A 43k Kernel for Planar Dominating Set using Computer-Aided Reduction Rule Discovery

Johan Torås Halseth

Department of Informatics

University of Bergen

A thesis submitted for the degree of

Master of Science

Supervisor: Daniel Lokshtanov

February 2016

I think everyone should learn how to
program a computer
because it teaches you how to think

- *Steve Jobs*

Acknowledgements

First and foremost I must thank my supervisor, Daniel. Your knowledge and guidance has been invaluable during the work with this thesis. You have always been available for questions; weekends, timezones and continents have never hindered me from reaching out. You are a great teacher, and have the ability to see simple and elegant solutions where I see problems; that is something I really admire.

I must also thank the students and faculty at the Department of Informatics, University of Bergen. You have created an environment of excitement and curiosity about the world of Computer Science, which has led me to love being a part of the field. A special thanks to the Algorithms group, whose free pizza got me interested in competitive programming and algorithms in the first place. Thanks to Simen, Magnar and Erik; you have been great team mates and mentors during programming competitions. It has been fun and extremely valuable experiences.

Of course my family must be mentioned. My dad got me interested in computers in the first place, by bringing computer magazines and old computers to tamper and play with, home from work. My parents have always let me do what I love, but reminded me to get some sleep when that is due. And Christina, my girlfriend, thank you for keeping up with me in the last week before the deadline. Now I am finally coming home.

Contents

Part I

Introduction and preliminaries

Chapter 1

Introduction

Imagine this scenario: You are the monarch in a country with several small towns scattered throughout the lands. The neighboring towns are connected by roads, which can be used to drive efficiently between them. Your people think you are an outstanding monarch, and you want to keep it that way, because it just feels so good being worshipped, as shown in Figure 1.1. Everything seems to be going smoothly, but suddenly the towns start having problems with occasional fires, and for some reason the houses burn to the ground quite fast. You decide to place fire stations in some of the towns, such that every town either has its own fire station, or a neighboring town with one. In that way you make sure that in case of fire, there's a fire truck ready in the town, or a savior can come from an adjacent town. However, building fire stations is expensive, and you need to save a good chunk of your taxpayers' money so you can build a new castle, in case you should need one. What is the minimum number of fire stations you can build to get rid of your problem?

Figure 1.1: Feels so good to be king.

An example of an instance of this problem is shown on the left in Figure 1.2. How hard can this be? You quickly realize that there are no obvious positions to place your fire stations, so you sort of need to try different placements, and convince yourself that your placement is optimal. On the right a solution is shown, proving that you can solve the problem using only 3 fire stations. In this particular case you can prove with relative ease that no smaller solution is possible, but this is not always that easy.

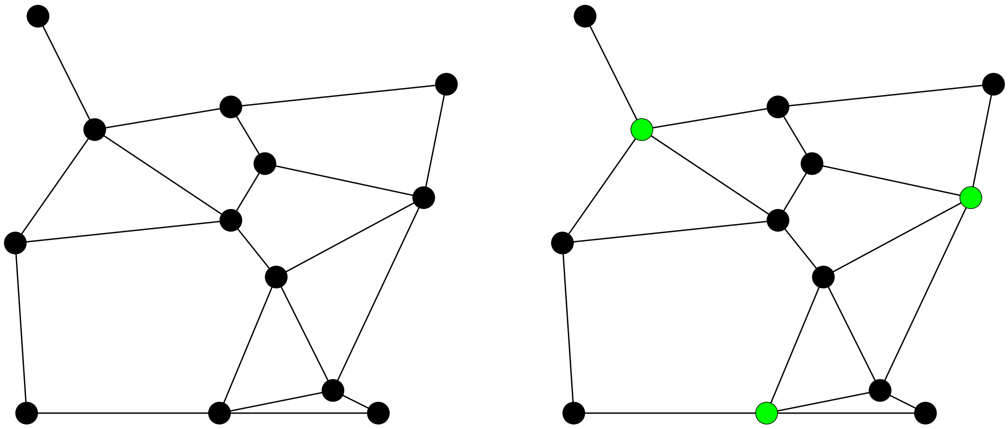

Figure 1.2: A graph and its minimal Dominating Set colored in green.

A bulletproof way to find the optimal solution is to try all possible placements of fire stations, and pick the smallest solution found. The problem with this approach is that the number of possible placements we have to try becomes astronomically big if the number of towns is large. This is what is known as a *brute-force* approach, and it is generally considered very bad for all other than small problem instances.

This particular problem is known as the DOMINATING SET problem, and has been heavily studied, both for its many practical applications and its theoretical aspects. The DOMINATING SET problem is important in many industries; a telephone operator placing radio towers at random hoping to achieve good coverage would quickly run out of money; in social network theory solving the problem can lead to insight in peoples' influence on each other[26]; and many problems in logistics, distribution and transportation can be modeled in this way.

Despite the great efforts of many people since the DOMINATING SET problem was first formalized in the 1950's[15] no efficient algorithm for solving it has been found. The reason for this was found in the 70's, when several researchers published papers on the theory on NP-completeness[7, 20, 18]. It turns out that DOMINATING SET is among these NP-complete problems, and it is believed that no efficient algorithms exist to solve them.

Even though we believe that it is not possible to solve the problem efficiently in the general case, many real-life instances of the problems have a structure that can make the problem more tractable. Looking back at the fire station example, we can observe that the instance in this case is a map showing the towns drawn in the plane, with no roads crossing each other. This information can be used to improve on our algorithms when trying to find a solution. As we shall see, the planar version of the problem, PLANAR DOMINATING SET, is considered easier than the general problem.

In some cases we can discover parts of the input instance that are easier to solve than the rest, and we can remove these parts efficiently from the instance. We will show that for a large instance to the PLANAR DOMINATING SET problem we can always reduce the instance to a size just depending on the size of the optimal Dominating Set. This remaining part we are left with is called the *kernel* of the instance.

1.1 Background and Thesis Overview

In this section, we will provide a brief overview of the thesis, including some background and results. This overview makes use of a few basic concepts from Parameterized Complexity. These concepts are explained in the preliminaries part.

In 2004, Alber, Fellows, and Niedermeier[2] presented a kernelization algorithm for PLANAR DOMINATING SET, and came up with the concept of Region Decomposition for planar graphs. This is a way of dividing planar graphs into regions, and several papers following theirs have made use of this particular technique for showing kernels for problems on planar graphs [1, 6, 11, 12, 13, 21, 23].

Unfortunately there are some ambiguities in the definitions of the core concepts of the Region Decomposition of Alber et al., and we do not see a way to resolve these ambiguities such that the proof of the lemma as published in [2, Lemma 6] is correct. For this reason, there has been some uncertainty [17] as to whether the statements made using Region Decomposition and its theorems are true, or whether they, and the results building on them, should be reconsidered. The first main contribution of this thesis is to show that minor modifications to the definitions, theorem statements and proofs of Alber et al. is enough to make their theorems and results go through. Our modifications are consistent with the use of the Region Decomposition technique in other work we are aware of, giving these results a solid foundation.

Our modified definition of Region Decomposition together with the proofs of Alber et al. show that if a planar graph has a dominating set of size k, then there exists a Region Decomposition of the graph having at most $3k$ regions. Together with the bounds of at most 55 vertices inside each region and $170k$ vertices in total outside regions from Alber et al., this results in a kernel on $335k$ vertices.

One of the papers that based their proofs on the Region Decomposition technique of Alber et al., was the 2007 paper by Chen, Fernau, Kanj, and Xia[6]. They introduce some additional reduction rules and by an improved analysis upper bound the number of vertices in every region by 16 and the vertices outside regions to $19k$. This yields a $67k$ kernel for PLANAR DOMINATING SET. Our second main contribution is to improve this result by introducing one extra reduction rule and by modifying the embedding of the graph. Doing this we are able to bound the number vertices outside regions to $7k$ in total, resulting in a $55k$ kernel.

Chen et al. were able to bound the size of the regions to 16 by using several reduction rules and an extensive case analysis. We automate this case analysis process by the use of a computer program that designs reduction rules by exhaustively searching for "reduced" regions, and proves the correctness of the generated reduction rules as it runs. In the process it keeps track of the biggest non-reducible regions found, arriving at the result that any region can be reduced to an equivalent region of size 12 or less. This gives rise to a $43k$ kernel. Be sure to read the Words of Warning at the start of Part III, as we finished the coding of the computer program a bit too close to the submission date.

Note that the search for reduced regions could have been done by naïvely enumerating all regions up to the size of 16, and keeping the reduced ones. But since this will take years, if not decades, even on a super-computer, we do an extensive analysis to speed up the process, making it possible to arrive at the result in a few days on a semi-powerful parallel computer.

Chapters 1-2 introduce the DOMINATING SET problem and necessary notation. In Chapters 3-5 we prove our Region Decomposition theorems and arrive at the $335k$ kernel. In Chapter 6 we improve this to $55k$. Finally, in Chapter 7 and out we describe our process for doing computer-aided reduction of regions, further improving the kernel to $43k$.

1.2 Terminology and Preliminaries

In this section we will introduce the reader to most of the core concepts we are touching on throughout the thesis. Many of the concepts can be found described in any good introductory books to Discrete Mathematics and Algorithms [25, 9] so we will only briefly explain the most basic ones, and spend most of our time describing those more particularly relevant for our discussion of the PLANAR DOMINATING SET problem.

1.2.1 Mathematical Notation

We will freely make use of standard set notation, as found in Rosen[25]. For the sake of convenience we will restate the most common ones here, defined tailored to our use.

Definition 1.1 (Multiset). *A* **multiset** *is an unordered collection of elements.*

Definition 1.2 (Set). *A* **set** *is a multiset where each element appears only once.*

Definition 1.3 (Universe). *A* **universe** *is the set that contains all the elements under consideration in a given situation.*

As an example, if we are looking at operations on sets of positive integers, the universe in this situation will precisely be the set of all positive integers.

Let A, B be two sets from some universe. We have the following definitions

Member of set If the element x is in the set A, we say that x is a ***member of*** A, and denote this by $x \in A$. A non-member y is denoted $y \notin A$.

Cardinality The number of members of A is called the cardinality of A, denoted $|A|$.

Empty set $A = \emptyset$ if $|A| = 0$.

Subset $A \subseteq B$, if for every $x \in A$, we have $x \in B$.

Equality $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Proper subset $A \subset B$ if $A \subseteq B$ and $A \neq B$.

Union $A \cup B = \{x \mid x \in A \vee x \in B\}$

Intersection $A \cap B = \{x \mid x \in A \wedge x \in B\}$

We will need the definitions for infimum and supremum on the set of real numbers:

Definition 1.4 (Infimum). *Let $S \subseteq \mathbb{R}$. The* **infimum of** *S, denoted $\inf(S)$, is a greatest element in $\mathbb{R}$ that is not greater than any element in S, if such an element exists.*

For instance, let $S = (0, 10]$. Then $\inf(S) = 0$. The definition of supremum is similar, but this one we will only need for the natural numbers:

Definition 1.5 (Supremum). *Let $S \subseteq \mathbb{N}$. The* **supremum of** *S, denoted $\sup(S)$, is a smallest element in $\mathbb{N}$ that is not smaller than any element in S, if such an element exists.*

1.2.2 Points in the Plane

To be able to talk about planarity of graphs later, we will need notation regarding points in the plane.

Definition 1.6 (Cartesian coordinate system $\mathbb{R}^2$). *We say that $\mathbb{R}^2$ is the set of all pairs (x, y) where $x, y \in \mathbb{R}$. $\mathbb{R}^2$ is commonly referred to as* **the plane**. *The members of $\mathbb{R}^2$ are called* **points**.

Note that we often will denote a set of points simply as a ***point set***.

Definition 1.7 (Point-to-point distance). *Let $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ be two points from $\mathbb{R}^2$. The* **Eucledian distance**, *or simply* **distance**, *between the two points is $dist(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.*

Definition 1.8 (Point set-to-point set distance). *Let $A, B \subseteq \mathbb{R}^2$ be two point sets. The* **distance between the sets** *is $dist(A, B) = \inf\{dist(a, b) \mid a \in A, b \in B\}$. We say that A and B are* **separated** *by the distance $dist(A, B)$.*

The reason we are using infimum for this definition is so we can say that two point sets that are infinitely close in R^2, but not intersecting, are separated by a distance of 0.

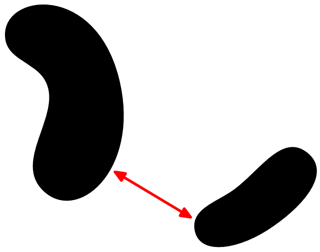

Figure 1.3: The distance between two point sets is the smallest distance between any two points, one from each set.

When doing addition and scalar multiplication on points, we treat them as regular vectors, and perform the operations as defined in Lay[19].

Definition 1.9 (Line segment)**.** *Let* p_1, p_2 *be two points in* $\mathbb{R}^2$*. The* **linesegment between** p_1 **and** p_2 *is the set of points* $L(p_1, p_2) = \{\lambda p_1 + (1-\lambda)p_2 \mid 0 \leq \lambda \leq 1\}$*.* p_1 *and* p_2 *are called the endpoints of the linesegment.*

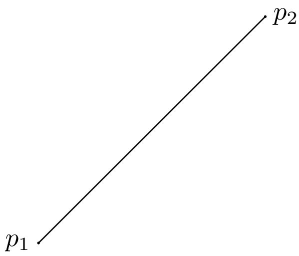

Figure 1.4: A linesegment is the set of all points on the line between the two endpoints p_1 and p_2.

Definition 1.10 (Simple polygon line)**.** *A* **simple polygon line** P *is a point set*

$$P = L(p_1, p_2) \cup L(p_2, p_3), \ldots, L(p_{t-1}, p_t)$$

such that the following properties are satisfied:

- $p_1, p_2, \ldots, p_t$ *is a finite sequence of points where* $p_i \neq p_j$ *for all* $i \neq j$
- *for* $1 < i < j < t$ *we have* $L(p_{i-1}, p_i) \cap L(p_j, p_{j+1}) = \emptyset$
- *for* $1 < i < t$ *we have* $L(p_{i-1}, p_i) \cap L(p_i, p_{i+1}) = \{p_i\}$

p_1 *and* p_t *are called the endpoints of* P*, and we write* $P = P(p_1, p_t)$*.*

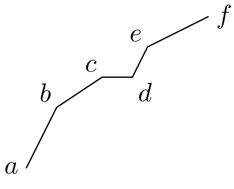

Figure 1.5: The simple polygon line is made up of line segments between the points a through f. a and f are the endpoints of the simple polygon line.

Definition 1.11 (Connected points)**.** *We say that* **a pair of points** p_1, p_2 **is connected** *in a point set* S *if there exists a simple polygon line* $P \subseteq S$ *such that* $p_1, p_2 \in P$*.*

Definition 1.12 (Connected pointset)**.** *A* **point set** S **is connected** *if every pair* $p_1, p_2 \in S$ *is connected in* S*.*

Definition 1.13 (Simple polygon)**.** *A* **simple polygon** P *is a set of points s.t. there exists a simple polygon line* P' *with endpoints* p_1, p_t*, and a linesegment* $L(p_1, p_t)$ *s.t.* $L(p_1, p_t) \cap P' = \{p_1, p_t\}$*, and we have*

$$P = P' \cup L(p_1, p_t)$$

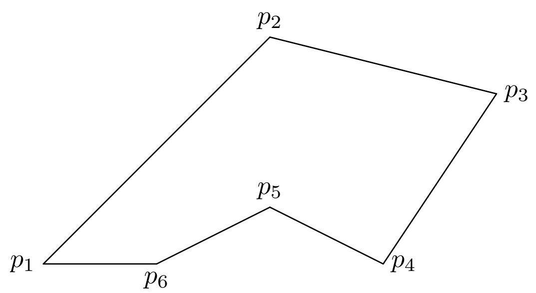

Figure 1.6: A simple polygon.

For convenience, when we say just ***polygon line*** or ***polygon***, we will refer to the definitions above, meaning simple polygon line and simple polygon.

Definition 1.14 (Face)**.** *Let* P *be a simple polygon. The inclusion-wise maximal connected point sets of* $\mathbb{R}^2 \setminus P$ *are called* **faces***. The infinite one is called the exterior face of* P*, while the finite one is called the interior face of the polygon.*

Definition 1.15 (Unit size disk)**.** *A* **unit size disk with center** $p \in \mathbb{R}^2$ *is the set of points*

$$\{p' \in \mathbb{R}^2 \mid dist(p, p') \leq 1\}$$

1.2.3 Graphs

Definition 1.16 (Vertex)**.** *A* **vertex** v *is a single element from some universe of elements* U*.*

Definition 1.17 (Edge)**.** *An* **edge** e *is an unordered pair of two vertices* u, v*, written* $e = (u, v)$*. We say that* e *is* **incident** *to* u *and* v*, also that* e **goes between** u *and* v*.* u *and* v *are the* **endpoints of** e*.*

Definition 1.18 (Simple graph)**.** *A* **simple graph** G *is a pair* V, E*, often written* $G = (V, E)$*.* V *is a set of vertices, and is called the* **vertices of** G*.* E *is a set of edges between vertices in* V*, and is called the* **edges of** G*. More specifically,* $E \subseteq \{(u, v) \mid u, v \in V, u \neq v\}$*.* V *and* E *are often written* $V(G)$ *and* $E(G)$ *respectively, when it is ambiguous which graph they belong to.*

Definition 1.19 (Multigraph)**.** *We say a graph* $G = (V, E)$ *is a* **multigraph** *if we allow* E *to be a multiset.*

As we will mostly work with multigraphs from this point forwards, we might sometimes just write ***graph***, meaning multigraph. Note that our definition of multigraph does not allow self-loops (i. e. an edge (u, v) s.t. $u = v$).

Definition 1.20 (Adjacent vertices)**.** *Let* $G = (V, E)$ *be a graph, and let* $u, v \in V$ *be two vertices in* G*. We say that* u *and* v *are* **adjacent** *if there is an edge in* G *incident to them both. That is, if* $(u, v) \in E$*.*

Definition 1.21 (Neighborhood of a vertex). *Let $G = (V, E)$ be a graph, and let $v \in V$ be a vertex in G. We denote* **the neighborhood of** *v as $N(v)$, and it consists of all the vertices adjacent to v in G. The* **closed neighborhood of** *v is denoted by $N[v] = \{v\} \cup N(v)$.*

Definition 1.22 (Neighborhood of a set of vertices). *Let $G = (V, E)$ be a graph, and let $S \subseteq V$ be a set of vertices in G. As for a single vertex, we denote* **the closed neighborhood of** *S as $N[S] = \bigcup_{s \in S} N[s]$ and* **the neighborhood of** *S as $N(S) = N[S] \setminus S$.*

Definition 1.23 (Degree of a vertex). *Let $G = (V, E)$ be a graph, and let $v \in V$ be a vertex in this graph. The* **degree of** *v is the number of edges from E incident to v, and we denote it by $deg(v)$.*

Note that since we in some cases will allow multiple edges, the degree of a vertex is not always the same as the number of neighbors it has, as is the case for simple graphs.

Definition 1.24 (Induced subgraph). *Let $G = (V, E)$ be a graph, and let $S \subseteq V$ be a set of vertices in G. The* **subgraph induced by** *S is the graph having vertices $V' = S$ and edges $E' = \{(u, v) \in E \mid u, v \in S\}$, and is denoted $G[S] = (V', E')$.*

Definition 1.25 (Deleting vertices of graph). *Let $G = (V, E)$ be a graph, and let $S \subseteq V$. The graph $G - S$ is the graph $G' = (V', E')$, where $V = V \setminus S$ and $E' = \{(v_1, v_2) \in E \mid v_1 \notin S \wedge v_2 \notin S\}$.*

Definition 1.26 (Disjoint union of graphs). *Let $G_1 = (V_1, E_2)$, $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. The* **disjoint union of** *G_1* **and** *G_2, denoted $G_1 + G_2$, is the graph $G = (V_1 \cup V_2, E_1 \cup E_2)$.*

Definition 1.27 (Adding edges to graph). *Let $G = (V, E)$ be a graph and let $E_1 \subseteq \{(u, v) \mid u, v \in V, u \neq v\}$ be a set of edges. The graph G' obtained by adding the edges E_1 to G is the graph $G' = G + E_1 = (V, E \cup E_1)$.*

Definition 1.28 (Supergraph). *Let $G = (V, E)$ be a graph. A graph $G' = (V', E')$ is a* **supergraph** *of G if $V = V'$ and $E \subseteq E'$.*

Definition 1.29 (Walk). *Let $G = (V, E)$ be a graph. A* **walk** *W of length k in G is a sequence $(v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1})$ where $v_i \in V$, $e_j \in E$, such that $\forall j \in \{1, \ldots, k\}, e_j = (v_j, v_{j+1})$.*

Definition 1.30 (Simple walk). *A* **simple walk** *is a walk where all the vertices in the sequence are distinct. This is commonly know as a* **path**.

Definition 1.31 (Cyclic walk). *A* **cyclic walk** *C is a walk $W = (v_1, e_1, \ldots, v_k, e_k, v_{k+1})$ such that $v_1 = v_{k+1}$.*

Definition 1.32 (Clique). *Let $G = (V, E)$ be a graph, and let $C \subseteq V$ be a set of vertices in G. We say that C is a* **clique** *if all vertices in C are adjacent, that is $\forall u \in C, \forall v \in C, (u, v) \in E$. A clique on n vertices is denoted K_n.*

Definition 1.33 (Independent set). *Let $G = (V, E)$ be a graph, and let $I \subseteq V$ be a set of vertices in G. We say that I is an* **independent set** *if no vertex in I is adjacent to any other vertex in I, i. e. $\forall u \in I, \forall v \in I, (u, v) \notin E$.*

Definition 1.34 (Partition of a graph). *Let $G = (V, E)$ be a graph, and let $A \subseteq V$ and $B = V \setminus A$ be two disjoint subsets of the vertices of G. We say that (A, B) is a* **partition** *of the graph.*

Definition 1.35 (Bipartite graph). *Let $G = (V, E)$ be a graph. We say that G is* **bipartite** *if there exists a partition (A, B) of $V(G)$ such that A is an independent set, and B is an independent set in G. (A, B) is called a* **bipartite partition of** *G.*

Definition 1.36 (Complete bipartite graph). *Let $G = (V, E)$ be a bipartite graph with bipartite partition (A, B). We say that G is a* **complete bipartite graph** *if every vertex in A is adjacent to every vertex in B, i. e. $\forall a \in A, \forall b \in B, (a, b) \in E$. A complete bipartite graph having partition (A, B) where $|A| = n$ and $|B| = m$ is denoted $K_{n,m}$.*

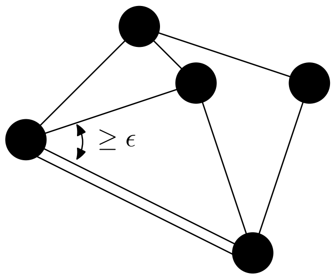

Figure 1.7: A planar graph can always be drawn with edges separated by a distance $\geq \epsilon > 0$.

1.2.4 Plane Graphs

The set of planar graphs is the type of graphs we'll turn most of our focus to in this thesis. We will now state some definitions regarding planar graphs that will be useful for the rest of the discussion.

Definition 1.37 (Plane embedding)**.** *A* **plane embedding** *of a graph $G = (V, E)$ is a four-tuple $\mathcal{E} = (\mathcal{U}_\mathcal{E}, \mathcal{P}_\mathcal{E}, f_\mathcal{E}, g_\mathcal{E})$, where $\mathcal{U}_\mathcal{E}$ is set of unit size disks, $\mathcal{P}_\mathcal{E}$ is a set of polygon lines, $f_\mathcal{E} : V \to \mathcal{U}_\mathcal{E}$ is a map that maps any vertex of V to a unit size disk in the plane, and $g_\mathcal{E} : E \to \mathcal{P}_\mathcal{E}$ is a map that maps any edge in E to a polygon line in the plane, such that there exists an $\epsilon > 0$ and the following conditions are satisfied:*

1. *For every pair of distinct vertices $v_1, v_2 \in V$ their unit disks are separated by ϵ, i. e.*
$$dist(f_\mathcal{E}(v_1), f_\mathcal{E}(v_2)) \geq \epsilon$$
2. *For every pair of distinct edges $e_1, e_2 \in E$, their polygon lines are separated by ϵ, i. e.*
$$dist(g_\mathcal{E}(e_1), g_\mathcal{E}(e_2)) \geq \epsilon$$
3. *For every edge $e \in E$ and for every vertex $v \in V$ not incident to e, the unit disk of v and the polygon line of e is separated by ϵ, i. e.*
$$dist(f_\mathcal{E}(v), g_\mathcal{E}(e)) \geq \epsilon$$
4. *For every edge $e \in E$ and for every vertex $v \in V$ incident to e, the unit disk of v and the the polygon line of e will touch exactly in the point p, where p is one of the endpoints of the polygon line of e, i. e.*
$$f_\mathcal{E}(v) \cap g_\mathcal{E}(e) = \{p\}$$

In simple terms the first condition says that no two vertices should be drawn intersecting in the plane, the second that no edges should cross in the drawing, and the last two that an edge should touch only the unit disks of its endpoints in the drawing.

Note that the definition of a plane embedding is tailored to our use later in the thesis, and might look a bit different from the standard way of defining it. Very often in the literature, vertices maps to points in the plane, and edges to straight lines or curves. It is however possible to show that these definitions are equivalent, but we'll omit the proofs, and refer to Mohar and Thomassen[22] for the standard way of defining graphs on surfaces.

Definition 1.38 (Planar graph)**.** *A graph is* **planar** *if it has a plane embedding.*

Definition 1.39 (Plane graph)**.** *A* **plane graph** *$\mathcal{G}$ is a graph G together with a plane embedding $\mathcal{E}$ of G: $\mathcal{G} = (G, \mathcal{E})$.*

Definition 1.40 (Drawing of plane graph)**.** *Let $G = (V, E)$ be a planar graph, and $\mathcal{E} = (\mathcal{U}_{\mathcal{E}}, \mathcal{P}_{\mathcal{E}}, f_{\mathcal{E}}, g_{\mathcal{E}})$ an embedding of G. The set of points $\mathcal{E}(G) = (\bigcup_{v\in V} f_{\mathcal{E}}(v)) \cup (\bigcup_{e\in E} g_{\mathcal{E}}(e))$ is called the* **drawing of** *G.*

Definition 1.41 (Plane supergraph)**.** *Let $\mathcal{G} = (G, \mathcal{E})$ be a plane graph. A* **plane supergraph of** $\mathcal{G}$ *is a supergraph G' of G and an embedding $\mathcal{E}'$ of G' such that for all $u \in V(G)$, $f_{\mathcal{E}}(u) = f_{\mathcal{E}'}(u)$ and for all $e \in E(G)$, $g_{\mathcal{E}}(e) = g_{\mathcal{E}'}(e)$.*

Definition 1.42 (Face in plane graph)**.** *The* **faces of a plane graph** *$(G, \mathcal{E})$ are the maximal connected sets of points of $\mathbb{R}^2 \setminus \mathcal{E}(G)$.*

The infinite face of a plane graph is called the ***external face*** or ***outer face***.

Definition 1.43 (Vertex incident to face)**.** *Let G be a graph with plane embedding $\mathcal{E} = (\mathcal{U}_{\mathcal{E}}, \mathcal{P}_{\mathcal{E}}, f_{\mathcal{E}}, g_{\mathcal{E}})$, and let f be a face in this graph. A vertex $v \in E$ is* **incident to the face** *f if $dist(f, f_{\mathcal{E}}(v)) = 0$.*

Definition 1.44 (Edge incident to face)**.** *Let G be a graph with plane embedding $\mathcal{E} = (\mathcal{U}_{\mathcal{E}}, \mathcal{P}_{\mathcal{E}}, f_{\mathcal{E}}, g_{\mathcal{E}})$, and let f be a face in this graph. An edge $e \in E$ is* **incident to the face** *f if $dist(f, g_{\mathcal{E}}(e)) = 0$.*

Observe that an edge can be incident to at most two faces.

Definition 1.45 (Size of face)**.** *The* **size of a face** *f in a plane graph is $|f| = \sum_{e\in E} w_f(e)$, where*

$$w_f(e) = \begin{cases} 0, & \textit{if } e \textit{ not incident to } f \\ 1, & \textit{if } e \textit{ is incident to } f \textit{ and some face other than } f \\ 2, & \textit{otherwise} \end{cases}$$

Observe that each edge in the graph will contribute 2 to the sum of the size of all faces in the graph, and we have that

$$\sum_{f\in F} |f| = 2|E|$$

where F is the set of faces in the embedding.

A well-known theorem that will be useful is the following.

Theorem 1.46 (Euler's formula [3, Thm 3.7])**.** *Let $(G = (V, E), \mathcal{E})$ be a connected plane graph, and let F be the set of faces in the embedding. We have that*

$$|V| + |F| - |E| = 2$$

Using the above theorem we can derive another useful one:

Theorem 1.47 (Bipartite Planar Graph Lemma)**.** *Let G be a simple planar graph with bipartition (A, B) where $\forall b \in B, deg(b) \geq 3$. Then $|B| \leq 2|A|$.*

Proof. Let $n = |V(G)| = |A| + |B|$ be the number of vertices, let $m = |E| \geq 3|B|$ be the number of edges, and let $f = |F|$ be the number of faces in G. Since the graph is simple and bipartite, a face in the graph must have a size of at least 4, meaning $4 \cdot f \leq 2m$ or $f \leq \frac{m}{2}$. Using this together with Euler's formula yields: $m = n + f - 2 \leq n + \frac{m}{2} - 2$ or $m \leq 2n$. From before we have $3|B| \leq m \leq 2n = 2|A| + 2|B| \implies |B| \leq 2|A|$. □

The famous theorem of Kuratowski says that a graph is planar if and only if it doesn't contain a subdivision of the clique on 5 vertices, K_5, or the complete bipartite graph $K_{3,3}$ as a subgraph. We won't define subdivisions here, and refer to Anderson[3] for this. The important fact is that this makes planarity checking relatively easy, and a number of efficient algorithms for checking planarity exist[16, 5]. This will be useful for us when we later want to generate many planar graphs on a computer, as we efficiently can check whether the generated graphs are planar.

A theorem we will use later is one very similar to Kuratowski's, regarding ***outerplanarity***:

Definition 1.48 (Outerplanar graph)**.** *A graph is* **outerplanar** *if it is has a plane embedding where every vertex is incident to the external face.*

Theorem 1.49. *A graph G is outerplanar if and only if it has no subdivision of K_4 or $K_{2,3}$ as a subgraph.*

We refer to Diestel[10] for the proof.

1.2.5 Decision Problems

Definition 1.50 (Decision Problem)**.** *A* **decision problem** *L is a subset $L \subseteq \Sigma^*$, where Σ is a fixed size alphabet, and Σ^* denotes the the set of all finite strings over Σ. A decision problem is also called a* **language***.*

Given an instance $I \in \Sigma^*$, what we want an algorithm to do is to decide whether $I \in L$ or $I \notin L$.

Definition 1.51 (Yes/no instances)**.** *Let $L \subseteq \Sigma^*$ be a language. An instance $I_1 \subseteq \Sigma^*$ such that $I_1 \in L$ is called a* **yes instance** *of L. An instance $I_2 \subseteq \Sigma^*$ such that $I_2 \notin L$ is called a* **no instance** *of L.*

In other words, an algorithm that solves a decision problem P will for a given instance I output *yes* if $I \in P$ and *no* otherwise.

Definition 1.52 (Instance size)**.** *The size of an instance $I \subseteq \Sigma^*$ is $|I| \cdot \log |\Sigma|$, i. e. the number of bits needed to encode I.*

When dealing with problems on graphs it is more convenient to denote the size of an instance graph $G = (V, E)$ as $|V| + |E|$. For general graphs this will be equivalent to Definition 1.52 up to a small polynomial factor and for planar graphs up to a constant factor[28] . Very often we will only consider the number of vertices when talking about the size of a graph, and for planar graphs this will again be equivalent up to a constant factor.

Decision problems are often related to what we call ***optimization problems***, but as we don't need that definition for our discussion here, we refer to Sipser[27, p. 393] for more info on this.

1.2.6 Algorithms and Runtime

Definition 1.53 (Big-O notation)**.** *Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be computable functions. We say that f* **is** *$O(g)$ if $f(n) \leq c \cdot g(n)$, for some constant c.*

This is like saying that f *will never get much bigger than* g, and is useful for describing runtimes of algorithms.

Definition 1.54 (Runtime)**.** *Let $\mathcal{A}$ be an algorithm that determines whether a given instance I is a yes or no instance to a problem P. Let $|I| = n$, and let the number of steps $\mathcal{A}$ uses to determine if $I \in P$ be bounded by a function $f(n)$. We then say that the* **runtime of $\mathcal{A}$** *is $O(f(n))$.*

Algorithms having a runtime $O(n^c)$, where c is a constant, we say are ***polynomial time*** algorithms. When we write $poly(n)$ or $n^{O(1)}$ it is equivalent to $O(n^c)$. Algorithms having a runtime $O(c^n)$ we call ***exponential time*** algorithms. When we talk about ***efficient algorithms***, we are talking about polynomial time algorithms. Sometimes we'll write ***polynomial algorithms*** and ***exponential algorithms***, meaning polynomial time and exponential time algorithms.

1.2.7 Dominating Set

Now let's turn back to the DOMINATING SET problem we introduced earlier. We start out by defining it formally:

Definition 1.55 (Dominating Set)**.** *A* **Dominating Set** D *of a graph* $G = (V, E)$ *is a subset of vertices* $D \subseteq V$ *such that* $N[D] = V$*.*

In other words, every vertex in G must either be in D or have a neighbor in D for D to be a Dominating Set of the graph. The decision problem now becomes

DOMINATING SET (DS) **Input:** Graph G, and integer k. **Question:** Is there a Dominating Set D of G of size at most k?

In terms of the definition of decision problems (Definition 1.50), this means we have a language DOMINATING SET where every yes instance is of the form (G, k), where G is a graph having a Dominating Set of size at most k.

We will often call the vertices in a given dominating set of G for ***dominators***.

The problem statement is pretty straightforward, but to find the solution in the general case is known to be hard. DOMINATING SET is one of the fundamental NP-complete problems. We'll have a look at what that means in the next section.

1.2.8 Efficient Algorithms and NP-hardness

When we design an algorithm we would naturally like it to be as fast, or *efficient*, as possible. We don't want to wait forever for the algorithm to output an answer to our question. In complexity theory, algorithms with a polynomial runtime is usually considered efficient. Algorithms with an exponential runtime are less desirable. The running time of these exponential algorithms increases very fast when the input size is increased.

Decision problems for which there exist polynomial time algorithms we say are in the complexity class P. All decision problems for which we can in polynomial time check if a given solution to the problem is valid or not, we say are in the class NP. Note that all problems in P are also in NP, but not necessarily the other way around. And some problems can be shown to be NP-hard, meaning that all problems in NP can be reduced to them [27, p. 304]. That is, if you can find an efficient algorithm for these problems, then all problems in NP can be solved efficiently. If such an NP-hard problem itself is in NP, it is called NP-complete and is considered among the hardest problems in NP. It is believed, but not proven, that P $\neq$ NP, and therefore that these NP-complete problems only have exponential time algorithms. For more on this, see Sipser[27].

1.2.9 Coping with NP-hardness

Even though we might have given up on finding polynomial time algorithms for NP-hard problems, we still want to solve them as fast as possible. Many of the NP-hard problems have a lot of practical applications (as the fire stations example for the DOMINATING SET problem), and we would like to be able to handle instances with special properties, or when the solution size in question (the parameter k) is small. Approximation algorithms that run in polynomial time is a common way of finding an answer that might be "good enough" in many practical cases. Next, we will look at one class of problems that can be solved quite efficiently when the parameter to the problem is small.

The parameter is a relevant secondary measurement k to the instance size, and for these problems we can find algorithms where the exponential factor no longer depends on n, but rather on k.

Chapter 2

Fixed Parameter Tractability and Kernelization

In the field of Parameterized Complexity we describe the running time of an algorithm in terms of a parameter to the problem, in addition to the size of a problem instance. We get the following definitions from Cygan, Fomin, Kowalik, Lokshtanov, Marx, Pilipczuk, Pilipczuk and Saurabh [8]:

Definition 2.1 (Parameterized problem). *[8, Def 1.1] A* **parameterized problem** *is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a fixed size alphabet. Given an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, k is called the* **parameter of the instance**.

Definition 2.2 (Instance size). *The* **size of an instance** *(x, k) of a parameterized problem is $|(x, k)| = |x| + k$.*

Definition 2.3 (Fixed Parameter Tractable problem). *[8, Def 1.2] Let L be a parameterized problem, and $(x, k) \in \Sigma^* \times \mathbb{N}$. L is called* **Fixed Parameter Tractable** *(FPT) if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$, a constant c, and an algorithm that can determine if $(x, k) \in L$ in time bounded by $f(k) \cdot |(x, k)|^c$.*

Observe that we give no bound on how fast the function $f(k)$ can grow, and it will in most cases be an exponential function. The upside is that the function is exponential in the parameter k, which very often can be small, while the total runtime is polynomial in n.

Analogous to the classes P and NP, the class of problems that are Fixed Parameter Tractable is called FPT. In the next section we will present a simple algorithm for the problem Vertex Cover that runs in FPT time.

2.1 Fixed Parameter Tractable Algorithm for Vertex Cover

Vertex Cover is an example of a problem that is known to be FPT, and actually can be solved quite efficiently if the parameter is small.

Definition 2.4 (Vertex Cover). *A* **Vertex Cover** *S of a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ s.t. $G - S$ has no edges.*

VERTEX COVER (VC)
Input: Graph G, and integer k.
Question: Is there a Vertex Cover S of G of size at most k?

The VERTEX COVER problem is known to be NP-hard [18], but luckily it is also known to be Fixed Parameter Tractable. We'll now show a simple FPT algorithm for determining the answer to the VC decision problem.

The one key observation we need for the algorithm is this:

Observation 2.5. For any vertex cover of a graph G, and for every edge of G, at least one of its endpoints must be in the vertex cover.

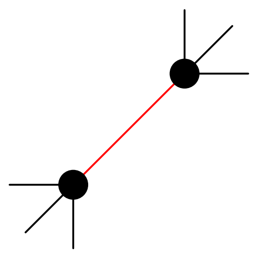

Figure 2.1: At least one of the red edge's endpoints must be in the vertex cover, so we try both choices.

This is the basic idea behind Algorithm 2.1.

Algorithm 2.1 Fixed Parameter Tractable algorithm forVERTEX COVER

Input: Graph G and integer $k \in \mathbb{N}$.
Output: *yes* if G has a vertex cover of size $\leq k$, *no* otherwise.

procedure $solve_{VC}(G,\, k)$
    if $E(G) = \emptyset$ **then**
        return *yes*
    else if $k = 0$ **then**
        return *no*
    else
        Pick any $(u, v) \in E(G)$
        return $solve_{VC}(G - u, k - 1) \vee solve_{VC}(G - v, k - 1)$
    end if
end procedure

As we can see, the algorithm picks any remaining edge in graph G of instance (G, k), and tries to remove one of it's endpoints, obtaining the graph G'. The resulting instance $(G', k - 1)$ is smaller, so it recursively invokes itself on this instance. If this call finds a solution to $(G', k - 1)$, we know there's also a solution to (G, k). If it is not, the algorithm tries removing the other endpoint instead, and if it again fails to find a solution, it can conclude there's no solution to (G, k).

We can find the runtime of the algorithm by observing that in the worst case every invocation will branch into two new invocations. Since we for every invocation decrease the parameter by 1, the recursion tree can become at most k levels deep. The rest of the work is all polynomial, and the resulting runtime is $2^k n^{O(1)}$, which is on the form $f(k) \cdot poly(n)$ and hence is FPT.

2.2 Kernels

Kernelization is a technique that leads to FPT running times for parameterized problems. Here a given input instance is reduced to an equivalent instance having a size bounded by the given parameter. The following definitions are adapted from Cygan et al[8, 2.1].

Definition 2.6 (Equivalent instances)**.** *Let L be a parameterized problem. Two instances $(x, k), (x', k') \in \Sigma^* \times \mathbb{N}$ are called* **equivalent** *if $(x, k) \in L \iff (x', k') \in L$.*

A preprocessing algorithm is an algorithm that given an instance (I, k) to a problem modifies this instance and output an equivalent instance (I', k').

Definition 2.7 (Output size of preprocessing algorithm)**.** *The size of the output from a preprocessing algorithm $\mathcal{A}$ is a function $size_{\mathcal{A}} : \mathbb{N} \to \mathbb{N} \cup \infty$:*

$$size_{\mathcal{A}}(k) = sup\{|I'| + k' : (I', k') = \mathcal{A}(I, k), I \in \Sigma^*\}$$

In other words we look at all possible instances with fixed parameter k to the algorithm, and measure the output size as the size of the largest output. The size is considered infinite if the size cannot be bounded by a function of k.

Definition 2.8 (Kernalization algorithm, kernel)**.** *Let L be a parameterized problem. A* **kernelization algorithm***, or simply a* **kernel***, is an algorithm $\mathcal{A}$ that takes as input a problem instance (I, k) of L, and in polynomial time creates a new equivalent instance (I', k') such that $|(I', k')| \leq f(k)$, where $f : \mathbb{N} \to \mathbb{N}$ is a computable function not dependent on $|I|$. $size_{\mathcal{A}}(k)$ is called the size of the kernel.*

Such kernelization algorithms very often proceed by applying reduction rules to the input instance. An example in our fire station case could be a town with no roads to neighboring towns. Of course we would have to put a fire station in this town, so we could ignore that town and decrease the number of fire stations to put by 1, resulting in an equivalent, smaller instance. Another simple observation is shown in Figure 2.2, where a town having at least one neighbor of degree 1, always is a good choice for a dominator. Therefore it is safe to remove all but one of these degree 1 towns from the instance.

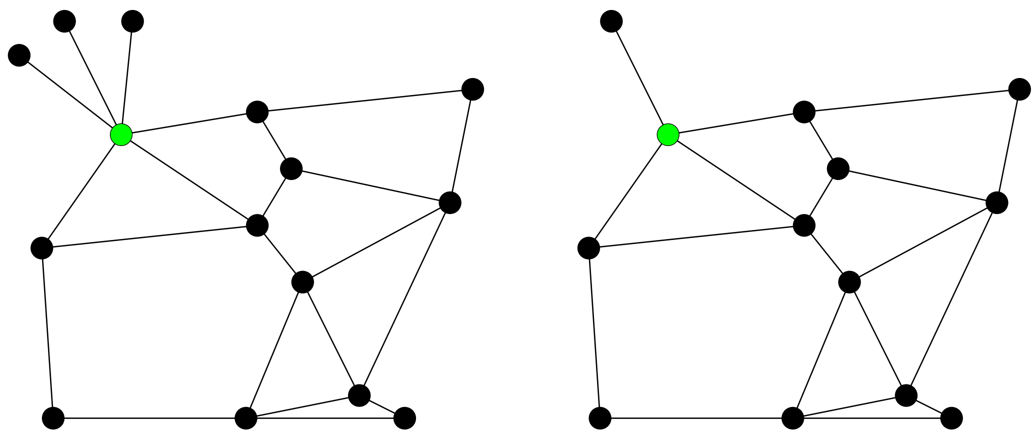

Figure 2.2: The degree 1 vertices forces the green vertex to be in a good dominating set. This is regardless of how many such degree 1 vertices there are, and we can remove all but one.

A problem having a kernelization algorithm is equivalent to the problem being fixed-parameter tractable, as the next two lemmas will show.

Lemma 2.9. *If a parameterized problem P is decidable and admits a kernel, then it can be solved in FPT time.*

Proof. Assume P is decidable in time $g(n)$ by using some algorithm $\mathcal{A}$, and let (x, k) be an instance of size n to P. Apply the polynomial time kernelization algorithm that outputs an equivalent instance (x', k') having a size bounded by some $f(k)$. We can now apply $\mathcal{A}$, which in time $O(g(f(k))) = O(f'(k))$ outputs an answer to the equivalent instance, and hence to the original instance. The total runtime is $O(f'(k) + poly(n))$, which is FPT. $\square$

Using this approach we can find FPT algorithms for all problems that have a kernelization algorithm. Surprisingly, the converse is also true:

Lemma 2.10. *If a parameterized problem P is solvable in FPT time, then it admits a kernel.*

The proof of Lemma 2.10 is out of scope and not necessary for this discussion, so we refer to Cygan et al.[8]. The two lemmas together gives us the following useful theorem.

Theorem 2.11. *A problem P is FPT if and only if it admits a kernel.*

The size of the kernel is important when we want to achieve efficient algorithms for solving a problem. If we can bound the size of the kernel to some linear function $f(k) = O(k)$, we say that the problem admits a ***linear kernel***. Similarly, if we can bound the size to some polynomial function $g(k) = O(k^c)$ for some constant c, we say the problem admits a ***polynomial kernel***. Some problems are harder to find small kernels for, and we might not be able to find any kernels better that exponential ones, $h(k) = 2^{O(k)}$, or even an exponential tower, $i(k) = 2^{2^{\cdot^{\cdot^{\cdot^{2^{O(k)}}}}}}$. The goal for any problem is to find as small kernels as possible.

2.2.1 Reduction Rules

A kernelization very often invoke small, polynomial time subroutines called ***reduction rules***. We will use these several times throughout the thesis. The following definitions are from Cygan et al.[8, p.18].

Definition 2.12 (Reduction rule)**.** *A* **reduction rule** *is a function $\phi : \Sigma^* \times \mathbb{N} \to \Sigma^* \times \mathbb{N}$ that maps an instance (x, k) to an equivalent instance (x', k') such that ϕ is computable in time polynomial in $|x|$ and k.*

Definition 2.13 (Soundness of rule)**.** *The property of a reduction rule that it translates an instance to an equivalent one, is called the* **soundness** *of the rule.*

When we create reduction rules, we will also prove their soundness, that is to show that the rules will produce equivalent instances.

Part II

A Linear Kernel for Planar Dominating Set

Chapter 3

Planar Dominating Set

We know Dominating Set is an NP-complete problem, so the next natural question to ask might be if it is FPT. In the general case it's unlikely to be, as that would imply a complexity theoretic result considered almost as unlikely as P = NP. We refer to Cygan et al.[8, ch.13] for more on this. But on planar graphs it is known to be FPT. It even admits a linear kernel.

In [2] Alber, Fellows & Niedermeier introduced the novel technique of ***Region Decomposition*** for giving kernels for problems on planar graphs. Using a set of reduction rules and this technique they achieved a kernel for Planar Dominating Set with $335k$ vertices. Later Chen, Fernau, Kanj, and Xia[6] improved on the reduction rules by Alber et al., and were able to show a kernel upper bounded by $67k$, which is the current smallest kernel known for this problem. In the following section we give definitions that classify the vertices inside the neighborhood of a vertex, as this is something we will make use of frequently.

3.1 Neighborhood of a Vertex

The following definitions are adopted from Alber et al.[2], and we will use them throughout the paper. Let $G = (V, E)$ be a graph.

Definition 3.1 (Neighborhood of a vertex)**.** *Let $v \in V$. We divide the neighborhood of v into 3 disjoint sets:*

- $N_1(v) = \{u \in N(v) \mid N(u) \setminus N[v] \neq \emptyset\}$
- $N_2(v) = \{u \in N(v) \setminus N_1(v) \mid N(u) \cap N_1(v) \neq \emptyset\}$
- $N_3(v) = N(v) \setminus (N_1(v) \cup N_2(v)))$

In other words, the vertices $N_1(v) \subseteq N(v)$ are those vertices having at least one neighbor outside the neighborhood of v, $N_2(v) \subseteq N(v)$ are those having at least one neighbor in $N_1(v)$, and $N_3(v) \subseteq N(v)$ are the rest of v's neighbors.

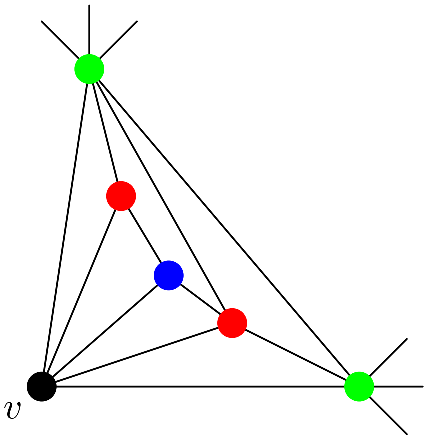

Figure 3.1: The neighborhood of a vertex v. Vertices from $N_1(v)$ are colored green, $N_2(v)$ are colored red, and $N_3(v)$ are colored blue.

3.2 Reduction Rules

Alber et al introduced two reduction rules in their paper. We won't need them in our discussion, so we refer to their paper for a description and proof of soundness [2, Rule 1, Rule 2].

However, we present two simple reduction rules here, inspired by their first rule. Let $G = (V, E)$:

Reduction Rule 3.1. Let $x, y \in N_2(v) \cup N_3(v)$ for some vertex $v \in V$ and $(x, y) \in E$, then remove the edge (x, y).

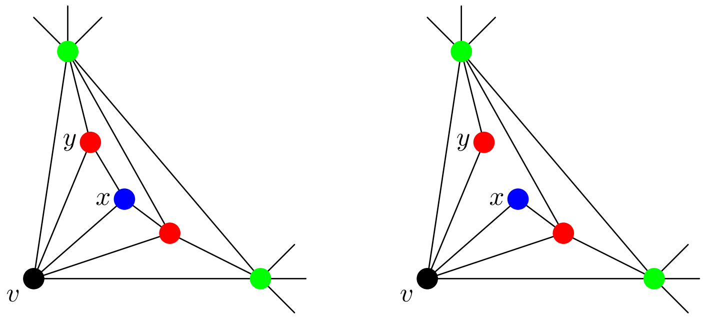

Figure 3.2: We can make $N_2(v) \cup N_3(v)$ vertices independent.

In this situation v is obviously a better choice for dominating the vertices in its neighborhood than x and y. This is why we can remove the edge between x and y, since none of them will be chosen as dominators. We now prove the soundness of this rule.

Lemma 3.2. *Reduction Rule 3.1 is sound.*

Proof. Let D be a dominating set of size k in G, and let G' be the graph obtained after performing the reduction rule. By the definition of $N_2(v)$ and $N_3(v)$, we know that $N[x] \subseteq N[v]$ and $N[y] \subseteq N[v]$, and hence v would always be a dominator at least as good as either of x, y. Therefore we can safely assume that neither x nor y is in D. Now D is also a dominating set of size k in G'.

For the other direction, let D' be a dominating set of size k in G'. D' will also be a dominating set of size k in G, since adding an edge to the graph cannot make D' non-dominating. $\square$

Reduction Rule 3.1 makes $N_2(v) \cup N_3(v)$ an independent set, which us useful for later analysis. Also note that the rule can be carried out in polynomial time.

Reduction Rule 3.2. Let $x, y \in N_2(v) \cup N_3(v)$ for some vertex $v \in V$ and $N(x) \subset N(y)$, then remove y from the graph.

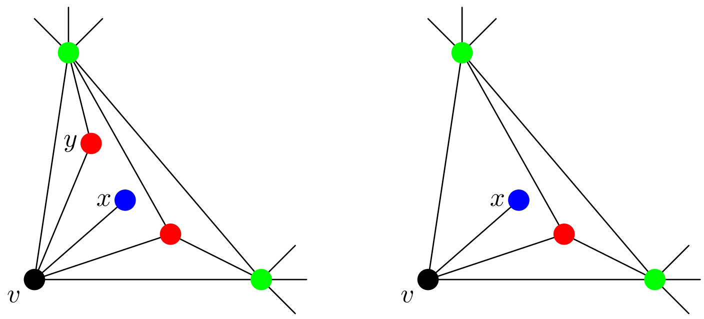

Figure 3.3: We can make remove vertices from $N_2(v) \cup N_3(v)$ that only serve the function of being dominated.

The intuition behind Rule 3.2 is that none of x, y will be useful as dominators, and therefore that the only way they affect the size of the dominating set is that they need to be dominated. But since y is dominated the moment x is, we can remove y.

Lemma 3.3. *Reduction Rule 3.2 is sound.*

Proof. Let D be a dominating set of size k in G, and let G' be the graph obtained after performing the reduction rule. We can assume (see the proof of Lemma 3.3) that neither x nor y is in D. Now D is also a dominating set of size k in G'.

For the other direction, let D' be a dominating set of size k in G'. D' will dominate x, and since $N[x] \subseteq N[v]$, we can assume $x \notin D'$ (if it was, putting v in instead would be at least as good). In G we know that $N(x) \subseteq N(y)$, and D' will also be a dominating set of size k in G. □

The effect of the two reduction rules is that $N_2(v) \cup N_3(v)$ will be independent, and that no vertex from this set will have a neighborhood that is a superset of another vertex' from the set. Note that in the case we have at least one $N_3(v)$ vertex, Rule 3.1 will reduce this to a degree 1 vertex, and any other vertex from the $N_2(v) \cup N_3(v)$ will be removed by Rule 3.2.

3.3 Regions and Region Decomposition as Treated by Alber et al.

After performing the reduction rules on the input instance, we want to upper bound the size of the resulting, equivalent, instance. This is where ***Region Decomposition*** comes in, by dividing the graph into several regions and then upper bounding the number of vertices outside the regions, upper bounding the number of such regions, and at last upper bounding the number of vertices inside such a region.

In this section we will describe regions and region decomposition as defined by Alber et al. [2], and point out the problem with these definitions. This should give an insight into why we want to redefine these concepts, and which special cases we must make sure to handle in our new definitions.

Given a reduced graph G, let $k = |D|$ be the size of a Dominating Set D in this graph. Alber et al. define *Regions* and *Region Decomposition* as follows:

Definition 3.4 ([2, Definition 2])**.** *Let $G = (V, E)$ be a plane graph. A region $R(v, w)$ between two vertices v, w is a closed subset of the plane with the following properties:*

1. *the boundary of $R(v, w)$ is formed by two simple paths P_1 and P_2 in V which connect v and w, and the length of each path is at most three (edges), and*
2. *all vertices which are strictly inside (i. e. lying in the region, but not sitting on the boundary) the region $R(v, w)$ are from $N(v, w)$.*

For a region $R = R(v, w)$, let $V(R)$ denote the vertices belonging to R, i. e.,

$$V(R) := \{u \in V \mid u \textit{ sits inside or on the boundary of } R\}$$

Definition 3.5 ([2, Definition 3])**.** *Let $G = (V, E)$ be a plane graph and $D \subseteq V$. A D-region decomposition of G is a set $\mathcal{R}$ of regions between pairs of vertices in D such that*

1. *for $R(v, w) \in \mathcal{R}$ no vertex from D (except for v, w) lies in $V(R(v, w))$ and*
2. *no two regions $R_1, R_2 \in \mathcal{R}$ do intersect (however, they may touch each other by having common boundaries).*

For a D-region decomposition $\mathcal{R}$ we define $V(\mathcal{R}) := \bigcup_{R \in \mathcal{R}} V(R)$. A D-region decomposition is called maximal if there is no region $R \notin \mathcal{R}$ such that $\mathcal{R}' := \mathcal{R} \cup \{R\}$ is a D-region decomposition with $V(\mathcal{R}) \subsetneq V(\mathcal{R}')$.

Definition 3.6 ([2, Definition 4])**.** *The induced graph $G_{\mathcal{R}} = (V_{\mathcal{R}}, E_{\mathcal{R}})$ of a D-region decomposition $\mathcal{R}$ of G is the graph with possible multiple edges which is defined by $V_{\mathcal{R}} := D$ and $E_{\mathcal{R}} := \{\{v, w\} \mid$ there is a region $R(v, w) \in \mathcal{R}$ between $v, w \in D\}$.*

Proposition 3.7 ([2, Proposition 1])**.** *For a reduced plane graph G with dominating set D, there exists a maximal D-region decomposition $\mathcal{R}$ such that $G_{\mathcal{R}}$ is thin.*

We don't state their definition of ***thin*** here, but for a graph to be thin, it has to be planar, among other things.

Alber et al. also show how to find such a maximal D-region decomposition of a plane graph, and they use this together with Definition 3.6 and Proposition 3.7 to prove that such a decomposition will have at most $3k$ regions. This require the decomposition graph $G_{\mathcal{R}}$ to be planar, if not Proposition 3.7 would not hold. For $G_{\mathcal{R}}$ to be planar, two regions can not be allowed to cross in the region decomposition, that is their drawing in the plane can't cross. This must also hold even when both are degenerated. Here a ***degenerated region*** $R(v, w)$ means a region where the paths P_1 and P_2 share at least one common vertex in addition to v, w.

Together with proofs bounding the number of vertices inside and outside of regions in a maximal D-region decomposition, this gives rise to a kernel of linear size.

However, there is a problem with the proof bounding the number of vertices outside regions. In the proof of [2, Lemma 6], the authors use that there are no crossing regions in a region decomposition to prove that all vertices from $N_1(v)$ are inside regions. But to accomplish this they say that for the degenerated region $R = \{v, u, u', w\}$ to cross in $\mathcal{R}$ without u already being in a region, the edge $\{u', w\}$ must cross a region $R(x, y) \in \mathcal{R}$, which implies that w is on the boundary or inside the crossing region. But consider the case where $R(x, y)$ is the degenerated region $\{x, u', y\}$, as shown in Figure 3.4. From Definition 3.5 it is hard to extract whether two such degenerated regions are considered crossing in this case. If we consider them crossing then $R(x, y)$ can cross R without w being inside or on the boundary of $R(x, y)$, making the proof of [2, Lemma 6] invalid. If the regions are not considered crossing then the decomposition graph from Definition 3.6 is no longer planar, which is crucial for Proposition 3.7.

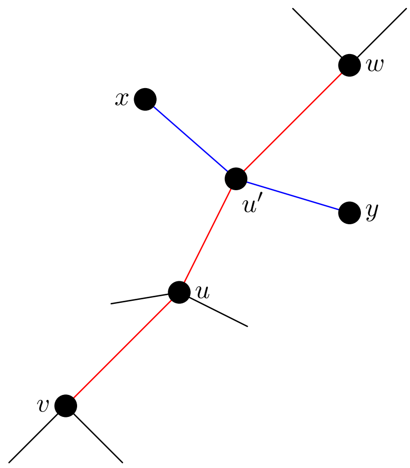

Figure 3.4: The degenerated region $R(x, y)$ can cross the degenerated region $R(v, w)$ without w being on the boundary of $R(x, y)$.

We will show that their decomposition of the graph into regions is correct up to minor modifications. These modifications are inconsequential for the use of the decomposition for later proofs. This means that the kernel size eventually obtained by Alber et al. is correct, which we will conclude in the next chapters.

Chapter 4

Decomposing the Graph

We will now make new definitions necessary for the discussion to follow. In Chapter 5 we will show that minor modifications to the theorems stated by Alber et al. are sufficient to resolve the errors in the proofs as discussed in Chapter 3. In Chapter 6 we will further lower the kernel size upper bound of Chen et al.[6] by an improved analysis.

4.1 Edges in Plane Embeddings

Definition 4.1 (Edge distance)**.** *Let $(G, \mathcal{E})$ be a plane graph, let $v \in V(G)$, and $e_1, e_2 \in E(G)$ be two edges both incident to v. Starting from the point where the polygon line of e_1 intersects the unit disk of v in the embedding, move clockwise along the disk. Let k be the number of polygon lines encountered before encountering the polygon line of e_2. Do the same, now starting from e_2 instead, let l be the number of polygon lines encountered before encountering e_1. The* **edge distance with respect to** *v of the two edges e_1, e_2, is denoted $dist_v(e_1, e_2) = \min(k, l)$.*

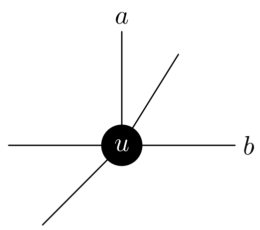

Figure 4.1: The edge distance of edges a and b with respect to u is 1.

Definition 4.2 (Consecutive edges)**.** *Edges having an edge distance of* 0 *with respect to $v \in V$, are said to be* **consecutive at** *v.*

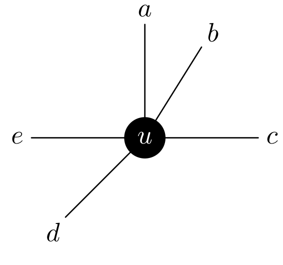

Figure 4.2: Consecutive pairs of edges at u are (a, b), (b, c), (c, d), (d, e) and (e, a).

Definition 4.3 (Parallel edges)**.** *We say that two edges $e_1, e_n \in E$ between vertices $v, w \in V(G)$ in a multigraph G are* **parallel** *if there exists a sequence of edges $e_1, \dots, e_n$ such that e_i and e_{i+1} are consecutive at both u and v for all $i \in \{1, \dots, n-1\}$.*

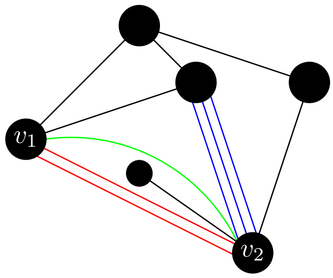

Figure 4.3: The red edges are parallel edges. So are the blue edges. Observe that the green edge is not parallel to the red edges since they are not consecutive at v_2.

4.2 Walks in Plane Embeddings

In the following definitions, let $(G = (V, E), \mathcal{E} = (\mathcal{U}, \mathcal{P}, f, g))$ be a plane graph.

Definition 4.4 (Join of edges)**.** *Let $e_1, e_2 \in E$ be two edges both incident to the vertex $v \in V$, and let $a = f(v) \cap g(e_1), b = f(v) \cap g(e_2)$ be the two points where the edges intersect with the vertex in the embedding. The* **join of** e_1 **and** e_2 **at** v*, is the linesegment $L(a, b)$. We write $join(e_1, v, e_2)$*

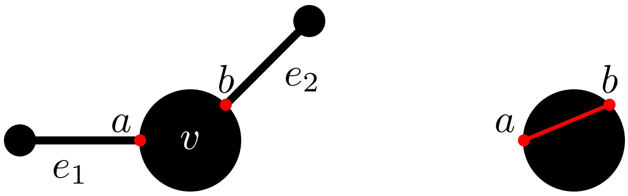

Figure 4.4: Edges e_1 and e_2 are both incident to some vertex v, and intersect with the unit disk of v in points a and b, marked with red on the left figure. $join(e_1, v, e_2)$ is the linesegment between a and b, colored in red on the right figure.

Definition 4.5 (Drawing of walk)**.** *Let $W = (v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_{k+1})$ be a walk in G. We obtain the* **drawing of** W*, denoted $draw(W)$, by taking the union of the points in $g(e)$ for every $e \in \{e_1, \dots, e_k\}$, together with the join of every e_i, e_{i+1} at v_{i+1}, i. e.*

$$draw(W) = g(e_1) \bigcup_{i \in \{2, \dots, k\}} \Big(g(e_i) \cup join(e_{i-1}, v_i, e_i) \Big)$$

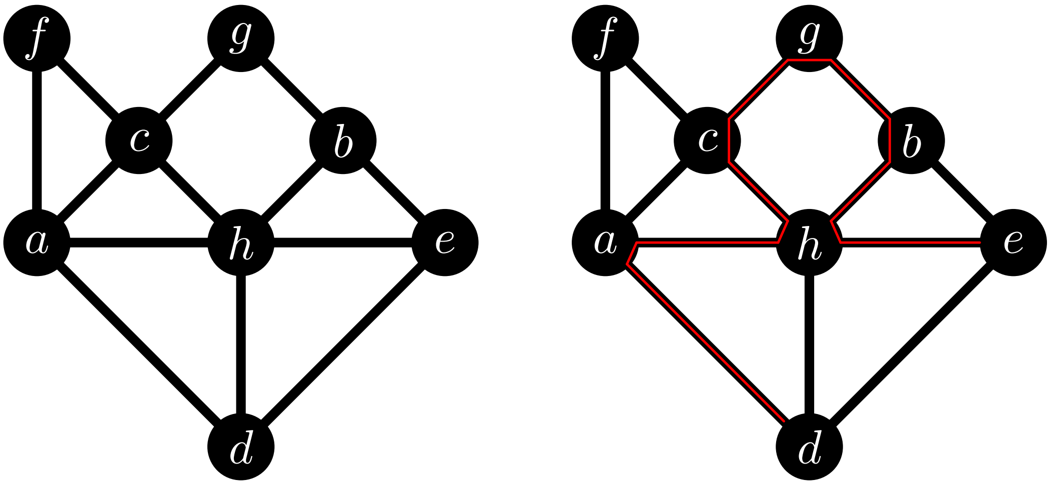

Figure 4.5: A graph (left), and a walk in this graph shown in red (right).

Definition 4.6 (Crossing walks)**.** *We say that two walks W_1, W_2 in G* **cross** *if $draw(W_1) \cap draw(W_2) \neq \emptyset$.*

Definition 4.7 (Simple cyclic walk)**.** *A* **simple cyclic walk** *C is a cyclic walk $W = (v_1, e_1, \ldots, v_k, e_k, v_1)$ in a plane graph such $draw(C) = draw(W) \cup join(e_1, v_1, e_k)$ is a simple polygon. We refer to $draw(C)$ as the drawing of C.*

Since we will only care about simple cyclic walks, we will often just refer to them as ***cyclic walks***.

Definition 4.8 (Interior of a cyclic walk)**.** *Let C be a cyclic walk in a plane graph. The interior face of $draw(C)$ is called the* **interior of the cyclic walk***, while the exterior face is called the* **exterior of the cyclic walk***.*

Definition 4.9 (Vertices in a cyclic walk)**.** *The vertices represented by unit disks being strictly in the interior of a cyclic walk, are called* **internal vertices** *of the walk. The vertices represented by unit disks being strictly in the exterior of a cyclic walk, are called* **external vertices***. The rest, having their unit disks intersect with the drawing of the walk, are called* **vertices on the walk***.*

Observe that by Definition 4.9, the vertices on the walk $C = (v_1, e_1, \ldots, v_k, e_k, v_1)$ are exactly the vertices $\{v_1, \ldots, v_k\}$.

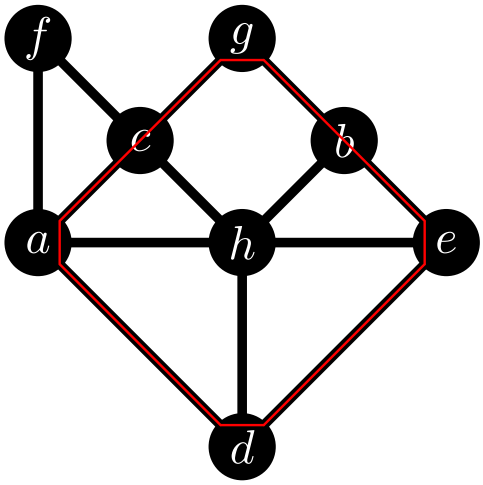

Figure 4.6: The drawing of a cyclic walk shown in red. h is the only internal vertex of the walk, while f is the only external vertex. a, b, c, d, e, g are vertices on the walk.

Definition 4.10 (Area of cyclic walk). *The* **area of a cyclic walk** C*, denoted* $area(C)$ *is the union of the points on the walk and all points in the interior.*

Definition 4.11 (Crossing cyclic walks). *We say that two cyclic walks* C_1, C_2 **cross** *if their areas intersect, i. e.* $area(C_1) \cap area(C_2) \neq \emptyset$*.*

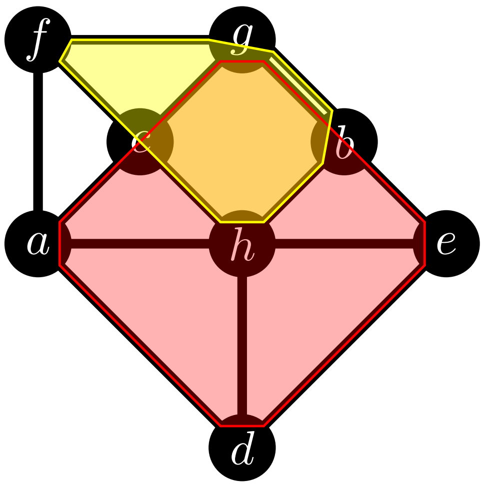

Figure 4.7: The two cyclic walks shown in red and yellow cross since their areas intersect.

Definition 4.12 (Cross at vertex). *We say that two cyclic walks* C_1, C_2 **cross at a vertex** v *if* $draw(C_1) \cap draw(C_2) \cap f(v) \neq \emptyset$*.*

Definition 4.13 (Cross at edge). *We say that two cyclic walks* C_1, C_2 **cross at an edge** e *if* $draw(C_1) \cap draw(C_2) \cap g(e) \neq \emptyset$*.*

We will use the next well-known fact without proof:

Fact 4.14. Let P_1 and P_2 be two simple polygons, and let A_1 and A_2 be their interior faces, respectively. If $A_1 \cap A_2 \neq \emptyset$, then $P_1 \cap P_2 \neq \emptyset$.

Lemma 4.15. *Let* C_1, C_2 *be two cyclic walks that cross. Then, either*

- $area(C_1) \subseteq area(C_2)$
- $area(C_2) \subseteq area(C_1)$
- *C_1 and C_2 cross at some vertex v*

Proof. Since C_1 and C_2 cross, then by definition $area(C_1) \cap area(C_2) \neq \emptyset$. This can happen if either cycle's area is the subset of the other, as in the first two cases. For the third case, assume that neither of the first two cases apply. In that case we must have $draw(C_1) \cap draw(C_2) \neq \emptyset$ by Fact 4.14, which means they either cross at an edge, or cross at some vertex. Assume that they cross at an edge $e = (v_1, v_2)$. Then by definition of a cyclic walk, v_1 and v_2 will be vertices on both walks, and the two walks will cross at a point in both of these vertices, namely $g(e) \cap f(v_1)$ and $g(e) \cap f(v_2)$. □

Observation 4.16. If $area(C_1) \cap area(C_2) \cap g(e) \neq \emptyset$, where C_1 and C_2 are cyclic walks and $e = (v_1, v_2)$, then $area(C_1) \cap area(C_2) \cap f(v_1) \neq \emptyset$ and $area(C_1) \cap area(C_2) \cap f(v_2) \neq \emptyset$ as well.

Lemma 4.17. *Let $C_1 = (v_1, e_1, \ldots, v_i, e_i, v_{i+1}, e_{i+1}, v_i \ldots, v_k, e_k, v_1)$ and C_2 be cyclic walks such that e_i and e_{i+1} are parallel edges going between v_i and v_{i+1}, and C_1 and C_2 don't cross at v_i. Then C_1 and C_2 cannot cross at v_{i+1}.*

Proof. Since C_1 and C_2 don't cross at v_i, $area(C_1) \cap area(C_2) \cap g(e_i) = \emptyset$ and $area(C_1) \cap area(C_2) \cap g(e_{i+1}) = \emptyset$. By Observation 4.16 this means that for C_1 and C_2 to cross at v_{i+1}, $J = join(e_i, v_{i+1}, e_{i+1})$ would need to cross C_2 at v_{i+1}. For this to happen there would need to exist a join $J' = join(e'_1, v_{i+1}, e'_2)$ s.t. $J' \cap J \neq \emptyset$, where $e'_1 \neq e_i, e'_2 \neq e_i, e'_1 \neq e_{i+1}, e'_2 \neq e_{i+1}$. Let the linesegments representing the joins be $L(a, b) = J$ and $L(a', b') = J'$. For J' to intersect J, a' and b' would need to separate a and b on the border of $f(v_{i+1})$, which contradicts that e_i and e_{i+1} are consecutive, see Figure 4.8. □

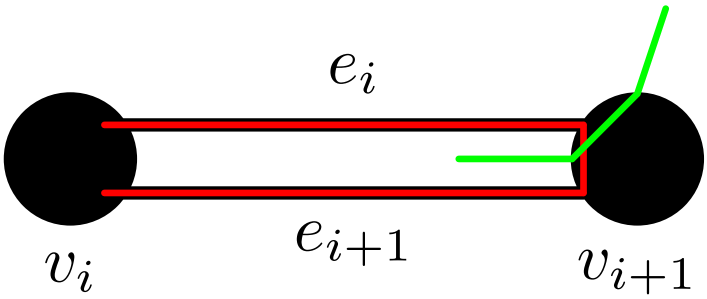

Figure 4.8: Two joins crossing contradicts that e_i and e_{i+1} is consecutive at v_{i+1}.

Definition 4.18 (Concatenation of simple walks)**.** *Let $W_1 = (v^1_1, e^1_1, \ldots, e^1_{k-1}, v^1_k)$ and $W_2 = (v^2_1, e^2_1, \ldots, e^2_{l-1}, v^2_l)$ be two simple walks in G such that $draw(W_1) \cap draw(W_2) = \emptyset$, and $v^1_1 = v^2_1$ and $v^1_k = v^2_l$. We say the* **concatenation of** W_1 **and** W_2 *is the cyclic walk having the drawing $draw(W_1) \cup draw(W_2) \cup join(e^1_1, v^1_1, e^2_1) \cup join(e^1_{k-1}, v^1_k, e^2_{l-1})$.*

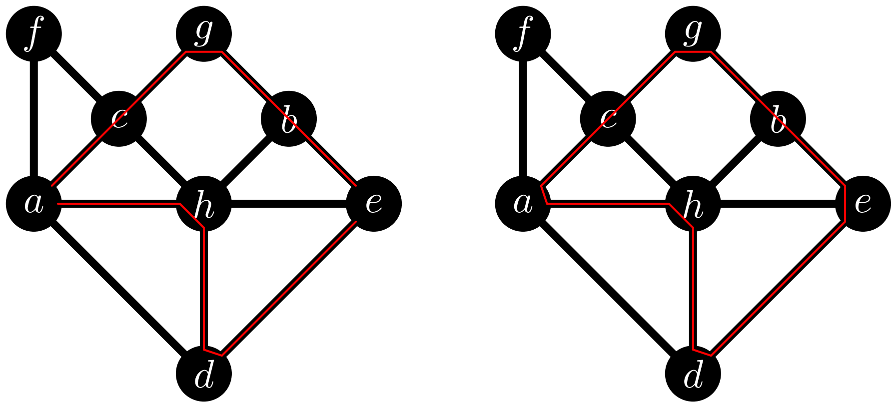

Figure 4.9: Concatenation of two simple walks that results in a cyclic walk.

4.3 Regions

Definition 4.19 (Region)**.** *Let $(G, \mathcal{E})$ be a plane graph, and let $v, w \in V(G)$ be two vertices in G. A* **region** *$R = R(v, w)$* **between** *v* **and** *w is defined by two simple, non-crossing walks W_1, W_2 of length at most 3 between v and w, and a set $V(R) \subseteq V(G)$ such that*

- *The concatenation C_R of W_1 and W_2 is a simple cyclic walk*
- *The vertices on C_R are denoted $\delta(R)$*
- *The internal vertices of C_R are denoted $I(R)$*
- $V(R) = I(R) \cup \delta(R) \setminus \{v, w\}$
- $V(R) \subseteq N(v, w)$

C_R is called the **boundary of** *R. $\delta(R)$ are the* **boundary vertices** *of R. $I(R)$ are called the* **internal vertices of** *R. $V(R)$ are called the* **vertices belonging to** *R. v, w are called the* **endpoints** *of the region.*

Notice that the two walks W_1, W_2 uniquely define which vertices are in $\delta(R)$ and $I(R)$, and hence $V(R)$.

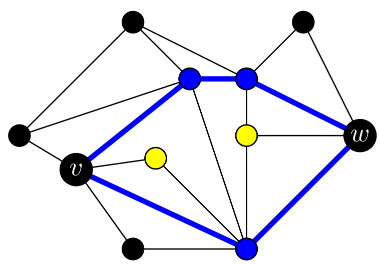

Figure 4.10: The region $R = R(v, w)$ has the endpoints v and w. The vertices $\delta(R) \setminus \{v, w\}$ are colored blue, and the internal vertices $I(R)$ yellow. These five vertices define the set $V(R)$.

Definition 4.20 (Area of a region)**.** *The* **area of a region** *R, $area(R)$, is the area of its boundary, $area(C_R)$.*

Definition 4.21 (Crossing regions)**.** *We say that two regions R_1, R_2* **cross** *if their areas intersect, that is $area(R_1) \cap area(R_2) \neq \emptyset$.*

Since the area of a region is defined by its cyclic walk, we will use Definition 4.12, Definition 4.13, Lemma 4.15 and Lemma 4.17 on regions later, referring to their boundary.

By definition, regions as those seen Figure 4.11a will be considered crossing. That is also the case for Figure 4.11b, as the two regions share an edge. However, in the latter case we would like regions to be able to "touch" each other in this way without crossing, without allowing the regions in a) to do that. This is the motivation for introducing ***graph enrichments***, which we will look at in the following section.

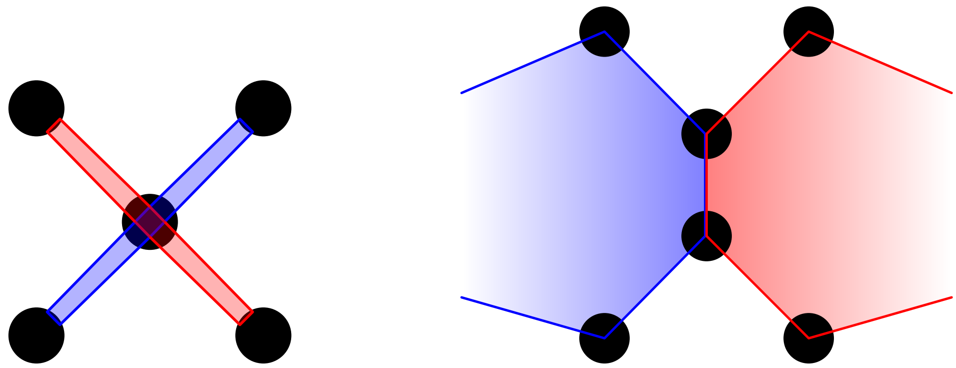

Figure 4.11: Crossing regions a) and b).

4.4 Graph Enrichment

Definition 4.22 (Copying an edge)**.** *Having a plane multigraph* $\mathcal{G} = (G = (V, E), \mathcal{E})$*, we define* **copying an edge** $e \in E$ *as the operation of adding a new edge* e' *to the graph, such that* e *and* e' *are parallel. When copying the edge* $e \in E$ *to* e'*, we set* $\epsilon' = \epsilon/2$*, and* $G' = (V, E \cup \{e'\})$ *can have the same embedding as* G*, only adding the polygon line of* e' *in a distance* ϵ' *from the polygon line of* e*. The result is a new plane graph* $\mathcal{G}' = (G', \mathcal{E}')$ *being structurally equal to* $\mathcal{G}$*, but having one extra edge.*

Definition 4.23 (Graph enrichment)**.** *Let* $\mathcal{G}$ *be a plane graph. We call the graph* $\mathcal{G}_e$ *an* **enrichment** *of* $\mathcal{G}$ *if it can be obtained from* $\mathcal{G}$ *only using the operation of copying edges.*

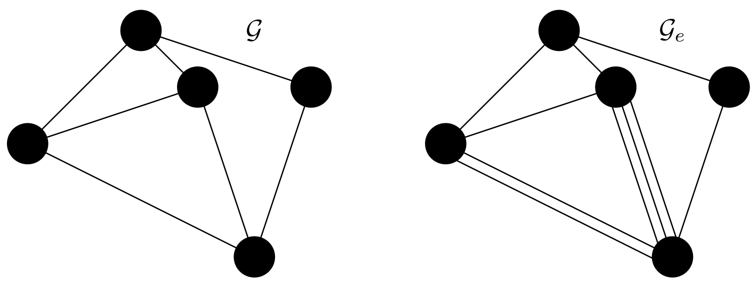

Figure 4.12: $\mathcal{G}_e$ is an enrichment of $\mathcal{G}$.

4.5 Region Decomposition

With the concept of graph enrichments in place, we can define a region decomposition of a plane graph.

Definition 4.24 (D-region decomposition). *A D-region decomposition of a plane graph $\mathcal{G} = (G, \mathcal{E})$ is pair $(\mathcal{G}_e, \mathcal{R})$, where $\mathcal{G}_e = (G_e, \mathcal{E}_e)$ is an enrichment of $(G, \mathcal{E})$, and $\mathcal{R}$ is a set of regions in $\mathcal{G}_e$ between pairs of vertices in $D \subseteq V(G_e)$ such that*

1. *for every $R \in \mathcal{R}$ no vertex from D is in $V(R)$*
2. *no two regions $R_1, R_2 \in \mathcal{R}$ cross in $\mathcal{G}_e$.*

We say that $\mathcal{R}$ **is realized on** *$\mathcal{G}_e$.*

Definition 4.25 (Vertices in regions). *For a D-region decomposition $(\mathcal{G}_e, \mathcal{R})$, we denote the* **vertices in regions** *as the set $V(\mathcal{R}) = \bigcup_{R \in \mathcal{R}} V(R)$. We also say that $\mathcal{R}$ covers the vertices $V(\mathcal{R})$.*

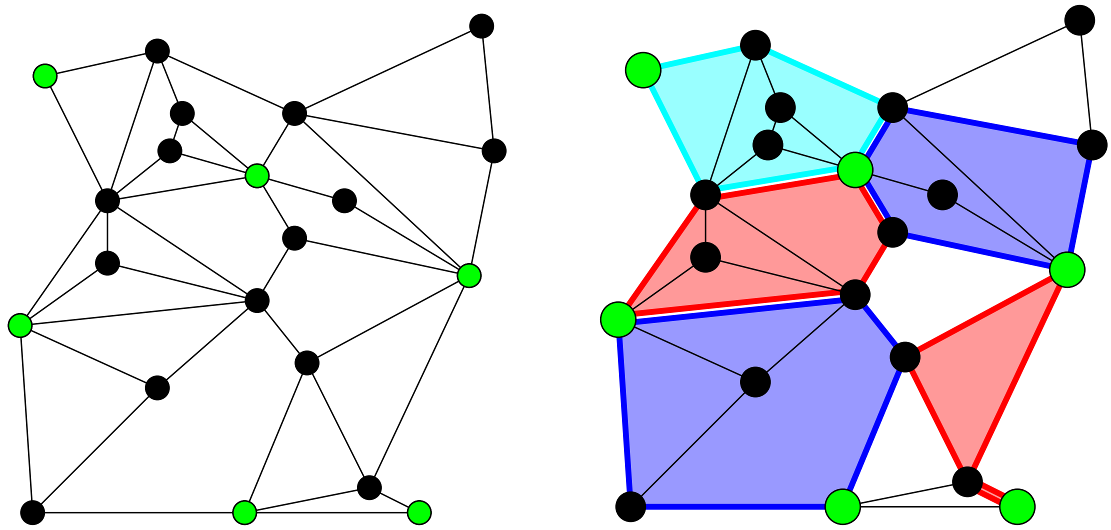

Figure 4.13: For a plane graph and a subset D of its vertices a possible D-region decomposition is pictured. The vertices of D are colored green, and the colors show the five chosen regions with boundary and area colored. Note that the decomposition is realized on an enrichment of $\mathcal{G}$.

We will now show that a D-region decomposition $(\mathcal{G}_e, \mathcal{R})$ of a plane graph $(G, \mathcal{E})$ gives rise to a planar graph $G_{\mathcal{R}}$ in a natural way.

Definition 4.26. *For a given D-region decomposition $(\mathcal{G}_e, \mathcal{R})$ of the plane graph $(G = (V, E), \mathcal{E})$, let*

$$V_{\mathcal{R}} = D$$
$$E_{\mathcal{R}} = \{(v, w) \mid \textit{there is a region } R(v, w) \in \mathcal{R} \textit{ between } v, w \in D\}$$

The graph $G_{\mathcal{R}} = (V_{\mathcal{R}}, E_{\mathcal{R}})$ is called **the decomposition graph of** $(\mathcal{G}_e, \mathcal{R})$.

Lemma 4.27. *The decomposition graph $G_{\mathcal{R}}$ of a D-region decomposition $(\mathcal{G}_e, \mathcal{R})$ has a plane embedding $\mathcal{E}_{\mathcal{R}}$, thus is planar.*

Proof. We construct a plane embedding $\mathcal{E}_{\mathcal{R}}$ of $G_{\mathcal{R}}$ as follows. Look at the embedding of the enrichment $\mathcal{G}_e = (G_e, \mathcal{E}_e)$ where $\mathcal{R}$ is realized. Now draw $G_{\mathcal{R}}$ in the plane by letting $V_{\mathcal{R}} = D$ have the same positions as in the drawing of G_e, and draw the edges $E_{\mathcal{R}}$ as the drawing of one of the walks of every region between the endpoints of that region. Since there are no two regions crossing each other in the D-region decomposition of G_e, the drawing of the edges will not cross, which means that $\mathcal{E}_{\mathcal{R}}$ is a plane embedding of $G_{\mathcal{R}}$. $\square$

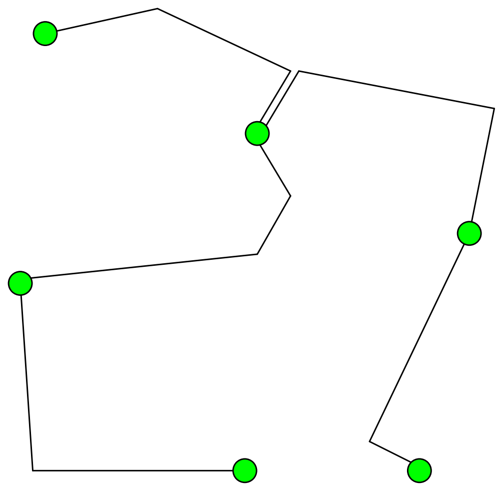

Figure 4.14: The decomposition graph for the D-region decomposition from Figure 4.13.

Figure 4.14 shows an embedding of the decomposition graph for the D-region decomposition from Figure 4.13.

Definition 4.28 (Co-areas of a decomposition)**.** *Let $G = (V, E)$ be a graph with embedding $\mathcal{E}$, and let $(\mathcal{G}_e, \mathcal{R})$ be a D-region decomposition of $(G, \mathcal{E})$, where $\mathcal{G}_e = (G_e, \mathcal{E}_e = (\mathcal{U}_e, \mathcal{P}_e, f_e, g_e)$. The family $\mathcal{A}$ of inclusion-wise maximal connected point sets in $\mathbb{R}^2 \setminus (\bigcup_{d\in D} f_{\mathcal{E}}(d) \bigcup_{R\in\mathcal{R}} area(R))$ is called the* **co-areas of the decomposition***, and each point set of $\mathcal{A}$ is called a* **co-area**.

Note that there is a one-to-one correspondence between the co-areas of a decomposition and the faces of the decomposition graph, meaning that for each face in the decomposition graph, there is a co-area in the decomposition, and vice versa.

Definition 4.29. *Let A be a co-area in a D-region decomposition $(\mathcal{G}_e, \mathcal{R})$ of $(G, \mathcal{E})$. We say that a vertex $v \in V(G)$ is* **incident to** *A if $dist(A, f_{\mathcal{E}}(v)) = 0$. We say v is* **in** *A if $f_{\mathcal{E}}(v) \subseteq A$.*

4.5.1 Maximal Region Decomposition

Definition 4.30 (0-maximal D-region decomposition)**.** *A* **0-maximal D-region decomposition** *is a decomposition $(\mathcal{G}_e, \mathcal{R})$ of G such that there does not exist an enrichment $\mathcal{G}'_e$ of $\mathcal{G}_e$ and a region R in $\mathcal{G}'_e$ s.t. $(\mathcal{G}'_e, \mathcal{R} \cup R)$ is a D-region decomposition covering more vertices than $(\mathcal{G}_e, \mathcal{R})$.*

Definition 4.31 (Maximal D-region decomposition)**.** *A* **maximal D-region decomposition** *is a 0-maximal decomposition $(\mathcal{G}_e, \mathcal{R})$ of G such that there does not exist an enrichment $\mathcal{G}'_e$ of $\mathcal{G}_e$ and a region R in $\mathcal{G}'_e$ and two regions $R_1, R_2 \in \mathcal{R}$ s.t. $\mathcal{R}' = (\mathcal{R} \setminus \{R_1, R_2\}) \cup R$ and $(\mathcal{G}'_e, \mathcal{R}')$ is a D-region decomposition covering at least as many vertices as $(\mathcal{G}_e, \mathcal{R})$.*

In other words, a D-region decomposition is maximal if one cannot add more regions containing new vertices, and one cannot find two regions to remove such that one can insert a new region that covers at least as many vertices.

Lemma 4.32. *Given a planar graph G and a set $D \subseteq V(G)$, there exists a maximal D-region decomposition (not necessarily unique), and we can find it in polynomial time.*

Proof. The algorithm will build a D-region decomposition in a greedy way. It picks a pair of vertices $v, w \in D$ and checks if there is a way to make a region between v, w such that it doesn't cross regions already found. This can be done in polynomial time by trying every possible pair of two simple, non-crossing walks of length at most 3 between the two vertices, and checking if the region made from these walks can be added to the region decomposition, copying edges if necessary, such that the number of vertices in regions increases. When this cannot be done anymore, the algorithm will look at every pair of regions in the decomposition, and check if they can be replaced with a single region covering at least as many vertices. When neither of these two operations can be performed, the found D-region decomposition is maximal, by definition. □

To bound the number of edges in the decomposition graph of a maximal D-region decomposition $(\mathcal{G}_e, \mathcal{R})$, we will need Definition 1.45 for size of a face in a plane graph together with the following:

Definition 4.33 (Thin graph)**.** *A plane graph $(G, \mathcal{E})$ is* **thin** *if all the faces in the embedding have size at least* 3*.*

The next lemma is a strengthening of Lemma 5 from Alber et al [2, Lemma 5].

Lemma 4.34. *Let $(G = (V, E), \mathcal{E})$ be a connected, plane, thin graph, and let F be the set of faces in this graph. We have $|E| = 3|V| - 6 - \sum_{f \in F}(|f| - 3)$.*

Proof. We know that $\sum\limits_{f \in F} |f| = 2|E|$, and since each face in a thin graph has size at least 3, we can rearrange the sum to become

$$2|E| = \sum_{f \in F} |f| = 3|F| + \sum_{f \in F}(|f| - 3)$$

Further reorganization yields

$$|F| = \frac{2}{3}|E| - \sum_{f \in F}(\frac{|f|}{3} - 1) \tag{4.1}$$

We now use Euler's formula (1.46)

$$|E| = |V| + |F| - 2$$

Insert (4.1) to get

$$|E| = |V| + \frac{2}{3}|E| - \sum_{f \in F}(\frac{|f|}{3} - 1) - 2$$
$$|E| = 3|V| - 6 - \sum_{f \in F}(|f| - 3)$$

□

In particular, we see that in a thin graph we always have $|E| \leq 3|V| - 6$

Lemma 4.35. *If all faces in a plane graph have size* 3*, then $|F| = 2|V| - 4$.*

Proof. By Lemma 4.34 we have that

$$|E| = 3|V| - 6 - \sum_{f \in F}(|f| - 3) = 3|V| - 6$$

if all faces are of size 3. Then by Euler's formula we have that

$$|F| = |E| - |V| + 2 = 2|V| - 4$$

□

Theorem 4.36. *Let D be a dominating set of the plane graph $\mathcal{G}$, and let $(\mathcal{G}_e, \mathcal{R})$ be a maximal D-region decomposition of $\mathcal{G}$. Then there exists a plane supergraph $\mathcal{G}'_{\mathcal{R}}$ of the decomposition graph $\mathcal{G}_{\mathcal{R}}$ s.t. $\mathcal{G}'_{\mathcal{R}}$ is connected and thin.*

Proof. We build $\mathcal{G}'_{\mathcal{R}}$ from $\mathcal{G}_{\mathcal{R}}$. Let $\mathcal{G}'_{\mathcal{R}} = \mathcal{G}_{\mathcal{R}}$ to begin with. Then as long as $\mathcal{G}'_{\mathcal{R}}$ is not connected, add an edge between two different components without violating planarity.

We claim $\mathcal{G}'_{\mathcal{R}}$ is thin. If not, there must be a face f of size ≤ 2 in the graph. Since $\mathcal{G}'_{\mathcal{R}}$ cannot have self-loops, f is of size 2 and is made from two edges between a pair of vertices $v, w \in D$. Note that f corresponds to a co-area A between two regions $R_1(v, w), R_2(v, w) \in \mathcal{R}$ in $\mathcal{G}_e$, because neither of the two edges could have been added by us connecting the graph. There cannot be another vertex $d \in D$ incident to f, since the graph is connected it would need to be adjacent to v or w, contradicting that the face had size 2.

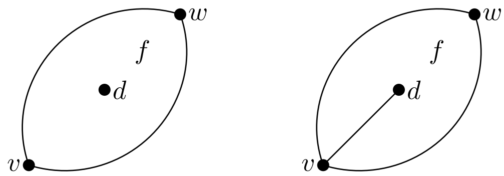

Figure 4.15: If there was a vertex $d \in D$ incident to the face between the two edges between v and w, it would need to be connected to v or w, and the face would have size ≥ 3.

Consider the case where there are no other vertices in the face f in $\mathcal{G}'_{\mathcal{R}}$. Since there are no vertices in f it means that there's either no vertices in the corresponding area A of $\mathcal{G}_e$, or that all vertices in this area are adjacent to v or w (since D is a dominating set, and v, w are the only vertices from D incident to this face). In either case we see that we could remove R_1 and R_2, adding one bigger region covering all their combined vertices together with any vertices in A, contradicting the maximality of $(\mathcal{G}_e, \mathcal{R})$.

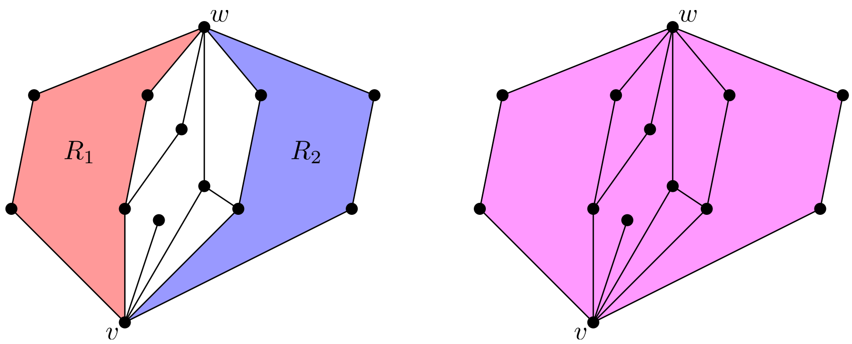

Figure 4.16: If there is no vertex $d \in D$ in the area between two regions between v and w, we can make a new, bigger region that combines the two and the vertices in the area.

This shows that the resulting graph $\mathcal{G}'_{\mathcal{R}}$ is connected and thin, and since we carefully added edges while still keeping the graph planar it will be a plane supergraph of $\mathcal{G}_{\mathcal{R}}$. ☐

Note that all faces in the decomposition graph $\mathcal{G}_{\mathcal{R}}$ is of size at least 3.

Theorem 4.36 together with Lemma 4.34 gives us the following two corollaries:

Corollary 4.37. *The number of edges in $G_{\mathcal{R}}$ is upper bounded by $3|V| - 6 - \sum_{f \in F}(|f| - 3)$, where F is the set of faces in $G_{\mathcal{R}}$.*

Corollary 4.38. *The number of regions in a maximal D-region decompositions $\mathcal{R}$ is upper bounded by $3|D| - 6$, that is $|\mathcal{R}| < 3|D|$.*

Using the bound on the number of regions in a maximal D-region decomposition, we will in the next chapter arrive at a linear kernel for PLANAR DOMINATING SET.

Chapter 5

A Linear Kernel

Using the definitions and lemmas from the previous section, we will now prove a lemma corresponding to [2, Lemma 6] of Alber et al., as discussed in Section 3.3.

Lemma 5.1. *Let $(G, \mathcal{E})$ be a plane graph with a dominating set D and let $(\mathcal{G}_e, \mathcal{R})$ be a maximal D-region decomposition of G. If $u \in N_1(v) \setminus D$ for some vertex $v \in D$ then $u \in V(\mathcal{R})$.*

Proof. Let $u \in N_1(v)$ for some $v \in D$, and assume that $u \notin V(\mathcal{R})$. Since $u \in N_1(v)$, it must be adjacent to a vertex $u' \notin N(v)$. We now have two cases to consider:

- $u' \in D$. Look at the region $R = R(v, u')$ created by the two walks $W_1 = (v, e_1, u, e_2, u')$ and $W_2 = (u', e_2', u, e_1', v)$ in $\mathcal{G}_e$, using only copied edges such that e_1 and e_1' are consecutive at v and u, and e_2 and e_2' are consecutive at u and u'. Note that this implies that R doesn't cross any other region at an edge. Since by assumption $(\mathcal{G}_e, \mathcal{R})$ is a maximal D-region decomposition, R cannot be added to the D-region decomposition of some enrichment $\mathcal{G}_e'$ of $\mathcal{G}_e$, and hence must cross a region in $\mathcal{R}$. If not $(\mathcal{G}_e', \mathcal{R} \cup \{R\})$ would be a D-region decomposition containing more vertices than $(\mathcal{G}_e, \mathcal{R})$, contradicting that $(\mathcal{G}_e, \mathcal{R})$ was maximal. Since by assumption $u \notin V(\mathcal{R})$, no region can cross R at u, and by Lemma 4.17 R cannot cross any other region in v or u', since the edges are consecutive. Then by Lemma 4.15 there must be a crossing region R' such that $area(R') \subseteq area(R)$ or $area(R) \subseteq area(R')$, and this cannot be the case since R and R' cannot cross at an edge.

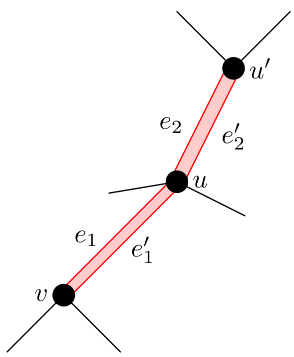

Figure 5.1: The only possible reason this region cannot be added to the D-region decomposition is if it would cross a region in u, but u is not in $V(\mathcal{R})$, a contradiction.

- $u' \notin D$. This means that there must be a vertex $w \in D$ that dominates u', where $w \neq v$. Now consider at the region $R = R(v, w)$ created by the walks $W_1 = (v, e_1, u, e_2, u', e_3, w)$ and $W_2 = (w, e_3', u', e_2', u, e_1', v)$ in $\mathcal{G}_e$, using only copied edges such that e_1 and e_1' are consecutive at v and u, e_2 and e'_2 are consecutive at u and u', and e_3 and e'_3 are consecutive at u' and w. Again, by assumption R must cross at least one region R' in $(\mathcal{G}_e, \mathcal{R})$. By the assumption that $v \notin V(\mathcal{R})$, R and R' don't cross at u. We claim that R and R' cross at u'. Suppose not, and then by Lemma 4.17 they neither cross at v or w, contradicting that R and R' cross.

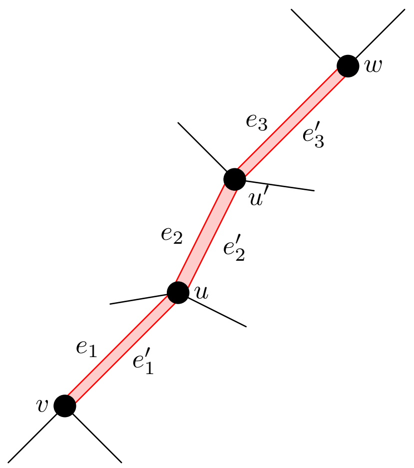

Figure 5.2: The region must cross an existing region at u'.

Let $R'' = R(x, y)$ be the region crossing at u' that has the edge with the smallest edge distance to edge $e_2 = (u, u')$. Observe that this is the region crossing at u' with the smallest edge distance to e_2 both clockwise and counterclockwise. Note that u' must be adjacent to x or y since it is in the region R''. Without loss of generality, u' is adjacent to x. Assume $x \neq w$ and look at the region $R^* = R(v, x)$ defined by the paths $(v, e_1, u, e_2, u', e_4, x)$ and $(x, e'_4, u', e'_2, u, e'_1, v)$ in an enrichment $\mathcal{G}'_e$ of $\mathcal{G}_e$. This region can be added to $\mathcal{R}$ such that $(\mathcal{G}'_e, \mathcal{R} \cup R^*)$ is a D-region decomposition of G, contradicting the maximality of $(\mathcal{G}_e, \mathcal{R})$.

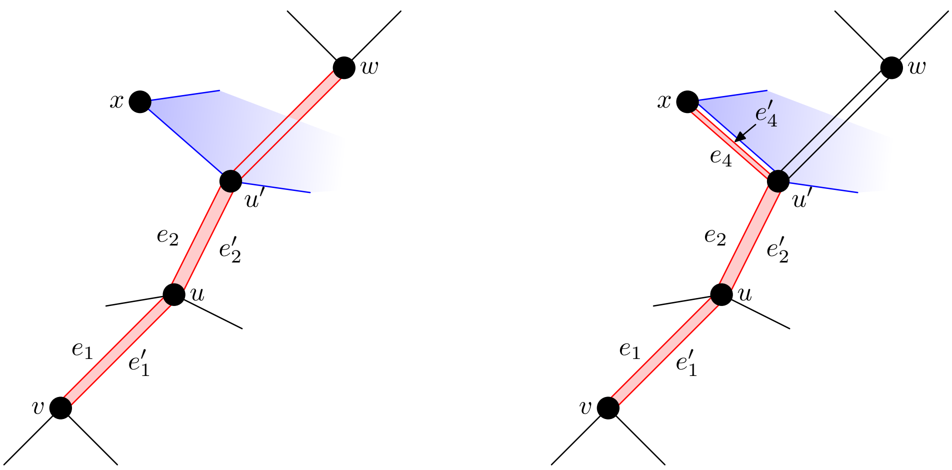

Figure 5.3: If we cannot add $R = R(v, w)$ because of a crossing region $R' = R(x, y)$, we can add $R^* = R(v, x)$ instead.

If $x = w$ we can add parallel edges creating an enrichment $\mathcal{G}'_e$ of $\mathcal{G}_e$ such that $R(v, w)$ and R'' do not cross in $\mathcal{G}'_e$, again contradicting the maximality of $(\mathcal{G}_e, \mathcal{R})$, see Figure 5.4.

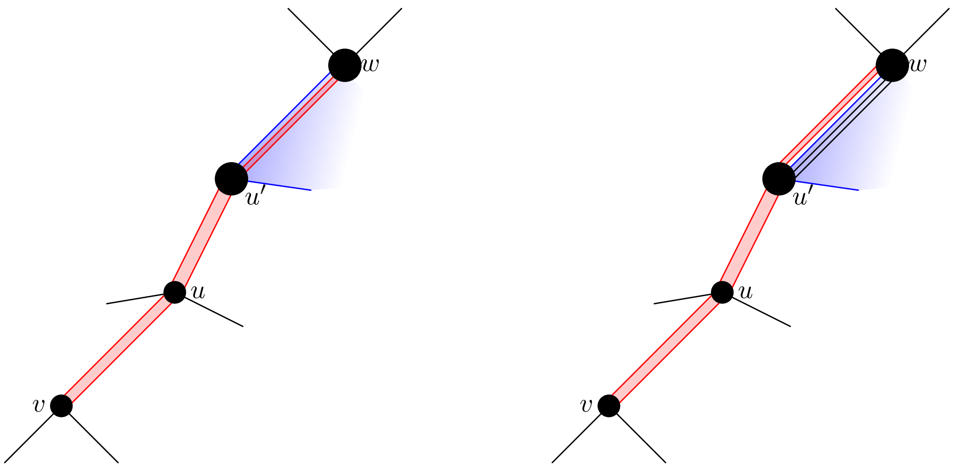

Figure 5.4: Adding edges to make the regions not cross.

□

5.1 The Kernel of Alber et al.

Alber et al. introduce two Reduction Rules [2, Rule 1, Rule 2], and in the rest of this section ***a reduced graph*** is a plane graph were neither of the two rules can be applied. Together with our definition of maximal D-region decompositions we obtain:

Proposition 5.2 ([2, Prop. 2]). *Let $G = (V, E)$ be a planar reduced graph and let D be a dominating set of G. If $(\mathcal{G}_e, \mathcal{R})$ is a maximal D-region decomposition of G then $|V \setminus V(\mathcal{R})| \leq 2|D| + 56|\mathcal{R}|$.*

Proposition 5.3 ([2, Prop. 3]). *A region R of a plane reduced graph contains at most 55 vertices, i.e., $|V(R)| \leq 55$.*

This leads to the reduced graph $G = (V, E)$ being a linear kernel:

Theorem 5.4 ([2, Theorem 2]). *For a reduced planar graph $G = (V, E)$ we get $|V| \leq 355k$, where k is the size of a dominating set D in G. In other words, the* Dominating Set *problem on planar graphs admits a linear kernel.*

Proof. By combining Proposition 5.2, Proposition 5.3 and Corollary 4.38 we get

$$\begin{aligned}
|V| &= |V(\mathcal{R})| + |V \setminus V(\mathcal{R})| \\
&= \sum_{R \in \mathcal{R}} |V(R)| + |V \setminus V(\mathcal{R})| \\
&\leq 55|\mathcal{R}| + 2|D| + 56|\mathcal{R}| \\
&= 2|D| + 111|\mathcal{R}| \\
&\leq 2|D| + 111 \cdot 3|D| \\
&= 335|D|
\end{aligned}$$

If the graph has more than $335k$ vertices after the rules are applied, we know there cannot be a dominating set of size at most k in the graph, and we can return a trivial no instance. If not we see that $|V| \leq 355k$. □

Chapter 6

A Smaller Kernel

In this chapter we will show that by allowing the kernelizaion algorithm to modify the plane embedding of the input graph, it is possible to achieve a smaller kernel.

6.1 Vertex Flipping

Definition 6.1 (Boundary adjacent)**.** *Let $(\mathcal{G}_e, \mathcal{R})$ be a region decomposition of the plane graph $(G, \mathcal{E})$. We say that $u, v \in V(\mathcal{G}_e)$ are boundary adjacent if there is a region $R \in \mathcal{R}$ with boundary C_R such that (u, v) is an edge in the walk C_R. The vertices boundary adjacent to v is denoted $N^*(v)$.*

Let $(\mathcal{G}_e, \mathcal{R})$ be a maximal D-region decomposition of a graph G where Reduction Rule 3.1 and 3.2 are exhaustively applied, together with its embedding $\mathcal{E}$. We want to bound the number of vertices in G. To achieve this we will modify the embedding $\mathcal{E}$. Performing the Embedding Rules and running the algorithm from Lemma 4.32 on the resulting embedding exhaustively, will leave us with a decomposition $(\mathcal{G}_e, \mathcal{R})$ that is maximal and reduced with respect to the Embedding Rules.

Definition 6.2 (Outside vertices)**.** *Vertices not in $V(\mathcal{R})$ or D are called outside vertices.*

Embedding Rule 6.1. Let $v \in D$, and let $u \in N^*(v)$ be a vertex boundary adjacent to v. If there is a vertex $v' \in N_2(v) \setminus V(\mathcal{R})$ having neighborhood $N(v') = \{v, u\}$, then change the embedding $\mathcal{E}$ such that v' is drawn on the opposite side of the edge (v, u), resulting in v' being in $V(\mathcal{R})$.

Figure 6.1: An $N_2(v)$ vertex of degree 2 will be flipped inside the region.

Claim 6.3. Embedding Rule 6.1 is sound.

Proof. Since the rule doesn't change the structure of the graph, what we must show is that the embedding is still plane after the change. Since u is boundary adjacent to v, there is an edge (v, u) in the embedding used by a region R. By definition this edge separates the interior of R from the exterior, and we can change $\mathcal{E}$ in such a way that v' and the edges (v, v'), (v', u) are drawn sufficiently close to this edge in the interior of R, keeping the embedding plane. ☐

Embedding Rule 6.2. If there is a vertex $v' \in N_3(v)$ for some $v \in D$, and v is the endpoint of a region R, then flip v' inside R by modifying $\mathcal{E}$.

Claim 6.4. Embedding Rule 6.2 is sound.

Proof. Since Reduction Rule 3.1 is exhaustively applied, we know v' must be of degree 1. The embedding is still plane after the change, since a vertex of degree 1 can be drawn sufficiently close to its neighbor in such a way that the edge between them doesn't cross anything. ☐

6.2 Upper Bound Outside Regions

Let D be a dominating set in the planar graph G, and consider the decomposition graph $\mathcal{G}_{\mathcal{R}}$ of D-region decomposition $(\mathcal{G}_e, \mathcal{R})$ of G. Let f be a face in $\mathcal{G}_{\mathcal{R}}$, and let $v \in D$ be one of the vertices incident to the face f. Let A be the co-area in $\mathcal{G}_e$ corresponding to f. We will now upper bound the number of vertices in the neighborhood of v we may find in this co-area.

Lemma 6.5. *There are no vertices from $N_1(v)$ in A.*

Proof. This follows from Lemma 5.1, which states that any vertex $u \in N_1(v)$ is in a $V(\mathcal{R})$. ☐

Lemma 6.6. *There are no vertices from $N_3(v)$ in A.*

Proof. This follows from Rule 6.2, where we flipped every vertex from $N_3(v)$ inside a region. ☐

It remains to bound the number of vertices from $N_2(v)$ found in A. We'll look at that next.

Lemma 6.7. *Let $v, w \in D$. Any vertex $u \in N(v)$ in A cannot be adjacent to a vertex from $N^*(w) \cup w$.*

Proof. Assume for sake of contradiction that u is adjacent to a vertex $a \in N^*(w) \cup w$. Consider the region defined by the two walks $(v, e_1, u, e_2, a, e_3, u)$ (or (v, e_1, u, e_2, u) if $a = w$) and $(u, e_3', a, e_2', u, e_1', v)$ $((u, e_2', u, e_1', v))$, made possible in an enrichment of $\mathcal{G}$. This region can be added to the D-region decomposition, contradicting its maximality.

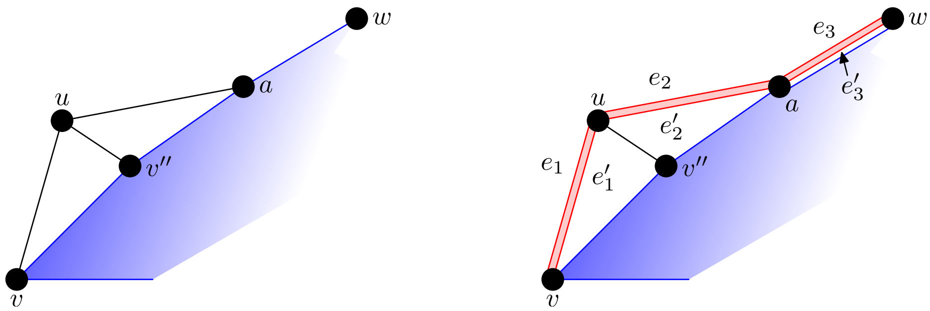

Figure 6.2: A vertex $u \in N_2(v)$ outside a region cannot be adjacent to any vertices from $N^*(w) \cup w$ in a maximal region decomposition.

☐

Corollary 6.8. *For a vertex $u \in N(v)$ in A we have $N(u) \subseteq N^*(v) \cup v$.*

Proof. By Lemma 6.5 and Lemma 6.6 we know that $u \in N_2(v)$, which by definition means that its neighborhood is a subset of $N_1(v) \cup N_2(v) \cup N_3(v) \cup \{v\}$. By Rule 3.1 we know that u cannot have neighbors in $N_3(v)$ or $N_2(v)$, which means that all its neighbors are from $N_1(v) \cup \{v\} \subseteq V(\mathcal{R}) \cup D$, by Lemma 5.1. Since the only vertices it can possibly be adjacent to are on the boundary of some region, we know that $N(u) \subseteq \bigcup_{w \in D} N^*(w) \cup w$, and by Lemma 6.7 we have $N(u) \subseteq N^*(v) \cup v$. $\square$

Lemma 6.9. *There is no vertex $u \in N_2(v)$ of degree 2 in A.*

Proof. For sake of contradiction, assume $u \in N_2(v)$ is in A and has degree 2. By Corollary 6.8 we know all of u's neighbors are from $N^*(v) \cup v$, and since it is of degree 2, it neighborhood must be exactly $\{v, v'\}$ for some vertex $v' \in N^*(v)$. By Rule 6.1 we know that such a vertex u is in $V(\mathcal{R})$. $\square$

Lemma 6.10. *Let f be a face of size 3 in $\mathcal{G}_{\mathcal{R}}$, and $v \in D$ a vertex incident to this face. Then the maximum number of vertices from $N_2(v)$ in A is 1.*

Proof. Take a vertex $u \in N_2(v)$ in A. By Corollary 6.8, all of u's neighbors come from $N^*(v) \cup v$. Since f is of size 3, there are at most two possible neighbors to u other than v, and they are the two vertices boundary adjacent to v and incident to A. And since u by Lemma 6.9 must have at least two neighbors other than v, we see it must be adjacent to both. And now we can have no more vertices from $N_2(v)$ in this face. $\square$

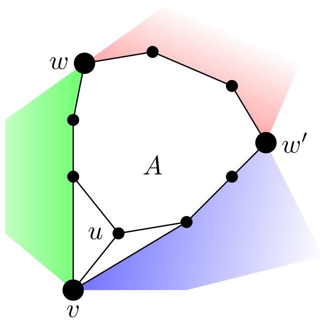

Figure 6.3: A vertex $u \in N_2(v)$ in an area corresponding to a face of size 3 must be adjacent to the two boundary adjacent vertices of v incident to this area.

Lemma 6.11. *Let f be a face of size > 3. The maximum number of vertices from $N_2 = \bigcup_{v \in D} N_2(v)$ in the corresponding area A is $6|f|$.*

Proof. The face f is made from edges in $\mathcal{G}_{\mathcal{R}}$ going between vertices of D. These edges corresponds to regions from the region decomposition $(\mathcal{G}_e, \mathcal{R})$, which makes up the area A. Observe that there are at most 2 nodes on the boundary of such a region in addition to vertices from D, meaning there are at most $3|f|$ vertices from $V(\mathcal{R}) \cup D$ incident to this area. By Lemma 6.9 vertices in A must have a degree of 3 or more, and then by Lemma 1.47 we know there can be at most $6|f|$ of them. $\square$

6.3 Bounding Region Size

Chen et al.[6] showed that we can apply a set of reduction rules in time $O(n^3)$ to the planar graph G, such that the resulting planar graph admits a D-region decomposition where the number of vertices inside a region R of the decomposition is upper bounded by $|V(R)| \leq 16$. This lets us state the following lemma:

Lemma 6.12. *There exists a polynomial time algorithm that given a region decomposition $(\mathcal{G}_e, \mathcal{R})$ and a region $R \in \mathcal{R}$ where $|V(R)| > 16$ returns an equivalent region decomposition $(\mathcal{G}'_e, \mathcal{R}')$ where $|V(\mathcal{G}'_e)| < |V(\mathcal{G}_e)|$.*

To achieve this result, they use a superset of the reduction rules we have used so far, so applying their additional rules will make sure we also can bound the size of the regions to 16.

6.4 An Improved Kernel Upper Bound

Using the results from the previous sections, we now arrive at the following results:

Lemma 6.13. *Let $\mathcal{G} = (G, \mathcal{E})$ be a plane graph reduced with respect to Reduction Rule 3.1 and 3.2 that has a dominating set D of size k. Let $(\mathcal{G}_e, \mathcal{R})$ be a maximal D-region decomposition of $\mathcal{G}$ where $\forall R \in \mathcal{R}, |V(R)| \leq r$, and $r \geq 6$. Then $|V(\mathcal{G})| \leq 7k + 3rk - 6r + 12$.*

Proof. Let F_3 be the set of faces of size 3, and let $F_{>3}$ be the set of faces of size > 3 in the decomposition graph $\mathcal{G}_\mathcal{R}$ of $(\mathcal{G}_e, \mathcal{R})$. Since $(\mathcal{G}_e, \mathcal{R})$ is maximal, we know that by Corollary 4.37 $\mathcal{G}_\mathcal{R}$ has at most $3k - 6 - \sum_{f \in F_{>3}} (|f| - 3)$ edges. By definition the number of edges in the decomposition graph is equal to the number of regions in the decomposition. Since $(G, \mathcal{E})$ is reduced, we can bound the number of vertices outside regions and in areas corresponding to faces of size 3 in $\mathcal{G}_\mathcal{R}$ to $3|F_3|$ by Lemma 6.10, and in areas corresponding to bigger faces to $\sum_{f \in F_{>3}} 6|f|$ by Lemma 6.11. Given that each region contains at most r vertices, we have the total number of vertices bounded by

$$
\begin{aligned}
&|D| + r \cdot |\mathcal{R}| + 3|F_3| + \sum_{f \in F_{>3}} 6|f| \\
&\leq k + r \cdot (3k - 6 - \sum_{f \in F_{>3}} (|f| - 3)) + 3|F_3| + \sum_{f \in F_{>3}} 6|f| \\
&= k + 3rk - 6r - r \sum_{f \in F_{>3}} |f| - \sum_{f \in F_{>3}} 3r + 3|F_3| + 6 \sum_{f \in F_{>3}} |f| \\
&= k + 3rk - 6r + 3|F_3| + 6 \sum_{f \in F_{>3}} |f| - r \sum_{f \in F_{>3}} |f| - \sum_{f \in F_{>3}} 3r \\
&= k + 3rk - 6r + 3|F_3| - \sum_{f \in F_{>3}} \big((r - 6)|f| + 3r\big) \\
&\leq k + 3rk - 6r + 3|F_3|
\end{aligned}
$$

We see that the number of vertices is maximum when all the faces have size 3, and by Lemma 4.35 that $|F| = |F_3| = 2|V| + 4 = 2k + 4$, which gives us

$$|V(G)| \leq k + 3rk - 6r + 3|F_3| = k + 3rk - 6r + 3(2k + 4) = 7k + 3rk - 6r + 12$$

□

In the following theorem, a graph being reduced means that Reduction Rules 3.1, 3.2 and the Reduction Rules from Chen et al.[6] are exhaustively applied.

Theorem 6.14. *Let $\mathcal{G}$ be a reduced plane graph having a dominating set D of size k. The number of vertices in $\mathcal{G}$ is upper bounded by $55k$.*

Proof. If $k \leq 1$, solve the problem in polynomial time and output a trivial kernel of constant size. If $k \geq 2$ then since the graph is reduced, we know by Lemma 6.12 that all regions have size at most 16. Set $r = 16$, and then by Lemma 6.13 the number of vertices in G is upper bounded by:

$$7k + 3rk - 6r + 12 = 7k + 3 \cdot 16 \cdot k - 6 \cdot 16 + 12 = 55k - 84 < 55k$$

$\square$

Part III

Computer-Aided Reduction of Regions

WORDS OF WARNING

The following part of the thesis explains a Computer-Aided technique for reducing the size of the planar graph instance. Unfortunately, we did finish the coding a bit late, and consider the concepts and proofs of correctness a bit lacking. We are confident the achieved results, the $43k$ kernel in particular, are correct, and we also arrived at the same results using a different technique. Especially Chapter 9-11 were written close up to the deadline for submission, and has not been vetted, corrected and cleaned up as much as we would have hoped before submission.

Chapter 7

A $43k$ kernel for Planar Dominating Set

So far we have arrived at kernels for Planar Dominating Set by performing reduction rules and upper bounding the number of vertices inside and outside regions in a maximal Region Decomposition of the resulting graph. Bodlander et al.[4] showed that for many problems, including Dominating Set, for parts of the graph that only have a constant set of vertices in common with the rest of the graph, there exists a finite set of ways this part can affect the solution for the whole graph. This means that all such small portions of the graph can be grouped into a finite set of equivalence classes. In each equivalence class there must be a smallest graph, in terms of the number of vertices, that can represent all graphs in that class.

Bodlander et al. don't give any specific constants or representatives for the problems they explore. In the case of Region Decomposition of plane graphs we have a natural boundary that separates a region from the rest of the graph, and we know the size of that boundary is a small constant, ≤ 6 in our case. This means that there must exist a small finite set of equivalent classes, and that each class must have a smallest representative. If we can find such a representative for each equivalence class, we can create a kernelization algorithm by changing all the regions in the graph by the representative of the equivalence class they belong to, and thereby creating an equivalent graph with as small regions as possible.

In this part of the thesis we analyze such regions, and describe how we use a computer program to search for small representatives for each equivalence class.

7.1 Definitions

Up until now, regions have just been parts of a bigger plane graph. To be able to look for reduced instances, we want to be able to talk about them as standalone objects. The next definitions let us do that. We will define quasi-regions as a sort of region where the boundary vertices don't have to be adjacent, as this will be useful in the analysis in later chapters. Quasi-regions are regions found in a graph. ***Detached quasi-regions*** are stand-alone graphs. Note that since a detached quasi-region is a graph, we might find quasi regions in a detached quasi-region.

Definition 7.1 (Noose)**.** *Let $(G, \mathcal{E})$ be a plane graph, and let $v, w \in V(G)$. A* **noose between** v **and** w *is a polygon line $N = P(a, b)$ such that*

- *$N \cap f_{\mathcal{E}}(v) = \{a\}$ and $N \cap f_{\mathcal{E}}(w) = \{b\}$*
- *For all $e \in E(G)$ we have $N \cap g_{\mathcal{E}}(e) = \emptyset$*

This means that a noose is a curve that can be drawn from v to w such that it only intersects with vertices, and not with edges.

Definition 7.2 (Vertices on a noose). *We say the* **vertices on a noose** *N, denoted $V(N)$ is the vertices it intersects.*

We often write $N = (n_1, n_2, \dots, n_k)$ to denote a noose intersecting vertices n_1 to n_k.

Definition 7.3 (Disjoint nooses). *Let $(G, \mathcal{E})$ be a plane graph, and let $N_1 = (a, n_1, \dots, n_k, b), N_2 = (c, m_1, \dots, m_l, d)$ be two nooses in this region. We say that N_1 and N_2 are disjoint if*

- $N_1 \cap N_2 = \emptyset$
- *for all $1 \leq i \leq k$ and $1 \leq j \leq l$, $n_i \neq m_j$.*

In other words, two disjoint nooses can start and end in the same vertex, but not in any other vertices.

Definition 7.4 (Length of a noose). *The* **length of a noose** *N is $|V(N)| - 1$.*

The reason for the -1 in the definition of length of noose, is to the familiarity of the length of a path being the number of edges on the path.

Definition 7.5 (Area of nooses). *Let $(G, \mathcal{E})$ be a plane graph, and let $N_1 = (a, n_1 \dots, n_k, b), N_2 = (a, m_1 \dots, m_k l, b)$ be two disjoint nooses such that they start in the same vertex, and end in the same vertex. Let $a_1 = f_{\mathcal{E}(a) \cap N_1}$ and $a_2 = f_{\mathcal{E}(a) \cap N_2}$ be the points where their polygon lines start, and let $b_1 = f_{\mathcal{E}(b) \cap N_1}$ and $b_2 = f_{\mathcal{E}(b) \cap N_2}$ be where they end. We say the area of the nooses is all the points in the interior of the simple polygon $N_1 \cup N_2 \cup L(b_1, b_2) \cup L(a_1, a_2)$, denoted $area(N_1, N_2)$.*

Note that the area of nooses is analogous to the area of a cyclic walk.

Definition 7.6 (Quasi-region). *Let $(G, \mathcal{E})$ be a plane graph. Let $v, w \in V(G)$, and let N_t, N_b be disjoint nooses of length at most 3 from v to w. The nooses define a* **quasi-region in** *G, denoted $(R_q = (V_q, E_q), \mathcal{E}_q)$ if the following holds:*

- $A = area(N_t, N_b)$, $\delta(R_q) = V(N_t) \cup V(N_b)$
- $V_q = \{v \in V(G) \mid f_{\mathcal{E}}(v) \cap A \neq \emptyset$
- $E_q = \{e \in E(G) \mid g_{\mathcal{E}}(e) \cap A \neq \emptyset$
- *for all $v \in V_q \setminus \delta(R)$, $f_{\mathcal{E}_q}(v) \subseteq A$*
- *for all $v \in V_q$, $v \in N[v, w]$*

A quasi-region is very similar to the definitions we have used for a region before, but now the "path" between the endpoint no longer need to have edges. A regular region as we have defined it earlier is therefore a special case of a quasi-region.

Definition 7.7 (Naming of quasi-region). *We denote a quasi-region R where $|N_t| = n$ and $|N_b| = m$ for a $(n, m)^q$-region. The vertices on the nooses, $\delta(R)$, are called* **boundary vertices of** *R. The vertices $I(R) = V \setminus \delta(R)$ are called the internal vertices of R. If we want to emphasize that there are edges between consecutive vertices on the N_t or N_b nooses, we denote this by $(n, m)^q_{(a_1, a_2, \dots)(b_1, b_2, \dots)}$-region, $a_i, b_i \in \{i, \bar{i}\}$, $a_i = i$ meaning there is an edge between vertex i and $i + 1$ on N_t, and $a_i = \bar{i}$ meaning there is no edge. It is similar for b_i.*

For shorthand, we say (n, m)-region for a quasi-region where all boundary edges are present.

Definition 7.8 (Detached quasi-region). *A* **detached quasi-region** *is a planar simple graph $R = (V, E)$ having endpoints $v, w \in V$ and an embedding $\mathcal{E}$ such that*

- *there are two simple, disjoint nooses N_t, N_b of length at most 3 from v to w*

- *$A = area(N_t, N_b)$, $\delta(R_q) = V(N_t) \cup V(N_b)$*
- *for all $e \in E$, $g_{\mathcal{E}}(e) \subseteq A$*
- *for all $v \in V \setminus \delta(R)$, $f_{\mathcal{E}}(v) \subseteq A$*
- *$N[v, w] = V$*
- *all vertices of $\delta(R)$ is incident to the exterior face in the embedding*

We see that a quasi-region in a graph defines a corresponding detached quasi-region.

Definition 7.9 (Detached quasi-region of quasi-region)**.** *Let R be a quasi-region with nooses N_t, N_b in the plane graph $(G, \mathcal{E})$. We say the* **corresponding detached quasi-region of** *R is the detached quasi-region obtained from N_t, N_b, removing vertices and edges that does not intersect $area(N_t, N_b)$, and removing any multi-edges.*

Observe that if there are edges among boundary vertices of a quasi-region that does not go "inside" the area of the quasi-region, then they are not present in the corresponding detached quasi-region.

Definition 7.10 (Equal boundaries)**.** *Let R_1 be a detached quasi-region with endpoints v^1, w^1 and nooses $N_t^1 = (t_1^1, \ldots, t_k^1)$ and $N_b^1 = (b_1^1, \ldots, b_l^1)$ such that $v^1 = t_1^1 = b_1^1$ and $w^1 = t_k^1 = b_l^1$. Let R_2 be a detached quasi-region with endpoints v^2, w^2 and nooses $N_t^2 = (t_1^2, \ldots, t_k^2)$ and $N_b^2 = (b_1^2, \ldots, b_l^2)$ such that $v^2 = t_1^2 = b_1^2$ and $w^2 = t_k^2 = b_l^2$. We say that R_1 and R_2 have equal boundaries, or are* **boundary equal***, if there exists a bijective function $f : \delta(R_1) \to \delta(R_2)$ such that*

- *For all $1 \le i \le k$, $f(t_i^1) = t_i^2$*
- *For all $1 \le i \le l$, $f(b_i^1) = b_i^2$*
- *For all $1 \le i < k$, $(t_i^1, t_{i+1}^1) \in E(R_1)$ iff $(t_i^2, t_{i+1}^2) \in E(R_2)$*
- *For all $1 \le i < l$, $(b_i^1, b_{i+1}^1) \in E(R_1)$ iff $(b_i^2, t_{b+1}^2) \in E(R_2)$*

We call f for the **boundary bijection between** *R_1* **and** *R_2, and say that R_1* **and** *R_2* **are boundary equal with boundary bijection** *f.*

We will later use the boundary bijection f on both single vertices and sets of vertices, so when we say $S' = f(S)$ for some set S, we mean $S' = \{f(s) \mid s \in S\}$.

Definition 7.11 (Type of detached quasi-regions)**.** *A* **type of detached quasi-regions** *is a set of detached quasi-regions being boundary equivalent.*

Definition 7.12 (Signature)**.** *Let $R = (V, E)$ be a detached quasi-region. We define the* **signature** *ζ_R* **of** *R as:*

$$\zeta_R : 2^{\delta(R)} \times 2^{\delta(R)} \to \mathbb{N}$$
$$\zeta_R(X, S) = \min_{D \subseteq V} |D| - |X|$$

s.t. $V \setminus S \subseteq N[D]$, $X \subseteq D$, $X, S \subseteq \delta(R)$ and $X \cap S = \emptyset$.

In other words, the signature encodes what the size of a minimum dominating set D of the detached quasi region is, such that the vertices of X are part of the dominating set, and the vertices of S don't have to be dominated by D. The intuition behind this is that if R is part of a bigger graph instance, then the boundary $\delta(R)$ of the region are the only vertices that can have neighbors outside the region. This means that two regions having the same signature will affect the minimum dominating set of the whole graph in the exact same way, and can be interchanged while keeping the problem instance equivalent!

Definition 7.13 (Valid input to signature)**.** *let R be a detached quasi-region. Two sets $X, S \subseteq V$ are called* **valid inputs to the signature of** *R, or simply* **valid inputs***, if $X, S \subseteq \delta(R)$ and $X \cap S = \emptyset$.*

Definition 7.14 (Satisfying set)**.** *Let $R = (V, E)$ be a detached quasi-region having signature ζ_R. Given valid inputs X, S let D be a set such that $V \setminus S \subseteq N[D]$, $X \subseteq D$ and $|D| = \zeta_R(X, S) + |X|$. We say that D* **is a set satisfying** *$\zeta_R(X, S)$, and we denote it by $\hat{\zeta}_R(X, S) = D$. We sometimes write $R(X, S)$, meaning $\hat{\zeta}_R(X, S)$.*

Definition 7.15 (Equal signatures)**.** *Let R_1, R_2 be detached quasi-regions with equal boundaries and boundary bijection $f : \delta(R_1) \to \delta(R_2)$. We say that R_1 and R_2 have the same signature, denoted $R_1 =_\zeta R_2$, or are* **signature equivalent***, if for all valid inputs X, S to the signature of R_1, $\zeta_{R_1}(X, S) = \zeta_{R_2}(f(X), f(S))$.*

Note that for all valid inputs X, S to the signature of R_1, $f(X), f(S)$ will be valid inputs to the signature of R_2 when R_1 and R_2 is boundary equivalent with boundary bijection f.

Definition 7.16 (Partial ordering of detached quasi-regions)**.** *Let R_1, R_2 be two detached quasi-regions. We say that R_1* **is smaller than or equal to** *R_2, denoted $R_1 \leq R_2$, if $R_1 =_\zeta R_2$ and $|V(R_1)| \leq |V(R_2)|$.*

Definition 7.17. *A detached quasi-region R is* **signature minimal** *if for all regions R' such that $R =_\zeta R'$, $R \leq R'$.*

For simplicity, we make this definition:

Definition 7.18 (Detached region)**.** *A* **detached region** *is a detached quasi-regions where all boundary edges are present.*

All detached quasi-regions can be put into different equivalence classes according to their signature. Since there is a finite set of different signatures, there is a finite number of such equivalence classes. Each equivalence class has one or more smallest representative, and these are exactly the signature minimal detached quasi-regions. Our goal will be to find one such representative for each type of detached regions. A signature minimal detached region is the "most reduced" region of a particular signature.

Note that we might say signature of a quasi-region, meaning the signature of the corresponding detached region of R.

Now we look back at our first definition of a region, the ones we can find in a region decomposition. As noted earlier, non-degenerate regions are also valid quasi-regions with all boundary edges present, hence we can find a corresponding detached region.

Definition 7.19 (Replacing a non-degenerate region with a detached region)**.** *Let $\mathcal{G}$ be a plane graph and let R be a non-degenerate region in $\mathcal{G}$. Let R_2 be a detached region boundary equal to R_1 with boundary bijection $f : \delta(R_1) \to \delta(R_2)$, where R_1 is the detached region corresponding to R. The graph obtained by* **replacing** *R* **with** *R_2* **in** *G is the plane graph $\mathcal{G}' = (G', \mathcal{E}')$, where $G' = ([G - V(R_1)] + R_2) + E_\delta + E_b$ with $E_\delta = \{(x, y) \mid x \in V \setminus V(R_1), y \in \delta(R_2), (x, f^{-1}(y)) \in E(G)\}$ and $E_b = \{(x, y) \mid x, y \in delta(R_2), (x, y) \notin E(R_2) \wedge (f^{-1}(x), f^{-1}(x)) \in E(R)\}$. We get the embedding $\mathcal{E}'$ of $\mathcal{G}'$ by noting that by definition R_1 and R_2 have embeddings such that their boundary vertices are drawn incident to the external face, and we can create $\mathcal{E}'$ by the union of the embedding $\mathcal{E}$ and the embedding of R_2, by mapping the unit disks of $\delta(R_2)$ to the unit disks of $\delta(R_1)$.*

Note that replacing a region R in $\mathcal{G}$ by its corresponding detached region doesn't do anything to a dominating set in the graph, since it is just removing multi-edges. This fact we will now use to show that replacing R with any signature equal detached region doesn't do anything to the size of a dominating set in the graph.

Lemma 7.20. *Let $\mathcal{G} = (G, \mathcal{E})$ be a plane graph having dominating set D. Let R be a region in $\mathcal{G}$, and let R_1 be the detached region corresponding to R. Let R_2 be a detached region that is signature equal to R_1 with boundary bijection $f : \delta(R_1) \to \delta(R_2)$. Replacing R with R_2 results in a graph $\mathcal{G}' = (G', \mathcal{E}')$ such that G' has a dominating set of size k if and only if G has a dominating set of size k.*

Proof. Let D be a dominating set of size k in G, and let

$$\begin{aligned} D_{boundary} &= D \cap \delta(R) \\ D_{inner} &= D \cap I(R) \\ D_R &= D_{boundary} \cup D_{inner} \\ D_{outside} &= D \setminus D_R \end{aligned}$$

Now set

$$\begin{aligned} X &= D_{boundary} \\ S &= (\delta(R) \cap N(D_{outside} \cup X)) \setminus X \end{aligned}$$

Set $X' = f(X)$, $S' = f(S)$. We claim that $D' = D_{outside} \cup X' \cup \hat{\zeta}_{R_2}(X', S')$ is a dominating set of size $|D'| \leq k$.

Note that $D = D_{outside} \cup X \cup D_{inner}$.

First, observe that $|D| \leq k$, since $|X'| = |X|$ and $|\hat{\zeta}_{R_2}(X', S')| = \zeta_{R_2}(X', S') = \zeta_{R_1}(X, S) = |D_{inner}|$ since R_1 and R_2 are signature equal.

Second, we show that D' is indeed a dominating set in G'.

- Let $v \in V(G') \setminus V(R_2)$. Note that v is also in $V(G) \setminus V(R_1)$. In G v is either in $D_{outside}$, adjacent to a vertex in $D_{outside}$, or adjacent to a vertex in X. If v was in $D_{outside}$, then it is also in D'. If it was adjacent to a vertex in $D_{outside}$, then it is also adjacent to a vertex in D'. If is was adjacent to to a vertex in X, then it is adjacent to a vertex in X' in G' by the definition of E_δ.
- $v \in \delta(R_2)$. If $v \in X'$ then it is also in D'. If v is not in X' and not in S', then it is dominated (either by being in the set or being adjacent to a vertex in the set) by $\hat{\zeta}_{R_2}(X', S')$. Last, if v is in S' then it is adjacent to a vertex in $D_{outside} \cup X$ since all vertices in S were adjacent to a vertex in $D_{outside}$, and by the definition of E_δ v will also be.
- $v \in I(R_2)$. By definition this vertex is dominated by $\hat{\zeta}_{R_2}(X', S') \subseteq D'$.

For the other direction let D' be a dominating set of size k in G'. Using the embedding $\mathcal{E}'$ of G', we see that we can create a region R' in an enrichment $(G'_e, \mathcal{E}'_e)$ of G' such that D' is still a dominating set. Using the exact same arguments, we can show that we can replace R' with R_1 in $(G'_e, \mathcal{E}'_e)$, to get a graph $(\hat{G}, \hat{\mathcal{E}})$, and find a dominating set D of size $\leq k$ in this graph. D is therefore also a dominating set in G. $\square$

7.2 A 43k Kernel

We will later prove the following two lemmas:

Lemma 7.21. *For any detached region R, we can in time $O(1)$ find a signature minimal detached region having the same signature as R.*

Lemma 7.22. *The biggest signature minimal detached regions consist of* 12 *region vertices and* 2 *endpoints.*

Aimed with these lemmas and the results from previous chapters, we give a kernelization algorithm as follows:

Theorem 7.23. Planar Dominating Set *admits a kernel of size* $43k$*.*

Proof. If the graph contains more than $43k$ vertices, then find a region R in the graph having more than 12 region vertices. We calculate the signature of R, and find a signature minimal detached region with the same signature as R by Lemma 7.21. By Lemma 7.20 we can replace R by the equivalent signature minimal region creating an equivalent graph G' of smaller size than G. After this is done exhaustively we know that the size of every region is upper bounded by $r \leq 12$, and by using Lemma 6.13 this yields a kernel size of:

$$7k + 3rk - 6r + 12 \leq 7k + 3 \cdot 12 \cdot k - 6 \cdot 12 + 12 = 43k - 60 < 43k$$

□

Chapter 8

Computer-Aided Reduction of Detached Regions

We will use the next chapters to prove Lemma 7.21 and Lemma 7.22.

8.1 More Definitions

Definition 8.1 (Vertex types). *Let $R = (V, E)$ be a detached quasi-region with endpoints v, w. Define $N_1(R) = \delta(R) \setminus \{v, w\}$, $N_2(R) = I(R) \cap N(N_1)$ and $N_3(R) = I(R) \setminus N_2$.*

Definition 8.2 (Outward pushed set). *Let $R = (V, E)$ be a detached quasi-region, $\mathcal{S}_i$ be a family of sets of size i s.t. $\forall S \in \mathcal{S}_i, S \subseteq V$ and $|S| = i$. A set $S \in \mathcal{S}_i$ is called* **outward pushed in** S_i *if there is no other set $S' \in \mathcal{S}_i$ such that $|S \cap \delta(R)| < |S' \cap \delta(R)|$.*

Definition 8.3. *Let R be a detached quasi-region with signature ζ_R. Let $\mathcal{D}$ be the family of sets that satisfy $\zeta_R(X, S)$. $\hat{\zeta}_R^+(X, S)$ denotes an outward pushed set $D \in \mathcal{D}$. We sometimes write $R^+(X, S)$, meaning $\hat{\zeta}_R^+(X, S)$.*

Definition 8.4 (Sub quasi-regions). *Let $(R, \mathcal{E})$ be a detached quasi-region and let R' be a quasi-region in $(R, \mathcal{E})$. R' is called a* **sub quasi-region of** *R.*

Definition 8.5 (Replacing a sub quasi-region). *R be a detached quasi-region with embedding $\mathcal{E}$ and with sub quasi-region R_1'. Let R_1 be the detached quasi-region corresponding to R_1'. Let R_2 be a detached quasi-region boundary equal to R_1 with boundary bijection $f : \delta(R_1) \to \delta(R_2)$. The graph obtained by* **replacing** *R_1* **with** *R_2* **in** *R is the detached quasi-region $R' = ([R - V(R_1)] + R_2) + E_\delta + E_b$ with $E_\delta = \{(x, y) \mid x \in V \setminus V(R_1), y \in \delta(R_2), (x, f^{-1}(y)) \in E(R)\}$ and $E_b = \{(x, y) \mid x, y \in delta(R_2), (x, y) \notin E(R_2) \wedge (f^{-1}(x), f^{-1}(x)) \in E(R)\}$. We get the embedding $\mathcal{E}'$ of R' by noting that by definition R_1 and R_2 have embeddings such that their boundary vertices are drawn incident to the external face, and we can create $\mathcal{E}'$ by the union of the embedding $\mathcal{E}$ and the embedding of R_2, by mapping the unit disks of $\delta(R_2)$ to the unit disks of $\delta(R_1)$.*

Lemma 8.6. *Let R be a detached quasi-region, and let R_1' be a sub quasi-region of R with corresponding detached region R_1. Let R_2 be a detached quasi-region having the same signature as R_1. Replacing R_1 with R_2 doesn't change the signature of R.*

Proof. Let R' be the detached quasi-region obtained by replacing R_1 with R_2.

For given sets $X, S \subseteq \delta(R)$, let $D = \hat{\zeta}_R(X, S)$ have size $k = |D|$. Set

$$
\begin{aligned}
D_{boundary} &= D \cap \delta(R_1) \\
D_{inner} &= D \cap I(R_1) \\
D_{R_1} &= D_{boundary} \cup D_{inner} \\
D_{outside} &= D \setminus D_{R_1}
\end{aligned}
$$

Now set

$$
\begin{aligned}
X_1 &= D_{boundary} \\
S_1 &= (\delta(R) \cap N(D_{outside} \cup X_1)) \setminus X_1
\end{aligned}
$$

Set $X_2 = f(X_1)$, $S_2 = f(S_1)$. We claim that $D' = D_{outside} \cup X_2 \cup \hat{\zeta}_{R_2}(X_2, S_2)$ is a set satisfying $\zeta_R(X, S)$.

Note that $D = D_{outside} \cup X_1 \cup D_{inner}$.

First, observe that $|D'| \leq k$, since $|X_2| = |X_1|$ and $|\hat{\zeta}_{R_2}(X_2, S_2)| = \zeta_{R_2}(X_1, S_1) = \zeta_{R_1}(X_1, S_1) = |D_{inner}|$ since R_1 and R_2 are signature equal.

Second, we show that D' is a set satisfying $\zeta_R(X, S)$.

- Let $v \in V(R') \setminus V(R_2)$. Note that v is also in $V(R) \setminus V(R_1)$. In R v is either in $D_{outside}$, adjacent to a vertex in $D_{outside}$, or adjacent to a vertex in X_1. If v was in $D_{outside}$, then it is also in D'. If it was adjacent to a vertex in $D_{outside}$, then it is also adjacent to a vertex in D'. If is was adjacent to to a vertex in X_1, then it is adjacent to a vertex in X_2 in R' by the definition of E_δ.
- $v \in \delta(R_2)$. If $v \in X_2$ then it is also in D'. If v is not in X_2 and not in S_2, then it is dominated (either by being in the set or being adjacent to a vertex in the set) by $\hat{\zeta}_{R_2}(X_2, S_2)$. Last, if v is in S_2 then it is adjacent to a vertex in $D_{outside} \cup X_2$ since all vertices in S_1 were adjacent to a vertex in $D_{outside} \cup X_1$, and by the definition of E_δ and E_b v will also be.
- $v \in I(R_2)$. By definition this vertex is dominated by $\hat{\zeta}_{R_2}(X_2, S_2) \subseteq D'$.

For the other direction let D' be a set satisfying $\zeta_{R'}(X, S)$ of size k. Doing the same replacement in the other direction, using the exact same arguments, we can find a set D satisfying $\zeta_R(X, S)$ of size $\leq k$. R and R' must therefore have the same signature. □

Lemma 8.7. *Let R be a detached quasi-region with signature ζ, and let $D = \hat{\zeta}^+(X, S)$. We have that $|D \cap I(R)| \leq 1$.*

Proof. Assume that $|D \cap I| \geq 2$ and let $x, y \in D \cap I$. Now $x, y \notin X$, and $D' = (D \setminus \{x, y\}) \cup \{v, w\}$ satisfies $\zeta(X, S)$, has size $|D'| \leq |D|$, and $X \subseteq D'$, contradicting that D was outward pushed. □

Lemma 8.8. *Let R be a detached quasi-region with signature ζ, and let $D = \hat{\zeta}^+(X, S)$. If $|D \cap I(R)| = 1$, then $|D \cap \{v, w\}| = 0$.*

Proof. Assume for sake of contradiction that $|D \cap \{v, w\}| \geq 1$, and without loss of generality that $v \in D$. Let $x \in D \cap I$. Now $D' = (D \setminus \{x\}) \cup \{w\}$ still satisfies $\zeta(X, S)$, and $X \subseteq D'$, contradicting that D was outward pushed. □

The following class of internal vertices will be useful in further discussion:

Definition 8.9 (Possible dominators)**.** *Let R be a detached quasi-region. The set $P \subseteq I(R)$ is the set of internal vertices where $\forall p \in P, \exists X, S \subseteq \delta(R)$ s.t. $p \in \hat{\zeta}_R^{+}(X,S)$. The set P is called* **possible dominators on** R.

Lemma 8.10. *Let $R(v,w)$ be a detached quasi-region, and let P be the set of possible dominators in R. A vertex $p \in P$ must meet the following criteria:*

1. $N_3(R) \subseteq N[p]$
2. $\exists u \in N(p)$ *s.t.* $u \notin N(v) \vee u \notin N(w)$

Proof. We will show a proof for each criteria.

1. Assume a vertex $p \in P$ that doesn't dominate all vertices in $N_3(R)$ is in the outward pushed set $D = \hat{\zeta}_R^{+}(X,S)$ for some $X, S \subseteq \delta(R)$. This means there must be at least one more vertex $u \in D$ to dominate N_3. The only vertices that can dominate vertices from N_3 are from $\{v,w\} \cup I$, but any vertex being in D together with p would violate Lemma 8.7 or Lemma 8.8.

2. Assume a vertex $p \in P$ is in the outward pushed set $D = \hat{\zeta}_R^{+}(X,S)$ for some $X, S \subseteq \delta(R)$, such that there's no $u \in N(p)$ where $u \notin N(v) \vee u \notin N(w)$. This means that $N(p) \subseteq N(v)$ and $N(p) \subseteq N(w)$. By Lemma 8.8, v and w cannot be in D. Since p is a vertex in the region, it must be adjacent to v or w. Without loss of generality, say it is adjacent to v. Now $(D \setminus \{p\}) \cup \{v\}$ still satisfies $\zeta_R(X,S)$, contradicting that D was outward pushed.

□

The reason the vertices in P are called possible dominators, is because they might actually be the best choice for dominating $N_3(R)$ given sets X, S, while this will never be the case for internal vertices not in P.

8.2 Reduction Rules Inside a Region

We will now show the reduction rules we will need to use inside a detached quasi-region. In the context of detached quasi-regions, a the soundness of a reduction rule means that the resulting detached quasi-region is signature equivalent to the original one. Note that these rules are ***not*** to be used with any kernelization algorithm. They are only to be used in analysis of signature minimal detached quasi-regions, since such detached quasi-regions have to be reduced.

Reduction Rule 8.1. Let $R(v,w)$ be a detached quasi-region, and let $u \in N_3$. If there are two vertices $x, y \in I$ such that $u \notin N[x]$ and $u \notin N[y]$, and $N(x) \subseteq N(y)$, then delete y.

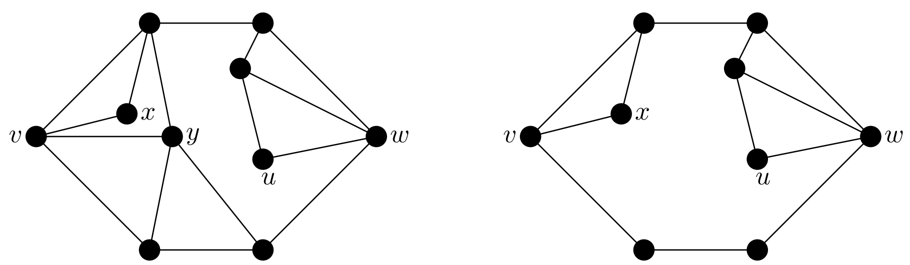

Figure 8.1: Reduction Rule 3.2 says we can remove y, since it will never be used as a dominator.

The intuition behind the reduction rule is that since there is a vertex u that cannot be dominated by x or y, neither of x, y will be possible dominators. Their only role is therefore to be dominated, but since the moment x is dominated y also is, y can be removed. Note the similarities between this rule and Reduction Rule 3.2. This rule is a bit stronger inside detached quasi-regions, since it will help us get rid of some vertices that wouldn't be taken care of by Reduction Rule 3.2. We now prove the soundness of this rule.

Lemma 8.11. *Reduction Rule 8.1 is sound.*

Proof. Let R' be the detached quasi-region obtained after performing the reduction rule.

For given sets $X, S \subseteq \delta(R)$, let $D = \hat{\zeta}_R^+(X, S)$. Since $x, y \notin P$, by definition $x, y \notin D$. D would therefore satisfy $\zeta_{R'}(X, S)$ also, and we have $\zeta_{R'}(X, S) \leq \zeta_R(X, S)$.

For the other direction, let $D' = \hat{\zeta}_{R'}^+(X, S)$. $x \notin P$ and $x \notin D'$, and must be dominated by D'. Since $N(x) \subseteq N(y)$, D' will also satisfy $\zeta_R(X, S)$ and $\zeta_R(X, S) \leq \zeta_{R'}(X, S)$. □

Reduction Rule 8.2. Let $x, y \in I \setminus P$. If (x, y) is an edge, remove it.

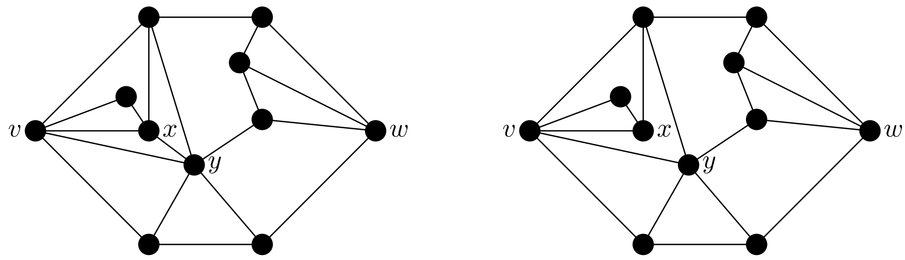

Figure 8.2: By Reduction Rule 8.2 we can remove the edge (x, y), none of them are possible dominators.

Again, note that this is a variant of Reduction Rule 3.1 tailored for our use inside a region.

Lemma 8.12. *Reduction Rule 8.2 is sound.*

Proof. Let R' be the region obtained after performing the reduction rule.

For given sets $X, S \subseteq \delta(R)$, let $D = \hat{\zeta}_R^+(X, S)$. Since $x, y \notin P$, we know they are not in D. Therefore D will also satisfy $\zeta_{R'}(X, S)$. The other direction is similar. □

Chapter 9

Bound Inside Inner and Single Regions

The strategy we choose for enumerating signature minimal detached regions, is to split them up into smaller parts, and building from these parts. We will now look at some of these parts, which all are sub quasi-regions.

9.1 Inner regions

Definition 9.1 (Inner noose)**.** *Let* $(R, \mathcal{E})$ *be a quasi-region, and let* $a, b \in V(R)$*. An* **inner noose from** a **to** b *is a noose* N *from* a *to* b *such that* $N \subseteq area(R)$*, and* $\forall d \in \delta(R) \setminus \{a, b\}, d \notin V(N)$*.*

In other words a noose that goes between vertices of the quasi-region, but doesn't touch boundary vertices except for its start and end vertex.

The next lemma follows from[24, 22] , so we will omit the formal proof:

Lemma 9.2. *Let* $R(v, w)$ *be a detached quasi-region with nooses* N_t, N_b*. Let* T *and* B *be the set of vertices on* N_t *and* N_b *(except* v, w*), respectively. The following are true:*

- *There is an inner noose from* v *to* w *in* R *if and only if there is no edge* $(t, b) \in E(R)$*, where* $t \in T$ *and* $b \in B$
- *There is two disjoint inner nooses from* v *to* w *in* R *if and only if there is no path* (t, u, b) *in* R*, where* $t \in T$*,* $b \in B$ *and* $u \in I(R)$

Lemma 9.2 will be helpful since it tells us that we can either find two inner nooses in a detached quasi-region, or there must be an edge or path dividing the detached quasi-region into smaller parts.

Lemma 9.3. *If there are two disjoint inner nooses from* v *to* w *in* R*, then there are also two disjoint inner nooses of length at most* 2 *from* v *to* w *in* R*.*

Proof. Assume there are no two disjoint inner nooses of length at most 2, and let N be the shortest noose of length more than 2. N must be of length 3 or more, which means that there are 3 or more vertices intersecting N. By definition of a detached quasi-region, all of these vertices must be adjacent to v or w. Let u be one such vertex such that it is not the first or last non-endpoint intersecting the noose, i. e. $N = (v, a, \dots, u, \dots, b, w)$, and assume without loss of generality that it is adjacent to v. Then $(v, u, \dots, b, w)$ is also a noose in R of length less than N, contradicting that N was the shortest one. Also note that this shorter noose cannot intersect with any noose that N didn't intersect with, since the edge (v, u) is present and cannot intersect with any noose. □

Definition 9.4 (Inner region). *Let N_1, N_2 be two inner nooses of length at most 3 in the interior of R, going between v and w. They define a quasi-region iR, called* **inner quasi-region of** *R, such that the boundary $\delta(iR)$ of iR are the vertices that intersects the nooses, and the internal vertices of iR are the vertices strictly inside the closed area defined by the nooses.*

Note that if you follow one of the inner nooses from v to w you can encounter up to two vertices different from v, w, and there may be actual edges between these vertices, but not necessarily. The important part is that there cannot be edges crossing these nooses, making the boundary of the inner region separate the internal vertices of iR from the rest of R.

Observation 9.5. All the internal vertices of an inner quasi-regions are from $N_3(R)$, and the boundary nodes, except v, w, are from $N_2(R) \cup N_3(R)$.

Definition 9.6 (Maximal inner region). *An inner region iR of R is* **inclusion-wise maximal** *if there is no other inner region iR' of R such that $E(iR) \subseteq E(iR')$ and $V(iR) \subset V(iR')$, or $E(iR) \subset E(iR')$ and $V(iR) \subseteq V(iR')$.*

In other words an inner quasi-region is maximal if you cannot "push" the inner nooses outwards to include more vertices or edges in the inner quasi-region.

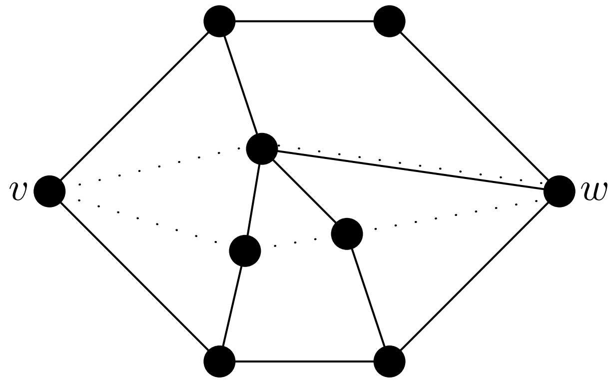

Figure 9.1: Two nooses that define a maximal inner region.

Let iR be a maximal inner quasi-region of R. How does the part of R outside iR now look like? It turns out that there is always an edge from every vertex in $\delta(iR) \setminus \{v, w\}$ to a vertex in $\delta(R) \setminus \{v, w\}$, as the next lemma shows.

Lemma 9.7. *Let R be a detached quasi-region having a maximal inner quasi-region iR. A vertex $b \in \delta(iR) \setminus \{v, w\}$ will be adjacent to at least one vertex from $\delta(R) \setminus \{v, w\}$.*

Proof. Assume this is not the case, and that there is a vertex $b \in \delta(iR) \setminus \{v, w\}$ that is not adjacent to any vertex from $\delta(R) \setminus \{v, w\}$, see Figure 9.2. Let $N_1 = (v, b, c, w)$ be the noose of the inner region that b is a part of, and let N_2 be the other noose of the iR. Since iR is maximal, this means that there cannot be a noose (v, c, w) that would make up a bigger inner region together with N_2. It must therefore be a path from b to a vertex in $\delta(R) \setminus \{v, w\}$. By assumption this path must be of length 2 or more. Let P be the path that has a vertex u that is incident to a face that c also is incident to. Such a vertex must exist, since if not there would need to be an edge from b to vertices in $\delta(R) \setminus \{v, w\}$. u must be adjacent to v or w, which means that either (v, u, c, w) or (v, b, u, w) is a noose in R, and it can be drawn on the outside of N_1, contradicting the maximality of iR. In the case $N_1 = (v, b, w)$ we see that any path from b to a vertex in $\delta(R) \setminus \{v, w\}$ would have a vertex u adjacent to b, and (v, b, u, w) or (v, u, b, w) would be a noose contradicting the maximality of iR. □

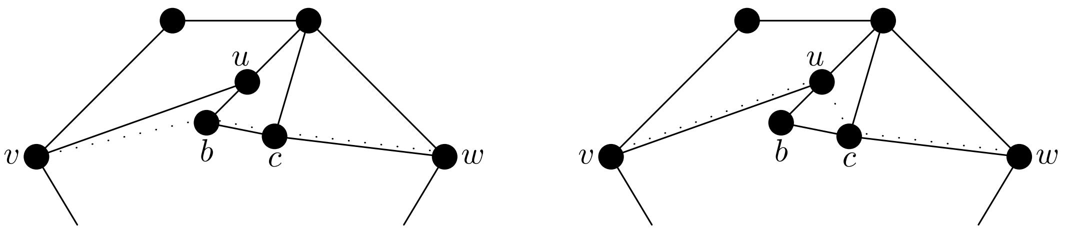

Figure 9.2: If a vertex b on the boundary of the inner region has no edge up to the boundary vertices of R, then the inner region cannot be maximal.

Lemma 9.8. *Let R be a detached quasi-region having top noose N_t, such that there are no edges between non-consecutive vertices on the noose, except maybe the start and end vertex. Let iR be a maximal inner quasi-region of R having top noose N of length ≤ 2. Then there is no vertices strictly in the area between N and N_t, $area(N, N_t)$.*

Proof. Assume that there is at least one vertex in this part, and look at a vertex u in this part that is incident to a face that N is incident to. Such a vertex must exist since there are no edges between any non-consecutive vertices on P_t (except maybe (u, v), that edge would be on the inside of the inner region). Without loss of generality assume u is adjacent to v. If $N = (v, w)$, then (v, u, w) is a noose contradicting that iR was maximal. If $N = (v, a, w)$, then (v, u, a, w) is a noose contradicting the maximality of iR. □

This means that we know a whole lot about how the parts in a detached quasi-region outside a maximal inner quasi-region look like. We know it is empty (no vertices, there can be edges) if the inner quasi-region has a noose of length ≤ 2, and that there must be an edge from each vertex on the boundary of the inner quasi-region to the boundary of the detached quasi-region. The edge between the boundary of R and the boundary of iR divides the outer part into small quasi-regions that we call ***outer regions***. They have a boundary size of either 3 or 4, and all their internal vertices must be dominated by a single endpoint, see Figure 9.3. These regions are a special case of what we call ***single regions***, which can have boundary size up to 6. We will look at single regions in Section 9.2. To speed up the computer program generating region, we will also look at how such single regions look like in signature minimal detached quasi-regions look inner quasi-regions are non-empty. This we do in Section 9.3.

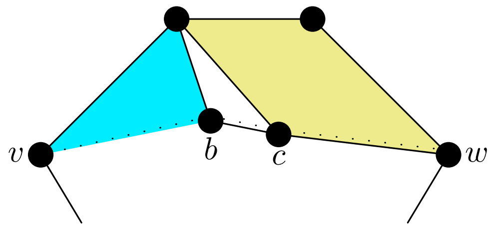

Figure 9.3: The vertices between the boundary of the maximal inner region have edges to vertices on the boundary of the region. This divides the part outside the inner region into small parts, (*outer regions*, colored in the figure) each having all their internal vertices dominated by a single endpoint. Note that the part between the two outer regions is empty.

9.1.1 Bounding the Size of an Inner Quasi-Region

Lemma 9.9. *An inner quasi-region of a signature minimal detached quasi-region has at most 4 internal vertices.*

Proof. Let $R = R(v, w)$ be a signature minimal detached quasi-region, and iR an inner quasi-region of R having at least one internal vertex. We'll look at several different cases. Let S be the set of internal vertices of iR that dominates all internal vertices of iR, called internal dominators. Let B be the vertices of $\delta(iR) \setminus \{v, w\}$ that dominates all internal vertices. Notice that vertices not in S or B cannot be in P, where P is the set of possible dominators of R.

- $|S| = 0$: Since there are no internal dominators, we can assume all internal vertices are independent by Reduction Rule 8.2. Also, we know that if there are vertices in B, then these must be adjacent to all internal vertices. All other vertices from $\delta(iR) \setminus \{v, w\}$ can be made non-adjacent to I by Rule 8.2. We can invoke Reduction Rule 8.1. Now all internal vertices are neighbors of the same vertices in $\delta(iR) \setminus \{v, w\}$, and either v or w, or both. There can be at most 2 such vertices after Rule 8.1 is exhaustively applied.

- $|S| = 1$:

  - $|B| \geq 1$**:** Let $b \in B$. There can be at most 3 internal vertices, as shown in Figure 9.4.

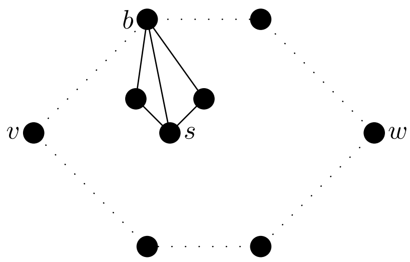

Figure 9.4: Any other vertex adjacent to both b and s would create a $K_{2,3}$, which is not outerplanar.

  - $|B| = 0$**:** Again we can remove all edges between $I \setminus S$ and vertices in $\delta(iR) \setminus \{v, w\}$, and invoke Reduction Rule 3.2. Now all vertices in $I \setminus S$ are adjacent to S, and to one or both in $\{v, w\}$, so there can be at most 2 of them after Rule 8.1 is exhaustively applied, resulting in at most 3 internal vertices.

- $|S| = 2$: In this case there can be at most 4 internal vertices, as shown in Figure 9.5.

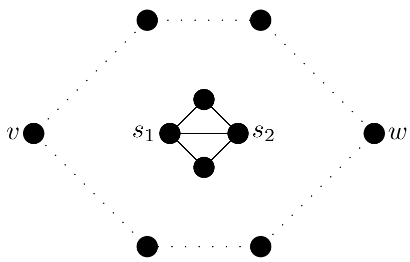

Figure 9.5: Any other vertex adjacent to s_1 and s_2 would create a $K_{2,3}$, which is not outerplanar.

- $|S| = 3$: These 3 internal dominators are the only possible internal vertices, because more internal vertices would create a K_4, which is not outerplanar, as shown in Figure 9.6.

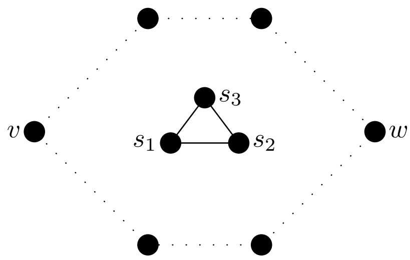

Figure 9.6: Any other vertex adjacent to the three internal vertices would create a K_4, which is not outerplanar.

- $|S| \geq 4$: Impossible because this would create a K_4.

□

9.2 Single Quasi-Regions

Definition 9.10 (Single region)**.** *Let R be a quasi-region with endpoints v, w. R is a* **single region** *if $I(R) \subseteq N(v)$. We denote a single $(n,m)^q$-region by $(n,m)^s$.*

As usual, a single region has a corresponding detached quasi-region, called ***a detached single quasi-region***.

As we will use these as building blocks later, it will be useful to be able to enumerate them easily. The next lemmas will be helpful for that purpose.

Lemma 9.11. *Let R be a detached $(2,1)^s_{(2,1)(*)}$-region having internal vertices I and nooses $N_t = (a,b,c), N_b = (a,c)$. If there is a vertex $p \in P(R)$, then $I \subseteq N[p]$.*

Proof. Let $D = R(X,S)$, and assume p is not adjacent to all the internal vertices of R. This means that if $p \in D$, there will be at least one more vertex from $A(R)$ in D. Call this vertex p'. If $p' = a$ or $p' = b$, we see that $D' = \{a,b\}$ satisfies $\zeta_R(X,S)$, contradicting that D was outward pushed. If $p' = c$ or $p' \in I$, then $D'' = \{a,c\}$ contradicts that D was outward pushed. □

So we know a possible dominator must dominate all internal vertices in the detached single quasi-region by Lemma 9.11, and that it must be adjacent to c by Lemma 8.10, as this is the only vertex of the region a may not be adjacent to (if (a,c) is not present). We use this to prove the following lemma.

Lemma 9.12. *A signature minimal detached $(2,1)^s_{(2,1)(*)}$-region has at most 4 internal vertices.*

Proof. Let $P = P(R)$. Look at the case where we have $|P| \geq 2$. This is shown in Figure 9.7. There cannot be more than two such universal vertices adjacent to c. And it is only possible to add one more vertex to the region such that they both still are universal.

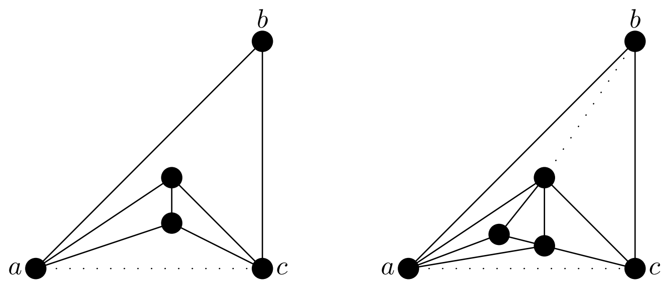

Figure 9.7: $|P| \geq 2$.

The case where $|P| = 1$ is shown in Figure 9.8. We see that we cannot add more vertices than this (4) without violating planarity or making the region reducible by Reduction Rule 8.1 or 8.2.

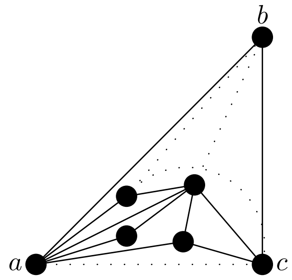

Figure 9.8: $|P| = 1$.

In the case $|P| = 0$ we can assume all internal vertices are independent by Reduction Rule 8.2, and Figure 9.9 shows the possible vertices. But note that if the internal degree 1 vertex is present, then we can remove the degree 3 vertex by Reduction Rule 8.1, so the maximum number of internal vertices in this case is also bounded by 4.

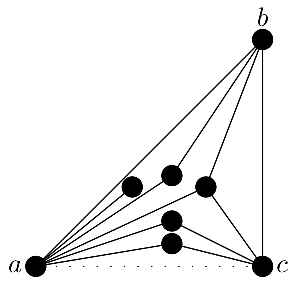

Figure 9.9: $|P| = 0$.

□

9.3 Outer Regions Without Possible Dominators

In the case of the inner quasi-region of a detached quasi-region being non-empty, we can simplify the structure of the area outside the inner quasi-region significantly. This is beneficial since it reduces the number of building blocks we have to put together and makes our computer program run much faster.

These "outer regions" is a special case of the single quasi-regions with boundary $(2, 1)$ and $(2, 2)$, and as for single quasi-regions we will look at cases where the boundary edges are present, and cases where some of them are not.

We denote these single regions by $(n, m)^n$.

This version of the outer regions is to be used together with a non-empty inner quasi-region. This let us apply Reduction Rule 8.2 and 8.1 to all internal vertices, since there is always an $N_3(R)$-vertex that's not adjacent to the internal vertices of the outer region. We can therefore assume that all internal vertices are independent, and that no internal vertex has a neighborhood that is a subset of the neighborhood of another internal vertex. Also, note that we can flip degree 1 vertices and degree 2 vertices between vertices on the boundary of the inner region, inside the inner region, so we can assume they are not in the outer region.

Consider the case of boundary size 3, shown in Figure 9.10. The only vertices we can have is one of degree 2 between a and b, or one of degree 3. Notice that both of these cannot be there in a signature minimal region by Reduction Rule 8.1.

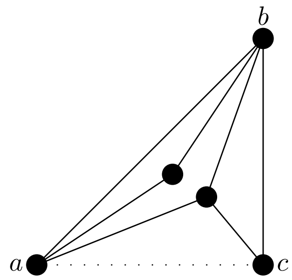

Figure 9.10: Outer 3-region.

We state that observation as a lemma:

Lemma 9.13. *A sub* $(2,1)^n_{(1,2)(*)}$*-region of signature minimal region can have most* 2 *internal vertices.*

For the case of boundary size 4, we have some more options, as shown in Figure 9.11. But observe that many of these cannot be there at once because of planarity constraints, and again some cannot appear together in a signature minimal region by Reduction Rule 8.1. Note that we omit the cases where the (a,c) edge or (b,d) edge are present, since when we will use these regions later, those edges cannot be there.

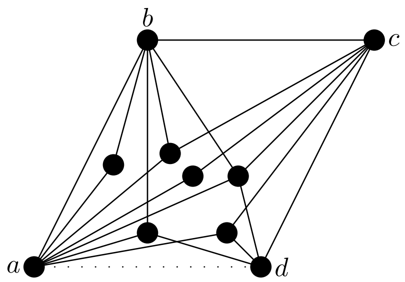

Figure 9.11: Outer 4-region.

Lemma 9.14. *A sub* $(2,2)^n_{(1,2)(*,2)}$*-region of signature minimal region can have most* 4 *internal vertices.*

9.4 Boundary Sizes and Vertices 2

Definition 9.15 (Sub quasi-region)**.** *Let R be a quasi-region with nooses N_t and N_b from v to w. A* **sub quasi-region** R' **of** *R is a quasi-region with nooses N'_t and N'_b such that $area(R') \subseteq area(R)$ and R' has at least one endpoint from $\{v,w\}$.*

Chapter 10

Splitting Regions Into Smaller Parts

We will in this chapter lay out the theory needed to see how we can combine small parts together to build bigger regions. In the next chapter we will set this theory into action, by explaining how our computer program that actually builds the detached quasi-regions work.

The goal is to enumerate signature minimal detached regions of different types, but to get there we will also need to enumerate some types of detached quasi-regions.

10.1 Fully Enumerated Representative Sets

We will for each type of detached quasi-regions find a ***fully enumerated representative set***.

Definition 10.1 (Fully enumerated representative set)**.** *Let T be a detached quasi-region type. We say a set $\mathcal{R}$ of detached quasi-regions is* **fully enumerated for** *T if for every R of type T, there is an $R' \in \mathcal{R}$ s.t. $R' \leq R$.*

We say a detached quasi-region type is ***fully enumerated*** if we can find a fully enumerated representative set for that type.

The next definition will be very important when we start to talk about figures, as it is much easier to explain how we decompose detached quasi-regions by a figure, than by words. Example of such figures can be seen starting from Figure 11.1

Definition 10.2 (Decomposing according to figure)**.** *Let $R = (V, E)$ be a detached quasi-region and let X be a figure. We say that R can be* **decomposed according to figure** *X if there is a bijection f between vertices on the figure and vertices in R such that*

- *if there is a vertex v_X on the figure, then $f(v_X) \in V$*
- *if there is an edge $e_X = (v_X, v'_X)$ on the figure, then $(f(v_X), f(v'_X)) \in E$*
- *for all dashed edges $\tilde{e}_X = (v_X, v'_X)$ on the figure, there is either an edge $(f(v_X), f(v'_X)) \in E$, or it would be possible to add that edge without violating planarity. Let the set of edges $\tilde{E} = \{(x, y) \in E \mid (f^{-1}(x), f^{-1}(x))$ is a dashed edge on the figure$\}$*
- *if there is a sub quasi-region with $N_t = (n_1, n_2, \ldots, n_n)$ and $N_b = (m_1, m_2, \ldots, m_m)$ of type T on the figure, then the corresponding sub quasi-region made from $(f(n_1), \ldots, f(n_n))$ and $(f(m_1), \ldots, f(m_m))$ in R is of type T, and the edges are consistent with those in $\tilde{E}$*

Note that we use a shorthand notation on the actual figures. Sub quasi-regions having boundary $(2,1)$ will be denoted 3 (since the total size of the boundary is 3), $(2,2)$ for 4 and so on. Single regions will be given the $\hat{}$ mark, as in $\hat{5}$. Inner quasi-regions are given the prefix "i". The edges present or not around the boundary can be seen on the figure by looking at the dashed edges. Parts of the region that have no vertices are marked "empty", or colored white. Single quasi-regions of the type used together with non-empty inner quasi-regions are marked with the suffix "nd" (for non-dominator). The empty inner quasi-regions are given the suffix "empty".

The definition of decomposing according to figure is not as complicated when looking at a figure. It basically says that it you are given detached quasi-region R and a figure X, R can be decomposed according to X if it is possible to draw R over the figure such that the sub quasi-region types, edges an vertices corresponds. Figure 11.3 shows the setup.

Definition 10.3 (Symmetry of a detached quasi-region)**.** *A symmetry sym of a detached quasi-region is a function that given a detached quasi-region outputs an isomorphic quasi-region where the names of the endpoints might have been interchanged, and the vertices on the noose N_t might have been renamed to the names on N_b, and vice versa.*

You can think of the symmetry of a quasi-region as drawing the detached quasi-region the plane and "flipping" it over one of its axis, see Figure 10.1 and 10.2. Notice how the direction of the nooses (the vertex names) changes. Each detached quasi-region will have up to 4 different symmetries.

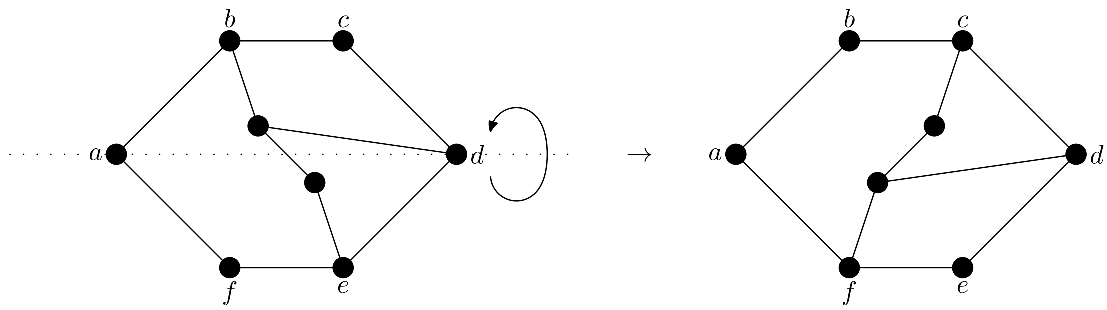

Figure 10.1: Symmetry around horizontal axis

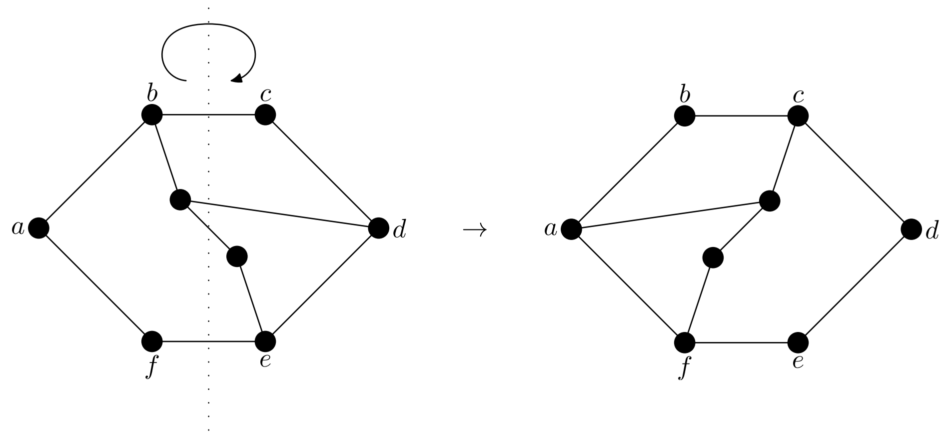

Figure 10.2: Symmetry around vertical axis

Fact 10.4 (Signatures of symmetries)**.** Let R_1, R_2 be two detached quasi-regions being signature equivalent, and let *sym* be a symmetry on the regions. Let $R'_1 = sym(R_1)$ and $R'_2 = sym(R_2)$.

Then R'_1 and R'_2 are signature equivalent.

Definition 10.5 (Symmetric closure)**.** *Let $\mathcal{R}$ be a set of detached quasi-regions. The* **symmetric closure of** *$\mathcal{R}$, denoted $sym(\mathcal{R})$ is the set containing all possible symmetries of regions in $\mathcal{R}$.*

Note that the identity is also considered a symmetry.

Definition 10.6 (Set enumerated by figure)**.** *We say that the set $\mathcal{R}$ of detached quasi-regions enumerated by a figure X, is the symmetric closure of exactly those detached quasi-regions R that can be decomposed according to figure X such that for each sub quasi-region $\tilde{R}$ of type T on the figure, $R[\tilde{R}]$ is in the fully enumerated representative set of T.*

Definition 10.7 (Reduced set)**.** *Let $\mathcal{R}$ be a set of detached quasi-regions. The* **reduced set of** *$\mathcal{R}$ is the set $\mathcal{R}' \subseteq \mathcal{R}$ where for each $R \in \mathcal{R}$, there is an $R' \in \mathcal{R}'$ such that*

- *$R' \leq R$*
- *for all $R'' \in \mathcal{R}' \setminus R'$, $R'' \neq_{\zeta} R$*

Definition 10.8 (Reduced set enumerated by figures)**.** *Let $\mathcal{X}$ be a set of figures. The* **reduced set enumerated by** *$\mathcal{X}$, denoted $enum(\mathcal{X})$, is the reduced set of $\mathcal{R}$, where $\mathcal{R}$ is the union of sets enumerated by the figures in $\mathcal{X}$.*

Lemma 10.9. *Let $\mathcal{X}$ be a set of figures, and let $\mathcal{R}$ be the reduced set enumerated by $\mathcal{X}$. Let R be a detached quasi-region that can be decomposed according to some figure $X \in \mathcal{X}$. Then there exists a quasi-region $R' \in \mathcal{R}$ such that $R' \leq R$.*

Proof. Assume that this is not the case, and that there is no such $R' \in \mathcal{R}$ such that $R' \leq R$. Consider the figure X that R can be decomposed according to. For each sub quasi-region $\tilde{R}$ of X, look at the detached quasi-region R_1 of $\tilde{R}$, having type T. According to Lemma 8.6 this sub quasi-region can be replaced in R with a quasi-region $R_2 \in enum(T)$ where $R_2 \leq R_1$ such that the resulting graph R^* is a quasi-region with $R^* \leq R$. After this is done for all sub quasi-regions $\tilde{R}$ of X, the detached quasi-region R_1^* of $\tilde{R}$ of type T is $R_1^* \in enum(T)$, meaning that R^* is a region enumerated by figure X, and hence there exists some $R' \in \mathcal{R}$ such that $R' \leq R^* \leq R$, a contradiction. □

Lemma 10.10. *Let $\mathcal{X}$ be a set of figures, and let $\mathcal{R}$ be the reduced set enumerated by $\mathcal{X}$. Let R be a detached quasi-region that can't be decomposed according to some figure $\mathcal{X}$. If there exists some $R' \leq R$ that can be decomposed according to some figure $X \in \mathcal{X}$, then there exists a quasi-region $R'' \in \mathcal{R}$ such that $R'' \leq R$.*

Proof. Since R' can be decomposed by Lemma 10.9 there is some $R'' \in \mathcal{R}$ such that $R'' \leq R' \leq R$, which proves the lemma. □

For a type T, what we want to find is a set of figures that enumerates a signature minimal representative for all equivalence classes of T. The previous two lemmas show that if we can find a set of figures such that all regions of that type either can be decomposed according to one of those figures, or we can reduce it such that it can be decomposed according to one of the figures, the set of figures will enumerate a fully enumerated reduced set for that type. We sum this up in the last lemma of this section:

Theorem 10.11. *Let T be a quasi-region type and let $\mathcal{X}$ be a set of figures. If there for all detached quasi-regions R of type T exists an $R' \leq R$ such that R' can be decomposed according to a figure $X \in \mathcal{X}$ such, then $enum(\mathcal{X})$ is a fully enumerated representative set for T.*

Proof. Since R' can be decomposed according to a figure $X \in \mathcal{X}$ there exists some $R'' \in enum(\mathcal{X})$ such that $R'' \leq R'$ by Lemma 10.9. We must then have $R'' \leq R' \leq R$, and the lemma follows. □

10.2 Order of Enumeration

In the last chapter we will explain how we actually enumerate these detached quasi-regions, by showing figures. Each type of detached quasi-regions will use fully enumerated representative sets of other types in their enumeration, so it will require them to already by enumerated. To give an overview, this will be the order we enumerate them in:

We will find fully enumerated representative sets for quasi-region types in the order

inner of types	$(n, m)^{i}_{(*)(*)}$ for $1 \leq n, m \leq 3$
single inner of types	$(1, 1)^{i,s}_{(*)(*)}, (2, 1)^{i,s}_{(1,*)(*)}, (2, 2)^{i,s}_{(1,*)(1,*)}$
single of types	$(2, 1)^{s}_{(1,2)(*)}, (2, 2)^{s}_{(1,*)(*,2)}, (3, 2)^{s}_{(1,2,3)(1,*)}, (3, 3)^{s}_{(1,2,3)(1,2,3)}$
single non-dominator of types	$(2, 1)^{s,n}_{(1,2)(*)}, (2, 2)^{s,n}_{(1,2)(*,2)}$
detached regions of types	$(2, 1), (2, 2), (3, 1), (3, 2), (3, 3)$

Chapter 11

Enumeration of Fully Enumerated Representative Sets

11.1 Inner regions

We will start with finding the fully enumerated representative set for detached inner quasi-regions of at most 4 internal vertices, as they don't require the fully enumerated sets of any other type. By Lemma 9.9 we know those are all the detached inner quasi-regions we need for signature minimal detached quasi-regions.

Let $I^{n,m}_{(*)(*)}$ be the reduced set of all inner regions that have at most 4 internal vertices. Now $I^{n,m}_{(*)(*)}$ is a fully enumerated representative set for $(n,m)^i$ by definition of a reduced set.

11.2 Enumerating Single Regions

We will now find fully enumerated representative sets for single regions.

11.2.1 $(2,1)^s$-regions

By Lemma 9.12 a signature minimal $(2,1)^s_{(1,2)(*)}$-region can have at most 4 internal vertices. Let $S^{2,1}_{(1,2)(*)}$ be the reduced set of $(2,1)^s_{(1,2)(*)}$-regions that have at most 4 internal vertices, and it is a fully enumerated representative set by definition.

11.2.2 $(2,2)^s$-regions

11.2.2.1 $(2,2)^s_{(1,*)(*,2)}$-regions

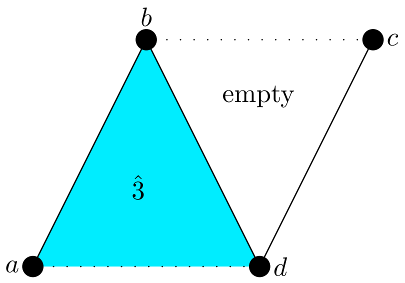

Figure 11.1

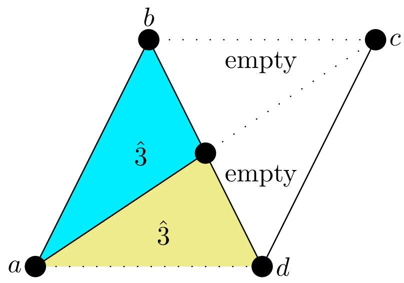

Figure 11.2

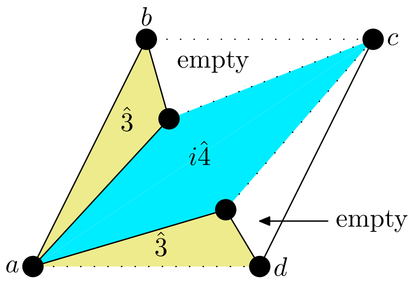

Figure 11.3

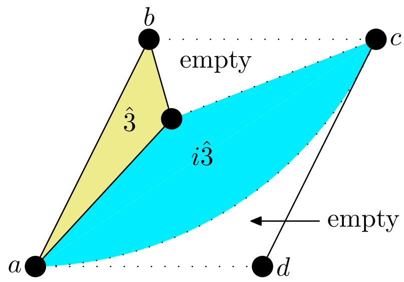

Figure 11.4

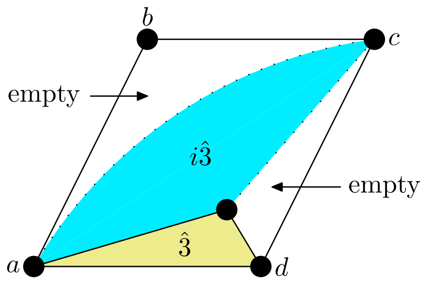

Figure 11.5

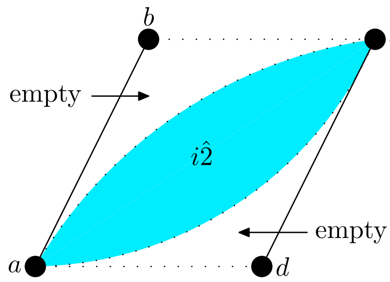

Figure 11.6

Lemma 11.1. *The reduced set* $S^{2,2}_{(1,*)(*,2)}$ *enumerated by the figures* $\mathcal{X}$ =\{*11.1, 11.2, 11.3, 11.5, 11.4, 11.6*\} *is a fully enumerated representative set for* $(2,2)^s_{(1,*)(*,2)}$.

Proof. Let R be a detached quasi-region of type $(2,2)^s_{(1,*)(*,2)}$, and let $N_t = (a,b,c)$, $N_b = (a,d,c)$. If there is a (b,d)-edge present, then it can be decomposed according to Figure 11.1. If there is a (b,d) path on two edges present, then it can be decomposed according to Figure 11.2.

If there is no (b,d)-edge or (b,d)-path present we can find two disjoint nooses from a to c by Lemma 9.2, defining an inner region. Because R is a single detached quasi-region, it is adjacent to any vertex on these nooses, except maybe c, so if any of the nooses is of length ≥ 3, we can also find a noose of length ≤ 2. Look at a maximal such inner region $\tilde{R}$ made from nooses of length ≤ 2:

- If $\tilde{R}$ is of $(2,2)^i_{(1)(1)}$-type, let the vertices on the inner nooses be $\tilde{N}_t = (a, \tilde{b}, c)$ and $\tilde{N}_b = (a, \tilde{d}, c)$. By Lemma 9.7 there must be edges $(b, \tilde{b})$ and $(d, \tilde{d})$ present. In this case R can be decomposed according to Figure 11.3.
- If $\tilde{R}$ is of $(2,1)^i_{(1)()}$-type, let the vertices on the inner nooses be $\tilde{N}_t = (a, \tilde{b}, c)$ and $\tilde{N}_b = (a, c)$. By Lemma 9.7 there must be an edge $(b, \tilde{b})$ present, and by Lemma 9.8 there cannot be vertices in the face made by $\tilde{N}_b$ and N_b. In this case R can be decomposed according to Figure 11.4.
- If $\tilde{R}$ is of $(1,2)^i_{()(1)}$-type, symmetric arguments show it can be decomposed according to Figure 11.5.
- If $\tilde{R}$ is of $(2,1)^i_{(1)()}$-type, let the the inner nooses be $\tilde{N}_t$ and $\tilde{N}_b$. By Lemma 9.8 there is no vertices in the face made by N_t and $\tilde{N}_t$, and no vertices in the face made by N_b and $\tilde{N}_b$. R can therefore be decomposed according to Figure 11.6.

This means that any quasi-region of type $(2,2)^s_{(1,2)(1,2)}$ can be decomposed by a figure in $\mathcal{X}$, and by Lemma 10.11 *enum*$(\mathcal{X})$ is a fully enumerated representative set for $(2,2)^s_{(1,2)(1,2)}$. □

11.3 $(3,2)^s$-regions

11.3.1 $(3,2)^s_{(1,2,3)(1,*)}$-regions

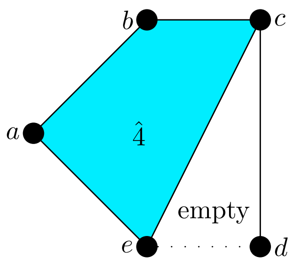

Figure 11.7

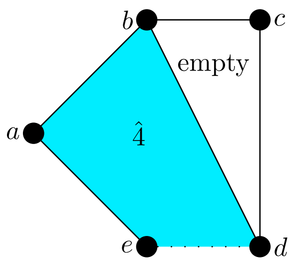

Figure 11.8

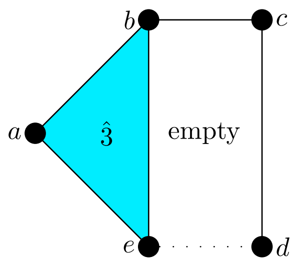

Figure 11.9

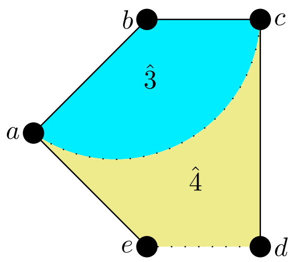

Figure 11.10

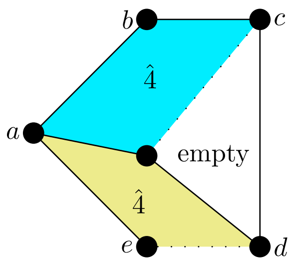

Figure 11.11

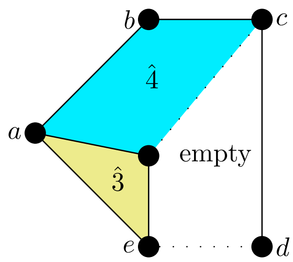

Figure 11.12

Lemma 11.2. *The reduced set* $S^{3,2}_{(1,2,3)(1,*)}$ *enumerated by the figures* $\mathcal{X}$ *=*{*11.7, 11.8, 11.9, 11.10, 11.11, 11.12*} *is a fully enumerated representative set for* $(3,2)^s_{(1,2,3)(1,*)}$.

Proof. Let R be a quasi-region of type $(3,2)^s_{(1,2,3)(1,*)}$, and let $N_t = (a, b, c, d)$, $N_b = (a, e, d)$. If there is an (c, e)-edge present, it can be decomposed according to Figure 11.7, if there is an (b, d)-edge present, it can be decomposed according to Figure 11.8, and if there is an (b, e)-edge present, it can be decomposed according to Figure 11.9.

If (b, d) and (b, e) edges are not present, then we know by Lemma 9.2 that we can find a noose from a to c. Notice that since a is adjacent to all internal vertices, if there is an (a, c)-noose, then there is also one of length ≤ 2. Consider the bottommost such noose. If this noose is of length 1, the region can be decomposed according to Figure 11.10. If the noose is of length 2, let this noose be $\tilde{N} = (a, \tilde{d}, c)$. $\tilde{d}$ must be adjacent to d or e, if not $\tilde{N}$ would not be the bottommost noose. If the edge $(\tilde{d}, d)$ is present, the R can be decomposed according to Figure 11.11, while if that is not the case then the edge $(\tilde{d}, e)$ must be present, and R can be decomposed according to Figure 11.12

This means that any quasi-region of type $(3, 2)^s_{(1,2,3)(1,2)}$ can be decomposed by a figure in $\mathcal{X}$, and by Lemma 10.11 *enum*$(\mathcal{X})$ is a fully enumerated representative set for $(3, 2)^s_{(1,2,3)(1,2)}$. $\square$

11.3.2 $(3, 3)^s_{(1,2,3)(1,2,3)}$-regions

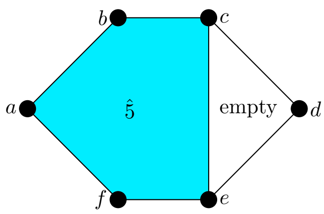

Figure 11.13

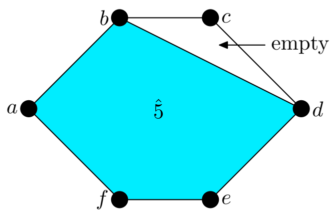

Figure 11.14

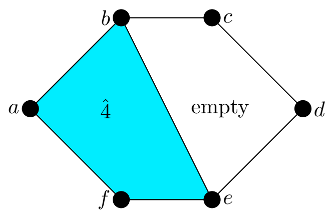

Figure 11.15

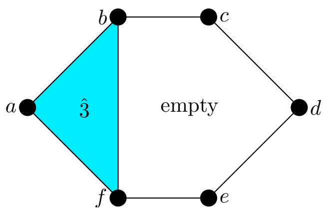

Figure 11.16

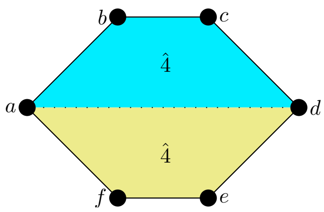

Figure 11.17

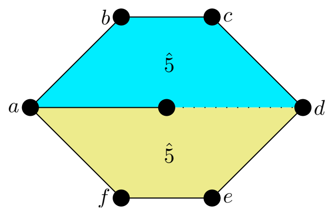

Figure 11.18

Lemma 11.3. *The reduced set* $S^{3,3}$ *enumerated by the figures* $\mathcal{X}$ =*{11.13, 11.14, 11.15, 11.16, 11.17, 11.18} is a fully enumerated representative set for* $(3, 3)^s$.

Proof. Let R be a quasi-region of type $(3, 3)^s$, and let $N_t = (a, b, c, d)$, $N_b = (a, f, e, d)$. If there is an (c, e)-edge present, it can be decomposed according to Figure 11.13, if there is an (b, d)-edge present, it can be decomposed according to Figure 11.14, if there is an (b, e)-edge present, it can be decomposed according to Figure 11.15, and if there is an (b, f)-edge present, it can be decomposed according to Figure 11.16.

If there are no (c, e), (b, d), (b, e) and (b, f) edges present, then by Lemma 9.2 we can find a noose from a to d. Since a is adjacent to every internal vertex of R, if there is such a noose then there's also a noose of length ≤ 2. Consider on such noose of length ≤ 2. If it has length 1, then R can be decomposed according to Figure 11.17. If it has length 2, then R can be decomposed according to Figure 11.18.

This means that any quasi-region of type $(3, 3)^s$ can be decomposed by a figure in $\mathcal{X}$, and by Lemma 10.11 *enum*$(\mathcal{X})$ is a fully enumerated representative set for $(3, 3)^s$. $\square$

11.4 $(2,1)^n$-regions

By Lemma 9.13 these regions can have at most 2 internal vertices. Let $N^{2,1}_{(1,2)(*)}$ be the reduced set of $(2,1)^n_{(1,2)(*)}$-regions that have at most 2 internal vertices. The set is a fully enumerated representative set by definition.

11.5 $(2,2)^{s,n}$-regions

By Lemma 9.14 these regions can have at most 4 internal vertices. Let $N^{2,2}_{(1,2)(*,2)}$ be the reduced set of $(2,2)^n_{(1,2)(*,2)}$-regions that have at most 4 internal vertices. The set is a fully enumerated representative set by definition.

11.6 $(2,1)$-regions

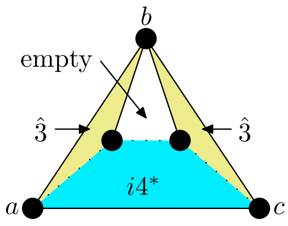

Figure 11.19

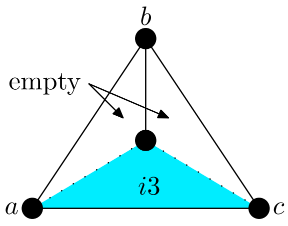

Figure 11.20

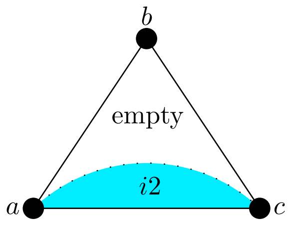

Figure 11.21

Lemma 11.4. *The reduced set $R^{2,1}$ enumerated by the figures $\mathcal{X}$ ={11.21, 11.20, 11.19} is a fully enumerated representative set for $(2,1)$.*

Proof. Let R be a quasi-region of type $(2,1)$. Notice that there will always be two $a-c$ nooses $\tilde{N}_t, \tilde{N}_b$, defining an inner region. Consider an maximal inner region:

- In the case of $|\tilde{N}_t| = 3$, let $\tilde{N}_t = (a, \tilde{b}, \tilde{c}, c)$, $\tilde{N}_b = (a, c)$. By Lemma 9.2 edges $(b, \tilde{b})$ and $(b, \tilde{c})$ must be present, and R can be decomposed according to Figure 11.19.
- In the case of $|\tilde{N}_t| = 2$, let $\tilde{N}_t = (a, \tilde{b}, c)$, $\tilde{N}_b = (a, c)$. By Lemma 9.2 edge $(b, \tilde{b})$ must be present, and by Lemma 9.8 there cannot be vertices in the face made up by N_t and $\tilde{N}_t$. In that case R can be decomposed according to Figure 11.20.
- In the case of $|\tilde{N}_t| = 1$, let $\tilde{N}_t = (a, c)$, $\tilde{N}_b = (a, c)$. By Lemma 9.8 there cannot be vertices in the face made up by N_t and $\tilde{N}_t$. In that case R can be decomposed according to Figure 11.21.

□

11.7 $(2,2)$-regions

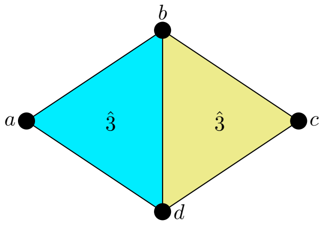

Figure 11.22

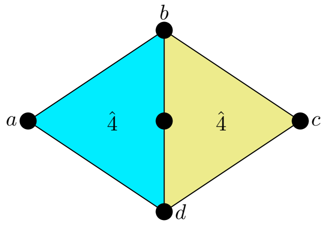

Figure 11.23

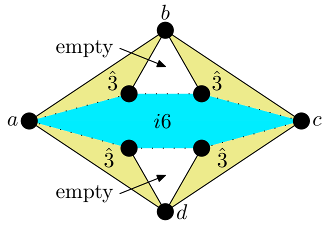

Figure 11.24

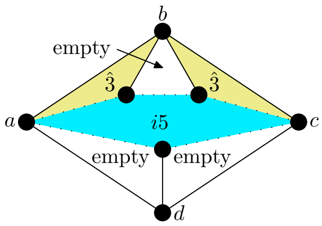

Figure 11.25

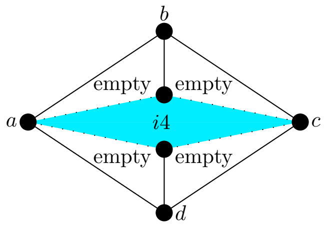

Figure 11.26

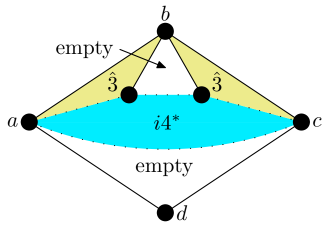

Figure 11.27

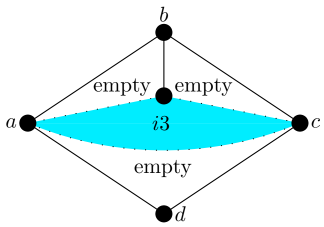

Figure 11.28

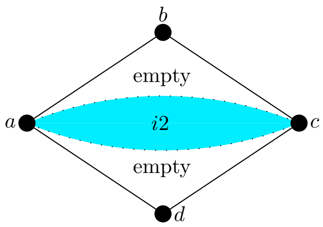

Figure 11.29

Lemma 11.5. *The reduced set $R^{2,2}$ enumerated by the figures $\mathcal{X}$ $=\{$11.22, 11.23, 11.24, 11.25, 11.26, 11.27, 11.28, 11.29$\}$ is a fully enumerated representative set for* $(2,2)$.

Proof. Let R be a quasi-region of type $(2,2)$, and $N_t = (a,b,c)$ and $N_b = (a,d,c)$. If there is an (b,d)-edge present it can be decomposed according to Figure 11.22. If there is an (b,d)-path of length 2 present R can be decomposed according to Figure 11.23.

If there is no (b,d)-edge or (b,d)-path present we can find two disjoint nooses $\tilde{N}_t$ and $\tilde{N}_b$ from a to c by Lemma 9.2, defining an inner region. Consider a maximal such inner region:

- If $|\tilde{N}_t| = 3$ let $\tilde{N}_t = (a,\tilde{b},\tilde{c},c)$. By Lemma 9.7 the edges $(b,\tilde{b})$ and $(b,\tilde{c})$ must be present.
  - If $|\tilde{N}_b| = 3$ let $\tilde{N}_b = (a,\tilde{f},\tilde{e},c)$. By Lemma 9.7 the edges $(d,\tilde{f})$ and $(d,\tilde{e})$ must be present, and R can be decomposed according to Figure 11.24.
  - If $|\tilde{N}_b| = 2$ let $\tilde{N}_b = (a,\tilde{e},c)$. By Lemma 9.7 the edge $(d,\tilde{e})$ must be present, and the face made up by $\tilde{N}_b$ and N_b must be empty by Lemma 9.8 and R can be decomposed according to Figure 11.25.
  - If If $|\tilde{N}_b| = 1$ let $\tilde{N}_b = (a,c)$. By Lemma 9.8 the face made up by $\tilde{N}_b$ and N_b can contain no vertices. R can be decomposed according to Figure 11.27.

- If $|\tilde{N}_t| = 2$ let $\tilde{N}_t = (a, \tilde{b}, c)$. By Lemma 9.7 the edge $(b, \tilde{b})$ must be present, and the face made up by $\tilde{N}_b$ and N_b must be empty by Lemma 9.8.
  - If $|\tilde{N}_b| = 2$ let $\tilde{N}_b = (a, \tilde{e}, c)$. By Lemma 9.7 the edge $(d, \tilde{e})$ must be present, and the face made up by $\tilde{N}_t$ and N_t can contain no vertices by Lemma 9.8. Then and R can be decomposed according to Figure 11.26.
  - If If $|\tilde{N}_b| = 1$ let $\tilde{N}_b = (a, c)$. By Lemma 9.8 the face made up by $\tilde{N}_t$ and N_t can contain no vertices. R can be decomposed according to Figure 11.28.
- If $|\tilde{N}_t| = 1$ let $\tilde{N}_t = (a, c)$. The face made up by $\tilde{N}_b$ and N_b must be empty by Lemma 9.8.
  - If If $|\tilde{N}_b| = 1$ let $\tilde{N}_b = (a, c)$. By Lemma 9.8 the face made up by $\tilde{N}_t$ and N_t can contain no vertices. R can be decomposed according to Figure 11.29.

This means that any quasi-region of type $(2, 2)$ can be decomposed by a figure in $\mathcal{X}$, and by Lemma 10.11 *enum*$(\mathcal{X})$ is a fully enumerated representative set for $(2, 2)$.

□

11.8 $(3, 1)$-regions

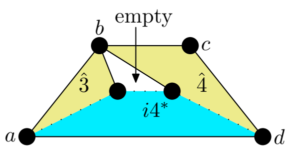

Figure 11.30

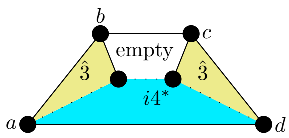

Figure 11.31

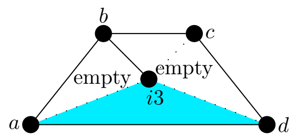

Figure 11.32

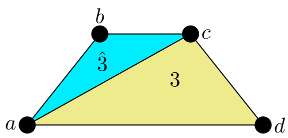

Figure 11.33

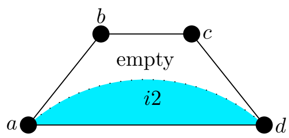

Figure 11.34

Lemma 11.6. *The reduced set $R^{3,1}$ enumerated by the figures $\mathcal{X}$ ={11.30, 11.31, 11.32, 11.33, 11.34} is a fully enumerated representative set for* $(3, 1)$.

Proof. Let R be a quasi-region of type $(3, 1)$, and $N_t = (a, b, c, d)$ and $N_b = (a, d)$. If there is a (a, c)-edge present, then R can be decomposed according to Figure 11.33.

If there is no (a, c)-edge, notice that there will always be two $a - d$ nooses $\tilde{N}_t, \tilde{N}_b$, defining an inner region. Consider an maximal inner region:

- If $|\tilde{N}_t| = 3$, let $\tilde{N}_t = (a, \tilde{b}, \tilde{c}, d)$. By Lemma 9.7 the edge $(b, \tilde{b})$ must be present, and at least one of $(b, \tilde{c})$ and $(c, \tilde{c})$. If $(b, \tilde{c})$ is present, then R ca be decomposed according to Figure 11.30. If not, then $(c, \tilde{c})$ must be present and R ca be decomposed according to Figure 11.31.
- If $|\tilde{N}_t| = 2$, let $\tilde{N}_t = (a, \tilde{b}, d)$. By Lemma 9.7 the edge $(b, \tilde{b})$ must be present, and by Lemma 9.8 the face made up by N_t and $\tilde{N}_t$ can contain no vertices. There might be an $(c, \tilde{b})$-edge, and R ca be decomposed according to Figure 11.32.
- If $|\tilde{N}_t| = 1$, let $\tilde{N}_t = (a, d)$. By Lemma 9.8 the face made up by N_t and $\tilde{N}_t$ can contain no vertices, and R ca be decomposed according to Figure 11.34.

This means that any quasi-region of type $(3, 1)$ can be decomposed by a figure in $\mathcal{X}$, and by Lemma 10.11 $enum(\mathcal{X})$ is a fully enumerated representative set for $(3, 1)$.

$\square$

11.9 $(3, 2)$-regions

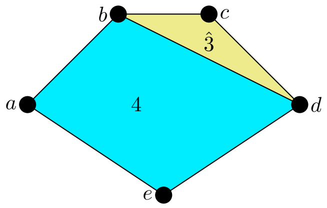

Figure 11.35

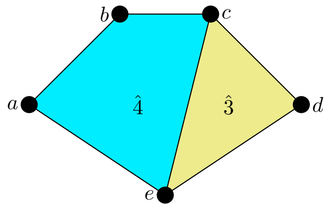

Figure 11.36

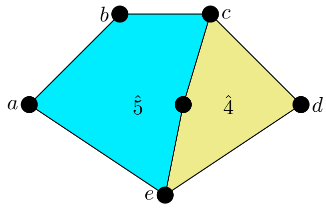

Figure 11.37

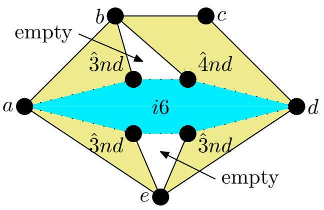

Figure 11.38

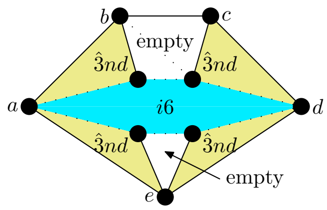

Figure 11.39

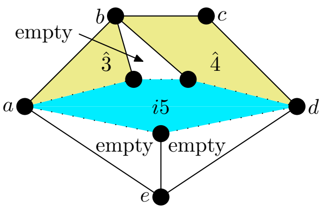

Figure 11.40

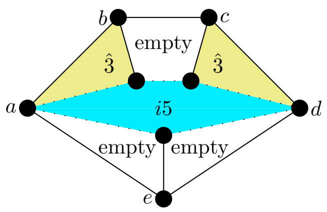

Figure 11.41

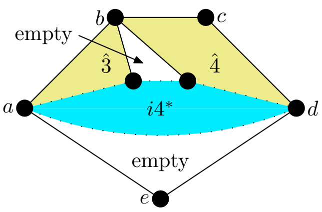

Figure 11.42

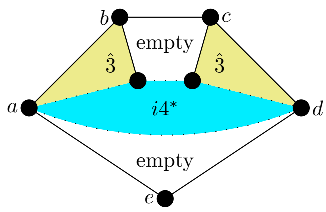

Figure 11.43

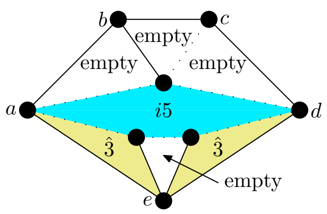

Figure 11.44

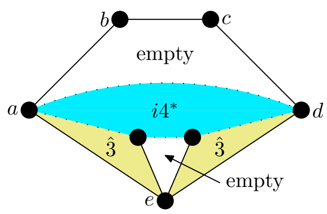

Figure 11.45

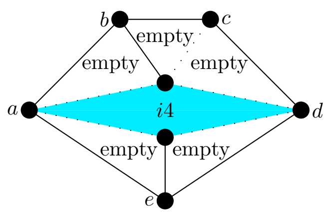

Figure 11.46

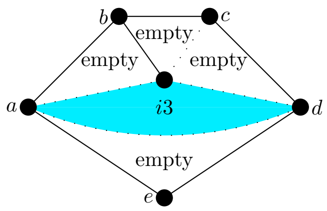

Figure 11.47

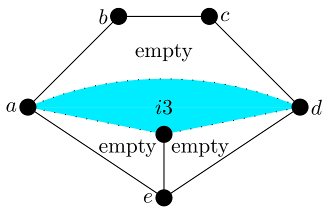

Figure 11.48

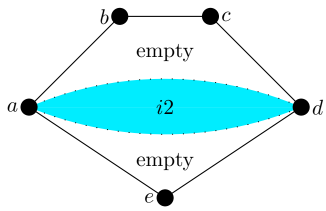

Figure 11.49

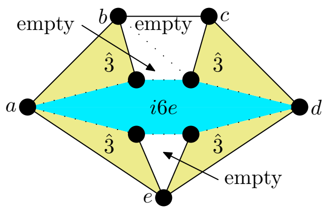

Figure 11.50

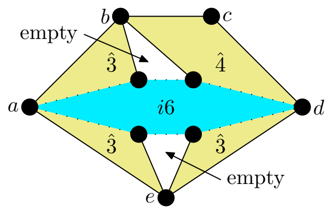

Figure 11.51

Lemma 11.7. *The reduced set $R^{3,2}$ enumerated by the figures $\mathcal{X}$ ={11.35,..., 11.51} is a fully enumerated representative set for* $(3, 2)$*.*

Proof. Let R' be a detached quasi-region of type $(3, 2)$, and let R be signature minimal such that $R \leq R'$. Note that if the inner quasi-regions are non-empty, then the outer single regions in such a signature minimal region is of the non-dominator type by the arguments in Section 9.3. Let $N_t = (a, b, c, d)$ and $N_b = (a, e, d)$ be the nooses of R. If there is an (b, d)-edge present (or by symmetry an (a, c)-edge) it can be decomposed according to Figure 11.35. If there is an (c, e)-edge present (or by symmetry an (b, e)-edge) it can be decomposed according to Figure 11.36. If there is an (c, e)-path of length 2 present (or by symmetry an (c, e)-path) R can be decomposed according to Figure 11.37.

If there is no (b, d)-edge, (c, e)-edge or (c, e)-path (or symmetries) present we can find two disjoint nooses $\tilde{N}_t$ and $\tilde{N}_b$ from a to d by Lemma 9.2, defining an inner region. Consider a maximal such inner quasi-region:

- If $|\tilde{N}_t| = 3$ and $|\tilde{N}_b| = 3$, let $\tilde{N}_t = (a, \tilde{b}, \tilde{c}, d)$ and $\tilde{N}_b = (a, \tilde{f}, \tilde{e}, c)$. By Lemma 9.7 the edges $(b, \tilde{b})$ (or by symmetry $(c, \tilde{c})$) , $(e, \tilde{f})$ and $(e, \tilde{e})$ must be present, and at least one of $(b, \tilde{c})$ and $(c, \tilde{c})$ must be present.
  - If $(c, \tilde{c})$ is present then there are two options:
    - ∗ If the inner region contains no vertices, the $(b, \tilde{c})$-edge might be there, and R can be decomposed according to Figure 11.50.
    - ∗ If the inner region contains vertices, the $(b, \tilde{c})$-edge might be there, then R can be decomposed according to Figure 11.39.
  - If $(c, \tilde{c})$ is not present then there are two options:
    - ∗ If the inner region contains no vertices, the $(b, \tilde{c})$-edge must be there, and R can be decomposed according to Figure 11.51.
    - ∗ If the inner region contains vertices, the $(b, \tilde{c})$-edge must be there, then R can be decomposed according to Figure 11.38.
- If $|\tilde{N}_t| = 3$ and $|\tilde{N}_b| = 2$, let $\tilde{N}_t = (a, \tilde{b}, \tilde{c}, d)$ and $\tilde{N}_b = (a, \tilde{e}, c)$. By Lemma 9.7 the edges $(b, \tilde{b})$ and $(e, \tilde{e})$ must be present, and at least one of $(b, \tilde{c})$ and $(c, \tilde{c})$ must be present. The face made up by N_b and $\tilde{N}_b$ cannot contain vertices by Lemma 9.8.

- If $(b, \tilde{c})$ is present then R can be decomposed according to Figure 11.40.
- If $(b, \tilde{c})$ is not present then $(c, \tilde{c})$ must be present, and R can be decomposed according to Figure 11.41.

- If $|\tilde{N}_t| = 3$ and $|\tilde{N}_b| = 1$, let $\tilde{N}_t = (a, \tilde{b}, \tilde{c}, d)$ and $\tilde{N}_b = (a, c)$. By Lemma 9.7 the edge $(b, \tilde{b})$ must be present, and at least one of $(b, \tilde{c})$ and $(c, \tilde{c})$ must be present. The face made up by N_b and $\tilde{N}_b$ cannot contain vertices by Lemma 9.8.
  - If $(b, \tilde{c})$ is present then R can be decomposed according to Figure 11.42.
  - If $(b, \tilde{c})$ is not present then $(c, \tilde{c})$ must be present, and R can be decomposed according to Figure 11.43.
- If $|\tilde{N}_t| = 2$ and $|\tilde{N}_b| = 3$, let $\tilde{N}_t = (a, \tilde{b}, d)$ and $\tilde{N}_b = (a, \tilde{e}, \tilde{d}, d)$. By Lemma 9.7 the edges $(b, \tilde{b})$, $(e, \tilde{e})$ and $(e, \tilde{d})$ must be present, and $(c, \tilde{b})$ might be present. R can be decomposed according to Figure 11.44.
- If $|\tilde{N}_t| = 2$ and $|\tilde{N}_b| = 2$, let $\tilde{N}_t = (a, \tilde{b}, d)$ and $\tilde{N}_b = (a, \tilde{d}, d)$. By Lemma 9.7 the edges $(b, \tilde{b})$ and $(e, \tilde{d})$ must be present, and $(c, \tilde{b})$ might be present. R can be decomposed according to Figure 11.46.
- If $|\tilde{N}_t| = 2$ and $|\tilde{N}_b| = 1$, let $\tilde{N}_t = (a, \tilde{b}, d)$ and $\tilde{N}_b = (a, d)$. By Lemma 9.7 the edge $(b, \tilde{b})$ must be present, and $(c, \tilde{b})$ might be present. R can be decomposed according to Figure 11.47.
- If $|\tilde{N}_t| = 1$ and $|\tilde{N}_b| = 3$, let $\tilde{N}_t = (a, d)$ and $\tilde{N}_b = (a, \tilde{d}, \tilde{c}, d)$. By Lemma 9.7 the edges $(e, \tilde{d})$ and $(e, \tilde{c})$ must be present, and R can be decomposed according to Figure 11.45.
- If $|\tilde{N}_t| = 1$ and $|\tilde{N}_b| = 2$, let $\tilde{N}_t = (a, d)$ and $\tilde{N}_b = (a, \tilde{c}, d)$. By Lemma 9.7 the edge $(e, \tilde{c})$ must be present, and R can be decomposed according to Figure 11.48.
- If $|\tilde{N}_t| = 1$ and $|\tilde{N}_b| = 1$, let $\tilde{N}_t = (a, d)$ and $\tilde{N}_b = (a, d)$. By Lemma 9.8 the face made up by N_t and $\tilde{N}_t$ and the face made up by N_t and $\tilde{N}_t$ can neither contain vertices, and R can be decomposed according to Figure 11.49.

Since we could find a region $R \leq R'$ that was decomposable by a figure in $\mathcal{X}$, by Lemma 10.11 $R^{3,2}$ is fully enumerated representative set for $(3, 2)$.

□

11.10 $(3, 3)$-regions

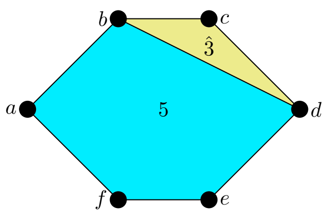

Figure 11.52

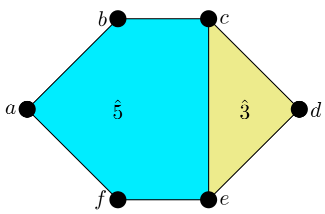

Figure 11.53

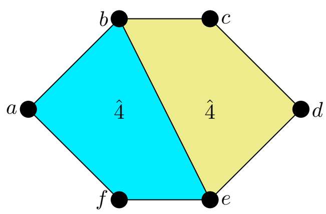

Figure 11.54

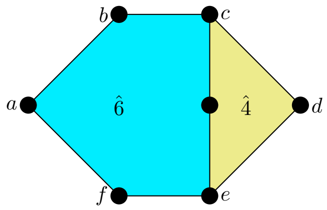

Figure 11.55

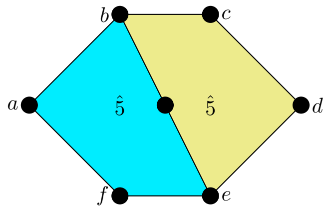

Figure 11.56

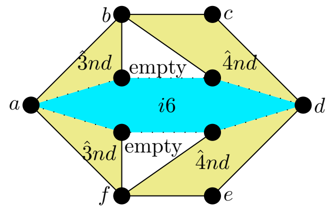

Figure 11.57

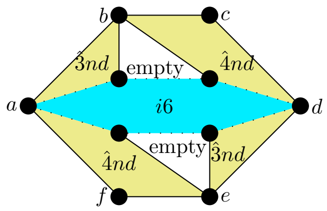

Figure 11.58

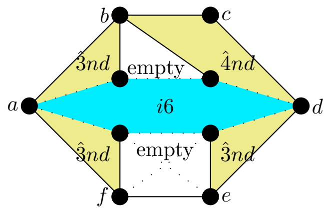

Figure 11.59

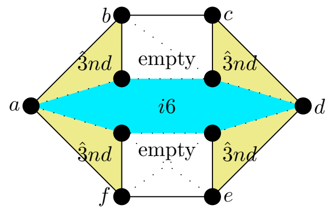

Figure 11.60

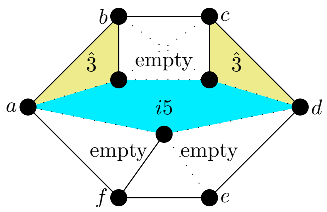

Figure 11.61

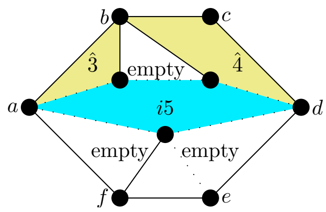

Figure 11.62

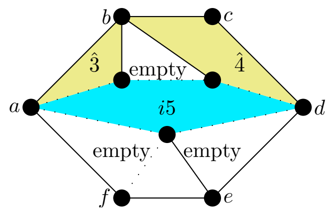

Figure 11.63

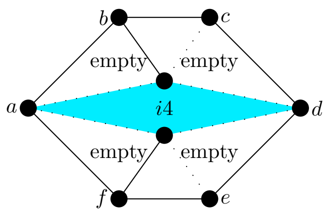

Figure 11.64

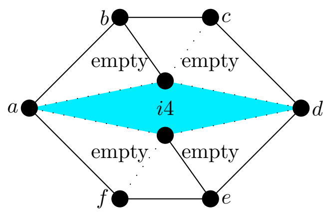

Figure 11.65

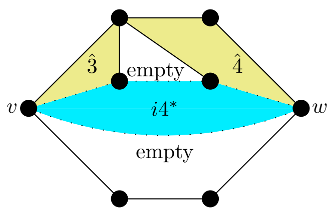

Figure 11.66

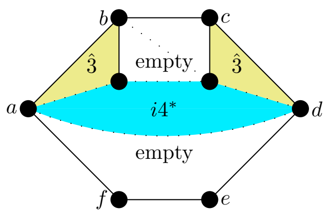

Figure 11.67

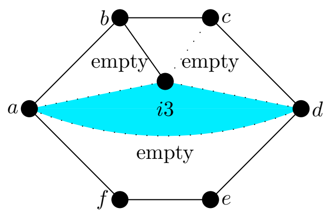

Figure 11.68

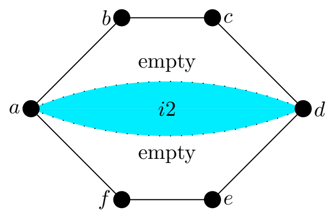

Figure 11.69

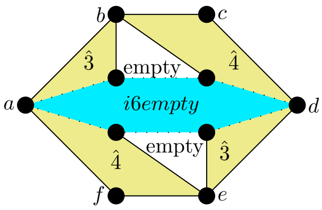

Figure 11.70

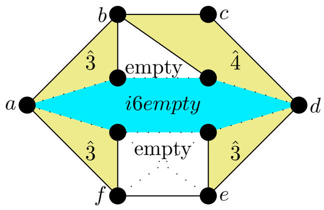

Figure 11.71

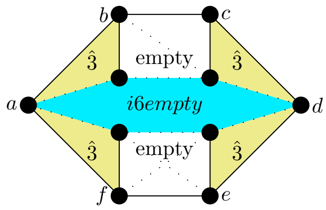

Figure 11.72

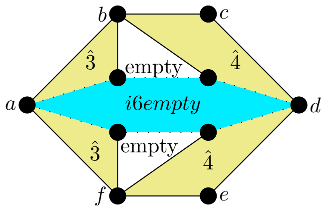

Figure 11.73

Lemma 11.8. *The reduced set $R^{3,3}$ enumerated by the figures $\mathcal{X}$ =[FIGURES] is a fully enumerated representative set for* $(3,2)$.

Proof. Let R' be a detached quasi-region of type $(3,3)$, and let R be signature minimal such that $R \leq R'$. Note that if the inner quasi-regions are non-empty, then the outer single regions in such a signature minimal region is of the non-dominator type by the arguments in Section 9.3. Let $N_t = (a,b,c,d)$ and $N_b = (a,f,e,d)$ be the nooses of R. If there is a (b,d)-edge present (or by symmetry a (a,c)-, (d,f)- or (a,e)-edge), then R can be decomposed according to 11.52. If there is a (c,e)-edge present (or a (b,f)-edge by symmetry) then R can be decomposed according to 11.53. If there is a (b,e)-edge present (or by symmetry a (c,f)-edge), then R can be decomposed according to 11.54.

If there is a path from c to e of length 2 (or from b to f by symmetry), then R can be decomposed according to 11.55. If there is a path from b to e of length 2 (or from c to f by symmetry), then R can be decomposed according to 11.56.

If none of the above are present we can find two disjoint nooses $\tilde{N}_t$ and $\tilde{N}_b$ from a to d by Lemma 9.2, defining an inner region. Consider a maximal such inner quasi-region. We will look at the area between N_t and $\tilde{N}_t$ first. We can consider all symmetries here.

- $|\tilde{N}_t| = 3$, let $\tilde{N}_t = (a,\tilde{b},\tilde{c},d)$. By Lemma 9.7 the edge $(b,\tilde{b})$ (or by symmetry $(c,\tilde{c})$) must be present, and at least one of $(b,\tilde{c})$ and $(c,\tilde{c})$ must be present.
  - If $(c,\tilde{c})$ is present, the edge $(b,\tilde{c})$ might be present. Either the inner quasi-region is empty, denote this case as CASE3-TOP-33-EMPTY, or it is non-empty, denote this case as CASE3-TOP-33-NONEMPTY.
  - If $(c,\tilde{c})$ is not present, the edge $(b,\tilde{c})$ must be present. Either the inner quasi-region is empty, denote this case as CASE3-TOP-34-EMPTY, or it is non-empty, denote this case as CASE3-TOP-34-NONEMPTY.
- $|\tilde{N}_t| = 2$, let $\tilde{N}_t = (a,\tilde{b},d)$. The edge $(b,\tilde{b})$ (or $(c,\tilde{b})$ by symmetry) must be present, and $(c,\tilde{b})$ might be. By Lemma 9.8 there are no edges in $area(N_t,\tilde{N}_t)$. Denote this CASE2-TOP.
- $|\tilde{N}_t| = 1$, let $\tilde{N}_t = (a,d)$. By Lemma 9.8 there are no edges in $area(N_t,\tilde{N}_t)$. Denote this case as CASE1-TOP.

Now we do the same for the area between N_b and $\tilde{N}_b$. This time we cannot rely on symmetry arguments, since we did that for the top part.

- $|\tilde{N}_b| = 3$, let $\tilde{N}_b = (a, \tilde{f}, \tilde{e}, d)$. By Lemma 9.7 at least one of the edges $(f, \tilde{f})$ and $e, \tilde{f})$ must be present.
  - If $(f, \tilde{f})$ is present, at least one of $(f, \tilde{e})$ and $(e, \tilde{e})$ must be present.
    - If $(e, \tilde{e})$ is present, one of the edges $(f, \tilde{e})$ and $(e, \tilde{f})$ might be present. Either the inner quasi-region is empty, denote this case as CASE3-BOT-33-EMPTY, or it is non-empty, denote this case as CASE3-BOT-33-NONEMPTY.
    - If $(e, \tilde{e})$ is not present, then $(f, \tilde{e})$ must be present. Either the inner quasi-region is empty, denote this case as CASE3-BOT-34-EMPTY, or it is non-empty, denote this case as CASE3-BOT-34-NONEMPTY.
  - If $(f, \tilde{f})$ is not present, then $(f, \tilde{e})$ and $(e, \tilde{e})$ must be present. Either the inner quasi-region is empty, denote this case as CASE3-BOT-43-EMPTY, or it is non-empty, denote this case as CASE3-BOT-43-NONEMPTY.
- $|\tilde{N}_b| = 2$, let $\tilde{N}_b = (a, \tilde{e}, d)$. At least one of the edges $(f, \tilde{e})$ and $e, \tilde{e})$ must be present. And the area between N_b and $\tilde{N}_b$ cannot contain any vertices.
  - If $(f, \tilde{e})$ is present, the edge $(e, \tilde{e})$ might be present. Denote this case as CASE2-BOT-fe.
  - If $(e, \tilde{e})$ is present, the edge $(f, \tilde{e})$ might be present. Denote this case as CASE2-BOT-ee.
- $|\tilde{N}_b| = 1$, let $\tilde{N}_b = (a, d)$. The area between N_b and $\tilde{N}_b$ cannot contain any vertices. CASE1-BOT.

Now we can combine the compatible top and bottom cases into all possible ways of decomposing R. Top cases with empty inner quasi-region are {CASE3-TOP-33-EMPTY, CASE3-TOP-34-EMPTY}, and for bottom cases we have {CASE3-BOT-33-EMPTY, CASE3-BOT-34-EMPTY, CASE3-BOT-43-EMPTY}. The combinations and figure that decompose R in that case are:

- CASE3-TOP-33-EMPTY + CASE3-BOT-33-EMPTY = Figure 11.72
- CASE3-TOP-33-EMPTY + CASE3-BOT-34-EMPTY = Figure 11.71
- CASE3-TOP-33-EMPTY + CASE3-BOT-43-EMPTY = Figure 11.71
- CASE3-TOP-34-EMPTY + CASE3-BOT-33-EMPTY = Figure 11.71
- CASE3-TOP-34-EMPTY + CASE3-BOT-34-EMPTY = Figure 11.73
- CASE3-TOP-34-EMPTY + CASE3-BOT-43-EMPTY = Figure 11.70

Cases where we use a general inner quasi-region (since it turned out to be fast enough for our program) are for top cases {CASE3-TOP-33-NONEMPTY, CASE3-TOP-34-NONEMPTY, CASE2-TOP-33, CASE1-TOP}. Bottom cases are {CASE3-BOT-33-NONEMPTY, CASE3-BOT-43-NONEMPTY, CASE3-BOT-34-NONEMPTY, CASE2-BOT-fe, CASE2-BOT-ee, CASE1-BOT}. This combines into

- CASE3-TOP-33-NONEMPTY + CASE3-BOT-33-NONEMPTY = Figure 11.60
- CASE3-TOP-33-NONEMPTY + CASE3-BOT-43-NONEMPTY = Figure 11.59
- CASE3-TOP-33-NONEMPTY + CASE3-BOT-34-NONEMPTY = Figure 11.59
- CASE3-TOP-33-NONEMPTY + CASE2-BOT-fe = Figure 11.61
- CASE3-TOP-33-NONEMPTY + CASE2-BOT-ee = Figure 11.61
- CASE3-TOP-33-NONEMPTY + CASE1-BOT = Figure 11.67
- CASE3-TOP-34-NONEMPTY + CASE3-BOT-33-NONEMPTY = Figure 11.59
- CASE3-TOP-34-NONEMPTY + CASE3-BOT-43-NONEMPTY = Figure 11.58

- CASE3-TOP-34-NONEMPTY + CASE3-BOT-34-NONEMPTY = Figure 11.57
- CASE3-TOP-34-NONEMPTY + CASE2-BOT-fe = Figure 11.62
- CASE3-TOP-34-NONEMPTY + CASE2-BOT-ee = Figure 11.63
- CASE3-TOP-34-NONEMPTY + CASE1-BOT = Figure 11.66
- CASE2-TOP + CASE2-BOT-fe = Figure 11.64
- CASE2-TOP + CASE2-BOT-ee = Figure 11.65
- CASE2-TOP + CASE1-BOT = Figure 11.68
- CASE1-TOP + CASE1-BOT = Figure 11.69

The rest of the combinations are symmetric to the above. Since we could find a region $R \leq R'$ that was decomposable by a figure in $\mathcal{X}$, by Lemma 10.11 $R^{3,3}$ is fully enumerated representative set for $(3,3)$.

$\square$

Chapter 12

Implementation and Results

The program performing the enumeration as described was implemented in C++ and is available on GitHub [14]. The program starts by enumerating the small regions and gradually build bigger regions. It will also save one signature minimal detached region for each signature found. Note that it is a parallel program, and should be run on a computer supporting many simultaneous threads to speed it up. In our case it was run on a 72 core computer, using about 4 days to find a signature minimal representative for the signatures of all detached region types.

An alternative implementation was also done by my supervisor Daniel Lokshtanov, using a different approach to finding signature minimal detached regions. This implementation gave the same results, that is the same set of signatures, and the same size of the signature minimal detached region corresponding to a given signature. Note that the two implementations don't necessarily find the exact same representative for a given signature, as there might be more than one signature minimal representative corresponding to the signature. The implementation done in this thesis keeps the first one found. Since it is parallel this even means that this particular implementation may find different signature minimal representatives each time it's run!

After running the signature generator and inspecting the results, the biggest signature minimal detached regions have size 14 vertices (including the two endpoints). There are 7 of them (not counting symmetric regions), and they are shown in Figure 12.1. In the case of single regions, the biggest ones of boundary size 4 are 7 (including the two endpoints). This is the proof for Lemma 7.22. For Lemma 7.21, note that the program takes no input, and by definition runs in constant, $O(1)$ time. In practical cases, note that one would save the results of the program to a file, and load it each time it is invoked, instead of actually running the whole enumeration each time.

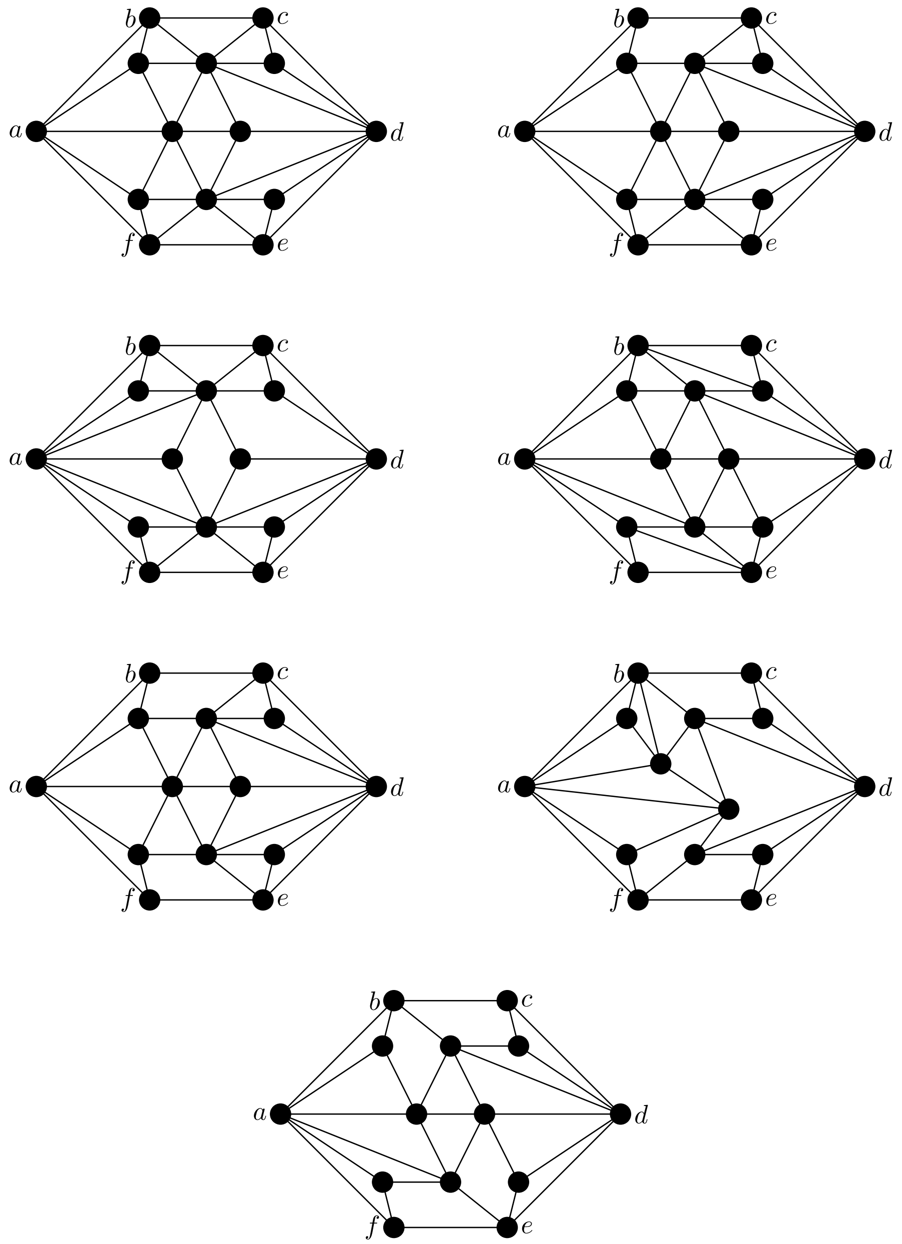

Figure 12.1: The largest signature minimal regions (symmetries not shown).

Chapter 13

Conclusions

In this thesis we explored the technique of Region Decomposition for finding kernels for Planar Dominating Set. We first redefined some concepts used in earlier work, fixing some ambiguities on the way. From those concepts we ended up at the $335k$ kernel upper bound from Alber et al[2]. We then built on the results of Chen et al.[6] to improve this upper bound to $55k$.

In the last part of the thesis we made use of a computer program to exhaustively search for reduced instances of regions, to be used together with the Region Decomposition technique. From the results from the computer program we were able to conclude that any region used in a Region Decomposition always can be reduced to an equivalent region having 12 vertices or less, in addition to its to endpoints. This let us arrive at a $43k$ kernel for Planar Dominating Set, but take into account the Words of Warning at the beginning of Part III.

13.1 Open Problems

How much smaller can the Planar Dominating Set kernel get? It has been shown a $2k$ lower bound on how small problem kernels one can hope to possibly find for Planar Dominating Set[6]. It would be interesting to see how close one could get the lower and upper bounds, and maybe even find tight bounds.

We showed that the worst-case size for fully reduced regions is 12, which means they cannot be made smaller than this without analyzing them together with the structure of the graph outside the region. A possible approach could be to instead of looking at pairs of dominating vertices making up a region, look at triples of dominators and make a decomposition where more vertices than those shared by a pair of dominators can be analyzed together.

The computer-aided reduction technique should be applicable to other problems on planar graphs as well. This approach should be possible to adapt as long as the problem instance can be decomposed into regions, and a region signature that encodes how the region interact with the rest of the graph can be defined.

Bibliography

[1] Jochen Alber, Britta Dorn, and Rolf Niedermeier. A general data reduction scheme for domination in graphs. In *SOFSEM 2006: Theory and Practice of Computer Science, 32nd Conference on Current Trends in Theory and Practice of Computer Science, Merín, Czech Republic, January 21-27, 2006, Proceedings*, pages 137–147, 2006.

[2] Jochen Alber, Michael R Fellows, and Rolf Niedermeier. Polynomial-time data reduction for dominating set. *Journal of the ACM (JACM)*, 51(3):363–384, 2004.

[3] Ian Anderson. *A First Course in Discrete Mathematics*. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2001.

[4] Hans L. Bodlaender, Fedor V. Fomin, Daniel Lokshtanov, Eelko Penninkx, Saket Saurabh, and Dimitrios M. Thilikos. (meta) kernelization. *CoRR*, abs/0904.0727, 2009.

[5] John M. Boyer and Wendy J. Myrvold. On the cutting edge: Simplified O(n) planarity by edge addition. *Journal of Graph Algorithms and Applications*, 8(3):241–273, 2004.

[6] Jianer Chen, Henning Fernau, Iyad A. Kanj, and Ge Xia. Parametric duality and kernelization: Lower bounds and upper bounds on kernel size. *SIAM Journal on Computing*, 37(4):1077–1106, 2007.

[7] Stephen A Cook. The complexity of theorem-proving procedures. In *Proceedings of the third annual ACM symposium on Theory of computing*, pages 151–158. ACM, 1971.

[8] Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Dániel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2016.

[9] Sanjoy Dasgupta, Christos H. Papadimitriou, and Umesh Vazirani. *Algorithms*. McGraw-Hill, Inc., New York, NY, USA, 1 edition, 2008.

[10] R. Diestel. *Graph Theory*. Electronic library of mathematics. Springer, 2006.

[11] Fedor V. Fomin and Dimitrios M. Thilikos. Fast parameterized algorithms for graphs on surfaces: Linear kernel and exponential speed-up. In *Automata, Languages and Programming: 31st International Colloquium, ICALP 2004, Turku, Finland, July 12-16, 2004. Proceedings*, pages 581–592, 2004.

[12] Jiong Guo and Rolf Niedermeier. Linear problem kernels for np-hard problems on planar graphs. In *Automata, Languages and Programming, 34th International Colloquium, ICALP 2007, Wroclaw, Poland, July 9-13, 2007, Proceedings*, pages 375–386, 2007.

[13] Jiong Guo, Rolf Niedermeier, and Sebastian Wernicke. Fixed-parameter tractability results for full-degree spanning tree and its dual. *Networks*, 56(2):116–130, 2010.

[14] Johan T. Halseth. Signature minimal region generator, github. `https://github.com/halseth/regions`, Feb 2016.

[15] Stephen T Hedetniemi and Renu C Laskar. Bibliography on domination in graphs and some basic definitions of domination parameters. *Annals of Discrete Mathematics*, 48:257–277, 1991.

[16] John Hopcroft and Robert Tarjan. Efficient planarity testing. *J. ACM*, 21(4):549–568, October 1974.

[17] Bart Jansen. Private communications, 2015.

[18] Richard M. Karp. *Reducibility Among Combinatorial Problems; Complexity of Computer Computations*. New York: Plenum, 1972.

[19] David C. Lay. *Linear Algebra and Its Applications*. Addison-Wesley, 4th edition, 2012.

[20] Leonid A Levin. Universal sequential search problems. *Problemy Peredachi Informatsii*, 9(3):115–116, 1973.

[21] Daniel Lokshtanov, Matthias Mnich, and Saket Saurabh. A linear kernel for a planar connected dominating set. *Theor. Comput. Sci.*, 412(23):2536–2543, 2011.

[22] B. Mohar and C. Thomassen. *Graphs on Surfaces*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, 2001.

[23] Hannes Moser and Somnath Sikdar. The parameterized complexity of the induced matching problem. *Discrete Applied Mathematics*, 157(4):715–727, 2009.

[24] Neil Robertson and Paul D. Seymour. Graph minors .xiii. the disjoint paths problem. *J. Comb. Theory, Ser. B*, 63(1):65–110, 1995.

[25] Kenneth H. Rosen. *Discrete Mathematics and Its Applications*. McGraw-Hill Higher Education, 6th edition, 2007.

[26] A Sasireka and AH Nandhu Kishore. Applications of dominating set of graph in computer networks.

[27] Michael Sipser. *Introduction to the Theory of Computation*. Cengage Learning, 3rd edition, 2013.

[28] György Turán. On the succinct representation of graphs. *Discrete Applied Mathematics*, 8(3):289 – 294, 1984.