= [\partial_z\phi_b - \mu\beta\frac{h}{h_b}\nabla_X b\cdot\nabla_X\phi_b - \mu\beta^2\varepsilon\zeta\frac{|\nabla_X b|^2}{h_b}\partial_z\phi_b]|_{z=-h_b}\\ &= [\partial_z\phi_b - \mu\beta\nabla_X b\cdot\nabla_X\phi_b]|_{z=-h_b} + \mu\varepsilon\beta B[\nabla_X,\partial_z]\phi_b|_{z=-h_b}.\end{aligned}$$

where

$$B[\nabla_X,\partial_z]\phi_b = -\frac{\zeta}{h_b}\nabla_X b\cdot\nabla_X\phi_b - \beta\zeta\frac{|\nabla_X b|^2}{h_b}\partial_z\phi_b.$$

To summarize the observations, we now have that ϕ_b solves

$$\begin{cases}\Delta^\mu_{X,z}\phi_b = -\mu\varepsilon A[\nabla_X,\partial_z]\phi_b \quad \text{in} \quad \mathcal{S}_b\\ \phi_b|_{z=0} = \psi, \quad [\partial_z\phi_b - \mu\beta\nabla_X b\cdot\nabla_X\phi_b]|_{z=-h_b} = \mu\varepsilon\beta B[\nabla_X,\partial_z]\phi_b|_{z=-h_b}.\end{cases} \tag{2.10}$$

Remark 2.8. *In the paper* [8]*, their strategy is to solve* (2.10) *first in the case $\varepsilon = 0$, where the solution is defined in terms of the inverse of a pseudo-differential operator. If we add the parameters μ and β then, in dimension one, this operator is given by*

$$\mathcal{L}^\mu[\beta b] = -\cosh\left((-1+\beta b(X))\sqrt{\mu}D\right)^{-1}\sinh\left(\beta b(X)\sqrt{\mu}D\right)\mathrm{sech}(\sqrt{\mu}D). \tag{2.11}$$

Formally, in dimension one, they obtain the first order approximation:

$$\mathcal{G}_0 = \sqrt{\mu}D\tanh(\sqrt{\mu}D) + \sqrt{\mu}D\mathcal{L}^\mu[\beta b].$$

At higher order they obtain the expansion of $\mathcal{G}^\mu$ given on the form

$$\frac{1}{\mu}\mathcal{G}^\mu = \frac{1}{\mu}\sum_{j=0}^n \varepsilon^j\mathcal{G}_j + \mathcal{O}(\varepsilon^{n+1}),$$

where $\mathcal{G}_j$ defined recursively for $j \geq 0$ and is the classical expansion for small amplitude waves when $\beta = 0$ [8] *(see also* [14] *where the approximation is proved with Sobolev bounds when $\beta = 0$). In this paper, our approach allow us to decouple the parameters μ, ε and β, writing expansions of the Dirichlet-Neumann operator which do not include the inversion of a pseudo-differential operator.*

2.3. **Multi-scale expansions of the velocity potential ϕ_b.** We will now use (2.10) to make multi-scale expansions of ϕ_b. But first, we state an important result to justify the procedure.

Proposition 2.9. *Let $d = 1,2$, $t_0 > \frac{d}{2}$, and $k \in \mathbb{N}$. Let $b \in C^\infty_c(\mathbb{R}^d)$ and $\zeta \in H^{\max\{t_0+2,k+1\}}(\mathbb{R}^d)$ such that* (1.5) *and* (1.13) *are satisfied. Let also $f \in H^{k,k}(\mathcal{S}_b)$ and $g \in H^k(\mathbb{R}^d)$ be two given functions. Then the boundary value problem*

$$\begin{cases}\nabla^\mu_{X,z}\cdot P(\Sigma_b)\nabla^\mu_{X,z}u = f \quad \text{in} \quad \mathcal{S}_b\\ u|_{z=0} = 0, \quad \partial^{P_b}_{n_b}u|_{z=-1+\beta b} = g,\end{cases} \tag{2.12}$$

admits a unique solution $u \in H^{k+1,0}(\mathcal{S}_b)$. Moreover, the solution satisfies the estimate

$$\|\nabla^\mu_{X,z}u\|_{H^{k,0}(\mathcal{S}_b)} \leq M(k+1)(|g|_{H^k} + \sum_{j=0}^k \|f\|_{H^{k-j,j}(\mathcal{S}_b)}). \tag{2.13}$$

The proof of Proposition 2.9 is similar to the one of Proposition 4.5 in [10] and is postponed for Appendix A, Subsection A.2 to ease the presentation. We may now use this result to construct an implicit function such that $\phi = \phi_b + \mathcal{O}(\mu\varepsilon)$.

Proposition 2.10. *Let $d = 1, 2$, $t_0 > \frac{d}{2}$, and $k \in \mathbb{N}$. Let $\psi \in \dot{H}^{k+3}(\mathbb{R}^d)$. Let also $b \in C_c^{\infty}(\mathbb{R}^d)$ and $\zeta \in H^{\max\{t_0+2,k+2\}}(\mathbb{R}^d)$ such that (1.5) and (1.13) are satisfied. Then there exists a unique solution $\phi_0 \in H^{k,0}(\mathcal{S}_b)$ solving*

$$\begin{cases} \Delta^{\mu}_{X,z}\phi = 0 \quad \text{in} \quad \mathcal{S}_b, \\ \phi|_{z=0} = \psi, \quad \big[\partial_z\phi - \mu\beta\nabla_X b \cdot \nabla_X\phi\big]\big|_{z=-1+\beta b} = 0, \end{cases} \tag{2.14}$$

where the solution satisfies the estimates

$$\|\nabla_{X,z}\phi\|_{H^{k,0}(\mathcal{S}_b)} \leq M(k+1)|\nabla_X\psi|_{H^k}, \tag{2.15}$$

and

$$\|\nabla^{\mu}_{X,z}(\phi_b - \phi)\|_{H^{k,0}(\mathcal{S}_b)} \leq \mu\varepsilon M(k+2)|\nabla_X\psi|_{H^{k+2}}. \tag{2.16}$$

Proof. The existence and uniqueness is a direct consequence Riesz representation Theorem, and the Poincaré inequality (A.6). Moreover, we know that $\tilde{\phi} = \phi(X, zh_b)$ defined is defined on the fixed strip $\mathcal{S} = \mathbb{R}^d \times [-1, 0]$ and satisfies

$$\|\nabla_{X,z}\tilde{\phi}\|_{H^{k,0}(\mathcal{S})} \leq M(k+1)|\nabla_X\psi|_{H^k}.$$

by Proposition 2.37 with $\varepsilon = 0$ in [14]. Then we may use this result, together with the relations $\nabla_{X,z}\phi(X,z) = \nabla_{X,z}\big(\tilde{\phi}(X, \frac{z}{h_b})\big) \in H^{k,0}(\mathcal{S}_b)$ and $\partial_z^2\phi = -\mu\Delta_X\phi$ to obtain the bound

$$\|\nabla_{X,z}\phi\|_{H^{k,0}(\mathcal{S}_b)} \leq M(k+1)|\nabla_X\psi|_{H^k}.$$

Consequently, estimate (2.16) follows from Proposition 2.9 and the observation that

$$\|\nabla^{\mu}_{X,z}(\phi_b - \phi)\|_{H^{k,0}(\mathcal{S}_b)} \leq \mu\varepsilon M(k+1)\|\frac{h_b}{h}A[\nabla_X, \partial_z]\phi\|_{H^{k,0}(\mathcal{S}_b)} + \|B[\nabla_X, \partial_z]\phi|_{z=-h_b}\|),$$

where $A[\nabla_X, \partial_z]$ is a second order differential operator and $B[\nabla_X, \partial_z]$ is a first order operator (second order after using the trace estimate (A.7)). Then we simply conclude by using (2.15) combined with product estimates for $H^k(\mathbb{R}^d)$ given by (A.9) and (A.10).

□

Next, we will make expansion with respect to $\mu\beta$ by splitting problem (2.10) into two parts. In particular, we will construct a function $\phi_0 = \phi_b + \mathcal{O}(\mu(\varepsilon + \beta))$ by solving the first part of the "straightened" Laplace problem with an explicit error of order $\mathcal{O}(\mu\beta)$, that will be canceled later, and an additional error of $\mathcal{O}(\mu\varepsilon)$.

Proposition 2.11. *Let $d = 1, 2$, $t_0 > \frac{d}{2}$, and $k \in \mathbb{N}$. Let $\psi \in \dot{H}^{k+3}(\mathbb{R}^d)$. Let also $b \in C_c^{\infty}(\mathbb{R}^d)$ and $\zeta \in H^{\max\{t_0+2,k+2\}}(\mathbb{R}^d)$ such that (1.5) and (1.13) are satisfied. If ϕ_0 satisfies the following Laplace problem:*

$$\begin{cases} \Delta^{\mu}_{X,z}\phi_0 = 0 \quad \text{in} \quad \mathcal{S}_b, \\ \phi_0|_{z=0} = \psi, \quad \big[\partial_z\phi_0 - \mu\beta\nabla_X b \cdot \nabla_X\phi_0\big]\big|_{z=-1+\beta b} = \mu\beta\nabla_X \cdot \mathcal{L}^{\mu}_1[\beta b]\nabla_X\psi, \end{cases} \tag{2.17}$$

where

$$\mathcal{L}^{\mu}_1[\beta b]\nabla_X\psi = -\frac{1}{\beta}\sinh\big(\beta b(X)\sqrt{\mu}|\mathrm{D}|\big)\mathrm{sech}(\sqrt{\mu}|\mathrm{D}|)\frac{1}{\sqrt{\mu}|\mathrm{D}|}\nabla_X\psi,$$

then for $z \in [-1+\beta b, 0]$ its expression is given by

$$\phi_0 = \frac{\cosh\big((z+1)\sqrt{\mu}|\mathrm{D}|\big)}{\cosh\big(\sqrt{\mu}|\mathrm{D}|\big)}\psi = \mathrm{F}_0\psi. \tag{2.18}$$

Moreover, the solution satisfies the estimate

$$\|\nabla^{\mu}_{X,z}(\phi_b - \phi_0)\|_{H^{k,0}(\mathcal{S}_b)} \leq \mu(\varepsilon + \beta)M(k+2)|\nabla_X\psi|_{H^{k+2}}. \tag{2.19}$$

Proof. Since ϕ_0 is given by the solution of the Laplace problem when the bottom is flat, we only need to verify the boundary condition at the bottom. In fact, we have that

$$\begin{aligned}
\text{LHS} :&= \big[\partial_z\phi_0 - \mu\beta\nabla_X b\cdot\nabla_X\phi_0\big]\big|_{z=-1+\beta b}\\
&= \mathcal{F}^{-1}\Big(\sqrt{\mu}|\xi|\sinh\big((z+1)\sqrt{\mu}|\xi|\big)\mathrm{sech}(\sqrt{\mu}|\xi|)\widehat{\psi}(\xi)\Big)(X)\big|_{z=-1+\beta b(X)}\\
&\quad - \mathcal{F}^{-1}\Big(\mu\beta\nabla_X b(X)\cdot i\xi\cosh\big((z+1)\sqrt{\mu}|\xi|\big)\mathrm{sech}(\sqrt{\mu}|\xi|)\widehat{\psi}(\xi)\Big)(X)\big|_{z=-1+\beta b(X)}\\
&= -\sqrt{\mu}\nabla_X\cdot\big(\sinh\big(\beta b(X)\sqrt{\mu}|\mathrm{D}|\big)\mathrm{sech}(\sqrt{\mu}|\mathrm{D}|)\frac{1}{|\mathrm{D}|}\nabla_X\psi\big).
\end{aligned}$$

The next step is to prove that ϕ_0 approximates ϕ_b with a precision of $\mathcal{O}(\mu(\varepsilon+\beta))$. To that end, we first note that $u=\phi_b-\phi_0$ solves the elliptic problem (2.12) with

$$f = -\mu\varepsilon\frac{h_b}{h}A[\nabla_X,\partial_z]\phi_0,$$

and

$$g = \mu\beta\frac{h_b}{h|\mathbf{n}_b|}\big(\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big)+\varepsilon B[\nabla_X,\partial_z]\phi_0\big)|_{z=-1+\beta b},$$

where the expressions of f and g are deduced from the decompositions of Observations 2.6 and 2.7 and the construction of ϕ_0. Moreover, since $-h_b(X) > -2$ (see (1.13)), we can extend the definition of ϕ_0 to the domain $\mathcal{S} := \mathbb{R}^d\times[-2,0]$. For any $(X,z)\in\mathcal{S}$, we write

$$\phi_0 = \frac{\cosh\big((z+1)\sqrt{\mu}|\mathrm{D}|\big)}{\cosh\big(\sqrt{\mu}|\mathrm{D}|\big)}\psi.$$

This extension is a Fourier multiplier depending on z, and we can use the estimates in Proposition A.4 together with the fact that $A[\nabla_X,\partial_z]\bullet$, given by (2.9), only depends on functions of X and is polynomial in z. Thus, combining the elliptic estimate (2.13) with (1.15), the non-cavitation conditions (1.5), (1.13), the product estimates for $H^k(\mathbb{R}^d)$ given by (A.9) and (A.10), we obtain that

$$\begin{aligned}
\|\nabla^\mu_{X,z}u\|_{H^{k,0}(\mathcal{S}_b)} &\leq \mu\varepsilon M(k+1)\|\frac{h_b}{h}A[\nabla_X,\partial_z]\phi_0\|_{H^{k,0}(\mathcal{S}_b)}\\
&\quad + \mu\varepsilon M(k+1)\big|\frac{\zeta}{h}\big|_{H^{k+2}}\big(|\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big)|_{H^k} + \big|\tilde{B}[\nabla_X,\partial_z]\phi_0|_{z=-h_b}\big|_{H^k}\big)\\
&\leq \mu(\varepsilon+\beta)M(k+2)\|\frac{h_b}{h}A[\nabla_X,\partial_z]\phi_0\|_{H^{k,0}(\mathcal{S})} + \mu(\varepsilon+\beta)M(k+2)|\nabla\psi|_{H^{k+1}}\\
&\lesssim \mu(\varepsilon+\beta)M(k+2)|\nabla\psi|_{H^{k+2}}.
\end{aligned}$$

□

Remark 2.12. *The source term $\mu\beta\nabla_X\cdot\mathcal{L}_1^\mu[\beta b]\nabla_X\psi$ in the Neumann condition of* (2.17) *is chosen so that the solution ϕ_0 of the system does not depend on the inverse of a pseudo-differential operator. Indeed, any other source term in the Neumann condition would induce the dependence of the solution on operators of this kind.*

We now construct the next order approximation by canceling the error of order $\mathcal{O}(\mu\beta)$. But first, we make an observation on the problem that needs to be solved.

Observation 2.13. *To make the next order approximation ϕ_1 such that $\phi_b = \phi_0+\mu\beta\phi_1+\mathcal{O}(\mu(\varepsilon+\mu\beta^2))$, we solve the problem*

$$\begin{cases}\Delta^\mu_{X,z}\phi_1 = \mu\beta F \quad \text{in} \quad \mathcal{S}_b,\\ \phi_1|_{z=0}=0, \quad \partial_z\phi_1|_{z=-1+\beta b} = -\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big),\end{cases}$$

where F is to be chosen and satisfies

$$\|F\|_{H^{k,k}(\mathcal{S}_b)} \leq M(k+2)|\nabla_X\psi|_{H^{k+2}}. \tag{2.20}$$

so that formally

$$\begin{cases} \frac{h}{h_b}\nabla_{X,z}\cdot P(\Sigma_b)\nabla_{X,z}(\phi_0+\mu\beta\phi_1) = \mathcal{O}(\mu\varepsilon+\mu^2\beta^2) \ \text{ in } \ \mathcal{S}_b, \\ (\phi_0+\mu\beta\phi_1)|_{z=0} = \psi, \quad \frac{h}{h_b}\partial_{n_b}^{P_b}(\phi_0+\mu\beta\phi_1)|_{z=-1+\beta b} = \mathcal{O}(\mu\varepsilon+\mu^2\beta^2). \end{cases}$$

Moreover, the presence of the source term $\mu\beta F$ is motivated by the fact that the boundary conditions require a function of the form

$$\phi_1 = -h_b\frac{\sinh(\frac{z}{h_b}\sqrt{\mu}|\mathrm{D}|)}{\cosh(\sqrt{\mu}|\mathrm{D}|)}\frac{1}{\sqrt{\mu}|\mathrm{D}|}\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big),$$

for $-h_b \leq z \leq 0$. Indeed, if we let $G = \nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big)$, then

$$\begin{aligned}\partial_z\phi_1|_{z=-h_b} &= -\mathcal{F}^{-1}\Big(\frac{\cosh(\frac{z}{h_b(X)}\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\hat{G}(\xi)\Big)(X)|_{z=-h_b(X)} \\ &= -G(X).\end{aligned}$$

Now, let us compute the Laplace operator. To do so, we introduce the notation

$$T_1(z)[X,\mathrm{D}]\bullet = \mathcal{F}^{-1}\Big(\frac{\sinh(\frac{z}{h_b(X)}\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\hat{\bullet}\Big)(X),$$

and

$$T_2(z)[X,\mathrm{D}]\bullet = \mathcal{F}^{-1}\Big(\frac{\cosh(\frac{z}{h_b(X)}\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\hat{\bullet}\Big)(X).$$

Using the identity $\Delta_X = -|\mathrm{D}|^2$, we observe that

$$\partial_z^2\phi_1 = \frac{\mu}{h_b}T_1(z)[X,\mathrm{D}]\frac{\Delta_X}{\sqrt{\mu}|\mathrm{D}|}G.$$

Similarly, after some computations we find

$$\begin{aligned}\mu\Delta_X\phi_1 &= -\mu h_bT_1(z)[X,\mathrm{D}]\frac{\Delta_X}{\sqrt{\mu}|\mathrm{D}|}G + \mu[h_bT_1(z)[X,D]\frac{1}{\sqrt{\mu}|D|},\Delta]G \\ &= -\mu T_1(z)[X,\mathrm{D}]\frac{\Delta_X}{\sqrt{\mu}|\mathrm{D}|}G + \mu[h_bT_1(z)[X,D]\frac{1}{\sqrt{\mu}|D|},\Delta]G + \mu\beta bT_1(z)[X,\mathrm{D}]\frac{\Delta_X}{\sqrt{\mu}|\mathrm{D}|}G.\end{aligned}$$

We define $\tilde{F}$ by

$$\tilde{F} = \mu\big[h_bT_1(z)[X,D]\frac{1}{\sqrt{\mu}|D|},\Delta\big]G + \mu\beta bT_1(z)[X,\mathrm{D}]\frac{\Delta_X}{\sqrt{\mu}|\mathrm{D}|}G$$

where $\mu\big[h_bT_1(z)[X,D]\frac{1}{\sqrt{\mu}|D|},\Delta\big]G = \mathcal{O}(\mu\beta)$ by direct calculation. From this expression, we identify F by

$$\begin{aligned}\Delta_{X,z}^\mu\phi_1 &= \mu\beta\tilde{F} + \mu(\frac{1}{h_b}-1)T_1(z)[X,\mathrm{D}]\frac{\Delta_X}{\sqrt{\mu}|\mathrm{D}|}G \\ &= \mu\beta\tilde{F} + \frac{\mu\beta b}{h_b}T_1(z)[X,\mathrm{D}]\frac{\Delta_X}{\sqrt{\mu}|\mathrm{D}|}G \\ &= \mu\beta F.\end{aligned}$$

The estimate (2.20) on F is a consequence of the boundedness of T_1 and T_2 for $z \in [-h_b, 0]$, given by Proposition A.5, while we estimate $\mathcal{L}_1^\mu$ in $H^{k+2}(\mathbb{R}^d)$ by Proposition 1.10 with inequality (1.15).

We summarize these observations in the next Proposition.

Proposition 2.14. *Let $d = 1, 2$, $t_0 > \frac{d}{2}$, and $k \in \mathbb{N}$. Let $\psi \in \dot{H}^{k+4}(\mathbb{R}^d)$. Let also $b \in C_c^\infty(\mathbb{R}^d)$ and $\zeta \in H^{\max\{t_0+2,k+2\}}(\mathbb{R}^d)$ such that (1.5) and (1.13) are satisfied. Then the function ϕ_1 given by*

$$\phi_1 = -h_b \frac{\sinh(\frac{z}{h_b}\sqrt{\mu}|\mathrm{D}|)}{\cosh(\sqrt{\mu}|\mathrm{D}|)} \frac{1}{\sqrt{\mu}|\mathrm{D}|}\nabla_X \cdot \big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big), \tag{2.21}$$

satisfies

$$\begin{cases} \Delta^\mu_{X,z}\phi_1 = \mu\beta F \quad \text{in} \quad \mathcal{S}_b, \\ \phi_1|_{z=0} = 0, \quad \partial_z\phi_1|_{z=-1+\beta b} = -\nabla_X \cdot \big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big), \end{cases} \tag{2.22}$$

where $F \in H^{k,k}(\mathcal{S}_b)$ is such that

$$\|F\|_{H^{k,k}(\mathcal{S}_b)} \leq M(k+2)|\nabla_X\psi|_{H^{k+2}}, \tag{2.23}$$

and

$$\mathcal{L}_1^\mu[\beta b]\nabla_X\psi = -\frac{1}{\beta}\sinh\big(\beta b(X)\sqrt{\mu}|\mathrm{D}|\big)\mathrm{sech}(\sqrt{\mu}|\mathrm{D}|)\frac{1}{\sqrt{\mu}|\mathrm{D}|}\nabla_X\psi.$$

Moreover, for ϕ_b satisfying (2.2) and ϕ_0 given by (2.18) there holds,

$$\|\nabla^\mu_{X,z}(\phi_b - (\phi_0 + \mu\beta\phi_1))\|_{H^{k,0}(\mathcal{S}_b)} \lesssim (\mu\varepsilon + \mu^2\beta^2)M(k+2)|\nabla\psi|_{H^{k+3}}. \tag{2.24}$$

Proof. By constriction of ϕ_1 given by (2.21), we know there exists an F such that (2.23) is satisfied. Now, let us prove (2.24). First, observe that the function

$$u = \phi_b - (\phi_0 + \mu\beta\phi_1)$$

solves

$$\begin{aligned} \frac{h}{h_b}\nabla^\mu_{X,z}P(\Sigma_b)\nabla^\mu_{X,z}u &= -\mu\varepsilon A[\nabla_X, \partial_z]\phi_0 - \mu^2\varepsilon\beta A[\nabla_X, \partial_z]\phi_1 - \mu^2\beta^2 F \\ &=: f. \end{aligned}$$

Moreover, at $z = -h_b$, we have the Neumann condition

$$\begin{aligned} \frac{h}{h_b}|\mathbf{n}_b|\partial^{P_b}_{n_b}u &= \partial_z\phi_0 - \mu\beta\nabla_X b \cdot \nabla_X\phi_0 + \mu\varepsilon\beta B[\nabla_X, \partial_z]\phi_0 + \mu\beta\partial_z\phi_1 + \mu^2\varepsilon\beta^2 B[\nabla_X, \partial_z]\phi_1 \\ &= \mu\varepsilon\beta B[\nabla_X, \partial_z]\phi_0 + \mu^2\varepsilon\beta^2 B[\nabla_X, \partial_z]\phi_1 \\ &=: g. \end{aligned}$$

Estimating each terms, noting that $A[\nabla_X, \partial_z]$ is a differential operator of order two and $B[\nabla_X, \partial_z]$ is of order one, while the error due to F is given by construction, we obtain that

$$\|\nabla^\mu_{X,z}u\|_{H^{k,0}(\mathcal{S}_b)} \leq \mu(\varepsilon + \varepsilon\beta + \mu\beta^2)M(k+2)|\nabla\psi|_{H^{k+2}}.$$

□

Observation 2.15. *We now construct an approximation of ϕ_b to the order $\mathcal{O}(\mu(\mu\varepsilon + \varepsilon\beta + \mu\beta^2))$. To do so, we add a term of order $\mu\varepsilon$ in the approximation of ϕ_b in order to cancel the terms of order $\mu\varepsilon$. In particular, we consider ϕ_2 solution of the problem*

$$\begin{cases} \partial_z^2\phi_2 = -\frac{\zeta}{h_b}(1 + \frac{h}{h_b})\Delta_X\psi \quad \text{in} \quad \mathcal{S}_b, \\ \phi_2|_{z=0} = 0, \quad \partial_z\phi_2|_{z=-1+\beta b} = 0. \end{cases}$$

Indeed, if we use the decomposition given by Observations (2.7) *and* (2.6)*, and the definitions of ϕ_0 and ϕ_1, we get:*

$$\frac{h}{h_b}\nabla^{\mu}_{X,z}\cdot P(\Sigma_b)\nabla^{\mu}_{X,z}(\phi_b-\phi_0-\mu\beta\phi_1-\mu\varepsilon\phi_2)=-\mu\varepsilon\partial_z^2\phi_2-\mu\varepsilon A[\nabla_X,\partial_z]\phi_0+\mathcal{O}(\mu^2\varepsilon),$$

and

$$\frac{h}{h_b}|\mathbf{n}_b|\partial^{P_b}_{n_b}(\phi_b-\phi_0-\mu\beta\phi_1-\mu\varepsilon\phi_2)|_{z=-h_b}=-\mu\varepsilon\partial_z\phi_2|_{z=-h_b}+\mathcal{O}(\mu(\mu\varepsilon+\varepsilon\beta+\mu\beta^2)).$$

Moreover, using the estimates in Proposition A.4 with $t_0>\frac{d}{2}$, one can deduce from the definition of $A[\nabla_X,\partial_z]\bullet$, given by (2.9)*, that*

$$\begin{aligned}\text{LHS} :&= \|A[\nabla_X,\partial_z]\phi_0-\frac{\zeta}{h_b}\big(1+\frac{h}{h_b}\big)\Delta_X\psi\|_{H^{k,0}(\mathcal{S}_b)}\\
&\lesssim \big|\frac{\zeta}{h_b}(1+\frac{h}{h_b})\big|_{H^{\max(t_0,k)}}\|\Delta_X(\phi_0-\psi)\|_{H^{k,0}(\mathcal{S})}\\
&\quad+\big|\frac{h}{h_b}\big|_{H^{\max(t_0,k)}}|\nabla_X\zeta|_{H^{\max(t_0,k)}}\|\nabla_X\partial_z\phi_0\|_{H^{k,0}(\mathcal{S})}\\
&\quad+\|\frac{h}{h_b}\nabla_X\cdot\big(\frac{1}{\varepsilon}\nabla_X\sigma\partial_z\phi_0\big)\|_{H^{k,0}(\mathcal{S})}+\|\partial_z\big(\frac{1}{\varepsilon}|\nabla_X\sigma|^2\partial_z\phi_0\big)\|_{H^{k,0}(\mathcal{S})}\\
&\leq \mu M(k+2)|\nabla_X\psi|_{H^{k+3}},\end{aligned}$$

for any $k\in\mathbb{N}$.

With this observation in mind, we can write the following result.

Proposition 2.16. *Let $d=1,2$, $t_0>\frac{d}{2}$ and $k\in\mathbb{N}$. Let $\psi\in\dot{H}^{k+4}(\mathbb{R}^d)$. Let also $b\in C_c^\infty(\mathbb{R}^d)$ and $\zeta\in H^{\max\{t_0+2,k+2\}}(\mathbb{R}^d)$ such that* (1.5) *and* (1.13) *are satisfied. If ϕ_2 satisfies the following Laplace problem*

$$\begin{cases}\partial_z^2\phi_2=-\dfrac{\zeta}{h_b}\big(1+\dfrac{h}{h_b}\big)\Delta_X\psi \quad \text{in} \quad \mathcal{S}_b,\\ \phi_2|_{z=0}=0,\quad \partial_z\phi_2|_{z=-1+\beta b}=0.\end{cases}$$

Then its expression is given by:

$$\phi_2=-\big(\frac{z^2}{2}+h_bz\big)\frac{\zeta}{h_b}\big(1+\frac{h}{h_b}\big)\Delta_X\psi.$$

Moreover, for ϕ_b satisfying (2.2)*, ϕ_0 given by* (2.18) *and ϕ_1 given by* (2.21)*, there holds*

$$\|\nabla^{\mu}_{X,z}(\phi_b-(\phi_0+\mu\beta\phi_1+\mu\varepsilon\phi_2))\|_{H^{k,0}(\mathcal{S}_b)}\lesssim\mu(\mu\varepsilon+\varepsilon\beta+\mu\beta^2)M(k+2)|\nabla\psi|_{H^{k+3}}. \quad (2.25)$$

Proof. The function ϕ_2 satisfies a simple ODE and is solved by integrating the equation two times in z:

$$\phi_2=\int_z^0\int_{-1+\beta b}^{z'}\frac{\zeta}{h_b}\big(1+\frac{h}{h_b}\big)\Delta_X\psi\,\mathrm{d}z''\mathrm{d}z'=-\big(\frac{z^2}{2}+h_bz\big)\frac{\zeta}{h_b}\big(1+\frac{h}{h_b}\big)\Delta_X\psi.$$

Then, by construction, we have that $u=\phi_b-(\phi_0+\mu\beta\phi_1+\mu\varepsilon\phi_2)$ satisfies

$$\begin{cases}\frac{h}{h_b}\nabla^{\mu}_{X,z}\cdot P(\Sigma_b)\nabla^{\mu}_{X,z}u=f \quad \text{in} \quad \mathcal{S}_b\\ u|_{z=0}=0,\quad \frac{h}{h_b}|n_b|\partial^{P_b}_{n_b}u|_{z=-h_b}=g,\end{cases} \quad (2.26)$$

with

$$f = -\mu\varepsilon[A[\nabla_X,\partial_z]\phi_0 - \frac{\zeta}{h_b}(1+\frac{h}{h_b})\Delta_X\psi] + \mu^2\beta^2 F - \mu^2\varepsilon\beta A[\nabla_X,\partial_z]\phi_1$$
$$-\mu^2\varepsilon(\Delta_X\phi_2 + \varepsilon A[\nabla_X,\partial_z]\phi_2),$$

and

$$g = -\mu\varepsilon\beta B[\nabla_X,\partial_z]\phi_0|_{z=-h_b} + \mu^2\beta^2\nabla_X b\cdot\nabla_X\phi_1|_{z=-h_b} - \mu^2\varepsilon\beta^2 B[\nabla_X,\partial_z]\phi_1|_{z=-h_b}$$
$$+\mu^2\varepsilon\beta\nabla_X b\cdot\nabla_X\phi_2|_{z=-h_b} - \mu^2\varepsilon^2\beta B[\nabla_X,\partial_z]\phi_2|_{z=-h_b}.$$

Then we use the elliptic estimate (2.13) to get that

$$\|\nabla^\mu_{X,z}u\|_{H^{k,0}(\mathcal{S}_b)} \leq \mu(\mu\varepsilon+\varepsilon\beta+\mu\beta^2)M(k+1)(|g|_{H^k} + \sum_{j=0}^k \|f\|_{H^{k-j,j}(\mathcal{S}_b)}),$$

with the the usual product estimates for $H^k(\mathbb{R}^d)$ combined with Observation 2.15, Proposition 2.16 and the fact that ϕ_2 is polynomial in z, we get

$$\|\nabla^\mu_{X,z}u\|_{H^{k,0}(\mathcal{S}_b)} \leq \mu(\mu\varepsilon+\varepsilon\beta+\mu\beta^2)M(k+2)|\nabla_X\psi|_{H^{s+3}}.$$

□

We will now make two observations that will further simplify the presentation.

Observation 2.17. *We may use Plancherel's identity and the Taylor series expansions:*

$$\cosh(x) = 1 + \frac{x^2}{2}\int_0^1 \cosh(tx)(1-t)\,dt$$
$$\frac{1}{\cosh(x)} = 1 + \frac{x^2}{2}\int_0^1 \Big(\frac{\tanh(tx)^2}{\cosh(tx)} - \frac{1}{\cosh(tx)^3}\Big)(1-t)\,dt,$$

for $x \in [0,1]$, to deduce that

$$\|(\phi_0-\psi) - \mu(\frac{z^2}{2}+z)|\mathrm{D}|^2\psi\|_{H^{k,0}(\mathcal{S}_b)} \lesssim \mu^2||\mathrm{D}|^4\psi|_{H^k} \leq \mu^2|\nabla_X\psi|_{H^{k+3}}, \tag{2.27}$$

with $z \in (-h_b, 0)$ and assumption (1.13) *on $\beta b(X)$.*

Observation 2.18. *From the second-order expansions given by the previous Observation 2.17 we have*

$$\phi_0 - \psi = \mu(\frac{z^2}{2}+z)|\mathrm{D}|^2\psi + \mu^2 z^2 R, \tag{2.28}$$

where R is some generic function satisfying the estimate

$$|R|_{H^k} \leq M(k)|\nabla_X\psi|_{H^{k+3}}. \tag{2.29}$$

It allows us to approximate the quantity $\phi_0 + \mu\varepsilon\phi_2$:

$$\begin{aligned}\phi_0 + \mu\varepsilon\phi_2 &= \phi_0 + \mu(\frac{z^2}{2}+h_b z)\frac{\varepsilon\zeta}{h_b}(1+\frac{h}{h_b})|\mathrm{D}|^2\psi\\ &= \phi_0 + (\phi_0-\psi)\big(\frac{h}{h_b}-1\big)\big(\frac{h}{h_b}+1\big) + \mu(\mu\varepsilon+\varepsilon\beta)R\\ &= \phi_0 + (\phi_0-\psi)\big(\frac{h^2}{h_b^2}-1\big) + \mu(\mu\varepsilon+\varepsilon\beta)R\\ &= \psi + \frac{h^2}{h_b^2}(\phi_0-\psi) + \mu(\mu\varepsilon+\varepsilon\beta)R.\end{aligned}$$

We can make the formal computations in Observation 2.18 rigorous.

Proposition 2.19. *Let $d=1,2$, $t_0>\frac{d}{2}$ and $k\in\mathbb{N}$ such that $k\geq t_0+1$. Let $\psi\in\dot{H}^{k+4}(\mathbb{R}^d)$. Let also $b\in C_c^\infty(\mathbb{R}^d)$ and $\zeta\in H^{k+3}(\mathbb{R}^d)$ such that (1.5) and (1.13) are satisfied. Lastly, let ϕ_{app} be defined by*

$$\phi_{\mathrm{app}}=\psi+\Big(\frac{h}{h_b}\Big)^2(\phi_0-\psi)+\mu\beta\phi_1, \tag{2.30}$$

with ϕ_1 given by (2.21). Then for ϕ_b satisfying (2.2) there holds,

$$\|\nabla^\mu_{X,z}(\phi_b-\phi_{\mathrm{app}})\|_{H^{k,0}(\mathcal{S}_b)}\lesssim\mu(\mu\varepsilon+\varepsilon\beta+\mu\beta^2)M(s+2)|\nabla\psi|_{H^{k+3}}. \tag{2.31}$$

Proof. We first use Proposition 2.16 to get the estimate

$$\begin{aligned}\|\nabla^\mu_{X,z}(\phi_b-\phi_{\mathrm{app}})\|_{H^{k,0}(\mathcal{S}_b)}&\lesssim\mu(\mu\varepsilon+\varepsilon\beta+\mu\beta^2)M(k+2)|\nabla\psi|_{H^{k+3}}\\&\quad+\|\nabla^\mu_{X,z}(\phi_0+\mu\varepsilon\phi_1-\phi_{\mathrm{app}})\|_{H^{k,0}(\mathcal{S}_b)}.\end{aligned}$$

Making the same approximations as in Observation 2.18 will complete the proof. In particular, accounting for the loss of derivatives given by (2.29) yields,

$$\|\nabla^\mu_{X,z}(\phi_0+\mu\beta\phi_1+\mu\varepsilon\phi_2-\phi_{\mathrm{app}})\|_{H^{k,0}(\mathcal{S})}\lesssim\mu(\mu\varepsilon+\varepsilon\beta)M(k+1)|\nabla\psi|_{H^{k+3}}.$$

Gathering these estimates concludes the proof. $\square$

2.4. **Multi-scale expansions of $\overline{V}$.** In this subsection we will use the expression of ϕ,ϕ_0, ϕ_1, and ϕ_{app} to construct approximations of $\overline{V}$. The first result is given in the following proposition.

Proposition 2.20. *Let $d=1,2$, $t_0>\frac{d}{2}$ and $s\geq0$. Let $b\in C_c^\infty(\mathbb{R}^d)$, $\psi\in\dot{H}^{k+3}(\mathbb{R}^d)$ and $\zeta\in H^{\max\{t_0+2,s+3\}}(\mathbb{R}^d)$ be such that (1.5) is satisfied. Then for ϕ defined by the solution of (2.14) and $\overline{V}[0,\beta b]$ defined by*

$$\overline{V}[0,\beta b]=\frac{1}{h_b}\int_{-h_b}^0\nabla_X\phi\,\mathrm{d}z \tag{2.32}$$

there holds,

$$|\overline{V}-\overline{V}[0,\beta b]|_{H^s}\leq\mu\varepsilon|\nabla_X\psi|_{H^{s+3}} \tag{2.33}$$

Proof. We will first prove the estimate on $\overline{V}-\overline{V}[0,\beta b]$ for $k\in\mathbb{N}$, and then use interpolation for $s\geq0$. By definition (2.8) and (2.32) we have that

$$|\overline{V}-\overline{V}[0,\beta b]|_{H^s}=\Big|\int_{-1+\beta b}^0\big[\frac{1}{h_b}\nabla_X(\phi_b-\phi)-\frac{1}{h}(\varepsilon\nabla_X\Big(\frac{\zeta}{h_b}\Big)z+\varepsilon\nabla_X\zeta)\partial_z\phi_b\big]\,\mathrm{d}z\Big|_{H^k}.$$

Now, note that h_b and h are only functions of X and satisfies (1.5) and (1.13), we can therefore use (A.9), (A.10), and (A.8) to get that

$$\begin{aligned}
|\overline{V} - \overline{V}[0,\beta b]|_{H^k} &\lesssim \Big|\frac{1}{h_b}\int_{-1+\beta b}^{0} \nabla_X(\phi_b - \phi)\,\mathrm{d}z\Big|_{H^k} + \varepsilon\Big|\frac{1}{h}\nabla_X\Big(\frac{\zeta}{h_b}\Big)\int_{-1+\beta b}^{0} z\partial_z\phi_b\,\mathrm{d}z\Big|_{H^k} \\
&\quad + \varepsilon\Big|\frac{1}{h}\nabla_X\zeta\int_{-1+\beta b}^{0}\partial_z\phi_b\,\mathrm{d}z\Big|_{H^k} \\
&\leq M(k)\|\nabla^{\mu}_{X,z}(\phi_b-\phi)\|_{H^{k+1,0}(\mathcal{S}_b)} + \varepsilon M(k+1)\|\partial_z\phi_b\|_{H^{k,0}(\mathcal{S}_b)} \\
&\quad + M(k)\sum_{j=1}^{k}\|\nabla^{\mu}_{X,z}\partial_z^{j-1}(\phi_b-\phi)\|_{H^{k-j+1,0}(\mathcal{S}_b)} \\
&\quad + \varepsilon M(k+1)\sum_{j=1}^{k}\|\partial_z^{j+1}\phi_b\|_{H^{k-j,0}(\mathcal{S}_b)} \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

We will now estimate each term. To estimate J_1, we apply (2.16) to get that

$$J_1 \leq \mu\varepsilon M(k+3)|\nabla_X\psi|_{H^{k+4}}.$$

To estimate J_2, we use Proposition A.4 to see that $|\partial_z\phi_0|_{H^k} \lesssim \mu|\nabla_X\psi|_{H^{k+1}}$ and combine it with (2.19) to get the estimate,

$$\begin{aligned}
J_2 &\leq \varepsilon M(k+1)(\|\nabla^{\mu}_{X,z}(\phi_b-\phi_0)\|_{H^{k+1,0}} + \|\partial_z\phi_0\|_{H^{k,0}}) \\
&\leq \mu\varepsilon M(s+3)|\nabla_X\psi|_{H^{k+3}}.
\end{aligned}$$

Finally, we will deal with J_3 and J_4. To that end, we need to trade the derivatives in ∂_z with derivatives in the horizontal variable by relating the functions with an elliptic problem. We introduce the notation

$$f \sim g \iff f(X,z) = r(X)g(X,z), \tag{2.34}$$

with $r \in H^k(\mathbb{R}^d)$ such that $|r|_{H^k} \leq M(k+1)$. Then, by construction, we have from (2.9) that

$$\begin{aligned}
(1+\mu|\nabla_X\sigma|^2)\partial_z^2\phi_b &= -\mu\Delta_X\phi_b - \mu\varepsilon(A[\nabla_X,\partial_z]\phi_b - \frac{1}{\varepsilon}|\nabla_X\sigma|^2\partial_z^2\phi_b) \\
&=: -\mu\Delta_X\phi_b - \mu\varepsilon\tilde{A}[\nabla_X,\partial_z]\phi_b,
\end{aligned}$$

where $\nabla_X\sigma$ is given by (2.6) and is of the form

$$\nabla_X\sigma \sim \varepsilon(1+z),$$

while $\tilde{A}[\nabla_X,\partial_z]$ is of the form

$$\tilde{A}[\nabla_X,\partial_z]\phi_b \sim \Delta_X\phi_b + (1+z)\nabla_X f\cdot\nabla_X\partial_z\phi_b + z\partial_z\phi_b,$$

for some function $f \in H^{k+3}(\mathbb{R}^d)$. Similarly, for ϕ defined by (2.14), we have the relation

$$\begin{aligned}
(1+\mu|\nabla_X\sigma|^2)\partial_z^2(\phi_b-\phi) &= \mu\Delta_X(\phi_b-\phi) - \mu\varepsilon\tilde{A}[\nabla_X,\partial_z](\phi_b-\phi) - \mu\varepsilon\tilde{A}[\nabla_X,\partial_z]\phi \\
&\quad - \mu|\nabla_X\sigma|^2\partial_z^2\phi.
\end{aligned}$$

Consequently, we can trade two derivatives in z by Δ_X, $\nabla_X\partial_z$, and ∂_z. From that point, we can deduce that for $k \geq 3$, we have

$$\partial_z^k(\phi_b - \phi) \sim \mu \sum_{\gamma\in\mathbb{N}^d \, |\gamma|\leq k-1} \partial_X^\gamma \partial_z\big((\phi_b - \phi) - \varepsilon\phi\big) + \sum_{j=1}^{k} \mu\varepsilon^2 \partial_z^j \phi,$$

For the last term we can use that $\partial_z^2\phi = -\mu\Delta_X\phi$. From these relations, and the control of the residual terms $r(X)$ in (2.34) with the product estimate (A.9), we may conclude from (2.16), (2.15), and (A.6) that

$$\begin{aligned} J_3 &\leq M(k+1)(\|\nabla^\mu_{X,z}(\phi_b - \phi)\|_{H^{k+1,0}(\mathcal{S}_b)} + \mu\varepsilon|\nabla_X\psi|_{H^{k+1}}) \\ &\leq \mu(\varepsilon + \varepsilon\beta + \mu\beta^2)M(k+3)|\nabla_X\psi|_{H^{k+3}}. \end{aligned}$$

To conclude, we estimate J_4. Since there is an ε appearing we only need to introduce ϕ_0 and we obtain

$$\begin{aligned} J_4 &= \varepsilon M(k+1)\sum_{j=1}^{k}\Big(\|\partial_z^{j+1}(\phi_b - \phi_0)\|_{H^{k-j,0}(\mathcal{S}_b)} + \|\partial_z^{j+1}\phi_0\|_{H^{k-j,0}(\mathcal{S}_b)}\Big) \\ &\leq \varepsilon(\mu\varepsilon + \mu\beta + \mu)M(k+3)|\nabla_X\psi|_{H^{k+3}}. \end{aligned}$$

□

The next result concerns the expansion of $\overline{V}$ with respect to $\mu\beta$:

Proposition 2.21. *Let $d = 1, 2$, $t_0 > \frac{d}{2}$ and $s \geq 0$. Let $b \in C_c^\infty(\mathbb{R}^d)$ and $\zeta \in H^{\max\{t_0+2,s+3\}}(\mathbb{R}^d)$ be such that (1.5) and (1.13) are satisfied. Let $\mathcal{L}_1^\mu[\beta b]$ and $\mathcal{L}_2^\mu[\beta b]$ be two pseudo-differential operators defined by*

$$\begin{aligned} \mathcal{L}_1^\mu[\beta b] &= -\frac{1}{\beta}\sinh\big(\beta b(X)\sqrt{\mu}|\mathrm{D}|\big)\mathrm{sech}(\sqrt{\mu}|\mathrm{D}|)\frac{1}{\sqrt{\mu}|\mathrm{D}|} \\ \mathcal{L}_2^\mu[\beta b] &= -(\mathcal{L}_1^\mu[\beta b] + b)\frac{1}{\mu|\mathrm{D}|^2}. \end{aligned}$$

Let also F_1, F_2, F_3, and F_4 be four Fourier multipliers defined by

$$\mathrm{F}_1 = \frac{\tanh\big(\sqrt{\mu}|\mathrm{D}|\big)}{\sqrt{\mu}|\mathrm{D}|}, \quad \mathrm{F}_2 = \frac{3}{\mu|\mathrm{D}|^2}(1-\mathrm{F}_1), \quad \mathrm{F}_3 = \mathrm{sech}(\sqrt{\mu}|D|), \quad \mathrm{F}_4 = \frac{2}{\mu|\mathrm{D}|^2}(1-\mathrm{F}_3).$$

Let $\psi \in \dot{H}^{s+5}(\mathbb{R}^d)$ and consider the approximation:

$$\begin{aligned} \overline{V}_0 = \frac{1}{h_b}\mathrm{F}_1\nabla_X\psi + \frac{\beta}{h_b}\mathcal{L}_1^\mu[\beta b]\nabla_X\psi + \frac{\mu\beta}{2}\nabla_X\mathrm{F}_4\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) \\ - \frac{\mu\beta^2}{2}\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\nabla_X\psi)\big) - \mu\beta^2(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\nabla_X\psi). \end{aligned} \tag{2.35}$$

Then for $\overline{V}$ defined by (1.8), there holds

$$|\overline{V} - \overline{V}_0|_{H^s} \leq (\mu\varepsilon + \mu^2\beta^2)M(s+3)|\nabla_X\psi|_{H^{s+4}}. \tag{2.36}$$

Furthermore, let $\overline{V}_{\mathrm{app}}$ be defined by the approximation:

$$\begin{aligned} \overline{V}_{\mathrm{app}} = \nabla_X\psi + \frac{\mu}{h}\nabla_X\Big(\frac{h^3}{h_b^3}\mathrm{F}_2\psi\Big) + \frac{\mu\beta}{h}\nabla_X\Big(\frac{h^3}{h_b^3}\mathcal{L}_2^\mu[\beta b]\psi\Big) + \frac{\mu\beta}{2}\nabla_X\mathrm{F}_4\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) \\ - \frac{\mu\beta^2}{2}\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\nabla_X\psi)\big) - \mu\beta^2(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\nabla_X\psi) \end{aligned} \tag{2.37}$$

Then there holds

$$|\overline{V}-\overline{V}_{\rm app}|_{H^s}\leq(\mu^2\varepsilon+\mu\varepsilon\beta+\mu^2\beta^2)M(s+3)|\nabla_X\psi|_{H^{s+4}}. \tag{2.38}$$

Proof. We give the proof in four steps.

Step 1. *Construction of* $\overline{V}_0$. To construct $\overline{V}_0$, we use the solution of ϕ_0 given by (2.18), the solution ϕ_1 given by (2.21), and formula (2.8), formally discarding terms of order $\mu\varepsilon$, to get that

$$\begin{aligned}h_b\overline{V}_0&=\int_{-1+\beta b(X)}^0\nabla_X\phi_0\,\mathrm{d}z+\mu\beta\int_{-1+\beta b(X)}^0\nabla_X\phi_1\,\mathrm{d}z\\&=I_1+I_2.\end{aligned}$$

Then by direct computations, we get

$$\begin{aligned}I_1&=\mathcal{F}^{-1}\Big(\int_{-1+\beta b(X)}^0\cosh\left((z+1)\sqrt{\mu}|\xi|\right)\mathrm{sech}(\sqrt{\mu}|\xi|)\;i\xi\hat{\psi}(\xi)\,\mathrm{d}z\Big)(X)\\&=\frac{\tanh\left(\sqrt{\mu}|\mathrm{D}|\right)}{\sqrt{\mu}|\mathrm{D}|}\nabla_X\psi-\sinh\left(\beta b(X)\sqrt{\mu}|\mathrm{D}|\right)\mathrm{sech}(\sqrt{\mu}|\mathrm{D}|)\frac{1}{\sqrt{\mu}|\mathrm{D}|}\nabla_X\psi.\end{aligned}$$

While for I_2, we simplify the notation by defining $G=\nabla_X\cdot\mathcal{L}_1^{\mu}[\beta b]\nabla_X\psi$ and then make the observation

$$\mu\beta\int_{-h_b}^0\nabla_X\phi_1(X,z)\mathrm{d}z=\mu\beta h_b\int_{-1}^0(\nabla_X\phi_1)(X,h_bz)\mathrm{d}z.$$

Then by the chain rule, we have the relation

$$\nabla_X(\phi_1(X,h_bz))=(\nabla_X\phi_1)(X,h_bz)-\beta\nabla_Xb(\partial_z\phi_1)(X,h_bz),$$

and

$$\partial_z(\phi_1(X,h_bz))=h_b(\partial_z\phi_1)(X,h_bz),$$

from which we obtain

$$\begin{aligned}I_2&=\mu\beta\int_{-h_b}^0\nabla_X\phi_1(X,z)\mathrm{d}z\\&=\mu\beta h_b\int_{-1}^0\nabla_X(\phi_1(X,h_bz))\mathrm{d}z+\mu\beta^2(\nabla_Xb)\int_{-1}^0\partial_z(\phi_1(X,h_bz))\mathrm{d}z\\&=\mu\beta h_b\nabla_X\big(h_b(1-\mathrm{sech}(\sqrt{\mu}|\mathrm{D}|))\frac{1}{\mu|\mathrm{D}|^2}G\big)-\mu\beta^2(\nabla_Xb)h_b\frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{\sqrt{\mu}|\mathrm{D}|}G.\end{aligned}$$

Adding these computations yields,

$$\overline{V}_0=\frac{1}{h_b}\mathrm{F}_1\nabla_X\psi+\frac{\beta}{h_b}\mathcal{L}_1^{\mu}[\beta b]\nabla_X\psi+\frac{\mu\beta}{2}\nabla_X(h_b\mathrm{F}_4G)-\mu\beta^2(\nabla_Xb)\mathrm{F}_1G,$$

To conclude this step, we use (1.18) to approximate $G=\nabla_X\cdot(b\nabla_X\psi)+\mu R_1$, where $|R_1|_{H^k}\leq M(k+1)|\nabla_X\psi|_{H^{k+4}}$. Then we have constructed the approximation:

$$\begin{aligned}\overline{V}_0=&\frac{1}{h_b}\mathrm{F}_1\nabla_X\psi+\frac{\beta}{h_b}\mathcal{L}_1^{\mu}[\beta b]\nabla_X\psi+\frac{\mu\beta}{2}\nabla_X\mathrm{F}_4\nabla_X\cdot\mathcal{L}_1^{\mu}[\beta b]\nabla_X\psi\\&-\frac{\mu\beta^2}{2}\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\nabla_X\psi)\big)-\mu\beta^2(\nabla_Xb)\mathrm{F}_1\nabla_X\cdot(b\nabla_X\psi)+\mu^2\beta^2R_2.\end{aligned}$$

For some R_2 satisfying $|R_2|_{H^k}\leq M(k+1)|\nabla_X\psi|_{H^{k+4}}$.

Step 2. We will now prove the estimate on $\overline{V} - \overline{V}_0$, where we argue as in the proof of Proposition 2.20. In particular, we do the estimates for $k \in \mathbb{N}$, and then use interpolation for $s \geq 0$. Also, we define the approximation

$$\phi_{\mathrm{app}}^1 = \phi_0 + \mu\beta\phi_1,$$

and let R be the function constructed in the previous step satisfying estimate $|R|_{H^k} \leq M(k+1)|\nabla_X\psi|_{H^{k+4}}$. Then we have that

$$|\overline{V} - \overline{V}_0|_{H^k} = \Big|\int_{-1+\beta b}^0 \big[\frac{1}{h_b}\nabla_X(\phi_b - \phi_{\mathrm{app}}^1) - \frac{1}{h}(\varepsilon\nabla_X\Big(\frac{\zeta}{h_b}\Big)z + \varepsilon\nabla_X\zeta)\partial_z\phi_b\big]\,\mathrm{d}z\Big|_{H^k} + \mu^2\beta^2|R|_{H^k}.$$

We now use (A.9), (A.10) and (A.8) to obtain

$$\begin{aligned}
|\overline{V} - \overline{V}_0|_{H^k} &\lesssim \Big|\frac{1}{h_b}\int_{-1+\beta b}^0 \nabla_X(\phi_b - \phi_{\mathrm{app}}^1)\,\mathrm{d}z\Big|_{H^k} + \varepsilon\Big|\frac{1}{h}\nabla_X\Big(\frac{\zeta}{h_b}\Big)\int_{-1+\beta b}^0 z\partial_z\phi_b\,\mathrm{d}z\Big|_{H^k} \\
&\quad + \varepsilon\Big|\frac{1}{h}\nabla_X\zeta\int_{-1+\beta b}^0 \partial_z\phi_b\,\mathrm{d}z\Big|_{H^k} + \mu^2\beta^2|R|_{H^k} \\
&\leq M(k)\|\nabla_{X,z}^\mu(\phi_b - \phi_{\mathrm{app}}^1)\|_{H^{k+1,0}(\mathcal{S}_b)} + \varepsilon M(k+1)\|\partial_z\phi_b\|_{H^{k,0}(\mathcal{S}_b)} \\
&\quad + M(k)\sum_{j=1}^k \|\nabla_{X,z}^\mu\partial_z^{j-1}(\phi_b - \phi_{\mathrm{app}}^1)\|_{H^{k-j+1,0}(\mathcal{S}_b)} + \varepsilon M(k+1)\sum_{j=1}^k \|\partial_z^{j+1}\phi_b\|_{H^{k-j,0}(\mathcal{S}_b)} \\
&\quad + \mu^2\beta^2 M(k+1)|\nabla_X\psi|_{H^{k+4}} \\
&= II_1 + II_2 + II_3 + II_4 + II_5.
\end{aligned}$$

We will now estimate each term. To estimate II_1, we apply (2.24) to get that

$$II_1 \leq \mu(\varepsilon + \varepsilon\beta + \mu\beta^2)M(k+3)|\nabla_X\psi|_{H^{k+4}}.$$

The estimate on II_2, is the same as for J_2 in the proof of Proposition 2.20:

$$\begin{aligned}
II_2 &\leq \varepsilon M(k+1)(\|\nabla_{X,z}^\mu(\phi_b - \phi_0)\|_{H^{k+1,0}} + \|\partial_z\phi_0\|_{H^{k,0}}) \\
&\leq \mu\varepsilon M(s+3)|\nabla_X\psi|_{H^{k+3}}.
\end{aligned}$$

Lastly, the estimates on II_3 and II_4 are similar to the estimates on J_3 and J_4 in the proof of Proposition 2.20. In particular, we trade the derivatives in ∂_z with derivatives in the horizontal variable by relating the functions with an elliptic problem. We recall the notation (2.34):

$$f \sim g \iff f(X,z) = r(X)g(X,z),$$

with $r \in H^k(\mathbb{R}^d)$ such that $|r|_{H^k} \leq M(k+1)$. Then for $\phi_{\mathrm{app}}^1 = \phi_0 + \mu\beta\phi_1$ defined by (2.17) and (2.22), we have the relation

$$\begin{aligned}
(1+\mu|\nabla_X\sigma|^2)\partial_z^2(\phi_b - \phi_{\mathrm{app}}^1) = \mu\Delta_X(\phi_b - \phi_{\mathrm{app}}^1) - \mu\varepsilon\tilde{A}[\nabla_X,\partial_z](\phi_b - \phi_{\mathrm{app}}^1) - \mu\varepsilon\tilde{A}[\nabla_X,\partial_z]\phi_{\mathrm{app}}^1 \\
- \mu|\nabla_X\sigma|^2\partial_z^2\phi_{\mathrm{app}}^1 + \mu^2\beta^2 F.
\end{aligned}$$

where F is some function satisfying (2.23) and goes into the rest. Consequently, we can trade two derivatives in z by Δ_X, $\nabla_X\partial_z$, and ∂_z. From that point, we can deduce that for $k \geq 3$, we have

$$\partial_z^k(\phi_b - \phi_{\mathrm{app}}^1) \sim \mu\sum_{\gamma\in\mathbb{N}^d\ |\gamma|\leq k-1} \partial_X^\gamma\partial_z\big((\phi_b - \phi_{\mathrm{app}}^1) - \varepsilon\phi_{\mathrm{app}}^1\big) + \sum_{j=1}^k \mu\varepsilon^2\partial_z^j\phi_{\mathrm{app}}^1,$$

From this estimate, where we control the residual terms $r(X)$ in (2.34) with the product estimate (A.9), then combine it with (2.24) and (A.6) to get

$$\begin{aligned} II_3 &\leq M(k+1)(\|\nabla^{\mu}_{X,z}(\phi_b - \phi^1_{\mathrm{app}})\|_{H^{k+1,0}(\mathcal{S}_b)} + \mu\varepsilon|\nabla_X\psi|_{H^{k+1}}) \\ &\leq \mu(\varepsilon + \varepsilon\beta + \mu\beta^2)M(k+3)|\nabla_X\psi|_{H^{k+3}}. \end{aligned}$$

To conclude, we need an estimate on II_4. But since $II_4 = J_4$, we have that

$$II_4 \leq \varepsilon(\mu\varepsilon + \mu\beta + \mu)M(k+3)|\nabla_X\psi|_{H^{k+3}}.$$

Step 3. *Construction of* $\overline{V}_{\mathrm{app}}$. The next step is to construct $\overline{V}_{\mathrm{app}}$ by replacing ϕ_b with ϕ_{app} in (2.8):

$$\overline{V}_{\mathrm{app}} = \int_{-1+\beta b}^{0} \big[\frac{1}{h_b}\nabla_X\phi_{\mathrm{app}} - \frac{1}{h}(\varepsilon\nabla_X\Big(\frac{\zeta}{h_b}\Big)z + \varepsilon\nabla_X\zeta)\partial_z\phi_{\mathrm{app}}\big]\,\mathrm{d}z. \tag{2.39}$$

Then using (2.30), we obtain that

$$\begin{aligned} \overline{V}_{\mathrm{app}} &= \int_{-1+\beta b}^{0}\frac{1}{h_b}\nabla_X\psi\,\mathrm{d}z + \int_{-1+\beta b}^{0}\frac{1}{h_b}\nabla_X\Big(\frac{h^2}{h_b^2}(\phi_0-\psi)\Big)\,\mathrm{d}z \\ &\quad - \int_{-1+\beta b}^{0}\frac{1}{h}(z\varepsilon\nabla_X(\frac{\zeta}{h_b}) + \varepsilon\nabla_X\zeta)\partial_z\Big(\frac{h^2}{h_b^2}(\phi_0-\psi)\Big)\,\mathrm{d}z \\ &\quad + \mu\beta\frac{1}{h_b}\int_{-1+\beta b}^{0}\nabla_X\phi_1\mathrm{d}z - \mu\varepsilon\beta\int_{-1+\beta b}^{0}\frac{1}{h}(z\nabla_X(\frac{\zeta}{h_b}) + \nabla_X\zeta)\partial_z\phi_1\,\mathrm{d}z \\ &= III_1 + III_2 + III_3 + III_4 + III_5. \end{aligned}$$

Clearly, $III_1 = \nabla_X\psi$ and to compute $III_2 + III_3$ we use formula (2.18) for ϕ_0:

$$\begin{aligned} III_2 + III_3 &= \frac{1}{h}\nabla_X\Big(\frac{h^3}{h_b^3}\frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{\sqrt{\mu}|\mathrm{D}|}\psi\Big) \\ &\quad - \frac{1}{h}\nabla_X\Big(\frac{h^3}{h_b^3}\Big(\sinh(\beta b(X)\sqrt{\mu}|\mathrm{D}|)\mathrm{sech}(\sqrt{\mu}|\mathrm{D}|)\frac{1}{\sqrt{\mu}|\mathrm{D}|}\psi - (-1+\beta b)\psi\Big)\Big) \\ &\quad - \varepsilon\beta\zeta h\frac{\nabla_X b}{h_b^3}\Big(\cosh(\beta b(X)\sqrt{\mu}|\mathrm{D}|)\mathrm{sech}(\sqrt{\mu}|\mathrm{D}|) - 1\Big)\psi \\ &= \frac{\mu}{3h}\nabla_X\Big(\frac{h^3}{h_b^3}\frac{3}{\mu|\mathrm{D}|^2}\Big(1 - \frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{\sqrt{\mu}|\mathrm{D}|}\Big)\Delta_X\psi\Big) \\ &\quad - \frac{1}{h}\nabla_X\Big(\frac{h^3}{h_b^3}\Big(\sinh(\beta b(X)\sqrt{\mu}|\mathrm{D}|)\mathrm{sech}(\sqrt{\mu}|\mathrm{D}|)\frac{1}{\sqrt{\mu}|\mathrm{D}|}\psi - \beta b\psi\Big)\Big) + \mu\varepsilon\beta R_5, \end{aligned}$$

where R_5 is given by

$$R_5 = -\zeta h\frac{\nabla_X b}{h_b^3}\Big(\cosh(\beta b(X)\sqrt{\mu}|\mathrm{D}|)\mathrm{sech}(\sqrt{\mu}|\mathrm{D}|) - 1\Big)\frac{1}{\mu|\mathrm{D}|^2}\Delta_X\psi.$$

Moreover, using the algebra property of the Sobolev spaces (A.9), (A.10), and estimate (1.17), we have that

$$|R_5|_{H^k} \leq M(k+1)|\nabla_X\psi|_{H^{k+1}}. \tag{2.40}$$

Next, we see that III_4 is already treated in Step 1. and satisfies:

$$III_4 = \frac{\mu\beta}{2}\nabla_X \mathrm{F}_4 \nabla_X \cdot \mathcal{L}_1^\mu[\beta b]\nabla_X\psi - \frac{\mu\beta^2}{2}\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\nabla_X\psi)\big) - \mu\beta^2(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\nabla_X\psi) + \mu^2\beta^2 R_6,$$

for some function R_6 satisfying $|R_6|_{H^k} \leq M(k+1)|\nabla_X\psi|_{H^{k+4}}$. Lastly, for the term III_5, we use integration by parts to find the expressions

$$\begin{aligned}
III_5 &= \mu\varepsilon\beta\int_{-1+\beta b}^0 \frac{1}{h}(z\nabla_X(\frac{\zeta}{h_b}) + \nabla_X\zeta)\partial_z\phi_1 \,\mathrm{d}z \\
&= -\mu\varepsilon\beta\frac{h_b^2}{h}\nabla_X(\frac{\zeta}{h_b})\Big(\frac{1}{2}\mathrm{F}_4\nabla_X\cdot(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi) + \frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{\sqrt{\mu}|\mathrm{D}|}\nabla_X\cdot(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi)\Big) \\
&\quad - \mu\varepsilon\beta\frac{h_b}{h}\nabla_X\zeta\frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{\sqrt{\mu}|\mathrm{D}|}\nabla_X\cdot(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi)
\end{aligned}$$

The multipliers are bounded on $H^k(\mathbb{R}^d)$ and combined with Proposition 1.10 we get that

$$|III_5|_{H^k} \leq \mu\varepsilon\beta M(k+1)|\nabla_X\psi|_{H^{k+1}}.$$

Adding these identities in the definition of $\overline{V}_{\mathrm{app}}$ we get that

$$\begin{aligned}
\overline{V}_{\mathrm{app}} &= \int_{-1+\beta b}^0 \big[\frac{1}{h_b}\nabla_X\phi_{\mathrm{app}} - \frac{1}{h}(z\varepsilon\nabla_X(\frac{\zeta}{h_b}) + \varepsilon\nabla_X\zeta)\partial_z\phi_{\mathrm{app}}\big]\,\mathrm{d}z + \mu\varepsilon\beta R_7 \\
&= \nabla_X\psi + \frac{\mu}{h}\nabla_X\Big(\frac{h^3}{h_b^3}\mathrm{F}_2\psi\Big) + \frac{\mu\beta}{h}\nabla_X\Big(\frac{h^3}{h_b^3}\mathcal{L}_2^\mu[\beta b]\psi\Big) + \frac{\mu\beta}{2}\nabla_X\mathrm{F}_4\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) \\
&\quad - \frac{\mu\beta^2}{2}\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\nabla_X\psi)\big) - \frac{\mu\beta^2}{2}(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\nabla_X\psi),
\end{aligned}$$

where R_7 is some generic function satisfying $|R_7|_{H^k} \leq M(k+1)|\nabla_X\psi|_{H^{k+4}}$.

Step 4. *Proof of* (2.38). We use the definition (2.39) of $\overline{V}_{\mathrm{app}}$ and (A.8) to identify the terms

$$\begin{aligned}
|\overline{V} - \overline{V}_{\mathrm{app}}|_{H^k} &= \Big|\int_{-1+\beta b}^0 \big[\frac{1}{h_b}\nabla_X(\phi_b - \phi_{\mathrm{app}}) - \frac{1}{h}(\varepsilon\nabla_X\Big(\frac{\zeta}{h_b}\Big)z + \varepsilon\nabla_X\zeta)\partial_z(\phi_b - \phi_{\mathrm{app}})\big]\,\mathrm{d}z\Big|_{H^k} \\
&\leq M(k)\|\nabla_{X,z}^\mu(\phi_b - \phi_{\mathrm{app}})\|_{H^{k+1,0}(\mathcal{S}_b)} + M(k+1)\|\partial_z(\phi_b - \phi_{\mathrm{app}})\|_{H^{k,0}(\mathcal{S}_b)} \\
&\quad + M(k)\sum_{j=1}^k \|\nabla_{X,z}^\mu\partial_z^{j-1}(\phi_b - \phi_{\mathrm{app}})\|_{H^{k-j+1,0}(\mathcal{S}_b)} \\
&\quad + \varepsilon M(k+1)\sum_{j=1}^k \|\partial_z^{j+1}(\phi_b - \phi_{\mathrm{app}})\|_{H^{k-j,0}(\mathcal{S}_b)} \\
&= IV_1 + IV_2 + IV_3 + IV_4.
\end{aligned}$$

For the two first terms we use estimate (2.31) to get that

$$IV_1 + IV_2 \leq (\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)M(k+3)|\nabla_X\psi|_{H^{k+4}}.$$

For the estimate of IV_3 and IV_4, we will use the same ideas that we used for I_3 and I_4. We first note that we only need to work with

$$\phi_{\mathrm{app}}^2 := \phi_0 + \mu\beta\phi_1 + \mu\varepsilon\phi_2,$$

constructed in Propositions 2.11, 2.14 and 2.16. Indeed, from Observation 2.18 we used the approximation (2.28) and depends polynomially on z. So formula (2.30) is related by ϕ^2_{app} through the relation

$$\partial_z^k(\phi^2_{\mathrm{app}} - \phi_{\mathrm{app}}) \sim \mu(\mu\varepsilon + \varepsilon\beta)\partial_z^k(z^2 R), \tag{2.41}$$

for $k \geq 1$ and where $R = R(X)$ satisfies (2.29). Then by definition of ϕ_0, ϕ_1, and ϕ_2 we have that

$$\begin{aligned}\partial_z^2(\phi_b - \phi^2_{\mathrm{app}}) &= -\mu\Delta_X(\phi_b - \phi_0 - \mu\beta\phi_1) - \mu\varepsilon A[\nabla_X, \partial_z](\phi_b - \phi_0) - \mu\varepsilon(A[\nabla_X, \partial_z]\phi_0 + \partial_z^2\phi_2)\\ &= -\mu\Delta_X(\phi_b - \phi_0 - \mu\beta\phi_1) - \mu\varepsilon\tilde{A}[\nabla_X, \partial_z](\phi_b - \phi_0) - \mu\varepsilon(A[\nabla_X, \partial_z]\phi_0 + \partial_z^2\phi_2)\\ &\quad - \mu|\nabla_X\sigma|^2\partial_z^2(\phi_b - \phi_0 - \mu\beta\phi_1 - \mu\varepsilon\phi_2)) - \mu^2\beta|\nabla_X\sigma|^2\partial_z^2\phi_1 + \mu^2\varepsilon|\nabla_X\sigma|^2\partial_z^2\phi_2,\end{aligned}$$

so that

$$\begin{aligned}(1 + \mu|\nabla_X\sigma|^2)\partial_z^2(\phi_b - \phi^2_{\mathrm{app}}) &= -\mu\Delta_X(\phi_b - \phi_0 - \mu\beta\phi_1) - \mu\varepsilon\tilde{A}[\nabla_X, \partial_z](\phi_b - \phi_0)\\ &\quad - \mu\varepsilon(A[\nabla_X, \partial_z]\phi_0 + \partial_z^2\phi_2) - \mu^2|\nabla_X\sigma|^2\partial_z^2(\beta\phi_1 + \varepsilon\partial_z^2\phi_2).\end{aligned}$$

Here derivatives of ϕ_1 is bounded using Proposition A.5 and by definition of σ, given by (2.6), we have that

$$\mu^2\beta|\nabla_X\sigma|^2\partial_z^2\phi_1 \sim \mu^2\varepsilon^2\beta\partial_z^2\phi_1.$$

Moreover, since ϕ_2 is only polynomial in z can use the notation above (2.34) to see the last term as

$$\mu^2\varepsilon|\nabla_X\sigma|^2\partial_z^2\phi_2 \sim \mu^2\varepsilon^3(1 + z + z^2).$$

Also, we see from observation 2.15 that

$$\mu\varepsilon(A[\nabla_X, \partial_z]\phi_0 + \partial_z^2\phi_2) \sim \mu\varepsilon\Delta_X(\phi_0 - \psi) + \mu\varepsilon(1 + z)\nabla_X f \cdot \nabla_X\partial_z\phi_0 + \mu\varepsilon z\partial_z\phi_0,$$

for some $f \in H^{k+3}(\mathbb{R}^d)$. Then arguing as in Step 2, we get the induction relation for $k \geq 3$:

$$\begin{aligned}\partial_z^k(\phi_b - \phi^2_{\mathrm{app}}) &\sim \mu \sum_{\gamma\in\mathbb{N}^d\ |\gamma|\leq k-1} \partial_X^\gamma\partial_z\Big((\phi_b - \phi^1_{\mathrm{app}}) + \varepsilon(\phi_b - \phi_0)\Big)\\ &\quad + \mu\varepsilon\sum_{j=1}^{k-2}\partial_z^j\big(\Delta_X\phi_0 + \nabla_X f \cdot \nabla_X\partial_z\phi_0 + \partial_z\phi_0\big) + \mu^2\varepsilon^2\beta\sum_{j=1}^{k}\partial_z^j\phi_1.\end{aligned}$$

Then as a result, we use these estimates with the product estimate (A.9), (A.4), and (A.5) to obtain the bound

$$\begin{aligned}\sum_{j=1}^{k}\|\partial_z^{j+1}(\phi_b - \phi^2_{\mathrm{app}})\|_{H^{k-j}(\mathcal{S}_b)} &\lesssim \mu M(k+1)\Big(\|\partial_z(\phi_b - \phi^1_{\mathrm{app}})\|_{H^{k,0}(\mathcal{S}_b)} + \varepsilon\|\partial_z(\phi_b - \phi_0)\|_{H^{k,0}(\mathcal{S}_b)}\\ &\quad + \mu^2\varepsilon|\nabla_X\psi|_{H^{k+1}} + \mu^2\varepsilon^2\beta|\nabla_X \cdot \mathcal{L}_1^\mu[\beta b]\nabla\psi|_{H^{k+1}}\Big),\end{aligned}$$

from which the estimate on IV_4 follows by (1.15), the relation (2.41) with estimate (2.29), and then (2.25) and (2.19):

$$\begin{aligned}IV_4 &\leq M(k)\sum_{j=1}^{k}\|\nabla^\mu_{X,z}\partial_z^{j-1}(\phi_b - \phi^2_{\mathrm{app}})\|_{H^{k-j+1,0}(\mathcal{S}_b)} + \mu(\mu\varepsilon + \varepsilon\beta)|\nabla_X\psi|_{H^{k+4}}\\ &\leq \mu(\mu\varepsilon + \varepsilon\beta)M(k+2)|\nabla_X\psi|_{H^{k+4}}.\end{aligned}$$

The same estimate holds for IV_5, and therefore completes the proof.

□

2.5. **Multi-scale expansions of $\mathcal{G}^\mu$.** In this section, we give the expansions of the Dirichlet-Neumann operator. We will use that $\mathcal{G}^\mu$ is directly related to $\overline{V}$ through (1.7) and (2.8). In particular, we have the following result two results.

Proposition 2.22. *Under the provisions of Proposition 2.20 we define*

$$\frac{1}{\mu}\mathcal{G}_b\psi = -\nabla_X \cdot \Big(\frac{h}{h_b}\int_{-h_b}^{0} \nabla_X \phi \, \mathrm{d}z\Big),$$

and for $\psi \in \dot{H}^{s+5}(\mathbb{R}^d)$ we have the estimate

$$\frac{1}{\mu}|\mathcal{G}^\mu\psi - \mathcal{G}_b\psi|_{H^s} \leq \mu\varepsilon M(s+3)|\nabla_X\psi|_{H^{s+4}}. \tag{2.42}$$

Proof. By definition of the Dirichlet-Neumann operator (1.9) and Proposition 2.20 we have the result

$$\begin{aligned}\frac{1}{\mu}|\mathcal{G}^\mu\psi - \mathcal{G}_b\psi|_{H^s} &= |\nabla_X \cdot (h(\overline{V} - \overline{V}[0,\beta b]\psi))|_{H^s} \\ &\leq \mu\varepsilon M(s+3)|\nabla_X\psi|_{H^{s+4}}.\end{aligned}$$

□

Proposition 2.23. *Under the provisions of Proposition 2.21, we can define the approximations*

$$\begin{aligned}\frac{1}{\mu}\mathcal{G}_0\psi = &-\mathrm{F}_1\Delta_X\psi - \beta(1+\frac{\mu}{2}\mathrm{F}_4\Delta_X)\nabla_X \cdot \big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) - \varepsilon\nabla_X \cdot \big(\zeta \mathrm{F}_1\nabla_X\psi\big) \\ &+ \frac{\mu\beta^2}{2}\nabla_X \cdot \big(\mathcal{B}[\beta b]\nabla_X\psi\big),\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\mu}\mathcal{G}_1\psi = &-\nabla_X \cdot (h\nabla_X\psi) - \frac{\mu}{3}\Delta_X\Big(\frac{h^3}{h_b^3}\mathrm{F}_2\Delta_X\psi\Big) - \mu\beta\Delta_X\big(\mathcal{L}_2^\mu[\beta b]\Delta_X\psi\big) \\ &- \frac{\mu\beta}{2}\mathrm{F}_4\Delta_X\nabla_X \cdot \big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) + \frac{\mu\beta^2}{2}\nabla_X \cdot \big(\mathcal{B}[\beta b]\nabla_X\psi\big),\end{aligned} \tag{2.43}$$

where

$$\begin{aligned}\mathcal{B}[\beta b]\nabla_X\psi = &\, b\mathrm{F}_4\nabla_X(\nabla_X \cdot (b\nabla_X\psi)) \\ &+ h_b\nabla_X\big(b\mathrm{F}_4\nabla_X \cdot (b\nabla_X\psi)) + 2h_b(\nabla_X b)\mathrm{F}_1\nabla_X \cdot (b\nabla_X\psi).\end{aligned} \tag{2.44}$$

Moreover, for $\psi \in \dot{H}^{s+6}(\mathbb{R}^d)$ we have the following estimates on the Dirichlet-Neumann operator

$$\frac{1}{\mu}|\mathcal{G}^\mu\psi - \mathcal{G}_0\psi|_{H^s} \leq (\mu\varepsilon + \mu^2\beta^2)M(s+3)|\nabla_X\psi|_{H^{s+5}} \tag{2.45}$$

$$\frac{1}{\mu}|\mathcal{G}^\mu\psi - \mathcal{G}_1\psi|_{H^s} \leq (\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)M(s+3)|\nabla_X\psi|_{H^{s+5}}. \tag{2.46}$$

Proof. To prove inequality (2.45), we introduce a generic function R such that

$$|R|_{H^s} \leq M(s+3)|\nabla_X\psi|_{H^{s+5}}. \tag{2.47}$$

Then note that the first two terms in $\mathcal{G}_0$ are obtained from the first two terms in $\overline{V}_0$. Indeed, let $G = \nabla_X \cdot \mathcal{L}_1^\mu[\beta b]\nabla_X\psi$ and use formula (2.35) to observe that

$$\begin{aligned}
\frac{1}{\mu}\mathcal{G}_0\psi &= -\nabla_X \cdot (h\overline{V}_0) \\
&= -\nabla_X \cdot \Big(\frac{h}{h_b}\mathrm{F}_1\nabla_X\psi\Big) - \beta\nabla_X \cdot \Big(\frac{h}{h_b}\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\Big) \\
&\quad - \frac{\mu\beta}{2}\nabla_X \cdot (h\mathrm{F}_4\nabla_X G) + \frac{\mu\beta^2}{2}\nabla_X \cdot \Big(h\big(\nabla_X\big(b\mathrm{F}_4\nabla_X \cdot (b\nabla_X\psi)\big) + 2(\nabla_X b)\mathrm{F}_1\nabla_X \cdot (b\nabla_X\psi)\big)\Big) \\
&= \mathrm{RHS}_1 + \mathrm{RHS}_2 + \mathrm{RHS}_3,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{RHS}_1 :&= -\nabla_X \cdot \Big(\frac{h}{h_b}\mathrm{F}_1\nabla_X\psi\Big) - \beta\nabla_X \cdot \Big(\frac{h}{h_b}\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\Big) \\
&= -\mathrm{F}_1\Delta_X\psi - \varepsilon\nabla_X \cdot \big(\frac{\zeta}{h_b}\mathrm{F}_1\nabla_X\psi\big) - \beta\nabla_X \cdot \big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) - \varepsilon\beta\nabla_X \cdot \big(\frac{\zeta}{h_b}\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big),
\end{aligned}$$

and we use (1.18) to get the approximation

$$\mathcal{L}_1^\mu[\beta b]\nabla_X\psi = -\beta b\nabla_X\psi + \mu R.$$

Then we obtain that

$$\begin{aligned}
\mathrm{RHS}_1 &= -\mathrm{F}_1\Delta_X\psi - \beta\nabla_X \cdot \big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) - \varepsilon\nabla_X \cdot \big(\frac{\zeta}{h_b}\mathrm{F}_1\nabla_X\psi\big) + \varepsilon\nabla_X \cdot \big(\frac{\zeta}{h_b}\beta b\nabla_X\psi\big) + \mu\varepsilon R \\
&= -\mathrm{F}_1\Delta_X\psi - \beta\nabla_X \cdot \big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) - \varepsilon\nabla_X \cdot \big(\zeta\mathrm{F}_1\nabla_X\psi\big) + \mu\varepsilon R.
\end{aligned}$$

For the remaining three terms, we first note that

$$\begin{aligned}
\mathrm{RHS}_2 :&= -\frac{\mu\beta}{2}\nabla_X \cdot (h\mathrm{F}_4\nabla_X G) \\
&= -\frac{\mu\beta}{2}\mathrm{F}_4\Delta_X G + \frac{\mu\beta^2}{2}\nabla_X \cdot (b\mathrm{F}_4\nabla_X G) + \frac{\mu\varepsilon\beta}{2}R_1,
\end{aligned}$$

where R_1 is given by

$$R_1 = -\nabla_X \cdot (\zeta\mathrm{F}_4\nabla_X G),$$

Using the estimates in Proposition A.7 and (1.18) allows us to put R_1 in the rest R satisfying (2.47). Moreover, since $G = \nabla_X \cdot \big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big)$, we obtain

$$\begin{aligned}
\mathrm{RHS}_2 = &-\frac{\mu\beta}{2}\mathrm{F}_4\Delta_X\nabla_X \cdot \big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) + \frac{\mu\beta^2}{2}\nabla_X \cdot \big(b\mathrm{F}_4\nabla_X(\nabla_X \cdot (b\nabla_X\psi))\big) \\
&+ (\mu\varepsilon\beta + \mu^2\beta^2)R.
\end{aligned}$$

To conclude, we identify the remaining terms with the ones in $\nabla_X \cdot \big(\mathcal{B}[\beta b]\nabla_X\psi\big)$ by (2.44), and we conclude by (2.36) that

$$\begin{aligned}
\frac{1}{\mu}|\mathcal{G}^\mu\psi - \mathcal{G}_0\psi|_{H^s} &= |\nabla_X \cdot (h(\overline{V} - \overline{V}_0))|_{H^s} \\
&\leq M(s+3)|\nabla_X\psi|_{H^{s+5}}.
\end{aligned}$$

The proof of inequality (2.46) is similar, where we first use formula (2.37) to get that

$$\begin{aligned}\frac{1}{\mu}\mathcal{G}_1\psi &= -\nabla_X\cdot(h\overline{V}_{\mathrm{app}})\\ &= -\nabla_X\cdot(h\nabla_X\psi) - \mu\Delta_X\Big(\frac{h^3}{h_b^3}\mathrm{F}_2\psi\Big) - \mu\beta\Delta_X\Big(\frac{h^3}{h_b^3}\mathcal{L}_2^\mu[\beta b]\psi\Big)\\ &\quad - \frac{\mu\beta}{2}\nabla_X\cdot\big(h\nabla_X\mathrm{F}_4\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big)\big)\\ &\quad + \frac{\mu\beta^2}{2}\nabla_X\cdot\Big(h(\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\nabla_X\psi)\big) + 2(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\nabla_X\psi))\Big).\end{aligned}$$

Then using the same arguments as for $\mathcal{G}_0$, for the last three terms, we know there is a function R such that

$$\begin{aligned}\frac{1}{\mu}\mathcal{G}_1\psi = &-\nabla_X\cdot(h\nabla_X\psi) - \mu\Delta_X\Big(\frac{h^3}{h_b^3}\mathrm{F}_2\psi\Big) - \mu\beta\Delta_X\Big(\frac{h^3}{h_b^3}\mathcal{L}_2^\mu[\beta b]\psi\Big)\\ &- \frac{\mu\beta}{2}\mathrm{F}_4\Delta_X\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) + \frac{\mu\beta^2}{2}\nabla_X\cdot\big(\mathcal{B}[\beta b]\nabla_X\psi\big) + (\mu\varepsilon\beta + \mu^2\beta^2)R,\end{aligned}$$

where R satisfies (2.47). Thus, we only use (1.16) to say

$$\mu\beta\mathcal{L}_2^\mu[\beta b] = \mu\beta R. \tag{2.48}$$

and combine it with the observation $\frac{h^3}{h_b^3} - 1 = \varepsilon R$, allowing us to neglect the term

$$\mu\beta\Delta_X\Big((\frac{h^3}{h_b^3} - 1)\mathcal{L}_2^\mu[\beta b]\Delta_X\psi\Big) = \mu\varepsilon\beta R.$$

By estimate (2.38) we conclude that (2.46) holds true.

□

3. Derivation of Boussinesq type systems with bathymetry

In this section, we derive a family of weakly dispersive Boussinesq systems in the shallow water regime with precision $\mathcal{O}(\mu\varepsilon)$ and $\mathcal{O}(\mu\varepsilon + \mu^2\beta^2)$.

3.1. Derivation of a Boussinesq type system with precision $\mathcal{O}(\mu\varepsilon)$.

We will now derive a system with precision $\mathcal{O}(\mu\varepsilon)$. This system is defined implicitly through the solution of an elliptic problem on a fixed domain with solution ϕ which depends on time through the Dirichlet data ψ.

Theorem 3.1. *Let $\mathcal{G}_b$ be defined by (2.22). Then for any $\mu \in (0,1]$, $\varepsilon \in [0,1]$, and $\beta \in [0,1]$ the water waves equations (1.4) are consistent, in the sense of Definition 1.15 with $n = 5$, at order $\mathcal{O}(\mu\varepsilon)$ with the Boussinesq type system:*

$$\begin{cases}\partial_t\zeta - \frac{1}{\mu}\mathcal{G}_b\psi = 0\\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}|\nabla_X\psi|^2 = 0,\end{cases} \tag{3.1}$$

Proof. For the first equation we use the approximation given in Proposition 2.22. While for the second equation we simply use (A.5) to replace the Dirichlet-Neumann operator by terms of order $\mathcal{O}(\mu\varepsilon)$. □

3.2. **Derivation of Boussinesq type systems with precision $\mathcal{O}(\mu\varepsilon + \mu^2\beta^2)$.** The system derived in this section will have the benefit of being explicit. This will reduce the computational cost from a numerical perspective, where the price we pay is given by an additional term of order $\mu^2\beta^2$. However, the system has improved dispersive properties when compared to classical models. Moreover, since the precision is of higher order in β, these systems can handle larger amplitude topography variations. The first result of this section reads:

Theorem 3.2. *Let* F_1 *and* F_4 *be the two Fourier multipliers given in Definition 1.6, and let* $\mathcal{L}_1^\mu$ *be given in Definition 1.10. Then for any* $\mu \in (0,1]$*,* $\varepsilon \in [0,1]$*, and* $\beta \in [0,1]$ *the water waves equations* (1.4) *are consistent, in the sense of Definition 1.15 with* $n = 6$*, at order* $\mathcal{O}(\mu\varepsilon + \mu^2\beta^2)$ *with the Boussinesq type system:*

$$\begin{cases} \partial_t\zeta + \mathrm{F}_1\Delta_X\psi + \beta(1 + \frac{\mu}{2}\mathrm{F}_4\Delta_X)\nabla_X \cdot (\mathcal{L}_1^\mu[\beta b]\nabla_X\psi) \\ \qquad\qquad +\varepsilon\mathrm{G}_1\nabla_X \cdot (\zeta\mathrm{G}_2\nabla_X\psi) - \frac{\mu\beta^2}{2}\nabla_X \cdot \big(\mathcal{B}[\beta b]\nabla_X\psi\big) = 0 \\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}(\mathrm{G}_1\nabla_X\psi)\cdot(\mathrm{G}_2\nabla_X\psi) = 0, \end{cases} \tag{3.2}$$

where

$$\mathcal{B}[\beta b]\bullet = b\mathrm{F}_4\nabla_X(\nabla_X \cdot (b\bullet)) + h_b\nabla_X\big(b\mathrm{F}_4\nabla_X \cdot (b\bullet)\big) + 2h_b(\nabla_X b)\mathrm{F}_1\nabla_X \cdot (b\bullet),$$

and $\mathrm{G}_1, \mathrm{G}_2$ *are any Fourier multipliers such that for any* $s \geq 0$ *and* $u \in H^{s+2}(\mathbb{R}^d)$*, we have*

$$|(\mathrm{G}_j - 1)u|_{H^s} \lesssim \mu|u|_{H^{s+2}}.$$

Proof. To start, we replace the Dirichlet-Neumann operator by (A.5) and its expansion given by (2.45) and discarding all the terms of order $\mathcal{O}(\mu(\varepsilon + \mu\beta^2))$ in the water waves equations (1.4) yields,

$$\begin{cases} \partial_t\zeta + \mathrm{F}_1\Delta_X\psi + \beta(1 + \frac{\mu}{2}\mathrm{F}_4\Delta_X)\nabla_X \cdot \big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) + \varepsilon\nabla_X \cdot \big(\zeta\mathrm{F}_1\nabla_X\psi\big) \\ \qquad\qquad\qquad - \frac{\mu\beta^2}{2}\nabla_X \cdot \big(\mathcal{B}[\beta b]\nabla_X\psi\big) = (\mu\varepsilon + \mu^2\beta^2)R, \\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}|\nabla_X\psi|^2 = \mu\varepsilon R. \end{cases}$$

where we introduced a generic function R such that

$$|R|_{H^s} \leq M(s+3)|\nabla_X\psi|_{H^{s+5}}. \tag{3.3}$$

To complete the proof, we use the assumption on G_j whenever there is the appearance of an ε. Then apply estimate (2.45) up to the rest R satisfying (3.3). □

The next result concerns a Boussinesq type system for which the first equation is exact and where the unknowns are given in terms of $(\zeta, \overline{V})$.

Theorem 3.3. *Let* F_1 *and* F_4 *be the two Fourier multipliers given in Definition 1.6, and let* $\mathcal{L}_1^\mu$ *be given in Definition 1.10. Then for any* $\mu \in (0,1]$*,* $\varepsilon \in [0,1]$*, and* $\beta \in [0,1]$ *the water waves equations* (1.4) *are consistent, in the sense of Definition 1.15 with* $n = 7$*, at order* $\mathcal{O}(\mu\varepsilon + \mu^2\beta^2)$ *with the Boussinesq type system:*

$$\begin{cases} \partial_t\zeta + \nabla_X \cdot (h\overline{V}) = 0 \\ \partial_t\overline{V} + \mathcal{T}_0^\mu[\beta b, \varepsilon\zeta]\nabla_X\zeta + \frac{\varepsilon}{2}\nabla_X|\overline{V}|^2 = \mathbf{0}, \end{cases} \tag{3.4}$$

where

$$\mathcal{T}_0^\mu[\beta b, \varepsilon\zeta]\bullet = \frac{1}{h}\Big(\mathrm{F}_1 \bullet + \beta\mathcal{L}_1^\mu[\beta b] \bullet + \varepsilon\zeta\mathrm{F}_1 \bullet\Big) + \frac{\mu\beta}{2}\nabla_X\mathrm{F}_4\nabla_X \cdot \big(\mathcal{L}_1^\mu[\beta b] \bullet\big)$$
$$- \frac{\mu\beta^2}{2}\nabla_X\big(b\mathrm{F}_4\nabla_X \cdot (b\bullet)\big) - \mu\beta^2(\nabla_X b)\mathrm{F}_1\nabla_X \cdot (b\bullet).$$

Proof. The first equation is exact by identity (1.9), and so we only work with the second equation of (1.4). However, using Theorem 1.18 we can work directly of on the second equation of (3.4) in the case $\mathrm{G}_1 = \mathrm{G}_2 = \mathrm{Id}$. Also, since we will take the gradient of ψ we need to increase the regularity of our rest function. In particular, let R be a generic function such that

$$|R|_{H^s} \leq M(s+3)|\nabla_X \psi|_{H^{s+6}}.$$

Then by (2.35) there holds

$$\begin{aligned} h\overline{V} = &\frac{h}{h_b}\mathrm{F}_1\nabla_X\psi + \beta\frac{h}{h_b}\mathcal{L}_1^{\mu}[\beta b]\nabla_X\psi + \frac{\mu\beta}{2}h_b\nabla_X\mathrm{F}_4\nabla_X\cdot\mathcal{L}_1^{\mu}[\beta b]\nabla_X\psi \\ &- \frac{\mu\beta^2}{2}h_b\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\nabla_X\psi)\big) - \mu\beta^2 h_b(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\nabla_X\psi) + (\mu\varepsilon+\mu^2\beta^2)R. \end{aligned}$$

Moreover, by (1.18) and (A.7) we make the observation

$$\frac{h}{h_b}\Big(\mathrm{F}_1\nabla_X\psi + \beta\mathcal{L}_1^{\mu}[\beta b]\nabla_X\psi\Big) = \mathrm{F}_1\nabla_X\psi + \beta\mathcal{L}_1^{\mu}[\beta b]\nabla_X\psi + \varepsilon\zeta\mathrm{F}_1\nabla_X\psi + \mu\varepsilon R,$$

so that

$$h\overline{V} = h\mathcal{T}_0^{\mu}[\beta b,\varepsilon\zeta]\nabla_X\psi + (\mu\varepsilon+\mu^2\beta^2)R.$$

From this expression, we can use the first equation to see that $\partial_t h = -\varepsilon\nabla_X\cdot(h\overline{V})$, and the estimates (A.7) together with the relation $\nabla_X\psi = \overline{V} + \mu R$ to get that:

$$\begin{aligned} h\partial_t\overline{V} &= (\partial_t h)\big(\mathrm{F}_1\nabla_X\psi - \overline{V}\big) + h\mathcal{T}_0^{\mu}[\beta b,\varepsilon\zeta]\nabla_X\partial_t\psi \\ &= h\mathcal{T}_0^{\mu}[\beta b,\varepsilon\zeta]\nabla_X\partial_t\psi. \end{aligned}$$

We may now use this relation in the second equation of (3.4) where we apply the gradient and $\mathcal{T}_0[\beta b,\varepsilon\zeta]$ to obtain that

$$h\partial_t\overline{V} + h\mathcal{T}_0^{\mu}[\beta b,\varepsilon\zeta]\nabla_X\zeta + \frac{\varepsilon}{2}h\mathcal{T}_0^{\mu}[\beta b,\varepsilon\zeta]\nabla_X|\overline{V}|^2 = (\mu\varepsilon+\mu^2\beta^2)R.$$

Then we conclude from the fact that $\mathcal{T}_0^{\mu}[\beta b,\varepsilon\zeta] = \mathrm{Id} + \mu R$.

□

3.2.1. *Hamiltonian structure.* We end this section by briefly commenting on the Hamiltonian structure of Boussinesq type systems with bathymetry. To do so, we recall the Hamiltonian of the water waves equations (1.4) [18]:

$$H(\zeta,\psi) = \frac{1}{2}\int_{\mathbb{R}^d}\zeta^2\,\mathrm{d}X + \frac{1}{2\mu}\int_{\mathbb{R}^d}\psi\mathcal{G}^{\mu}\psi\,\mathrm{d}X, \tag{3.5}$$

with $H(\zeta,\psi)$ satisfying the system

$$\begin{cases} \partial_t\zeta &= \delta_\psi H \\ \partial_t\psi &= -\delta_\zeta H, \end{cases} \tag{3.6}$$

where δ_ψ and δ_ζ are functional derivatives. Then replacing the Dirichlet-Neumann operator in (3.5) with its approximation its approximation (2.45) we obtain

$$\begin{aligned} H(\zeta,\psi) = & \frac{1}{2}\int_{\mathbb{R}^d}\zeta^2\,\mathrm{d}X + \frac{1}{2}\int_{\mathbb{R}^d}\mathrm{F}_1\nabla_X\psi\cdot\nabla_X\psi\,\mathrm{d}X \\ & + \frac{\varepsilon}{2}\int_{\mathbb{R}^d}\zeta\mathrm{G}\nabla_X\psi\cdot\mathrm{G}\nabla_X\psi\,\mathrm{d}X + \frac{\beta}{2}\int_{\mathbb{R}^d}\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\cdot\nabla_X\psi\,\mathrm{d}X \\ & + \frac{\mu\beta}{4}\int_{\mathbb{R}^d}\mathrm{F}_4\Delta_X\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\cdot\nabla_X\psi\,\mathrm{d}X - \frac{\mu\beta^2}{4}\int_{\mathbb{R}^d}\nabla_X\psi\cdot\mathcal{B}[\beta b]\nabla_X\psi\,\mathrm{d}X \\ & + \mathcal{O}(\mu\varepsilon+\mu^2\beta^2), \end{aligned} \tag{3.7}$$

for some Fourier multiplier G of the form $\mathrm{G} = 1+\mathcal{O}(\mu)$.

Now, to compute the functional derivatives in system (3.6) we note that the Fourier multipliers that appear are self-adjoint. While for the pseudo-differential operator of order zero, $\mathcal{L}_1^\mu$, one can use the fact that there exists an adjoint. However, a simpler approach is to approximate it by (1.19) and gives

$$\mathcal{L}_1^\mu[\beta b] = -b\mathrm{F}_3 - \frac{\mu\beta^2}{6}b^3|\mathrm{D}|^2\mathrm{F}_3 + \mathcal{O}(\mu^2\beta^4).$$

Using this relation implies

$$\begin{aligned} \mathrm{RHS}_1 : = & \frac{\beta}{2}\int_{\mathbb{R}^d}\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\cdot\nabla_X\psi\,\mathrm{d}X \\ = & -\frac{\beta}{2}\int_{\mathbb{R}^d}b\mathrm{F}_3\nabla_X\psi\cdot\nabla_X\psi\,\mathrm{d}X + \frac{\mu\beta^3}{12}\int_{\mathbb{R}^d}b^3\Delta_X\mathrm{F}_3\nabla_X\psi\cdot\nabla_X\psi\,\mathrm{d}X + \mathcal{O}(\mu^2\beta^5), \end{aligned}$$

and

$$\begin{aligned} \mathrm{RHS}_2 : = & \frac{\mu\beta}{4}\int_{\mathbb{R}^d}\mathrm{F}_4\Delta_X\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\cdot\nabla_X\psi\,\mathrm{d}X \\ = & -\frac{\mu\beta}{4}\int_{\mathbb{R}^d}\mathrm{F}_4\Delta_X(b\nabla_X\psi)\cdot\nabla_X\psi\,\mathrm{d}X + \mathcal{O}(\mu^2\beta^3). \end{aligned}$$

In particular, the first equation in (3.6) is given by

$$\delta_\psi H = -\mathrm{F}_1\Delta_X\psi + \nabla_X\cdot(\mathcal{A}^\mu[\beta b]\nabla_X\psi) - \varepsilon\mathrm{G}\nabla_X\cdot(\zeta\mathrm{G}\nabla_X\psi) + \frac{\mu\beta^2}{4}\nabla_X\cdot\Big(\big(\mathcal{B}[\beta b]+\mathcal{B}[\beta b]^*\big)\nabla_X\psi\Big),$$

where

$$\mathcal{A}^\mu[\beta b]\bullet = \frac{\beta}{2}\big(\mathrm{F}_3(b\bullet)+b\mathrm{F}_3\bullet\big) + \frac{\mu\beta}{2}\big(\mathrm{F}_4\Delta_X(b\bullet)+b\mathrm{F}_4\Delta_X\bullet\big) - \frac{\mu\beta^3}{12}\Big(b^3\Delta_X\mathrm{F}_3\bullet+\Delta_X\mathrm{F}_3(b^3\bullet)\Big),$$

and where $\mathcal{B}[\beta b]^*$ stands for the adjoint of $\mathcal{B}[\beta b]$ and reads

$$\mathcal{B}[\beta b]^*\nabla_X\psi = b\mathrm{F}_4\nabla_X\big(\nabla_X\cdot(b\nabla_X\psi)\big) + b\mathrm{F}_4\nabla_X(b\nabla_X\cdot(h_b\nabla_X\psi)) + 2b\mathrm{F}_1\nabla_X(h_b\nabla_X b\cdot\nabla_X\psi).$$

Similarly for the second equation:

$$\delta_\zeta H = -\zeta - \frac{\varepsilon}{2}|\mathrm{G}\nabla_X\psi|^2.$$

Then using (3.6), we will arrive at the following system

$$\begin{cases} \partial_t\zeta + \mathrm{F}_1\Delta_X\psi - \nabla_X\cdot\big(\mathcal{A}^\mu[\beta b]\nabla_X\psi\big) + \varepsilon\mathrm{G}\nabla_X\cdot(\zeta\mathrm{G}\nabla_X\psi) \\ \qquad\qquad -\frac{\mu\beta^2}{4}\nabla_X\cdot\Big(\big(\mathcal{B}[\beta b]+\mathcal{B}[\beta b]^*\big)\nabla_X\psi\Big) = 0 \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\mathrm{G}\nabla_X\psi|^2 = 0, \end{cases} \tag{3.8}$$

where its Hamiltonian reads:

$$H(\zeta,\psi)=\frac{1}{2}\int_{\mathbb{R}^d}\zeta^2\,\mathrm{d}X+\frac{1}{2}\int_{\mathbb{R}^d}\mathrm{F}_1\nabla_X\psi\cdot\nabla_X\psi\,\mathrm{d}X$$
$$+\frac{\varepsilon}{2}\int_{\mathbb{R}^d}\zeta\mathrm{G}\nabla_X\psi\cdot\mathrm{G}\nabla_X\psi\,\mathrm{d}X-\frac{\beta}{2}\int_{\mathbb{R}^d}(1+\frac{\mu}{2}\mathrm{F}_4\Delta_X)b\mathrm{F}_3\nabla_X\psi\cdot\nabla_X\psi\,\mathrm{d}X$$
$$-\frac{\mu\beta^2}{4}\int_{\mathbb{R}^d}\mathcal{B}[\beta b]\nabla_X\psi\cdot\nabla_X\psi\,\mathrm{d}X,$$

and is preserved by smooth solutions of (3.8).

Remark 3.4. *If we neglect terms of order $\mathcal{O}(\mu\varepsilon+\mu\beta)$, using $\mathrm{F}_3=1+\mathcal{O}(\mu)$, we obtain the system derived in* [10].

4. Derivation of Green-Naghdi type systems with bathymetry

In this section, we derive weakly dispersive Green-Naghdi systems in the shallow water regime with precision $\mathcal{O}(\mu^2\varepsilon+\mu\varepsilon\beta+\mu^2\beta^2)$. The following Green-Naghdi type system may be derived from the water waves equations:

Theorem 4.1. *Let F_2 and F_4 be the two Fourier multipliers given in Definition 1.6, and let $\mathcal{L}_2^\mu$ be given in Definition 1.10. Then for any $\mu\in(0,1]$, $\varepsilon\in[0,1]$, and $\beta\in[0,1]$ the water waves equations* (1.4) *are consistent, in the sense of Definition 1.15 with $n=6$, at order $\mathcal{O}(\mu^2\varepsilon+\mu\varepsilon\beta+\mu^2\beta^2)$ with the Green-Naghdi type system:*

$$\begin{cases}\partial_t\zeta+\nabla_X\cdot(h\mathcal{T}_1^\mu[\beta b,\varepsilon\zeta]\nabla_X\psi)-\frac{\mu\beta^2}{2}\nabla_X\cdot\big(\mathcal{B}[\beta b]\nabla_X\psi\big)=0\\ \partial_t\psi+\zeta+\dfrac{\varepsilon}{2}|\nabla_X\psi|^2-\dfrac{\mu\varepsilon}{2}h^2(\sqrt{\mathrm{F}_2}\Delta_X\psi)^2=0,\end{cases}\tag{4.1}$$

where

$$\mathcal{B}[\beta b]\bullet=b\mathrm{F}_4\nabla_X(\nabla_X\cdot(b\bullet))+h_b\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\bullet)\big)+2h_b(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\bullet),$$

and

$$\mathcal{T}_1^\mu[\beta b,\varepsilon\zeta]\bullet=\mathrm{Id}+\frac{\mu}{3h}\nabla_X\sqrt{\mathrm{F}_2}\Big(\frac{h^3}{h_b^3}\sqrt{\mathrm{F}_2}\nabla_X\cdot\bullet\Big)+\frac{\mu\beta}{h}\nabla_X\Big(\mathcal{L}_2^\mu[\beta b]\nabla_X\cdot\bullet\Big)$$
$$+\frac{\mu\beta}{2h}\mathrm{F}_4\nabla_X\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\bullet\big),$$

and $\sqrt{\mathrm{F}_2}$ is the square root of F_2.

Proof. We see that the first equation can be deduced by trading the Dirichlet-Neumann operator with its approximation (2.46). Indeed, we obtain that

$$\partial_t\zeta+\nabla_X\cdot(h\nabla_X\psi)+\frac{\mu}{3}\Delta_X\Big(\frac{h^3}{h_b^3}\mathrm{F}_2\Delta_X\psi\Big)+\mu\beta\Delta_X\big(\mathcal{L}_2^\mu[\beta b]\Delta_X\psi\big)$$
$$+\frac{\mu\beta}{2}\mathrm{F}_4\Delta_X\nabla_X\cdot\mathcal{L}_1^\mu[\beta b]\nabla_X\psi-\frac{\mu\beta^2}{2}\mathcal{B}[\beta b]\nabla_X\psi=(\mu^2\varepsilon+\mu\varepsilon\beta+\mu^2\beta^2)R,$$

where we introduce a generic function R such that

$$|R|_{H^s}\leq M(s+3)|\nabla_X\psi|_{H^{s+5}}.\tag{4.2}$$

Therefore we need to approximate the term

$$\frac{\mu}{3}\Delta_X\Big(\frac{h^3}{h_b^3}\mathrm{F}_2\Delta_X\psi\Big)$$

at order $\mathcal{O}(\mu^2\varepsilon + \mu\varepsilon\beta)$. Indeed, using (A.7) to say $\mathrm{F}_2 = 1 + \mu R$ and $\sqrt{\mathrm{F}_2} = 1 + \mu R$, we obtain

$$\frac{\mu}{3}\Delta_X\Big((\frac{h^3}{h_b^3} - 1)\mathrm{F}_2\Delta_X\psi\Big) = \frac{\mu}{3}\Delta_X\sqrt{\mathrm{F}_2}\Big((\frac{h^3}{h_b^3} - 1)\sqrt{\mathrm{F}_2}\Delta_X\psi\Big) + \mu^2\varepsilon R.$$

Gathering these observations yields

$$\frac{\mu}{3}\Delta_X\Big(\frac{h^3}{h_b^3}\mathrm{F}_2\Delta_X\psi\Big) = \frac{\mu}{3}\Delta_X\sqrt{\mathrm{F}_2}\Big(\frac{h^3}{h_b^3}\sqrt{\mathrm{F}_2}\Delta_X\psi\Big) + \mu^2\varepsilon R. \tag{4.3}$$

For the second equation, we use (A.5) to make the observation

$$\begin{aligned}
&\frac{(\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \varepsilon\nabla_X\zeta\cdot\nabla_X\psi)^2}{1+\varepsilon^2\mu|\nabla_X\zeta|^2} - \Big(\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \varepsilon\nabla_X\zeta\cdot\nabla_X\psi\Big)^2\\
&= \frac{\mu\varepsilon^2|\nabla_X\zeta|^2(\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \varepsilon\nabla_X\zeta\cdot\nabla_X\psi)^2}{1+\varepsilon^2\mu|\nabla_X\zeta|^2}\\
&= \mu\varepsilon^2 R.
\end{aligned}$$

Meaning that we only need to make an approximation of

$$\Big(\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \varepsilon\nabla_X\zeta\cdot\nabla_X\psi\Big)^2,$$

at order $\mathcal{O}(\mu)$. In particular, we use (A.5) to simplify the second equation in the water waves equations (1.4) to get that

$$\partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla_X\psi|^2 - \frac{\mu\varepsilon}{2}\Big(\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \varepsilon\nabla_X\zeta\cdot\nabla_X\psi\Big)^2 = \mu^2\varepsilon R. \tag{4.4}$$

Then using (A.5) we have that

$$\begin{aligned}
\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi &= -\nabla_X\cdot(h\nabla_X\psi) + \mu R\\
&= -h\Delta_X\psi - \varepsilon\nabla_X\zeta\cdot\nabla_X\psi + (\mu+\beta)R,
\end{aligned}$$

and we may use this expression to simplify (4.4) where we again use that $\sqrt{\mathrm{F}_2} = 1 + \mu R$. Thus, we conclude the proof of this theorem with estimate (A.5) up to a rest R satisfying (4.2).

□

One may also derive a system with unknowns $(\zeta, \overline{V})$ instead of (ζ,ψ), for which the first equation is exact. The new system reads:

Theorem 4.2. *Let* F_2 *and* F_4 *be the two the Fourier multipliers given in Definition 1.6, let* $\mathcal{L}_1^\mu$ *and* $\mathcal{L}_2^\mu$ *be given in Definition 1.10. Then for any* $\mu \in (0,1]$, $\varepsilon \in [0,1]$, *and* $\beta \in [0,1]$ *the water waves equations* (1.4) *are consistent, in the sense of Definition 1.15 with* $n = 7$, *at order* $\mathcal{O}(\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)$ *with the Green-Naghdi type system:*

$$\begin{cases}\partial_t\zeta + \nabla_X\cdot(h\overline{V}) = 0,\\ \partial_t(\mathcal{I}^\mu[h]\overline{V}) + \mathcal{I}[h]\mathcal{T}_2^\mu[\beta b, h]\nabla_X\zeta + \frac{\varepsilon}{2}\nabla_X\big(|\overline{V}|^2\big) + \mu\varepsilon\nabla_X\mathcal{R}_1^\mu[\beta b, h, \overline{V}] = \mathbf{0},\end{cases} \tag{4.5}$$

where $\overline{V}$ *defined by* (1.8),

$$\mathcal{I}^\mu[h]\bullet = \mathrm{Id} - \frac{\mu}{3h}\sqrt{\mathrm{F}_2}\nabla_X\Big(h^3\sqrt{\mathrm{F}_2}\nabla_X\cdot\bullet\Big), \tag{4.6}$$

$$\mathcal{T}_2^\mu[\beta b,\varepsilon\zeta]\bullet = \mathrm{Id} + \frac{\mu}{3h}\sqrt{\mathrm{F}_2}\nabla_X\Big(\frac{h^3}{h_b^3}\sqrt{\mathrm{F}_2}\nabla_X\cdot\bullet\Big) + \frac{\mu\beta}{h}\nabla_X\Big(\mathcal{L}_2^\mu[\beta b]\nabla_X\cdot\bullet\Big)$$
$$+ \frac{\mu\beta h_b}{2h}\nabla_X\mathrm{F}_4\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\bullet\big) - \frac{\mu\beta^2 h_b}{2h}\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\bullet)\big)$$
$$- \frac{\mu\beta^2 h_b}{h}(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\bullet),$$

and

$$\mathcal{R}_1^\mu[\beta b,h,\overline{V}] = -\frac{h^2}{2}(\nabla_X\cdot\overline{V})^2 - \frac{1}{3h}\big(\nabla_X(h^3\nabla_X\cdot\overline{V})\big)\cdot\overline{V} - \frac{1}{2}h^3\Delta_X(|\overline{V}|^2) + \frac{1}{6h}h^3\Delta_X(|\overline{V}|^2). \tag{4.7}$$

Proof. The first equation is exact so we only need to work on the second equation. Also, since $\overline{V}$ is related to the gradient of ψ we need to increase the regularity of the rest function R. In particular, we introduce a generic function R satisfying

$$|R|_{H^s} \leq M(s+3)|\nabla_X\psi|_{H^{s+6}}. \tag{4.8}$$

Then from the estimate (2.37), (2.48) and the argument in the previous proof, we know that

$$h\overline{V} = h\nabla_X\psi + \frac{\mu}{3}\nabla_X\sqrt{\mathrm{F}_2}\Big(\frac{h^3}{h_b^3}\sqrt{\mathrm{F}_2}\Delta_X\psi\Big) + \mu\beta\nabla_X\Big(\mathcal{L}_2^\mu[\beta b]\Delta_X\psi\Big) \tag{4.9}$$
$$+ \frac{\mu\beta h_b}{2}\nabla_X\mathrm{F}_4\nabla_X\cdot\big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big) - \frac{\mu\beta^2 h_b}{2}\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\nabla_X\psi)\big)$$
$$- \mu\beta^2 h_b(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\nabla_X\psi) + (\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)R.$$

Deriving this equality in time and using the definition of $\mathcal{T}_2[\beta b,\varepsilon\zeta]\bullet$ we obtain the relation

$$h\partial_t\overline{V} = \partial_t h(\nabla_X\psi - \overline{V}) + h\mathcal{T}_2^\mu[\beta b,\varepsilon\zeta]\nabla_X\partial_t\psi + \frac{\mu}{3}\nabla_X\sqrt{\mathrm{F}_2}\Big(\partial_t\Big(\frac{h^3}{h_b^3}\Big)\sqrt{\mathrm{F}_2}\Delta_X\psi\Big) \tag{4.10}$$
$$+ (\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)R.$$

Moreover, noting that $\dfrac{1}{h_b} = 1 + \beta R$ we can deduce that

$$\frac{\mu}{3}\nabla_X\sqrt{\mathrm{F}_2}\Big(\partial_t\Big(\frac{h^3}{h_b^3}\Big)\sqrt{\mathrm{F}_2}\Delta_X\psi\Big) = -\mu\varepsilon\nabla_X\sqrt{\mathrm{F}_2}\big(h^2\big(\nabla_X\cdot(h\overline{V})\big)\sqrt{\mathrm{F}_2}\Delta_X\psi\big) + \mu\varepsilon\beta R,$$

and using the first equation of system (4.5) we have that (4.10) is approximated by

$$h\mathcal{T}_2^\mu[\beta b,\varepsilon\zeta]\nabla_X\partial_t\psi = h\partial_t\overline{V} + \varepsilon\big(\nabla_X\cdot(h\overline{V})\big)(\nabla_X\psi - \overline{V}) \tag{4.11}$$
$$+ \mu\varepsilon\nabla_X\sqrt{\mathrm{F}_2}\big(h^2\big(\nabla_X\cdot(h\overline{V})\big)\sqrt{\mathrm{F}_2}\Delta_X\psi\big) + (\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2)R.$$

To conclude we simply need to approximate $\nabla_X\psi$ by $\overline{V}$ where we use (2.37) to get the classical approximation:

$$\begin{cases}\nabla_X\psi = \overline{V} + \mu R\\ \nabla_X\psi = \overline{V} - \frac{\mu}{3h}\nabla_X(h^3\nabla_X\cdot\overline{V}) + \mu^2 R.\end{cases} \tag{4.12}$$

Furthermore, using (4.12) and (4.11) we obtain that

$$h\mathcal{T}_2^\mu[\beta b, \varepsilon\zeta]\partial_t\nabla_X\psi = h\partial_t\overline{V} - \frac{\mu\varepsilon}{3h}\big(\nabla_X\cdot(h\overline{V})\big)\nabla_X(h^3\nabla_X\cdot\overline{V}) \tag{4.13}$$
$$+\mu\varepsilon\nabla_X\sqrt{\mathrm{F}_2}\big(h^2\big(\nabla_X\cdot(h\overline{V})\big)\sqrt{\mathrm{F}_2}\nabla_X\cdot\overline{V}\big)$$
$$+(\mu^2\varepsilon+\mu\varepsilon\beta+\mu^2\beta^2)R.$$

We will now simplify the second equation of the water waves system (1.4) at order $\mathcal{O}(\mu^2\varepsilon+\mu\varepsilon\beta+\mu^2\beta^2)$. Using Theorem 1.24 allows us to work with the second equation of (4.1). First use (4.12) to deduce that

$$|\nabla_X\psi|^2 = |\overline{V}|^2 - \frac{2\mu}{3h}\big(\nabla_X(h^3\nabla_X\cdot\overline{V})\big)\cdot\overline{V} + \mu^2 R.$$

With this relation, we may apply the gradient to the second equation of (4.1), and then apply the operator $\mathcal{T}_2^\mu[\beta b, \varepsilon\zeta]\bullet$, using the approximation (4.12), and discarding all the terms of order $\mathcal{O}(\mu^2\varepsilon+\mu\varepsilon\beta+\mu^2\beta^2)$ to get

$$\mathcal{T}_2^\mu[\beta b, \varepsilon\zeta]\nabla_X\partial_t\psi + \mathcal{T}_2^\mu[\beta b, \varepsilon\zeta]\nabla_X\zeta + \frac{\varepsilon}{2}\mathcal{T}_2^\mu[\beta b, \varepsilon\zeta]\Big(\nabla_X(|\overline{V}|^2 - \frac{2\mu}{3h}\big(\nabla_X(h^3\nabla_X\cdot\overline{V})\big)\cdot\overline{V})\Big)$$
$$-\frac{\mu\varepsilon}{2}h^2|\nabla_X\overline{V}|^2 = (\mu^2\varepsilon+\mu\varepsilon\beta+\mu^2\beta^2)R.$$

Then we apply (4.13), neglecting F_2 whenever there are terms with $\mu\varepsilon$ and together with the observation

$$\mathcal{T}_2^\mu[\beta b, \varepsilon\zeta]\bullet = \mathrm{Id} + \frac{\mu}{3h}\nabla_X\Big(h^3\nabla_X\cdot\bullet\Big) + \mu\beta R,$$

to deduce that

$$\partial_t\overline{V} + \mathcal{T}_2^\mu[\beta b, \varepsilon\zeta]\nabla_X\zeta + \frac{\varepsilon}{2}\nabla_X(|\overline{V}|^2) - \frac{\mu\varepsilon}{3h^2}\big(\nabla_X\cdot(h\overline{V})\big)\nabla_X(h^3\nabla_X\cdot\overline{V}) \tag{4.14}$$
$$+\frac{\mu\varepsilon}{h}\nabla_X\big(h^2\big(\nabla_X\cdot(h\overline{V})\big)\nabla_X\cdot\overline{V}\big) + \frac{\mu\varepsilon}{3h}\nabla_X\big(h^3\Delta_X(|\overline{V}|^2)\big)$$
$$-\frac{\mu\varepsilon}{3}\nabla_X\Big(\frac{1}{h}\big(\nabla_X(h^3\nabla_X\cdot\overline{V})\big)\cdot\overline{V}\Big) - \frac{\mu\varepsilon}{2}\nabla_X\big(h^2|\nabla_X\overline{V}|^2\big)$$
$$=(\mu^2\varepsilon+\mu\varepsilon\beta+\mu\beta^2)R.$$

Now, using $\sqrt{\mathrm{F}_2} = 1+\mu R$ and $\partial_t h = \varepsilon\nabla_X\cdot(h\overline{V})$ remark that

$$\partial_t(\overline{V} - \frac{\mu}{3h}\sqrt{\mathrm{F}_2}\nabla_X(h^3\sqrt{\mathrm{F}_2}\nabla_X\cdot\overline{V}))$$
$$= \partial_t\overline{V} - \frac{\mu}{3h}\sqrt{\mathrm{F}_2}\nabla_X(h^3\sqrt{\mathrm{F}_2}\nabla_X\cdot\partial_t\overline{V}) - \frac{\mu\varepsilon}{3h^2}\nabla_X\cdot(h\overline{V})\nabla_X(h^3\nabla_X\cdot\overline{V})$$
$$+\frac{\mu\varepsilon}{h}\nabla_X\big(h^2\nabla_X\cdot(h\overline{V})F_2\nabla_X\cdot\overline{V}\big)$$

So that from (4.14), and discarding all the terms of order $\mathcal{O}(\mu^2\varepsilon+\mu\varepsilon\beta+\mu\beta^2)$, we get

$$\partial_t(\overline{V} - \frac{\mu}{3h}\sqrt{\mathrm{F}_2}\nabla_X(h^3\sqrt{\mathrm{F}_2}\nabla_X\cdot\overline{V}))$$
$$= -\mathcal{T}_2^\mu[\beta b, h]\nabla_X\zeta - \frac{\varepsilon}{2}\nabla_X\big(|\overline{V}|^2\big) - \frac{\mu\varepsilon}{3h}\nabla_X\Big(h^3\Delta_X(|\overline{V}|^2)\Big) + \frac{\mu\varepsilon}{2}\nabla_X(h^2(\nabla_X\cdot\overline{V})^2)$$
$$+\frac{\mu\varepsilon}{3}\nabla_X\Big(\frac{1}{h}\big(\nabla_X(h^3\nabla_X\cdot\overline{V})\big)\cdot\overline{V}\Big) + \frac{\mu}{3h}\nabla_X(h^3\nabla_X\cdot(\mathcal{T}_2[\beta b, h]\nabla_X\zeta))$$
$$+\frac{\mu\varepsilon}{2}\nabla_X(h^3\Delta_X(|\overline{V}|^2)) + (\mu^2\varepsilon+\mu\varepsilon\beta+\mu\beta^2)R,$$

which at the end gives:

$$\partial_t(\mathcal{I}^\mu[h]\overline{V}) + \mathcal{I}[h]\mathcal{T}_2^\mu[\beta b, h]\nabla_X\zeta + \frac{\varepsilon}{2}\nabla_X\big(|\overline{V}|^2\big) + \mu\varepsilon\nabla_X\mathcal{R}_1^\mu[\beta b, h, \overline{V}] = (\mu^2\varepsilon + \mu\varepsilon\beta + \mu\beta^2)R.$$

□

4.0.1. *Hamiltonian structure.* We end this section by briefly commenting on the Hamiltonian structure of the Green-Naghdi type systems with bathymetry. Starting from the Hamiltonian of the water waves equations (3.5) and replacing the Dirichlet-Neumann operator by the approximation (2.43), using also (4.3), we get

$$\begin{aligned} H(\zeta,\psi) = \frac{1}{2}\int_{\mathbb{R}^d}\zeta^2\,\mathrm{d}X + \frac{1}{2}\int_{\mathbb{R}^d}h|\nabla_X\psi|^2\,\mathrm{d}X - \frac{\mu}{6}\int_{\mathbb{R}^d}\psi\Delta_X\sqrt{\mathrm{F}_2}\big(\frac{h^3}{h_b^3}\sqrt{\mathrm{F}_2}\Delta_X\psi\big)\,\mathrm{d}X \\ -\frac{\mu\beta}{2}\int_{\mathbb{R}^d}(\Delta_X\psi)\mathcal{L}_2^\mu[\beta b]\Delta_X\psi\,\mathrm{d}X + \frac{\mu\beta}{4}\int_{\mathbb{R}^d}\nabla_X\psi\cdot\mathrm{F}_4\Delta_X\big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big)\,\mathrm{d}X \\ +\frac{\mu\beta^2}{4}\int_{\mathbb{R}^d}\psi\nabla_X\cdot\big(\mathcal{B}[\beta b]\nabla_X\psi\big)\,\mathrm{d}X + \mathcal{O}(\mu^2\varepsilon + \mu\varepsilon\beta + \mu^2\beta^2). \end{aligned}$$

Then, we make use of the two expansions given by (1.19) and (1.20)

$$\mathcal{L}_1^\mu[\beta b] = -b\mathrm{F}_3 + \mathcal{O}(\mu\beta^2), \quad \mathcal{L}_2^\mu[\beta b] = -\frac{1}{2}b\mathrm{F}_4 + \frac{\beta^2}{6}b^3\mathrm{F}_3 + \mathcal{O}(\mu\beta^4),$$

and write

$$\begin{aligned} \mathrm{RHS}_1 &:= -\frac{\mu\beta}{2}\int_{\mathbb{R}^d}(\Delta_X\psi)\mathcal{L}_2^\mu[\beta b]\Delta_X\psi\,\mathrm{d}X \\ &= \frac{\mu\beta}{4}\int_{\mathbb{R}^d}b(\Delta_X\psi)\mathrm{F}_3\Delta_X\psi\,\mathrm{d}X - \frac{\mu\beta^3}{12}\int_{\mathbb{R}^d}b^3(\Delta_X\psi)\mathrm{F}_3\Delta_X\psi\,\mathrm{d}X + \mathcal{O}(\mu^2\beta^5) \\ &= \frac{\mu\beta}{4}\int_{\mathbb{R}^d}b(\Delta_X\psi)\mathrm{F}_3\Delta_X\psi\,\mathrm{d}X - \frac{\mu\beta^3}{12}\int_{\mathbb{R}^d}b^3\big(\sqrt{\mathrm{F}_3}\Delta_X\psi\big)^2\,\mathrm{d}X + \mathcal{O}(\mu^2\beta^5), \end{aligned}$$

where for the last equality, we used $\mathrm{F}_2 = \sqrt{\mathrm{F}_2} + \mathcal{O}(\mu)$, and

$$\begin{aligned} \mathrm{RHS}_2 :=& \frac{\mu\beta}{4}\int_{\mathbb{R}^d}\nabla_X\psi\cdot\mathrm{F}_4\Delta_X\big(\mathcal{L}_1^\mu[\beta b]\nabla_X\psi\big)\,\mathrm{d}X \\ =& -\frac{\mu\beta}{4}\int_{\mathbb{R}^d}\nabla_X\psi\cdot\mathrm{F}_4\Delta_X\big(b\nabla_X\psi\big)\,\mathrm{d}X + \mathcal{O}(\mu^2\beta^3). \end{aligned}$$

Deriving the equations associated to this approximated Hamiltonian thus obtained, we get the Green-Naghdi type system

$$\begin{cases} \partial_t\zeta + \nabla_X\cdot(h\nabla_X\psi) + \frac{\mu}{3}\Delta_X\sqrt{\mathrm{F}_2}\big(\frac{h^3}{h_b^3}\sqrt{\mathrm{F}_2}\Delta_X\psi\big) - \frac{\mu\beta}{4}\nabla_X\cdot\big(\mathcal{Q}^\mu[b]\nabla_X\psi\big) \\ \qquad -\frac{\mu\beta^2}{4}\nabla_X\cdot\Big(\big(\mathcal{B}[\beta b] + \mathcal{B}[\beta b]^*\big)\nabla_X\psi\Big) + \frac{\mu\beta^3}{6}\Delta_X\sqrt{\mathrm{F}_3}(b^3\sqrt{\mathrm{F}_3}\Delta_X\psi) = 0 \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla_X\psi|^2 - \frac{\mu\varepsilon}{2}h^2(\sqrt{\mathrm{F}_2}\Delta_X\psi)^2 = 0 \end{cases}$$

where

$$\mathcal{Q}^\mu[b]\bullet = \Big(\mathrm{F}_3\nabla_X(b\nabla_X\cdot\bullet) + \nabla_X(b\mathrm{F}_3\nabla_X\cdot\bullet)\Big) + \Big(\mathrm{F}_4\nabla_X\nabla_X\cdot\big(b\,\bullet\,\big) + b\mathrm{F}_4\Delta_X\,\bullet\,\Big),$$

and

$$\mathcal{B}[\beta b]\bullet = b\mathrm{F}_4\nabla_X(\nabla_X\cdot(b\bullet)) + h_b\nabla_X\big(b\mathrm{F}_4\nabla_X\cdot(b\bullet)\big) + 2h_b(\nabla_X b)\mathrm{F}_1\nabla_X\cdot(b\bullet).$$

Appendix A

A.1. **On the properties of pseudo-differential operators.** In this section, we will give a rigorous meaning to the pseudo-differential operators given in Proposition 1.10. Before turning to the proof, we recall the definition of a symbol.

Definition A.1. *Let $d = 1, 2$ and $m \in \mathbb{R}$. We say $L \in S^m$ is a symbol of order m if $L(X, \xi)$ is $C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and satisfies*

$$\forall \alpha \in \mathbb{N}^d, \quad \forall \gamma \in \mathbb{N}^d, \quad \langle \xi \rangle^{-(m-|\gamma|)} |\partial_X^\alpha \partial_\xi^\gamma L(X, \xi)| < \infty.$$

We also introduce the seminorm

$$\mathcal{M}_m(L) = \sup_{|\alpha| \leq \lceil \frac{d}{2} \rceil + 1} \sup_{|\gamma| \leq \lceil \frac{d}{2} \rceil + 1} \sup_{(X,\xi) \in \mathbb{R}^d \times \mathbb{R}^d} \Big\{ \langle \xi \rangle^{-(m-|\gamma|)} |\partial_X^\alpha \partial_\xi^\gamma L(X, \xi)| \Big\}. \tag{A.1}$$

Moreover, we recall the main tool we will use to justify the pseudo-differential operators in Sobolev spaces:

Theorem A.2. *Let $d = 1, 2$, $s \geq 0$, and $L \in S^m$. Then formula (1.10) defines a bounded pseudo-differential operator from $H^{s+m}(\mathbb{R}^d)$ to $H^s(\mathbb{R})$ and satisfies*

$$|\mathcal{L}[X, D]u|_{H^s} \leq \mathcal{M}_m(L) |u|_{H^{s+m}}. \tag{A.2}$$

With this Theorem at hand, we can now give the proof.

Proof of Proposition 1.10. We will first prove that for $s \geq 0$ the operators $\mathcal{L}_i^\mu$ are a uniformly bounded on $H^s(\mathbb{R}^d)$. To prove this point we need to verify that the symbols:

$$\begin{aligned}
L_1^\mu(\beta b(X), \xi) &= -\frac{1}{\beta} \sinh(\beta b(X) \sqrt{\mu}|\xi|) \mathrm{sech}(\sqrt{\mu}|\xi|) \frac{1}{\sqrt{\mu}|\xi|} \\
L_2^\mu(\beta b(X), \xi) &= \frac{1}{\beta} (\sinh(\beta b(X) \sqrt{\mu}|\xi|) \mathrm{sech}(\sqrt{\mu}|\xi|) \frac{1}{\sqrt{\mu}|\xi|} - \beta b) \frac{1}{\mu|\xi|^2} \\
L_3^\mu(\beta b(X), \xi) &= -\big(\cosh(\beta b(X) \sqrt{\mu}|\xi|) \mathrm{sech}(\sqrt{\mu}|\xi|) - 1 \big) \frac{1}{\mu|\xi|^2}
\end{aligned}$$

are elements of S^0 where the constants $\mathcal{M}_0(L_i)$ are independent of μ and β. We treat each symbol separately.

We start by proving that the symbol L_1^μ is in S^0. To do so, we will split the frequency domain into three regions. First, let $\beta\sqrt{\mu}|\xi| \leq 1$ and $\sqrt{\mu}|\xi| \leq 1$. Then since $L_1^\mu(\beta b(X), \xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and the Taylor expansion around $(X, 0)$ gives us

$$|\partial_X^\alpha \partial_\xi^\gamma L_1^\mu(\beta b(X), \xi)| \lesssim 1.$$

Next, consider the region $\beta\sqrt{\mu}|\xi| > 1$. Then we also have that $|\xi| \geq \sqrt{\mu}|\xi| > 1$. With this in mind, we can prove the necessary decay estimate. Indeed, since $b \in C_c^\infty(\mathbb{R}^d)$ satisfies (1.13), i.e. for $h_{b,\max} \in (0, 1)$:

$$0 < h_{b,\min} \leq 1 - \beta b(X),$$

combined with $\mathrm{sech}(x) \sim e^{-x}$ and $\sinh(x) \sim e^x$ for $x \in \mathbb{R}$, we have that

$$\begin{aligned}
\Big| \partial_X^\alpha \partial_\xi^\gamma \Big(\frac{\sinh(\beta b(X) \sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)} \Big) \Big| &\lesssim \mu^{\frac{|\gamma|}{2}} (1 + \sqrt{\mu}|\xi|)^{|\alpha|} e^{-h_{b,\min}\sqrt{\mu}|\xi|} \\
&\lesssim (\frac{2}{h_{b,\min}})^{|\alpha|} \mu^{\frac{|\gamma|}{2}} e^{-\frac{1}{2} h_{b,\min}\sqrt{\mu}|\xi|}.
\end{aligned}$$

Additionally, there holds

$$\mu^{\frac{|\gamma|}{2}}(1+|\xi|)^{|\gamma|}e^{-\sqrt{\mu}|\xi|} \lesssim 1,$$

and so we obtain the estimate

$$\Big|\partial_X^\alpha \partial_\xi^\gamma\Big(\frac{\sinh(\beta b(X)\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\Big)\Big| \lesssim (1+|\xi|)^{-|\gamma|}. \tag{A.3}$$

Then combining this estimate with the Leibniz rule we obtain that

$$\begin{aligned}|\partial_X^\alpha \partial_\xi^\gamma L_1^\mu(\beta b(X),\xi)| &\lesssim \frac{1}{\beta}\sum_{\gamma_1\in\mathbb{N}^d\,:\,\gamma_1\leq\gamma}\Big|\partial_X^\alpha\partial_\xi^{\gamma_1}\Big(\frac{\sinh(\beta b(X)\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\Big)\Big|\,\Big|\partial_\xi^{\gamma-\gamma_1}\Big((\sqrt{\mu}|\xi|)^{-1}\Big)\Big| \\ &\lesssim (\beta\sqrt{\mu}|\xi|)^{-1}\sum_{\gamma_1\in\mathbb{N}^d\,:\,\gamma_1\leq\gamma}(1+|\xi|)^{-|\gamma_1|}|\xi|^{|\gamma_1|-|\gamma|} \\ &\lesssim |\xi|^{-|\gamma|}.\end{aligned}$$

However, since we have that $|\xi|\geq\sqrt{\mu}|\xi|>1$, we obtain the desired result

$$|\partial_X^\alpha \partial_\xi^\gamma L_1^\mu(\beta b(X),\xi)| \lesssim \langle\xi\rangle^{-|\gamma|}. \tag{A.4}$$

Lastly, let $\beta\sqrt{\mu}|\xi|\leq 1$ and $\sqrt{\mu}|\xi|\geq 1$. In this case, we expand $x\mapsto\sinh(x)$ to obtain

$$\begin{aligned}L_1^\mu(\beta b(X),\xi) &= -b(X)\mathrm{sech}(\sqrt{\mu}|\xi|) \\ &\quad -\frac{1}{6\beta}\Big(\beta^3 b(X)^3\int_0^1\cosh(t\beta b(X)\sqrt{\mu}|\xi|)(1-t)^2\,\mathrm{d}t\Big)(\sqrt{\mu}|\xi|)^2\mathrm{sech}(\sqrt{\mu}|\xi|).\end{aligned}$$

To conclude, we observe that

$$|\partial_X^\alpha\partial_\xi^\gamma\big(b(X)\mathrm{sech}(\sqrt{\mu}|\xi|)\big)| \lesssim \mu^{\frac{|\gamma|}{2}}e^{-\sqrt{\mu}|\xi|} \lesssim \langle\xi\rangle^{-|\gamma|}.$$

For the second term, we let $t\in[0,1]$ and observe that we only need to consider

$$\frac{\cosh(t\beta b(X)\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}(\sqrt{\mu}|\xi|)^2,$$

for which the decay estimate follows similarly to (A.3). Indeed, we first observe that

$$\begin{aligned}\Big|\partial_X^\alpha\partial_\xi^\gamma\Big(\frac{\cosh(t\beta b(X)\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\Big)\Big| &\lesssim \mu^{\frac{|\gamma|}{2}}t^{|\alpha|}(1+\sqrt{\mu}t|\xi|)^{|\alpha|}e^{-h_{b,\min}\sqrt{\mu}|\xi|} \\ &\lesssim \mu^{\frac{|\gamma|}{2}}e^{-\frac{h_{b,\min}}{2}\sqrt{\mu}|\xi|},\end{aligned}$$

since $t\in[0,1]$. Then by the Leibniz rule, we get

$$\begin{aligned}\Big|\partial_X^\alpha\partial_\xi^\gamma\Big(\frac{\cosh(t\beta b(X)\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\mu|\xi|^2\Big)\Big| &\lesssim \sum_{\gamma_1\in\mathbb{N}^d\,:\,\gamma_1\leq\gamma}\mu^{\frac{|\gamma_1|}{2}+1}e^{-\frac{h_{b,\min}}{2}\sqrt{\mu}|\xi|}|\xi|^{2+|\gamma_1|-|\gamma|} \\ &\lesssim |\xi|^{-|\gamma|},\end{aligned}$$

for $|\xi|\geq\sqrt{\mu}|\xi|>1$. Consequently, we can conclude this case. By Theorem (A.2) there holds,

$$|\mathcal{L}_1^\mu[\beta b]u|_{H^s}\leq M(s)|u|_{H^s}.$$

For the symbol L_2^μ, we observe for $\beta\sqrt{\mu}|\xi|\leq 1$ and $\sqrt{\mu}|\xi|\leq 1$ and a Taylor expansion that it is smooth and bounded. Moreover, for frequencies such that $\beta\sqrt{\mu}|\xi|>1$, we can argue as we did for L_1^μ to get sufficient decay in the frequency variable at infinity. Lastly, in the case $\beta\sqrt{\mu}|\xi| \leq 1$ and $\sqrt{\mu}|\xi| \geq 1$ we use the following expansion

$$L_2^\mu(\beta b(X),\xi) = b(X)\big(\mathrm{sech}(\sqrt{\mu}|\xi|) - 1\big)\frac{1}{\mu|\xi|^2} + \frac{1}{6\beta}\Big(\beta^3 b(X)^3 \int_0^1 \cosh(t\beta b(X)\sqrt{\mu}|\xi|)(1-t)^2 \mathrm{d}t\Big)\mathrm{sech}(\sqrt{\mu}|\xi|),$$

and again argue as we did for L_1^μ.

The estimates on L_3^μ is simpler since it does not depend on $\frac{1}{\beta}$. Thus, using similar arguments we can prove the necessary decay at infinity, and a Taylor series to prove the boundedness for small frequencies.

The estimate (1.18) follows directly from the boundedness on $H^s(\mathbb{R}^d)$ of $\mathcal{L}_2^\mu[\beta b]$ since its symbol is in S^0 and that

$$L_1^\mu(\beta b(X),\xi) + b(X) = -\mu L_2^\mu(\beta b(X),\xi)|\xi|^2.$$

Indeed, the symbol $r_1(X,\xi) = L_2^\mu(\beta b(X),\xi)|\xi|^2$ is an element of S^2 and by Theorem A.2 we deduce that $\mathcal{R}_1[X,\mathrm{D}]u(X) = \mathcal{F}^{-1}(r_1(X,\xi)\hat{u}(\xi))(X)$ satisfies

$$|\mathcal{R}_1[X,\mathrm{D}]u|_{H^s} \lesssim |u|_{H^{s+2}},$$

so that

$$|\mathcal{L}_1^\mu[\beta b]u + bu|_{H^s} = \mu|\mathcal{R}[X,\mathrm{D}]u|_{H^s} \lesssim \mu|u|_{H^{s+2}}.$$

The next estimate, given by (1.19), is deduced from the Taylor expansion of the symbol L_1^μ given by:

$$L_1^\mu(\beta b(X),\xi) = -\Big(b(X) + \frac{\mu\beta^2}{6}b(X)^3|\xi|^2\Big)\mathrm{sech}(\sqrt{\mu}|\xi|) - \frac{\mu^2\beta^4}{120}b(X)^5|\xi|^4 r_2(X,\xi),$$

where the rest r_2 is given by

$$r_2(X,\xi) = \int_0^1 \cosh(t\beta b(X)\sqrt{\mu}|\xi|)(1-t)^4 \mathrm{d}t\,\mathrm{sech}(\sqrt{\mu}|\xi|),$$

and is an element of S^0 by arguing as above. By extension, the symbol $b(X)^5|\xi|^4 r_2(X,\xi) \in S^4$ and we conclude by Theorem A.2.

Lastly, we consider estimate (1.20). Again by a Taylor series expansion, we observe that

$$L_2^\mu(\beta b(X),\xi) = \Big(b(X)(\mathrm{sech}(\sqrt{\mu}|\xi|) - 1)\frac{1}{\mu|\xi|^2} + \frac{\beta^2}{6}b(X)^3\Big)\mathrm{sech}(\sqrt{\mu}|\xi|) + \frac{\mu\beta^4}{120}b(X)^5|\xi|^2 r_2(X,\xi),$$

where $b(X)^5|\xi|^2 r_2(X,\xi) \in S^2$, allowing us to conclude by Theorem A.2.

□

Remark A.3. *We note that we could improve the estimates in the proof above. For instance, we can get $L_1 \in S^{-1}$. However, the constant $\mathcal{M}_{-1}(L_1)$ would be singular with respect to β and μ.*

A.2. **Technical estimates.** In this section we give a series of multiplier estimates. To start, we recall the Fourier multiplier depending on the transverse variable:

$$\mathrm{F}_0 u(X) = \mathcal{F}^{-1}\Big(\frac{\cosh((z+1)\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\hat{u}(\xi)\Big)(X).$$

Then the first result reads:

Proposition A.4. *Let $s \in \mathbb{R}$ and take $u \in \mathscr{S}(\mathbb{R}^d)$, then there holds*

$$\begin{aligned}|\mathrm{F}_0 u|_{H^s} &\lesssim |u|_{H^s}\\ |\partial_z \mathrm{F}_0 u|_{H^s} &\lesssim \mu|\nabla_X u|_{H^{s+1}}\\ |\partial_z^2 \mathrm{F}_0 u|_{H^s} &\lesssim \mu|\nabla_X u|_{H^{s+1}}.\end{aligned}$$

Moreover, for $k \in \mathbb{N}$ and under condition (1.13) *we have similar estimates on the domain $\mathcal{S}_b = \mathbb{R}^d \times [-1+\beta b, 0]$:*

$$\begin{aligned}\|\mathrm{F}_0 u - u\|_{H^{k,0}(\mathcal{S}_b)} &\lesssim \mu|\nabla_X u|_{H^{k+1}}\\ \|\partial_z \mathrm{F}_0 u\|_{H^{k,0}(\mathcal{S}_b)} &\lesssim \mu|\nabla_X u|_{H^{k+1}}\\ \|\partial_z^2 \mathrm{F}_0 u\|_{H^{k,0}(\mathcal{S}_b)} &\lesssim \mu|\nabla_X u|_{H^{k+1}}.\end{aligned}$$

Proof. The estimates on $H^s(\mathbb{R}^d)$ are a direct consequence of Plancherel's identity and the Taylor expansion formula for $x \in \mathbb{R}$:

$$\cosh(x) = 1 + \frac{x^2}{2}\int_0^1 \cosh(tx)(1-t)\,\mathrm{d}t.$$

For the estimates on $\mathcal{S}_b$, we use that $-h_b(X) > -2$, by assumption (1.13), then extend the definition of F_0 to the domain $\mathcal{S} := \mathbb{R}^d \times [-2, 0]$. The first estimate on $\mathcal{S}_b$ is a consequence of

$$\|\mathrm{F}_0 u - u\|_{H^{k,0}(\mathcal{S}_b)} \leq \|\mathrm{F}_0 u - u\|_{H^{k,0}(\mathcal{S})} = \Big\|\frac{\cosh\left((z+1)\sqrt{\mu}|\mathrm{D}|\right)}{\cosh\left(\sqrt{\mu}|\mathrm{D}|\right)}u - u\Big\|_{H^{k,0}(\mathcal{S})} \lesssim \mu|\nabla_X u|_{H^{k+1}}.$$

The remaining estimates are proved similarly. $\square$

The next result concerns the following operators:

$$T_1(z)[X,\mathrm{D}]u(X) = \mathcal{F}^{-1}\Big(\frac{\sinh(\frac{z}{h_b(X)}\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\hat{u}(\xi)\Big)(X),$$

and

$$T_2(z)[X,\mathrm{D}]u(X) = \mathcal{F}^{-1}\Big(\frac{\cosh(\frac{z}{h_b(X)}\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\hat{u}(\xi)\Big)(X).$$

We should note that we will apply these operators to functions depending only on X, which makes the dependence in $z \in [-h_b, 0]$ easier to deal with.

Proposition A.5. *Let $k \in \mathbb{N}$ and take $u \in \mathscr{S}(\mathbb{R}^d)$, then under condition* (1.13) *we have*

$$\begin{aligned}\|T_1 u\|_{H^{k,0}(\mathcal{S}_b)} &\leq M(k)|u|_{H^k}\\ \|T_2 u\|_{H^{k,0}(\mathcal{S}_b)} &\leq M(k)|u|_{H^k}.\end{aligned}$$

Proof. We first observe that T_1 is well-defined on $\mathscr{S}(\mathbb{R}^d)$. Indeed, for $t_0 > \frac{d}{2}$ there holds

$$\begin{aligned}
|T_1 u(X)| &\leq \sup_{z\in[-h_b(X),0]} \Big|\mathcal{F}^{-1}\Big(\frac{\sinh(\frac{z}{h_b(X)}\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\hat{u}(\xi)\Big)(X)\Big| \\
&\leq \sup_{\xi\in\mathbb{R}^d} \tanh(\sqrt{\mu}|\xi|)\langle\xi\rangle^{t_0}|\hat{u}(\xi)| \int_{\mathbb{R}^d}\langle\xi\rangle^{-t_0}\,\mathrm{d}\xi \\
&< \infty.
\end{aligned}$$

Moreover, using similar arguments one can prove $T_1 u \in \mathscr{S}(\mathbb{R}^d)$. The same is true for T_2.

Next, we prove the estimates. To do so, we first let $k = 0$ and use a change of variable, Hölder's inequality, the Sobolev embedding, and Plancherel's identity to make the observation:

$$\begin{aligned}
\|T_1 u\|^2_{L^2(\mathcal{S}_b)} &= \int_{\mathbb{R}^d} h_b(X)\int_{-1}^0 |\mathcal{F}^{-1}\Big(\frac{\sinh(z\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\hat{u}(\xi)\Big)(X)|^2\,\mathrm{d}z\mathrm{d}X \\
&\leq |h_b|_{L^\infty}\int_{-1}^0\int_{\mathbb{R}^d} |\mathcal{F}^{-1}\Big(\frac{\sinh(z\sqrt{\mu}|\xi|)}{\cosh(\sqrt{\mu}|\xi|)}\hat{u}(\xi)\Big)(X)|^2\,\mathrm{d}X\mathrm{d}z \\
&\leq M_0|u|^2_{L^2}.
\end{aligned}$$

For higher derivatives, the proof is the same after an application of the chain rule. The same is true for T_2.

□

The next result is on the Dirichlet-Neumann operator (Theorem 3.15 in [14]):

Proposition A.6. *Let $s \geq 0$. Let $\zeta \in H^{s+3}(\mathbb{R}^d)$ be such that (1.5) is satisfied, and take $\psi \in \dot{H}^{s+3}(\mathbb{R}^d)$. Then one has*

$$\frac{1}{\mu}|\mathcal{G}^\mu\psi|_{H^{s+1}} \leq M(s+3)|\nabla_X\psi|_{H^{s+2}}. \tag{A.5}$$

Lastly, we have the following estimates on the multipliers:

$$\mathrm{F}_1 = \frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{\sqrt{\mu}|\mathrm{D}|},\quad \mathrm{F}_2 = \frac{3}{\mu|\mathrm{D}|^2}(1-\mathrm{F}_1),\quad \mathrm{F}_3 = \mathrm{sech}(\sqrt{\mu}|D|),\quad \mathrm{F}_4 = \frac{2}{\mu|\mathrm{D}|^2}(1-\mathrm{F}_3).$$

Proposition A.7. *Let $s \in \mathbb{R}$ and take $u \in \mathscr{S}(\mathbb{R}^d)$, then for $i \in \{1,2,3,4\}$ there holds*

$$|(\mathrm{F}_i - 1)u|_{H^s} \lesssim \mu|\nabla_X u|_{H^{s+1}}.$$

Proof. The estimates are a direct consequence of Plancherel's identity and the Taylor expansion formulas:

$$\begin{aligned}
\cosh(x) &= 1 + \frac{x^2}{2}\int_0^1 \cosh(tx)(1-t)\,\mathrm{d}t \\
\sinh(x) &= x + \frac{x^3}{6}\int_0^1 \cosh(tx)(1-t)^2\,\mathrm{d}t, \\
\frac{1}{\cosh(x)} &= 1 - \frac{x^2}{2} + \frac{x^4}{24}\int_0^1 \Big(\mathrm{sech}(tx) - 20\mathrm{sech}^3(tx) + 24\mathrm{sech}^5(tx)\Big)(1-t)^3\,\mathrm{d}t
\end{aligned}$$

for $0 \leq x \leq 1$.

□

A.3. **Classical estimates.** In this section, we recall some classical estimates that will be used throughout the paper. Finally, we end the section with the proof of Proposition 2.9.

Lemma A.8. *Let $\beta \in [0,1]$, $b \in C^\infty_c(\mathbb{R}^d)$, $h_b = 1-\beta b$, $\mathcal{S}_b = (-h_b, 0)\times \mathbb{R}^d$, and assume (1.13) holds true. Then for $u \in H^1(\mathcal{S}_b)$ satisfying $u|_{z=0} = 0$, there holds*

$$\|u\|_{L^2(\mathcal{S}_b)} \lesssim \|\nabla^\mu_{X,z} u\|_{L^2(\mathcal{S}_b)}, \tag{A.6}$$

and

$$|u|_{z=-h_b}|_{L^2} \lesssim \|\nabla^\mu_{X,z} u\|_{L^2(\mathcal{S}_b)}. \tag{A.7}$$

Moreover, if we further suppose $u \in H^{k,k}(\mathcal{S}_b)$ then

$$\Big|\int_{-1+\beta b(\cdot)}^0 u(\cdot, z)\,\mathrm{d}z\Big|^2_{H^k} \leq M(k)\big(\|\nabla^\mu_{X,z}u\|^2_{H^{k,0}(\mathcal{S}_b)} + \sum_{j=1}^k \|\partial_z^j u\|^2_{H^{k-j,0}(\mathcal{S}_b)}\big). \tag{A.8}$$

Proof. For the proof of (A.6) we use assumption $u|_{z=0} = 0$ and the Fundamental Theorem of Calculus combined with Cauchy-Schwarz inequality we get that

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{-1+\beta b(X)}^0 |u(X,z)|^2\,\mathrm{d}z\mathrm{d}X &= \int_{\mathbb{R}^d}\int_{-1+\beta b(X)}^0 |\int_z^0 (\partial_z u)(X,z')\,\mathrm{d}z'|^2\,\mathrm{d}z\mathrm{d}X \\ &\leq (1+\beta|b|_{L^\infty})^2 \int_{\mathbb{R}^d}\int_{-1+\beta b(X)}^0 |(\partial_z u)(X,z')|^2\,\mathrm{d}z'\mathrm{d}X.\end{aligned}$$

For the proof of (A.7), we first use the assumption $u|_{z=0} = 0$ with the Fundamental Theorem of Calculus and Young's inequality to get that

$$\begin{aligned}\int_{\mathbb{R}} u(X,-h_b(X))^2\,\mathrm{d}X &= \int_{\mathbb{R}}\int_{-1+\beta b(X)}^0 \partial_z\big(u(X,z)^2\big)\,\mathrm{d}z\mathrm{d}X \\ &\leq \int_{\mathbb{R}}\int_{-1+\beta b(X)}^0 \partial_z u(X,z)^2\,\mathrm{d}z\mathrm{d}X + \int_{\mathbb{R}}\int_{-1+\beta b(X)}^0 u(X,z)^2\,\mathrm{d}z\mathrm{d}X.\end{aligned}$$

Then by (A.6) we conclude that

$$|u|_{-h_b}|_{L^2} \lesssim \|\nabla^\mu_{X,z}u\|_{L^2(\mathcal{S}_b)}.$$

For the proof (A.8), we first consider the estimate with one derivative to fix the idea. In particular, we perform a change of variable and then use the chain rule and Hölder's inequality to get

$$\begin{aligned}\Big|\nabla_X \int_{-1+\beta b(\cdot)}^0 u(\cdot,z)\,\mathrm{d}z\Big|^2_{L^2} &= \int_{\mathbb{R}^d}\big|\nabla_X\int_{-1}^0 u(X, zh_b(X))h_b(X)\,\mathrm{d}z\big|^2\,\mathrm{d}X \\ &\leq \int_{\mathbb{R}^d}\big(\int_{-1}^0 |(\nabla_X u)(X,zh_b(X))|h_b(X)\,\mathrm{d}z\big)^2\,\mathrm{d}X \\ &\quad + \int_{\mathbb{R}^d}\big(\int_{-1}^0 |z|\beta|\nabla_X b|\,|(\partial_z u)(X,zh_b(X))|h_b(X)\,\mathrm{d}z\big)^2\,\mathrm{d}X \\ &\quad + \beta|\nabla_X b|_{L^\infty}\int_{\mathbb{R}^d}\big(\int_{-1}^0 |u(X,zh_b(X))|\,\mathrm{d}z\big)^2\,\mathrm{d}X.\end{aligned}$$

Next, we can transform the integral back to its original domain using (1.13), and then apply Cauchy-Schwarz and Hölder's inequality to obtain

$$\Big|\nabla_X\int_{-1+\beta b(\cdot)}^0 u(\cdot,z)\,\mathrm{d}z\Big|^2_{L^2} \leq M(k+1)(\|u\|^2_{H^{1,0}(\mathcal{S}_b)} + \|\partial_z u\|^2_{L^2(\mathcal{S}_b)})$$

Repeating this process for any $k \in \mathbb{N}$, using the Leibniz rule, gives us

$$\Big| \int_{-1+\beta b(\cdot)}^{0} u(\cdot, z)\,\mathrm{d}z \Big|_{H^k}^2 = \sum_{\gamma \in \mathbb{N}^d : |\gamma| \leq k} \int_{\mathbb{R}^d} \big| \partial_X^\gamma \int_{-1}^{0} \big(u(X, zh_b(X)) h_b(X) \big)\,\mathrm{d}z \big|^2 \,\mathrm{d}X$$

$$\leq M(k)(\|u\|_{H^{k,0}(\mathcal{S}_b)}^2 + \sum_{j=0}^{k} \|\partial_z^j u\|_{H^{k-j,0}(\mathcal{S}_b)}^2).$$

To conclude our observation, we use the assumption $u|_{z=0} = 0$ to apply the Poincaré inequality (A.6) on the first terms. $\square$

Before proving the main result, we need some classical estimates (see Proposition $B.2$ and Proposition $B.4$ in [14]).

Lemma A.9. *Let $t_0 \geq \frac{d}{2}$, $s \geq -t_0$, , $f \in H^{\max\{t_0,s\}}(\mathbb{R}^d)$, and take $g \in H^s(\mathbb{R}^d)$ then*

$$|fg|_{H^s} \lesssim |f|_{H^{\max\{t_0,s\}}} |g|_{H^s}. \tag{A.9}$$

Moreover, if there exist $c_0 > 0$ and $1 + g \geq c_0$ then

$$\Big| \frac{f}{1+g} \Big|_{H^s} \lesssim C(c_0, |g|_{L^\infty})(1 + |f|_{H^s}) |g|_{H^s}. \tag{A.10}$$

Lastly, we will prove the main result of this section:

Proof of Proposition 2.9. We first establish the existence and uniqueness of variational solutions to (2.12). Here the variational formulation associated with (2.12) is given by

$$\int_{\mathcal{S}_b} P(\Sigma_b) \nabla_{X,z}^\mu u \cdot \nabla_{X,z}^\mu \varphi \,\mathrm{d}z\mathrm{d}X = \int_{\mathcal{S}_b} f\varphi \,\mathrm{d}z\mathrm{d}X + \int_{\mathbb{R}^d} g\, \varphi|_{z=-h_b} \,\mathrm{d}X, \tag{A.11}$$

for $\varphi \in H^1(\mathcal{S}_b)$. Then using the coercivity estimate (2.5) and the Poincaré inequality (A.6) to get that

$$c\|\varphi\|_{H^1} \leq \int_{\mathcal{S}_b} P(\Sigma_b) \nabla_{X,z}^\mu \varphi \cdot \nabla_{X,z}^\mu \varphi \,\mathrm{d}z\mathrm{d}X.$$

While the right-hand side of (A.11) is continuous by Cauchy-Schwarz and the trace inequality (A.7). As a result, by Riesz representation Theorem, there exists a unique variational solution $u \in H^{1,0}(\mathcal{S}_b)$.

Next, we will prove that $u \in H^{k,0}(\mathcal{S}_b)$ by considering the problem on the fixed strip $\mathcal{S} = \mathbb{R}^d \times [-1, 0]$, where we define

$$\Sigma(X, z) = (X, hz + \varepsilon\zeta).$$

Then we have that

$$(u \circ \Sigma_b^{-1}) \circ \Sigma(X, z) = u(X, zh_b) := \tilde{u}(X, z),$$

and through a change of variable, we obtain the equation

$$\int_{\mathcal{S}} \tilde{P}(\Sigma) \nabla_{X,z}^\mu \tilde{u} \cdot \nabla_{X,z}^\mu \tilde{\varphi} \,\mathrm{d}z\mathrm{d}X = \int_{\mathcal{S}} \tilde{f}\tilde{\varphi} \,\mathrm{d}z\mathrm{d}X + \int_{\mathbb{R}^d} g\, \tilde{\varphi}|_{z=-1} \,\mathrm{d}X, \tag{A.12}$$

where $\tilde{f}(X, z) = f(X, zh_b(X))$, $\tilde{\varphi}(X, z) = \varphi(X, zh_b(X))$, and $\tilde{P}(\Sigma)$ is an elliptic matrix given by

$$\tilde{P}(\Sigma) = \begin{pmatrix} (1 + \partial_z\theta)\mathrm{Id} & -\sqrt{\mu}\nabla_X\theta \\ -\sqrt{\mu}(\nabla_X\theta)^T & \dfrac{1 + \mu|\nabla_X\theta|^2}{1 + \partial_z\theta} \end{pmatrix}.$$

with $\theta(X,z) = (\varepsilon\zeta - \beta b)z + \varepsilon\zeta$. At this point, the problem is classical, and we refer to Proposition 4.5 in [10] to deduce that $\nabla^{\mu}_{X,z}\tilde{u} \in H^{k,0}(\mathcal{S})$ for $k \in \mathbb{N}$ and satisfying

$$\begin{aligned}\|\nabla^{\mu}_{X,z}\tilde{u}\|_{H^{k,0}(\mathcal{S})} &\leq M(k+1)(|g|_{H^k} + \|\tilde{f}\|_{H^{k,0}})\\ &\leq M(k+1)(|g|_{H^k} + \sum_{j=0}^{k}\|\partial_z^j f\|_{H^{k-j,0}}).\end{aligned}$$

In the last inequality, we used the chain rule and the product estimate (A.9). Moreover, for $k \geq 1 + t_0$ we have $\tilde{u} \in C^2(\mathcal{S})$ and is a classical solution of

$$\begin{cases}\nabla^{\mu}_{X,z}\tilde{P}(\Sigma)\nabla^{\mu}_{X,z}\tilde{u} = \tilde{f}\\ v|_{z=0} = 0, \quad \partial_n^{\tilde{P}}\tilde{u}|_{z=-1} = g.\end{cases}$$

Then using the equation, we can control the partial derivatives in z by the derivatives in X through

$$\frac{1+|\nabla_X\theta|^2}{h}\partial_z^2\tilde{u} = \tilde{f} - \mu\nabla_X\cdot(h\nabla_X\tilde{u}) + \mu\nabla_X\cdot(\nabla_X\theta\partial_z\tilde{u}) + \mu\partial_z(\nabla_X\theta\cdot\nabla_X\tilde{u}) - \frac{(\partial_z|\nabla_X\theta|^2)}{h}\partial_z\tilde{u},$$

and the regularity and positivity of $\frac{1+|\nabla_X\theta|^2}{h}$. Indeed, there holds,

$$\|\partial_z^k\tilde{u}\|_{L^2(\mathcal{S})} \leq M(k+1)(\|\tilde{u}\|_{H^{k,0}(\mathcal{S})} + \|\partial_z\tilde{u}\|_{H^{k,0}(\mathcal{S})} + \|\partial_z^k\tilde{f}\|_{L^2(\mathcal{S}_b)}).$$

Having the desired regularity, we may relate these observations with the original problem u on $\mathcal{S}_b$. In particular, by (1.13) we have that

$$\nabla^{\mu}_{X,z}u(X,z) = \nabla^{\mu}_{X,z}\big(\tilde{u}(X,\frac{z}{h_b})\big) \in H^{k,0}(\mathcal{S}_b),$$

using the chain rule, the regularity of h_b, and a change of variable to get that

$$\begin{aligned}\|\nabla^{\mu}_{X,z}u\|_{H^{k,0}(\mathcal{S}_b)} &\leq M(k+1)(\|\nabla^{\mu}_{X,z}\tilde{u}\|_{H^{k,0}(\mathcal{S})} + \sum_{j=0}^{k}\|\partial_z^j\tilde{u}\|_{H^{k,0}(\mathcal{S})})\\ &\leq M(k+1)(|g|_{H^k} + \sum_{j=0}^{k}\|\partial_z^j f\|_{H^{k-j,0}(\mathcal{S}_b)}).\end{aligned}$$

□

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Paper IV

2.4 Justification of the Benjamin-Ono equation as an internal water waves model

M. O. Paulsen.

Submitted for publication.

JUSTIFICATION OF THE BENJAMIN-ONO EQUATION AS AN INTERNAL WATER WAVES MODEL

MARTIN OEN PAULSEN

Abstract. In this paper, we give the first rigorous justification of the Benjamin-Ono equation:

$$\partial_t\zeta + c(1-\frac{\gamma}{2}\sqrt{\mu}|\mathrm{D}|)\partial_x\zeta + c\frac{3\varepsilon}{2}\zeta\partial_x\zeta = 0, \tag{BO}$$

as an internal water wave model on the physical time scale. To be precise, we first prove the existence of a solution to the internal water wave equations for a two-layer fluid with surface tension, where one layer is of shallow depth and the other is of infinite depth. The existence time is of order $\mathcal{O}(\frac{1}{\varepsilon})$ for $\varepsilon \leq \sqrt{\mu}$ and with a small amount of surface tension $\mathrm{bo}^{-1} = \varepsilon\sqrt{\mu}$. Here, ε and μ denote the nonlinearity and shallowness parameters, and bo is the Bond number. Then, we show that these solutions are close, on the same time scale, to the solutions of the BO equation with a precision of order $\mathcal{O}(\mu)$.

The long-time well-posedness of the two-layer fluid problem was first studied by Lannes [Arch. Ration. Mech. Anal., 208(2):481-567, 2013] in the case where both fluids have finite depth. Here, we adapt this work to the case where one of the fluid domains is of finite depth, and the other one is of infinite depth. The novelties of the proof are related to the geometry of the problem, where the difference in domains alters the functional setting for the Dirichlet-Neumann operators involved. In particular, we study the various compositions of these operators that require a refined symbolic analysis of the Dirichlet-Neumann operator on infinite depth and derive new pseudo-differential estimates that might be of independent interest.

1. Introduction

1.1. **The Benjamin-Ono equation.**

The Benjamin-Ono (BO) equation is a nonlocal asymptotic model for the unidirectional propagation of weakly nonlinear, long internal waves in a two-layer fluid. The equation is given by

$$\partial_t\zeta + c(1-\frac{\gamma}{2}\sqrt{\mu}|\mathrm{D}|)\partial_x\zeta + c\frac{3\varepsilon}{2}\zeta\partial_x\zeta = 0, \tag{1.1}$$

where $x \in \mathbb{R}$, $t > 0$ and $\zeta = \zeta(x,t)$ denotes the free surface, which is a real-valued function. Here, ε is a small parameter measuring the weak nonlinearity of the waves, μ is the shallowness parameter, $c > 0$ is the wave speed, and $\gamma \in (0,1)$ is the ratio between the densities of the two fluids. The operator $|\mathrm{D}|$ is a Fourier multiplier defined by

$$|\mathrm{D}|f(x) = \mathcal{F}^{-1}\big(|\xi|\mathcal{F}(f)(\xi)\big)(x).$$

The BO equation was introduced formally by Benjamin [12] in 1967 and at the same time independently by Davis and Acrivos [22]. We also refer the reader to the book by Klein and Saut [45], Chapter 3, for a detailed state-of-the-art. The studies in [12, 22] showed that the BO equation admits solitary waves with mere algebraic decay, as opposed to the exponential decay exhibited for the solitary waves of the KdV equation. Davis and Acrivos

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also gave experimental results. The experiments were carried out in a wave tank with a stratified solution of salt and water, where almost any disturbance to the surface would, after a short time, produce a wave with a fixed shape that propagates stably. It was later noted by Ono [63] that the ease with which they could generate solitary waves indicates that they are solitons.

The paper by Ono sparked much interest in studying the dynamics of the BO equation. It was proved that the solitary waves are unique (up to translation) [11], and the stability of these objects is studied in [13, 72, 43] (see the references for a precise definition). Moreover, the stability of these waves is strong enough to preserve its own identity upon nonlinear interactions. The strong interaction between several solitary waves is studied in [54, 61] and relies on explicit formulas (see also [43] for the asymptotic stability of one soliton and $N-$solitons).

The fact that explicit solutions like the soliton (or multi-solitons) exist is a consequence of the complete integrability of the BO equation. Nakamura [60] proved the existence of an infinite number of conserved quantities and discovered a Lax pair structure (see also [14, 25, 31]). This insight is proven to be crucial for the study of the dynamics of the BO equation and was further developed by Gérard and Kapeller [31]. They constructed a nonlinear Fourier transform for the BO equation on the torus, which has several applications to low regularity well-posedness of the initial value problem, the long-time behavior of solutions, and stability of traveling waves (see [28] for a survey on this topic). More recently, Gérard [29] derived an explicit formula for the BO equation based on the Lax pair structure with remarkable consequences, for instance, the zero-dispersion limit problem [30] (see also [27, 26]).

The Cauchy problem for BO has been extensively studied in the last 40 years. It was first proved to be globally well-posed in $H^s(\mathbb{R})$ for $s > \frac{3}{2}$ using an energy method, see [1, 38]. We also refer to the results [65, 46, 42] for an improvement by including the dispersive smoothing effects in the energy estimates. One of the main difficulties in improving the result further is that the flow map fails to be C^2 in any Sobolev space $H^s(\mathbb{R})$ [59] (see also [47]). A breakthrough was achieved by Tao [70], where he introduced a clever change of variables (the gauge transform) to improve the structure of the nonlinearity. As a consequence, he obtained a global well-posedness result for data in $H^1(\mathbb{R})$. Several papers expanded on these ideas. We refer the interested reader to [17, 37, 58, 34] for results on the line and [56, 57, 58] in the periodic case culminating in the global well-posedness in L^2. So far, the theory is based on PDE methods. However, by actively using the integrable structure, Gérard, Kappeler, and Topalov [32] proved the sharp global well-posedness result in $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$ on the torus. Also, still relying on the integrability, Killip, Laurens, and Vişan [44] recently proved the global well-posedness in $H^s(\mathbb{R})$ for $s > -\frac{1}{2}$ on the line.

1.1.1. *Full justification.* Despite the rich well-posedness theory for the BO equation, it is still an open question whether its solutions are close to the ones of the original physical system. In the rigorous derivation of any asymptotic model, it is fundamental to know whether its solutions converge to the solutions of the reference model from which it is derived. The BO equation's reference model is a coupled system of Euler equations for two fluids that are joined with an interface, as in Figure 1.1. Under the irrotationality condition, we will call the reference model the "internal water waves system". To prove that BO is a valid approximation, we shall compare their solutions to the physical parameters:

$$\varepsilon = \frac{a}{\lambda} \quad , \quad \mu = \frac{H^2}{\lambda^2} \quad \text{and} \quad \mathrm{bo} = \frac{\rho^+ g\lambda^2}{\sigma},$$

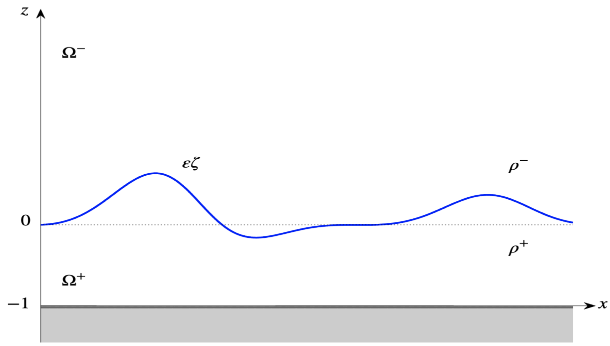

FIGURE 1. The blue line denotes the surface elevation $z = \varepsilon\zeta$ and separates two fluids with density $0 < \rho^- < \rho^+$.

where a is the typical amplitude of the waves, H is the still water depth in the lower fluid, λ is the typical wavelength, ρ^+ is the density of the lower fluid, g is the acceleration of gravity, σ is the surface tension parameter. Since surface tension is only relevant for short waves, we will suppose that bo^{-1} is small. To be precise, we will let $\varepsilon \lesssim \sqrt{\mu}$ and $\mathrm{bo}^{-1} = \varepsilon\sqrt{\mu}$, where we answer the following three questions:

1. The solutions of the internal water wave equations exist on the relevant time scale $\mathcal{O}(\frac{1}{\varepsilon})$.
2. The solutions of the BO equation exist (at least) on the time scale $\mathcal{O}(\frac{1}{\varepsilon})$.
3. Lastly, we must establish the *consistency* between the BO equation and the internal water wave equations and then show that the error is of order $\mathcal{O}(\mu t)$ when comparing the two solutions.

The first point is the most challenging step of this paper, where we need to prove that the internal water waves equations are long-time well-posed for regular initial data. Moreover, we are confined to the specific geometry where one fluid layer is of shallow depth and the other of infinite depth. The main obstacle to constructing such solutions is the tendency of internal waves to break down due to Kelvin-Helmholtz instabilities. This issue was resolved in the case of a single fluid (i.e., $\rho^- = 0$), where stable solutions are deduced by imposing the *Rayleigh-Taylor criterion*:

$$-\partial_z P^+|_{z=\varepsilon\zeta} > 0, \tag{1.2}$$

where z is the vertical coordinate, and $P^+|_{z=\varepsilon\zeta}$ is the pressure at the surface. The physical relevance of this criterion can be seen by considering the Euler equations for a trivial flow $-\partial_z P^+|_{z=\varepsilon\zeta} = \rho^+ g$, where gravity $g > 0$ is the restoring force. In this sense, the criterion is a natural condition to ensure that the pressure force is restoring and does not amplify the waves [71]. From a mathematical perspective, the criterion ensures the hyperbolicity of the water waves equation. Moreover, it is proved that under this condition, the water waves system in finite depth is locally well-posed by Lannes [48] and then long-time well-posed by Alvarez-Samaniego and Lannes [8]. We also refer the reader to the pioneering work of Wu [73, 74] in the case of infinite depth where one of the key observations was the use of the Rayleigh-Taylor criterion (1.2) to remove a smallness condition on the data (see also the more recent work on extended life span and improved regularity results [75, 76, 77, 7, 5, 6, 3, 2]).

In the case of two fluid systems, the internal water waves system produces Kelvin-Helmholtz instabilities and becomes ill-posed unless there is surface tension $\sigma > 0$ [23, 36, 39]. There are several results on the well-posedness of the internal water wave systems in different configurations of the fluid domain where the time of existence depends on σ [9, 10, 18, 68, 69]. On the other hand, the strength of surface tension is only relevant for very small values in water waves theory. Therefore, we must add surface tension to exploit its regularizing effect, but obtain a uniform local well-posedness result with respect to $\sigma > 0$, allowing it to be taken small. Lannes solved this problem in [50] for two fluid layers of finite depth, where he derived a new stability criterion depending on σ. A crucial point is that surface tension could be taken small enough in the criterion such that it does not affect the main dynamics of the equation. It is also noted in the paper that the criterion depends strongly on the geometry of the problem. Many of the technical difficulties in this paper are related to this observation. In this work, we consider one of the layers to be of infinite depth. This is the key point of the paper, which will require a symbolic analysis of the Dirichlet-Neumann operator in a new functional setting. To achieve this goal, we derive several pseudo-differential estimates for symbols with limited smoothness.

The second point is well-known since the BO equation is globally well-posed for regular data (see the discussion above). However, we will consider two intermediate models to derive the BO equation. We will first derive a weakly dispersive BO-type system from the internal water waves system that is consistent with a precision of order $\mathcal{O}(\mu)$. Then, we will consider unidirectional solutions of this system, with the same precision, to deduce a weakly dispersive BO equation, which is also consistent with the BO equation. To derive these models, we closely follow the work of Bona, Lannes, and Saut in [15]. In this paper, they derive several shallow water models for internal fluids and comment on the formal derivation of the BO equation.

Finally, we comment on several works that are closely related to the derivation of the BO equation. In [19], Craig, Guyenne, and Kalisch used a Hamiltonian perturbation approach to formally derive asymptotic models from the two-layer system. Among the models is the BO equation. The benefit of this approach is that the systems inherit the Hamiltonian structure, but as noted in [45], the process could lead to ill-posed systems. In particular, the BO system they derive, which links the BO equation, is linearly ill-posed. In [33], Ifrim, Rowan, Tataru, and Wan show that the BO equation can also be viewed as an asymptotic model from the water waves equations in infinite depth in the case of constant vorticity. The approximation they obtain is rigorously justified but, of course, not related to the asymptotic description of internal waves. Lastly, in [62], Ohi and Iguchi proved the well-posedness of the internal water waves for one fluid of infinite depth to derive the BO equation. However, in their paper, the existence time is of order $\mathcal{O}(\mathrm{bo}^{-\frac{1}{2}})$, which is too short to justify the BO equation on the physical time scale. The technique is based on the one of Wu [73], where the reference model is given in holomorphic coordinates. We will instead use a version of the Zakharov-Craig-Sulem formulation and closely follow the work of Lannes [50]. *In particular, the goal of this paper is to prove the long-time existence of the internal water waves equations with one fluid of infinite depth and positive surface tension. Then we show that the difference between two regular solutions of the internal water waves equations and the BO equation, which evolves from the same initial datum, is bounded by $\mathcal{O}(\mu)$ for all $0 \leq t \lesssim \varepsilon^{-1}$ for any $\varepsilon, \mu \in (0,1)$* such that $\varepsilon \lesssim \mu$ and $\mathrm{bo}^{-1} = \varepsilon\sqrt{\mu}$.

1.2. **The governing equations.** The basis of this study is the Euler equations for an irrotational two-layer fluid written in the Zakharov-Craig-Sulem formulation [79, 20, 21].

For the upper layer, we consider the following set of equations

$$\begin{cases} \partial_t\zeta - \mathcal{G}^-[\zeta]\psi^- = 0 \\ \rho^-\Big(\partial_t\psi^- + g\zeta + \frac{1}{2}(\partial_x\psi^-)^2 - \frac{1}{2}\frac{(\mathcal{G}^-[\zeta]\psi^- + \partial_x\zeta\partial_x\psi^-)^2}{1+(\partial_x\zeta)^2}\Big) = -P^-|_{z=\zeta}. \end{cases} \tag{1.3}$$

Here, the free surface elevation is the graph of $\zeta(t,x) \in \mathbb{R}$, the function $P^-|_{z=\epsilon\zeta}$ is the pressure force at the free surface. The function $\psi^-(t,x) \in \mathbb{R}$ is the trace at the surface of the velocity potential solving the elliptic problem

$$\begin{cases} (\partial_x^2 + \partial_z^2)\Phi^- = 0 \quad \text{for} \quad \Omega^- = \{(x,z) \, : \, z > \zeta\} \\ \Phi^-|_{z=\zeta} = \psi^-, \end{cases} \tag{1.4}$$

and $\mathcal{G}^-$ is the negative Dirichlet-Neumann operator defined by

$$\mathcal{G}^-[\zeta]\psi^- = (\partial_z\phi^- - \partial_x\zeta\partial_x\phi^-)|_{z=\zeta}.$$

For the fluid in the lower layer, the governing equations are given in terms of (ζ, ψ^+) and read

$$\begin{cases} \partial_t\zeta - \mathcal{G}^+[\zeta]\psi^+ = 0 \\ \rho^+\Big(\partial_t\psi^+ + g\zeta + \frac{1}{2}(\partial_x\psi^+)^2 - \frac{1}{2}\frac{(\mathcal{G}^+[\zeta]\psi^+ + \partial_x\zeta\partial_x\psi^+)^2}{1+(\partial_x\zeta)^2}\Big) = -P^+|_{z=\zeta}, \end{cases} \tag{1.5}$$

where the elliptic problem in the lower fluid is given by

$$\begin{cases} (\partial_x^2 + \partial_z^2)\Phi^+ = 0 \quad \text{for} \quad \Omega^+ = \{(x,z) \, : \, -H < z < \zeta\} \\ \Phi^+|_{z=\zeta} = \psi^+ \quad \partial_z\Phi^+|_{z=-H} = 0, \end{cases} \tag{1.6}$$

and the positive Dirichlet-Neumann operator is defined by

$$\mathcal{G}^+[\zeta]\psi^+ = (\partial_z\phi^+ - \partial_x\zeta\partial_x\phi^+)|_{z=\zeta}.$$

To ease the notation, we make the following simplifications

$$\gamma = \frac{\rho^-}{\rho^+}, \quad \rho^- < \rho^+ = 1, \quad g = 1.$$

Moreover, we recall that the difference in pressure at the interface is proportional to the mean curvature of the interface:

$$P^+|_{z=\zeta} - P^-|_{z=\zeta} = \sigma\kappa(\zeta),$$

where $\sigma \in (0,1)$ is the surface tension parameter and $\kappa(\zeta)$ is defined by

$$\kappa(\zeta) = -\partial_x\Big(\frac{\partial_x\zeta}{\sqrt{1+(\partial_x\zeta)^2}}\Big). \tag{1.7}$$

We will now collect all these equations into one system, where we reduce the number of unknowns by using the first equation in (1.5) and (1.3) to see that

$$\mathcal{G}^-[\zeta]\psi^- = \mathcal{G}^+[\zeta]\psi^+. \tag{1.8}$$

In particular, we will prove later that we can write ψ^- as a function of ψ^+ through the inverse relation

$$\psi^- = (\mathcal{G}^-[\zeta])^{-1}\mathcal{G}^+[\zeta]\psi^+.$$

Then, we can define the new variable ψ by the formula

$$\begin{aligned} \psi &= \psi^+ - \gamma\psi^- \\ &= \big(1 - \gamma(\mathcal{G}^-[\zeta])^{-1}\mathcal{G}^+[\zeta]\big)\psi^+ \\ &= \mathcal{J}[\zeta]\psi^+. \end{aligned}$$

The unknowns ζ and ψ are the primary variables. We follow the work of Lannes [50] to show that we can use them to recover the velocity potentials $\Phi^{\pm}$ through the transmission problem:

$$\begin{cases} \Delta_{x,z}\Phi^{\pm} = 0 & \text{in} \quad \Omega^{\pm} \\ \Phi^{+}|_{z=\zeta} - \gamma\Phi^{-}|_{z=\zeta} = \psi \\ \partial_n \Phi^{-}|_{z=\zeta} = \partial_n \Phi^{+}|_{z=\zeta}, \quad \partial_z \Phi^{+}|_{z=-H} = 0, \end{cases} \tag{1.9}$$

with $\psi^{\pm} = \Phi^{\pm}|_{z=\zeta}$ and the normal condition on $z = \zeta$ is the same as (1.8) where ∂_n stands for the upwards normal derivative. From these relations, it will be possible to reduce the two-fluid equations into a set of equations defined by ζ and ψ where we formally define a new Dirichlet-Neumann operator that links the two fluids through the relation,

$$\mathcal{G}[\zeta] = \mathcal{G}^{+}[\zeta](\mathcal{J}[\zeta])^{-1}. \tag{1.10}$$

From the above expressions, we have the main governing equations (in dimensional form) that we will study throughout this paper:

$$\begin{cases} \partial_t \zeta - \mathcal{G}[\zeta]\psi = 0 \\ \partial_t \psi + (1-\gamma)\zeta + \frac{1}{2}\big((\partial_x \psi^{+})^2 - \gamma(\partial_x \psi^{-})^2\big) + \mathcal{N}[\zeta, \psi^{\pm}] = -\sigma\kappa(\zeta), \end{cases} \tag{1.11}$$

where

$$\mathcal{N}[\zeta, \psi^{\pm}] = \frac{\gamma(\mathcal{G}^{-}[\zeta]\psi^{-} + \partial_x\zeta\partial_x\psi^{-})^2 - (\mathcal{G}^{+}[\zeta]\psi^{+} + \partial_x\zeta\partial_x\psi^{+})^2}{2(1+(\partial_x\zeta)^2)}.$$

1.2.1. *Nondimensionalization of the equations.* To derive an asymptotic model from (1.11), we will compare every variable and function with physical characteristic parameters of the same dimension H, a, or λ. Since the BO equation describes long waves, it is natural to consider the scaling:

$$x = \lambda x', \quad \zeta = a\zeta',$$

where the prime notation denotes a nondimensional quantity. To identify the remaining variables ψ' and $t = \frac{\lambda}{c_{\text{ref}}} t'$ one needs information on the reference velocity c_{ref}. To do so, we look at the linearized equations (with $\sigma = 0$):

$$\begin{cases} \partial_t \zeta - \mathcal{G}[0]\psi = 0 \\ \partial_t \psi + (1-\gamma)\zeta = 0, \end{cases} \tag{1.12}$$

where $\mathcal{G}[0]$ is a Fourier multiplier given by[1]

$$\mathcal{G}[0]\psi(x) = \mathcal{F}^{-1}\Big(|\xi| \frac{\tanh(H|\xi|)}{1+\gamma\tanh(H|\xi|)}\hat{\psi}(\xi)\Big)(x).$$

For a wave with typical wavelength λ, the frequencies are concentrated around $|\xi| = \frac{2\pi}{\lambda}$. Therefore, if we suppose that the depth of lower fluid is small compared to the wavelength, then we have by a Taylor expansion that

$$\mathcal{G}[0]\psi = -H\partial_x^2\psi,$$

up to higher order terms in μ. From this simplification we can reduce (1.12) to a wave equation where we make the identification $c_{\text{ref}}^2 = H(1-\gamma)$, and from the second equation we find the dimensions of ψ:

$$\psi = \frac{a\lambda}{\sqrt{H}}\psi'.$$

[1]See Remark 2.3

Lastly, we also choose to scale the transverse variable with H (i.e., $z = Hz'$) to have a reference domain in the lower fluid of unitary depth. Then applying these changes of variables and dropping the prime notation, we find that the nondimensional internal water waves system (1.11) is given by:

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta]\psi = 0 \\ \partial_t\psi + (1-\gamma)\zeta + \frac{1}{2}\big(\varepsilon(\partial_x\psi^+)^2 - \gamma\varepsilon(\partial_x\psi^-)^2\big) + \varepsilon\mathcal{N}[\varepsilon\zeta,\psi^\pm] = -\frac{1}{\mathrm{bo}}\frac{1}{\varepsilon\sqrt{\mu}}\kappa(\varepsilon\sqrt{\mu}\zeta), \end{cases} \tag{1.13}$$

where

$$\mathcal{N}[\varepsilon\zeta,\psi^\pm] = \frac{1}{2\mu}\frac{\gamma(\mathcal{G}_\mu^-[\varepsilon\zeta]\psi^- + \varepsilon\mu\partial_x\zeta\partial_x\psi^-)^2 - (\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+ + \varepsilon\mu\partial_x\zeta\partial_x\psi^+)^2}{(1+\varepsilon^2\mu(\partial_x\zeta)^2)}.$$

The operators $\mathcal{G}_\mu^\pm[\varepsilon\zeta]$ are defined by

$$\mathcal{G}_\mu^\pm[\varepsilon\zeta]\psi^\pm = \sqrt{1+\varepsilon^2(\partial_x\zeta)^2}\partial_n\Phi^\pm|_{z=\varepsilon\zeta},$$

through the solutions of the scaled Laplace equations:

$$\begin{cases} (\mu\partial_x^2 + \partial_z^2)\Phi^+ = 0 \quad \text{for} \quad \Omega^+ = \{(x,z) \,:\, -1 < z < \varepsilon\zeta\} \\ \Phi^+|_{z=\varepsilon\zeta} = \psi^+ \quad \partial_z\Phi^+|_{z=-1} = 0, \end{cases}$$

and

$$\begin{cases} (\mu\partial_x^2 + \partial_z^2)\Phi^- = 0 \quad \text{for} \quad \Omega^- = \{(x,z) \,:\, z > \varepsilon\zeta\} \\ \Phi^-|_{z=\varepsilon\zeta} = \psi^-. \end{cases}$$

1.3. **Main results.** In this paper we will first prove the well-posedness of (1.13) on a time scale $\mathcal{O}(\frac{1}{\varepsilon})$ for $\varepsilon \lesssim \sqrt{\mu}$ and $\mathrm{bo}^{-1} = \varepsilon\sqrt{\mu}$. To state this result, there are two fundamental assumptions that we need to make.

Definition 1.1 (Non-cavitation condition). *Let $\varepsilon \in (0,1)$, $s > \frac{1}{2}$ and take $\zeta_0 \in H^s(\mathbb{R})$. We say ζ_0 satisfies the "non-cavitation condition" if there exist $h_{\min} \in (0,1)$ such that*

$$h = 1 + \varepsilon\zeta_0(x) \geq h_{\min}, \quad \textit{for all} \quad x \in \mathbb{R}. \tag{1.14}$$

The second condition is to ensure the solutions do not break down due to Kelvin-Helmholtz instabilities and is key to showing the long-time existence for solutions of (1.13). The criterion is enforced for data in the energy space, which we will now define.

Definition 1.2 (Energy space). *Let $\varepsilon, \mu, \gamma, \mathrm{bo}^{-1} \in (0,1)$ and $N \in \mathbb{N}$. Then we define the function space $H^{N+1}_{\gamma,\mathrm{bo}}(\mathbb{R})$ by*

$$H^{N+1}_{\gamma,\mathrm{bo}}(\mathbb{R}) = H^{N+1}(\mathbb{R}),$$

with norm

$$|u|^2_{H^{N+1}_{\gamma,\mathrm{bo}}} = (1-\gamma)|u|^2_{H^N} + \mathrm{bo}^{-1}|\partial_x u|^2_{H^N}.$$

We define $\dot{H}^{s+\frac{1}{2}}_\mu(\mathbb{R})$ as a Beppo-Levi space

$$\dot{H}^{s+\frac{1}{2}}_\mu(\mathbb{R}) = \dot{H}^{s+\frac{1}{2}}(\mathbb{R}) = \{u \in L^2_{loc}(\mathbb{R}) \,:\, \partial_x u \in H^{s-\frac{1}{2}}(\mathbb{R})\},$$

endowed with

$$|u|_{\dot{H}^{s+\frac{1}{2}}_\mu} = \Big|\frac{|\mathrm{D}|}{(1+\sqrt{\mu}|\mathrm{D}|)^{\frac{1}{2}}}u\Big|_{H^s}.$$

Moreover, let $\alpha \in \mathbb{N}^2$ and define the "good unknowns" by

$$\zeta_{(\alpha)} = \partial^\alpha_{x,t}\zeta, \quad \psi_{(\alpha)} = \partial^\alpha_{x,t}\psi - \varepsilon\underline{w}\partial^\alpha_{x,t}\zeta.$$

Then the natural energy space $\mathscr{E}^N_{\mathrm{bo},T}$ *associated to* (1.13) *is defined for functions* $\mathbf{U} = (\zeta_{(\alpha)}, \psi_{(\alpha)})$ *in*

$$\mathscr{E}^N_{\mathrm{bo},T} = \{\mathbf{U} \in C([0,T]; H^N(\mathbb{R}) \times \dot{H}^{t_0+3}(\mathbb{R}), \quad \sup_{t\in[0,T]} \mathcal{E}^{N,t_0}_{\mathrm{bo},\mu}(U(t)) < \infty\}, \tag{1.15}$$

whose norm is the square root of

$$\mathcal{E}^{N,t_0}_{\mathrm{bo},\mu}(\mathbf{U}) = |\partial_x\psi|^2_{H^{t_0+2}} + \sum_{\alpha\in\mathbb{N}^2, |\alpha|\leq N} |\zeta_{(\alpha)}|^2_{H^1_{\gamma,\mathrm{bo}}} + |\psi_{(\alpha)}|^2_{\dot{H}^{\frac{1}{2}}_{\mu}}. \tag{1.16}$$

Remark 1.3. *Here, the energy depends on both time derivatives and spatial derivatives. This is the method put forward by* [66, 67, 55] *to control the surface tension term for the water wave equations (see Remark 5.1 for the specifics on this point). This method was later used for the internal water waves equations with surface tension in the case of two fluids of finite depth* [50]*, which is one of the primary references of this paper.*

Definition 1.4 (Stability criterion)**.** *Let* $\mathbf{U}_0 = (\zeta_0, \psi_0) \in L^2(\mathbb{R}) \times \dot{H}^{\frac{1}{2}}_{\mu}(\mathbb{R})$ *and* $\mathcal{E}^{N,t_0}_{\mathrm{bo},\mu}(\mathbf{U}_0) < \infty$*. Then we define the "stability criterion" by*

$$0 < \mathfrak{d}(\mathbf{U}) := \inf_{\mathbb{R}} \mathfrak{a} - \Upsilon \mathfrak{c}(\zeta) |[\![\underline{V}^{\pm}]\!]|^4_{H^{t_0+1}}, \quad at \ \ t = 0, \tag{1.17}$$

where

$$\Upsilon = \frac{\mathrm{bo}}{4}(1-\gamma)^2\gamma^2\varepsilon^2\mu,$$

and

$$\mathfrak{a} = \Big((1-\gamma) + \varepsilon\big((\partial_t + \varepsilon\underline{V}^+\partial_x)\underline{w}^+ - \gamma(\partial_t + \varepsilon\underline{V}^-\partial_x)\underline{w}^-\big)\Big)$$

$$\mathfrak{e}(\zeta) = \sup_{f\in H^{\frac{1}{2}}(\mathbb{R}), f\neq 0} \frac{\big((\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}\partial_x f, \partial_x f\big)_{L^2}}{|1+\sqrt{\mu}|\mathrm{D}|^{\frac{1}{2}} f|^2_{L^2}}$$

$$\mathfrak{c}(\zeta) = \mathfrak{e}(\zeta)^2(1+\varepsilon^2\mu|\partial_x\zeta|^2_{L^\infty})^{\frac{3}{2}}.$$

The quantities $\underline{V}^{\pm}$*,* $\underline{w}^{\pm}$ *describe the horizontal and vertical velocity field in the fluids and are given in Definition 4.1. See also Corollary A.18 in the Appendix, where they are given in terms of* ζ *and* ψ*.*

Remark 1.5. *The stability criterion* (1.17) *can be seen as a two-layer generalization of the Rayleigh-Taylor criterion where*

$$\mathfrak{a} = -\big(\partial_z P^+ - \gamma\partial_z P^-\big)|_{z=\varepsilon\zeta} > \Upsilon\mathfrak{c}(\zeta)|[\![\underline{V}^{\pm}]\!]|^4_{H^{t_0+1}}.$$

We will let bo^{-1} *to be of order* $\mathcal{O}(\varepsilon\sqrt{\mu})$*. In this case, the size of the quantity* Υ *is of order* $\mathcal{O}(\varepsilon\sqrt{\mu})$ *and is neglected in the BO regime.*

Remark 1.6. *One key difference with the work of Lannes* [50] *for the internal water waves on finite depth is in the symbolic analysis of* $\mathcal{G}^-_\mu[\varepsilon\zeta]$*. The operator depends on the solution of an elliptic problem on a domain with infinite depth. This alters the functional setting, where we also need precise estimates depending on the parameters* $\varepsilon, \mu \in (0,1)$*.*

Theorem 1.7. *Let* $t_0 = 1$*,* $N \geq 5$*,* $\varepsilon, \mu, \gamma \in (0,1)$ *such that* $\varepsilon \leq \sqrt{\mu}$ *and* $\mathrm{bo}^{-1} = \varepsilon\sqrt{\mu}$*. Assume that* $\mathbf{U}_0 = (\zeta_0, \psi_0)^T \in L^2(\mathbb{R}) \times \dot{H}^{\frac{1}{2}}_{\mu}(\mathbb{R})$ *such that* $\mathcal{E}^{N,t_0}_{\mathrm{bo},\mu}(\mathbf{U}_0) < \infty$*. Suppose further that* $\mathbf{U}_0$ *satisfies the non-cavitation condition* (1.14) *and the stability criterion* (1.17)*. Then there exists a constant* $C = C(h^{-1}_{\min}, \gamma, \frac{1}{\mathfrak{d}(\mathbf{U}_0)}) > 0$ *and a time*

$$T = T(C\mathcal{E}^{N,t_0}_{\mathrm{bo},\mu}(\mathbf{U}_0)) > 0,$$

which is a nonincreasing function of its argument, and a unique solution $\mathbf{U} = (\zeta, \psi)^T \in \mathscr{E}^N_{\mathrm{bo},\varepsilon^{-1}T}$ *of* (1.13)*. Moreover, the solution satisfies*

$$\sup_{t\in[0,\varepsilon^{-1}T]} \mathcal{E}^{N,t_0}_{\mathrm{bo},\mu}(\mathbf{U}) \leq \mathcal{E}^{N,t_0}_{\mathrm{bo},\mu}(\mathbf{U}_0). \tag{1.18}$$

Remark 1.8. *For notational convenience we shall write* $\mathcal{E}^N(\mathbf{U})$ *instead of* $\mathcal{E}^{N,t_0}_{\mathrm{bo},\mu}(\mathbf{U})$*. We also consider the case of one horizontal dimension since our primary goal is to justify the BO equation, which is a model that does not include transverse effects. We will deal with the higher dimensional case in a forthcoming paper.*

Remark 1.9. *The local well-posedness of* (1.13) *was first proved by Ohi and Iguchi* [62]*. However, in their paper, the existence time is of order* $\mathcal{O}(\mathrm{bo}^{-\frac{1}{2}})$*, which is far too short to justify the BO equation on the physical time scale. In fact, Theorem 1.7 is the first proof of the long-time well-posedness of the internal water waves in the case where one layer is of infinite depth.*

Remark 1.10. *The surface tension term is regularizing and plays a fundamental role in the well-posedness of* (1.13)*. However, as noted in Remark 1.5, it does not affect the dynamics of the BO equation.*

Having defined a solution of the reference model (1.13) on a long time scale, the next step is to derive the asymptotic models. Here we follow the road map in [15], where they derived several internal water wave models in finite depth and gave comments on the formal derivation of the BO equation. In particular, it is convenient to write (1.13) in terms of $\psi^+ \in \dot{H}^{\frac{3}{2}}_{\mu}(\mathbb{R})$ through the interface operator:

$$\mathbf{H}_\mu[\varepsilon\zeta]\psi^+ = \partial_x\psi^- \in H^{\frac{1}{2}}(\mathbb{R}), \tag{1.19}$$

where $\psi^- = \Phi^-|_{z=\varepsilon\zeta} \in \mathring{H}^{\frac{3}{2}}(\mathbb{R})$ and $\Phi^- \in \dot{H}^2(\Omega^-)$ is the unique solution[2] (up to a constant) of

$$\begin{cases} (\mu\partial_x^2 + \partial_z^2)\Phi^- = 0 & \text{in} \quad \Omega^- \\ \partial_n\Phi^- = (1+\varepsilon^2(\partial_x\zeta)^2)^{-\frac{1}{2}}\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+ & \text{on} \quad z = \varepsilon\zeta. \end{cases} \tag{1.20}$$

Then we may define the the velocity variable

$$\begin{aligned} v &= \partial_x\psi \\ &= \partial_x\psi^+ - \gamma\mathbf{H}_\mu[\varepsilon\zeta]\psi^+, \end{aligned} \tag{1.21}$$

and apply a derivative to the second equation of (1.13) to find that

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+ = 0 \\ \partial_t v + (1-\gamma)\partial_x\zeta + \frac{\varepsilon}{2}\partial_x\big((\partial_x\psi^+)^2 - \gamma(\mathbf{H}_\mu[\varepsilon\zeta]\psi^+)^2\big) + \varepsilon\partial_x\mathcal{N}[\varepsilon\zeta,\psi^\pm] = -\frac{1}{\mathrm{bo}}\frac{1}{\varepsilon\sqrt{\mu}}\partial_x\kappa(\varepsilon\sqrt{\mu}\zeta). \end{cases} \tag{1.22}$$

As noted in the introduction, we will first derive a system from (1.22), where we will show that a solution of the internal water waves equations, with regular data (ζ_0, v_0), solves a weakly dispersive BO system:

$$\begin{cases} \partial_t\zeta + (1-\gamma\tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x v + \varepsilon\partial_x(\zeta v) = 0 \\ \partial_t v + c^2\partial_x\zeta + \varepsilon v\partial_x v = 0, \end{cases}$$

[2]See Proposition 2.4.

up to an error of order $\mathcal{O}(\mu+\varepsilon\sqrt{\mu})$. Then under an additional assumption on the data (for right-moving waves), we will show that we can approximate this system with the solutions of

$$\partial_t\zeta + c\big(1-\frac{\gamma}{2}\tanh(\sqrt{\mu}|\mathrm{D}|)\big)\partial_x\zeta + c\frac{3\varepsilon}{2}\zeta\partial_x\zeta = 0.$$

Remark 1.11. *An alternative approach would be to rigorously derive the regularized BO system given by:*

$$\begin{cases}\big(1+\alpha\sqrt{\mu}\gamma|\mathrm{D}|\big)\partial_t\zeta + \big(1+(\alpha-1)\gamma\sqrt{\mu}|\mathrm{D}|\big)\partial_x v + \varepsilon\partial_x(\zeta v) = 0\\ \partial_t v + c^2\partial_x\zeta + \varepsilon v\partial_x v = 0,\end{cases} \tag{1.23}$$

for $\alpha \geq 0$. This is the model that was formally derived in [15], *and moreover we can use it to derive the "regularized Benjamin-Ono equation"* [21]*:*

$$\big(1+\alpha\sqrt{\mu}\gamma|\mathrm{D}|\big)\partial_t\zeta + c\partial_x\zeta + (2\alpha-1)\frac{\gamma}{2}\sqrt{\mu}|\mathrm{D}|\partial_x\zeta + c\frac{3\varepsilon}{2}\zeta\partial_x\zeta = 0.$$

The rigorous derivation of these models is straightforward when having Theorem 1.7 at hand. However, the result would depend on α since the long-time well-posedness of (1.23) *requires $\alpha > 1$* [78].

Before we proceed, we establish the long-time well-posedness of the weakly dispersive models introduced above. To do so, we will have to sharpen the non-cavitation condition (1.14) to define an energy associated with (1.17).

Definition 1.12 ($\gamma-$dependent surface condition)**.** *Let $\varepsilon,\gamma \in (0,1)$ and $s > \frac{1}{2}$. We say the initial surface elevation $\zeta_0 \in H^s(\mathbb{R}^d)$ satisfy the "$\gamma-$dependent surface condition" if there exist $h_{\min,\gamma} \in (0,1)$ such that*

$$1+\varepsilon\zeta_0(x)-\gamma \geq h_{\min,\gamma}, \quad \textit{for all } \ x\in\mathbb{R}. \tag{1.24}$$

Remark 1.13. *The main difference with* (1.23) *is that we can impose a physical constraint on the data instead of a constraint on the parameter in the equation.*

Theorem 1.14. *Let $\varepsilon,\mu,\gamma,c \in (0,1)$ and $s > \frac{3}{2}$. Assume that $(\zeta_0,v_0) \in H^s(\mathbb{R})\times H^s(\mathbb{R})$. Then there exists a constant $C = C(h_{\min,\gamma}^{-1},\gamma)$ and a time*

$$T = T(C|(\zeta_0,v_0)|_{H^s\times H^s}) > 0,$$

which is a nonincreasing function of its argument such that:

1. *There exists a unique solution $\zeta^{wBO} \in C\big([0,\varepsilon^{-1}T] : H^s(\mathbb{R})\big)$ to equation*

$$\partial_t\zeta^{wBO} + c\big(1-\frac{\gamma}{2}\tanh(\sqrt{\mu}|\mathrm{D}|)\big)\partial_x\zeta^{wBO} + c\frac{3\varepsilon}{2}\zeta^{wBO}\partial_x\zeta^{wBO} = 0, \tag{1.25}$$

that satisfies

$$\sup_{t\in[0,\varepsilon^{-1}T]}|\zeta^{wBO}|_{H^s} \leq C|\zeta_0|_{H^s}. \tag{1.26}$$

2. *Suppose further that ζ_0 satisfies the $\gamma-$dependent surface condition* (1.24)*. Then there exist a unique solution $(\zeta^{BOs},v^{BOs}) \in C([0,\varepsilon^{-1}T] : H^s\big(\mathbb{R}) \times H^s(\mathbb{R})\big)$ to system*

$$\begin{cases}\partial_t\zeta^{BOs} + (1-\gamma\tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x v^{BOs} + \varepsilon\partial_x(\zeta^{BOs}v^{BOs}) = 0\\ \partial_t v^{BOs} + c^2\partial_x\zeta^{BOs} + \varepsilon v^{BOs}\partial_x v^{BOs} = 0,\end{cases} \tag{1.27}$$

that satisfies

$$\sup_{t\in[0,\varepsilon^{-1}T]}|(\zeta^{BOs},v^{BOs})|_{H^s\times H^s} \leq C|(\zeta_0,v_0)|_{H^s\times H^s}. \tag{1.28}$$

Remark 1.15. *System* (1.27) *and* (1.25) *are new and are chosen such that it is easy to deduce the long-time existence. The choice was based on the observations made in* [64]*, where weakly dispersive shallow water models are considered and can, in some cases, give rise to well-posed systems while their strongly dispersive versions are not.*

Remark 1.16. *For the data* $\zeta_0 \in H^s(\mathbb{R})$ *with* $s > \frac{3}{2}$ *the long-time (global) well-posedness of BO is classical. The result can easily be extended for* (1.25) *with* $\varepsilon \in (0,1)$*. See, for instance,* [53] *in the case of the BO equation on a fixed time with* $\varepsilon = 1$*.*

With this result in hand, we can use it as a link to prove the consistency between the BO equation and the internal water waves equations.

Theorem 1.17. *Let* $\varepsilon, \mu, \gamma \in (0,1)$*,* $c^2 = (1-\gamma)$ *such that* $\varepsilon \leq \sqrt{\mu}$ *and* $\mathrm{bo}^{-1} = \varepsilon\sqrt{\mu}$*. Assume that* $\mathbf{U}_0 = (\zeta_0, \psi_0)^T$ *satisfies the assumptions of Theorem 1.7 and define* $v_0 = \partial_x \psi_0$*. Suppose also that* ζ_0 *satisfies the* $\gamma-$*dependent surface condition* (1.24)*. Then there exists a time* $T > 0$ *such that for some generic function* R *satisfying*

$$|R|_{H^{N-5}} \leq C(\mathcal{E}^N(\mathbf{U}_0)),$$

for all $t \in [0, \varepsilon^{-1}T]$ *we have the following results:*

1. *There exists a unique solution* $(\zeta, v)^T \in C\big([0,\varepsilon^{-1}T] : H^{N-\frac{1}{2}}(\mathbb{R}) \times H^{N-\frac{1}{2}}(\mathbb{R})\big)$ *to* (1.22)*, where* $v = \partial_x \psi$*. Moreover, on the same time interval, the same solution also satisfies*

$$\begin{cases} \partial_t \zeta + (1 - \gamma \tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x v + \varepsilon \partial_x(\zeta v) = (\mu + \varepsilon\sqrt{\mu})R \\ \partial_t v + c^2 \partial_x \zeta + \varepsilon v \partial_x v = (\mu + \varepsilon\sqrt{\mu})R. \end{cases}$$

2. *There exist a unique solution* $\zeta^{wBO} \in C\big([0,\varepsilon^{-1}T] : H^{N-\frac{1}{2}}(\mathbb{R})\big)$ *that solves*

$$\partial_t \zeta^{wBO} + c\big(1 - \frac{\gamma}{2}\tanh(\sqrt{\mu}|\mathrm{D}|)\big)\partial_x \zeta^{wBO} + c\frac{3\varepsilon}{2}\zeta^{wBO}\partial_x \zeta^{wBO} = 0.$$

   Suppose further that the data v_0 *is given by*

$$v_0 = (1 + \frac{\gamma}{2}\tanh(\sqrt{\mu}|\mathrm{D}|))\zeta_0 - \frac{\varepsilon}{4}\zeta_0^2, \tag{1.29}$$

   and define $v^{wBO} \in C\big([0,\varepsilon^{-1}T] : H^{N-\frac{1}{2}}(\mathbb{R})\big)$ *by*

$$v^{wBO} = (1 + \frac{\gamma}{2}\tanh(\sqrt{\mu}|\mathrm{D}|))\zeta^{wBO} - \frac{\varepsilon}{4}(\zeta^{wBO})^2.$$

   Then the solution (ζ^{wBO}, v^{wBO}) *also satisfies*

$$\begin{cases} \partial_t \zeta^{wBO} + (1 - \gamma \tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x v^{wBO} + \varepsilon \partial_x(\zeta^{wBO} v^{wBO}) = \mu R \\ \partial_t v^{wBO} + c^2 \partial_x \zeta^{wBO} + \varepsilon v^{wBO} \partial_x v^{wBO} = \mu R. \end{cases}$$

3. *There exist a unique solution* $\zeta^{BO} \in C\big([0,\varepsilon^{-1}T] : H^{N-\frac{1}{2}}(\mathbb{R})\big)$ *that solves*

$$\partial_t \zeta^{BO} + c\big(1 - \frac{\gamma}{2}\sqrt{\mu}|\mathrm{D}|\big)\partial_x \zeta^{BO} + c\frac{3\varepsilon}{2}\zeta^{BO}\partial_x \zeta^{BO} = 0,$$

   and on the same time interval it satisfies

$$\partial_t \zeta^{BO} + c\big(1 - \frac{\gamma}{2}\tanh(\sqrt{\mu}|\mathrm{D}|)\big)\partial_x \zeta^{BO} + c\frac{3\varepsilon}{2}\zeta^{BO}\partial_x \zeta^{BO} = \mu R.$$

A consequence of the above results is the full justification of the BO equation.

Theorem 1.18. *Let* $\varepsilon, \mu, \gamma \in (0,1)$*,* $c^2 = (1-\gamma)$ *such that* $\varepsilon \leq \sqrt{\mu}$ *and* $\mathrm{bo}^{-1} = \varepsilon\sqrt{\mu}$*. Assume that* $\mathbf{U}_0 = (\zeta_0, \psi_0)^T$ *satisfies the assumptions of Theorem 1.7 with* $N \geq 7$ *and define* $v_0 = \partial_x \psi_0$*. Suppose further that* ζ_0 *satisfies the* $\gamma-$*dependent surface condition* (1.24) *and that* v_0 *satisfies* (1.29)*. Then there exists* $T > 0$ *such that:*

1. *There exist a unique solution* $(\zeta, v) \in C\big([0, \varepsilon^{-1}T] : H^{N-\frac{1}{2}}(\mathbb{R}) \times H^{N-\frac{1}{2}}(\mathbb{R})\big)$ *to* (1.22) *where* $v = \partial_x \psi$*.*

2. *From the same initial data:*
   2.1. *There exists a unique solution* $(\zeta^{BOs}, v^{BOs}) \in C([0, \varepsilon^{-1}T] : H^s\big(\mathbb{R}) \times H^s(\mathbb{R})\big)$ *to the weakly dispersive BO system*
   $$\begin{cases} \partial_t \zeta^{BOs} + (1 - \gamma \tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x v^{BOs} + \varepsilon \partial_x(\zeta^{BOs} v^{BOs}) = 0 \\ \partial_t v^{BOs} + c^2 \partial_x \zeta^{BOs} + \varepsilon v^{BOs} \partial_x v^{BOs} = 0, \end{cases}$$
   and for any $t \in [0, \varepsilon^{-1}T]$ *there holds,*
   $$|(\zeta - \zeta^{BOs}, v - v^{BOs})|_{L^\infty([0,t]\times\mathbb{R})} \leq (\mu + \varepsilon\sqrt{\mu}) t C(\mathcal{E}^N(\mathbf{U}_0)). \tag{1.30}$$
   2.2. *There exists a unique solution* $\zeta^{wBO} \in C\big([0, \varepsilon^{-1}T] : H^{N-\frac{1}{2}}(\mathbb{R})\big)$ *to the weakly dispersive BO equation*
   $$\partial_t \zeta^{wBO} + c(1 - \frac{\gamma}{2}\tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x \zeta^{wBO} + c\frac{3\varepsilon}{2}\zeta^{wBO}\partial_x \zeta^{wBO} = 0,$$
   and for any $t \in [0, \varepsilon^{-1}T]$ *there holds,*
   $$|\zeta - \zeta^{wBO}|_{L^\infty([0,t]\times\mathbb{R})} \leq \mu t C(\mathcal{E}^N(\mathbf{U}_0)). \tag{1.31}$$
   2.3. *There exists a unique solution* $\zeta^{BO} \in C\big([0, \varepsilon^{-1}T] : H^{N-\frac{1}{2}}(\mathbb{R})\big)$ *to the BO equation*
   $$\partial_t \zeta^{BO} + c(1 - \frac{\gamma}{2}\sqrt{\mu}|\mathrm{D}|)\partial_x \zeta^{BO} + c\frac{3\varepsilon}{2}\zeta^{BO}\partial_x \zeta^{BO} = 0,$$
   and for any $t \in [0, \varepsilon^{-1}T]$ *there holds,*
   $$|\zeta - \zeta^{BO}|_{L^\infty([0,t]\times\mathbb{R})} \leq \mu t C(\mathcal{E}^N(\mathbf{U}_0)). \tag{1.32}$$

Remark 1.19. *Estimates* (1.30) *and* (1.31) *together with the well-posedness theory imply the full justification of their respective systems as internal water waves equations. These are new results, but their primary purpose is to serve as an intermediate step for the derivation of the BO equation and are added here for the sake of completeness.*

1.3.1. *Strategy and outline of the proofs.* The main body of the paper is devoted to the proof of Theorem 1.7, which relies on energy estimates similar to the ones provided by Lannes [50]. To do so, we first need to prove that the operators involved in the main system (1.13) are well-defined and can be formulated solely in terms of ζ and ψ. We start by studying the operator $\mathcal{G}_\mu$ in Section 2, which will be given by the expression:

$$\mathcal{G}_\mu[\varepsilon\zeta]\psi = \mathcal{G}_\mu^+[\varepsilon\zeta]\Big(1 - \gamma(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\Big)^{-1}\psi. \tag{1.33}$$

The main difference with the work of Lannes is that we have the composition of two operators $\mathcal{G}_\mu^+$ and $\mathcal{G}_\mu^-$ that act on different space. This is a consequence of having one fluid of finite depth and the other of infinite depth. In particular, to define the composition $(\mathcal{G}_\mu^-)^{-1}$ with $\mathcal{G}_\mu^+$ we need to work on a homogeneous[3] target space.

[3]The function space with norm $|f|_{\mathring{H}^{s+\frac{1}{2}}} = |\mathrm{D}^{\frac{1}{2}} f|_{H^s}$.

The next step is to give a symbolic description of the operators involved in the expression of $\mathcal{G}_\mu$. This involves some of the key estimates that will be used to close the energy estimates. To do so, we need a symbolic description of each of the operators involved in (1.33). The key estimate, which is proved in Section 3 reads,

$$|\mathcal{G}_\mu^-[\varepsilon\zeta]\psi^- - \big(-\sqrt{\mu}|\mathrm{D}|\psi^-\big)|_{H^{s+\frac{1}{2}}} \leq \varepsilon\mu C\big(|\zeta|_{H^{t_0+3}}\big)|\psi^-|_{\mathring{H}^{s+\frac{1}{2}}}.$$

The symbolic approximation of $\mathcal{G}_\mu^-$ is well-known; see the collection of work by Lannes, Alazard, Metivier, Burq, and Zuily [48, 7, 5, 6] for similar estimates. Their results are sharper in the sense that they require less regularity on ζ. However, the symbolic description they provide is without the parameters ε, μ, and more importantly, is given on the space $\psi^- \in H^{s+\frac{1}{2}}(\mathbb{R})$. More precisely, the main difference is that we need to account for the small parameters and arrive at the homogeneous space $\mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$ to close the estimates later. This result seems new and could be of independent interest. We refer the reader to Remark 3.3 for the novelties of the proof and possible extensions.

The last step before providing the a priori estimates is the quasilinearization of the internal water wave system. This is done in Section 4, where we only give detailed proofs of the steps unique to the current setting. In fact, we can reduce some of the proofs to the estimates performed in [50]. We also provide details on the derivation of the stability criterion in Proposition 4.6, where we obtain coercivity type estimates. From these results, we establish a priori bounds on the solution in Section 5. Then we use them to prove Theorem 1.7 in Section 6.

For the remainder of the paper, we give the details on the derivation of the BO equation. We also provide a complete justification of the models that link the BO equation with the internal water waves equations. In Section 8, we follow the work of Bona, Lannes, and Saut [15] to derive the intermediate systems provided in Theorem 1.17 and the BO equation. Then, in Section 7, we provide a short proof of Theorem 1.14, which gives the long-time well-posedness of a weakly-dispersive BO system. Lastly, in Section 9, we conclude the paper by proving Theorem 1.18, i.e., the full justification of each of the systems derived in Theorem 1.17.

1.4. **Definition and notations.**

- We define the gradient by

$$\nabla_{x,z}^\mu = (\sqrt{\mu}\partial_x, \partial_z)^T$$

  and we introduce the scaled Laplace operator

$$\Delta_{x,z}^\mu = \nabla_{x,z}^\mu \cdot \nabla_{x,z}^\mu = \mu\partial_x^2 + \partial_z^2.$$

- We let c denote a positive constant independent of μ, ε, and bo that may change from line to line. Also, as a shorthand, we use the notation $a \lesssim b$ to mean $a \leq c\, b$.
- Let $L^2(\mathbb{R})$ be the usual space of square integrable functions with norm $|f|_{L^2} = \sqrt{\int_{\mathbb{R}} |f(x)|^2\, dx}$. Also, for any $f, g \in L^2(\mathbb{R})$ we denote the scalar product by $\big(f, g\big)_{L^2} = \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx$.
- Let $f : \mathbb{R} \to \mathbb{R}$ be a tempered distribution, let $\hat{f}$ or $\mathcal{F}f$ be its Fourier transform. Let $F : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then the Fourier multiplier associated with $F(\xi)$ is denoted F and defined by the formula:

$$\mathcal{F}\big(\mathrm{F}(\mathrm{D})f(x)\big)(\xi) = F(\xi)\hat{f}(\xi).$$

- For any $s \in \mathbb{R}$ we call the multiplier $\widehat{|\mathrm{D}|^s f}(\xi) = |\xi|^s \hat{f}(\xi)$ the Riesz potential of order $-s$.
- For any $s \in \mathbb{R}$ we call the multiplier $\Lambda^s = (1 + \mathrm{D}^2)^{\frac{s}{2}} = \langle \mathrm{D} \rangle^s$ the Bessel potential of order $-s$.
- The Sobolev space $H^s(\mathbb{R})$ is equivalent to the weighted L^2-space with $|f|_{H^s} = |\Lambda^s f|_{L^2}$.
- For any $s \geq 0$ we will denote $\mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$ the homogeneous Sobolev space with $|f|_{\mathring{H}^{s+\frac{1}{2}}} = |\mathrm{D}^{\frac{1}{2}} f|_{H^s}$. One should note that $|\mathrm{D}| = \mathcal{H}\partial_x$, where $\widehat{\mathcal{H}f}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$ is the Hilbert transform.
- For any $s \geq 0$ we will denote $\dot{H}^{s+1}(\mathbb{R})$ the Beppo-Levi space with $|f|_{\dot{H}^{s+1}} = |\Lambda^s \partial_x f|_{L^2}$.
- For any $s \geq 0$ we will denote $\dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R}) = \dot{H}^{s+\frac{1}{2}}(\mathbb{R})$ with $|f|_{\dot{H}_\mu^{s+\frac{1}{2}}} = |\mathfrak{B} f|_{L^2}$ and where $\mathfrak{B}$ is a Fourier multiplier defined in frequency by:
$$\mathcal{F}(\mathfrak{B}f)(\xi) = \frac{|\xi|}{(1 + \sqrt{\mu}|\xi|)^{\frac{1}{2}}} \hat{f}(\xi).$$
- We say f is a Schwartz function $\mathscr{S}(\mathbb{R})$, if $f \in C^\infty(\mathbb{R})$ and satisfies for all $j, k \in \mathbb{N}$,
$$\sup_x |x^j \partial_x^k f| < \infty.$$
- Let $a < b$ be real numbers and consider the domain $\mathcal{S} = (a, b) \times \mathbb{R}$. Then the space $\dot{H}^s(\mathcal{S})$ is endowed with the seminorm
$$\|f\|^2_{\dot{H}^{s+1}(\mathcal{S})} = \int_a^b |\nabla_{x,z} f(\cdot, z)|^2_{H^s} \,\mathrm{d}z.$$
- If A and B are two operators, then we denote the commutator between them to be $[A, B] = AB - BA$.
- Let $t_0 > \frac{1}{2}$, $s \geq 0$, and $h_{\min} \in (0, 1)$. Then for $C(\cdot)$ being a positive non-decreasing function of its argument, we define the constants
$$M = C(\frac{1}{h_{\min}}, |\zeta|_{H^{t_0+2}}),$$
and
$$M(s) = C(M, |\zeta|_{H^s}).$$

1.4.1. *Diffeomorphisms.* In many instances it is convenient to "straighten" the fluid domain. In particular, instead of working on the fluid domain $\Omega_t^\pm$ we introduce the two strips:
$$\mathcal{S}^+ = \{(x, z) \in \mathbb{R}^2 \,:\, -1 < z < 0\} \quad \text{and} \quad \mathcal{S}^- = \{(x, z) \in \mathbb{R}^2 \,:\, 0 < z\}.$$
The mapping between $\mathcal{S}^\pm$ and $\Omega_t^\pm$ will be given by the trivial diffeomorphisms defined below.

Definition 1.20. *Let $t_0 > \frac{1}{2}$ and $\zeta \in H^{t_0+2}(\mathbb{R})$ such that the non-cavitation assumptions (1.14) is satisfied. For the lower domain, we have that:*

1. *We define the time-dependent diffeomorphism mapping the lower domain $\mathcal{S}^+$ onto the water domain Ω_t^+ through*
$$\Sigma^+ : \begin{cases} \mathcal{S}^+ & \longrightarrow \quad \Omega_t^+ \\ (x, z) & \mapsto \quad (x, z(1 + \varepsilon\zeta) + \varepsilon\zeta). \end{cases}$$

2. The Jacobi matrix J_{Σ^+} *is given by*

$$J_{\Sigma^+} = \begin{pmatrix} 1 & 0 \\ \varepsilon(1+z)\partial_x\zeta & (1+\varepsilon\zeta) \end{pmatrix},$$

and is bounded on $\mathcal{S}^+$*. Moreover the determinant is given by* $1+\varepsilon\zeta$ *and is bounded below due to the non-cavitation condition* (1.14)*.*

3. The matrix associated with the change of variable for the Laplace problem is given by

$$P(\Sigma^+) = \begin{pmatrix} (1+\varepsilon\zeta) & -\varepsilon\sqrt{\mu}(z+1)\partial_x\zeta \\ -\varepsilon\sqrt{\mu}(z+1)\partial_x\zeta & \frac{1+\varepsilon^2\mu(z+1)^2|\partial_x\zeta|^2}{1+\varepsilon\zeta} \end{pmatrix},$$

and is uniformly coercive. In fact, one can verify that it satisfies for all $\theta \in \mathbb{R}^{d+1}$ *and any* $(X,z) \in \mathcal{S}^+$ *that*

$$P(\Sigma^+)\theta\cdot\theta \geq \frac{1}{1+M}|\theta|^2 \quad \textit{and} \quad \|P^+(\Sigma^+)\|_{L^\infty} \leq M. \tag{1.34}$$

Similarly, for the upper domain, we have that:

1. We define the time-dependent diffeomorphism mapping the upper-half plane $\mathcal{S}^-$ *onto the water domain* Ω_t^- *through the transformation*

$$\Sigma^- : \begin{cases} \mathcal{S}^- & \longrightarrow \quad \Omega_t^- \\ (x,z) & \mapsto \quad (x, z+\varepsilon\zeta). \end{cases}$$

2. The Jacobi matrix J_{Σ^-} *is given by*

$$J_{\Sigma^-} = \begin{pmatrix} 1 & 0 \\ \varepsilon\partial_x\zeta & 1 \end{pmatrix},$$

and is bounded on $\mathcal{S}^-$*. Moreover, the determinant is given by* 1*.*

3. The matrix associated with the change of variable for the Laplace problem is given by

$$P(\Sigma^-) = \begin{pmatrix} 1 & -\varepsilon\sqrt{\mu}\partial_x\zeta \\ -\varepsilon\sqrt{\mu}\partial_x\zeta & 1+\varepsilon^2\mu(\partial_x\zeta)^2 \end{pmatrix},$$

and is uniformly coercive. In fact, one can verify that it satisfies for all $\theta \in \mathbb{R}^2$ *and any* $(x,z) \in \mathcal{S}^-$ *that*

$$P(\Sigma^-)\theta\cdot\theta \geq \frac{1}{1+M}|\theta|^2 \quad \textit{and} \quad \|P^-(\Sigma^-)\|_{L^\infty} \leq M. \tag{1.35}$$

2. Properties of $\mathcal{G}_\mu$

In this section, we aim to give a rigorous meaning to the operator $\mathcal{G}_\mu$ introduced formally in equation (1.10) and study its properties. The main results in this section will now be stated.

Proposition 2.1. *Let* $t_0 \geq 1$, $s \in [0, t_0+1]$, *and* $\zeta \in H^{t_0+2}(\mathbb{R})$ *be such that* (1.14) *is satisfied. Then the mapping*

$$\mathcal{G}_\mu[\varepsilon\zeta] : \begin{cases} \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R}) & \to H^{s-\frac{1}{2}}(\mathbb{R}) \\ \psi & \mapsto \mathcal{G}_\mu^+[\varepsilon\zeta]\Big(1-\gamma(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\Big)^{-1}\psi, \end{cases} \tag{2.1}$$

is well-defined and satisfies the following properties:

1. *For any* $0 \leq s \leq t_0 + 1$ *,there holds,*

$$|\mathcal{G}_\mu[\varepsilon\zeta]\psi|_{H^{s-\frac{1}{2}}} \leq \mu^{\frac{3}{4}} M |\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}, \tag{2.2}$$

and

$$|\mathcal{G}_\mu[\varepsilon\zeta]\psi|_{H^{s-\frac{1}{2}}} \leq \mu^{\frac{1}{2}} M |\partial_x\psi|_{H^{s-\frac{1}{2}}}. \tag{2.3}$$

2. *For any* $0 \leq s \leq t_0 + \frac{1}{2}$ *there holds,*

$$|\mathcal{G}_\mu[\varepsilon\zeta]\psi|_{H^{s-\frac{1}{2}}} \leq \mu M |\psi|_{\dot{H}_\mu^{s+1}}, \tag{2.4}$$

and

$$|\mathcal{G}_\mu[\varepsilon\zeta]\psi|_{H^{s-\frac{1}{2}}} \leq \mu^{\frac{3}{4}} M |\partial_x\psi|_{H^{s+\frac{1}{2}}}. \tag{2.5}$$

3. *The operator is uniformly coercive on on* $\psi \in \dot{H}_\mu^{\frac{1}{2}}(\mathbb{R})$*,*

$$|\psi|^2_{\dot{H}_\mu^{\frac{1}{2}}} \leq M\big(\psi, \frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta]\psi\big)_{L^2}. \tag{2.6}$$

4. *The bilinear form is symmetric on* $\dot{H}_\mu^{\frac{1}{2}}(\mathbb{R}) \times \dot{H}_\mu^{\frac{1}{2}}(\mathbb{R})$*,*

$$\big(\mathcal{G}_\mu[\varepsilon\zeta]\psi, \psi\big)_{L^2} = \big(\psi, \mathcal{G}_\mu[\varepsilon\zeta]\psi\big)_{L^2}. \tag{2.7}$$

5. *For all* $s \in [0, t_0 + 1]$ *and* $f, g \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$ *there holds,*

$$|\big(\Lambda^s\mathcal{G}_\mu[\varepsilon\zeta]f, \Lambda^s g\big)_{L^2}| \leq \mu M |f|_{\dot{H}_\mu^{s+\frac{1}{2}}} |g|_{\dot{H}_\mu^{s+\frac{1}{2}}}. \tag{2.8}$$

Remark 2.2. *Here we take* $t_0 \geq 1$ *since we will apply the result at high regularity. If one would take* $t_0 > \frac{1}{2}$ *then we would need* $s \in [\max\{0, 1 - t_0\}, t_0 + \frac{1}{2}]$ *to enforce Remark A.9, which specifies the needed regularity to define* $\mathcal{G}_\mu^-$*.*

Remark 2.3. *For the undisturbed case, we can use formulas* (A.3) *and* (A.19) *to find that*

$$\mathcal{G}_\mu[0] = \sqrt{\mu}|\mathrm{D}|\frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{1 + \gamma\tanh(\sqrt{\mu}|\mathrm{D}|)}. \tag{2.9}$$

We will follow the same strategy as in [50], where we study the operators involved in the expression for $\mathcal{G}_\mu$ and prove that we can define them by ψ through the transition problem (1.9). We study each part individually in separate subsections. The main difference from the previous work is that we have to carefully track the dependence on the parameters for the current regime. Moreover, the functional setting of the upper fluid is fundamentally different from the one in the lower fluid domain.

2.1. Properties of $(\mathcal{G}_\mu^-)^{-1}\mathcal{G}_\mu^+$.

For the description of the operator $(\mathcal{G}_\mu^-)^{-1}\mathcal{G}_\mu^+$ we will follow the proof of Proposition 1 in [50]. The main difference is that we have an interaction of two operators that act on different scales, and this is seen in the estimates given below:

Proposition 2.4. *Let* $t_0 \geq 1$*,* $s \in [0, t_0 + 1]$ *and* $\zeta \in H^{t_0+2}(\mathbb{R})$ *be such that* (1.14) *is satisfied. Then the mapping*

$$(\mathcal{G}_\mu[\varepsilon\zeta]^-)^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta] \; : \; \begin{cases} \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R}) & \to \mathring{H}^{s+\frac{1}{2}}(\mathbb{R}) \\ \psi^+ & (\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+ \end{cases} \tag{2.10}$$

is well-defined and satisfies

$$|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+|_{\mathring{H}^{s+\frac{1}{2}}} \leq \mu^{\frac{1}{4}} M |\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}}, \tag{2.11}$$

and

$$|\partial_x\big((\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+\big)|_{H^s} \leq M|\partial_x\psi^+|_{H^s}. \tag{2.12}$$

Remark 2.5. *If we change the role of $\pm$ the result is not true. Indeed, consider the case $\varepsilon\zeta = 0$, then we have by direct computations that*

$$(\mathcal{G}_\mu^+[0])^{-1}\mathcal{G}_\mu^-[0] = (\tanh(\sqrt{\mu}|\mathrm{D}|)^{-1},$$

and we obtain the estimate

$$|(\mathcal{G}_\mu^+[0])^{-1}\mathcal{G}_\mu^-[0]\psi^-|_{\dot{H}_\mu^{\frac{1}{2}}} \lesssim \frac{1}{\sqrt{\mu}}|\psi^-|_{H^{\frac{1}{2}}},$$

which is incompatible with having $\psi^- \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$.

Before we turn to the proof, we need a Lemma that will be used to justify some of the computations that will be made.

Lemma 2.6. *Suppose the provisions of Proposition 2.4, and further that $\phi^- = \Phi^- \circ \Sigma^- \in \dot{H}^{s+1}(\mathcal{S}^-)$ is a variational solution to*

$$\begin{cases} \nabla_{x,z}^\mu \cdot P(\Sigma^-)\nabla_{x,z}^\mu\phi^- = 0 & \text{in} \quad \mathcal{S}^- \\ \partial_n^{P^-}\phi^- = \mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+ & \text{on} \quad z = 0. \end{cases} \tag{2.13}$$

Then there is a number $R > 0$ such that for $\mathcal{S}_R^- = \{(x,z) \in \mathcal{S}^- \ : \ z > R\}$ and any $\alpha + \beta$ integer > 0 there holds,

$$\partial_z^\alpha\partial_x^\beta\phi^- \in L^2(\mathcal{S}_R^-) \quad \text{and} \quad \lim_{z\to\infty}\sup_{x\in\mathbb{R}}\big|\partial_z^\alpha\partial_x^\beta\phi^-(x,z)\big| = 0. \tag{2.14}$$

Moreover, we have the following trace inequalities:

$$|\phi^-|_{z=0}|_{\mathring{H}^{s+\frac{1}{2}}} \leq \|\Lambda^s\nabla_{x,z}\phi^-\|_{L^2(\mathcal{S}^-)}, \tag{2.15}$$

and

$$|\phi^-|_{z=0}|_{\mathring{H}_\mu^{s+\frac{1}{2}}} \leq \mu^{-\frac{1}{4}}\|\Lambda^s\nabla_{x,z}\phi^-\|_{L^2(\mathcal{S}^-)}. \tag{2.16}$$

Remark 2.7. *For the proof of* (2.15)*, we need to work on the upper-half plane to prove the estimate, and this is the technical reason why we do not study the reverse composition: $(\mathcal{G}_\mu^+)^{-1}\mathcal{G}_\mu^-$.*

Remark 2.8. *The proof of estimate* (2.14) *is given in* [4] *for $\phi|_{z=0} \in H^{s+\frac{1}{2}}(\mathbb{R})$ and we will briefly explain the changes for $\phi|_{z=0} \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$.*

Assuming for a moment that Lemma 2.6 holds true, we can give the proof of Proposition 2.4.

Proof of Proposition 2.4. We divide the proof into four main steps.

<u>Step 1.</u> *$(\mathcal{G}_\mu^-)^{-1} \circ \mathcal{G}_\mu^+$ is well-defined on $\mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$.* It is sufficient to prove that there exists a unique variational solution $\phi^- \in \dot{H}^{s+1}(\mathcal{S}^-)$ to the system (2.13) for any $\psi^+ \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$. Indeed, assuming there is such a solution, then by (2.15), we can define

$$\psi^- = \phi^-|_{z=0} \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R}),$$

so that for any $\psi^+ \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$ there is a ψ^- where Proposition A.13 implies

$$\mathcal{G}_\mu^-[\varepsilon\zeta]\psi^- = \mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+.$$

To prove the claim we first let $s=0$, and use (2.14) to define the variational problem associated to (2.13) by

$$\begin{aligned} a(\phi^-,\varphi) &= \int_{\mathcal{S}^-} P(\Sigma^-)\nabla^\mu_{x,z}\phi^- \cdot \nabla^\mu_{x,z}\varphi \,\mathrm{d}x\mathrm{d}z \\ &= -\int_{\{z=0\}} (\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+)\varphi \,\mathrm{d}x \\ &= L(\varphi), \end{aligned} \tag{2.17}$$

for any $\varphi \in C^\infty(\overline{\mathcal{S}^-}) \cap \dot{H}^1(\mathcal{S}^-)$. We will now verify the assumptions of Lax-Milgram's Theorem to deduce a variational solution in $\dot{H}^1(\mathcal{S}^-)$ in two steps (extending the result to $\dot{H}^{s+1}(\mathcal{S}^-)$ is classical).

The first step is to show that the application $\varphi \mapsto L(\varphi)$ is continuous on $\dot{H}^1(\mathcal{S}^-)$. To do so, we note by estimate (A.6) that

$$\begin{aligned} |L(\varphi)| &\leq |\big(\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+, \varphi|_{z=0}\big)_{L^2}| \\ &\leq \mu|\psi^+|_{\dot{H}^{\frac{1}{2}}_\mu}|\varphi|_{z=0}|_{\dot{H}^{\frac{1}{2}}_\mu}. \end{aligned}$$

Then use (2.16) to obtain the needed bound

$$\mu^{\frac{3}{4}}|\varphi|_{z=0}|_{\dot{H}^{\frac{1}{2}}_\mu} \leq \|\nabla^\mu_{x,z}\varphi\|_{L^2(\mathcal{S}^-)}. \tag{2.18}$$

Lastly, The bilinear form $a(\cdot,\cdot)$ is continuous and coercive on $\dot{H}^1(\mathcal{S}^-)$ by using estimate (1.35). We may therefore conclude that there is a unique solution $\phi^- \in \dot{H}^1(\mathcal{S}^-)$.

Step 2. *Estimate* (2.11) *holds true.* Let $\psi^- = \phi^-|_{z=0} \in \mathring{H}^{s+\frac{1}{2}}$ defined by the solution of (2.13). Then we can use estimate (A.22) to get that

$$\begin{aligned} |(\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+|_{\mathring{H}^{s+\frac{1}{2}}} &= |\psi^-|_{\mathring{H}^{s+\frac{1}{2}}} \\ &\leq M\|\Lambda^s\nabla^\mu_{x,z}\phi^-\|_{L^2(\mathcal{S}^+)}. \end{aligned}$$

Thus, we need to verify

$$\|\Lambda^s\nabla^\mu\phi^-\|_{L^2(\mathcal{S}^-)} \leq \mu^{\frac{1}{4}}|\psi^+|_{\dot{H}^{s+\frac{1}{2}}_\mu}.$$

To this end, we apply Λ^s to (2.13) and study:

$$\begin{cases} \nabla^\mu_{x,z}\cdot P(\Sigma^-)\nabla^\mu_{x,z}\Lambda^s\phi^- = -\nabla_{x,z}[P(\Sigma^-),\Lambda^s]\nabla^\mu_{x,z}\phi^- & \text{in} \quad \mathcal{S}^- \\ \partial^{P^-}_n\Lambda^s\phi^- = -\mathbf{e}_z\cdot[P(\Sigma^-),\Lambda^s]\nabla^\mu_{x,z}\phi^- + \Lambda^s\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+ & \text{on} \quad z=0. \end{cases}$$

Then from the variational formulation, we find that

$$\begin{aligned} \int_{\mathcal{S}^-} P(\Sigma^-)\nabla^\mu_{x,z}\Lambda^s\phi^- \cdot \nabla^\mu_{x,z}\Lambda^s\phi^- \,\mathrm{d}x\mathrm{d}z = &-\int_{\{z=0\}} \Lambda^s(\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+)\Lambda^s\phi^- \,\mathrm{d}x \\ &+ \int_{\mathcal{S}^-} [P(\Sigma^-),\Lambda^s]\nabla^\mu_{x,z}\phi^- \cdot \nabla^\mu_{x,z}\Lambda^s\phi^- \,\mathrm{d}x\mathrm{d}z. \end{aligned}$$

Now, by the coercivity estimate (1.35), Cauchy-Schwarz, (A.6), (2.16), and the commutator estimate (B.8) we get that

$$\|\Lambda^s\nabla^\mu_{x,z}\phi^-\|_{L^2(\mathcal{S}^-)} \leq M\big(\mu^{\frac{1}{4}}|\psi^+|_{\dot{H}^{s+\frac{1}{2}}_\mu} + \|\Lambda^{s-1}\nabla^\mu_{x,z}\phi^-\|_{L^2(\mathcal{S}^-)}\big). \tag{2.19}$$

Also, using the estimates in Step 1, we obtain the base case

$$\|\nabla^\mu_{x,z}\phi^-\|_{L^2(\mathcal{S}^-)} \leq \mu^{\frac{1}{4}} M |\psi^+|_{\dot{H}^{\frac{1}{2}}_\mu}. \tag{2.20}$$

We may now conclude the proof of this step by continuous induction.

Step 3. To prove estimate (2.12) we first observe that

$$\langle\xi\rangle^s|\xi| \lesssim \langle\xi\rangle^{s+\frac{1}{2}}|\xi|^{\frac{1}{2}} \quad \text{and} \quad \langle\xi\rangle^{s+\frac{1}{2}}\frac{|\xi|}{(1+\sqrt{\mu}|\xi|)^{\frac{1}{2}}} \lesssim \mu^{-\frac{1}{4}}\langle\xi\rangle^s|\xi|.$$

Then use Plancherel's identity and (2.11) to find that

$$\begin{aligned}|\partial_x\big((\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}\circ\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+\big)|_{H^s} &\lesssim |(\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}\circ\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+|_{\mathring{H}^{(s+\frac{1}{2})+\frac{1}{2}}}\\ &\leq \mu^{\frac{1}{4}}M|\psi^+|_{\dot{H}_\mu^{(s+\frac{1}{2})+\frac{1}{2}}}\\ &\leq M|\partial_x\psi^+|_{H^s}.\end{aligned}$$

□

To close this subsection, we give the proof of Lemma 2.6.

Proof of Lemma 2.6. We first give a proof of the trace inequalities on $C^\infty(\overline{\mathcal{S}^-})\cap\dot{H}^1(\mathcal{S}^-)$. For the proof of (2.15), we define a multiplier being a smooth cut-off function in frequency $\chi:[0,\infty)\to[0,1]$ such that $\chi(0)=1$, $\chi(\xi)=0$ for $\xi>1$, and $\chi,\chi'\in L^\infty(\mathbb{R})$. Then by the Fundamental Theorem of Calculus and Young's inequality, we obtain that

$$\begin{aligned}||\mathrm{D}|^{\frac{1}{2}}\phi^-|_{z=0}|^2_{L^2} &= -\int_{\mathbb{R}}\int_0^{\frac{2}{|\xi|}}\partial_z\big(\chi(z|\xi|)\big||\xi|^{\frac{1}{2}}\hat{\phi}^-\big|^2\big)\,\mathrm{d}z\mathrm{d}\xi\\ &\lesssim |\chi'|_{L^\infty}\int_{\mathbb{R}}\int_0^\infty\big||\xi|\hat{\phi}^-\big|^2\,\mathrm{d}z\mathrm{d}\xi + |\chi|_{L^\infty}\int_{\mathbb{R}}\int_0^\infty\big|(|\xi|+\partial_z)\hat{\phi}^-\big|^2\,\mathrm{d}z\mathrm{d}\xi.\end{aligned}$$

Then to conclude, we use Plancherel's identity to obtain (2.15) for $s=0$. The general case is proved similarly.

For the proof of (2.16) we let $\eta\in C^1_b([0,\infty))$ such that $\eta(0)=1$ and $\eta(z)=0$ for $z>1$. Then use the Fundamental Theorem of Calculus and Young's inequality to get that

$$\begin{aligned}|\phi^-|_{z=0}|^2_{\dot{H}^{\frac{1}{2}}_\mu} &= \int_{\mathbb{R}}\frac{|\xi|^2}{1+\sqrt{\mu}|\xi|}|\hat{\phi}^-|_{z=0}|^2\,\mathrm{d}\xi\\ &\lesssim |\eta'|_{L^\infty}\int_{\mathbb{R}}\int_0^\infty|\xi|^2|\hat{\phi}^-|^2\,\mathrm{d}z\mathrm{d}\xi + |\eta|_{L^\infty}\frac{1}{\sqrt{\mu}}\int_{\mathbb{R}}\int_0^\infty|(|\xi|+\partial_z)\hat{\phi}^-|^2\,\mathrm{d}z\mathrm{d}\xi.\end{aligned}$$

The result follows from the use of Plancherel's identity.

For the proof of (2.14), we note from Definition 1.20 that $\Phi^-=\phi^-\circ\Sigma^{-1}\in\dot{H}^{s+1}(\Omega^-)$ is harmonic, and by Sobolev embedding we have that ζ is bounded above by some constant $R>0$. From these observations, we can use (2.15) to see that $\Phi^-|_{z=R}\in\mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$. Moreover, we consider the harmonic extension on $\mathcal{S}^-_R$ given by the Poisson formula:

$$\Phi^-_R(x,z)=\mathrm{e}^{-(z-R)\sqrt{\mu}|\mathrm{D}|}\Phi^-(x,R).$$

From this formula, we can verify (2.14) for Φ^-_R by direct computations as in [4]. To conclude, we use that both functions are harmonic and agree on the line $z=R$.

□

We should note that there are several results that follow from Proposition 2.4, and that will be used throughout the paper. However, to ease the presentation, we postponed these results for the Appendix in Section A.3 since the proofs are technical and not needed in this section.

2.2. **Properties of** $(1-\gamma(\mathcal{G}_\mu^-)^{-1}\mathcal{G}_\mu^+)^{-1}$**.** Following the road map provided in [50], we can recover the velocity potentials $\phi^\pm$ from the knowledge of ζ and a trace ψ defined though the transmission problem:

$$\begin{cases} \nabla_{x,z}^{\mu^\pm}\cdot P(\Sigma^\pm)\nabla_{x,z}^{\mu^\pm}\phi^\pm = 0 & \text{in} \quad \mathcal{S}^\pm \\ \phi^+|_{z=0} - \gamma\phi^-|_{z=0} = \psi \\ \partial_n^{P^-}\phi^-|_{z=0} = \partial_n^{P^+}\phi^+|_{z=0}, \quad \partial_n^{P^+}\phi^+|_{z=-1} = 0, \end{cases} \tag{2.21}$$

where the solvability of this problem is ensured in the next result:

Proposition 2.9. *Let $t_0 \geq 1$, $s \in [0, t_0+1]$ and $\zeta \in H^{t_0+2}(\mathbb{R})$ be such that* (1.14) *is satisfied. Then for all $\psi \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$, there exist a unique solution $\phi^\pm \in \dot{H}^{s+1}(\mathcal{S}^\pm)$ to* (2.21) *and that satisfies*

$$\|\Lambda^s\nabla_{x,z}^\mu\phi^\pm\|_{L^2(\mathcal{S}^+)} \leq \sqrt{\mu}M|\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}.$$

The proof of this result is a consequence of the following Lemma:

Lemma 2.10. *Let $t_0 \geq 1$, $s \in [0, t_0+1]$. Then the mapping*

$$\mathcal{J}_\mu[\varepsilon\zeta] \; : \; \begin{cases} \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R}) & \to \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R}) \\ \psi^+ & \mapsto (1-\gamma(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta])\psi^+ \end{cases} \tag{2.22}$$

is one-to-one and onto. Moreover, it satisfies the estimate

$$|(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}} \leq M|\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}}. \tag{2.23}$$

Remark 2.11. *For any $\psi \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$ we can define ψ^+ by*

$$\psi^+ = (\mathcal{J}_\mu[\varepsilon\zeta])^{-1}\psi \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R}),$$

from which we define ψ^- by

$$\psi^- = (\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+ \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R}).$$

Also, note that from these identities and (2.11) *that there holds,*

$$\begin{aligned} |\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}} &= |\mathcal{J}_\mu[\varepsilon\zeta]\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}} \\ &\leq |\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}} + \gamma|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}} \\ &\leq |\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}} + \gamma\mu^{-\frac{1}{4}}|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+|_{\mathring{H}^{s+\frac{1}{2}}} \\ &\leq M|\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}}. \end{aligned} \tag{2.24}$$

Proof of Lemma 2.10. To prove estimate (2.23), we first consider the case $s=0$. Then we use the definition of $\mathcal{J}_\mu[\varepsilon\zeta]$, the construction $\psi^- = (\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+$ to get that

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{J}_\mu[\varepsilon\zeta]\psi^+\, \mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+\,\mathrm{d}x &= \int_{\mathbb{R}} \psi^+\,\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+\,\mathrm{d}x - \gamma\int_{\mathbb{R}}(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+\,\mathcal{G}^+\psi^+\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \psi^+\,\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+\,\mathrm{d}x - \gamma\int_{\mathbb{R}}\psi^-\,\mathcal{G}_\mu^-[\varepsilon\zeta]\psi^-\,\mathrm{d}x. \end{aligned}$$

Then apply Proposition A.4 and A.13 combined with the coercivity of $P(\Sigma^{\pm})$ to obtain the estimate

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{J}_\mu[\varepsilon\zeta]\psi^+ \, \mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+ \,\mathrm{d}x &= \int_{\mathcal{S}^+} P(\Sigma^+)\nabla_{x,z}^\mu \phi^+ \cdot \nabla_{x,z}^\mu \phi^+ \,\mathrm{d}x\mathrm{d}z \\ &\quad + \gamma \int_{\mathcal{S}^-} P(\Sigma^-)\nabla_{x,z}^\mu \phi^- \cdot \nabla_{x,z}^\mu \phi^- \,\mathrm{d}x\mathrm{d}z \\ &\geq \frac{1}{1+M}\|\nabla_{x,z}^\mu \phi^+\|_{L^2(\mathcal{S}^+)}^2.\end{aligned}$$

Moreover, since $\mathcal{J}_\mu[\varepsilon\zeta]\psi^+ \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R}) \subset \dot{H}_\mu^{\frac{1}{2}}(\mathbb{R})$ we have (A.6) at hand. In particular, we obtain from the above estimates and (A.4) that

$$|\psi^+|_{\dot{H}_\mu^{\frac{1}{2}}} \leq M|\mathcal{J}_\mu[\varepsilon\zeta]\psi^+|_{\dot{H}_\mu^{\frac{1}{2}}}. \tag{2.25}$$

Equivalently, estimate (2.23) holds in the case $s = 0$. We may also use this estimate to prove the invertibility as in [50]. In particular, we have from the lower bound (2.25) and Proposition 2.4 with estimate (2.11) that $\mathcal{J}_\mu$ is an injective and closed operator since,

$$\gamma|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}} \leq \mu^{-\frac{1}{4}}\gamma|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+|_{\mathring{H}^{s+\frac{1}{2}}} \leq \gamma M|\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}}.$$

Moreover, from the lower bound, (2.25), on $\mathcal{J}_\mu$ we know that it is also semi-Fredholm. Then since for small enough values of $\gamma \in (0,1)$ it is invertible by a Neumann series expansion, we have from the homotopic invariance of the index that the operator is in fact Fredholm of index zero [40]. Consequently, the operator is also surjective and therefore invertible.

For the general case of $s \in [0, t_0+1]$, the proof is the same as Lemma 2 in [50]. □

We may now use Lemma 2.10 to prove Proposition 2.9.

Proof of Proposition 2.9. We first consider the existence of a unique solution ϕ^+ in the lower domain. Since $\psi \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$ we can use Proposition 2.10 to make the definition:

$$\psi^+ = (\mathcal{J}_\mu[\varepsilon\zeta])^{-1}\psi \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R}) \quad \text{and} \quad \psi^- = (\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+ \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R}) \subset \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R}),$$

where we let $\phi^{\pm}|_{z=0} = \psi^{\pm}$. Then we can use the first point of Proposition A.4 to deduce a unique solution ϕ^+ in the lower domain, where the estimate follows from (A.5) and (2.23):

$$\begin{aligned}\|\Lambda^s\nabla^\mu\phi^+\|_{L^2(\mathcal{S}^{\pm})} &\leq \sqrt{\mu}M|\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}} \\ &= \sqrt{\mu}M|(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}} \\ &\leq \sqrt{\mu}M|\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}.\end{aligned}$$

For the upper half plane, we use Proposition 2.4 we use Remark A.9 together with the first point of Proposition A.13 to deduce a unique solution ϕ^-. The estimate is a consequence of estimates (A.23), (2.11), and then (2.23):

$$\begin{aligned}\|\Lambda^s\nabla_{x,z}^\mu\phi^-\|_{L^2(\mathcal{S}^-)} &\leq \mu^{\frac{1}{4}}M|\psi^-|_{\mathring{H}^{s+\frac{1}{2}}} \\ &= \mu^{\frac{1}{4}}\gamma M|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+|_{\mathring{H}^{s+\frac{1}{2}}} \\ &\leq \sqrt{\mu}M|\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}} \\ &\leq \sqrt{\mu}M|\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}.\end{aligned}$$

□

We may now give the proof of the main result of the section.

Proof of Proposition 2.1. We prove each point individually in four separate steps.

Step 1. The proof of estimate (2.2) and (2.3) follows by (A.7), (2.23), and Plancherel's identity:

$$|\mathcal{G}_\mu[\varepsilon\zeta]\psi|_{H^{s-\frac{1}{2}}} = |\mathcal{G}_\mu^+[\varepsilon\zeta](\mathcal{J}_\mu[\varepsilon\zeta])^{-1}\psi|_{H^{s-\frac{1}{2}}} \leq \mu^{\frac{3}{4}}|(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}} \leq \mu^{\frac{3}{4}}|\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}} \leq \sqrt{\mu}|\partial_x\psi|_{H^s}.$$

Step 2. The proof of (2.4) and (2.5), is proved the same way as in Step 1, but we instead use (A.8).

Step 3. The coercivity estimate (2.6) follows by construction where use the identities in Remark 2.11 to get that

$$\big(\psi, \mathcal{G}_\mu[\varepsilon\zeta]\psi\big)_{L^2} = \big(\psi, \mathcal{G}_\mu^+[\varepsilon\zeta](\mathcal{J}_\mu[\varepsilon\zeta])^{-1}\psi\big)_{L^2} = \big(\mathcal{J}_\mu[\varepsilon\zeta]\psi^+, \mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+\big)_{L^2}.$$

Now, argue as in the proof Proposition 2.10 where estimates (A.4) and (2.24) implies

$$\big(\psi, \mathcal{G}_\mu[\varepsilon\zeta]\psi\big)_{L^2} \geq \frac{\mu}{1+M}|\psi^+|^2_{\dot{H}_\mu^{\frac{1}{2}}} \geq \frac{\mu}{1+M}|\psi|^2_{\dot{H}_\mu^{\frac{1}{2}}}.$$

Step 4. The symmetry follows by the second point in Propostions A.4 and A.13.

Step 5. Finally, for the proof estimate (2.8) we use (A.6) and Lemma 2.10.

□

3. Symbolic analysis of the Dirichlet-Neumann operator

In this section, we will give a symbolic description of the operator $\mathcal{G}_\mu$ defined given by (2.1). The estimates need to be precise in terms of the parameters μ, ε, and where we carefully track the Sobolev regularity with respect to ψ and ζ. One reason for these expressions is that we need to have an estimate of the type:

$$|\big(\mathcal{G}_\mu[\varepsilon\zeta]\circ(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\big(\partial_x(fg)\big), f\big)_{L^2(\mathbb{R})}| \leq M(t_0+3)|g|_{H^{t_0+1}}|f|^2_{L^2}, \tag{3.1}$$

which appears naturally in the energy estimates and the quasilinearisation of the main equations. As we can see from (3.1), one needs to absorb a derivative and be uniform with respect to the small parameters. Also, recall that

$$\mathcal{G}_\mu[\varepsilon\zeta]\psi = \mathcal{G}_\mu^+[\varepsilon\zeta]\Big(1-\gamma(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\Big)^{-1}\psi,$$

which means we need a good symbolic description of each of the operators involved in the expression. This is the strategy that was implemented in [48, 50], where we first will consider the symbolic description of $\mathcal{G}_\mu^\pm$, then we treat the inverse operators that are involved.

3.1. Symbolic analysis of $\mathcal{G}_\mu^\pm$.

For the symbolic description of $\mathcal{G}_\mu^+$, we know that the operator coincides with the ones studied in [50]. In particular, we can use Theorem 4, in dimension one, which is one of the key estimates of the paper:

Proposition 3.1 (Theorem 4 in [50])**.** *Let $t_0 > \frac{1}{2}$ and $\zeta \in H^{t_0+3}(\mathbb{R})$ be such that (1.14) is satisfied. Then for all $0 \leq s \leq t_0$ and $\psi \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$, one can approximate the positive Dirichlet-Neumann operator by*

$$\mathrm{Op}(S^+)\psi(x) = \sqrt{\mu}\mathcal{F}^{-1}\Big(|\xi|\tanh(\sqrt{\mu}t(x,\xi)\hat{\psi}(\xi)\Big)(x),$$

where we define the "tail" by the symbol

$$t(X,\xi) = (1+\varepsilon\zeta)\frac{\arctan(\varepsilon\sqrt{\mu}\partial_x\zeta)}{\varepsilon\sqrt{\mu}\partial_x\zeta}|\xi|. \tag{3.2}$$

Moreover, for $k=0,1$, the operator satisfy

$$|\mathcal{G}_\mu^+[\varepsilon\zeta]\psi - \mathrm{Op}(S^+)\psi|_{H^{s+\frac{k}{2}}} \leq \varepsilon\mu^{1-\frac{k}{4}}M(t_0+3)|\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}. \tag{3.3}$$

In the case of infinite depth, it is pointed out in Remark 17 in [50] that the tail effects vanish (formally) since the hyperbolic tangent is a bottom effect. In particular, we have the following result that proves this fact.

Proposition 3.2. *Let $t_0 \geq 1$ and $\zeta \in H^{t_0+3}(\mathbb{R})$. Then for all $0 \leq s \leq t_0$ and $\psi \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$, one can approximate the negative Dirichlet-Neumann operator by*

$$S^-(\mathrm{D}) = -\sqrt{\mu}|\mathrm{D}|,$$

where we have the following estimate

$$|\mathcal{G}_\mu^-[\varepsilon\zeta]\psi - \mathrm{Op}(S^-)\psi|_{H^{s+\frac{1}{2}}} \leq \varepsilon\mu M(t_0+3)|\psi|_{\mathring{H}^{s+\frac{1}{2}}}. \tag{3.4}$$

Remark 3.3. *The proof can be seen as a modified version of the proof presented in [50], but where we need to make two approximations of an elliptic problem depending on the frequencies and weighted estimates to deal with some integrability issues in $\mathcal{S}^- = \mathbb{R} \times [0,\infty)$. The weights and the cut-off functions are adapted to the approximation and deal with separate issues. We also note that the proof in [50] also extends to two dimensions. In fact, Proposition 3.2 is mainly the result that restricts Theorem 1.7 to one horizontal dimension in this paper. However, we expect the result to be true for $X \in \mathbb{R}^2$ by modifying the ansatz in the proof and letting $\mathrm{Op}(S^-)$ be given by*

$$\mathrm{Op}(S^-)\psi(X) = -\sqrt{\mu}\mathcal{F}^{-1}\Big(\sqrt{|\xi|^2 + \varepsilon^2\mu\big((|\nabla_X\zeta|^2|\xi|^2) - (\nabla_X\zeta\cdot\xi)^2\big)}\hat{\psi}(\xi)\Big)(X),$$

for $X,\xi \in \mathbb{R}^2$.

Proof. The proof will be given in several steps, where we first decompose the estimate into two main parts depending on the frequencies. In particular, let χ_1 be a Fourier multiplier with a smooth symbol and equal to one around zero. Also, let $\chi_2 = 1-\chi_1$ be the high-frequency part. Then we have by duality

$$\begin{aligned} |\mathcal{G}_\mu^-[\varepsilon\zeta]\psi + \sqrt{\mu}|\mathrm{D}|\psi|_{H^{s+\frac{1}{2}}} &= \sup_{|\varphi|_{L^2}=1}\Big|\int_{\mathbb{R}}\Lambda^{s+\frac{1}{2}}\big(\mathcal{G}_\mu^-[\varepsilon\zeta]\psi + \sqrt{\mu}|\mathrm{D}|\psi\big)\chi_1(\mathrm{D})\varphi\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}}\Lambda^{s+\frac{1}{2}}\big(\mathcal{G}_\mu^-[\varepsilon\zeta]\psi + \sqrt{\mu}|\mathrm{D}|\psi\big)\chi_2(\mathrm{D})\varphi\,\mathrm{d}x\Big| \\ &\leq \sup_{|\varphi|_{L^2}=1}\Big(|I_1(\varphi)| + |I_2(\varphi)|\Big). \end{aligned} \tag{3.5}$$

We will now make two approximations of the elliptic problem (A.20), where the estimate on I_1 can be made using the Poisson kernel. On the other hand, the estimate on I_2 requires an approximate solution of (A.20) that accounts for the principal part of the elliptic operator and is "well-behaved" in high frequency on weighted Sobolev spaces.

<ins>Step 1.</ins> *Approximate solutions for I_1.* We know that $\phi_{\mathrm{app}}^1 = e^{-z\sqrt{\mu}|\mathrm{D}|}\psi$ solves

$$\begin{cases} (\mu\partial_x^2 + \partial_z^2)\phi_{\mathrm{app}}^1 = 0 \quad \text{in} \quad \mathcal{S}^- \\ \phi_{\mathrm{app}}^1|_{z=0} = \psi, \end{cases} \tag{3.6}$$

and

$$\partial_z \phi^1_{\mathrm{app}}|_{z=0} = -\sqrt{\mu}|\mathrm{D}|\psi.$$

Now, let ϕ^- be the solution of (A.20) and define $u_1 = \phi^- - \phi^1_{\mathrm{app}}$. Then u_1 solves

$$\begin{cases} \nabla^\mu_{x,z} \cdot P(\Sigma^-)\nabla^\mu_{x,z} u_1 = \varepsilon\mu r_1 \quad \text{in} \quad \mathcal{S}^- \\ u_1|_{z=0} = 0, \quad \lim_{z\to\infty} |\nabla^\mu_{x,z} u_1| = 0, \end{cases} \tag{3.7}$$

where r_1 reads

$$r_1 = \partial_x\big((\partial_x\zeta)\partial_z\phi^1_{\mathrm{app}}\big) + (\partial_x\zeta)\partial_x\partial_z\phi^1_{\mathrm{app}} - \varepsilon\mu(\partial_x\zeta)^2\partial_z^2\phi^1_{\mathrm{app}}.$$

Moreover, we have by construction that

$$\partial_n^{P^-} u_1|_{z=0} = \mathcal{G}^-_\mu[\varepsilon\zeta]\psi + \sqrt{\mu}|\mathrm{D}|\psi + \varepsilon\mu r_2,$$

where the second rest is given by

$$r_2 = (\partial_x\zeta)\partial_x\psi + \varepsilon\mu(\partial_x\zeta)^2|\mathrm{D}|\psi.$$

Step 1.1. *Estimate on* I_1. From the previous step, we deduce that

$$\begin{aligned} |I_1(\varphi)| &\leq \Big|\int_{\mathbb{R}} (\Lambda^{s+\frac{1}{2}}\partial_n^{P^-} u_1|_{z=0})\chi_1(\mathrm{D})\varphi \,\mathrm{d}x\Big| + \varepsilon\mu\Big|\int_{\mathbb{R}} (\Lambda^{s+\frac{1}{2}} r_2)\chi_1(\mathrm{D})\varphi \,\mathrm{d}x\Big| \\ &= |I_1^1(\varphi)| + |I_1^2(\varphi)|. \end{aligned}$$

We will now estimate each piece separately. We start with the estimate of $I_1^2(\varphi)$. Here we use the definition of r_2 to find that

$$\begin{aligned} |I_1^2(\varphi)| &\leq \varepsilon\mu\Big|\int_{\mathbb{R}} (\partial_x\zeta)(\partial_x\psi)(\Lambda^{s+\frac{1}{2}}\chi_1(\mathrm{D})\varphi)\,\mathrm{d}x\Big| + \varepsilon^2\mu^{\frac{3}{2}}\Big|\int_{\mathbb{R}} (\partial_x\zeta)^2(|\mathrm{D}|\psi)(\Lambda^{s+\frac{1}{2}}\chi_1(\mathrm{D})\varphi)\,\mathrm{d}x\Big| \\ &= |I_1^{2,1}(\varphi)| + |I_1^{2,2}(\varphi)|. \end{aligned}$$

Each term is estimated similarly, and for the estimate of the first term, we recall $\partial_x = -\mathcal{H}|\mathrm{D}|$, where $\mathcal{H}$ is the Hilbert transform. Consequently, we have from Hölder's inequality, Sobolev embedding, and commutator estimate (B.9) that

$$\begin{aligned} |I_1^{2,1}(\varphi)| &\leq \varepsilon\mu\Big|\int_{\mathbb{R}} [\partial_x\zeta, |\mathrm{D}|^{\frac{1}{2}}]\mathcal{H}|\mathrm{D}|^{\frac{1}{2}}\psi)(\Lambda^{s+\frac{1}{2}}\chi_1(\mathrm{D})\varphi)\,\mathrm{d}x\Big| \\ &\quad + \varepsilon\mu\Big|\int_{\mathbb{R}} (\partial_x\zeta)(\mathcal{H}|\mathrm{D}|^{\frac{1}{2}}\psi)(\Lambda^{s+\frac{1}{2}}|\mathrm{D}|^{\frac{1}{2}}\chi_1(\mathrm{D})\varphi)\,\mathrm{d}x\Big| \\ &\leq \varepsilon\mu M|\psi|_{\mathring{H}^{\frac{1}{2}}}|\varphi|_{L^2}. \end{aligned}$$

Then using the same estimates on $|I_1^{2,2}(\varphi)|$, we have that

$$|I_1^2(\varphi)| \leq \varepsilon\mu M|\psi|_{\mathring{H}^{\frac{1}{2}}}|\varphi|_{L^2}.$$

For the estimate on $|I_1^1(\varphi)|$, we use the Divergence Theorem to deduce that

$$\begin{aligned} I_1^1(\varphi) &= \int_{\mathcal{S}^-} P(\Sigma^-)\nabla^\mu_{x,z}u_1 \cdot (\Lambda^{s+\frac{1}{2}}\chi_1(\mathrm{D})\nabla^\mu_{x,z}\varphi^{\mathrm{ext}_1})\,\mathrm{d}x\mathrm{d}z + \varepsilon\mu\int_{\mathcal{S}^-} r_1(\Lambda^{s+\frac{1}{2}}\chi_1(\mathrm{D})\varphi^{\mathrm{ext}_1})\,\mathrm{d}x\mathrm{d}z \\ &= I_1^{1,1}(\varphi) + I_1^{1,2}(\varphi). \end{aligned}$$

Here we let φ^{ext_1} be the extension of φ onto the upper half-plane defined by

$$\varphi^{\mathrm{ext}_1}(x,z) = \mathrm{e}^{-z\sqrt{\mu}|\mathrm{D}|}\varphi(x),$$

and it satisfies the smoothing estimate

$$\||\mathrm{D}|^{\frac{1}{2}}\mathrm{e}^{-z\sqrt{\mu}|\mathrm{D}|}\varphi\|_{L^2(\mathcal{S}^-)} \leq \mu^{-\frac{1}{4}}|\varphi|_{L^2}. \tag{3.8}$$

Then we estimate each term separately. Since $I_1^{1,2}(\varphi)$ contains a rest term, its estimate is similar to the one on $I_1^2(\varphi)$ where we have to estimate the terms:

$$\begin{aligned}|I_1^{1,2}(\varphi)| \leq \varepsilon\mu\Big(&\Big|\int_{\mathcal{S}^-}(\partial_x\zeta)(\partial_z\phi^1_{\mathrm{app}})\partial_x(\Lambda^{s+\frac{1}{2}}\chi_1(\mathrm{D})\varphi^{\mathrm{ext}_1})\,\mathrm{d}x\mathrm{d}z\Big|\\
&+\Big|\int_{\mathcal{S}^-}(\partial_z\partial_x\phi^1_{\mathrm{app}})\big((\partial_x\zeta)\Lambda^{s+\frac{1}{2}}\chi_1(\mathrm{D})\varphi^{\mathrm{ext}_1}\big)\,\mathrm{d}x\mathrm{d}z\Big|\\
&+\varepsilon\mu\Big|\int_{\mathcal{S}^-}(\partial_z^2\phi^1_{\mathrm{app}})\big((\partial_x\zeta)^2\Lambda^{s+\frac{1}{2}}\chi_1(\mathrm{D})\varphi^{\mathrm{ext}_1}\big)\,\mathrm{d}x\mathrm{d}z\Big|\Big)\\
=\ &|I_1^{1,2,1}(\varphi)|+|I_1^{1,2,2}(\varphi)|+|I_1^{1,2,3}(\varphi)|.\end{aligned}$$

For the estimate on $|I_1^{1,2,1}(\varphi)|$, we simply use the cut-off χ_1 and (3.8) to obtain that

$$|I_1^{1,2,1}(\varphi)| \leq \varepsilon\mu M|\psi|_{\mathring{H}^{\frac{1}{2}}}|\varphi|_{L^2},$$

while for $|I_1^{1,2,2}(\varphi)|$, we can integrate in z to find that

$$\begin{aligned}|I_1^{1,2,2}(\varphi)| \leq\ &\varepsilon\mu\Big|\int_{\mathbb{R}}(\partial_x\psi)\big((\partial_x\zeta)\Lambda^{s+\frac{1}{2}}\chi_1(\mathrm{D})\varphi\big)\,\mathrm{d}x\Big|\\
&+\varepsilon\mu\Big|\int_{\mathcal{S}^-}(\partial_x\phi^1_{\mathrm{app}})\big((\partial_x\zeta)\Lambda^{s+\frac{1}{2}}\chi_1(\mathrm{D})\partial_z\varphi^{\mathrm{ext}_1}\big)\,\mathrm{d}x\mathrm{d}z\Big|.\end{aligned}$$

The first term is estimated as we did for $I_1^2(\varphi)$, while the second term is estimated as $|I_1^{1,2,1}(\varphi)|$. The same can be done for $|I_1^{1,2,3}(\varphi)|$. Therefore, we have that

$$|I_1^{1,2}(\varphi)| \leq \varepsilon\mu M|\psi|_{\mathring{H}^{\frac{1}{2}}}|\varphi|_{L^2}.$$

Next, we estimate $I_1^{1,1}(\varphi)$. To do so, we introduce the multiplier $\mathrm{m}(\mathrm{D}) = \Lambda^{-\frac{1}{2}}|\mathrm{D}|^{\frac{1}{2}}$, and use (3.8), Hölder inequality, and Sobolev embedding to find that

$$|I_1^{1,1}(\varphi)| \leq \mu^{\frac{1}{4}}M\big(\|[\mathrm{m}(\mathrm{D}),P(\Sigma^-)]\nabla^\mu_{x,z}u_1\|_{L^2(\mathcal{S}^-)}+\|\mathrm{m}(\mathrm{D})\nabla^\mu_{x,z}u_1\|_{L^2(\mathcal{S}^-)}\big)|\varphi|_{L^2}. \tag{3.9}$$

Here the first term is estimated directly, by using the definition of $P(\Sigma^-)$, that $\mathrm{m}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bounded, and the definition of $u_1 = \phi - \phi^1_{\mathrm{app}}$ with estimate (A.23), to deduce that

$$\begin{aligned}\|[\mathrm{m}(\mathrm{D}),P(\Sigma^-)]\nabla^\mu_{x,z}u_1\|_{L^2(\mathcal{S}^-)} &\leq \varepsilon\sqrt{\mu}M\|\nabla^\mu_{x,z}u_1\|_{L^2(\mathcal{S}^-)}\\
&\leq \varepsilon\mu^{\frac{3}{4}}M|\psi|_{\mathring{H}^{\frac{1}{2}}}.\end{aligned} \tag{3.10}$$

To conclude this estimate, we treat the second part in (3.9) by using the coercivity of $P(\Sigma^-)$ given by (1.35), then introduce a commutator, and apply the Divergence Theorem to make the following observation:

$$\begin{aligned}\|\mathrm{m}(\mathrm{D})\nabla^\mu_{x,z}u_1\|^2_{L^2(\mathcal{S}^-)} &\leq M\Big|\int_{\mathcal{S}^-}P(\Sigma^-)\mathrm{m}(\mathrm{D})\nabla^\mu_{x,z}u_1\cdot\mathrm{m}(\mathrm{D})\nabla^\mu_{x,z}u_1\,\mathrm{d}x\mathrm{d}z\Big|\\
&\leq \varepsilon\mu M\Big|\int_{\mathcal{S}^-}r_1(\mathrm{m}(\mathrm{D}))^2u_1\,\mathrm{d}x\mathrm{d}z\Big|\\
&\quad+\|[\mathrm{m}(\mathrm{D}),P(\Sigma^-)]\nabla^\mu_{x,z}u_1\|_{L^2(\mathcal{S}^-)}\|\mathrm{m}(\mathrm{D})\nabla^\mu_{x,z}u_1\|_{L^2(\mathcal{S}^-)}.\end{aligned}$$

Then use the definition of r_1 to gain precision of $\sqrt{\mu}$ and is treated as we did for $I_1^{1,2}(\varphi)$, while the second term is the same as for (3.10), and we find that

$$\begin{aligned}\|\mathrm{m}(\mathrm{D})\nabla^\mu_{x,z}u_1\|^2_{L^2(\mathcal{S}^-)} &\leq \varepsilon M\big(\mu\|\mathrm{e}^{-z\sqrt{\mu}|\mathrm{D}|}|\mathrm{D}|\psi\|_{L^2(\mathcal{S}^-)} + \mu^{\frac{3}{4}}|\psi|_{\mathring{H}^{\frac{1}{2}}}\big)\|\mathrm{m}(\mathrm{D})\nabla^\mu_{x,z}u_1\|_{L^2(\mathcal{S}^-)}\\ &\leq \varepsilon\mu^{\frac{3}{4}}M|\psi|_{\mathring{H}^{\frac{1}{2}}}\|\mathrm{m}(\mathrm{D})\nabla^\mu_{x,z}u_1\|_{L^2(\mathcal{S}^-)}.\end{aligned}$$

Adding all these estimates gives us a control of (3.9) by

$$|I_1^{1,1}(\varphi)| \leq \varepsilon\mu M|\psi|_{\mathring{H}^{\frac{1}{2}}}|\varphi|_{L^2},$$

and allows us to conclude this step:

$$|I_1(\varphi)| \leq \varepsilon\mu M|\psi|_{\mathring{H}^{\frac{1}{2}}}|\varphi|_{L^2}.$$

Step 2. *Approximate solutions for I_2.* By using the approach in [48, 50] we can show that the function:

$$\phi^2_{\mathrm{app}}(x,z) = (\mathrm{Op}(L)\psi)(x,z) = \mathcal{F}^{-1}\Big(L(x,\xi,z)\hat{\psi}(\xi)\Big)(x), \tag{3.11}$$

where L is a symbol given by

$$L(x,\xi,z) = e^{-z\big(\frac{\sqrt{\mu}|\xi|}{1+\varepsilon^2\mu(\partial_x\zeta)^2} - i\frac{\varepsilon\mu\partial_x\zeta\xi}{1+\varepsilon^2\mu(\partial_x\zeta)^2}\big)},$$

provides a good approximation of the elliptic problem (A.20). Moreover, we have by direct computations that

$$(\partial_n^{P^-}\phi^2_{\mathrm{app}})|_{z=0} = -\sqrt{\mu}|\mathrm{D}|\psi.$$

In fact, the approximation ϕ^2_{app} is constructed from the solution of the following ODE:

$$(1+\varepsilon^2\mu(\partial_x\zeta)^2)\partial_z^2 L - 2i\varepsilon\mu(\partial_x\zeta)\xi\partial_z L - \mu\xi^2 L = 0,$$

which corresponds to the principal part of (A.20) with "frozen coefficients". As a result, we have that $u_2 = \phi^- - \phi^2_{\mathrm{app}}$ solves

$$\begin{cases}\nabla^\mu_{x,z}\cdot P(\Sigma^-)\nabla^\mu_{x,z}u_2 = \varepsilon\mu r_3 \quad \text{in} \quad \mathcal{S}^-\\ u_2|_{z=0} = 0, \quad \lim\limits_{z\to\infty}\omega(z)|\nabla^\mu_{x,z}u_2| = 0,\end{cases} \tag{3.12}$$

where we simply take $\omega(z) = e^{-\frac{z}{2}}$ and the rest term can be computed explicitly:

$$r_3 = 2\mathrm{Op}((\partial_x\zeta)\partial_x\partial_z L)\psi - 2i\frac{1}{\varepsilon}\mathrm{Op}(\partial_x L\xi)\psi - \frac{1}{\varepsilon}\mathrm{Op}(\partial_x^2 L)\psi + \mathrm{Op}\big((\partial_x^2\zeta)\partial_z L\big)\psi.$$

One should note that derivatives in x on L give rise to $\varepsilon\mu$ and is polynomial in z. The polynomial dependence in z will require weighted estimates in high frequency to get an estimate for $\psi \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$.

Step 2.2. *Estimate on I_2.* By construction and the Divergence Theorem, we have that

$$\begin{aligned}I_2(\varphi) &= \int_{\mathbb{R}}(\Lambda^{s+\frac{1}{2}}\partial_n^{P^-}u_2|_{z=0})\chi_2(\mathrm{D})\varphi\,\mathrm{d}x\\ &= \int_{\mathcal{S}^-}\Lambda^{s+1}P(\Sigma^-)\nabla^\mu_{x,z}u_2\cdot(\Lambda^{-\frac{1}{2}}\chi_2(\mathrm{D})\nabla^\mu_{x,z}\varphi^{\mathrm{ext}_2})\,\mathrm{d}x\mathrm{d}z\\ &\quad + \varepsilon\mu\int_{\mathcal{S}^-}(\Lambda^s r_3)(\Lambda^{\frac{1}{2}}\chi_2(\mathrm{D})\varphi^{\mathrm{ext}_2})\,\mathrm{d}x\mathrm{d}z\\ &= I_2^1(\varphi) + I_2^2(\varphi).\end{aligned}$$

We now need to incorporate a weight in the estimates. To do so, we let φ^{ext_2} be the extension of φ onto the upper half-plane defined by

$$\varphi^{\mathrm{ext}_2}(x,z) = (\omega_\mu(z))^2 e^{-z\sqrt{\mu}|\mathrm{D}|}\varphi(x).$$

Here we let $\omega_\mu(z) = e^{-\frac{\sqrt{\mu}z}{2M}}$. For the estimate on $I_2^1(\varphi)$ we first make the decomposition

$$\begin{aligned}
I_2^1(\varphi) &= \int_{\mathcal{S}^-} \omega_\mu \chi_2(\mathrm{D})\Lambda^{s+1}P(\Sigma^-)\nabla^\mu_{x,z}u_2 \cdot (\Lambda^{-\frac{1}{2}}(\omega_\mu)^{-1}\nabla^\mu_{x,z}\varphi^{\mathrm{ext}_2})\,\mathrm{d}x\mathrm{d}z \\
&= \int_{\mathcal{S}^-} [\chi_2(\mathrm{D})\Lambda^{s+1}, P(\Sigma^-)]\omega_\mu\nabla^\mu_{x,z}u_2 \cdot (\Lambda^{-\frac{1}{2}}(\omega_\mu)^{-1}\nabla^\mu_{x,z}\varphi^{\mathrm{ext}_2})\,\mathrm{d}x\mathrm{d}z \\
&\quad + \int_{\mathcal{S}^-} P(\Sigma^-)\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla^\mu_{x,z}u_2 \cdot (\Lambda^{-\frac{1}{2}}(\omega_\mu)^{-1}\nabla^\mu_{x,z}\varphi^{\mathrm{ext}_2})\,\mathrm{d}x\mathrm{d}z \\
&= I_2^{1,1}(\varphi) + I_2^{1,2}(\varphi).
\end{aligned}$$

For the control of $I_2^{1,1}(\varphi)$, we use Cauchy-Schwarz, commutator estimate (B.8), and the half-derivative smoothing from the Poisson kernel (3.8), and the weight estimate:

$$\|\omega_\mu(z)\varphi\|_{L^2(\mathcal{S}^-)} \leq \mu^{-\frac{1}{4}}M|\varphi|_{L^2},$$

to get that

$$\begin{aligned}
|I_2^{1,1}(\varphi)| &\leq \varepsilon\sqrt{\mu}M\|\omega_\mu\Lambda^s\nabla^\mu_{x,z}u_2\|_{L^2(\mathcal{S}^-)}\|\Lambda^{-\frac{1}{2}}(\omega_\mu)^{-1}\nabla^\mu_{x,z}\varphi^{\mathrm{ext}_2})\|_{L^2(\mathcal{S}^-)} \\
&\leq \varepsilon\sqrt{\mu}M\|\omega_\mu\Lambda^s\nabla^\mu_{x,z}u_2\|_{L^2(\mathcal{S}^-)}\mu^{\frac{1}{4}}|\varphi|_{L^2}.
\end{aligned}$$

We also use the definition $u_2 = \phi^- - \phi^2_{\mathrm{app}}$, the decay of the weight, estimates (A.23) and the ones provided in Lemma B.10 to obtain

$$\begin{aligned}
\|\omega_\mu\Lambda^s\nabla^\mu_{x,z}u_2\|_{L^2(\mathcal{S}^-)} &\leq \|\Lambda^s\nabla^\mu_{x,z}\phi\|_{L^2(\mathcal{S}^-)} + \|\frac{1}{z}\Lambda^s\mathrm{Op}(\partial_x L)\psi\|_{L^2(\mathcal{S}^-)} \\
&\quad + \sqrt{\mu}\|\Lambda^s\mathrm{Op}(L)\partial_x\psi\|_{L^2(\mathcal{S}^-)} + \|\Lambda^s\mathrm{Op}(\partial_z L)\psi\|_{L^2(\mathcal{S}^-)} \\
&\leq \mu^{\frac{1}{4}}M|\psi|_{\mathring{H}^{s+\frac{1}{2}}}.
\end{aligned}$$

We therefore have

$$|I_2^{1,1}(\varphi)| \leq \varepsilon\mu M|\psi|_{\mathring{H}^{s+\frac{1}{2}}}|\varphi|_{L^2}.$$

For the estimate on $I_2^{1,2}(\varphi)$, we use Hölder's inequality and Sobolev embedding to find that

$$|I_2^{1,2}(\varphi)| \leq \mu^{\frac{1}{4}}M\|\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla^\mu_{x,z}u_2\|_{L^2(S^-)}|\varphi|_{L^2}.$$

So we have again reduced the problem to an elliptic estimate. Using the coercivity of $P(\Sigma^-)$ and integration by parts we find that

$$\begin{aligned}
A &= \|\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla^\mu_{x,z}u_2\|^2_{L^2(S^-)} \qquad (3.13)\\
&\leq M\Big|\int_{\mathcal{S}^-} P(\Sigma^-)\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla^\mu_{x,z}u_2 \cdot \Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla^\mu_{x,z}u_2\,\mathrm{d}x\mathrm{d}z\Big| \\
&\leq M\Big|\int_{\mathcal{S}^-} (\nabla^\mu_{x,z}\cdot P(\Sigma^-)\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla^\mu_{x,z}u_2)(\Lambda^{s+1}\chi_2(\mathrm{D})u_2)\,\mathrm{d}x\mathrm{d}z\Big| \\
&\quad + M\Big|\int_{\mathcal{S}^-} (\mathbf{e}_z\cdot P(\Sigma^-)\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla^\mu_{x,z}u_2)(\Lambda^{s+1}\chi_2(\mathrm{D})\omega'_\mu u_2)\,\mathrm{d}x\mathrm{d}z\Big| \\
&= M\cdot(A_1 + A_2).
\end{aligned}$$

Before estimating A_1, we will decompose it into three pieces:

$$\begin{aligned}
A_1 =& \Big| \int_{\mathcal{S}^-} (\nabla^\mu_{x,z} \cdot [P(\Sigma^-), \Lambda^{s+1}\chi_2(\mathrm{D})]\omega_\mu \nabla^\mu_{x,z} u_2)(\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu u_2)\,\mathrm{d}x\mathrm{d}z \Big| \\
&+ \varepsilon\mu\Big| \int_{\mathcal{S}^-} (\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu r_3)(\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu u_2)\,\mathrm{d}x\mathrm{d}z\Big| \\
&+ \Big| \int_{\mathcal{S}^-} (\Lambda^{s+1}\chi_2(\mathrm{D})(\omega_\mu)'\mathbf{e}_z \cdot P(\Sigma^-)\nabla^\mu_{x,z} u_2)(\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu u_2)\,\mathrm{d}x\mathrm{d}z\Big| \\
=& A_1^1 + A_1^2 + A_1^3.
\end{aligned}$$

For the estimate on A_1^1, we write out each term explicitly to clearly see the terms we need to treat. In particular, we have

$$\begin{aligned}
\nabla^\mu_{x,z} &\cdot [P(\Sigma^-), \Lambda^{s+1}\chi_2(\mathrm{D})]\omega_\mu \nabla^\mu_{x,z} u_2 \\
&= -\varepsilon\mu\partial_x[\partial_x\zeta, \Lambda^{s+1}\chi_2(\mathrm{D})]\omega_\mu\partial_z u_2 - \varepsilon\mu\partial_z[\partial_x\zeta, \Lambda^{s+1}\chi_2(\mathrm{D})]\omega_\mu\partial_x u_2 \\
&\quad + \varepsilon^2\mu\partial_z[(\partial_x\zeta)^2, \Lambda^{s+1}\chi_2(\mathrm{D})]\omega_\mu\partial_z u_2.
\end{aligned}$$

Then using this decomposition together with Plancherel's identity, integration by parts, commutator estimate (B.8), and the support of χ_2, we find that

$$\begin{aligned}
A_1^1 \leq& \varepsilon\mu\Big| \int_{\mathcal{S}^-} (\Lambda^{-1}\partial_x[\partial_x\zeta, \Lambda^{s+1}\chi_2(\mathrm{D})]\omega_\mu\partial_z u_2)(\Lambda^{s+2}\chi_2(\mathrm{D})\omega_\mu u_2)\,\mathrm{d}x\mathrm{d}z\Big| \\
&+ \varepsilon\mu\Big| \int_{\mathcal{S}^-} ([\partial_x^2\zeta, \Lambda^{s+1}\chi_2(\mathrm{D})]\omega_\mu\partial_z u_2)(\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu u_2)\,\mathrm{d}x\mathrm{d}z\Big| \\
&+ \varepsilon\mu\Big| \int_{\mathcal{S}^-} ([\partial_x\zeta, \Lambda^{s+1}\chi_2(\mathrm{D})]\omega_\mu\partial_z u_2)(\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu \partial_x u_2)\,\mathrm{d}x\mathrm{d}z\Big| \\
&+ \varepsilon^2\mu\Big| \int_{\mathcal{S}^-} ([\partial_x\zeta, \Lambda^{s+1}\chi_2(\mathrm{D})]\omega_\mu\partial_z u_2)(\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu \partial_z u_2)\,\mathrm{d}x\mathrm{d}z\Big| \\
\leq& \varepsilon\mu M\|\Lambda^s\partial_z u_2\|_{L^2(\mathcal{S}^-)}\|\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla_{x,z}u_2\|_{L^2(S^-)}.
\end{aligned}$$

Then to conclude this part, we simply use that $u_2 = \phi^- - \phi^2_{\mathrm{app}}$, together with the estimates (A.23) and Lemma B.10:

$$\begin{aligned}
A_1^1 &\leq \varepsilon\mu^{1+\frac{1}{4}}M|\psi|_{\mathring{H}^{s+\frac{1}{2}}}\|\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla_{x,z}u_2\|_{L^2(S^-)} \\
&\leq \varepsilon\mu^{\frac{3}{4}}M|\psi|_{\mathring{H}^{s+\frac{1}{2}}}\|\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla^\mu_{x,z}u_2\|_{L^2(S^-)}.
\end{aligned}$$

Next, we make an estimate on A_1^2 using the definition of r_3 in the previous step:

$$\begin{aligned}
A_1^2 \leq& 2\varepsilon\mu\Big| \int_{\mathcal{S}^-} (\Lambda^s\chi_2(\mathrm{D})\omega_\mu\mathrm{Op}((\partial_x\zeta)\partial_x\partial_z L)\psi)(\Lambda^{s+2}\chi_2(\mathrm{D})\omega_\mu u_2)\,\mathrm{d}x\mathrm{d}z\Big| \\
&+ 2\mu\Big| \int_{\mathcal{S}^-} (\Lambda^s\chi_2(\mathrm{D})\omega_\mu\mathrm{Op}(\partial_x L)\partial_x\psi)(\Lambda^{s+2}\chi_2(\mathrm{D})\omega_\mu u_2)\,\mathrm{d}x\mathrm{d}z\Big| \\
&+ \varepsilon\mu\Big| \int_{\mathcal{S}^-} (\Lambda^s\chi_2(\mathrm{D})\omega_\mu\mathrm{Op}\big((\partial_x^2\zeta)\partial_z L\big)\psi)(\Lambda^{s+2}\chi_2(\mathrm{D})\omega_\mu u_2)\,\mathrm{d}x\mathrm{d}z\Big| \\
&+ \mu\Big| \int_{\mathcal{S}^-} (\Lambda^s\chi_2(\mathrm{D})\omega_\mu\mathrm{Op}(\partial_x^2 L)\psi)(\Lambda^{s+2}\chi_2(\mathrm{D})\omega_\mu u_2)\,\mathrm{d}x\mathrm{d}z\Big| \\
=& A_1^{2,1} + A_1^{2,2} + A_1^{2,3} + A_1^{2,4}.
\end{aligned}$$

The first three terms are easily estimated by Lemma B.10, with Remark B.11, and the support of χ_2 where we find that

$$\begin{aligned}A_1^{2,1}+A_1^{2,2}+A_1^{2,3} &\leq \varepsilon\mu^{1+\frac{1}{4}}M|\psi|_{\mathring{H}^{s+\frac{1}{2}}}\|\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla_{x,z}u_2\|_{L^2(\mathcal{S}^-)}\\ &\leq \varepsilon\mu^{\frac{3}{4}}M|\psi|_{\mathring{H}^{s+\frac{1}{2}}}\|\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla^\mu_{x,z}u_2\|_{L^2(\mathcal{S}^-)}.\end{aligned}$$

For the last term, we use the weight to gain decay in z. Then apply the same estimates as above to find that

$$\begin{aligned}A_1^{2,4} &\leq \sqrt{\mu}\|\Lambda^s\chi_2(\mathrm{D})\mathrm{Op}(\frac{1}{z}\partial_x^2L)\psi\|_{L^2(\mathcal{S}^-)}\|\Lambda^{s+2}\chi_2(\mathrm{D})\omega_\mu u_2\|_{L^2(\mathcal{S}^-)}\\ &\leq \varepsilon\mu^{\frac{3}{4}}M(t_0+3)|\psi|_{\mathring{H}^{s+\frac{1}{2}}}\|\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla^\mu_{x,z}u_2\|_{L^2(\mathcal{S}^-)}.\end{aligned}$$

As a result, we have the same estimate on A_1^2, and so we proceed with the estimate on A_1^3. We observe by definition of $P(\Sigma^-)$ that we have one term without ε that need special attention:

$$\begin{aligned}A_1^3 \leq &\frac{\varepsilon\mu^{\frac{3}{2}}}{2M}\Big|\int_{\mathcal{S}^-}(\Lambda^s\chi_2(\mathrm{D})\omega_\mu(\partial_x\zeta)\partial_x u_2)(\Lambda^{s+2}\chi_2(\mathrm{D})\omega_\mu u_2)\,\mathrm{d}x\mathrm{d}z\Big|\\ &+\frac{\varepsilon^2\mu^{\frac{3}{2}}}{2M}\Big|\int_{\mathcal{S}^-}(\Lambda^s\chi_2(\mathrm{D})\omega_\mu(\partial_x\zeta)^2\partial_z u_2)(\Lambda^{s+2}\chi_2(\mathrm{D})\omega_\mu u_2)\,\mathrm{d}x\mathrm{d}z\Big|\\ &+\frac{\sqrt{\mu}}{2M}\Big|\int_{\mathcal{S}^-}(\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\partial_z u_2)(\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu u_2)\,\mathrm{d}x\mathrm{d}z\Big|\\ =\ &A_1^{3,1}+A_1^{3,2}+A_1^{3,3}.\end{aligned}$$

Here we see that the first two terms are easily estimated by using the weight to gain decay and then combining it with the estimates in Lemma B.10,

$$A_1^{3,1}+A_1^{3,2}\leq\varepsilon\mu^{\frac{3}{4}}M(t_0+1)|\psi|_{\mathring{H}^{s+\frac{1}{2}}}\|\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla^\mu_{x,z}u_2\|_{L^2(\mathcal{S}^-)}.$$

Lastly, for $A_1^{3,3}$ we use integration by parts and the support of χ_2 to find that

$$\begin{aligned}A_1^{3,3} &\leq \frac{\mu}{4M}\Big|\int_{\mathcal{S}^-}(\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu u_2)(\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu u_2)\,\mathrm{d}x\mathrm{d}z\Big|\\ &\leq\frac{\mu}{4M}\|\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla_{x,z}u_2\|^2_{L^2(\mathcal{S}^-)},\end{aligned}$$

and therefore can be reabsorbed into A. The only remaining estimate is now on A_2. However, we note that it is similar to A_1^3, and the same estimates apply. In conclusion, we may gather all these estimates and use (3.13) to find that

$$(1-\frac{1}{4})\|\Lambda^{s+1}\chi_2(\mathrm{D})\omega_\mu\nabla^\mu_{x,z}u_2\|_{L^2(\mathcal{S}^-)}\leq\varepsilon\mu^{\frac{3}{4}}M(t_0+3)|\psi|_{\mathring{H}^{s+\frac{1}{2}}}.$$

Returning to I_2^1, we obtain the bound

$$|I_2^1(\varphi)|\leq\varepsilon\mu M(t_0+3)|\psi|_{\mathring{H}^{s+\frac{1}{2}}}|\varphi|_{L^2}.$$

To conclude this step, it only remains to estimate $I_2^2(\varphi)$. However, this is just a simple version of A_1^2 and the following estimate is easy to deduce

$$\begin{aligned}|I_2(\varphi)| &\leq |I_2^1(\varphi)|+|I_2^2(\varphi)|\\ &\leq\varepsilon\mu M(t_0+3)|\psi|_{\mathring{H}^{s+\frac{1}{2}}}|\varphi|_{L^2}.\end{aligned}$$

Step 3. *Conclusion of proof.* We have from Step 1. and Step 2. that the result holds:

$$\begin{aligned}|\mathcal{G}_\mu^-[\varepsilon\zeta]\psi + \sqrt{\mu}|\mathrm{D}|\psi|_{H^{s+\frac{1}{2}}} &\leq \sup_{|\varphi|_{L^2}=1}\Big(|I_1(\varphi)| + |I_2(\varphi)|\Big)\\ &\leq \varepsilon\mu M(t_0+3)|\psi|_{\mathring{H}^{s+\frac{1}{2}}}.\end{aligned}$$

□

3.2. **Symbolic analysis of $(\mathcal{G}_\mu^-)^{-1}\mathcal{G}_\mu^+$.** The next step in studying the symbolic behaviour of $\mathcal{G}_\mu$ is to understand the composition of $(\mathcal{G}_\mu^-)^{-1}$ with $\mathcal{G}_\mu^+$.

Corollary 3.4. *Let $t_0 \geq 1$ and $\zeta \in H^{t_0+3}(\mathbb{R})$ be such that* (1.14) *is satisfied. Then for all $0 \leq s \leq t_0$ and $f \in \mathscr{S}(\mathbb{R})$ one can approximate the opertor $(\mathcal{G}_\mu^-)^{-1}\mathcal{G}_\mu^+$ by*

$$\mathrm{Op}\Big(\frac{S^+}{S^-}\Big)f(x) = -\mathcal{F}^{-1}\big(\tanh(\sqrt{\mu}t(x,\xi))\hat{f}(\xi)\big)(x),$$

where t is defined by (3.2)*. Moreover, there holds,*

$$|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]f - \mathrm{Op}\Big(\frac{S^+}{S^-}\Big)f|_{\mathring{H}^{s+\frac{1}{2}}} \leq \varepsilon\sqrt{\mu}M(t_0+3)|f|_{H^{s-\frac{1}{2}}}. \tag{3.14}$$

Remark 3.5. *From the symmetry of the Dirichlet-Neumann operators $\mathcal{G}^\pm$ they can be defined on Sobolev spaces of negative order by duality. See Remark* 3.17 *in* [51]*.*

Proof. We define $\tilde{\psi}^- = \mathrm{Op}\Big(\frac{S^+}{S^-}\Big)f$ and let $\tilde{\phi}^-$ be a solution of

$$\begin{cases}\nabla^\mu_{x,z}\cdot P(\Sigma^-)\nabla^\mu_{x,z}\tilde{\phi}^- = 0 & \text{in} \quad \mathcal{S}^-\\ \tilde{\phi}^- = \tilde{\psi}^- & \text{on} \quad z=0.\end{cases}$$

Then we can make the definition

$$\mathcal{G}_\mu^-[\varepsilon\zeta]\tilde{\psi}^- = \partial_n\tilde{\phi}^-|_{z=0}.$$

Now, let ϕ^- be defined by the solution of

$$\begin{cases}\nabla^\mu_{x,z}\cdot P(\Sigma^-)\nabla^\mu_{x,z}\phi^- = 0 & \text{in} \quad \mathcal{S}^-\\ \partial_n^{P^-}\phi^- = \mathcal{G}_\mu^+[\varepsilon\zeta]f & \text{on} \quad z=0.\end{cases} \tag{3.15}$$

Then the difference $u = \phi^- - \tilde{\phi}^-$ satisfies

$$\begin{cases}\nabla^\mu_{x,z}\cdot P(\Sigma^-)\nabla^\mu_{x,z}u = 0 & \text{in} \quad \mathcal{S}^-\\ \partial_n^{P^-}u = \mathcal{G}_\mu^+[\varepsilon\zeta]f - \mathcal{G}_\mu^-[\varepsilon\zeta]\tilde{\psi}^- & \text{on} \quad z=0.\end{cases} \tag{3.16}$$

It is straightforward to prove that these problems are well-defined where both ϕ^- and $\tilde{\phi}^-$ satisfy (2.14). Consequently, we can consider the variational equation:

$$\begin{aligned}\int_{\mathcal{S}^-} P(\Sigma^-)\nabla^\mu_{x,z}\Lambda^s u\cdot\nabla^\mu_{x,z}\Lambda^s u\,\mathrm{d}x\mathrm{d}z = &-\int_{\{z=0\}}\Lambda^s(\mathcal{G}_\mu^+[\varepsilon\zeta]\psi - \mathcal{G}_\mu^-[\varepsilon\zeta]\tilde{\psi}^-)\Lambda^s u\,\mathrm{d}x\\ &+\int_{\mathcal{S}^-}[\Lambda^s, P(\Sigma^-)]\nabla^\mu_{x,z}u\cdot\nabla^\mu_{x,z}\Lambda^s u\,\mathrm{d}x\mathrm{d}z,\end{aligned}$$

where $u_0 = u|_{z=0}$ and trace inequality (2.15) implies

$$\sqrt{\mu}|u_0|_{\mathring{H}^{s+\frac{1}{2}}} \leq \|\Lambda^s\nabla^\mu_{x,z}u\|_{L^2(\mathcal{S}^-)}.$$

To conclude the proof, we need the following inequality

$$\Big|\big(\Lambda^s(\mathcal{G}_\mu^+[\varepsilon\zeta]f - \mathcal{G}_\mu^-[\varepsilon\zeta]\tilde{\psi}^-), \Lambda^s u_0\big)_{L^2}\Big| \leq \varepsilon\sqrt{\mu}M(t_0+3)|f|_{H^{s-\frac{1}{2}}}|u_0|_{\mathring{H}^{s+\frac{1}{2}}}. \tag{3.17}$$

Assuming the inequality holds, we can argue as in the proof of Proposition 2.4, Step 2. to obtain the result.

To prove (3.17), we will decompose the left-hand side in several pieces. In particular, we first observe that

$$\big(\Lambda^s\mathcal{G}_\mu^+[\varepsilon\zeta]f,\Lambda^s u_0\big)_{L^2}=\Big(\mathrm{Op}\Big(\frac{S^+}{|\mathrm{D}|}\Big)^*\Lambda^s f,\Lambda^s|\mathrm{D}|u_0\Big)_{L^2}+R_1+R_2,$$

where the rest is given by

$$\begin{aligned}R_1&=\big(\big[\Lambda^s,\mathcal{G}_\mu^+[\varepsilon\zeta]\big]f,\Lambda^s u_0\big)_{L^2},\\ R_2&=\big(\Lambda^{s-\frac12}f,\Lambda^{\frac12}\big(\mathcal{G}_\mu^+[\varepsilon\zeta]-\mathrm{Op}(S^+)\big)\Lambda^s u_0\big)_{L^2}.\end{aligned}$$

Here we estimate R_1 by Cauchy-Schwarz, (A.9), and (B.3) to get that

$$\begin{aligned}|R_1|&\leq\varepsilon\mu M|f|_{\dot{H}_\mu^{s-\frac12}}|u_0|_{\dot{H}_\mu^{s+\frac12}}\\ &\leq\varepsilon\sqrt{\mu}M|f|_{H^{s-\frac12}}|u_0|_{\mathring{H}^{s+\frac12}}.\end{aligned}$$

While for R_2 we use (3.3) (with $k=1$) and (B.3) to get that

$$|R_2|\leq\varepsilon\sqrt{\mu}M(t_0+3)|f|_{H^{s-\frac12}}|u_0|_{\dot{H}_\mu^{s+\frac12}}.$$

Next, we decompose the remaining part where we get that

$$\big(\Lambda^s\mathcal{G}_\mu^-[\varepsilon\zeta]\tilde{\psi}^-,\Lambda^s u_0\big)_{L^2}=\Big(\mathrm{Op}\Big(\frac{S^+}{|\mathrm{D}|}\Big)^*\Lambda^s f,\Lambda^s|\mathrm{D}|u_0\Big)_{L^2}+R_3+R_4+R_5,$$

where the rest terms are defined by

$$\begin{aligned}R_3&=\big(\big[\Lambda^s,\mathcal{G}_\mu^-[\varepsilon\zeta]\big]\tilde{\psi}^-,\Lambda^s u_0\big)_{L^2},\\ R_4&=\big(\Lambda^{s-\frac12}\tilde{\psi}^-,\Lambda^{\frac12}\big(\mathcal{G}_\mu^-[\varepsilon\zeta]-\mathrm{Op}(S^-)\big)\Lambda^s u_0\big)_{L^2},\\ R_5&=\big(\Lambda^{\frac12}[\Lambda^s,\mathrm{Op}(\frac{S^+}{S^-})]f,\Lambda^{-\frac12}\mathrm{Op}(S^-)\Lambda^s u_0\big)_{L^2}.\end{aligned}$$

For R_3, the commutator estimate (A.26), and (B.22) yields,

$$\begin{aligned}|R_3|&\leq\varepsilon\sqrt{\mu}M|\tilde{\psi}^-|_{\mathring{H}^{s-\frac12}}|u_0|_{\mathring{H}^{s+\frac12}}\\ &\leq\varepsilon\sqrt{\mu}M|f|_{H^{s-\frac12}}|u_0|_{\mathring{H}^{s+\frac12}}.\end{aligned}$$

For R_4, we argue similarly but use Proposition 3.2 with estimate (3.4) to get that

$$|R_4|\leq\varepsilon\mu M(t_0+3)|f|_{H^{s-\frac12}}|u_0|_{\mathring{H}^{s+\frac12}}.$$

Finally, for R_5, we apply Cauchy-Schwarz, the commutator estimate (B.23) to get that

$$\begin{aligned}|R_5|&\leq|[\Lambda^s,\mathrm{Op}(\frac{S^+}{S^-})]f|_{H^{\frac12}}|\Lambda^{s-\frac12}|\mathrm{D}|u_0|_{L^2}\\ &\leq\varepsilon\sqrt{\mu}M|f|_{H^{-\frac12}}|u_0|_{\mathring{H}^{s+\frac12}}.\end{aligned}$$

Gathering all these observations, we can estimate the left-hand side of (3.17) by

$$\mathrm{LHS}_{(3.17)}\leq\Big|\big(\Lambda^{\frac12}(\mathrm{Op}(\frac{S^+}{|\mathrm{D}|})^*-\mathrm{Op}(\frac{S^+}{|\mathrm{D}|}))\Lambda^s f,\Lambda^{s-\frac12}|\mathrm{D}|u_0\big)_{L^2}\Big|+\sum_{i=1}^5|R_i|.$$

Then to conclude the proof, we use Cauchy-Schwarz, the adjoint estimate (B.24) to get that

$$\mathrm{LHS}_{(3.17)}\leq\varepsilon\sqrt{\mu}M(t_0+3)|f|_{H^{s-\frac12}}|u_0|_{\mathring{H}^{s+\frac12}}.$$

□

3.3. **Symbolic analysis of $(\mathcal{G}_\mu^-)^{-1}\mathcal{G}_\mu$.** Next we will study the symbolic behaviour of $(\mathcal{G}_\mu^-)^{-1}\mathcal{G}_\mu$. To do so, we recall that we may write $\mathcal{G}_\mu = \mathcal{G}_\mu^+(\mathcal{J}_\mu)^{-1}$ where the symbolic behaviour of $\mathcal{J}_\mu$ is captured by the symbol

$$S_J = 1 - \gamma\frac{S^+}{S^-}. \tag{3.18}$$

Corollary 3.6. *Let $t_0 \geq 1$ and $\zeta \in H^{t_0+3}(\mathbb{R})$ be such that (1.14) is satisfied. Then for all $0 \leq s \leq t_0$ and $f \in \mathscr{S}(\mathbb{R})$ one can approximate the operator $(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta]$ by*

$$\mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)f(x) = -\mathcal{F}^{-1}\Big(\frac{\tanh(\sqrt{\mu}t(x,\xi))}{1+\gamma\tanh(\sqrt{\mu}t(x,\xi))}\hat{f}(\xi)\Big)(x),$$

where t is defined by (3.2). Moreover, there holds,

$$|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta]f - \mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)f|_{\mathring{H}^{s+\frac{1}{2}}} \leq \varepsilon\sqrt{\mu}M(t_0+3)|f|_{H^{s-\frac{1}{2}}}. \tag{3.19}$$

Proof. We will give the proof in three steps. First, we will prove that S_J provides a good symbolic description of $\mathcal{J}_\mu$. Then we will show that $1/S_J$ describes the inverse, and lastly, we combine each step with the previous results of this section to prove (3.19).

Step 1. We can approximate $\mathcal{J}_\mu$ with $\mathrm{Op}(S_J)$. Indeed, by definition and estimate (3.14) we get that

$$\begin{aligned}|\mathcal{J}_\mu[\varepsilon\zeta]f - \mathrm{Op}(S_J)f|_{\mathring{H}^{s+\frac{1}{2}}} &= \gamma|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]f - \mathrm{Op}\big(\frac{S^+}{S^-}\big)f|_{\mathring{H}^{s+\frac{1}{2}}} \\ &\leq \varepsilon\sqrt{\mu}M(t_0+3)|f|_{H^{s-\frac{1}{2}}}.\end{aligned} \tag{3.20}$$

Step 2. We can approximate $(\mathcal{J}_\mu)^{-1}$ with $\mathrm{Op}(1/S_J)$. To prove this fact, we simply employ Proposition 2.10 with inequality (2.23) to deduce that

$$\begin{aligned}|(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}f - \mathrm{Op}(1/S_J)f|_{\dot{H}_\mu^{s+\frac{1}{2}}} &= |(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}\big(1-\mathcal{J}_\mu[\varepsilon\zeta]\mathrm{Op}(1/S_J)\big)f|_{\dot{H}_\mu^{s+\frac{1}{2}}} \\ &\leq M|\big(1-\mathcal{J}_\mu[\varepsilon\zeta]\mathrm{Op}(1/S_J)\big)f|_{\dot{H}_\mu^{s+\frac{1}{2}}} \\ &\leq M\Big(|\big(1-\mathrm{Op}(S_J)\mathrm{Op}(1/S_J)\big)f|_{\dot{H}_\mu^{s+\frac{1}{2}}} \\ &\quad + |\big(\mathcal{J}_\mu[\varepsilon\zeta]-\mathrm{Op}(S_J)\big)\mathrm{Op}(1/S_J)f|_{\dot{H}_\mu^{s+\frac{1}{2}}} \\ &=: R_1 + R_2.\end{aligned}$$

Here R_1 is estimated by (B.26):

$$R_1 \leq \mu^{-\frac{1}{4}}|\big(1-\mathrm{Op}(S_J)\mathrm{Op}(1/S_J)\big)f|_{H^{s+\frac{1}{2}}} \leq \varepsilon\mu^{\frac{1}{4}}|f|_{H^{s-\frac{1}{2}}}.$$

While R_2 we use the estimate in Step 1. and (B.25) to obtain

$$\begin{aligned}R_2 &\leq \mu^{-\frac{1}{4}}|\big(\mathcal{J}_\mu[\varepsilon\zeta]-\mathrm{Op}(S_J)\big)\mathrm{Op}(1/S_J)f|_{\mathring{H}^{s+\frac{1}{2}}} \\ &\leq \varepsilon\mu^{\frac{1}{4}}M(t_0+3)|\mathrm{Op}(1/S_J)f|_{H^{s-\frac{1}{2}}} \\ &\leq \varepsilon\mu^{\frac{1}{4}}M(t_0+3)|f|_{H^{s-\frac{1}{2}}}.\end{aligned}$$

As a result, we have the following estimate

$$|(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}f - \mathrm{Op}(1/S_J)f|_{\dot{H}_\mu^{s+\frac{1}{2}}} \leq \varepsilon\mu^{\frac{1}{4}}M(t_0+3)|f|_{H^{s-\frac{1}{2}}}. \tag{3.21}$$

Step 3. Estimate (3.19) holds true. To prove this, we first use the definition $\mathcal{G}_\mu = \mathcal{G}_\mu^+ \mathcal{J}_\mu^{-1}$ to make the decomposition:

$$\begin{aligned}
\mathrm{LHS}_{(3.19)} &:= |(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta](\mathcal{J}_\mu[\varepsilon\zeta])^{-1}f - \mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)f|_{\mathring{H}^{s+\frac{1}{2}}} \\
&\leq |(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\big((\mathcal{J}_\mu[\varepsilon\zeta])^{-1} - \mathrm{Op}\Big(\frac{1}{S_J}\Big)\big)f|_{\mathring{H}^{s+\frac{1}{2}}} \\
&\quad + |(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\mathrm{Op}\Big(\frac{1}{S_J}\Big)f - \mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)f|_{\mathring{H}^{s+\frac{1}{2}}} \\
&=: R_3 + R_4.
\end{aligned}$$

For R_3 we use Proposition 2.4 with inequality (2.11) and then Step 2. to get that

$$R_3 \leq \mu^{\frac{1}{4}}M|\big((\mathcal{J}_\mu[\varepsilon\zeta])^{-1} - \mathrm{Op}\Big(\frac{1}{S_J}\Big)\big)f|_{\dot{H}_\mu^{s+\frac{1}{2}}} \leq \varepsilon\sqrt{\mu}M(t_0+3)|f|_{H^{s-\frac{1}{2}}}.$$

For R_4, we decompose it further:

$$\begin{aligned}
R_4 &= |\Big((\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta] - \mathrm{Op}\Big(\frac{S^+}{S^-}\Big)\Big)\mathrm{Op}\Big(\frac{1}{S_J}\Big)f|_{\mathring{H}^{s+\frac{1}{2}}} \\
&\quad + |\Big(\mathrm{Op}\Big(\frac{S^+}{S^-}\Big)\mathrm{Op}\Big(\frac{1}{S_J}\Big) - \mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)\Big)f|_{\mathring{H}^{s+\frac{1}{2}}} \\
&=: R_5 + R_6.
\end{aligned}$$

For R_5 we use (3.14) and (B.25) to get that

$$R_5 \leq \varepsilon\sqrt{\mu}M(t_0+3)|\mathrm{Op}\Big(\frac{1}{S_J}\Big)f|_{H^{s-\frac{1}{2}}} \leq \varepsilon\sqrt{\mu}M(t_0+3)|f|_{H^{s-\frac{1}{2}}}.$$

While for R_6 we simply employ estimate (B.31) which yields

$$R_6 = \varepsilon\sqrt{\mu}M|f|_{H^{s-\frac{1}{2}}}.$$

Gathering all these estimates concludes the proof.

□

3.4. **Symbolic description of $\mathcal{G}_\mu$.** For the next result, we will give a simple approximation of $\mathcal{G}_\mu$. We will allow for a loss of derivatives in the estimate. However, the loss in regularity is translated into precision with respect to the small parameters.

Corollary 3.7. *Let $t_0 \geq 1$ and $\zeta \in H^{t_0+2}(\mathbb{R})$ be such that (1.14) is satisfied. Then for all $0 \leq s \leq t_0$ and $\psi \in \dot{H}^{s+2}(\mathbb{R})$ one can approximate the operator $\mathcal{G}_\mu[\varepsilon\zeta]$ by the Fourier multiplier $\mathcal{G}_\mu[0]$ defined by*

$$\mathcal{G}_\mu[0] = \sqrt{\mu}|\mathrm{D}|\frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{1+\gamma\tanh(\sqrt{\mu}|\mathrm{D}|)},$$

and it satisfies the estimate

$$|\mathcal{G}_\mu[\varepsilon\zeta]\psi - \mathcal{G}_\mu[0]\psi|_{H^s} \leq \varepsilon\mu M|\partial_x\psi|_{H^{s+1}}. \tag{3.22}$$

Proof. For the proof of inequality (3.22), we let $\mathcal{G}_\mu[0] = \mathcal{G}_\mu^+[0](\mathcal{J}_\mu[0])^{-1}$ and make the following decomposition

$$\begin{aligned}
|\mathcal{G}_\mu[\varepsilon\zeta]\psi - \mathcal{G}_\mu^+[0](\mathcal{J}_\mu[0])^{-1}\psi|_{H^s} &\leq |\big(\mathcal{G}_\mu^+[\varepsilon\zeta] - \mathcal{G}_\mu^+[0]\big)(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}\psi|_{H^s} \\
&\quad + |\mathcal{G}_\mu^+[0]\big((\mathcal{J}_\mu[\varepsilon\zeta])^{-1} - (\mathcal{J}_\mu[0])^{-1}\big)\psi|_{H^s} \\
&= B_1 + B_2.
\end{aligned}$$

For the estimate on B_1, we use the classical small amplitude expansion of $\mathcal{G}_\mu^+$ provided by Proposition 3.44 (with $n=1$, $k=1$, $s'=s+\frac{1}{2}$) in [51], and then (2.23):

$$B_1 \leq \varepsilon\mu M|\psi|_{\dot{H}_\mu^{s+\frac{3}{2}}} \leq \varepsilon\mu M|\partial_x\psi|_{H^{s+1}}.$$

For an estimate on B_2, we first observe that

$$\sqrt{\mu}|\xi|\tanh(\sqrt{\mu}|\xi|) \leq \mu\frac{|\xi|^2}{(1+\sqrt{\mu}|\xi|)^{\frac{1}{2}}}.$$

We may therefore use the product rule, combined with (2.23) and the shape derivative formulas (A.55) and (A.56) to find that

$$\begin{aligned}
B_2 &\leq \mu|\partial_x\big((\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(1-\mathcal{J}_\mu[\varepsilon\zeta](\mathcal{J}_\mu[0])^{-1})\psi\big)|_{\dot{H}_\mu^{s+\frac{1}{2}}}\\
&\leq \mu M\Big(|\mathrm{d}_\zeta(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\partial_x\zeta)(1-\mathcal{J}_\mu[\varepsilon\zeta](\mathcal{J}_\mu[0])^{-1})\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}\\
&\quad+|\mathrm{d}_\zeta\mathcal{J}_\mu[\varepsilon\zeta](\partial_x\zeta)(\mathcal{J}_\mu[0])^{-1}\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}+|\partial_x\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}\Big)\\
&\leq \varepsilon\mu M|\partial_x\psi|_{H^{s+1}}.
\end{aligned}$$

□

3.5. **Symbolic analysis of** $(\mathcal{J}_\mu)^{-1}(\mathcal{G}_\mu^-)^{-1}\partial_x$**.** We end this section with the study of an operator that appears in the quasilinearisation of the equations. In particular, we will need the symbolic description for the energy estimates later (see inequality (5.12)). The operator is well-defined by the first point in Corollary A.17, and can be approximated by a first order operator with a skew-symmetric principal symbol.

Corollary 3.8. *Let $t_0 \geq 1$ and $\zeta \in H^{t_0+3}(\mathbb{R})$ be such that* (1.14) *is satisfied. Then for all $0 \leq s \leq t_0$ and $f \in H^{s+\frac{1}{2}}(\mathbb{R})$, one can approximate the operator $\mathfrak{B}^2(\mathcal{J}_\mu)^{-1}(\mathcal{G}_\mu^-)^{-1}\partial_x$ by*

$$\mathrm{Op}\Big(\frac{\mathfrak{B}^2}{S_JS^-}\Big)\partial_x f(x) = -\frac{1}{\sqrt{\mu}}\mathcal{F}^{-1}\Big(\frac{|\xi|(1+\sqrt{\mu}|\xi|)^{-1}}{1+\gamma\tanh(\sqrt{\mu}t(x,\xi))}i\xi\hat{f}(\xi)\Big)(x),$$

and for $k=0,1$, there holds,

$$|\big[\mathfrak{B}^2(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1} - \mathrm{Op}(\frac{\mathfrak{B}^2}{S_JS^-})\big]\partial_x f|_{H^{s+\frac{k}{2}}} \leq \varepsilon\mu^{-1-\frac{k}{4}}M(t_0+3)|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac{1}{2}}f|_{H^s}. \tag{3.23}$$

Proof. We first decompose the main operator into three pieces:

$$\mathfrak{B}^2(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x f = \mathrm{Op}(\frac{\mathfrak{B}^2}{S_JS^-})\partial_x f + R_1 + R_2, \tag{3.24}$$

where we define $f^\sharp = (\mathcal{J}_\mu)^{-1}(\mathcal{G}_\mu^-)^{-1}\partial_x f$

$$R_1 = \Big(1-\mathrm{Op}(\frac{\mathfrak{B}^2}{S_JS^-})\mathrm{Op}(\frac{S_JS^-}{\mathfrak{B}^2})\Big)\mathfrak{B}^2 f^\sharp,$$

and

$$R_2 = \mathrm{Op}(\frac{\mathfrak{B}^2}{S_JS^-})\big(\mathrm{Op}(S^-S_J) - \mathcal{G}_\mu^-[\varepsilon\zeta]\mathcal{J}_\mu[\varepsilon\zeta]\big)f^\sharp.$$

For the estimate on R_1 we use (B.33) to get,

$$|R_1|_{H^{s+\frac{k}{2}}} \leq \varepsilon\mu^{-\frac{k}{4}}M|\mathfrak{B}f^\sharp|_{H^s}.$$

Then apply (2.23) and(A.34) to find that

$$|\mathfrak{B}f^\sharp|_{L^2} = |(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x f|_{\dot{H}_\mu^{\frac{1}{2}}} \tag{3.25}$$
$$\leq \mu^{-\frac{1}{4}}M|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x f|_{\mathring{H}^{\frac{1}{2}}}$$
$$\leq \mu^{-\frac{3}{4}}M|f|_{\mathring{H}^{\frac{1}{2}}},$$

which yields the following estimate on R_1:

$$|R_1|_{H^{s+\frac{k}{2}}} \leq \varepsilon\mu^{-1-\frac{k}{4}}M|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac{1}{2}}f|_{H^s}.$$

For the estimate on R_2 we would like to employ the symbolic description of $\mathcal{G}_\mu^{\pm}$. However, since estimate (3.4) arrives on $\mathring{H}^{\frac{1}{2}}(\mathbb{R})$, we need to carefully decompose this term so that we can estimate the operators in $f^\sharp$. First, we observe that

$$\begin{aligned}\mathrm{Op}(S^-S_J) - \mathcal{G}_\mu^-[\varepsilon\zeta]\mathcal{J}_\mu[\varepsilon\zeta]\big)f^\sharp &= \Big(\big(\mathrm{Op}(S^-) - \mathcal{G}_\mu^-[\varepsilon\zeta]\big)\mathcal{J}_\mu[\varepsilon\zeta]\Big)f^\sharp \\ &\quad + \big(\mathrm{Op}(S^-S_J) - \mathrm{Op}(S^-)\mathrm{Op}(S_J)\big)f^\sharp \\ &\quad + \Big(\mathrm{Op}(S^-)\big(\mathrm{Op}(S_J) - \mathcal{J}_\mu[\varepsilon\zeta]\big)\Big)f^\sharp \\ &= r_1 + r_2 + r_3.\end{aligned}$$

Let $R_2^i = \Lambda^{\frac{k}{2}}\mathrm{Op}(\frac{\mathfrak{B}^2}{S_JS^-})r_i$ with $i = 1,2,3$. Then for the contribution of R_2^1, we first use estimates (B.25) and (B.5) to find that

$$|R_2^1|_{H^s} \leq \mu^{-\frac{3}{4}-\frac{k}{4}}M(t_0+3)|r_1|_{\mathring{H}^{s+\frac{1}{2}}}.$$

Then apply Proposition 3.2, the definition of $f^\sharp$, and argue as in the proof of estimate (3.25) to deduce,

$$\begin{aligned}|R_2^1|_{H^s} &\leq \varepsilon\mu^{\frac{1}{4}-\frac{k}{4}}M(t_0+3)|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x f|_{\mathring{H}^{s+\frac{1}{2}}} \\ &\leq \varepsilon\mu^{-1}M(t_0+3)|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac{1}{2}}f|_{H^s}.\end{aligned}$$

For R_2^2, we apply estimates (B.25), (B.5), (B.30), and (3.25):

$$|R_2^2|_{H^s} \leq \varepsilon\mu^{-1}M|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac{1}{2}}f|_{H^s}.$$

Lastly, for R_2^3, we will decompose it further:

$$\begin{aligned}R_2^3 &= -\gamma\Lambda^{\frac{k}{2}}\mathrm{Op}\big(\frac{\mathfrak{B}^2}{S_JS^-}\big)\Big(\mathrm{Op}(S^-)\mathrm{Op}\big(\frac{S^+}{S^-}\big) - \mathrm{Op}(S^+)\Big)f^\sharp \\ &\quad - \gamma\Lambda^{\frac{k}{2}}\mathrm{Op}\big(\frac{\mathfrak{B}^2}{S_JS^-}\big)\Big(\mathrm{Op}(S^+) - \mathcal{G}_\mu^+[\varepsilon\zeta]\Big)f^\sharp \\ &\quad - \gamma\Lambda^{\frac{k}{2}}\mathrm{Op}\big(\frac{\mathfrak{B}^2}{S_JS^-}\big)\Big(\mathcal{G}_\mu^-[\varepsilon\zeta] - \mathrm{Op}(S^-)\Big)(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]f^\sharp \\ &= R_2^{3,1} + R_2^{3,2} + R_2^{3,3}.\end{aligned}$$

For $R_2^{3,1}$, we use (B.25), (B.5), (B.30), and (3.25) to find that

$$|R_2^{3,1}|_{H^s} \leq \varepsilon\mu^{-1}M|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac{1}{2}}f|_{H^s}.$$

For $R_2^{3,2}$, we use (B.25), (B.5), then estimate (3.3) to find that

$$|R_2^{3,2}|_{H^s} \leq \varepsilon\mu^{1-\frac{k}{4}}M(t_0+3)|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac{1}{2}}f|_{H^s}$$

For $R_2^{3,3}$, we use estimates (B.25), (B.5), (3.4), (2.11), (3.25) to find that

$$\begin{aligned}
|R_2^{3,3}|_{H^s} &\leq \mu^{-1}M\Big|\Big(\mathrm{Op}(S^-) - \mathcal{G}_\mu^-[\varepsilon\zeta]\Big)(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]f^\sharp\Big|_{H^{s+\frac12}}\\
&\leq \varepsilon M(t_0+3)|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]f^\sharp|_{\mathring{H}^{s+\frac12}}\\
&\leq \varepsilon \mu^{\frac14} M(t_0+3)|f^\sharp|_{\dot{H}_\mu^{s+\frac12}}\\
&\leq \varepsilon\mu^{-1}M(t_0+3)|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}f|_{H^s}.
\end{aligned}$$

Gathering all these estimates implies

$$|R_2|_{L^2} \leq \varepsilon\mu^{-1-\frac{k}{4}}M(t_0+3)|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}f|_{H^s},$$

and adding the estimates for R_1 and R_2 completes the proof.

□

4. Quasilinearization of the internal water wave system

In this section, we will put the internal water waves system (1.13) in a quasilinear form. This is done by applying time and space derivatives to the system and taking care of the principal terms. In particular, on need shape derivative formulas for the Dirichlet-Neumann operator (2.1), where we prove in Lemma A.22 that

$$\mathrm{d}_\zeta\mathcal{G}_\mu[\varepsilon\zeta](h)\psi = -\varepsilon\mathcal{G}_\mu[\varepsilon\zeta]\big(h(\underline{w}^+ - \gamma\underline{w}^-)\big) - \varepsilon\mu\mathcal{I}[\mathbf{U}]h,$$

where $\underline{w}^\pm$ is defined below together with the operator $\mathcal{I}[\mathbf{U}]h$. In fact, the operator $\mathcal{I}[\mathbf{U}]h$ is one of the main quantities that we need to understand, and is where we will use the symbolic descriptions from the previous section. Before stating the main result, we formally introduce the main operators involved, and that will be studied in detail later.

Definition 4.1. *Let the functions $\psi^\pm$ serve as Dirichlet data for the elliptic problems* (1.6) *and* (1.4)*, then we define the horizontal components of the velocities at the surface by*

$$\underline{w}^\pm = \frac{\mathcal{G}_\mu^\pm[\varepsilon\zeta]\psi^+ + \varepsilon\mu\partial_x\zeta\partial_x\psi^\pm}{1+\varepsilon^2\mu(\partial_x\zeta)^2}.$$

We define the vertical component of the velocity at the surface by

$$\underline{V}^\pm = \partial_x\psi^\pm - \varepsilon(\underline{w}^\pm\psi)\partial_x\zeta,$$

and

$$[\![\underline{V}^\pm]\!] = \underline{V}^+ - \underline{V}^-.$$

We refer to Corollary A.18 for the precise definition. Moreover, we define the quantity related to the Rayleigh-Taylor criterion by

$$\mathfrak{a} = \Big((1-\gamma) + \varepsilon\big((\partial_t + \varepsilon\underline{V}^+\partial_x)\underline{w}^+ - \gamma(\partial_t+\varepsilon\underline{V}^-\partial_x)\underline{w}^-\big)\Big), \tag{4.1}$$

and a quantity related to the presence of surface tension:

$$\mathcal{K}[\varepsilon\sqrt{\mu}\partial_x\zeta]\bullet = (1+\varepsilon^2\mu(\partial_x\zeta)^2)^{-\frac32}\bullet.$$

From these quantities, we let $\mathbf{U} = (\zeta,\psi)^T$ and define the linear operator, $\mathcal{I}[\mathbf{U}]$, of order one by,

$$\mathcal{I}[\mathbf{U}]\bullet = \partial_x(\bullet\underline{V}^+) + \gamma\mathcal{G}_\mu[\varepsilon\zeta](\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x\Big(\bullet[\![\underline{V}^\pm]\!]\Big), \tag{4.2}$$

and its adjoint reads,

$$\mathcal{I}[\mathbf{U}]^*\bullet = -\underline{V}^+\partial_x\bullet - \gamma[\![\underline{V}^\pm]\!]\partial_x\big((\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta]\bullet\big). \tag{4.3}$$

Moreover, we define the instability operator:

$$\mathfrak{Ins}[\mathbf{U}]\bullet = \mathfrak{a}\bullet - (1-\gamma)\gamma\varepsilon^2\mu[\![\underline{V}^{\pm}]\!]\mathfrak{E}_\mu[\varepsilon\zeta]\big(\bullet[\![\underline{V}^{\pm}]\!]\big) - \mathrm{bo}^{-1}\partial_x\mathcal{K}[\varepsilon\sqrt{\mu}\partial_x\zeta]\partial_x\bullet, \tag{4.4}$$

where

$$\mathfrak{E}_\mu[\varepsilon\zeta]\bullet = \partial_x\circ(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\circ\partial_x\bullet\,. \tag{4.5}$$

Then to put the internal water waves system in matrix form, it is convenient to introduce the notation

$$\mathcal{A}[U] = \begin{pmatrix} 0 & -\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta] \\ \mathfrak{Ins}[\mathbf{U}] & 0\end{pmatrix}, \qquad \mathcal{B}[\mathbf{U}] = \begin{pmatrix}\varepsilon\mathcal{I}[\mathbf{U}] & 0 \\ 0 & -\varepsilon\mathcal{I}[\mathbf{U}]^*\end{pmatrix},$$

where $\mathcal{A}$ and $\mathcal{B}$ corresponds to the principal part of the system. To account for surface tension, one also needs to track the dependence in the sub-principal part, which is defined for $\alpha = (\alpha_1,\alpha_2)\in\mathbb{N}^2$:

$$\mathcal{C}_\alpha[U] = \begin{pmatrix} 0 & -\frac{1}{\mu}\mathcal{G}_{\mu,(\alpha)}[\varepsilon\zeta] \\ \mathrm{bo}^{-1}\mathcal{K}_{(\alpha)}[\varepsilon\sqrt{\mu}\partial_x\zeta] & 0\end{pmatrix},$$

where we let $(\partial_1,\partial_2) = (\partial_x,\partial_t)$, $F=(f_1,f_2)$ to define

$$\mathcal{G}_{\mu,(\alpha)}[\varepsilon\zeta]F = \sum_{j=1}^{2}\alpha_j\mathrm{d}_\zeta\mathcal{G}[\varepsilon\zeta](\partial_j\zeta)f_j,$$

and we have:

$$\mathcal{K}_{(\alpha)}[\partial_x\zeta]F = -\partial_x\Big(\sum_{j=1}^{2}\mathrm{d}_\zeta\mathcal{K}[\partial_x\partial_j\zeta]\partial_x f_j + \mathcal{K}[\partial_x f_j]\partial_x\partial_j\zeta\Big).$$

With these formulas, we can state the main result of the section.

Proposition 4.2. *Let $T>0$, $t_0\geq 1$ and $N\in\mathbb{N}$ be such that $N\geq 5$. Furthermore, let $\mathbf{U} = (\zeta,\psi)^T\in\mathscr{E}^N_{\mathrm{bo},T}$ be such that (1.14) is satisfied on $[0,T]$. Also, for any $\alpha=(\alpha^1,\alpha^2)\in\mathbb{N}^2$, $\check{\alpha}^j = \alpha - \mathbf{e}_j$, with $1\leq|\alpha|\leq N$ we define $\partial^\alpha_{x,t} = \partial_x^{\alpha_1}\partial_t^{\alpha_2}$, and let $\underline{w}^{\pm}$ be defined as in (A.41). Moreover, define $\underline{w}$ by*

$$\underline{w} = \underline{w}^+ - \gamma\underline{w}^-,$$

and

$$\zeta_{(\alpha)} = \partial^\alpha_{x,t}\zeta, \quad \psi_{(\alpha)} = \partial^\alpha_{x,t}\psi - \varepsilon\underline{w}\partial^\alpha_{x,t}\zeta, \quad \psi_{\langle\check{\alpha}\rangle} = (\psi_{(\check{\alpha}^1)},\psi_{(\check{\alpha}^2)})^T.$$

Then for $\mathbf{U}_{(\alpha)} = (\zeta_{(\alpha)},\psi_{(\alpha)})^T$ and $\mathbf{U}_{\langle\check{\alpha}\rangle} = (\zeta_{(\alpha)},\psi_{\langle\check{\alpha}\rangle})^T$, there holds

$$\text{if } 1\leq|\alpha|<N: \qquad \partial_t\mathbf{U}_{(\alpha)} + \mathcal{A}[\mathbf{U}]\mathbf{U}_{(\alpha)} = \varepsilon(R_\alpha,S_\alpha)^T, \tag{4.6}$$

$$\text{if } |\alpha| = N: \quad \partial_t\mathbf{U}_{(\alpha)} + \mathcal{A}[\mathbf{U}]\mathbf{U}_{(\alpha)} + \mathcal{B}[\mathbf{U}]\mathbf{U}_{(\alpha)} + \mathcal{C}[\mathbf{U}]\mathbf{U}_{\langle\check{\alpha}\rangle} = \varepsilon(R_\alpha,S_\alpha)^T \tag{4.7}$$

The rest functions satisfy the estimates

$$|R_\alpha|^2_{H^1_{\gamma,\mathrm{bo}}} + |S_\alpha|^2_{\dot{H}^{\frac{1}{2}}_\mu} \leq C\mathcal{E}^N(\mathbf{U})\big(1+\varepsilon^2\sqrt{\mu}|[\![\underline{V}^{\pm}]\!]|^2_{L^\infty}|\zeta|^2_{<N+\frac{1}{2}>}\big), \tag{4.8}$$

for some $C>0$ and where $|\zeta|_{<N+\frac{1}{2}>}$ is defined by

$$|\zeta|_{<N+\frac{1}{2}>} = \sum_{\alpha\in\mathbb{N}^2,|\alpha|=N}|\partial^\alpha_{x,t}\zeta|_{\mathring{H}^{\frac{1}{2}}}.$$

For the proof, we need to carefully track the dependencies in the small parameters. However, this part is very similar to the one in [50] and can therefore be considered a technical point. We give the details and point out the differences in the Appendix A, Section A.5. Now, before we proceed with the energy estimates, we need to give a rigorous meaning to the operators given in Definition 4.1. This will be done in separate subsections.

4.0.1. *Properties of $\mathcal{I}[\mathbf{U}]\bullet$.* First, we study the operator $\mathcal{I}[\mathbf{U}]\bullet$ which appears later in the shape derivative formulas for $\mathcal{G}_\mu$.

Proposition 4.3. *Let $t_0 \geq 1$ and $\mathbf{U} = (\zeta, \psi)^T$, with $\zeta \in H^{t_0+3}(\mathbb{R})$ satisfying (1.14) and $\psi \in \dot{H}^{t_0+2}(\mathbb{R})$. Then we may define the operator*

$$\mathcal{I}[\mathbf{U}]\bullet = \partial_x(\bullet \underline{V}^+) + \gamma \mathcal{G}_\mu[\varepsilon\zeta](\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x\Big(\bullet [\![\underline{V}^\pm]\!]\Big),$$

and it satisfies the following properties:

1. *For all $0 \leq s \leq t_0$ and $f \in H^{s+\frac{1}{2}}(\mathbb{R})$ there holds,*
$$|\mathcal{I}[\mathbf{U}]f|_{H^{s-\frac{1}{2}}} \leq M|f|_{H^{s+\frac{1}{2}}}|\partial_x\psi|_{H^{t_0+\frac{1}{2}}}, \tag{4.9}$$
and
$$|\mathcal{I}[\mathbf{U}]f|_{H^{s-\frac{1}{2}}} \leq M|f|_{H^{t_0+\frac{1}{2}}}|\partial_x\psi|_{H^{s+\frac{1}{2}}}. \tag{4.10}$$
2. *Let $\mathfrak{a} = 1 + \mathfrak{b}$, with $\mathfrak{b} \in H^{t_0+1}(\mathbb{R})$, and for all $f \in L^2(\mathbb{R})$ there holds,*
$$\big(\mathfrak{a}\mathcal{I}[\mathbf{U}]f, f\big)_{L^2} \leq M(t_0+3)(1+|\mathfrak{b}|_{H^{t_0+1}})|\partial_x\psi|_{H^{t_0+1}}|f|_{L^2}^2. \tag{4.11}$$
3. *Let $K \in H^{t_0+1}(\mathbb{R})$, then for all $f \in H^1(\mathbb{R})$ there holds,*
$$\big(\partial_x(K\partial_x\mathcal{I}[\mathbf{U}]f), f\big)_{L^2} \leq M(t_0+3)|K|_{H^{t_0+1}}|\partial_x\psi|_{H^{t_0+1}}|f|_{H^1}^2. \tag{4.12}$$
4. *For all $f \in \dot{H}_\mu^{\frac{1}{2}}(\mathbb{R})$, $g \in H^{\frac{1}{2}}(\mathbb{R})$, one has*
$$\big(\mathcal{I}[\mathbf{U}]^* f, g\big)_{L^2} \leq M|\partial_x\psi|_{H^{t_0+\frac{1}{2}}}|f|_{\dot{H}_\mu^{\frac{1}{2}}}(|g|_{L^2} + \mu^{\frac{1}{4}}|g|_{H^{\frac{1}{2}}}). \tag{4.13}$$

Proof. We prove each point in separate steps.

Step 1. To prove the first point, we observe that the first part of $\mathcal{I}[\mathbf{U}]f$ is estimated by the product estimate (B.6) and (A.45):

$$\begin{aligned}|\partial_x(f\underline{V}^+)|_{H^{s-\frac{1}{2}}} &\leq |f|_{H^{s+\frac{1}{2}}}|\underline{V}^+|_{H^{\max\{s+\frac{1}{2},t_0\}}} \\ &\leq M|f|_{H^{s+\frac{1}{2}}}|\partial_x\psi|_{H^{t_0+\frac{1}{2}}}.\end{aligned}$$

To estimate the remaining part we first estimate $\mathcal{G}_\mu$ with (2.2) to find that

$$|\mathcal{G}_\mu[\varepsilon\zeta](\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\big(\partial_x(f[\![\underline{V}^\pm]\!])\big)|_{H^{s-\frac{1}{2}}} \leq \mu^{\frac{3}{4}}|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\big(\partial_x(f[\![\underline{V}^\pm]\!])\big)|_{\dot{H}_\mu^{s+\frac{1}{2}}}.$$

Then use (A.34), the product estimate (B.6), and (A.45) to see that

$$\begin{aligned}\mu^{\frac{3}{4}}|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\big(\partial_x(f[\![\underline{V}^\pm]\!])\big)|_{\dot{H}_\mu^{s+\frac{1}{2}}} &\leq \mu^{\frac{1}{2}}|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\big(\partial_x(f[\![\underline{V}^\pm]\!])\big)|_{\mathring{H}^{s+\frac{1}{2}}} \\ &\leq |f[\![\underline{V}^\pm]\!]|_{H^{s+\frac{1}{2}}} \\ &\leq M|f|_{H^{s+\frac{1}{2}}}|\partial_x\psi|_{H^{t_0+\frac{1}{2}}}.\end{aligned}$$

For the proof of (4.10), it is proved similarly where we only need to modify the part when we apply the product estimate.

Step 2. In this estimate, we note that the instability operator is a first-order differential operator acting on $f \mapsto \mathcal{I}[\mathbf{U}]f$. But the principal symbol is skew-adjoint. Indeed, for the first part, we use integration by parts

$$\big(\mathfrak{a}\partial_x(f\underline{V}^+), f\big)_{L^2} = \big(\mathfrak{a}(\partial_x\underline{V}^+)f, f\big)_{L^2} - \frac{1}{2}\big(\big(\partial_x(\mathfrak{a}\underline{V}^+)\big)f, f\big)_{L^2}.$$

Then Hölder's inequality, the Sobolev embedding $H^{t_0}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, the product estimate (B.6), and (A.44) gives

$$\begin{aligned}|\big(\mathfrak{a}\partial_x(f\underline{V}^+), f\big)_{L^2}| &= (1+|b|_{H^{t_0+1}})|\underline{V}^+|_{H^{t_0+1}}|f|_{L^2}^2 \\ &\leq (1+|b|_{H^{t_0+1}})|\partial_x\psi|_{H^{t_0+1}}|f|_{L^2}^2.\end{aligned}$$

For the second part, we use integration by parts together with the fact that $\mathcal{G}_\mu$ and $\mathcal{G}_\mu^-$ are symmetric to make the following decomposition

$$\begin{aligned}\big(\mathfrak{a}\mathcal{G}_\mu[\varepsilon\zeta](\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\big(\partial_x(f[\![\underline{V}^\pm]\!])\big), f\big)_{L^2} &= -\big(f[\![\underline{V}^\pm]\!], \partial_x\Big((\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta](\mathfrak{a}f)\Big)\big)_{L^2} \\ &= -\big(f[\![\underline{V}^\pm]\!], \partial_x\Big((\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta] - \mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)\Big)(\mathfrak{a}f)\big)_{L^2} \\ &\quad + \big(\partial_x(f[\![\underline{V}^\pm]\!]), \mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)(\mathfrak{a}f)\big)_{L^2} \\ &=: R_1 + R_2.\end{aligned}$$

For R_1 we use Cauchy-Schwarz and estimate (3.19) to deduce that

$$\begin{aligned}|R_1| &\leq |f[\![\underline{V}^\pm]\!]|_{L^2}|((\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta] - \mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big))(\mathfrak{a}f)|_{\mathring{H}^1} \\ &\leq \varepsilon\sqrt{\mu}M(t_0+3)|f[\![\underline{V}^\pm]\!]|_{L^2}|\mathfrak{a}f|_{L^2}.\end{aligned}$$

Then use the product estimate (B.6) and (A.45) to get that

$$|R_1| \leq \varepsilon\sqrt{\mu}M(t_0+3)(1+|b|_{H^{t_0}})|\partial_x\psi|_{H^{t_0}}|f|_{L^2}^2.$$

For R_2, we will split it into several parts:

$$\begin{aligned}R_2 &= \Big(\Big(\mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)^* - \mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)\Big)\partial_x(f[\![\underline{V}^\pm]\!]), (\mathfrak{a}f)\Big)_{L^2} + \Big(\mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)(f\partial_x[\![\underline{V}^\pm]\!]), (\mathfrak{a}f)\Big)_{L^2} \\ &\quad + \Big([\mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big), [\![\underline{V}^\pm]\!]]\partial_x f, (\mathfrak{a}f)\Big)_{L^2} + \Big([\![\underline{V}^\pm]\!]\mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)\partial_x f, (\mathfrak{a}f)\Big)_{L^2} \\ &= R_2^1 + R_2^2 + R_2^3 + R_2^4.\end{aligned}$$

For R_2^1 we use estimate (B.29), for R_2^2 we use (B.27), for R_2^3 we use (B.28), combined with (B.6) and (A.45) we get that

$$|R_2^1| + |R_2^2| + |R_2^3| \leq \sqrt{\mu}M(1+|b|_{H^{t_0}})|\partial_x\psi|_{H^{t_0+1}}|f|_{L^2}^2.$$

On the other hand, we can use integration by parts on R_2 to cancel R_2^4. Indeed, we obtain that

$$\begin{aligned}R_2 &= -\Big((f[\![\underline{V}^\pm]\!]), \mathrm{Op}\Big(\partial_x\frac{S^+}{S^-S_J}\Big)(\mathfrak{a}f)\Big)_{L^2} - \Big((f[\![\underline{V}^\pm]\!]), \mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)(f\partial_x\mathfrak{b})\Big)_{L^2} \\ &\quad - \Big((f[\![\underline{V}^\pm]\!]), [\mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big), \mathfrak{b}]\partial_x f\Big)_{L^2} - \Big((f[\![\underline{V}^\pm]\!]), \mathfrak{a}\mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)\partial_x f\Big)_{L^2} \\ &= R_2^5 + R_2^6 + R_2^7 + R_2^8.\end{aligned}$$

Here we estimate R_2^5, R_2^6, R_2^7 as above where we also use estimate (B.32) for R_2^5. Adding these two decompositions implies

$$2|R_2| \leq \big|\sum_{j=1}^{8} R_2^j\big| \leq \sqrt{\mu}M(1+|b|_{H^{t_0+1}})|\partial_x\psi|_{H^{t_0+1}}|f|_{L^2}^2.$$

Step 3. Since K is symmetric and $f \mapsto \mathcal{I}[\mathbf{U}]f$ is skew-symmetric, we can absorb one derivative by integrating by parts as we did in the previous step.

Step 4. From the definition (4.3) we have that

$$\begin{aligned}\big(\mathcal{I}[\mathbf{U}]^*f, g\big)_{L^2} &= -\big(\underline{V}^+\partial_x f, g\big)_{L^2} - \gamma\big([\![\underline{V}^\pm]\!]\partial_x\big((\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta]f\big), g\big)_{L^2}\\ &= A_1 + \gamma A_2.\end{aligned}$$

For the first term, we introduce a commutator, then apply Hölder's inequality, Sobolev embedding, estimate (A.45), and (B.10) to find that

$$\begin{aligned}|A_1| &\leq |[\underline{V}^+, (1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}]\frac{\partial_x}{(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}}f|_{L^2}|g|_{L^2}\\ &\quad + |\underline{V}^+|_{L^\infty}|\frac{\partial_x}{(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}}f||(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}g|_{L^2}\\ &\leq M|\partial_x\psi|_{H^{t_0+\frac12}}|f|_{\dot{H}_\mu^{\frac12}}(|g|_{L^2}+\mu^{\frac14}|g|_{H^{\frac12}}).\end{aligned}$$

For the second term, we instead use commutator estimate (B.9) and (2.11) to obtain

$$\begin{aligned}|A_2| &\leq \big|\big[[\![\underline{V}^\pm]\!], |\mathrm{D}|^{\frac12}\big]\mathcal{H}|\mathrm{D}|^{\frac12}(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta]f|_{L^2}|g|_{L^2} + |[\![\underline{V}^\pm]\!]|_{L^\infty}|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta]f|_{\mathring{H}^{\frac12}}|g|_{\mathring{H}^{\frac12}}\\ &\leq M\mu^{\frac14}|\partial_x\psi|_{H^{t_0+\frac12}}|f|_{\dot{H}_\mu^{\frac12}}(|g|_{L^2}+|g|_{H^{\frac12}}).\end{aligned}$$

□

4.1. **Properties of $\mathfrak{Ins}[\mathbf{U}]\bullet$.** The instability operator is defined in terms of $\mathfrak{E}_\mu$ which is given by

$$\mathfrak{E}_\mu[\varepsilon\zeta]\bullet = \partial_x \circ (\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1} \circ \partial_x \bullet\,.$$

Meaning we first need to study its properties, which is the topic of the next Proposition.

Proposition 4.4. *Let $t_0 \geq 1$ and $\zeta \in H^{t_0+2}(\mathbb{R})$ be such that* (1.14) *is satisfied. Then we have the following results:*

1. *There exist a constant $c \leq M$ such that for all $f \in H^{\frac12}(\mathbb{R})$ there holds,*

$$0 \leq \big(\mathfrak{E}_\mu[\varepsilon\zeta]f, f\big)_{L^2} \leq \frac{c}{\mu}|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}f|_{L^2}^2. \tag{4.14}$$

2. *If suppose further that ζ is time dependent and satisfies* (1.14) *uniformly in time, then for all $f \in H^{\frac12}(\mathbb{R})$ there holds,*

$$|\big(\big[\partial_t, \mathfrak{E}_\mu[\varepsilon\zeta]\big]f, f\big)_{L^2}| \leq \frac{\varepsilon}{\mu}M|\partial_t\zeta|_{H^{t_0+1}}|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}f|_{L^2}^2. \tag{4.15}$$

Proof. We give the proof in two separate points.

Step 1. For the proof of the first point, we deduce the positivity by integrating parts:

$$\begin{aligned}\big(\mathfrak{E}_\mu[\varepsilon\zeta]f,f\big)_{L^2} &= -\big((\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x f,\partial_x f\big)_{L^2}\\ &= -\big(g,(\mathcal{G}_\mu^-[\varepsilon\zeta])\mathcal{J}_\mu[\varepsilon\zeta]g\big)_{L^2}\\ &\geq 0,\end{aligned}$$

where we defined g by

$$g=(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x f,$$

and used the fact that $\mathcal{G}_\mu^-$ is negative and $\mathcal{J}_\mu$ is positive.

For the upper bound, we use Plancherel's identity, and Cauchy-Schwarz inequality with the estimates (2.23) and (A.34) to find that

$$\begin{aligned}|\big(\mathfrak{E}_\mu[\varepsilon\zeta]f,f\big)_{L^2}| &\leq |(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x f|_{\dot{H}_\mu^{\frac12}}|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}f|_{L^2}\\ &\leq \mu^{-\frac14}M|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x f|_{\mathring{H}^{\frac12}}|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}f|_{L^2}\\ &\leq \mu^{-\frac34}M|f|_{\mathring{H}^{\frac12}}|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}f|_{L^2}.\end{aligned}$$

Step 2. By direct computations, we need to control the following two terms:

$$\begin{aligned}\big[\partial_t,\mathfrak{E}_\mu[\varepsilon\zeta]\big]f &= \partial_x\circ(\mathrm{d}_\zeta(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\partial_t\zeta))\circ(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\circ\partial_x f\\ &\quad+\partial_x\circ(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}\circ\mathrm{d}_\zeta(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}(\partial_t\zeta)\circ\partial_x f\\ &= f_1+f_2.\end{aligned}$$

For the contribution of the first term, we use Plancherel's identity, Cauchy-Schwarz, and estimates (A.56) and (A.34):

$$\begin{aligned}|\big(f_1,f\big)_{L^2}| &\leq |\mathrm{d}_\zeta(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\partial_t\zeta)(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\circ\partial_x f|_{\dot{H}_\mu^{\frac12}}|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}f|_{L^2}\\ &\leq \varepsilon\mu^{-\frac14}M|\partial_t\zeta|_{H^{t_0+1}}|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x f|_{\mathring{H}^{\frac12}}|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}f|_{L^2}\\ &\leq \varepsilon\mu^{-\frac14}M|\partial_t\zeta|_{H^{t_0+1}}|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}f|_{L^2}^2.\end{aligned}$$

For the estimate on f_2 we use we use Plancherel's identity, Cauchy-Schwarz, (2.23), (A.37), and then (A.34) to obtain

$$\begin{aligned}|\big(f_2,f\big)_{L^2}| &\leq \mu^{-\frac14}|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\circ\mathrm{d}_\zeta\mathcal{G}_\mu^-[\varepsilon\zeta](\partial_t\zeta)\circ(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\circ\partial_x f|_{\mathring{H}^{\frac12}}|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}f|_{L^2}\\ &\leq \varepsilon\mu^{-1}M|\partial_t\zeta|_{H^{t_0+1}}|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac12}f|_{L^2}^2.\end{aligned}$$

□

Remark 4.5. *In the first point, we can define the smallest constant* $c\leq M$ *such that* (4.14) *holds by*

$$\mathfrak{c}(\zeta)=\sup_{f\in H^{\frac12}(\mathbb{R}),f\neq0}\frac{\big((\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x f,\partial_x f\big)_{L^2}}{|1+\sqrt{\mu}|\mathrm{D}|^{\frac12}f|_{L^2}^2}. \tag{4.16}$$

For the next result, we will treat the properties of the instability operator. In particular, we will show that under the stability criterion, we can have a coercivity-type estimate. This is essential for the well-posedness theory to work and relies on the surface tension parameter through the Bond number $\mathrm{bo}^{-1}>0$. To be clear, we restate the stability criterion in the introduction:

$$0<\mathfrak{d}(\mathbf{U}):=\inf_{\mathbb{R}}\mathfrak{a}-\Upsilon\mathfrak{c}(\zeta)|[\![\underline{V}^{\pm}]\!]|_{H^{t_0+1}}^4,$$

where

$$\Upsilon = \frac{\mathrm{bo}}{4}(1-\gamma)^2\gamma^2\mu\varepsilon^2,$$

and $\mathfrak{a}$ is given by (4.18), $\mathfrak{e}$ is given in (4.16), and we define

$$\mathfrak{c}(\zeta) = \mathfrak{e}(\zeta)^2(1+\varepsilon^2\mu|\partial_x\zeta|^2_{L^\infty})^{\frac{3}{2}}.$$

Proposition 4.6. *Let $\varepsilon, \mu, \mathrm{bo}^{-1} \in (0,1)$, $T > 0$, $t_0 \geq 1$, and $\mathbf{U} = (\zeta, \psi) \in \mathscr{E}^N_{\mathrm{bo},T}$ be such that the non-cavitation condition* (1.14) *holds and satisfies the stability criterion* (1.17) *on* $[0,T]$*. Then we have the following set of inequalities on the same time interval:*

1. *For all $u \in H^1_{\gamma,\mathrm{bo}}(\mathbb{R})$, $C > 0$, and $\mathfrak{Ins}(\mathbf{U})$ defined by* (4.4)*, one has*

$$\big(u, \mathfrak{Ins}(\mathbf{U})u\big)_{L^2} \leq C(\mathcal{E}^1(\mathbf{U}))|u|^2_{H^1_{\gamma,\mathrm{bo}}}. \tag{4.17}$$

2. *For $a(\mathbf{U})$ defined by*

$$a(\mathbf{U}) = (1-\gamma)\gamma\varepsilon^2\mathfrak{e}(\zeta)|[\![\underline{V}^\pm]\!]|^2_{L^\infty}, \tag{4.18}$$

and there is some $C_1 > 0$ and $b(\mathbf{U})$ defined by

$$b(\mathbf{U}) = (1-\gamma)\gamma\varepsilon^2\sqrt{\mu}C_1\mathfrak{e}(\zeta)|[\![\underline{V}^\pm]\!]|^2_{H^{t_0+1}}, \tag{4.19}$$

such that for $\mathfrak{d}$ defined by (1.17) *there holds,*

$$\frac{1}{2}\mathfrak{d}(\mathbf{U})|u|_{H^1_{\gamma,\mathrm{bo}}} \leq \big(u, \mathfrak{Ins}(\mathbf{U})u\big)_{L^2} + a(\mathbf{U})|u|^2_{L^2} + b(\mathbf{U})|u|_{H^{-\frac{1}{2}}}. \tag{4.20}$$

3. *Lastly, there is a control on $u \in \mathring{H}^{\frac{1}{2}}(\mathbb{R})$ through the inequality*

$$\varepsilon\mu^{\frac{1}{4}}|[\![\underline{V}^\pm]\!]|^2_{H^{t_0+1}}|u|_{\mathring{H}^{\frac{1}{2}}} \leq \varepsilon C(\mathcal{E}^1(\mathbf{U}))|u|_{H^1_{\gamma,\mathrm{bo}}}. \tag{4.21}$$

Proof. The proof is similar to the one of Lemma 11 in [50]. However, we give a short proof for the convenience of the reader to track the constants that are responsible for the definitions above. We divide the proof into three steps

Step 1. For the proof of (4.17), we have to deal with the terms:

$$\begin{aligned}\big(u, \mathfrak{Ins}(\mathbf{U})u\big)_{L^2} &= \big(u, \mathfrak{a}u\big)_{L^2} - (1-\gamma)\gamma\varepsilon^2\mu\big(u, [\![\underline{V}^\pm]\!]\mathfrak{E}_\mu[\varepsilon\zeta]\big(u[\![\underline{V}^\pm]\!]\big)\big)_{L^2} \\ &\quad - \mathrm{bo}^{-1}\big(u, \partial_x\mathcal{K}[\varepsilon\sqrt{\mu}\partial_x\zeta]\partial_x u\big)_{L^2} \\ &= I_1 + I_2 + I_3.\end{aligned}$$

For I_1, we have that

$$\big(\inf_{x\in\mathbb{R}} \mathfrak{a}\big)|u|^2_{L^2} \leq I_1 \leq C(\mathcal{E}^1(\mathbf{U}))(1-\gamma)|u|^2_{L^2}, \tag{4.22}$$

where the upper bound is direct since the energy includes time derivatives. For the estimate on I_2, we employ (4.14) to find that

$$-(1-\gamma)\gamma\varepsilon^2\mathfrak{e}(\zeta)|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac{1}{2}}(u[\![\underline{V}^\pm]\!])|^2_{L^2} \leq I_2 \leq 0.$$

For the estimate on I_3, we have by integration by parts that

$$\mathrm{bo}^{-1}(1+\varepsilon^2\mu|\partial_x\zeta|_{L^\infty})^{-\frac{3}{2}}|\partial_x u|^2_{L^2} \leq I_3 \leq \mathrm{bo}^{-1}|\partial_x u|^2_{L^2}. \tag{4.23}$$

So the upper bound in (4.17) is proved.

Step 2. For the proof of (4.20), we need to work on the lower bound in I_2. First, we can replace the lower bound using the commutator estimate (B.8) and Hölder's inequality to find that

$$|\langle\sqrt{\mu}\mathrm{D}\rangle^{\frac{1}{2}}(u[\![\underline{V}^{\pm}]\!])|_{L^2}^2 \leq C_1\sqrt{\mu}|[\![\underline{V}^{\pm}]\!]|_{H^{t_0+1}}^2|u|_{H^{-\frac{1}{2}}}^2 + \sqrt{\mu}|[\![\underline{V}^{\pm}]\!]|_{L^\infty}^2||\mathrm{D}|^{\frac{1}{2}}u|_{L^2}^2 + |[\![\underline{V}^{\pm}]\!]|_{L^\infty}^2|u|_{L^2}^2, \tag{4.24}$$

for some constant $C_1 > 0$. Then we may define $b(\mathbf{U})$ by (4.19) and use the expression for $a(\mathbf{U})$ to find that

$$I_3 \geq -b(\mathbf{U})|u|_{H^{-\frac{1}{2}}} - a(\mathbf{U})\big(|u|_{L^2}^2 + \sqrt{\mu}|u|_{\mathring{H}^{\frac{1}{2}}}^2\big).$$

Then adding all these estimates, one finds that

$$\begin{aligned}
a(\mathbf{U})|u|_{L^2}^2 + b(\mathbf{U})|u|_{H^{-\frac{1}{2}}} &+ \big(u, \mathfrak{Ins}(\mathbf{U})u\big)_{L^2}\\
&= a(\mathbf{U})|u|_{L^2}^2 + b(\mathbf{U})|u|_{H^{-\frac{1}{2}}} + I_1 + I_2 + I_3\\
&\geq \big(\inf_{x\in\mathbb{R}} \mathfrak{a}\big)|u|_{L^2}^2 + \mathrm{bo}^{-1}(1+\varepsilon^2\mu|\partial_x\zeta|_{L^\infty})^{-\frac{3}{2}}|\partial_x u|_{L^2}^2 - a(\mathbf{U})\sqrt{\mu}|u|_{\mathring{H}^{\frac{1}{2}}}^2.
\end{aligned}$$

To conclude, we first use interpolation and Young's inequality to find that

$$\begin{aligned}
\sqrt{\mu}a(\mathbf{U})|u|_{\mathring{H}^{\frac{1}{2}}}^2 &\leq \sqrt{\mu}a(\mathbf{U})|u|_{L^2}|\partial_x u|_{L^2}\\
&\leq \mathrm{bo}\frac{\mu}{2}(1+\varepsilon^2\mu|\partial_x\zeta|_{L^\infty})^{\frac{3}{2}}a(\mathbf{U})^2|u|_{L^2}^2 + \frac{1}{2\mathrm{bo}}(1+\varepsilon^2\mu|\partial_x\zeta|_{L^\infty})^{-\frac{3}{2}}|\partial_x u|_{L^2}^2.
\end{aligned}$$

Clearly, the proof is over if we have the following inequality

$$\inf_{x\in\mathbb{R}} \mathfrak{a} - \Upsilon\mathfrak{c}(\zeta)|[\![\underline{V}^{\pm}]\!]|_{L^\infty}^4 > 0,$$

which holds by (1.17) on $[0, T]$.

Step 3. To prove the third point, we again use interpolation and the full stability criterion (1.17) to obtain

$$\begin{aligned}
\varepsilon^2\sqrt{\mu}|[\![\underline{V}^{\pm}]\!]|_{H^{t_0+1}}^4|u|_{\mathring{H}^{\frac{1}{2}}}^2 &\leq \Upsilon\mathfrak{c}(\zeta)|[\![\underline{V}^{\pm}]\!]|_{H^{t_0+1}}^4|u|_{L^2}^2 + \mathrm{bo}^{-1}(1+\mu|\partial_x\zeta|_{L^\infty})^{-\frac{3}{2}}|\partial_x u|_{L^2}^2\\
&\leq \varepsilon^2 M\big((\inf_{x\in\mathbb{R}} \mathfrak{a})|u|_{L^2}^2 + \mathrm{bo}^{-1}|\partial_x u|_{L^2}^2\big).
\end{aligned}$$

□

5. A priori estimates

We are now in the position to derive energy estimates for the internal water waves system (1.13). To define a natural energy to the system, we distinguish between the cases $\alpha = 0$ and $1 \leq |\alpha| \leq N$. For $\alpha = 0$, we have enough regularity on the data to control the solutions with the energy:

$$E^0(\mathbf{U}) = |\Lambda^{t_0+\frac{5}{2}}\zeta|_{H^1_{\gamma,\mathrm{bo}}}^2 + \frac{1}{\mu}\big(\Lambda^{t_0+\frac{5}{2}}\psi, \mathcal{G}_\mu[0]\Lambda^{t_0+\frac{5}{2}}\psi\big)_{L^2}, \tag{5.1}$$

where $\mathcal{G}[0]\psi$ is defined by formula (2.9):

$$\mathcal{G}[0]\psi = \sqrt{\mu}|\mathrm{D}|\frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{1+\gamma\tanh(\sqrt{\mu}|\mathrm{D}|)}\psi.$$

For $1 \leq |\alpha| \leq N$, we use Proposition 4.2 and exploit its quaslinear structure to define a suitable symmetrizer. In particular, we will need to cancel specific terms in the energy estimates, and it will be done by introducing the symmetrizer:

$$Q(\mathbf{U}) = Q^{(1)}(\mathbf{U}) + Q^{(2)}(\mathbf{U}) = \begin{pmatrix} \mathfrak{Ins}[\mathbf{U}] & 0 \\ 0 & \frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta] \end{pmatrix} + \begin{pmatrix} a(\mathbf{U}) + b(\mathbf{U})\Lambda^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.2}$$

where $a(U)$ and $b(U)$ are defined by (4.18) and (4.19). With this symmetrizer at hand, we define energies by

$$E_\alpha(\mathbf{U}) = \big(\mathbf{U}_{(\alpha)}, Q(\mathbf{U})\mathbf{U}_{(\alpha)}\big)_{L^2}. \tag{5.3}$$

Finally, taking the sum over all α will be the main quantity used to control the principal part of (1.13):

$$E^k(\mathbf{U}) = E^0(\mathbf{U}) + \sum_{1\leq|\alpha|\leq k} E_\alpha(\mathbf{U}). \tag{5.4}$$

Before proceeding, we make three comments on the choice of the energy.

Remark 5.1.

1. *The term $Q_1[\mathbf{U}]$ in (5.2), is chosen specifically to cancel the principal part:*

$$\big(\mathcal{A}[\mathbf{U}]\mathbf{U}_{(\alpha)}, Q_1(\mathbf{U})\mathbf{U}_{(\alpha)}\big)_{L^2} = 0,$$

which appears naturally in the energy estimates. See estimate of B_2^1 in the proof of Proposition 5.2 below.

2. *The role of the term $Q_2[\mathbf{U}]$ is to make the energy equivalent to the energy norm. In particular, for (5.4) to be coercive, we need $Q_2[\mathbf{U}]$ and the stability criterion (1.17) to have a lower bound on $\mathfrak{Ins}[\mathbf{U}]$ in the energy space.*

3. *To close the energy estimates, we actually need to modify (5.4). This is because $\mathfrak{Ins}[\mathbf{U}]$ is a second-order operator and therefore makes a contribution to the sub-principal part of the equation (when $|\alpha| = N$). In particular, we correct the energy to cancel the terms*

$$\big(\mathcal{C}[\mathbf{U}]\mathbf{U}_{\langle\check{\alpha}\rangle}, Q_1(\mathbf{U})\mathbf{U}_{(\alpha)}\big)_{L^2},$$

which will appear in the a priori estimates. To do so, we define the quantity

$$\tilde{Q}(\mathbf{U}) = \begin{pmatrix} \mathrm{bo}^{-1}\mathcal{K}_{(\alpha)}[\varepsilon\sqrt{\mu}\partial_x\zeta] & 0 \\ 0 & \frac{1}{\mu}\mathcal{G}_{\mu,(\alpha)} \end{pmatrix},$$

where we define the modified energy for the internal water waves equation by

$$E^N_{IWW}(\mathbf{U}) = E^N(\mathbf{U}) + C_2 C E^{N-1}(\mathbf{U}) + \sum_{|\alpha|=N} \big(\mathbf{U}_{(\alpha)}, \tilde{Q}(\mathbf{U})\mathbf{U}_{\langle\check{\alpha}\rangle}\big)_{L^2}, \tag{5.5}$$

for some constants $C, C_2 > 0$ to be fixed in the proof.

Proposition 5.2. *Let $\varepsilon, \mu, \gamma \in (0,1)$, $\mathrm{bo}^{-1} = \varepsilon\sqrt{\mu}$, $t_0 = 1$, $N \geq 5$, and $\mathbf{U} = (\zeta, \psi)^T \in \mathscr{E}^{N,t_0}_{\mathrm{bo},\gamma}$ be a solution to (1.13) on a time interval $[0,T]$ for some $T > 0$. Assume that $\mathbf{U}$ satisfies the non-cavitation condition (1.14) and the stability criterion (1.17) on $[0,T]$. Then, for the modified energy defined by (5.5) there is a constant $C = C(|\zeta|_{H^5}, \frac{1}{h_{\min}}, \gamma, \frac{1}{\mathfrak{d}(\mathbf{U})}) > 0$ such that,*

$$\frac{d}{dt}E^N_{IWW}(\mathbf{U}) \leq \varepsilon C E^N_{IWW}(\mathbf{U}), \tag{5.6}$$

for all $0 < t < T$. Furthermore, for the energy defined by (5.4) there holds,

$$\frac{1}{C}\mathcal{E}^N(\mathbf{U}) \leq E^N(\mathbf{U}) \leq C\mathcal{E}^N(\mathbf{U}), \tag{5.7}$$

and

$$\frac{1}{C}\mathcal{E}^N(\mathbf{U}) \leq E^N_{IWW}(\mathbf{U}) \leq C\mathcal{E}^N(\mathbf{U}), \tag{5.8}$$

for all $0 < t < T$.

Proof. We first prove (5.7) and (5.8) in two separate steps before turning to the proof of (5.6).

Proof of (5.7). First consider the case $k = 0$. Let $(\mathcal{G}_\mu[0])^{\frac{1}{2}}$ be the square root of the symbol associated to $\mathcal{G}_\mu[0]$ and then use Plancherel's identity to see that

$$\frac{1}{\mu}\big(\Lambda^{\frac{7}{2}}\psi, \mathcal{G}_\mu[0]\Lambda^{\frac{7}{2}}\psi\big)_{L^2} = \frac{1}{\mu}|(\mathcal{G}_\mu[0])^{\frac{1}{2}}\psi|^2_{H^{\frac{7}{2}}}.$$

Then use (B.2) and the definition (5.1) to obtain that

$$|\Lambda^{\frac{7}{2}}\zeta|^2_{H^1_{\gamma,\mathrm{bo}}} + \frac{1}{C}|\partial_x\psi|^2_{H^3} \leq E^0(\mathbf{U}) \leq C|\psi|^2_{\dot{H}^4_\mu} + \mathcal{E}^N(\mathbf{U}).$$

The lower bound will be absorbed in $\mathcal{E}^N(\mathbf{U})$ when summing over all k. While for the upper bound, we relate ψ with the definition of $\psi_{(\alpha)}$ through the estimate

$$|\psi|_{\dot{H}^4_\mu} \leq \sum_{\alpha\in\mathbb{N}^2\,:\,|\alpha|\leq N-1} |\psi_{(\alpha)}|_{\dot{H}^{\frac{1}{2}}_\mu} + \varepsilon|\underline{w}\zeta_{(\alpha)}|_{\dot{H}^{\frac{1}{2}}_\mu}.$$

Then (B.6) and (A.42) yields,

$$\varepsilon|\underline{w}\zeta_{(\alpha)}|_{\dot{H}^{\frac{1}{2}}_\mu} \leq \varepsilon\mu^{-\frac{1}{4}}|\underline{w}\zeta_{(\alpha)}|_{H^{\frac{1}{2}}} \leq \varepsilon\sqrt{\mu}C|\psi|_{\dot{H}^2}|\zeta_{(\alpha)}|_{H^1}.$$

Combining these estimates implies

$$|\psi|^2_{\dot{H}^4_\mu} \leq C\mathcal{E}^N(\mathbf{U}). \tag{5.9}$$

For the case $k = N$, we use the definition of the individual energies (5.3) to find that

$$E_\alpha(\mathbf{U}) = \big(\zeta_{(\alpha)}, \mathfrak{Ins}[U]\zeta_{(\alpha)}\big)_{L^2} + \big(\zeta_{(\alpha)}, (a(\mathbf{U}) + b(\mathbf{U})\Lambda^{-1})\zeta_{(\alpha)}\big)_{L^2} + \frac{1}{\mu}\big(\psi_{(\alpha)}, \mathcal{G}_\mu[\varepsilon\zeta]\psi_{(\alpha)}\big)_{L^2}.$$

The first two terms are controlled by (4.20) and (4.17). While the last term is controlled by (2.6) and (2.8), which implies the result

$$\frac{1}{C}\mathcal{E}^N(\mathbf{U}) \leq E^N(\mathbf{U}) \leq C\mathcal{E}^N(\mathbf{U}).$$

Proof of (5.8). Next, we prove (5.8). To estimate the additional term when $|\alpha| = N$, we first observe that (suppressing the argument in ζ):

$$\begin{aligned}\frac{1}{\mu}\big|\big(\psi_{(\alpha)}, \mathcal{G}_{\mu,(\alpha)}\psi_{\langle\check{\alpha}\rangle}\big)_{L^2}\big| \leq \frac{1}{\mu}\sum_{j=1}^2 \alpha_j\Big(&\big|\big(\psi_{(\alpha)}, \mathrm{d}\mathcal{G}^+_\mu(\partial_j\zeta)(\mathcal{J}_\mu)^{-1}\psi_{(\check{\alpha}^j)}\big)_{L^2}\big| \\ &+ \big|\big(\psi_{(\alpha)}, \mathcal{G}^+_\mu(\mathcal{J}_\mu)^{-1}(\mathrm{d}\mathcal{J}_\mu(\partial_j\zeta))(\mathcal{J}_\mu)^{-1}\psi_{(\check{\alpha}^j)}\big)_{L^2}\big|\Big),\end{aligned}$$

where $\check{\alpha}^j = \alpha - \mathbf{e}_j$. For the first term we use (A.16) with (2.23), while the for the second term we also use (A.6), (2.23), and then (A.55) to find that

$$\frac{1}{\mu}\big|\big(\psi_{(\alpha)}, \mathcal{G}_{\mu,(\alpha)}\psi_{\langle\check{\alpha}\rangle}\big)_{L^2}\big| \leq \varepsilon M|\psi_{(\alpha)}|_{\dot{H}^{\frac{1}{2}}_\mu}\sum_{j=1}^2|\psi_{(\check{\alpha}^j)}|_{\dot{H}^{\frac{1}{2}}_\mu}, \tag{5.10}$$

Then since the surface tension term can be treated directly by integration by parts, we use Young's inequality to find that

$$\sum_{|\alpha|=N} |\big(\mathbf{U}_{(\alpha)}, \tilde{Q}(\mathbf{U})\mathbf{U}_{\langle\check{\alpha}\rangle}\big)_{L^2}| \leq \sum_{|\alpha|=N} \mathrm{bo}^{-1}|\big(\zeta_{(\alpha)}, \mathcal{K}_{(\alpha)}\zeta_{\langle\check{\alpha}\rangle}\big)_{L^2}| + \frac{1}{\mu}|\big(\psi_{(\alpha)}, \mathcal{G}_{\mu,(\alpha)}\psi_{\langle\check{\alpha}\rangle}\big)_{L^2}|$$
$$\leq \frac{1}{2C_2}\mathcal{E}^N(\mathbf{U}) + \frac{C_2}{2}\mathcal{E}^{N-1}(\mathbf{U}),$$

for any $C_2 > 0$. Then let $C_2 \geq C$ and use (5.7) with the definition of the modified energy (5.5) to see that

$$\frac{1}{2}E^N(\mathbf{U}) \leq E^N_{IWW}(\mathbf{U}) \leq C_2 C E^N(\mathbf{U}).$$

Lastly, we will prove energy estimate (5.6) by considering two cases.

An energy estimate in the case $k = 0$. We first use (1.13) to make the decomposition

$$\partial_t\zeta = \frac{1}{\mu}\mathcal{G}_\mu[0]\psi + \varepsilon\mathcal{N}_1(\mathbf{U})$$
$$\partial_t\psi = -\big((1-\gamma) - \mathrm{bo}^{-1}\partial_x^2\big)\zeta + \varepsilon\mathcal{N}_2(\mathbf{U}),$$

where

$$\mathcal{N}_1(\mathbf{U}) = \frac{1}{\mu\varepsilon}(\mathcal{G}_\mu[\varepsilon\zeta] - \mathcal{G}_\mu[0])\psi,$$

and

$$\mathcal{N}_2(\mathbf{U}) = \frac{1}{\varepsilon\mathrm{bo}}\partial_x\Big(\big(\frac{1}{\sqrt{1+\varepsilon^2\mu(\partial_x\zeta)^2}} - 1\big)\partial_x\zeta\Big) - \frac{1}{2}\big((\partial_x\psi^+)^2 + \gamma(\partial_x\psi^-)^2\big) - \mathcal{N}(\mathbf{U}).$$

The decomposition emphasizes the terms of order ε, while the linear terms are canceled by our choice of the energy:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E^0(U) = \big(\Lambda^{\frac{7}{2}}\partial_t\zeta, \big((1-\gamma) - \mathrm{bo}^{-1}\partial_x^2\big)\Lambda^{\frac{7}{2}}\zeta\big)_{L^2} + \frac{1}{\mu}\big(\Lambda^{\frac{7}{2}}\partial_t\psi, \mathcal{G}_\mu[0]\Lambda^{\frac{7}{2}}\psi\big)_{L^2}$$
$$= \varepsilon\big(\Lambda^{\frac{7}{2}}\mathcal{N}_1(\mathbf{U}), \big((1-\gamma) - \mathrm{bo}^{-1}\partial_x^2\big)\Lambda^{\frac{7}{2}}\zeta\big)_{L^2} + \frac{\varepsilon}{\mu}\big(\Lambda^{\frac{7}{2}}\mathcal{N}_2(\mathbf{U}), \mathcal{G}_\mu[0]\Lambda^{\frac{7}{2}}\psi\big)_{L^2}.$$
$$= A_1 + A_2.$$

For the estimate on A_1, we use Plancherel's identity, Cauchy-Schwarz inequality, and (3.22) to find that

$$|A_1| \leq \varepsilon C|\partial_x\psi|_{H^{\frac{7}{2}}}|\Lambda^{\frac{9}{2}}\zeta|_{H^1_{\gamma,\mathrm{bo}}}.$$

Then to estimate $\partial_x\psi$, we use (5.9). Combining these estimates implies

$$|A_1| \leq \varepsilon C\mathcal{E}^N(\mathbf{U}).$$

For the estimate on A_2, we integrate by parts and use the algebra property of $H^{\frac{7}{2}}(\mathbb{R})$ to find that

$$|A_2| \leq \frac{\varepsilon}{\mu}\mathrm{bo}^{-1}C|\partial_x\zeta|_{H^{\frac{7}{2}}}|\mathcal{G}_\mu[0]\psi|_{H^{\frac{7}{2}}} + \frac{\varepsilon}{\mu}(|\partial_x\psi^+|_{H^{\frac{7}{2}}} + |\partial_x\psi^-|_{H^{\frac{7}{2}}} + |\mathcal{N}(\mathbf{U})|_{H^{\frac{7}{2}}})|\mathcal{G}_\mu[0]\psi|_{H^{\frac{7}{2}}},$$

where we recall the definition

$$\mathcal{N}[\varepsilon\zeta, \psi] = \frac{1}{2\mu}\frac{\gamma(\mathcal{G}^-_\mu[\varepsilon\zeta]\psi^- + \varepsilon\mu\partial_x\zeta\partial_x\psi^-)^2 - (\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+ + \varepsilon\mu\partial_x\zeta\partial_x\psi^+)^2}{(1+\varepsilon^2\mu(\partial_x\zeta)^2)}.$$

To conclude, we simply use (B.1) to control $\frac{1}{\mu}\mathcal{G}_\mu[0]\psi$, while for the remaining quantities we use Corollary A.18 and estimates (A.40) with (5.9) to get that

$$|\partial_x\psi^+|^2_{H^{\frac{7}{2}}} + |\partial_x\psi^-|^2_{H^{\frac{7}{2}}} \leq C|\partial_x\psi|^2_{H^{\frac{7}{2}}} \leq C\mathcal{E}^N(\mathbf{U}).$$

Moreover, to deal with the terms in $\mathcal{N}(\mathbf{U})$ we use the algebra property of $H^{\frac{7}{2}}(\mathbb{R})$, estimate (A.42) to obtain

$$|\mathcal{N}(\mathbf{U})|_{H^{\frac{7}{2}}} \leq C(\frac{\gamma}{\mu}|w^-|^2_{H^{\frac{7}{2}}} + \frac{1}{\mu}|w^+|^2_{H^{\frac{7}{2}}}) \leq C|\partial_x\psi|^2_{H^{\frac{9}{2}}} \leq C\mathcal{E}^N(\mathbf{U}),$$

where in the last estimate we argued as in (5.9) and used the assumption $\varepsilon\sqrt{\mu} = \mathrm{bo}^{-1}$. Gathering all these estimates, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}E^0(U) \leq \varepsilon C\mathcal{E}^N(\mathbf{U}). \tag{5.11}$$

An energy estimate in the case $1 \leq k \leq N$. We first derive an estimate on each $E_\alpha(\mathbf{U})$ defined by (5.3) for $1 \leq |\alpha| \leq N$. To do so, we use the self-adjointness of $Q(\mathbf{U})$ to find that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha(\mathbf{U}) &= \frac{1}{2}\big(\mathbf{U}_{(\alpha)}, \big(\partial_t Q(\mathbf{U})\big)\mathbf{U}_{(\alpha)}\big)_{L^2} + \big(\partial_t\mathbf{U}_{(\alpha)}, Q(\mathbf{U})\mathbf{U}_{(\alpha)}\big)_{L^2}\\ &= B_1 + B_2.\end{aligned}$$

Control of B_1. By definition of $Q(\mathbf{U})$ given by (5.2), we find that

$$\begin{aligned}B_1 &= \frac{1}{2}\big([\partial_t, \mathfrak{Ins}(\mathbf{U})]\zeta_{(\alpha)}, \zeta_{(\alpha)}\big)_{L^2} + \frac{1}{2\mu}\big([\partial_t, \mathcal{G}]\psi_{(\alpha)}, \psi_{(\alpha)}\big)_{L^2}\\ &\quad + \frac{1}{2}\big((\partial_t a)\zeta_{(\alpha)}, \zeta_{(\alpha)}\big)_{L^2} + \frac{1}{2}\big((\partial_t b)\zeta_{(\alpha)}, \Lambda^{-1}\zeta_{(\alpha)}\big)_{L^2}\\ &= B_1^1 + B_1^2 + B_1^3 + B_1^4.\end{aligned}$$

For the estimate on B_1^1, we have by definition and integration by parts that

$$\begin{aligned}B_1^1 &= \frac{1}{2}\big(\zeta_{(\alpha)}, (\partial_t\mathfrak{a})\zeta_{(\alpha)}\big)_{L^2} + \frac{1}{2\mathrm{bo}}\big(\partial_x\zeta_{(\alpha)}, \big[\partial_t, \mathcal{K}[\varepsilon\sqrt{\mu}\partial_x\zeta]\big]\partial_x\zeta_{(\alpha)}\big)_{L^2}\\ &\quad - (1-\gamma)\gamma\varepsilon^2\mu\big(\big(\partial_t[\![\underline{V}^\pm]\!]\big)\zeta_{(\alpha)}, \mathfrak{E}_\mu[\varepsilon\zeta]\big(\zeta_{(\alpha)}[\![\underline{V}^\pm]\!]\big)\big)_{L^2}\\ &\quad - \frac{1}{2}(1-\gamma)\gamma\varepsilon^2\mu\big([\![\underline{V}^\pm]\!]\zeta_{(\alpha)}, \big[\partial_t, \mathfrak{E}_\mu[\varepsilon\zeta]\big]\big(\zeta_{(\alpha)}[\![\underline{V}^\pm]\!]\big)\big)_{L^2}\\ &= B_1^{1,1} + B_1^{1,2} + B_1^{1,3} + B_1^{1,4}.\end{aligned}$$

For the estimate on $B_1^{1,1}$, we have by definition of $\mathfrak{a}$ given by (4.1) and estimates: Hölder's inequality, Sobolev embedding with $s > \frac{1}{2}$, and (B.6) to find that

$$\begin{aligned}|B_1^{1,1}| &\leq \frac{1}{2}|\partial_t\mathfrak{a}|_{H^s}|\zeta_{(\alpha)}|^2_{L^2}\\ &\leq \varepsilon(|\partial_t^2\underline{w}^+|_{H^s} + |\underline{V}^+|_{H^s}|\partial_t\partial_x\underline{w}^+|_{H^s})|\zeta_{(\alpha)}|^2_{L^2}\\ &\quad + \varepsilon(|\partial_t^2\underline{w}^-|_{H^s} + |\underline{V}^-|_{H^s}|\partial_t\partial_x\underline{w}^-|_{H^s})|\zeta_{(\alpha)}|^2_{L^2}.\end{aligned}$$

For $\partial_t^2 \underline{w}^+$ we can use that it also satisfies estimate (A.12) and use it with (A.42) and (A.40) to find that

$$|\partial_t^2 \underline{w}^+|_{H^1} \leq \sum_{j=1}^{2} \mu^{\frac{3}{4}} |\partial_t^j \zeta|_{H^2} |\partial_t^{2-j}\psi|_{\dot{H}^2_\mu}.$$

Then estimate the time derivative of ψ by using the definition of $\psi_{(\alpha)}$ and argue as we did for the estimate on (5.9) to find that

$$|\partial_t^2 \psi|^2_{\dot{H}^2_\mu} \leq \sum_{\alpha\in\mathbb{N}^2\,:\,|\alpha|\leq N-1} |\psi_{(\alpha)}|^2_{\dot{H}^{\frac{1}{2}}_\mu} + \varepsilon|\underline{w}\zeta_{(\alpha)}|^2_{\dot{H}^{\frac{1}{2}}_\mu} \leq C\mathcal{E}^N(\mathbf{U}).$$

The same can be done for $\underline{w}^-$, and combine it (A.45) we conclude that

$$|B_1^{1,1}| \leq \varepsilon C\mathcal{E}^N(\mathbf{U}).$$

For the estimate on $B_1^{1,2}$, it only depends on $\partial_t\mathcal{K}[\varepsilon\sqrt{\mu}\partial_x\zeta]$ which is in $L^\infty(\mathbb{R})$. Indeed, it follows from an estimate on $\sqrt{\mu}\partial_x\partial_t\zeta$, where we let $s > \frac{1}{2}$ and use the Sobolev embedding and (2.4):

$$\sqrt{\mu}|\partial_x\partial_t\zeta|_{H^s} \leq \frac{1}{\sqrt{\mu}}|\mathcal{G}_\mu\psi|_{H^{s+1}} \leq C|\partial_x\psi|_{H^{s+1}}.$$

As a result, we obtain by Hölder's inequality and the definition of the energy that

$$|B_1^{1,2}| \leq \varepsilon C\mathcal{E}^N(\mathbf{U}).$$

For the estimate on $B_1^{1,3}$, we use estimates (4.14), (4.24), and (4.21):

$$\begin{aligned} B_1^{1,3} &\leq (1-\gamma)\gamma\varepsilon^2\mu|(1+\sqrt{\mu}|\mathrm{D}|)^{\frac{1}{2}}(\zeta_{(\alpha)}[\![\underline{V}^\pm]\!])|^2_{L^2} \\ &\leq \varepsilon C\mathcal{E}^N(\mathbf{U}). \end{aligned}$$

Likewise, the estimate on $B_1^{1,4}$ we simply use (4.15) instead of (4.14) to obtain the same bound and we deduce that

$$|B_1^1| \leq \varepsilon C\mathcal{E}^N(\mathbf{U}).$$

Clearly, the estimates on B_1^3 and B_1^4 is estimated similarly to $B_1^{1,1}$, while we estimate B_1^2 we have by direct computations that

$$|B_1^2| \leq \frac{1}{2\mu}\big([\partial_t, \mathcal{G}_\mu^+](\mathcal{J}_\mu)^{-1}\psi_{(\alpha)}, \psi_{(\alpha)}\big)_{L^2} + \frac{1}{2\mu}\big(\mathcal{G}_\mu^+(\mathcal{J}_\mu)^{-1}(\mathrm{d}\mathcal{J}_\mu(\partial_t\zeta))(\mathcal{J}_\mu)^{-1}\psi_{(\alpha)}, \psi_{(\alpha)}\big)_{L^2}.$$

For the first term we use Proposition 3.6 of [8] with (2.23), while the for the second term we also use (A.6), (2.23), and then (A.55) to find that

$$|B_1^2| \leq \varepsilon C\mathcal{E}^N(\mathbf{U}).$$

Collecting each estimate so far gives

$$|B_1| \leq \varepsilon C\mathcal{E}^N(\mathbf{U}).$$

Control of B_2. We use system (4.7) to find that

$$\begin{aligned} B_2 &= -\big(\mathcal{A}[\mathbf{U}]\mathbf{U}_{(\alpha)}, Q(\mathbf{U})\mathbf{U}_{(\alpha)}\big)_{L^2} - \big(\mathcal{B}[\mathbf{U}]\mathbf{U}_{(\alpha)}, Q(\mathbf{U})\mathbf{U}_{(\alpha)}\big)_{L^2} \\ &\quad - \big(\mathcal{C}[\mathbf{U}]\mathbf{U}_{\langle\check{\alpha}\rangle}, Q(\mathbf{U})\mathbf{U}_{(\alpha)}\big)_{L^2} + \varepsilon\big((R_\alpha, S_\alpha)^T, Q(\mathbf{U})\mathbf{U}_{(\alpha)}\big)_{L^2} \\ &= B_2^1 + B_2^2 + B_2^3 + B_2^4. \end{aligned}$$

Control of B_2^1. From Definition 4.1 and (5.2) we have that

$$\big(\mathcal{A}[\mathbf{U}]\mathbf{U}_{(\alpha)}, Q_1(\mathbf{U})\mathbf{U}_{(\alpha)}\big)_{L^2} = 0.$$

We may therefore decompose B_2^1 into two pieces:

$$\begin{aligned} B_2^1 &= -\big(\mathcal{A}[\mathbf{U}]\mathbf{U}_{(\alpha)}, Q_2(\mathbf{U})\mathbf{U}_{(\alpha)}\big)_{L^2} \\ &= a(\mathbf{U})\big(\frac{1}{\mu}\mathcal{G}_\mu\psi_{(\alpha)}, \zeta_{(\alpha)}\big)_{L^2} + b(\mathbf{U})\big(\frac{1}{\mu}\mathcal{G}_\mu\psi_{(\alpha)}, \Lambda^{-1}\zeta_{(\alpha)}\big)_{L^2}. \end{aligned}$$

Both terms are estimated directly by (2.8) and then use (4.21):

$$\begin{aligned} |B_2^1| &\leq M|\psi_{(\alpha)}|_{\dot{H}^{\frac{1}{2}}_\mu}\big(a(\mathbf{U})|\zeta_{(\alpha)}|_{\dot{H}^{\frac{1}{2}}_\mu} + b(\mathbf{U})|\Lambda^{-1}\zeta_{(\alpha)}|_{\dot{H}^{\frac{1}{2}}_\mu}\big) \\ &\leq M|\psi_{(\alpha)}|_{\dot{H}^{\frac{1}{2}}_\mu}\big(\varepsilon^2|\partial_x\zeta_{(\alpha)}|_{L^2} + b(\mathbf{U})|\zeta_{(\alpha)}|_{L^2}\big) \\ &\leq \varepsilon C\mathcal{E}^N(\mathbf{U}), \end{aligned}$$

where we used that $\varepsilon^2 \leq \varepsilon(\varepsilon\sqrt{\mu})^{\frac{1}{2}} = \varepsilon \mathrm{bo}^{-\frac{1}{2}}$.

Control of B_2^2. We first decompose B_2^2 into four parts

$$\begin{aligned} B_2^2 = {} & \varepsilon\big(\mathcal{I}[\mathbf{U}]\zeta_{(\alpha)}, \mathfrak{Ins}[\mathbf{U}]\zeta_{(\alpha)}\big)_{L^2} - \frac{\varepsilon}{\mu}\big(\mathcal{I}^*[\mathbf{U}]\psi_{(\alpha)}, \mathcal{G}_\mu\psi_{(\alpha)}\big)_{L^2} \\ & + \varepsilon a(\mathbf{U})\big(\mathcal{I}[\mathbf{U}]\zeta_{(\alpha)}, \zeta_{(\alpha)}\big)_{L^2} + \varepsilon b(\mathbf{U})\big(\mathcal{I}[\mathbf{U}]\zeta_{(\alpha)}, \Lambda^{-1}\zeta_{(\alpha)}\big)_{L^2} \\ = {} & B_2^{2,1} + B_2^{2,2} + B_2^{2,3} + B_2^{2,4}. \end{aligned}$$

We first observe that the two last terms are easily treated with an estimate of the type (4.11) to get that

$$|B_2^{2,3}| + |B_2^{2,4}| \leq \varepsilon C\mathcal{E}^N(\mathbf{U}).$$

For the remaining two terms, we will need to work some more.

Control of $B_2^{2,1}$. Clearly, by (4.21) it is enough to prove that

$$|B_2^{2,1}| \leq \mathcal{E}^N(\mathbf{U})\big(1 + \varepsilon^2\sqrt{\mu}|[\![\underline{V}^\pm]\!]|^2_{W^{1,\infty}}|\zeta|^2_{<N+\frac{1}{2}>}\big). \tag{5.12}$$

Since the quantities involved in this depend on $(\mathcal{G}_\mu^-)^{-1}$, we need to adapt the proof of [50]. However, the proof relies on the symbolic expression of $(\mathcal{J}_\mu)^{-1}(\mathcal{G}_\mu^-)^{-1}\partial_x$, and this estimate is provided in Corollary 3.8 which has the same outcome as the one in [50]. In particular, we refer the reader to the proof of estimate (5.15) in this paper.

Control of $B_2^{2,2}$. This estimate is similar to estimate (5.16) in [50], but we give the details here to account for the difference in $\mathcal{G}_\mu^-$. In particular, we have by definition (4.3), that we must provide an estimate on the terms

$$\begin{aligned} B_2^{2,2} &= \frac{\varepsilon}{\mu}\big(\underline{V}^+\partial_x\psi_{(\alpha)}, \mathcal{G}_\mu\psi_{(\alpha)}\big)_{L^2} + \gamma\frac{\varepsilon}{\mu}\big([\![\underline{V}^\pm]\!]\partial_x\big((\mathcal{G}_\mu^-)^{-1}\mathcal{G}_\mu\psi_{(\alpha)}\big), \mathcal{G}_\mu\psi_{(\alpha)}\big)_{L^2} \\ &= B_2^{2,2,1} + B_2^{2,2,2}. \end{aligned}$$

We let $g = \mathcal{J}_\mu^{-1}\psi_{(\alpha)}$ and use the definition $\mathcal{G}_\mu = \mathcal{G}_\mu^+\mathcal{J}_\mu^{-1}$ to find that

$$\begin{aligned}
B_2^{2,2,1} &= \frac{\varepsilon}{\mu}\big(\underline{V}^+\partial_x\mathcal{J}_\mu g, \mathcal{G}_\mu^+ g\big)_{L^2} \\
&= \frac{\varepsilon}{\mu}\big(\underline{V}^+\partial_x g, \mathcal{G}_\mu^+ g\big)_{L^2} - \gamma\frac{\varepsilon}{\mu}\big(\underline{V}^+\partial_x(\mathcal{G}_\mu^-)^{-1}\mathcal{G}_\mu^+ g, \mathcal{G}_\mu^+ g\big)_{L^2} \\
&= B_2^{2,2,1,1} + B_2^{2,2,1,2}.
\end{aligned}$$

For the estimate on $B_2^{2,2,1,1}$, we use (A.10), combined with Sobolev embedding, (2.23), and (A.45) to obtain,

$$\begin{aligned}
|B_2^{2,2,1,1}| &\leq \varepsilon M|\underline{V}^+|_{W^{1,\infty}}|g|^2_{\dot{H}_\mu^{\frac{1}{2}}} \\
&\leq \varepsilon M|\underline{V}^+|_{H^2}|\psi_{(\alpha)}|^2_{\dot{H}_\mu^{\frac{1}{2}}} \\
&\leq \varepsilon C\mathcal{E}^N(\mathbf{U}).
\end{aligned}$$

For the estimate on $B_2^{2,2,1,2}$, we let $\tilde{g} = (\mathcal{G}_\mu^-)^{-1}\mathcal{G}_\mu^+\psi_{(\alpha)}$ and use (A.27). It is then straightforward to obtain the desired bound using also (2.11):

$$\begin{aligned}
|B_2^{2,2,1,1}| &\leq \gamma\frac{\varepsilon}{\mu}|\big(\underline{V}^+\partial_x\tilde{g}, \mathcal{G}_\mu^-\tilde{g}\big)_{L^2}| \\
&\leq \varepsilon\mu^{-\frac{1}{2}}M|\underline{V}^+|_{W^{1,\infty}}|\tilde{g}|^2_{\mathring{H}^{\frac{1}{2}}} \\
&\leq \varepsilon C\mathcal{E}^N(\mathbf{U}).
\end{aligned}$$

Lastly, for the estimate on $B_2^{2,2,2}$ we let $g^\sharp = (\mathcal{G}_\mu^-)^{-1}\mathcal{G}_\mu\psi_{(\alpha)}$, and again use (A.27) with (2.4) and (2.23):

$$\begin{aligned}
|B_2^{2,2,2}| &= \gamma\frac{\varepsilon}{\mu}|\big([\![\underline{V}^\pm]\!]\partial_x g^\sharp, \mathcal{G}_\mu^- g^\sharp\big)_{L^2}| \\
&\leq \varepsilon\mu^{-\frac{1}{2}}M|(\mathcal{G}_\mu^-)^{-1}\mathcal{G}_\mu^+\mathcal{J}_\mu^{-1}\psi_{(\alpha)}|^2_{\mathring{H}^{\frac{1}{2}}} \\
&\leq \varepsilon C\mathcal{E}^N(\mathbf{U}).
\end{aligned}$$

From these estimates, we conclude the bound on B_2^2 where we find that

$$|B_2^2| \leq \varepsilon C\mathcal{E}^N(\mathbf{U}).$$

Control of B_2^3. To get an estimate of order ε we need to cancel the following term (when $k = N$):

$$B_2^{3,1} := \big(\mathcal{C}[\mathbf{U}]\mathbf{U}_{\langle\check{\alpha}\rangle}, Q_1(\mathbf{U})\mathbf{U}_{(\alpha)}\big)_{L^2} = -\big(\mathcal{A}[\mathbf{U}]\mathbf{U}_{(\alpha)}, \tilde{Q}(\mathbf{U})\mathbf{U}_{\langle\check{\alpha}\rangle}\big)_{L^2}.$$

We will cancel $B_2^{3,1}$ later by introducing $\tilde{Q}$ into the energy (see Remark 5.1). Therefore, we may decompose $(B_2^3 - B_2^{3,1})$ into two parts:

$$\begin{aligned}
(B_2^3 - B_2^{3,1}) &= \frac{1}{\mu}a(\mathbf{U})\big(\mathcal{G}_{\mu,(\alpha)}\psi_{\langle\check{\alpha}\rangle}, \zeta_{(\alpha)}\big)_{L^2} + \frac{1}{\mu}b(\mathbf{U})\big(\mathcal{G}_{\mu,(\alpha)}\psi_{\langle\check{\alpha}\rangle}, \Lambda^{-1}\zeta_{(\alpha)}\big)_{L^2} \\
&= B_2^{3,2} + B_2^{3,3}.
\end{aligned}$$

However, these terms are easily dealt with using estimates on $\mathcal{G}_\mu^+$ and $(\mathcal{J}_\mu)^{-1}$ (similar to (5.10)) and that $\varepsilon \leq \mathrm{bo}^{-\frac{1}{2}}$ to obtain

$$|B_2^{3,2}| + |B_2^{3,3}| \leq \varepsilon C\mathcal{E}^N(\mathbf{U}).$$

Control of B_2^4. For B_2^4 we have to estimate the following terms,

$$B_2^4 = \varepsilon\big(R_\alpha, \mathfrak{Ins}[\mathbf{U}]\zeta_{(\alpha)}\big)_{L^2} + \frac{\varepsilon}{\mu}\big(S_\alpha, \mathcal{G}_\mu\psi_{(\alpha)}\big)_{L^2}$$
$$+ \varepsilon a(\mathbf{U})\big(R_\alpha, \zeta_{(\alpha)}\big)_{L^2} + \varepsilon b(\mathbf{U})\big(R_\alpha, \Lambda^{-1}\zeta_{(\alpha)}\big)_{L^2}.$$

For the first term, we use (4.17) and (4.8) to obtain,

$$\varepsilon|\big(R_\alpha, \mathfrak{Ins}[\mathbf{U}]\zeta_{(\alpha)}\big)_{L^2}| \leq \varepsilon C\mathcal{E}^1(\mathbf{U})|R_\alpha|_{H^1_{\gamma,\mathrm{bo}}} \leq \varepsilon C\mathcal{E}^N(\mathbf{U}).$$

For the second term is treated by (2.8), (4.8), and (4.21) to get

$$\frac{\varepsilon}{\mu}|\big(S_\alpha, \mathcal{G}_\mu\psi_{(\alpha)}\big)_{L^2}| \leq \varepsilon M|S_\alpha|_{\dot{H}^{\frac{1}{2}}}|\psi_{(\alpha)}|_{\dot{H}^{\frac{1}{2}}} \leq \varepsilon C\mathcal{E}^N(\mathbf{U}).$$

The remaining two estimates satisfy the same upper bound, and so gathering all these estimates, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha(\mathbf{U}) - B_1^{3,1} \leq \varepsilon C\mathcal{E}^N(\mathbf{U}). \tag{5.13}$$

As pointed out above, we need to cancel $B_1^{3,1}$ and is done by modifying the energy by

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{U}_{(\alpha)}, \tilde{Q}(\mathbf{U})\mathbf{U}_{\langle\check{\alpha}\rangle}\big)_{L^2} &= \frac{1}{2}\big(\mathbf{U}_{(\alpha)}, \big(\partial_t\tilde{Q}(\mathbf{U})\big)\mathbf{U}_{\langle\check{\alpha}\rangle}\big)_{L^2} - \big(\mathcal{A}[\mathbf{U}]\mathbf{U}_{(\alpha)}, \tilde{Q}(\mathbf{U})\mathbf{U}_{\langle\check{\alpha}\rangle}\big)_{L^2}\\ &\quad - \big(\mathcal{B}[\mathbf{U}]\mathbf{U}_{(\alpha)}, \tilde{Q}(\mathbf{U})\mathbf{U}_{\langle\check{\alpha}\rangle}\big)_{L^2} - \big(\mathcal{C}[\mathbf{U}]\mathbf{U}_{\langle\check{\alpha}\rangle}, \tilde{Q}(\mathbf{U})\mathbf{U}_{\langle\check{\alpha}\rangle}\big)_{L^2}\\ &\quad + \varepsilon\big((R_\alpha, S_\alpha)^T, \tilde{Q}(\mathbf{U})\mathbf{U}_{\langle\check{\alpha}\rangle}\big)_{L^2}\\ &= D_1 + D_2^1 + D_2^2 + D_2^3 + D_2^4.\end{aligned}$$

Where we observe that $D_2^1 = -B_2^{3,1}$ and gives the desired cancellation (when $k = N$), while the remaining terms are treated as above and satisfy

$$|D_1| + |D_2^2| + |D_2^3| + |D_2^4| \leq \varepsilon C\mathcal{E}^N(\mathbf{U}).$$

We therefore find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(E_\alpha(\mathbf{U}) + \big(\mathbf{U}_{(\alpha)}, \tilde{Q}(\mathbf{U})\mathbf{U}_{\langle\check{\alpha}\rangle}\big)_{L^2}\Big) \leq \varepsilon C\mathcal{E}^N(\mathbf{U}), \tag{5.14}$$

where we recall that $\tilde{Q} = 0$ when $|\alpha| < N$.

Proof of estimate (5.6). To conclude, we use (5.11) combined with (5.14) where we sum over $0 \leq |\alpha| \leq N$ and then apply estimate (5.8) to find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E^N_{\mathrm{IWW}}(\mathbf{U}) \leq \varepsilon CE^N_{\mathrm{IWW}}(\mathbf{U}).$$

□

6. Proof of Theorem 1.7

The strategy for the proof of Theorem 1.7 is classical, and we refer the reader to [51] Chapter 4 for a detailed proof in the case of the water waves equations. See also Chapter 9 in the same book in the case of the water waves equations with surface tension. We will now give the main steps involved.

Proof. The first step is to regularize the internal water waves equations (1.13), and use the Fixed Point Theorem to deduce the existence of a solution. In fact, the solution is smooth. However, the existence time depends on the regularization parameter and shrinks to zero when taking the limit. To extend the existence time, we verify non-cavitatation condition and the stability criteria on a long-time scale using the Fundamental Theorem of Calculus. Then one applies Proposition 5.2 and uses compactness to deduce a limit.

For uniqueness, one needs to have an estimate of the difference between two solutions, which can be proved with estimates similar to the ones used for the proof of Proposition 5.2. □

7. Proof of Theorem 1.14

Since the long-time well-posedness of the unidirectional model (1.25) is classical (see Remark 1.16) we only give the proof for (1.27). The strategy of the proof is the same as for (1.13). It relies on the energy method, where we need to find a suitable symmetrizer for the system. For simplicity, let $\mathbf{U} = (\zeta, v)^T$, and we write (1.27) on the compact form:

$$\partial_t \mathbf{U} + \mathcal{M}(\mathbf{U})\mathbf{U} = \mathbf{0}, \tag{7.1}$$

with

$$\mathcal{M}(\mathbf{U}) = \begin{pmatrix} \varepsilon v \partial_x & (h - \gamma \tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x \\ (1-\gamma)\partial_x & \varepsilon v \partial_x \end{pmatrix}, \tag{7.2}$$

where $h = 1 + \varepsilon\zeta$. To define an energy, we introduce the symmetrizer

$$\mathcal{Q}(\mathbf{U}) = \begin{pmatrix} (1-\gamma) & 0 \\ 0 & (h - \gamma \tanh(\sqrt{\mu}|\mathrm{D}|)) \end{pmatrix}. \tag{7.3}$$

Then the energy associated with the weakly dispersive BO system (1.27) can be written as

$$E^s_{\mathrm{BOs}}(\mathbf{U}) = \big(\Lambda^s \mathbf{U}, \mathcal{Q}(\mathbf{U})\Lambda^s \mathbf{U}\big)_{L^2}.$$

We will use this energy to deduce a bound on the solutions of (7.2). As noted in the proof Theorem 1.7, this is not enough. One also needs a uniqueness type estimate to establish the well-posedness with the energy method. This estimate is derived similarly, but since we will need it for the proof of the full justification, we also state it in the next result. For convenience, we let two solutions of (7.2) be given by $\mathbf{U}_1 = (\zeta_1, v_1)^T$ and $\mathbf{U}_2 = (\zeta_2, v_2)^T$ where we define $(\eta, w) = (\zeta_1 - \zeta_2, u_1 - u_2)$. Then $\mathbf{W} = (\eta, w)^T$ solves

$$\partial_t \mathbf{W} + \mathcal{M}(\mathbf{U}_1, D)\mathbf{W} = \mathbf{F}, \tag{7.4}$$

with $\mathcal{M}$ defined as in (7.2) and the source term is given by

$$\mathbf{F} = -\big(\mathcal{M}(\mathbf{U}_1) - \mathcal{M}(\mathbf{U}_2)\big)\mathbf{U}_2. \tag{7.5}$$

The energy associated to (7.4) is given in terms of the symmetrizer $\mathcal{Q}(\mathbf{U}_1)$ defined in (7.3) and reads

$$\tilde{E}^s_{\mathrm{BOs}}(\mathbf{W}) := \big(J^s \mathbf{W}, \mathcal{Q}(\mathbf{U}_1) J^s \mathbf{W}\big)_{L^2}. \tag{7.6}$$

From these energies, we have the following *a priori* estimate and an estimate on the difference between two solutions.

Proposition 7.1. *Let $\varepsilon, \mu, \gamma \in (0,1)$, $s > \frac{3}{2}$, and $(\zeta, v) \in C([0,T]; H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ be a solution to* (7.1) *on a time interval $[0,T]$ for some $T > 0$. Moreover, assume that* (1.24) *holds uniformly in time and define $K(s) = C_1(h^{-1}_{\min,\gamma}, |(\zeta, v)|_{H^s \times H^s}) > 0$. Then, for the energy given in Definition 5.4, there holds,* *1. For all $0 < t < T$, we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t}E^s_{BOs}(\mathbf{U}) \leq \varepsilon K(s) E_s(\mathbf{U}). \tag{7.7}$$

2. For all $0 < t < T$, there exist $C_2 > 0$ such that

$$C_2^{-1}|(\zeta, v)|^2_{H^s\times H^s} \leq E^s_{BOs}(\mathbf{U}) \leq C_2|(\zeta, v)|^2_{H^s\times H^s}. \tag{7.8}$$

Moreover, let $(\zeta_1, v_1), (\zeta_1, v_1) \in C([0,T]; H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ be solutions to (7.1) on a time interval $[0,T]$ and satisfying the condition (1.24) uniformly in time. Define the difference to be $\mathbf{W} = (\eta, w) = (\zeta_1 - \zeta_2, v_1 - v_2)$ and let $\tilde{K}(s) = C_3(h^{-1}_{\min,\gamma}, |(\zeta_i, v_i)|_{H^s\times H^s}) > 0$ for $i = 1, 2$. Then, for the energy defined by (7.6), there holds

3. For all $0 < t < T$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}^0_{BOs}(\mathbf{W}) \leq \varepsilon \tilde{K}(s)\tilde{E}^0_{BOs}(\mathbf{W}). \tag{7.9}$$

4. For all $0 < t < T$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}^s_{BOs}(\mathbf{W}) \leq \varepsilon \tilde{K}(s+1)\tilde{E}^s_{BOs}(\mathbf{W}). \tag{7.10}$$

5. For all $0 < t < T$ and $r \geq 0$, there exist $C_3 > 0$ such that

$$C_3^{-1}|(\eta, w)|^2_{H^r\times H^r} \leq \tilde{E}^r_{BOs}(\mathbf{W}) \leq C_3|(\eta, w)|^2_{H^r\times H^r}. \tag{7.11}$$

Remark 7.2. *In estimate (7.10), we observe that there is a loss of derivative. This is not sufficient for the proof of the continuous dependence with respect to initial data. In fact, one would have to refine this estimate using a Bona-Smith argument [16]. Since we are not concerned with the details of the well-posedness of this system, we simply allow this rough estimate and use it to deduce a the convergence estimate when comparing its solution with the ones of the internal water waves system.*

Proof of Proposition 7.1. The proof of point $3-5$ is similar to the proof of the first two points. We will therefore only prove (7.7) and (7.8).

We first prove estimate (7.8). By definition, we have that

$$E^s_{\mathrm{BOs}}(\mathbf{U}) = (1-\gamma)|\Lambda^s\zeta|^2_{L^2} + \big(\Lambda^s v, (h - \gamma\tanh(\sqrt{\mu}|\mathrm{D}|))\Lambda^s v\big)_{L^2}.$$

Then, as a result of the γ-dependent surface condition (1.24) and Plancherel's identity

$$\big(\Lambda^s v, (h - \gamma\tanh(\sqrt{\mu}|\mathrm{D}|))\Lambda^s v\big)_{L^2} \geq h_{\min,\gamma}|v|^2_{H^s},$$

since $0 \leq \tanh(\sqrt{\mu}|\xi|) \leq 1$. The reverse inequality is a consequence of Hölder's inequality, the Sobolev embedding with $s > \frac{3}{2}$:

$$\begin{aligned} E^s_{\mathrm{BOs}}(\mathbf{U}) &\leq |\zeta|^2_{H^s} + |\sqrt{\tanh(\sqrt{\mu}|\mathrm{D}|)}v|^2_{H^s} + (1 + \varepsilon|\zeta|_{L^\infty})|v|^2_{H^s} \\ &\leq K(s)|(\zeta, v)|^2_{H^s\times H^s}. \end{aligned}$$

Next, we prove (7.7). By using (7.1) and the fact that $\mathcal{Q}(\mathbf{U})$ is self-adjoint, we compute

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E^s_{\mathrm{BOs}}(\mathbf{U}) &= -\big(\Lambda^s\mathcal{M}(\mathbf{U})\mathbf{U}, \mathcal{Q}(\mathbf{U})\Lambda^s\mathbf{U}\big)_{L^2} + \frac{1}{2}\big(\Lambda^s\mathbf{U}, (\partial_t\mathcal{Q}(\mathbf{U}))\Lambda^s\mathbf{U}\big)_{L^2} \\ &= -I + II. \end{aligned}$$

<u>Control of I</u>. We may write

$$\begin{aligned} I &= \big([\Lambda^s, \mathcal{M}(\mathbf{U})]\mathbf{U}, \mathcal{Q}(\mathbf{U})\Lambda^s\mathbf{U}\big)_{L^2} + \big(\mathcal{Q}(\mathbf{U})\mathcal{M}(\mathbf{U})\Lambda^s\mathbf{U}, \Lambda^s\mathbf{U}\big)_{L^2} \\ &=: I_1 + I_2. \end{aligned}$$

Control of I_1. It follows from the Cauchy-Schwarz inequality that

$$|I_1| \leq |[\Lambda^s, \mathcal{M}(\mathbf{U})]\mathbf{U}|_{L^2}|\mathcal{Q}(\mathbf{U})\Lambda^s\mathbf{U}|_{L^2},$$

where the first term is treated using the commutator estimate (B.8) yields

$$\begin{aligned}|[\Lambda^s, \mathcal{M}(\mathbf{U})]\mathbf{U}|_{L^2} &\leq \varepsilon|[\Lambda^s, v]\partial_x\zeta|_{L^2} + \varepsilon|[\Lambda^s, \zeta]\partial_x v|_{L^2} + \varepsilon|[\Lambda^s, v]\partial_x v|_{L^2} \\ &\leq \varepsilon K(s)|(\zeta, v)|_{H^s\times H^s}.\end{aligned}$$

The second term is easily treated by Hölder's inequality, the Sobolev embedding with $s > \frac{1}{2}$:

$$|\mathcal{Q}(\mathbf{U})\Lambda^s\mathbf{U}|_{L^2} \lesssim |\Lambda^s\zeta|_{L^2} + \varepsilon|\zeta|_{L^\infty}|\Lambda^s v|_{L^2} \leq K(s)|(\zeta, v)|_{H^s\times H^s}.$$

The desired bound on I_1 then follows:

$$|I_1| \leq \varepsilon K(s)|(\zeta, v)|^2_{H^s\times H^s}.$$

Control of I_2. By definition, we must estimate the following terms,

$$\begin{aligned}&\big(\mathcal{Q}(\mathbf{U}, D)\mathcal{M}(\mathbf{U})\Lambda^s\mathbf{U}, \Lambda^s\mathbf{U}\big)_{L^2} \\ &= (1-\gamma)\varepsilon\big(v\partial_x\Lambda^s\zeta, \Lambda^s\zeta\big)_{L^2} + (1-\gamma)\big((h - \gamma\tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x\Lambda^s v, \Lambda^s\zeta\big)_{L^2} \\ &\quad + (1-\gamma)\big((h - \gamma\tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x\Lambda^s\zeta, \Lambda^s v\big)_{L^2} + \big((h - \gamma\tanh(\sqrt{\mu}|\mathrm{D}|))\Lambda^s v, \Lambda^s v\big)_{L^2} \\ &=: I_1^1 + I_1^2 + I_1^3 + I_2^4.\end{aligned}$$

Control of I_1^1. Using integration by part and the Sobolev embedding yields

$$|I_1^1| \leq (1-\gamma)\frac{\varepsilon}{2}\big|\big(\partial_x v\Lambda^s\zeta, \Lambda^s\zeta\big)_{L^2}\big| \leq \varepsilon K(s)|(\zeta, v)|^3_{H^s\times H^s}.$$

Control of $I_1^1 + I_1^2$. Observe, after integration by parts that

$$I_1^1 = -I_1^2 - \varepsilon\big((\partial_x\zeta)\Lambda^s v, \Lambda^s v\big)_{L^2}.$$

The first term cancels with I_1^2, while the Sobolev embedding easily controls the remaining part,

$$\varepsilon|\big((\partial_x\zeta)\Lambda^s v, \Lambda^s v\big)_{L^2}| \leq \varepsilon K(s)|(\zeta, v)|^3_{H^s\times H^s}.$$

<u>Control of II.</u> Using equation (7.1) gives us the following terms to estimate,

$$\begin{aligned}II &= \varepsilon\big(\Lambda^s v, (\partial_t\zeta)\Lambda^s v\big)_{L^2} \\ &= -\varepsilon\big(\Lambda^s v, \big((h - \gamma\tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x v\big)\Lambda^s v\big)_{L^2} - \varepsilon^2\big(\Lambda^s v, (\partial_x(\zeta v))\Lambda^s v\big)_{L^2}.\end{aligned}$$

Consequently, the desired estimate follows from Hölder's inequality and the Sobolev embedding

$$|II| \leq \varepsilon K(s)|(\zeta, v)|^2_{H^s\times H^s}. \tag{7.12}$$

Adding together all the estimates, combined with (7.8) yields,

$$\frac{\mathrm{d}}{\mathrm{d}t}E^s_{\mathrm{BOs}}(\mathbf{U}) \leq \varepsilon K(s)E^s_{\mathrm{BOs}}(\mathbf{U}),$$

and completes the proof of Proposition 7.1.

□

Proof of Theorem 1.14. The proof is an application of the energy method where we use the estimates in Proposition 7.1 and Remark 7.2. We refer the reader to [64] for a similar proof. □

8. Proof of Theorem 1.17

For the derivation of (1.1), we follow the strategy given in [15], where we formulated (1.13) in terms of the (ζ, v) solving system (1.22). The convenience of this formulation is apparent since we can use the shallow water expansion of $\mathcal{G}_\mu^+$ derived by Emerald in [24][4]:

$$|\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+ - \partial_x\big(-\mu(1+\varepsilon\zeta)\mathrm{T}(\mathrm{D})\partial_x\psi^+\big)|_{H^s} \leq \mu^2\varepsilon M(s+3)|\partial_x\psi^+|_{H^{s+3}}, \tag{8.2}$$

where $\mathrm{T}(\mathrm{D}) = \frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{\sqrt{\mu}|\mathrm{D}|}$ and for $s \geq 0$. Finally, the last ingredient before we turn to the proof is an expansion of the interface operator defined by (1.19).

Proposition 8.1. *Let $t_0 \geq 1$, $s \geq 0$, and $\zeta \in H^{s+3}(\mathbb{R})$ such that it satisfies (1.14). Then for all $\psi^+ \in \dot{H}^{s+4}(\mathbb{R})$ there holds,*

$$\Big|\mathbf{H}_\mu[\varepsilon\zeta]\psi^+ + \sqrt{\mu}|\mathrm{D}|\partial_x\psi^+\Big|_{H^{s-\frac{1}{2}}} \leq (\mu^{\frac{3}{2}} + \sqrt{\mu}\varepsilon)M(s+3)|\partial_x\psi^+|_{H^{s+3}}. \tag{8.3}$$

Remark 8.2. *The expansion of $\mathbf{H}_\mu[\varepsilon\zeta]$ is the same as the one mentioned in* [15] *(see Remark* 20 *of this paper). Even though the expansion they gave was formal, it is straightforward to adapt their method in finite depth to the current configuration.*

Proof of Proposition 8.1. The proof relies on making an approximate solution of $\phi^- = \Phi^- \circ \Sigma^-$ solving (1.20). In particular, we know that

$$\phi^-_{\mathrm{app}} = -\sqrt{\mu}|\mathrm{D}|\mathrm{e}^{-z\sqrt{\mu}|\mathrm{D}|}\psi^+,$$

solves

$$\begin{cases} (\mu\partial_x^2 + \partial_z^2)\phi^-_{\mathrm{app}} = 0, \\ \partial_z\phi^-_{\mathrm{app}\ |z=0} = -\mu\partial_x^2\psi^+, \end{cases} \tag{8.4}$$

and satisfies the decay estimates (2.14) by using Plancherel's identity. Then defining the difference $u = \phi^- - \phi^-_{\mathrm{app}}$, we have that it solves

$$\begin{cases} \nabla^\mu_{x,z}\cdot P(\Sigma^-)\nabla^\mu_{x,z}u = \varepsilon\mu r_1 \quad \text{in} \quad \mathcal{S}^- \\ \partial_n^{P^-}u|_{z=0} = \mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+ + \mu\partial_x^2\psi^+ + \varepsilon\mu r_2, \quad \lim\limits_{z\to\infty}|\nabla^\mu_{x,z}u| = 0, \end{cases}$$

where r_1 reads

$$r_1 = \partial_x\big((\partial_x\zeta)\partial_z\phi^-_{\mathrm{app}}\big) + (\partial_x\zeta)\partial_x\partial_z\phi^-_{\mathrm{app}} - \varepsilon\mu(\partial_x\zeta)^2\partial_z^2\phi^-_{\mathrm{app}},$$

and the second rest is given by

$$r_2 = (\partial_x\zeta)\partial_x\psi^+ + \varepsilon\mu(\partial_x\zeta)^2|\mathrm{D}|\psi^+.$$

Then arguing similarly as in Step 2. of the proof of Proposition 2.4, we find that

$$\|\Lambda^s\nabla^\mu_{x,z}u\|_{L^2(\mathcal{S}^-)} \leq M\big(\mu\varepsilon\|\Lambda^s r_1\|_{L^2(\mathcal{S}^-)} + |\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+ + \mu\partial_x^2\psi^+|_{H^s} + \mu\varepsilon|r_2|_{H^s}\big).$$

For the estimate on r_1 we use (3.8) to deal with the integrability on $\mathcal{S}^-$, while the estimate for r_2 is straightforward since we allow for loss of derivatives. Lastly, we use (8.1) to approximate $\mathcal{G}_\mu^+$. From these estimates, we get the following bound on the gradient

$$\|\Lambda^s\nabla^\mu_{x,z}u\|_{L^2(\mathcal{S}^-)} \leq (\mu^2 + \mu\varepsilon)|\partial_x\psi^+|_{H^{s+3}}.$$

[4] The classical expansion of $\mathcal{G}_\mu^+$ is provided by Proposition 3.8 in [8]:

$$|\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+ - \partial_x\big(-\mu(1+\varepsilon\zeta)\partial_x\psi^+\big)|_{H^s} \leq \mu^2 M(s+3)|\partial_x\psi^+|_{H^{s+3}}. \tag{8.1}$$

To conclude, we use trace estimate (2.15) and the construction to find that

$$\begin{aligned}|\partial_x u|_{z=0}|_{H^{s-\frac{1}{2}}} &\leq |u|_{z=0}|_{\mathring{H}^{s+\frac{1}{2}}}\\ &\leq \mu^{-\frac{1}{2}}\|\Lambda^s\nabla^\mu_{x,z}u\|_{L^2(\mathcal{S}^-)}\\ &\leq (\mu^{\frac{3}{2}}+\sqrt{\mu}\varepsilon)|\partial_x\psi^+|_{H^{s+3}}.\end{aligned}$$

□

Having these two expansions at hand, we can turn to the main proof of the section.

Proof of Theorem 1.17. We first observe that we can have bound on $v = \partial_x\psi$ in terms of the intial data. Indeed, using the definition of $\psi_{(\alpha)}$ with (B.6) and (A.42) yields,

$$\begin{aligned}|\partial_x\psi|_{H^{N-\frac{1}{2}}} \leq |\psi|_{\dot{H}^{N+\frac{1}{2}}_\mu} &\leq \sum_{\alpha\in\mathbb{N}^2\,:\,|\alpha|\leq N} |\psi_{(\alpha)}|_{\dot{H}^{\frac{1}{2}}_\mu} + \varepsilon|\underline{w}\zeta_{(\alpha)}|_{\dot{H}^{\frac{1}{2}}_\mu}\\ &\leq \sum_{\alpha\in\mathbb{N}^2\,:\,|\alpha|\leq N} |\psi_{(\alpha)}|_{\dot{H}^{\frac{1}{2}}_\mu} + M|\psi|_{\dot{H}^2}|\zeta_{(\alpha)}|_{H^1_{\gamma,\mathrm{bo}}}\\ &\leq C(\mathcal{E}^N(\mathbf{U})).\end{aligned}$$

Then Theorem 1.7 with estimate (1.18) implies

$$\sup_{t\in[0,\varepsilon^{-1}T]}\big(|\zeta|^2_{H^{N-\frac{1}{2}}} + |v|^2_{H^{N-\frac{1}{2}}}\big) \leq C(\mathcal{E}^N(\mathbf{U}^0)). \tag{8.5}$$

With this estimate in mind, we now give the proof in two steps where we first derive (1.17) in the sense of consistency.

Step 1. To derive (1.17) we let R be some generic function satisfying

$$|R|_{H^{N-5}} \leq C(\mathcal{E}^N(\mathbf{U}_0)). \tag{8.6}$$

Then we can simplify (1.22). In particular, for the second equation in (1.13), we first estimate the nonlinear terms. For $\mathcal{N}[\varepsilon\zeta,\psi^\pm]$ we apply estimates (B.6), (A.25), (A.7), and (1.18) to obtain,

$$|\mathcal{N}[\varepsilon\zeta,\psi^\pm]|_{H^{N-5}} \leq \mu C(\mathcal{E}^N(\mathbf{U}_0)).$$

For the first equation, we use (8.2), and we have that

$$\begin{cases}\partial_t\zeta + \partial_x^2\mathrm{T}(\mathrm{D})\psi^+ + \varepsilon\partial_x(\zeta\partial_x\mathrm{T}(\mathrm{D})\psi^+) = \varepsilon\mu R\\ \partial_t v + \big((1-\gamma)-\mathrm{bo}^{-1}\partial_x^2\big)\partial_x\zeta + \frac{\varepsilon}{2}\partial_x\big((\partial_x\psi^+)^2 - \gamma(\mathbf{H}_\mu[\varepsilon\zeta]\psi^+)^2\big) = \mu\varepsilon R,\end{cases} \tag{8.7}$$

To relate the equation to v, we observe by (8.3) that

$$\begin{aligned}\partial_x\psi^+ &= \gamma\mathbf{H}_\mu[\varepsilon\zeta]\psi^+ + v\\ &= -\gamma\sqrt{\mu}|\mathrm{D}|\partial_x\psi^+ + v + (\mu^{\frac{3}{2}}+\sqrt{\mu}\varepsilon)R\\ &= (1-\gamma\sqrt{\mu}|\mathrm{D}|)v + (\mu^{\frac{3}{2}}+\sqrt{\mu}\varepsilon)R.\end{aligned}$$

From this relation, we can express the first equation in terms of ζ and v also using the expansions of $\mathcal{G}^+_\mu$ given by (8.2) and the expansion of T(D):

$$|(\mathrm{T}(\mathrm{D})-1)f|_{H^s} \leq \mu C|\partial_x f|_{H^{s+1}},$$

which is a direct consequence of Plancherel's identity and a Taylor expansion in low frequencies. Indeed, from these observations there holds,

$$\begin{aligned}\partial_x^2 \mathrm{T}(\mathrm{D})\psi^+ + \varepsilon\partial_x(\zeta\partial_x \mathrm{T}(\mathrm{D})\psi^+) = (1-\gamma\sqrt{\mu}|\mathrm{D}|\mathrm{T}(\mathrm{D}))\partial_x v + \varepsilon\partial_x(\zeta v)\\ + (\mu + \varepsilon\sqrt{\mu} + \varepsilon\mu)R.\end{aligned}$$

For the second equation, we have that

$$(\partial_x\psi^+)^2 = v^2 - 2\gamma v\sqrt{\mu}|\mathrm{D}|v + \mu R,$$

and

$$(\mathbf{H}_\mu[\varepsilon\zeta]\psi^+)^2 = \mu R.$$

Consequently, if we let $\mathrm{bo}^{-1} = \varepsilon\sqrt{\mu}$ we find the desired system

$$\begin{cases}\partial_t\zeta + (1-\gamma\tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x v + \varepsilon\partial_x(\zeta v) = (\mu+\varepsilon\sqrt{\mu})R\\ \partial_t v + (1-\gamma)\partial_x\zeta + \varepsilon v\partial_x v = (\mu+\varepsilon\sqrt{\mu})R.\end{cases}$$

Step 2. To derive (1.25) we can make an approximation v^{wBO} in terms of ζ^{wBO} at order $\mathcal{O}(\mu+\varepsilon^2)$ solving (1.25). Then we will show that this solution is consistent with the weakly dispersive BO system (1.27). The proof is given in two steps, where we first make formal computations and then prove the consistency once we have constructed v^{wBO}. To simplify the presentation further, we let $c = (1-\gamma)^{\frac{1}{2}}$ and introduce the change of variable $x = c\tilde{x}$ and $v = c\tilde{v}$ to obtain:

$$\begin{cases}\partial_t\zeta + (1-\gamma\tanh(\sqrt{\mu}|\mathrm{D}|))\partial_{\tilde{x}}\tilde{v} + \varepsilon\partial_{\tilde{x}}(\zeta\tilde{v}) = 0\\ \partial_t\tilde{v} + \partial_{\tilde{x}}\zeta + \varepsilon\tilde{v}\partial_{\tilde{x}}\tilde{v} = 0.\end{cases}$$

For simplicity, we also omit the tilde notation in the proof.

Step 2.1. *Formal derivation:* To construct the lowest order approximation of v^{wBO} in terms of the solution ζ^{wBO}, we let $\sqrt{\mu} = \mathcal{O}(\varepsilon)$ and first consider system (1.27) at order ε:

$$\begin{cases}\partial_t\zeta^{\mathrm{wBO}} + \partial_x v^{\mathrm{wBO}} = 0\\ \partial_t v^{\mathrm{wBO}} + \partial_x\zeta^{\mathrm{wBO}} = 0.\end{cases}$$

The system is reduced to a wave equation with speed one. We therefore choose the right-moving solution where v^{wBO} is equal to ζ^{wBO} at first order. Then having an approximation at first order we can now make a correction at higher order:

$$v^{\mathrm{wBO}} = \zeta^{\mathrm{wBO}} + \sqrt{\mu}A + \varepsilon B,$$

for some functions A and B depending on the solution ζ^{wBO}. In particular, we construct A and B by plugging the ansatz into (1.27):

$$\begin{cases}\partial_t\zeta^{\mathrm{wBO}} + \partial_x\zeta^{\mathrm{wBO}} + \big(\sqrt{\mu}\partial_x A - \gamma\tanh(\sqrt{\mu}|\mathrm{D}|)\partial_x\zeta^{\mathrm{wBO}}\big) + \varepsilon\big(\partial_x B + \partial_x((\zeta^{\mathrm{wBO}})^2)\big) = 0\\ \partial_t\zeta^{\mathrm{wBO}} + \partial_x\zeta^{\mathrm{wBO}} + \sqrt{\mu}\partial_t A + \varepsilon\big(\partial_t B + \zeta^{\mathrm{wBO}}\partial_x\zeta^{\mathrm{wBO}}\big) = 0.\end{cases}$$

Now using the transport equations to see that $\partial_x A = -\partial_t A$ and $\partial_x B = -\partial_t B$ (up to an order $\mathcal{O}(\varepsilon+\sqrt{\mu})$), we find the equations:

$$\begin{cases}2\sqrt{\mu}\partial_x A = \gamma\tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x\zeta^{\mathrm{wBO}}\\ 2\partial_x B = -\zeta^{\mathrm{wBO}}\partial_x\zeta^{\mathrm{wBO}}.\end{cases}$$

We therefore let v^{wBO} be given by

$$v^{\mathrm{wBO}} = (1 + \frac{\gamma}{2}\tanh(\sqrt{\mu}|\mathrm{D}|))\zeta^{\mathrm{wBO}} - \frac{\varepsilon}{4}(\zeta^{\mathrm{wBO}})^2. \tag{8.8}$$

Step 2.2. *Rigorous derivation:* For any $(\zeta_0, v_0) \in H^{N-\frac{1}{2}}(\mathbb{R}) \times H^{N-\frac{1}{2}}(\mathbb{R})$ where v_0 satisfies

$$v_0 = (1 + \frac{\gamma}{2}\tanh(\sqrt{\mu}|\mathrm{D}|))\zeta_0 - \frac{\varepsilon}{4}(\zeta_0)^2,$$

then there is a time $T_1 > 0$ and a unique solution $\zeta^{\mathrm{wBO}} \in C\big([0, \varepsilon^{-1}T_1] : H^{N-\frac{1}{2}}(\mathbb{R})\big)$ associated to ζ_0 solving the equation

$$\partial_t \zeta^{\mathrm{wBO}} + (1 - \frac{\gamma}{2}\tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x \zeta^{\mathrm{wBO}} + \frac{3\varepsilon}{2}\zeta^{\mathrm{wBO}}\partial_x\zeta^{\mathrm{wBO}} = 0.$$

Moreover, we can define $v^{\mathrm{wBO}} \in C\big([0, \varepsilon^{-1}T_1] : H^{N-\frac{1}{2}}(\mathbb{R})\big)$ by (8.8). Then by Plancherel's identity and a Taylor expansion:

$$|\tanh(\sqrt{\mu}|\xi|)f - \sqrt{\mu}|\xi|f|_{H^s} \leq \mu^{\frac{3}{2}}C|\partial_x^3 f|_{H^s}, \tag{8.9}$$

we deduce from (1.26) that

$$\begin{cases} \partial_t\zeta^{\mathrm{wBO}} + (1 - \gamma\tanh(\sqrt{\mu}|\mathrm{D}|))\partial_x v^{\mathrm{wBO}} + \varepsilon\partial_x(\zeta^{\mathrm{wBO}}v^{\mathrm{wBO}}) = (\mu + \varepsilon\sqrt{\mu} + \varepsilon^2)R \\ \partial_t v^{\mathrm{wBO}} + \partial_x\zeta^{\mathrm{wBO}} + \varepsilon v^{\mathrm{wBO}}\partial_x v^{\mathrm{wBO}} = (\mu + \varepsilon\sqrt{\mu} + \varepsilon^2)R, \end{cases}$$

for R satisfying (8.6). Now, rescaling the equation back to its original variables concludes the proof of this step.

Step 3. For the consistency result in the case of the BO equation we simply use (8.9).

□

9. Proof of Theorem 1.18

Proof. First, we let $N \geq 7$ and take initial data (ζ_0, ψ_0) satisfying the assumptions of Theorem 1.7. Then using (8.5), we can define the solutions of the internal water waves equations (1.22) with variables

$$(\zeta, v) \in C([0, \varepsilon^{-1}T_1]; H^{N-\frac{1}{2}}(\mathbb{R}) \times H^{N-\frac{1}{2}}(\mathbb{R})),$$

from the data $(\zeta_0, \partial_x\psi_0) \in H^{N-\frac{1}{2}}(\mathbb{R}) \times H^{N-\frac{1}{2}}(\mathbb{R})$ for some $T_1 > 0$. Next, we use Theorem 1.17 and the matrix notation (7.2) to say that on the same time interval the functions $\mathbf{U} = (\zeta, v)^T$ solves

$$\partial_t\mathbf{U} + \mathcal{M}(\mathbf{U})\mathbf{U} = (\mu + \varepsilon^2)(R, R)^T,$$

for any $t \in [0, \varepsilon^{-1}T_1]$ and where the rest satisfies

$$|R|_{H^{N-5}} \leq C(\mathcal{E}^N(\mathbf{U}_0)).$$

We will now use this to prove estimates (1.30), (1.31), and (1.31) in three separate steps.

Step 1. *Proof of estimate* (1.30). We take the same data (ζ_0, v_0) with $v_0 = \partial_x\psi_0$, and use Theorem 1.27 to deduce the existence of $T_2 > 0$ such that

$$\mathbf{U}^{\mathrm{BOs}} = (\zeta^{\mathrm{BOs}}, v^{\mathrm{BOs}}) \in C([0, \varepsilon^{-1}T_2]; H^{N-\frac{1}{2}}(\mathbb{R}) \times H^{N-\frac{1}{2}}(\mathbb{R})),$$

solves system (7.1):

$$\partial_t\mathbf{U}^{\mathrm{BOs}} + \mathcal{M}(\mathbf{U}^{\mathrm{BOs}})\mathbf{U}^{\mathrm{BOs}} = \mathbf{0},$$

for any $t \in [0, \varepsilon^{-1}T_2]$. We may now take the difference between the two solutions

$$\mathbf{W} = (\eta, w)^T = \mathbf{U} - \mathbf{U}^{\mathrm{BOs}},$$

to find that

$$\partial_t \mathbf{W} + \mathcal{M}(\mathbf{U})\mathbf{W} = \mathbf{F} + (\mu + \varepsilon^2)(R, R)^T,$$

for any $t \in [0, \varepsilon^{-1}\min\{T_1, T_2\}]$ where $\mathbf{F}$ is defined by (7.5). Then using the estimate (7.10) with $N - 5 > \frac{3}{2}$ and adding the rest term we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}_{N-5}(\mathbf{W}) \leq (\mu + \varepsilon^2)|\big(\Lambda^{N-5}(R,R)^T, \mathcal{Q}(\mathbf{U}_1)\Lambda^{N-5}\mathbf{W}\big)_{L^2}| + \tilde{K}(N-4)\tilde{E}^{N-5}(\mathbf{W}).$$

Furthermore, by definition of $\mathcal{Q}(\mathbf{U}_1)$, the Hölder inequality, the Sobolev embedding, and (7.11) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}_{N-5}(\mathbf{W}) \leq (\mu + \varepsilon^2)C(\mathcal{E}^N(\mathbf{U}_0))(\tilde{E}_{N-5}(\mathbf{W}))^{\frac{1}{2}} + \varepsilon\tilde{K}(N-4)\tilde{E}_{N-5}(\mathbf{W}).$$

Then Grönwall's inequality and (7.11) yields

$$|(\eta, w)|_{H^{N-5}\times H^{N-5}} \leq (\mu + \varepsilon^2)C(\mathcal{E}^N(\mathbf{U}_0))e^{\varepsilon\tilde{K}(N-4)t}.$$

Finally, we observe that

$$\varepsilon\tilde{K}(N-4)t \leq C\mathcal{E}^N(\mathbf{U}_0)\min\{T_1, T_2\},$$

and use it together with the embedding $H^{N-5}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for $N \geq 7$ to find:

$$\begin{aligned}|\mathbf{U} - \mathbf{U}^{\mathrm{BOs}}|_{L^\infty([0,t];\mathbb{R})} &\lesssim |(\eta, w)|_{L^\infty([0,t]:H^{N-5}\times H^{N-5})} \\ &\leq (\mu + \varepsilon\sqrt{\mu})tC(\mathcal{E}^N(\mathbf{U}_0)),\end{aligned}$$

for all $t \in [0, \varepsilon^{-1}\min\{T_1, T_2\}]$. This completes the proof of estimate (1.30).

Step 2. *Proof of estimate* (1.30). By Theorem 1.27 we deduce the existence of $T_3 > 0$ such that

$$\zeta^{\mathrm{wBO}} \in C([0, \varepsilon^{-1}T_3]; H^{N-\frac{1}{2}}(\mathbb{R})),$$

solves (1.25) and from it we define

$$v^{\mathrm{wBO}} = (1 + \frac{\gamma}{2}\tanh(\sqrt{\mu}|\mathrm{D}|))\zeta^{\mathrm{wBO}} - \frac{\varepsilon}{4}(\zeta^{\mathrm{wBO}})^2.$$

Moreover, by estimate (1.26) we have that

$$\sup_{t\in[0,\varepsilon^{-1}T_3]} |(\zeta^{\mathrm{wBO}}, v^{\mathrm{wBO}})|_{H^{N-\frac{1}{2}}\times H^{N-\frac{1}{2}}} \leq C(|\zeta_0|_{H^{N-\frac{1}{2}}}).$$

Then if we define $\mathbf{U}^{\mathrm{wBO}} = (\zeta^{\mathrm{wBO}}, v^{\mathrm{wBO}})^T$ we can use Theorem 1.17 and argue as above to find that

$$\begin{aligned}|\mathbf{U} - \mathbf{U}^{\mathrm{wBO}}|_{L^\infty([0,t];\mathbb{R})} &\leq |\mathbf{U} - \mathbf{U}^{\mathrm{BOs}}|_{L^\infty([0,t];\mathbb{R})} + |\mathbf{U}^{\mathrm{BOs}} - \mathbf{U}^{\mathrm{wBO}}|_{L^\infty([0,t];\mathbb{R})} \\ &\leq \mu tC(\mathcal{E}^N(\mathbf{U}_0)),\end{aligned}$$

for all $t \in [0, \varepsilon^{-1}\min\{T_1, T_2, T_3\}]$.

Step 3. *Proof of estimate* (1.30). Let $T = \min\{T_1, T_2, T_3\}$. Then from the data $\zeta_0 \in H^{N-\frac{1}{2}}(\mathbb{R})$ there exist a unique solution $\zeta^{\mathrm{BO}} \in C([0, \varepsilon^{-1}T]; H^{N-\frac{1}{2}}(\mathbb{R}))$ that is bounded by its initial data. Moreover, by Theorem 1.17 the solution satisfies

$$\partial_t\zeta^{\mathrm{BO}} + c\big(1 - \frac{\gamma}{2}\tanh(\sqrt{\mu}|\mathrm{D}|)\big)\partial_x\zeta^{\mathrm{BO}} + c\frac{3\varepsilon}{2}\zeta^{\mathrm{BO}}\partial_x\zeta^{\mathrm{BO}} = \mu R.$$

Then if we define the difference $\tilde{\eta} = (\zeta^{\mathrm{wBO}} - \zeta^{\mathrm{BO}})$ it is straightforward to deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\eta}|^2_{H^{N-5}} \leq \varepsilon C(|\zeta_0|_{H^{N-4}})|\tilde{\eta}|^2_{H^{N-5}} + \mu|R|_{H^{N-5}}|\tilde{\eta}|_{H^{N-5}}.$$

The result is then a direct consequence of Grönwall's inequality and the previous steps. □

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Appendix A. Properties of the Dirichlet-Neumann operators

In this section we will give several results on $\mathcal{G}^{\pm}_{\mu}[\varepsilon\zeta]$ and $\mathcal{G}_{\mu}[\varepsilon\zeta]$. Then we will use these results to share some details on the proof Proposition 4.2. But as we will shortly see, the main quantities in (1.13) can be estimated in terms of the principal unknown (ζ, ψ) where the estimates are very similar to the ones in [50]. We will therefore give more details when there is a difference, and when the estimates are the same, we will simply refer to the results in [50].

A.1. **Properties of $\mathcal{G}^{+}_{\mu}$.**

We start this section by giving a precise definition of the positive Dirichlet-Neumann operator $\mathcal{G}^{+}_{\mu}$.

Definition A.1. *Let $t_0 > \frac{1}{2}$, $\psi^+ \in \dot{H}^{\frac{3}{2}}_{\mu}(\mathbb{R})$, and $\zeta \in H^{t_0+2}(\mathbb{R})$ be such that (1.14) is satisfied. Let Φ^+ be the unique solution in $\dot{H}^2(\Omega^+_t)$ of the boundary value problem*

$$\begin{cases} \Delta^{\mu}_{x,z}\Phi^+ = 0 & \text{in} \quad \Omega^+_t \\ \Phi^+ = \psi^+ & \text{on} \quad z = \varepsilon\zeta \\ \partial_z\Phi^+ = 0 & \text{on} \quad z = -1, \end{cases}$$

then $\mathcal{G}^{+}_{\mu}[\varepsilon\zeta]\psi^+ \in H^{\frac{1}{2}}(\mathbb{R})$ is defined by

$$\mathcal{G}^{+}_{\mu}[\varepsilon\zeta]\psi^+ = (\partial_z\Phi^+ - \mu\varepsilon\partial_x\zeta\partial_x\Phi^+)|_{z=\varepsilon\zeta}.$$

Remark A.2. *Under the provisions of Definition A.1 and let $\phi^+ = \Phi \circ \Sigma^+$ where Σ^+ is the diffeomorphisim from the fixed domain $\mathcal{S}^+$ onto Ω^+_t given in Definition 1.20, then we have that*

$$\begin{cases} \nabla^{\mu}_{x,z} \cdot P(\Sigma^+)\nabla^{\mu}_{x,z}\phi^+ = 0 & \text{in} \quad \mathcal{S}^+ \\ \phi^+ = \psi^+ & \text{on} \quad z = 0, \\ \partial^{P^+}_n\phi^+ = 0 & \text{on} \quad z = -1, \end{cases} \tag{A.1}$$

where

$$\partial^{P^+}_n = \mathbf{e}_z \cdot P(\Sigma^+)\nabla^{\mu}_{x,z}.$$

Moreover, we have an equivalent expression of $\mathcal{G}^+[\zeta]\psi^+$ given by

$$\mathcal{G}^{+}_{\mu}[\varepsilon\zeta]\psi^+ = \partial^{P^+}_n\phi^+|_{z=0}. \tag{A.2}$$

Remark A.3. *For $\varepsilon = 0$ we have that $\mathcal{G}^+$ becomes*

$$\mathcal{G}_\mu^-[0]\psi^+ = \sqrt{\mu}|\mathrm{D}|\tanh(\sqrt{\mu}|\mathrm{D}|)\psi^+. \tag{A.3}$$

Next, we will state several results on the Dirichlet-Neumann operator that are taken from [51, 50].

Proposition A.4. *Let $t_0 > \frac{1}{2}$, $s \in [0, t_0+1]$ and $\zeta \in H^{t_0+2}(\mathbb{R})$ be such that* (1.14) *is satisfied. Then we have the following properties:*

1. *For $\psi^+ \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$ there is a (variational) solution of* (A.1) *satisfying the estimates*
$$\sqrt{\mu}|\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}} \leq M\|\Lambda^s\nabla_{x,z}^\mu\phi^+\|_{L^2(\mathcal{S}^+)}, \tag{A.4}$$
and
$$\|\Lambda^s\nabla_{x,z}^\mu\phi^+\|_{L^2(\mathcal{S}^+)} \leq \sqrt{\mu}M|\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}}. \tag{A.5}$$
2. *We may extend definition A.1 for*
$$\mathcal{G}_\mu^+[\varepsilon\zeta] : \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R}) \to H^{s-\frac{1}{2}}(\mathbb{R}),$$
where for all $\psi_1, \psi_2 \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$, there holds,
$$\int_{\mathbb{R}} \psi_1\mathcal{G}_\mu^+[\varepsilon\zeta]\psi_2 \,\mathrm{d}x = \int_{\mathcal{S}^+} \nabla_{x,z}^\mu\phi_1^+ \cdot P^+(\Sigma^+)\nabla_{x,z}^\mu\phi_2^+ \,\mathrm{d}x\mathrm{d}z.$$
3. *For all $\psi_1, \psi_2 \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$, there holds,*
$$\big(\Lambda^s\psi_1, \Lambda^s\mathcal{G}_\mu^+[\varepsilon\zeta]\psi_2\big)_{L^2} \leq \mu M|\psi_1|_{\dot{H}_\mu^{s+\frac{1}{2}}}|\psi_2|_{\dot{H}_\mu^{s+\frac{1}{2}}}. \tag{A.6}$$
4. *For $\psi^+ \in H_\mu^{s+\frac{1}{2}}(\mathbb{R})$ the following estimates hold*
$$\forall s \in [0, t_0 + \frac{3}{2}], \quad |\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+|_{H^{s-\frac{1}{2}}} \leq \mu^{\frac{3}{4}}M|\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}, \tag{A.7}$$
$$\forall s \in [0, t_0 + 1], \quad |\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+|_{H^{s-\frac{1}{2}}} \leq \mu M|\psi|_{\dot{H}_\mu^{s+1}}. \tag{A.8}$$
5. *For all $\psi_1 \in \dot{H}_\mu^{s-\frac{1}{2}}(\mathbb{R})$ and $\psi_2 \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$, there holds*
$$\big(\big[\Lambda^s, \mathcal{G}_\mu^+[\varepsilon\zeta]\big]\psi_1, \Lambda^s\psi_2\big)_{L^2} \leq \mu\varepsilon M|\psi_1|_{\dot{H}_\mu^{s-\frac{1}{2}}}|\psi_2|_{\dot{H}_\mu^{s+\frac{1}{2}}}. \tag{A.9}$$
6. *For all $V \in H^{t_0+1}(\mathbb{R})$ and $\psi^+ \in H_\mu^{\frac{1}{2}}(\mathbb{R})$ there holds,*
$$\big((V\partial_x\psi^+), \frac{1}{\mu}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+\big)_{L^2} \leq M|V|_{W^{1,\infty}}|\psi^+|_{\dot{H}_\mu^{\frac{1}{2}}}. \tag{A.10}$$

Remark A.5. *The regularity on ζ here is not optimal. Specifically, the dependence on $|\zeta|_{H^{t_0+2}}$ in the definition of M can be lowered in the estimates above. However, we do not give these estimates since we will in other instances require more regularity on the free surface (as an example, see estimate* (3.3)*).*

Remark A.6. *The last estimate* (A.9) *is taken from* [50] *(see equation number* (2.24)*). However, in* (2.24)*, there is an ε missing due to a typo and has been added here* [52].

The next result concerns the shape derivative of $\mathcal{G}_\mu^+$. The result is found in [51, 50].

Proposition A.7. *Let $t_0 > \frac{1}{2}$, $s \in [0, t_0+1]$, and for any $\zeta \in H^{t_0+2}(\mathbb{R})$ satisfying* (1.14) *we have the following properties:*

1. *For* $\psi^+ \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$ *the shape derivative of* $\mathcal{G}_\mu^+[\varepsilon\zeta]$ *at* $\zeta \in H^{t_0+2}(\mathbb{R})$ *in the direction of* $h \in H^{t_0+2}(\mathbb{R})$ *is given by the formula*
$$\mathrm{d}_\zeta \mathcal{G}_\mu^+[\varepsilon\zeta](h)\psi^+ = -\varepsilon \mathcal{G}_\mu^+[\varepsilon\zeta](h\underline{w}^+) - \varepsilon\mu\partial_x(h\underline{V}^+). \tag{A.11}$$
2. *For all* $0 \le s \le t_0+1$, $j \ge 1$ *there holds*
$$|\mathrm{d}_\zeta^j \mathcal{G}_\mu^+[\varepsilon\zeta](h)\psi^+|_{H^{s-\frac{1}{2}}} \le \varepsilon^j \mu^{\frac{3}{4}} M \prod_{m=1}^{j} |h_m|_{H^{\max\{s,t_0\}+1}} |\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}}. \tag{A.12}$$
3. *For all* $0 \le s \le t_0+\frac{1}{2}$, $j \ge 1$, *and* $\psi^+ \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$, *there holds,*
$$|\mathrm{d}_\zeta^j \mathcal{G}_\mu^+[\varepsilon\zeta](h)\psi^+|_{H^{s-\frac{1}{2}}} \le \varepsilon^j \mu M \prod_{m=1}^{j} |h_m|_{H^{\max\{s+\frac{1}{2},t_0\}+1}} |\psi^+|_{\dot{H}_\mu^{s+1}}. \tag{A.13}$$
4. *For all* $0 \le s \le t_0+\frac{1}{2}$, $j \ge 1$, $\psi^+ \in \dot{H}_\mu^{\max\{s,t_0\}+1}(\mathbb{R})$, *there holds,*
$$|\mathrm{d}_\zeta^j \mathcal{G}_\mu^+[\varepsilon\zeta](h)\psi^+|_{H^{s-\frac{1}{2}}} \le \varepsilon^j \mu M |h_k|_{H^{s+\frac{1}{2}}} \prod_{m\neq k}^{j} |h_m|_{H^{\max\{s,t_0\}+\frac{3}{2}}} |\psi|_{\dot{H}_\mu^{\max\{s,t_0\}+1}}. \tag{A.14}$$
5. *For all* $0 \le s \le t_0$, $j \ge 1$, $\psi^+ \in \dot{H}_\mu^{\max\{s+\frac{1}{2},t_0\}+1}(\mathbb{R})$, *there holds,*
$$|\mathrm{d}_\zeta^j \mathcal{G}_\mu^+[\varepsilon\zeta](h)\psi^+|_{H^{s-\frac{1}{2}}} \le \varepsilon^j \mu M |h_k|_{H^{s+1}} \prod_{m\neq k}^{j} |h_m|_{H^{\max\{s+\frac{1}{2},t_0\}+\frac{3}{2}}} |\psi^+|_{\dot{H}_\mu^{\max\{s+\frac{1}{2},t_0\}+1}}. \tag{A.15}$$
6. *For all* $0 \le s \le t_0+1$, $j \ge 1$, $\psi_1, \psi_2 \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$, *there holds,*
$$\big|\big(\Lambda^s \mathrm{d}^j \mathcal{G}_\mu^+[\varepsilon\zeta](\mathbf{h})\psi_1, \Lambda^s\psi_2\big)_{L^2}\big| \le \varepsilon^j \mu M \prod_{m=1}^{j} |h_m|_{H^{\max\{s,t_0\}+1}} |\psi_1|_{\dot{H}^{s+\frac{1}{2}}} |\psi_2|_{\dot{H}_\mu^{s+\frac{1}{2}}}. \tag{A.16}$$
7. *For for all* $0 \le s \le t_0+\frac{1}{2}$, $j \ge 1$, $\psi_1 \in \dot{H}_\mu^{\max\{s,t_0\}+1}(\mathbb{R}), \psi_2 \in \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$, *there holds,*
$$\big|\big(\Lambda^s \mathrm{d}^j \mathcal{G}_\mu^+[\varepsilon\zeta](\mathbf{h})\psi_1, \Lambda^s\psi_2\big)_{L^2}\big| \le \varepsilon^j \mu M |h_l|_{H^{s+\frac{1}{2}}} \prod_{m\neq l}^{j} |h_m|_{H^{\max\{s,t_0\}+\frac{3}{2}}} |\psi_1|_{\dot{H}^{\max\{s,t_0\}+1}} |\psi_2|_{\dot{H}_\mu^{s+\frac{1}{2}}}. \tag{A.17}$$

A.2. **Properties of $\mathcal{G}_\mu^-$.** In this section will define and give the main properties of the negative Dirichlet-Neumann operator.

Definition A.8. *Let* $t_0 \ge 1$, $\psi \in \mathring{H}^{\frac{3}{2}}(\mathbb{R})$, *and* $\zeta \in H^{t_0+2}(\mathbb{R})$ *be such that* (1.14) *is satisfied. Let* Φ^- *be the unique solution in* $\dot{H}^2(\Omega_t^-)$ *of the boundary value problem*
$$\begin{cases} \Delta_{x,z}^\mu \Phi^- = 0 & \text{in} \quad \Omega_t^- \\ \Phi^- = \psi & \text{on} \quad z = \varepsilon\zeta, \end{cases}$$
then $\mathcal{G}_\mu^-[\varepsilon\zeta]\psi^- \in H^{\frac{1}{2}}(\mathbb{R})$ *is defined by*
$$\mathcal{G}_\mu^-[\varepsilon\zeta]\psi^- = (\partial_z\Phi^- - \varepsilon\mu\partial_x\zeta\partial_x\Phi^-)|_{z=\varepsilon\zeta}. \tag{A.18}$$

Remark A.9. *As noted in Remark* 3.50 (2) *in* [51], *we can define the negative Dirichlet-Neumann operator by formula* (A.18) *(or formula* (A.21) *below) for* $\psi^- \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$ *if* $t_0 > \frac{1}{2}$ *and* $s \ge \max\{0, 1-t_0\}$, *where the restriction is a consequence of* (B.6). *We therefore put* $t_0 \ge 1$ *for simplicity.*

Remark A.10. *The scaling for $\mathcal{G}_\mu^-[\varepsilon\zeta]$ is different from the one used in [51], where the author used the scaling natural for infinite depth. In that case one would change $\mu = 1$ and ε by $\epsilon = \frac{a}{\lambda}$. In our case, the internal water wave model depends on both scales and is explained in detail in Subsection 1.2.1.*

Remark A.11. *For $\varepsilon\zeta = 0$ we have that $\mathcal{G}_\mu^-$ becomes*

$$\mathcal{G}_\mu^-[0]\psi^- = \sqrt{\mu}|\mathrm{D}|\psi^-. \tag{A.19}$$

Remark A.12. *Under the provisions of Definition A.8 and let $\phi^- = \Phi \circ \Sigma^-$ where Σ^+ is the diffeomorphisim from the fixed domain $\mathcal{S}^-$ onto Ω_t^- given in Definition 1.20, then we have that*

$$\begin{cases} \nabla_{x,z}^\mu \cdot P(\Sigma^-)\nabla_{x,z}^\mu \phi^- = 0 & \text{in} \quad \mathcal{S}^- \\ \phi^- = \psi & \text{on} \quad z = 0, \end{cases} \tag{A.20}$$

and we have an equivalent expression of $\mathcal{G}_\mu^-[\varepsilon\zeta]\psi^-$ given by

$$\mathcal{G}_\mu^-[\varepsilon\zeta]\psi^- = \partial_n^{P^-}\phi^-|_{z=0}, \tag{A.21}$$

where $\partial_n^{P^-} = \mathbf{e}_z \cdot P(\Sigma^-)\nabla_{x,z}^\mu$.

Here we use the results in [51] adapted to the current scaling.

Proposition A.13. *Let $t_0 \geq 1$, $s \in [0, t_0 + 1]$ and $\zeta \in H^{t_0+2}(\mathbb{R})$ be such that (1.14) is satisfied. We have the following properties:*

1. *For $\psi \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$ there is a (variational) solution of (A.1) satisfying the estimate*
$$\sqrt{\mu}|\psi^-|_{\mathring{H}^{s+\frac{1}{2}}} \leq M\|\Lambda^s \nabla_{x,z}^\mu \phi^-\|_{L^2(\mathcal{S}^-)}, \tag{A.22}$$
and
$$\|\Lambda^s \nabla_{x,z}^\mu \phi^-\|_{L^2(\mathcal{S}^-)} \leq \mu^{\frac{1}{4}} M|\psi^-|_{\mathring{H}^{s+\frac{1}{2}}}. \tag{A.23}$$
2. *By remark A.9 we may extend Definition A.8 for*
$$\mathcal{G}_\mu^-[\varepsilon\zeta] : \mathring{H}^{s+\frac{1}{2}}(\mathbb{R}) \to H^{s-\frac{1}{2}}(\mathbb{R}),$$
where for all $\psi_1, \psi_2 \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$, there holds,
$$\int_{\mathbb{R}} \psi_1 \mathcal{G}_\mu^-[\varepsilon\zeta]\psi_2 \,\mathrm{d}x = -\int_{\mathcal{S}^-} \nabla_{x,z}^\mu \phi_1 \cdot P^-(\Sigma^-)\nabla_{x,z}^\mu \phi_2 \,\mathrm{d}x\mathrm{d}z.$$
3. *For all $\psi_1, \psi_2 \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$, there holds,*
$$\big(\Lambda^s \psi_1, \Lambda^s \mathcal{G}_\mu^-[\varepsilon\zeta]\psi_2\big)_{L^2} \leq \sqrt{\mu}M|\psi_1|_{\mathring{H}^{s+\frac{1}{2}}}|\psi_2|_{\mathring{H}^{s+\frac{1}{2}}}. \tag{A.24}$$
4. *For $\psi^- \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$ the following estimates hold*
$$|\mathcal{G}_\mu^-[\varepsilon\zeta]\psi^-|_{H^{s-\frac{1}{2}}} \leq \sqrt{\mu}M|\psi^-|_{\mathring{H}^{s+\frac{1}{2}}}. \tag{A.25}$$
5. *For all $\psi_1 \in \mathring{H}^{s-\frac{1}{2}}(\mathbb{R})$ and $\psi_2 \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$, there holds,*
$$\big(\big[\Lambda^s, \mathcal{G}_\mu^-[\varepsilon\zeta]\big]\psi_1, \Lambda^s\psi_2\big)_{L^2} \leq \varepsilon\sqrt{\mu}M|\psi_1|_{\mathring{H}^{s-\frac{1}{2}}}|\psi_2|_{\mathring{H}^{s+\frac{1}{2}}}. \tag{A.26}$$
6. *For all $V \in H^{t_0+1}(\mathbb{R})$ and $\psi^- \in \mathring{H}^{\frac{1}{2}}(\mathbb{R})$ there holds,*
$$\big((V\partial_x\psi^-), \frac{1}{\mu}\mathcal{G}_\mu^-[\varepsilon\zeta]\psi^-\big)_{L^2} \leq \mu^{-\frac{1}{2}}M|V|_{W^{1,\infty}}|\psi^-|_{\mathring{H}^{\frac{1}{2}}}^2. \tag{A.27}$$

Remark A.14. *If we compare the estimates for $\mathcal{G}_\mu^+$ with the ones above, we note that there is a $\sqrt{\mu}$ missing. However, this is a simple consequence of the functional setting where there is an additional gain in μ from the observation that*

$$\sqrt{\mu}|\xi|\tanh(\sqrt{\mu}|\xi|) \sim \mu\frac{|\xi|^2}{(1+\sqrt{\mu}|\xi|)}.$$

Remark A.15. *The inequality* (A.26) *is a direct extension of inequality* (2.24) *in* [50]. *While the last inequality* (A.27) *is the extension of Proposition* 3.30 *in* [51] *to infinite depth. The extension is straightforward and is explained on page* 88 *of this book.*

Lastly, we will need a shape derivative formula for $\mathcal{G}_\mu^-$ and estimates on higher order shape derivatives [51]:

Proposition A.16. *Let $t_0 \geq 1$, $s \in [0, t_0+1]$, and take $\zeta \in H^{t_0+2}(\mathbb{R})$. Then we have the following properties:*

1. *For $\psi^- \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$ the shape derivative of $\mathcal{G}_\mu^-[\varepsilon\zeta]$ at $\zeta \in H^{t_0+2}(\mathbb{R})$ in the direction of $h \in H^{t_0+2}(\mathbb{R})$ is given by the formula*
$$\mathrm{d}_\zeta \mathcal{G}_\mu^-[\varepsilon\zeta](h)\psi^- = -\varepsilon\mathcal{G}_\mu^-[\varepsilon\zeta](h\underline{w}^-) - \varepsilon\mu\partial_x(h\underline{V}^-). \tag{A.28}$$
2. *For all $0 \leq s \leq t_0+1$, $j \geq 1$ and $\psi_1, \psi_2 \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$, there holds,*
$$\left|\left(\Lambda^s \mathrm{d}^j \mathcal{G}_\mu^-[\varepsilon\zeta](\mathbf{h})\psi_1, \Lambda^s\psi_2\right)_{L^2}\right| \leq \varepsilon^j\sqrt{\mu}M\prod_{m=1}^{j}|h_m|_{H^{\max\{s,t_0\}+1}}|\psi_1|_{\mathring{H}^{s+\frac{1}{2}}}|\psi_2|_{\mathring{H}^{s+\frac{1}{2}}}. \tag{A.29}$$
3. *For all $0 \leq s \leq t_0+\frac{1}{2}$, $j \geq 1$, $\psi_1 \in \mathring{H}^{\max\{s,t_0\}+\frac{3}{2}}(\mathbb{R}), \psi_2 \in \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$, there holds,*
$$\left|\left(\Lambda^s \mathrm{d}^j \mathcal{G}_\mu^-[\varepsilon\zeta](\mathbf{h})\psi_1, \Lambda^s\psi_2\right)_{L^2}\right| \leq \varepsilon^j\sqrt{\mu}M|h_l|_{H^{s+\frac{1}{2}}}\prod_{m\neq l}^{j}|h_m|_{H^{\max\{s,t_0\}+\frac{3}{2}}}|\psi_1|_{\mathring{H}^{\max\{s,t_0\}+1}}|\psi_2|_{\mathring{H}^{s+\frac{1}{2}}}. \tag{A.30}$$

A.3. **Corollaries from the results in Section 2.** We will give an important generalization of Proposition 2.4 where we prove that we can trade $\mathcal{G}_\mu^+$ with any operator $\mathrm{Op}(A) : X \to H^{s-\frac{1}{2}}(\mathbb{R})$ satisfying
$$\left|\left(\Lambda^s\mathrm{Op}(A)f_1, \Lambda^s f_2\right)_{L^2}\right| \leq MM_A(f_1)\sqrt{\mu}|f_2|_{\mathring{H}^{s+\frac{1}{2}}}, \tag{A.31}$$
where $M_A(\psi)$ is some positive number defined by the norm on X. This can be seen from Step 1.1 in the proof, which is the only place where we use that $(\mathcal{G}_\mu^-)^{-1}$ is composed with $\mathcal{G}_\mu^+$. We have the following result.

Corollary A.17. *Let $t_0 \geq 1$, $s \in [0, t_0+1]$, and $\zeta \in H^{t_0+2}(\mathbb{R})$ be such that* (1.14) *is satisfied. Then for $f \in \mathscr{S}(\mathbb{R})$ and $\mathrm{Op}(A)$ satisfying condition* (A.31)*, the mapping*
$$(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathrm{Op}(A) \ : \begin{cases} X & \to \mathring{H}^{s+\frac{1}{2}}(\mathbb{R}) \\ f & \mapsto (\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathrm{Op}(A)f \end{cases} \tag{A.32}$$
is well-defined and satisfies
$$|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathrm{Op}(A)f|_{\mathring{H}^{s+\frac{1}{2}}} \leq MM_A(f). \tag{A.33}$$
Moreover, we have the particular cases of $\mathrm{Op}(A)$:

1. If $\mathrm{Op}(A) = \partial_x$*, then* $X = \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$ *and there holds,*

$$|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x f|_{\mathring{H}^{s+\frac{1}{2}}} \leq \mu^{-\frac{1}{2}} M |f|_{\mathring{H}^{s+\frac{1}{2}}}. \tag{A.34}$$

2. If $\mathrm{Op}(A) = \mathrm{d}_\zeta^j \mathcal{G}_\mu^+[\varepsilon\zeta]$ *for* $j \geq 1$*, then* $X = \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$ *and for* $\mathbf{h} = (h_1, ..., h_j) \in H^{t_0+2}(\mathbb{R})^j$ *there holds,*

$$|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathrm{d}_\zeta^j \mathcal{G}_\mu^+[\varepsilon\zeta] f|_{\mathring{H}^{s+\frac{1}{2}}} \leq \varepsilon^j \mu^{\frac{1}{4}} M \prod_{m=1}^{j} |h_m|_{H^{\max\{s,t_0\}+1}} |f|_{\dot{H}_\mu^{s+\frac{1}{2}}}, \tag{A.35}$$

and for $X = \dot{H}_\mu^{\max\{s,t_0\}+1}(\mathbb{R})$ *there holds,*

$$|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathrm{d}_\zeta^j \mathcal{G}_\mu^+[\varepsilon\zeta] f|_{\mathring{H}^{s+\frac{1}{2}}} \leq \varepsilon^j \mu^{\frac{1}{4}} M |h_k|_{H^{s+\frac{1}{2}}} \prod_{m \neq k}^{j} |h_m|_{H^{\max\{s,t_0\}+\frac{3}{2}}} |f|_{\dot{H}_\mu^{\max\{s,t_0\}+1}}. \tag{A.36}$$

3. If $\mathrm{Op}(A) = \mathrm{d}^j \mathcal{G}_\mu^-[\varepsilon\zeta]$ *for* $j \geq 1$*, then* $X = \mathring{H}^{s+\frac{1}{2}}(\mathbb{R})$ *and for* $\mathbf{h} = (h_1, ..., h_j) \in H^{t_0+2}(\mathbb{R})^j$ *there holds,*

$$|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathrm{d}_\zeta^j \mathcal{G}_\mu^-[\varepsilon\zeta] f|_{\mathring{H}^{s+\frac{1}{2}}} \leq \varepsilon^j M \prod_{m=1}^{j} |h_m|_{H^{\max\{s,t_0\}+1}} |f|_{\mathring{H}_\mu^{s+\frac{1}{2}}}, \tag{A.37}$$

and for $X = \mathring{H}^{\max\{s,t_0\}+1}(\mathbb{R})$ *there holds,*

$$|(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathrm{d}_\zeta^j \mathcal{G}_\mu^+[\varepsilon\zeta] f|_{\mathring{H}^{s+\frac{1}{2}}} \leq \varepsilon^j M |h_k|_{H^{s+\frac{1}{2}}} \prod_{m \neq k}^{j} |h_m|_{H^{\max\{s,t_0\}+\frac{3}{2}}} |f|_{\mathring{H}^{\max\{s,t_0\}+1}}. \tag{A.38}$$

Proof. As explained above, we only need to consider the specific cases of $\mathrm{Op}(A)$. Moreover, since we have (A.33) at hand, we simply verify inequality (A.31) and identify the constant $M_A(f_1)$.

<u>Step 1.</u> For the proof of (A.34) we use that $\partial_x = -\mathcal{H}|\mathrm{D}|$, where $\mathcal{H}$ is the Hilbert transform and then use Plancherel's identity together with Cauchy-Schwarz inequality to deduce the bound

$$\big|\big(\Lambda^s \partial_x f_1, \Lambda^s f_2\big)_{L^2}\big| \leq M |f_1|_{\mathring{H}^{s+\frac{1}{2}}} |f_2|_{\mathring{H}^{s+\frac{1}{2}}}.$$

<u>Step 2.</u> For the proof of (A.35), we use estimate (A.16) and(B.3) to get the estimate

$$\begin{aligned}\big|\big(\Lambda^s \mathrm{d}^j \mathcal{G}_\mu^+[\varepsilon\zeta](\mathbf{h}) f_1, \Lambda^s f_2\big)_{L^2}\big| &\leq \varepsilon^j \mu M |f_1|_{\dot{H}_\mu^{s+\frac{1}{2}}} |f_2|_{\dot{H}_\mu^{s+\frac{1}{2}}} \prod_{m=1}^{j} |h_m|_{H^{\max\{s,t_0\}+1}} \\ &\leq \varepsilon^j \mu^{\frac{3}{4}} M |f_1|_{\dot{H}_\mu^{s+\frac{1}{2}}} |f_2|_{\mathring{H}^{s+\frac{1}{2}}} \prod_{m=1}^{j} |h_m|_{H^{\max\{s,t_0\}+1}}.\end{aligned}$$

While the proof of (A.36) is obtained by (A.17).

<u>Step 3.</u> This step is a direct consequence of estimates (A.29) and (A.30), where estimate <u>(A.29)</u> implies

$$\big|\big(\Lambda^s \mathrm{d}^j \mathcal{G}_\mu^-[\varepsilon\zeta](\mathbf{h}) f_1, \Lambda^s f_2\big)_{L^2}\big| \leq \varepsilon^j \sqrt{\mu} M |f_1|_{\mathring{H}^{s+\frac{1}{2}}} |f_2|_{\mathring{H}^{s+\frac{1}{2}}} \prod_{m=1}^{j} |h_m|_{H^{\max\{s,t_0\}+1}}.$$

□

A.4. **Basic definitions.** We start this section by defining the main quantities involved in (1.13) and relating them to the primary variables (ζ, ψ).

Corollary A.18. *Let $t_0 \geq 1$, $s \in [0, t_0 + 1]$, and $\zeta \in H^{t_0+2}(\mathbb{R})$ satisfying* (1.14)*. Moreover define $\psi^{\pm}$ as in Remark 2.11 and as the trace of $\phi^{\pm}$ satisfying* (2.21)*. Then there holds:*

1. *The tangential velocity is given by the mapping*

$$\mathcal{V}_{\|}^{\pm} \ : \ \begin{cases} \dot{H}_{\mu}^{s+\frac{1}{2}}(\mathbb{R}) & \to H^{s-\frac{1}{2}}(\mathbb{R}) \\ \psi & \mapsto \partial_x \psi^{\pm} \end{cases} \tag{A.39}$$

is well-defined and satisfies

$$\forall s \in [0, t_0 + 1], \quad |\mathcal{V}_{\|}^{\pm}|_{H^s} \leq M|\partial_x \psi|_{H^s}. \tag{A.40}$$

2. *The horizontal component of the velocity is given by the mappings*

$$\underline{w}^{\pm} \ : \ \begin{cases} \dot{H}_{\mu}^{s+\frac{1}{2}}(\mathbb{R}) & \to H^{s-\frac{1}{2}}(\mathbb{R}) \\ \psi & \mapsto \frac{\mathcal{G}_{\mu}^{\pm}[\varepsilon\zeta]\psi^{\pm} + \varepsilon\mu \partial_x \zeta \partial_x \psi^{\pm}}{1 + \varepsilon^2 \mu (\partial_x \zeta)^2}, \end{cases} \tag{A.41}$$

is well-defined and satisfies

$$s \in [0, t_0 + 1], \quad |\underline{w}^{\pm}\psi|_{H^{s-\frac{1}{2}}} \leq \mu^{\frac{3}{4}} M |\psi|_{\dot{H}_{\mu}^{s+\frac{1}{2}}}, \tag{A.42}$$

and

$$s \in [0, t_0 + \frac{1}{2}], \quad |\underline{w}^{\pm}\psi|_{H^{s-\frac{1}{2}}} \leq \mu M |\psi|_{\dot{H}_{\mu}^{s+1}}. \tag{A.43}$$

3. *The vertical component of the velocity is given by the mappings*

$$\underline{V}^{\pm} \ : \ \begin{cases} \dot{H}_{\mu}^{s+\frac{1}{2}}(\mathbb{R}) & \to H^{s-\frac{1}{2}}(\mathbb{R}) \\ \psi & \mapsto \mathcal{V}_{\|}^{\pm} - \varepsilon(\underline{w}^{\pm}\psi)\partial_x \zeta \end{cases} \tag{A.44}$$

is well-defined and satisfies

$$s \in [0, t_0 + \frac{1}{2}], \quad |\underline{V}^{\pm}|_{H^s} \leq M|\partial_x \psi|_{H^s}. \tag{A.45}$$

Proof. We prove each point in separate steps.

Step 1. For $\mathcal{V}_{\|}^{+}$ we will use formula $\psi^{+} = (\mathcal{J}_{\mu})^{-1}\psi$ and tn Proposition 2.10 with estimate (2.23) to get

$$\begin{aligned} |\mathcal{V}_{\|}^{+}|_{H^s} &\leq M|(\mathcal{J}_{\mu}[\varepsilon\zeta])^{-1}\psi|_{\dot{H}_{\mu}^{s+\frac{1}{2}}} + \mu^{\frac{1}{4}} M |(\mathcal{J}_{\mu}[\varepsilon\zeta])^{-1}\psi|_{\dot{H}_{\mu}^{(s+\frac{1}{2})+\frac{1}{2}}} \\ &\leq M|\partial_x \psi|_{H^s}. \end{aligned}$$

For $\mathcal{V}_{\|}^{-}$ we use formula $\psi^{-} = (\mathcal{G}_{\mu}^{-})^{-1}\mathcal{G}_{\mu}^{+}\psi^{+}$, then Proposition 2.4 with estimate (2.12), and the above estimates to deduce:

$$|\mathcal{V}_{\|}^{-}|_{H^s} = |\partial_x \big((\mathcal{G}_{\mu}[\varepsilon\zeta]^{-})^{-1} \mathcal{G}_{\mu}^{+}[\varepsilon\zeta]\psi^{+}\big)|_{H^s} \leq M|\partial_x \psi^{+}|_{H^s} \leq M|\partial_x \psi|_{H^s}.$$

Step 2. We consider the first estimate of $\underline{w}^+$, where we use estimate (A.7), the product estimates (B.6) and definition of ψ^+ with (2.23) to get:

$$\begin{aligned}
|\underline{w}^+|_{H^{s-\frac{1}{2}}} &\leq |\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+|_{H^{s-\frac{1}{2}}} + \mu\varepsilon|\partial_x\zeta|_{H^{\max\{t_0,s-\frac{1}{2}\}}}|\partial_x\psi^+|_{H^{s-\frac{1}{2}}}\\
&\leq \mu^{\frac{3}{4}}M|\psi^+|_{\dot{H}^{s+\frac{1}{2}}_\mu}\\
&\leq \mu^{\frac{3}{4}}M|\psi|_{\dot{H}^{s+\frac{1}{2}}_\mu}.
\end{aligned}$$

For the second estimate on w^+, we argue similarly using (A.8) instead of (A.7).

For $\underline{w}^-$ we first use the definition of $\psi^- = (\mathcal{G}^-_\mu)^{-1}\mathcal{G}^+_\mu\psi^+$ and apply the same estimates, combined with (2.11):

$$\begin{aligned}
|\underline{w}^-|_{H^{s-\frac{1}{2}}} &\leq |\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+|_{H^{s-\frac{1}{2}}} + \varepsilon\mu|\partial_x\zeta|_{H^{\max\{t_0,s-\frac{1}{2}\}}}|\partial_x\big((\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+\big)|_{H^{s-\frac{1}{2}}}\\
&\leq \mu^{\frac{3}{4}}M|\psi|_{\dot{H}^{s+\frac{1}{2}}_\mu} + \varepsilon\mu M|(\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}\mathcal{G}^+_\mu[\varepsilon\zeta]\psi^+|_{\mathring{H}^{s+\frac{1}{2}}}\\
&\leq \mu^{\frac{3}{4}}M|\psi|_{\dot{H}^{s+\frac{1}{2}}_\mu}.
\end{aligned}$$

For the second estimate on $\underline{w}^-$, we argue as above, where we use (A.8) instead of (A.7).

Step 3. We prove first (A.45). For the first part of $V^\pm$ we use estimates (A.40), while the second part is estimated by the product estimate (B.6) and then (A.42) to see that

$$\begin{aligned}
|V^\pm|_{H^s} &\leq |\mathcal{V}^\pm_\parallel|_{H^s} + \varepsilon|\underline{w}^\pm\partial_x\zeta|_{H^s}\\
&\leq M|\partial_x\psi|_{H^s} + \varepsilon|\partial_x\zeta|_{H^{\max\{t_0,s\}}}|\underline{w}^\pm|_{H^{(s+\frac{1}{2})-\frac{1}{2}}}\\
&\leq M(|\partial_x\psi|_{H^s} + \mu^{\frac{3}{4}}|\psi|_{\dot{H}^{s+1}_\mu}),\\
&\leq M(|\partial_x\psi|_{H^s}.
\end{aligned}$$

□

A.5. **Proof of Proposition 4.2.** We will in this section give the details on the proof of Proposition 4.2. Since the strategy in the proof is exactly the same as in [50] and the main quantities involved satisfy the same estimates (see Corollary A.18), we only prove the steps that are unique to the current regime. The first step is to derive linearization formulas. for the Dirichlet-Neumann operator $\mathcal{G}_\mu$.

A.5.1. *Linearization formulas.* The main step in the quasilinearisation of the internal water waves system (1.13) is to get linearization formulas for

$$\mathcal{G}_\mu[\varepsilon\zeta] = \mathcal{G}^+_\mu[\varepsilon\zeta]\Big(1-\gamma(\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}\mathcal{G}^+_\mu[\varepsilon\zeta]\Big)^{-1}\psi$$

Here it will be important to get an explicit formula for the shape derivative of $\mathcal{G}_\mu$ with respect to ζ and track the dependencies in the small parameters.

Proposition A.19. *Let $T>0$, $t_0\geq 1$ and $N\in\mathbb{N}$ be such that $N\geq 5$. Furthermore, let $U=(\zeta,\psi)^T\in\mathscr{E}^{N,t_0}_{\mathrm{bo},T}$ be such that* (1.14) *is satisfied on* $[0,T]$*. For all $\alpha=(\alpha^1,\alpha^2)\in\mathbb{N}^2$, $\check{\alpha}^j=\alpha-\mathbf{e}_j$, with $1\leq|\alpha|\leq N$ and define $\partial^\alpha_{x,t}=\partial^{\alpha_1}_x\partial^{\alpha_2}_t$, and let $\underline{w}^\pm$ be as defined in* (A.41)*. Moreover, define $\underline{w}$ by*

$$\underline{w} = \underline{w}^+ - \gamma\underline{w}^-,$$

and

$$\zeta_{(\alpha)} = \partial_{x,t}^{\alpha}\zeta, \quad \psi_{(\alpha)} = \partial_{x,t}^{\alpha}\psi - \varepsilon\underline{w}\partial_{x,t}^{\alpha}\zeta, \quad \psi_{\langle\check{\alpha}\rangle} = (\psi_{\check{\alpha}^1}), \psi_{(\check{\alpha}^2)}).$$

Then for all $1 \leq |\alpha| \leq N$*, one has*

$$\text{if } \alpha < N: \quad \frac{1}{\mu}\partial_{x,t}^{\alpha}(\mathcal{G}_{\mu}[\varepsilon\zeta]\psi) = \frac{1}{\mu}\mathcal{G}_{\mu}[\varepsilon\zeta]\psi_{(\alpha)} + \varepsilon R_{(\alpha)},$$
$$\text{if } \alpha \leq N: \quad \frac{1}{\mu}\partial_{x,t}^{\alpha}(\mathcal{G}_{\mu}[\varepsilon\zeta]\psi) = \frac{1}{\mu}\mathcal{G}_{\mu}[\varepsilon\zeta]\psi_{(\alpha)} - \varepsilon\mathcal{I}[U]\zeta_{(\alpha)} + \frac{1}{\mu}\mathcal{G}_{\mu,(\alpha)}[\varepsilon\zeta]\psi_{\langle\check{\alpha}\rangle} + \varepsilon R_{\alpha},$$

where the linear operators $\mathcal{I}[U]$ *and* $\mathcal{G}_{\mu,(\alpha)}$ *are given in Definition 4.1, while* R_α *is a function that satisfies the estimate*

$$|R_{\alpha}(t)|^2_{H^1_{\gamma,\mathrm{bo}}} \leq C\mathcal{E}^N(\mathbf{U}(t)), \tag{A.46}$$

for some $C > 0$ *and for all* $t \in [0, T]$*.*

Remark A.20. *The principal part of the linearization formula for* $\mathcal{G}_\mu$ *is given in terms of* $\mathcal{G}_\mu\psi_{(\alpha)}$ *and* $\mathcal{I}[U]\zeta_\alpha$*. While the additional term* $\mathcal{G}_{\mu,(\alpha)}\psi_{\langle\check{\alpha}\rangle}$ *is sub-principal that offers no difficulty in the proof, but is needed to deal with surface tension in the energy estimates (see the third point in Remark 5.1).*

We will now give the main ingredients in proving the linearization formulas in Lemma A.19. These formulas involve the precise formulation of the directional derivative of $\mathcal{G}_\mu$, and we therefore give the definition.

Remark A.21. *Let* $\psi \in H^{s+\frac{1}{2}}(\mathbb{R})$ *and* $\zeta \in H^{t_0+2}(\mathbb{R})$*, then* $\zeta \mapsto \mathcal{G}_\mu[\varepsilon\zeta]\psi$ *is smooth and the directional derivative, in the direction of* $h \in H^{t_0+2}(\mathbb{R})$*, is given by*

$$\mathrm{d}_{\zeta}\mathcal{G}_{\mu}[\varepsilon\zeta](h)\psi = \lim_{\nu\to 0}\frac{\mathcal{G}_{\mu}[\varepsilon\zeta + \nu h]\psi - \mathcal{G}_{\mu}[\varepsilon\zeta]\psi}{\nu}.$$

The smoothness follows from the shape analyticity of $\mathcal{G}_\mu^{\pm}$ [51].

Lemma A.22. *Let* $t_0 \geq 1$ *and* $\zeta \in H^{t_0+2}(\mathbb{R})$ *be such that* (1.14) *is satisfied.*

1. *For all* $\psi \in \dot{H}_{\mu}^{\frac{1}{2}}(\mathbb{R})$*,* $h \in H^{t_0+2}(\mathbb{R})$*, and* $U = (\zeta, \psi)^T$*, one has*
$$\mathrm{d}_{\zeta}\mathcal{G}_{\mu}[\varepsilon\zeta](h)\psi = -\varepsilon\mathcal{G}_{\mu}[\varepsilon\zeta]\big(h(\underline{w}^{+} - \gamma\underline{w}^{-})\big) - \varepsilon\mu\mathcal{I}[U]h. \tag{A.47}$$
where $\mathcal{I}[U]h$ *is defined by*
$$\mathcal{I}[U]h = \partial_x(h\underline{V}^{+}) + \gamma\mathcal{G}_{\mu}[\varepsilon\zeta](\mathcal{G}_{\mu}^{-}[\varepsilon\zeta])^{-1}\partial_x\Big(h(\underline{V}^{+} - \underline{V}^{-})\Big).$$
2. *For all* $0 \leq s \leq t_0 + 1$*,* $j \geq 1$*, there holds,*
$$|\mathrm{d}_{\zeta}^{j}\mathcal{G}_{\mu}[\varepsilon\zeta](h)\psi|_{H^{s-\frac{1}{2}}} \leq \varepsilon^{j}\mu^{\frac{3}{4}}M\prod_{m=1}^{j}|h_m|_{H^{\max\{s,t_0\}+1}}|\psi|_{\dot{H}_{\mu}^{s+\frac{1}{2}}}. \tag{A.48}$$
3. *For all* $0 \leq s \leq t_0 + \frac{1}{2}$*,* $j \geq 1$*, there holds,*
$$|\mathrm{d}_{\zeta}^{j}\mathcal{G}_{\mu}[\varepsilon\zeta](h)\psi|_{H^{s-\frac{1}{2}}} \leq \varepsilon^{j}\mu M\prod_{m=1}^{j}|h_m|_{H^{\max\{s+\frac{1}{2},t_0\}+1}}|\psi|_{\dot{H}_{\mu}^{s+1}}. \tag{A.49}$$
4. *For all* $0 \leq s \leq t_0 + \frac{1}{2}$*,* $j \geq 1$*, there holds,*
$$|\mathrm{d}_{\zeta}^{j}\mathcal{G}_{\mu}[\varepsilon\zeta](h)\psi|_{H^{s-\frac{1}{2}}} \leq \varepsilon^{j}\mu^{\frac{3}{4}}M|h_k|_{H^{s+\frac{1}{2}}}\prod_{m\neq k}^{j}|h_m|_{H^{\max\{s,t_0\}+\frac{3}{2}}}|\psi|_{\dot{H}_{\mu}^{\max\{s,t_0\}+1}}. \tag{A.50}$$

5. For all $0 \leq s \leq t_0$, $j \geq 1$, *there holds,*

$$|\mathrm{d}^j_\zeta \mathcal{G}_\mu[\varepsilon\zeta](h)\psi|_{H^{s-\frac{1}{2}}} \leq \varepsilon^j \mu M |h_k|_{H^{s+1}} \prod_{m\neq k}^{j} |h_m|_{H^{\max\{s+\frac{1}{2},t_0\}+\frac{3}{2}}} |\psi|_{\dot{H}_\mu^{\max\{s+\frac{1}{2},t_0\}+1}}. \tag{A.51}$$

Proof. The proof is similar to the one in [50], but we have to track the dependence in the small parameters for the current scaling and adapt them to a different functional setting. We prove each point separately.

Step 1. We do the details here, where we note that it is important not to compose the inverse of $\mathcal{G}_\mu^+$ with $\mathcal{G}_\mu^-$ (see Remark 2.5). With this in mind, we observe by direct computations (suppressing the argument in ζ) that

$$\mathrm{d}(\mathcal{G}_\mu \circ \mathcal{J}_\mu)(h) = \mathrm{d}\mathcal{G}_\mu(h) \circ \mathcal{J}_\mu + \mathcal{G}_\mu \circ \mathrm{d}\mathcal{J}_\mu(h).$$

Then use definition (2.1) and compose the above identity with $\mathcal{J}_\mu^{-1}$ from the right to obtain that

$$\mathrm{d}\mathcal{G}_\mu(h) = \mathrm{d}\mathcal{G}_\mu^+(h) \circ (\mathcal{J}_\mu)^{-1} - \mathcal{G}_\mu \circ \mathrm{d}\mathcal{J}_\mu(h) \circ (\mathcal{J}_\mu)^{-1}. \tag{A.52}$$

Then we note by Proposition 2.11 that $(\mathcal{J}_\mu)^{-1}\psi = \psi^+$ allowing us to use formula (A.11) and express the first term by

$$\begin{aligned}\mathrm{d}\mathcal{G}_\mu^+(h)(\mathcal{J}_\mu)^{-1}\psi &= \mathrm{d}\mathcal{G}_\mu^+(h)\psi^+ \\ &= -\varepsilon\mathcal{G}_\mu^+(h\underline{w}^+) - \varepsilon\mu\partial_x(h\underline{V}^+).\end{aligned}$$

For the second term of (A.52), we first make the observation

$$\begin{aligned}(\mathcal{G}_\mu^-)^{-1} \circ \mathrm{d}\mathcal{G}_\mu^+(h) &= (\mathcal{G}_\mu^-)^{-1} \circ \mathrm{d}(\mathcal{G}_\mu^- \circ ((\mathcal{G}_\mu^-)^{-1} \circ \mathcal{G}_\mu^+))(h) \\ &= (\mathcal{G}_\mu^-)^{-1} \circ \Big(\mathrm{d}\mathcal{G}_\mu^-(h) \circ ((\mathcal{G}_\mu^-)^{-1} \circ \mathcal{G}_\mu^+) - \gamma^{-1}\mathcal{G}_\mu^- \circ \mathrm{d}\mathcal{J}_\mu(h)\Big) \\ &= (\mathcal{G}_\mu^-)^{-1} \circ \mathrm{d}\mathcal{G}_\mu^-(h) \circ (\mathcal{G}_\mu^-)^{-1} \circ \mathcal{G}_\mu^+ + \gamma^{-1}\mathrm{d}\mathcal{J}_\mu(h),\end{aligned}$$

so that

$$\mathrm{d}\mathcal{J}_\mu(h) = \gamma(\mathcal{G}_\mu^-)^{-1} \circ \Big(\mathrm{d}\mathcal{G}_\mu^+(h) - \mathrm{d}\mathcal{G}_\mu^-(h) \circ (\mathcal{G}_\mu^-)^{-1} \circ \mathcal{G}_\mu^+\Big).$$

Then by Remark 2.11 we have that $\psi^- = (\mathcal{G}_\mu^-)^{-1} \circ \mathcal{G}_\mu^+\psi^+$, together with the shape derivative formulas (A.11) and (A.28), we deduce that

$$\begin{aligned}\mathrm{d}\mathcal{J}_\mu(h) \circ (\mathcal{J}_\mu)^{-1}\psi &= \mathrm{d}\mathcal{J}_\mu(h)\psi^+ \\ &= \gamma(\mathcal{G}_\mu^-)^{-1}\Big(\mathrm{d}\mathcal{G}_\mu^+(h)\psi^+ - \mathrm{d}\mathcal{G}_\mu^-(h)\psi^-\Big) \\ &= -\varepsilon\gamma(\mathcal{G}_\mu^-)^{-1}\Big(\mathcal{G}_\mu^+(hw^+) - \mathcal{G}_\mu^-(hw^-)\Big) - \gamma\varepsilon\mu(\mathcal{G}_\mu^-)^{-1}\Big(\partial_x(hV^+) - \partial_x(hV^-)\Big).\end{aligned}$$

From (A.52) and definition of $\mathcal{I}[U]h$ given by (4.2), we may collect the above identities and factorize the leading terms together to find that

$$\begin{aligned}\mathrm{d}\mathcal{G}_\mu(h)\psi &= -\varepsilon\mathcal{G}_\mu^+(hw^+) - \varepsilon\gamma\mathcal{G}_\mu \circ (\mathcal{G}_\mu^-)^{-1}\Big(\mathcal{G}_\mu^+(hw^+) - \mathcal{G}_\mu^-(hw^-)\Big) \\ &\quad - \varepsilon\mu\Big(\partial_x(hV^+) + \gamma\mathcal{G}_\mu \circ (\mathcal{G}^-)^{-1}\Big(\partial_x(hV^+) - \partial_x(hV^-)\Big)\Big) \\ &= -\varepsilon\Big(1 + \gamma\mathcal{G}_\mu \circ (\mathcal{G}^-)^{-1}\Big) \circ \mathcal{G}_\mu^+(hw^+) + \varepsilon\gamma\mathcal{G}_\mu(hw^-) - \mu\varepsilon\mathcal{I}(U)h.\end{aligned}$$

Clearly, the proof is a consequence of the following relation

$$\Big(1 + \gamma\mathcal{G}_\mu \circ (\mathcal{G}_\mu^-)^{-1}\Big) \circ \mathcal{G}_\mu^+ = \mathcal{G}_\mu. \tag{A.53}$$

and can be seen in the identity

$$\mathcal{G}_\mu^+ = \mathcal{G}_\mu \circ \mathcal{J}_\mu$$
$$= \mathcal{G}_\mu - \gamma \mathcal{G}_\mu \circ (\mathcal{G}_\mu^-)^{-1} \circ \mathcal{G}_\mu^+.$$

Step 2. We first use the previous step and (A.52) to write

$$\mathrm{d}\mathcal{G}_\mu(h)\psi = \mathrm{d}\mathcal{G}_\mu^+(h)\psi^+ - \mathcal{G} \circ \mathrm{d}\mathcal{J}_\mu(h)\psi^+. \tag{A.54}$$

Then for the first term, we use (A.12) and Proposition (2.10) with estimate (2.23) to get that

$$|\mathrm{d}^j \mathcal{G}_\mu^+(h)\psi^+|_{H^{s-\frac{1}{2}}} \leq \varepsilon^j \mu^{\frac{3}{4}} M \prod_{m=1}^j |h_m|_{H^{\max\{s,t_0\}+1}} |\mathcal{J}_\mu^{-1}\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}},$$
$$\leq \varepsilon^j \mu^{\frac{3}{4}} M \prod_{m=1}^j |h_m|_{H^{\max\{s,t_0\}+1}} |\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}.$$

For the second term, we first let $j = 1$ and observe by direct calculations (or see previous step) that

$$|\mathrm{d}\mathcal{J}_\mu\psi^+|_{\mathring{H}^{s+\frac{1}{2}}} \leq |(\mathcal{G}_\mu^-)^{-1}\mathrm{d}\mathcal{G}_\mu^+(h_1)\psi^+|_{\mathring{H}^{s+\frac{1}{2}}} + |(\mathcal{G}_\mu^-)^{-1}\mathrm{d}\mathcal{G}_\mu^-(h_1)\psi^-|_{\mathring{H}^{s+\frac{1}{2}}}$$
$$=: R_1 + R_2.$$

Here we treat R_1 by (A.35) and then (2.23) to get that

$$R_1 \leq \varepsilon \mu^{\frac{1}{4}} M |h_1|_{H^{\max\{s,t_0\}+1}} |\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}.$$

For R_2 we use (A.37), then the relation $\psi^- = (\mathcal{G}_\mu^-)^{-1}\mathcal{G}_\mu^+\psi^+$ together with estimates (2.11) and (2.23) to obtain,

$$R_2 \leq \varepsilon M |h_1|_{H^{\max\{s,t_0\}+1}} |\psi^-|_{\mathring{H}^{s+\frac{1}{2}}}$$
$$\leq \varepsilon \mu^{\frac{1}{4}} M |h_1|_{H^{\max\{s,t_0\}+1}} |\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}.$$

As a result, for $j = 1$, we have by (2.2) (for $0 \leq s \leq t_0 + 1$) and the above estimates that

$$|\mathcal{G} \circ \mathrm{d}\mathcal{J}(h)\psi^+|_{H^{s-\frac{1}{2}}} \leq \mu^{\frac{3}{2}} M |\mathrm{d}\mathcal{J}(h)\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}}$$
$$\leq \sqrt{\mu} M |\mathrm{d}\mathcal{J}(h)\psi^+|_{\mathring{H}^{s+\frac{1}{2}}}$$
$$\leq \varepsilon \mu^{\frac{3}{4}} M |h_1|_{H^{\max\{s,t_0\}+1}} |\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}.$$

The remaining cases follow recursively and can be proved by induction.

Step 3−5. The proof follows similarly, where for the first term in (A.54) we instead use (A.13) or (A.14), or (A.15). While for the second term, we can gain precision in μ using (2.4).

□

Remark A.23. *In Step 1. of the proof, we found a formula for the shape derivative of $\mathcal{J}_\mu$:*

$$\mathrm{d}_\zeta \mathcal{J}_\mu[\varepsilon\zeta](h) = \gamma(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1} \circ \Big(\mathrm{d}_\zeta \mathcal{G}_\mu^+[\varepsilon\zeta](h) - \mathrm{d}_\zeta \mathcal{G}_\mu^-[\varepsilon\zeta](h) \circ (\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1} \circ \mathcal{G}_\mu^+[\varepsilon\zeta]\Big).$$

Furthermore, in Step 2.*, we proved the estimate*

$$|\mathrm{d}_\zeta \mathcal{J}_\mu[\varepsilon\zeta](\partial_x\zeta)\psi|_{\mathring{H}^{s+\frac{1}{2}}} \leq \varepsilon\mu^{\frac{1}{4}} M|\partial_x\zeta|_{H^{\max\{s,t_0\}+1}}|\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}. \tag{A.55}$$

From these expressions, we can deduce an estimate on the inverse:

$$\begin{aligned}|\mathrm{d}_\zeta(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\partial_x\zeta)\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}} &= |(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}\circ \mathrm{d}_\zeta\mathcal{J}_\mu[\varepsilon\zeta](\partial_x\zeta)\circ(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}\psi^+|_{\dot{H}_\mu^{s+\frac{1}{2}}} \\ &\leq \varepsilon M|\partial_x\zeta|_{H^{\max\{s,t_0\}+1}}|\psi|_{\dot{H}_\mu^{s+\frac{1}{2}}}.\end{aligned} \tag{A.56}$$

For an application of these estimates, see proof of (3.22)*.*

Remark A.24. *In step* 1. *in the proof we also have some convenient identities for* $\mathcal{G}_\mu$*. We saw that*

$$\Big(1+\gamma\mathcal{G}_\mu[\varepsilon\zeta]\circ(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\Big)\circ\mathcal{G}_\mu^+[\varepsilon\zeta] = \mathcal{G}_\mu[\varepsilon\zeta].$$

Composing this identity with $(\mathcal{G}_\mu^+[\varepsilon\zeta])^{-1}$ *on the right, we get a quantity we will use later:*

$$\big(1+\gamma\mathcal{G}_\mu[\varepsilon\zeta](\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\big) = \mathcal{G}_\mu[\varepsilon\zeta](\mathcal{G}_\mu^+[\varepsilon\zeta])^{-1} \tag{A.57}$$

Furthermore, we have that

$$\begin{aligned}-\gamma(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta] &= -\gamma(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta](\mathcal{J}_\mu[\varepsilon\zeta])^{-1} \\ &= 1-(\mathcal{J}_\mu[\varepsilon\zeta])^{-1},\end{aligned}$$

and implies

$$(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}+\gamma(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta] = 1+2\gamma(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta]. \tag{A.58}$$

Lastly, we will make use of the following identities:

$$\begin{aligned}\gamma(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta](\mathcal{J}_\mu[\varepsilon\zeta])^{-1} &= -(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}+\gamma(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta](\mathcal{J}_\mu[\varepsilon\zeta])^{-1}+(\mathcal{J}_\mu[\varepsilon\zeta])^{-1} \\ &= (\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\mathcal{J}_\mu[\varepsilon\zeta]-1) \\ &= -\gamma(\mathcal{J}_\mu[\varepsilon\zeta])^{-1}(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta],\end{aligned}$$

which implies

$$\gamma(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\mathcal{G}_\mu^+[\varepsilon\zeta]\mathcal{J}_\mu[\varepsilon\zeta]^{-1}(\mathcal{G}_\mu^+[\varepsilon\zeta])^{-1} = -\gamma\mathcal{J}_\mu[\varepsilon\zeta]^{-1}(\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}. \tag{A.59}$$

Proof of Proposition A.19. The proof relies on the estimates provided by Lemma A.22. However, these estimates are the same as in [50], and so we refer the reader to the proof of Proposition 6 in this paper. □

To prove the main result of this section, we also need the following Lemma:

Lemma A.25. *Under the assumptions of Proposition A.19, one has for* $|\alpha|\leq N-1$ *that*

$$\mathcal{G}_\mu^\pm[\varepsilon\zeta]\psi_{(\alpha)} = r_\alpha^\pm,$$

In the case when $|\alpha| = N$*, then*

$$\mathcal{G}_\mu^\pm[\varepsilon\zeta]\psi_{(\alpha)}^\pm = \mathcal{G}_\mu[\varepsilon\zeta]\psi_{(\alpha)} - \gamma\varepsilon\mu\mathcal{G}_\mu[\varepsilon\zeta](\mathcal{G}_\mu^\mp[\varepsilon\zeta])^{-1}\partial_x\Big(\zeta_{(\alpha)}(\underline{V}^+-\underline{V}^-)\Big),$$

where the residual terms $r_\alpha^\pm$ *satisfies*

$$|(\mathcal{G}_\mu^\pm[\varepsilon\zeta])^{-1}r_\alpha^+|^2_{\dot{H}_\mu^{\frac{3}{2}}} \leq C\mathcal{E}^N(\mathbf{U})(1+\gamma\varepsilon^2\sqrt{\mu}|\zeta|^2_{<N+\frac{1}{2}>}),$$

with $|\zeta|_{<N+\frac{1}{2}>} = \sum_{\alpha\in\mathbb{N}^2,|\alpha|=N}|\partial_{x,t}^\alpha\zeta|_{\mathring{H}^{\frac{1}{2}}}$.

Remark A.26. *Here it is crucial to use that* $\mathring{H}^{s+\frac{1}{2}}(\mathbb{R}) \subset \dot{H}_\mu^{s+\frac{1}{2}}(\mathbb{R})$ *where we have the estimate*

$$|f|_{\dot{H}_\mu^{s+\frac{1}{2}}} \leq \mu^{-\frac{1}{4}}|f|_{\mathring{H}^{s+\frac{1}{2}}}.$$

To regain the precision in μ*, we note that is provided in Proposition 2.4 and Corollary A.17.*

Proof. The proof is similar to the proof of Proposition 7 in [50], so we will just point out the differences. To that end, we focus on the case $|\alpha| = N$ and apply $\partial_{x,t}^\alpha$ to the relation $\mathcal{G}_\mu^\pm[\varepsilon\zeta] = \mathcal{G}_\mu[\varepsilon\zeta]$. Then using identities (A.11), (A.11), and (A.47) (or Proposition A.19 without sub-principal terms) to find that

$$\mathcal{G}_\mu^\pm[\varepsilon\zeta]\psi_{(\alpha)} - \varepsilon\mu\partial_x(\zeta_{(\alpha)}\underline{V}^\pm) = \mathcal{G}_\mu[\varepsilon\zeta]\psi_{(\alpha)} - \varepsilon\mu\mathcal{I}[U]\zeta_{(\alpha)} + r_\alpha^\pm,$$

where $r_\alpha^\pm$ is on the form

$$\mathrm{d}_\zeta^j\mathcal{G}_\mu[\varepsilon\zeta](\partial_{x,t}^{l^1}\zeta, ..., \partial_{x,t}^{l^j}\zeta)\partial_{x,t}^\delta\psi - \mathrm{d}_\zeta^j\mathcal{G}_\mu[\varepsilon\zeta]^\pm(\partial_{x,t}^{l^1}\zeta, ..., \partial_{x,t}^{l^j}\zeta)\partial_{x,t}^\delta\psi^\pm,$$

with $\sum\limits_{i=1}^{j}|l^j| + |\delta| = N$, $0 \leq |\delta| \leq N-1$ and $|l^i| \leq N$. The proof of estimate is the same as in the Proposition 7 in [50] after using Remark A.26. Moreover, applying the definition of $\mathcal{I}[U]\zeta_{(\alpha)}$ we find that

$$\mathcal{G}_\mu^+[\varepsilon\zeta]\psi_{(\alpha)}^+ = \mathcal{G}_\mu[\varepsilon\zeta]\psi_{(\alpha)} - \gamma\varepsilon\mu\mathcal{G}_\mu[\varepsilon\zeta](\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\partial_x\Big(\zeta_{(\alpha)}(\underline{V}^+ - \underline{V}^-)\Big).$$

Similarly, for $\mathcal{G}_\mu^-$ we observe that

$$\mathcal{G}_\mu^-[\varepsilon\zeta]\psi_{(\alpha)}^- = \mathcal{G}_\mu[\varepsilon\zeta]\psi_{(\alpha)} - \varepsilon\mu\big(1 + \gamma\mathcal{G}_\mu[\varepsilon\zeta](\mathcal{G}_\mu^-[\varepsilon\zeta])^{-1}\big)\partial_x\Big(\zeta_{(\alpha)}(\underline{V}^+ - \underline{V}^-)\Big).$$

Then we conclude by relation (A.57).

□

We will now turn to the proof of Proposition 4.2. However, since the quantities involved satisfy the same estimates as in [50], we only point out the main differences (see also for a similar approach [35]).

Proof of Proposition 4.2. Let $\psi^\pm$ and ψ be defined as in the transmission problem (2.21) throughout the proof. Then for the first equation, we simply apply $\partial_{x,t}^\alpha$ and conclude by Proposition A.19.

For the second equation, we need to prove that

$$\begin{aligned} &\text{if } 1 \leq |\alpha| < N: & \partial_t\psi_{(\alpha)} + \mathfrak{Ins}[\mathbf{U}]\zeta_{(\alpha)} = \varepsilon S_\alpha, \\ &\text{if } \quad |\alpha| = N: & \partial_t\psi_{(\alpha)} + \mathfrak{Ins}[\mathbf{U}]\zeta_{(\alpha)} - \varepsilon\mathcal{I}[\mathbf{U}]^*\psi_{(\alpha)} = \varepsilon S_\alpha. \end{aligned} \tag{A.60}$$

We will only prove the most difficult case $|\alpha| = N$, but first need an observation. Let ∂ denote either the derivative with respect to x or t. Then the following identity holds,

$$\begin{aligned} &\partial_t\partial\psi + (1-\gamma)\partial\zeta - \frac{\varepsilon}{\mu}(\underline{w}^+ - \gamma\underline{w}^-)\partial_t\partial\zeta \\ &+ \varepsilon\underline{V}^+(\partial_x\partial\psi^+ - \varepsilon\underline{w}^+\partial_x\partial\zeta) - \gamma\varepsilon\underline{V}^-(\partial_x\partial\psi^- - \varepsilon\underline{w}^-\partial_x\partial\zeta) \\ &= -\frac{1}{\mathrm{bo}}\frac{1}{\varepsilon\sqrt{\mu}}\partial\kappa(\varepsilon\sqrt{\mu}\zeta). \end{aligned} \tag{A.61}$$

To prove the claim (A.61), we apply ∂ to the second equation (1.13) to find that

$$\partial_t\partial\psi+(1-\gamma)\partial\zeta+\varepsilon\big((\partial_x\psi^+)(\partial_x\partial\psi^+)-\gamma(\partial_x\psi^-)(\partial_x\partial\psi^-)\big)+\varepsilon\partial\mathcal{N}[\varepsilon\zeta,\psi^\pm]=-\frac{1}{\mathrm{bo}}\frac{1}{\varepsilon\sqrt{\mu}}\partial\kappa(\varepsilon\sqrt{\mu}\zeta), \tag{A.62}$$

where

$$\begin{aligned}\varepsilon\partial\mathcal{N}[\varepsilon\zeta,\psi^\pm] &= -\partial\Bigg(\frac{\varepsilon}{2\mu}(1+\varepsilon^2\mu(\partial_x\zeta)^2)\Big((\underline{w}^+)^2-\gamma(\underline{w}^-)^2\Big)\Bigg)\\ &= -\varepsilon^3(\partial_x\zeta)(\partial_x\partial\zeta)\Big((\underline{w}^+)^2-\gamma(\underline{w}^-)^2\Big)\\ &\quad -\frac{1}{\mu}(1+\varepsilon^2\mu(\partial_x\zeta)^2)\Big((\underline{w}^+)(\partial\underline{w}^+)-\gamma(\underline{w}^-)(\partial\underline{w}^-)\Big).\end{aligned}$$

To conclude, we use observe by definition that (or use Lemma 4.13 [51]):

$$\begin{aligned}&\varepsilon(\partial_x\psi^\pm)(\partial_x\partial\psi^\pm)-\varepsilon^3(\partial_x\zeta)(\partial_x\partial\zeta)(\underline{w}^\pm)^2-\frac{\varepsilon}{\mu}(1+\varepsilon^2\mu(\partial_x\zeta)^2)(\underline{w}^\pm)(\partial\underline{w}^\pm)\\ &\quad=\varepsilon\underline{V}^\pm(\partial_x\partial\psi^\pm-\varepsilon\underline{w}^\pm\partial_x\partial\zeta)-\frac{\varepsilon}{\mu}\underline{w}^\pm\partial\big(\mathcal{G}_\mu^\pm[\varepsilon\zeta]\psi^\pm\big).\end{aligned}$$

By adding these observations and trading $\frac{1}{\mu}\mathcal{G}_\mu^+[\varepsilon\zeta]\psi^+$ and $\frac{1}{\mu}\mathcal{G}_\mu^-[\varepsilon\zeta]\psi^-$ with $\partial_t\zeta$, we deduce the identity (A.61).

The next step is to let $\alpha=\beta+\delta$ with $|\delta|=1$, where we trade ∂ with $\partial_{x,t}^\delta$ in (A.61). Then we apply $\partial_{x,t}^\beta$ to this equation, where we claim that it will result in the following equation:

$$\begin{aligned}&\partial_t\partial_{x,t}^\alpha\psi+(1-\gamma)\partial_{x,t}^\alpha\zeta-\frac{\varepsilon}{\mu}(\underline{w}^+-\gamma\underline{w}^-)\partial_t\partial_{x,t}^\alpha\zeta\\ &\quad+\varepsilon\underline{V}^+(\partial_x\psi_{(\alpha)}^++\varepsilon(\partial_x\underline{w}^+)\partial_x\partial_{x,t}^\alpha\zeta)-\gamma\varepsilon\underline{V}^-(\partial_x\psi_{(\alpha)}^-+\varepsilon(\partial_x\underline{w}^-)\partial_x\partial_{x,t}^\alpha\zeta)\\ &\quad=-\frac{1}{\mathrm{bo}}\frac{1}{\varepsilon\sqrt{\mu}}\partial_{x,t}^\alpha\kappa(\varepsilon\sqrt{\mu}\zeta)+\varepsilon S_\alpha,\end{aligned}\tag{A.63}$$

where S_α is some generic function, satisfying

$$|S_\alpha|^2_{\dot{H}_\mu^{\frac12}}\leq C\mathcal{E}^N(U)\big(1+\varepsilon^2\sqrt{\mu}|\underline{V}^+-\underline{V}^-|^2_{L^\infty}|\zeta|^2_{<N+\frac12>}\big).\tag{A.64}$$

The proof of this fact is the same as in [50], where the estimate relies on the following inequalities

$$|\psi_{(\alpha)}^\pm|_{\dot{H}_\mu^{\frac12}}\leq 1+\gamma\varepsilon\mu^{\frac14}|\underline{V}^+-\underline{V}^-|_{L^\infty}|\zeta|_{<N+\frac12>},\tag{A.65}$$

and

$$\frac{1}{\sqrt{\mu}}|\partial^\beta\underline{w}^\pm|_{\dot{H}_\mu^{\frac12}}\leq 1+\gamma\varepsilon\mu^{\frac14}|\underline{V}^+-\underline{V}^-|_{L^\infty}|\zeta|_{<N+\frac12>}.\tag{A.66}$$

This can be seen in the proof of Lemma 9 in [50]. In our case, the proof is a consequence of Lemma A.25 and Corollary A.18. However, the quantities involved satisfy the same estimates and therefore complete the estimate on S_α.

We may now further decompose (A.63), where we may now use the definition of $\psi_{(\alpha)}$ to find that

$$\partial_t\psi_{(\alpha)}+\mathfrak{a}\zeta_{(\alpha)}+\varepsilon\underline{V}^+\partial_x\psi_{(\alpha)}^+-\gamma\varepsilon\underline{V}^-\partial_x\psi_{(\alpha)}^-=-\frac{1}{\mathrm{bo}}\frac{1}{\varepsilon\sqrt{\mu}}\partial^\alpha\kappa(\varepsilon\sqrt{\mu}\zeta)+\varepsilon S_\alpha,$$

where we identify $\mathfrak{a}$ by

$$\mathfrak{a}=\Big((1-\gamma)+\varepsilon\big((\partial_t+\varepsilon\underline{V}^+\partial_x)\underline{w}^+-\gamma(\partial_t+\varepsilon\underline{V}^-\partial_x)\underline{w}^-\big)\Big).$$

To conclude, we simply need to work on the terms:

$$\varepsilon\underline{V}^+\partial_x\psi^+_{(\alpha)} - \gamma\varepsilon\underline{V}^-\partial_x\psi^-_{(\alpha)} = \frac{\varepsilon}{2}\big(\underline{V}^+ + \underline{V}^-\big)\partial_x\psi_{(\alpha)} + \frac{\varepsilon}{2}\big(\underline{V}^+ - \underline{V}^-\big)\partial_x\big(\psi^+_{(\alpha)} + \gamma\psi^-_{(\alpha)}\big). \quad \text{(A.67)}$$

Then by identities (A.58), (A.59), and Lemma A.25 implies

$$\begin{aligned}
&\psi^+_{(\alpha)} + \gamma\psi^-_{(\alpha)} \\
&= \big((\mathcal{J}_\mu[\varepsilon\zeta]^{-1} + \gamma(\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta]\big)\psi_{(\alpha)} \\
&\quad - \gamma\varepsilon\mu\Big(\mathcal{J}_\mu[\varepsilon\zeta]^{-1}(\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1} + \gamma(\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta](\mathcal{G}^+_\mu[\varepsilon\zeta])^{-1}\Big)\partial_x\big(\zeta_{(\alpha)}(\underline{V}^+ - \underline{V}^-)\big) + \tilde{r}_\alpha \\
&= \big(1 + 2\gamma(\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta]\big)\psi_{(\alpha)} - (1-\gamma)\gamma\varepsilon\mu\mathcal{J}_\mu[\varepsilon\zeta]^{-1}(\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}\circ\partial_x\big(\zeta_{(\alpha)}(\underline{V}^+ - \underline{V}^-)\big) + \tilde{r}_\alpha,
\end{aligned}$$

with

$$\partial_x\tilde{r}_\alpha = \partial_x\big((\mathcal{G}^+_\mu[\varepsilon\zeta])^{-1}r^+_\alpha + (\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}r^-_\alpha\big).$$

Here $r^\pm_\alpha$ is the term in Lemma A.25, which is of lower order. So that (A.67) becomes

$$\varepsilon\underline{V}^+\partial_x\psi^+_{(\alpha)} - \gamma\varepsilon\underline{V}^-\partial_x\psi^-_{(\alpha)} = -\varepsilon\mathcal{I}[\mathbf{U}]^*\psi_{(\alpha)} - (1-\gamma)\gamma\varepsilon^2\mu(\underline{V}^+ - \underline{V}^-)\mathfrak{E}_\mu[\varepsilon\zeta]\big(\zeta_{(\alpha)}(\underline{V}^+ - \underline{V}^-)\big),$$

where we identify $\mathcal{I}[\mathbf{U}]^*$ by

$$\mathcal{I}[\mathbf{U}]^*\bullet = -\underline{V}^+\partial_x\bullet - \gamma(\underline{V}^+ - \underline{V}^-)\partial_x\big((\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}\mathcal{G}_\mu[\varepsilon\zeta]\bullet\big).$$

The surface tension term is linearized with the formula

$$\frac{1}{\mathrm{bo}}\frac{1}{\varepsilon\sqrt{\mu}}\partial^\alpha\kappa(\varepsilon\sqrt{\mu}\zeta) = \mathrm{bo}^{-1}\partial_x\mathcal{K}(\varepsilon\sqrt{\mu}\partial_x\zeta)\partial_x\zeta_{(\alpha)} + \mathcal{K}_{(\alpha)}[\sqrt{\mu}\varepsilon\partial_x\zeta]\zeta_{\langle\check{\alpha}\rangle} + \varepsilon S_\alpha,$$

where $\mathcal{K}$ is given in Definition 4.1. We also identify $\mathfrak{Ins}[\mathbf{U}]$ by

$$\mathfrak{Ins}[\mathbf{U}]\bullet = \mathfrak{a}\bullet - (1-\gamma)\gamma\varepsilon^2\mu[\![\underline{V}^\pm]\!]\mathfrak{E}_\mu[\varepsilon\zeta]\big(\bullet[\![\underline{V}^\pm]\!]\big) - \mathrm{bo}^{-1}\partial_x\mathcal{K}[\varepsilon\sqrt{\mu}\partial_x\zeta]\partial_x\bullet,$$

where $\mathfrak{E}_\mu[\varepsilon\zeta]$ reads

$$\mathfrak{E}_\mu[\varepsilon\zeta]\bullet = \partial_x\circ\mathcal{J}_\mu[\varepsilon\zeta]^{-1}(\mathcal{G}^-_\mu[\varepsilon\zeta])^{-1}\circ\partial_x\bullet.$$

□

Appendix B. Tools

B.1. **Estimates on Fourier multipliers and classical estimates.**

In this section, we will give basic multiplier estimates. To be precise, we will give a definition of the Fourier multipliers.

Definition B.1. *We say that a Fourier multiplier* F *is of order* s ($s \in \mathbb{R}$) *and write* $\mathrm{F} \in \mathcal{S}^s$ *if* $\xi \in \mathbb{R} \mapsto F(\xi) \in \mathbb{C}$ *is smooth and satisfies*

$$\forall\xi \in \mathbb{R}, \forall\beta \in \mathbb{N}, \qquad \sup_{\xi\in\mathbb{R}} \langle\xi\rangle^{\beta-s}|\partial^\beta F(\xi)| < \infty.$$

We also introduce the seminorm

$$\mathcal{N}^s(\mathrm{F}) = \sup_{\beta\in\mathbb{N},\beta\leq 4}\ \sup_{\xi\in\mathbb{R}} \langle\xi\rangle^{\beta-s}|\partial^\beta F(\xi)|.$$

The first result is several basic multiplier estimates that are used throughout the paper.

Proposition B.2. *Let* $\mu, \gamma \in (0,1)$ *and* $f \in \mathscr{S}(\mathbb{R})$. *Then there exist a universal constant* $C > 0$ *such that:*

1. For the symbol

$$\mathcal{G}_\mu[0] = \sqrt{\mu}|\mathrm{D}|\frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{1+\gamma\tanh(\sqrt{\mu}|\mathrm{D}|)},$$

there holds,

$$\frac{1}{\mu}|\mathcal{G}_\mu[0]f|_{L^2} \leq C|\partial_x f|_{H^1}, \tag{B.1}$$

2. For the symbol

$$(\mathcal{G}_\mu[0])^{\frac{1}{2}} = \Big(\sqrt{\mu}|\mathrm{D}|\frac{\tanh(\sqrt{\mu}|\mathrm{D}|)}{1+\gamma\tanh(\sqrt{\mu}|\mathrm{D}|)}\Big)^{\frac{1}{2}},$$

there holds,

$$\frac{1}{C}|f|_{\dot{H}^{\frac{1}{2}}_\mu} \leq \frac{1}{\sqrt{\mu}}|(\mathcal{G}_\mu[0])^{\frac{1}{2}}f|_{L^2} \leq C|f|_{\dot{H}^{\frac{1}{2}}_\mu}. \tag{B.2}$$

3. There holds,

$$|f|_{\dot{H}^{s+\frac{1}{2}}_\mu} \leq \mu^{-\frac{1}{4}}|\partial_x f|_{H^{s-\frac{1}{2}}} \leq \mu^{-\frac{1}{4}}||\mathrm{D}|^{\frac{1}{2}}f|_{H^s}, \tag{B.3}$$

and

$$|f|_{\dot{H}^{s+\frac{1}{2}}_\mu} \leq |\partial_x f|_{H^s}. \tag{B.4}$$

Moreover, for $\mathfrak{B} = |\mathrm{D}|(1+\sqrt{\mu}|\mathrm{D}|)^{-\frac{1}{2}}$ *and* $S^- = -\sqrt{\mu}|\mathrm{D}|$ *there holds,*

$$|\frac{\mathfrak{B}^2}{S^-}f|_{H^{s+\frac{1}{2}}} \leq \mu^{-\frac{3}{4}}|f|_{\dot{H}^{s+\frac{1}{2}}_\mu}. \tag{B.5}$$

Proof. We first consider the square of the symbol in frequency, where the elementary inequality holds

$$\frac{1}{2}\sqrt{\mu}|\xi|\tanh(\sqrt{\mu}|\xi|) \leq \sqrt{\mu}|\xi|\frac{\tanh(\sqrt{\mu}|\xi|)}{1+\gamma\tanh(\sqrt{\mu}|\xi|)} \leq \sqrt{\mu}|\xi|\tanh(\sqrt{\mu}|\xi|).$$

Then by splitting in high and low frequency, we can prove that there is a number $C > 0$ such that

$$\frac{\mu}{C}\frac{|\xi|^2}{(1+\sqrt{\mu}|\xi|)} \leq \sqrt{\mu}|\xi|\tanh(\sqrt{\mu}|\xi|) \leq \mu C\frac{|\xi|^2}{(1+\sqrt{\mu}|\xi|)}.$$

To conclude, use Plancherel's identity for (B.1), and take the square root of the inequalities above to deduce (B.2).

The proof of the estimates in point three follows directly by Plancherel's identity and elementary inequalities. □

We will also use the following product estimates (see Proposition $B.2$ and Proposition $B.4$ in [51]).

Lemma B.3. *Let* $t_0 > \frac{d}{2}$, $s \geq -t_0$, $f \in H^{\max\{t_0,s\}}(\mathbb{R}^d)$, *and take* $g \in H^s(\mathbb{R}^d)$ *then*

$$|fg|_{H^s} \lesssim |f|_{H^{\max\{t_0,s\}}}|g|_{H^s}. \tag{B.6}$$

Moreover, if there exist $c_0 > 0$ *and* $1+g \geq c_0$ *then*

$$\Big|\frac{f}{1+g}\Big|_{H^s} \lesssim C(c_0,|g|_{L^\infty})(1+|f|_{H^s})|g|_{H^s}. \tag{B.7}$$

Lastly, we will use several commutator estimates for Fourier multipliers. The first result is a generalization of the classical Kato-Ponce estimate[5] and is given in [51]:

[5]See [41] for the case $\mathrm{F} = \Lambda^s$.

Proposition B.4. *Let $t_0 > \frac{1}{2}$, $s \geq 0$ and $\mathrm{F} \in \mathcal{S}^s$. If $f \in H^{\max\{t_0+1,s\}}(\mathbb{R})$ then, for all $g \in H^{s-1}(\mathbb{R})$,*

$$|[\mathrm{F}, f]g|_{L^2} \leq \mathcal{N}^s(\mathrm{F})|f|_{H^{\max\{t_0+1,s\}}}|g|_{H^{s-1}}. \tag{B.8}$$

Moreover, we will also need some commutator estimates on multipliers with non-smooth symbol.

Proposition B.5. *Let $f, g \in \mathscr{S}(\mathbb{R})$, $t_0 > \frac{1}{2}$ then there is a universal constant $C > 0$ such that*

$$|[f, |\mathrm{D}|^{\frac{1}{2}}]g| \leq C|f|_{H^{t_0+\frac{1}{2}}}|g|_{L^2} \tag{B.9}$$

and for $\mu \in (0, 1)$ there holds,

$$|[f, (1 + \sqrt{\mu}|\mathrm{D}|)^{\frac{1}{2}}]g|_{L^2} \leq C|f|_{H^{t_0+\frac{1}{2}}}|g|_{L^2}. \tag{B.10}$$

Proof. For the proof of (B.9), we write the commutator as a convolution product in frequency:

$$\big|[f, |\mathrm{D}|^{\frac{1}{2}}]g\big|_{L^2} = \bigg| \int_{\mathbb{R}} \Big(|\xi|^{\frac{1}{2}} - |\rho|^{\frac{1}{2}}\Big)\hat{f}(\xi - \rho)\widehat{\partial_x g}(\rho)\, d\rho\bigg|_{L^2_\xi}.$$

Then the proof is a direct consequence of the estimate $|\xi|^{\frac{1}{2}} - |\rho|^{\frac{1}{2}} \leq 1 + |\xi - \rho|^{\frac{1}{2}}$ when combined with Minkowski integral inequality and Cauchy-Schwarz inequality.

Inequality (B.10) is proved similarly. □

B.2. **Estimates on pseudo-differential operators.** In this section, we will give estimates of pseudo-differential operators whose symbol depends on the free surface. In particular, the framework needs to handle symbols of limited smoothness which is developed in [49]. To be precise, we give the definition of the objects we will study.

Definition B.6. *Let $m \in \mathbb{R}$ and $t_0 > \frac{1}{2}$. A symbol $\sigma(x, \xi)$ belongs to the class $\Gamma^m_{t_0}$ if and only if*

$$\sigma|_{\mathbb{R}\times\{|\xi|\leq 1\}} \in L^\infty(\{|\xi| \leq 1\} \,:\, H^{t_0}(\mathbb{R}))$$

and for all $\beta \in \mathbb{N}$ one has

$$\sup_{\xi\in\mathbb{R}} \langle\xi\rangle^{\beta-m}|\partial^\beta_\xi \sigma(\cdot, \xi)|_{H^{t_0}} < \infty.$$

We also introduce the seminorm

$$\mathcal{L}^m_{k,s}(\sigma) := \sup_{\beta\leq k}\ \sup_{|\xi|\geq\frac{1}{4}} \langle\xi\rangle^{\beta-m}|\partial^\beta_\xi \sigma(\cdot, \xi)|_{H^s},$$

Moreover, let $l_{k,s}$ be a measure of the information in low frequencies given by

$$l_{k,s}(\sigma) := \sup_{\beta\leq k, |\xi|\leq 1} |\partial^\beta_\xi \sigma(\cdot, \xi)|_{H^s},$$

and define

$$|\sigma|_{H^s_{(m)}} = l_{0,s}(\sigma) + \mathcal{L}^m_{3,s}(\sigma).$$

The main tool we will use to derive our estimate is Theorem 1, 8, and Corollary 43 in [49] (in dimension one).

Proposition B.7. *Let $f \in \mathscr{S}(\mathbb{R})$, $t_0 \in (\frac{1}{2}, s_0]$. Then we have the following estimates:*

1. *For* $\sigma \in \Gamma^m_{s_0}$*,* $s \in (-t_0, t_0)$*, and* $m \in \mathbb{R}$*, there holds,*

$$|\mathrm{Op}(\sigma)f|_{H^s} \leq |\sigma|_{H^{s_0}_{(m)}} |f|_{H^{s+m}}, \tag{B.11}$$

   Moreover, for $-t_0 < s+m \leq t_0+1$ *and* $-t_0 < s \leq t_0+1$*, there holds,*

$$|[\Lambda^s, \mathrm{Op}(\sigma)]f|_{H^{\frac{1}{2}}} \leq |\sigma|_{H^{s_0}_{(m)}} |f|_{H^{s+m-\frac{1}{2}}}, \tag{B.12}$$

2. *Let* $m_1, m_2 \in \mathbb{R}$*,* $\sigma_1 \in \Gamma^{m_1}_{s_0}$ *and* $\sigma_2 \in \Gamma^{m_2}_{s_0}$ *such that* $-t_0 < s + m_j \leq t_0 + 1$ *and* $-t_0 < s \leq t_0 + 1$*. Moreover, let* $(v_1, v_2) \in W^{1,\infty}(\mathbb{R})$ *and define*

$$\sigma_j(x,\xi) = \Sigma(v_j(x), \xi) \in C^\infty(\mathbb{R}, L^\infty(|\xi| \leq 1)),$$

   such that for all $\alpha, \beta \in \mathbb{N}$ *there is a positive nondecreasing function* $C_{\alpha,\beta}(\cdot)$ *satisfying,*

$$\sup_{\xi\in\mathbb{R}} \langle\xi\rangle^{\beta-m} |\partial_v^\alpha \partial_\xi^\beta \Sigma(v_j,\xi)| \leq C_{\alpha,\beta}(|v_j|).$$

   Then one has,

$$|\big(\mathrm{Op}(\sigma_1)^* - \mathrm{Op}(\overline{\sigma}_1)\big)f|_{H^s} \leq |v_1|_{W^{1,\infty}} |f|_{H^{s+m_1-1}} \tag{B.13}$$

$$|[\mathrm{Op}(\sigma_1), \mathrm{Op}(\sigma_2)]f|_{H^s} \leq |v_1|_{W^{1,\infty}} |v_2|_{W^{1,\infty}} |f|_{H^{s+m_1+m_2-1}} \tag{B.14}$$

$$|\mathrm{Op}(\sigma_1\sigma_2) - \mathrm{Op}(\sigma_1)\circ\mathrm{Op}(\sigma_2)f|_{H^s} \leq |v_1|_{W^{1,\infty}} |v_2|_{W^{1,\infty}} |f|_{H^{s+m_1+m_2-1}}. \tag{B.15}$$

Remark B.8. *The adjoint estimate* (B.13) *and* (B.15) *is not given in* [49]*, but can be deduced using similar methods. This fact is stated in footnote* 8 *and* 9 *of* [50].

Remark B.9. *The estimate in Theorem* 8 *is more general than* (B.14)*, but here we only state the result for a symbol that is bounded in the origin and smooth for positive frequencies.*

Lemma B.10. *Let* $t_0 > \frac{1}{2}$*,* $s \geq 0$*,* $\zeta \in H^{t_0+3}(\mathbb{R})$ *and define the symbol*

$$L(x,\xi,z) = e^{-z\left(\frac{\sqrt{\mu}|\xi|}{1+\varepsilon^2\mu(\partial_x\zeta)^2} - i\frac{\varepsilon\mu\partial_x\zeta\xi}{1+\varepsilon^2\mu(\partial_x\zeta)^2}\right)}.$$

Then for any $f \in \mathscr{S}(\mathbb{R})$ *there holds,*

$$\|\Lambda^s \mathrm{Op}(L)\partial_x f\|_{L^2(\mathcal{S}^-)} \leq \mu^{-\frac{1}{4}} C(|\zeta|_{H^{t_0+1}}) |f|_{\mathring{H}^{s+\frac{1}{2}}} \tag{B.16}$$

$$\|\Lambda^s \mathrm{Op}(\partial_z L) f\|_{L^2(\mathcal{S}^-)} \leq \mu^{\frac{1}{4}} C(|\zeta|_{H^{t_0+1}}) |f|_{\mathring{H}^{s+\frac{1}{2}}} \tag{B.17}$$

$$\|\Lambda^s \mathrm{Op}(\frac{1}{z}\partial_x L) f\|_{L^2(\mathcal{S}^-)} \leq \varepsilon\mu^{\frac{3}{4}} C(|\zeta|_{H^{t_0+2}}) |f|_{\mathring{H}^{s+\frac{1}{2}}} \tag{B.18}$$

$$\|\Lambda^s \mathrm{Op}(\frac{1}{z}\partial_x^2 L) f\|_{L^2(\mathcal{S}^-)} \leq \varepsilon\mu^{\frac{3}{4}} C(|\zeta|_{H^{t_0+3}}) |f|_{\mathring{H}^{s+\frac{1}{2}}} \tag{B.19}$$

$$\|\Lambda^s \mathrm{Op}(\partial_x\partial_z L) f\|_{L^2(\mathcal{S}^-)} \leq \varepsilon\mu^{\frac{3}{4}} C(|\zeta|_{H^{t_0+2}}) |f|_{\mathring{H}^{s+\frac{1}{2}}} \tag{B.20}$$

$$\|\Lambda^s \mathrm{Op}(\partial_x L)\partial_x f\|_{L^2(\mathcal{S}^-)} \leq \varepsilon\mu^{\frac{3}{4}} C(|\zeta|_{H^{t_0+2}}) |f|_{\mathring{H}^{s+\frac{1}{2}}}. \tag{B.21}$$

Remark B.11. *Estimate* (B.17) *also holds for the symbol* $\mathrm{Op}\big((\partial_x^2\zeta)\partial_z L\big)$*, but with constant depending on* $|\zeta|_{H^{t_0+2}}$*.*

Proof. The inequalities are proved similarly, where we first consider the proof of (B.16). To employ Proposition B.7, we define the symbol:

$$\sigma_z(x,\xi) = e^{-z(\frac{1}{1+\varepsilon^2\mu(\partial_x\zeta)^2} - \frac{1}{2})\sqrt{\mu}|\xi|} e^{-iz\frac{\varepsilon\mu\partial_x\zeta\xi}{1+\varepsilon^2\mu(\partial_x\zeta)^2}}.$$

Then for $z \in (0,\infty)$ and any $|\xi| \geq \frac{1}{4}$ we have that

$$|\partial_\xi^\beta \sigma_z(x,\xi)|_{H^{t_0}} \lesssim C(|\partial_x \zeta|_{H^{t_0}})(z\sqrt{\mu})^\beta e^{-\frac{z}{2}\sqrt{\mu}|\xi|}$$
$$\leq C(|\partial_x \zeta|_{H^{t_0}})(z\sqrt{\mu}|\xi|)^\beta e^{-\frac{z}{2}\sqrt{\mu}|\xi|}\langle \xi\rangle^{-\beta},$$

by the algebra property of $H^{t_0}(\mathbb{R})$, so that

$$\mathcal{L}^0_{3,t_0}(\sigma_z) \lesssim C(|\zeta|_{H^{t_0+1}}).$$

While in low frequencies, the symbol is bounded, which implies

$$l_{0,t_0}(\sigma_z) \leq C(|\zeta|_{H^{t_0+1}}).$$

Since $\sigma \in \Gamma^0_{t_0}$ we may use Proposition B.7 with inequality (B.11) and Plancherel's identity to say that

$$|\mathrm{Op}(L)f|_{H^s} \leq C(|\zeta|_{H^{t_0+1}})|\mathrm{e}^{-\frac{z}{2}\sqrt{\mu}|\mathrm{D}|}f|_{H^s}.$$

To conclude, we use this inequality with Plancherel's identity and Fubini's Theorem to see that

$$\|\Lambda^s \mathrm{Op}(L)\partial_x f\|^2_{L^2(\mathcal{S}^-)} = \int_0^\infty |(\mathrm{Op}(L)\partial_x f)(\cdot,z)|^2_{H^s}\,\mathrm{d}z$$
$$\leq C(|\zeta|_{H^{t_0+1}})\int_0^\infty |e^{-\frac{z}{2}\sqrt{\mu}|\mathrm{D}|}\partial_x f|^2_{H^s}\,\mathrm{d}z$$
$$\leq C(|\zeta|_{H^{t_0+1}})\int_{\mathbb{R}} ||\xi|^{\frac{1}{2}}\langle\xi\rangle^s \hat{f}(\xi)|^2\int_0^\infty e^{-z\sqrt{\mu}|\xi|}|\xi|\,\mathrm{d}z\mathrm{d}x$$
$$\leq \mu^{-\frac{1}{2}}C(|\zeta|_{H^{t_0+1}})|f|^2_{\mathring{H}^{s+\frac{1}{2}}}.$$

The proof of the remaining inequalities essentially boils down to having one more polynomial power in $|\xi|$ when compared to z. In particular, for the proof of (B.17), we see that there is a gain of $\sqrt{\mu}$ and a $|\xi|$ that appears after computing the derivative with respect to z. Therefore, the proof follows as it did for (B.16).

For the proof of (B.18), there is also a gain in the small parameters. However, there is a polynomial dependence in z and an additional $|\xi|$. Since we need $|\xi|$ for the integrability, we define

$$\tilde{\sigma}_z(x,\xi) = |\xi|^{-1}\partial_x\sigma_z(x,\xi),$$

and make the computation for $|\xi| > \frac{1}{4}$:

$$|\partial_\xi^\beta \tilde{\sigma}_z(x,\xi)|_{H^{t_0}} \leq \varepsilon C(|\zeta|_{H^{t_0+2}})(z\mu)(z\sqrt{\mu})^\beta e^{-\frac{z}{2}\sqrt{\mu}|\xi|}$$
$$\leq z\varepsilon\mu C(|\partial_x\zeta|_{H^{t_0+2}})(z\sqrt{\mu}|\xi|)^\beta e^{-\frac{z}{2}\sqrt{\mu}|\xi|}\langle\xi\rangle^{-\beta}.$$

Then we use Proposition B.7 to find that

$$\|\Lambda^s\mathrm{Op}(\frac{1}{z}\partial_x L)f\|^2_{L^2(\mathcal{S}^-)} \leq (\varepsilon\mu)^2 C(|\zeta|_{H^{t_0+2}})\int_0^\infty |e^{-\frac{z}{2}\sqrt{\mu}|\mathrm{D}|}|\mathrm{D}|f|^2_{H^s}\,\mathrm{d}z$$
$$\leq \varepsilon^2\mu^{\frac{3}{2}}C(|\zeta|_{H^{t_0+1}})|f|^2_{\mathring{H}^{s+\frac{1}{2}}}.$$

For the estimate of (B.18), the proof is similar where we define

$$\tilde{\tilde{\sigma}}_z(x,\xi) = |\xi|^{-1}\partial_x^2\sigma_z(x,\xi),$$

and find that

$$|\partial_\xi^\beta\tilde{\tilde{\sigma}}_z(x,\xi)|_{H^{t_0}} \leq z\varepsilon\mu C(|\partial_x\zeta|_{H^{t_0+3}})\langle\xi\rangle^{-\beta}.$$

At this point, the proof is the same.

The remaining two estimates are proved similar to the ones above.

□

The next estimates are given on $H^s(\mathbb{R})$ and are simpler to deal with. They are versions of estimates used in [50], but are listed here for the sake of clarity.

Proposition B.12. *Let $\varepsilon, \mu, \gamma \in (0,1)$, $t_0 > \frac{1}{2}$, and $f, g \in \mathscr{S}(\mathbb{R})$. Then there exist a universal constant $C > 0$ such that:*

1. *Define the operator*

$$\mathrm{Op}(\frac{S^+}{S^-})f(x) = -\mathcal{F}^{-1}\Big(\tanh(\sqrt{\mu}t(x,\xi)\hat{f}(\xi)\Big)(x),$$

where

$$t(X,\xi) = (1+\varepsilon\zeta)\frac{\arctan(\varepsilon\sqrt{\mu}\partial_x\zeta)}{\varepsilon\sqrt{\mu}\partial_x\zeta}|\xi|.$$

Then for $s \in (-t_0, t_0)$ there holds,

$$|\mathrm{Op}(\frac{S^+}{S^-})f|_{H^s} \leq C(|\zeta|_{H^{t_0+1}})|f|_{H^s}, \tag{B.22}$$

and for $s \in (-t_0, t_0+1]$ there holds,

$$|[\Lambda^s, \mathrm{Op}(\frac{S^+}{S^-})]f|_{H^{\frac{1}{2}}} \leq \varepsilon\sqrt{\mu}C(|\zeta|_{H^{t_0+2}})|f|_{H^{s-\frac{1}{2}}} \tag{B.23}$$

$$|(\mathrm{Op}(\frac{S^+}{S^-})^* - \mathrm{Op}(\frac{S^+}{S^-}))f|_{H^s} \leq \varepsilon\sqrt{\mu}C(|\zeta|_{H^{t_0+2}})|f|_{H^{s-1}}. \tag{B.24}$$

2. *Define the operators*

$$\mathrm{Op}(S_J)f(x) = \mathcal{F}^{-1}\Big(\big(1+\gamma\tanh(\sqrt{\mu}t(x,\xi)\big)\hat{f}(\xi)\Big)(x)$$

$$\mathrm{Op}(\frac{1}{S_J})f(x) = \mathcal{F}^{-1}\Big(\frac{\hat{f}(\xi)}{1+\gamma\tanh(\sqrt{\mu}t(x,\xi))}\Big)(x).$$

Then for $s \in (-t_0, t_0)$ there holds,

$$|\mathrm{Op}(\frac{1}{S_J})f|_{H^s} \leq C(|\zeta|_{H^{t_0+1}})|f|_{H^s}, \tag{B.25}$$

and for $s \in (-t_0, t_0+1]$ there holds,

$$|\big(1 - \mathrm{Op}(S_J)\mathrm{Op}(\frac{1}{S_J})\big)f|_{H^s} \leq \varepsilon\sqrt{\mu}C(|\zeta|_{H^{t_0+2}})|f|_{H^{s-1}}. \tag{B.26}$$

3. *Define the operator*

$$\mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)f(x) = -\mathcal{F}^{-1}\Big(\frac{\tanh(\sqrt{\mu}t(x,\xi))}{1+\gamma\tanh(\sqrt{\mu}t(x,\xi))}\hat{f}(\xi)\Big)(x).$$

Then for $s \in (-t_0, t_0)$ there holds,

$$|\mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)f|_{H^s} \leq C(|\zeta|_{H^{t_0+1}})|f|_{H^s}, \tag{B.27}$$

and for $s \in (-t_0, t_0+1]$ there holds,

$$|[\mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big), f]\partial_x g|_{H^s} \leq C(|\zeta|_{H^{t_0+1}})|g|_{H^s}|f|_{H^s} \tag{B.28}$$

$$|\Big(\mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)^* - \mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)\Big)f|_{H^s} \leq \varepsilon C(|\zeta|_{H^{t_0+2}})|f|_{H^{s-1}} \tag{B.29}$$

$$|\big(\mathrm{Op}(S^-S_J) - \mathrm{Op}(S^-)\mathrm{Op}(S_J)\big)f|_{H^s} \leq \varepsilon\mu C(|\zeta|_{H^{t_0+2}})|f|_{\dot{H}^s_\mu} \tag{B.30}$$

$$|\Big(\mathrm{Op}\Big(\frac{S^+}{S^-}\Big)\mathrm{Op}\Big(\frac{1}{S_J}\Big) - \mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)\Big)f|_{H^s} \leq \varepsilon\sqrt{\mu}C(|\zeta|_{H^{t_0+2}})|f|_{H^{s-1}}. \tag{B.31}$$

4. Define the operators

$$\mathrm{Op}\Big(\partial_x\frac{S^+}{S^-S_J}\Big)f(x) = -\mathcal{F}^{-1}\Big(\partial_x\frac{\tanh(\sqrt{\mu}t(x,\xi))}{1+\gamma\tanh(\sqrt{\mu}t(x,\xi))}\hat{f}(\xi)\Big)(x)$$

$$\mathrm{Op}\Big(\frac{\mathfrak{B}^2}{S_JS^-}\Big)f(x) = -\frac{1}{\sqrt{\mu}}\mathcal{F}^{-1}\Big(\frac{|\xi|(1+\sqrt{\mu}|\xi|)^{-1}}{1+\gamma\tanh(\sqrt{\mu}t(x,\xi))}\hat{f}(\xi)\Big)(x)$$

$$\mathrm{Op}\Big(\frac{S_JS^-}{\mathfrak{B}^2}\Big)\mathfrak{B}f(x) = -\sqrt{\mu}\mathcal{F}^{-1}\Big(\frac{1+\gamma\tanh(\sqrt{\mu}t(x,\xi))}{|\xi|(1+\sqrt{\mu}|\xi|)^{-1}}\widehat{\mathfrak{B}f}(\xi)\Big)(x).$$

Then for $s \in (-t_0, t_0)$ there holds,

$$|\mathrm{Op}\Big(\partial_x\frac{S^+}{S^-S_J}\Big)f|_{H^s} \leq C(|\zeta|_{H^{t_0+2}})|f|_{H^s}. \tag{B.32}$$

Moreover, let $k = 0, 1$, then for $s \in (-t_0, t_0)$ there holds,

$$|\big(1 - \mathrm{Op}(\frac{\mathfrak{B}^2}{S_JS^-})\mathrm{Op}(\frac{S_JS^-}{\mathfrak{B}^2})\big)\mathfrak{B}f\big|_{H^{s+\frac{k}{2}}} \leq \varepsilon\mu^{-\frac{k}{4}}C(|\zeta|_{H^{t_0+2}})|f|_{H^s}. \tag{B.33}$$

Proof. For the proof of (B.22), we note that

$$\frac{S^+}{S^-} = \tanh(\sqrt{\mu}t(x,\xi)) \in \Gamma^0_{t_0},$$

and the proof follows by Theorem B.7 with estimate (B.11):

$$|\mathrm{Op}(\frac{S^+}{S^-})f|_{H^s} \leq C(|\zeta|_{H^{t_0+1}})|f|_{H^s}.$$

For the proof of (B.23), we use (B.12) and the estimates above. While for the proof of (B.24), we observe that for $\sigma = \frac{S^+}{S^-}$ we have

$$|\sigma|_{W^{1,\infty}} \leq \varepsilon\sqrt{\mu}C(|\partial_x\zeta|_{W^{1,\infty}}),$$

and we conclude by (B.13) and the Sobolev embedding.

The estimates in point 3 are proved similarly. The only exception is (B.30). We let $\chi(\sqrt{\mu}|\xi|)$ be a smooth multiplier with compact support and equal to one around zero. Then observe that we need a Soblev estimate on

$$\begin{aligned}\mathrm{Op}(S^+) - \mathrm{Op}(S^-)\mathrm{Op}(\frac{S^+}{S^-}) &= \big(\mathrm{Op}(\frac{S^+}{S^-}) - \mathrm{Op}(S^-)\mathrm{Op}(\frac{S^+}{(S^-)^2})\big)\chi S^- \\ &\quad + \big(\mathrm{Op}(S^+) - \mathrm{Op}(S^-)\mathrm{Op}(\frac{S^+}{S^-})\big)(1-\chi).\end{aligned}$$

Note that the operators are of order $\sqrt{\mu}$ and with symbols that are elements in $\Gamma^1_{t_0}$. Then using (B.15), where the derivative of the symbols give rise to another $\varepsilon\sqrt{\mu}$ we obtain that

$$\begin{aligned}|\big(\mathrm{Op}(S^- S_J) - \mathrm{Op}(S^-)\mathrm{Op}(S_J)\big)f|_{H^s} &\leq \varepsilon\mu C(|\zeta|_{H^{t_0+2}})\big(\sqrt{\mu}|\chi\partial_x f|_{H^s} + |(1-\chi)f|_{H^s}\big)\\ &\leq \varepsilon\mu C(|\zeta|_{H^{t_0+2}})|f|_{\dot{H}^s_\mu}.\end{aligned}$$

where in the last inequality we also use that $\mu^{\frac{1}{4}}\chi\langle\xi\rangle^{\frac{1}{2}} \lesssim 1$ and $(1-\chi)\frac{\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}}\mu^{\frac{1}{4}}\langle\xi\rangle^{\frac{1}{2}}}{\sqrt{\mu}|\xi|} \lesssim 1$.

The estimates in point 4, we note that the symbol in (B.32) is of order zero and we conclude by (B.11). While for the proof of (B.33), we need to work on the domain of $\mathfrak{B}f$ for the composition to be well-defined in low frequency. In particular, we have that

$$\sigma_1 = \frac{\mathfrak{B}^2}{S_J S^-} \in \Gamma^0_{t_0}, \quad \sigma_2 = \langle \mathrm{D}\rangle^{-\frac{1}{2}}\frac{|\mathrm{D}|S_J S^-}{\mathfrak{B}^2} \in \Gamma^{\frac{1}{2}}_{t_0},$$

and the product $\sigma_1\sigma_2 = |\mathrm{D}|\langle\mathrm{D}\rangle^{-\frac{1}{2}} \in \Gamma^{\frac{1}{2}}_{t_0}$, allowing us to deduce by (B.15) for $-t_0 < s' \leq t_0 + \frac{1}{2}$ that,

$$\begin{aligned}|\big(1 - \mathrm{Op}(\frac{\mathfrak{B}^2}{S_J S^-})\mathrm{Op}(\frac{S_J S^-}{\mathfrak{B}^2})\big)\mathfrak{B}f|_{H^{s'}} &\leq C|\big(\mathrm{Op}(\sigma_1\sigma_2) - \mathrm{Op}(\sigma_1)\mathrm{Op}(\sigma_2)\big)\frac{\langle\mathrm{D}\rangle^{\frac{1}{2}}}{\langle\sqrt{\mu}\mathrm{D}\rangle^{\frac{1}{2}}}f|_{H^{s'}}\\ &\leq \varepsilon C(|\zeta|_{H^{t_0+2}})|\frac{\langle\mathrm{D}\rangle^{\frac{1}{2}}}{\langle\sqrt{\mu}\mathrm{D}\rangle^{\frac{1}{2}}}f|_{H^{s'-\frac{1}{2}}},\end{aligned}$$

where the ε is a consequence of the estimate on the derivative of the symbol S_J. Now for the case $k = 1$, we can compensate the half-derivative with the symbol $(1 + \sqrt{\mu}|\mathrm{D}|)^{-\frac{1}{2}}$ at a price of $\mu^{-\frac{1}{4}}$. For $k = 0$, the symbol is uniformly bounded and thus completes the proof.

□

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