A minimum requiring angle trisection

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For Audun Holme,
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1 Paul Nahin's envelope-folding problem

The solution to a problem can be *too* perfect, winning our admiration, but not our engagement. In contrast, a solution that is still rough and ready round the edges retains the power to draw us in, making us want to try our own hand. This was our response to reading Paul Nahin's account of what he calls the "envelope-folding problem" and to which he devotes a six-page section in his recent book *When Least is Best* [10, §3.3]. Perhaps attempting to share our enjoyment with readers of *Normat* is ill-advised, merely depriving them of similar pleasurable diversion. But the stimulus we should most like to transmit, in the hope that readers can resolve matters further, is that of discovering more than we can explain.

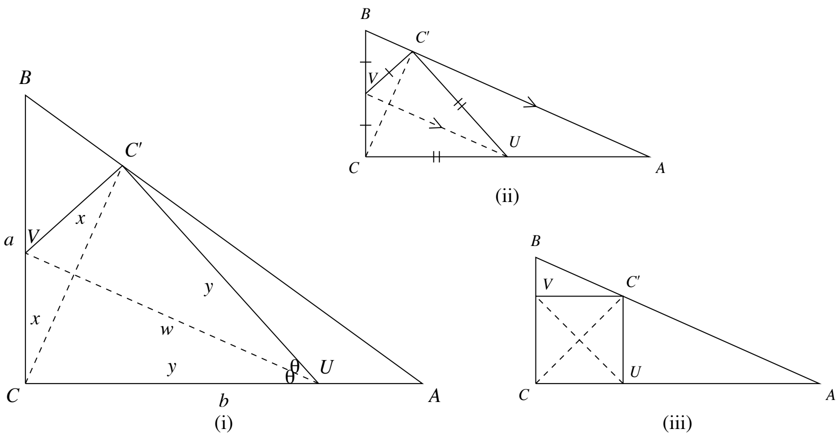

Figure 1: Folding a right triangle

Nahin's problem seems simple enough. We have a right triangle $\triangle ABC$ with right angle at C, with legs a and b and hypotenuse c, where $a^2+b^2=c^2$ and without loss of generality we may suppose that $a \leq b$, as indicated in Figure 1(i). The vertex C is folded onto the hypotenuse AB at C' to create a crease UV. The envelope-folding problem asks: what is the least possible area of the folded right triangle $\triangle UVC$?

For instance, we can always fold parallel to the hypotenuse, as in Figure 1(ii). In this case, $\triangle UVC$ has an area one quarter that of $\triangle ABC$, that is, $ab/8$. Again, we can also fold along a diagonal on the inscribed square, as in Figure 1(iii). Since the inscribed square of $\triangle ABC$ is well-known to have side $s = ab/(a+b)$, this gives a folded triangle $\triangle UVC$ with area $s^2/2$. The inequality between the arithmetic and geometric means implies that this second area is always at least as large as the first, with equality only in the case of the isosceles right triangle. Symmetry considerations suggest that, in the isosceles case, the common fold does give the minimum area of the folded right triangle $\triangle UVC$.

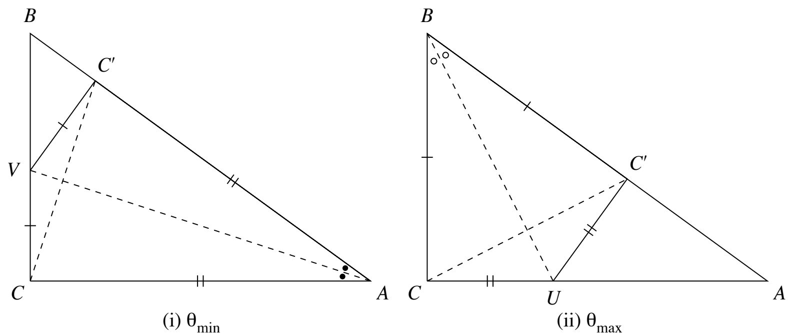

Figure 2: Angle bisectors as extreme folds

But otherwise there seems little in the way of intuition to guide us, so we must have recourse to a more analytical investigation. To this end, Nahin lets $\theta = \angle CUV$ be the angle that the crease UV makes with AC (see Figure 1(i)). If the crease is to cut off a triangle $\triangle UVC$, then it varies from the internal angle bisector of $\angle BAC$ to the internal angle bisector of $\angle ABC$ (compare Figure 2). Thus, writing as usual $\alpha = \angle BAC$,

$$\frac{\alpha}{2} \leq \theta \leq \frac{\pi}{4} + \frac{\alpha}{2}. \tag{1}$$

Nahin then builds expectation through a circuitous derivation of the area Δ of $\triangle UVC$ in terms of the parameter θ:

$$\Delta = \Delta(\theta) = \frac{(ab)^2}{8\cos\theta\sin\theta[a\sin\theta + b\cos\theta]^2}. \tag{2}$$

Now, it is possible to derive (2) much more directly, as we do in the next section. But that is not the point. Rather, the minimisation of (2) is easier than Nahin seems to suggest, amounting to the maximisation of the denominator. Thus, recalling only double and then treble angle formulae, we go on to locate the minimum of (2) subject to (1) with startling simplicity by *angle trisection*:

$$\theta = \frac{1}{3}(\pi/2 + \alpha). \tag{3}$$

Now it is well-known that, in general, angle trisection involves the solution of a cubic equation, and is therefore not amenable to construction by straightedge and compasses. However, any angle can be trisected by paper-folding — an introduction [11] to paper-folding, including this result, was presented by Benedetto Scimemi in *Normat* in 1998. David Cox, who contributes an article to the present issue, provides related results in his book [2, §10.3]. So, Nahin's envelope-folding problem has the happy outcome that the minimising solution can itself be obtained by folding. Readers who follow *Normat* closely may also be interested to note that the results in [11] also lead to foldings for the roots of the quartic equations arising in the ladder problems reviewed by Kent Holing in his series [8, esp. Pt. III] (compare [10, §3.4] and [5, Ex. 18]).

Unfortunately, Nahin misses these developments. For, his purpose in introducing the problem was to illustrate "how a computer can play a highly useful role in minimization analyses". So, having achieved (2) — and challenged his readers to try the "nasty business" of setting $d\Delta/d\theta = 0$ — he opts immediately to use a computer to study the behaviour of $\Delta(\theta)$ directly, plotting it in the isosceles case $a/b = 1$, where we already intuit the answer by symmetry, and in the case $a/b = 1/2$, where an answer is less obvious.

Is the neatness of (3) something special to Nahin's problem? To find out, we investigate some related minimisation problems. For example, in Section 3, we consider minimising variously the displacement CC' of C and the width UV of the resulting crease. The former is nice enough, as the displacement is minimised by folding parallel to the hypotenuse, as in Figure 1(ii), when the displacement is the altitude and the width is half the hypotenuse. But the latter is of the same order of difficulty as Nahin's problem, in that it, too, requires the solution of a cubic equation, but without an answer as neat as (3). Curiously enough, if we switch to folding A onto BC, then minimising either the area of folded triangle or the width of the crease require only the solution of quadratic equations, as we show in Section 4. In terms of the functions involved in these problems, some might have served Nahin's interests in computer-aided studies as well as, if not better than, minimising the area Δ in the envelope-folding problem. However, Nahin's choice seems inspired in the combination it offers of arresting challenge and attractive outcome.

So, we are left with the puzzle: what is the meaning of the angle trisection in (3) for minimisation? To point up this question, in Section 5, we leave the reader to consider what is in effect a limiting case of Nahin's problem where, for fixed a, the right triangle becomes a strip of width a — but now, following [5], in *doing without calculus*.

2 Minimising the area

Our first task is to provide a straightforward derivation of (2). Naturally, we should keep in mind the key property of folding, that the crease UV is the perpendicular bisector of the displacement CC'. Thus, $\angle C'UV = \angle CUV = \theta$, so that $\angle CUC' = 2\theta$ (see Figure 3).

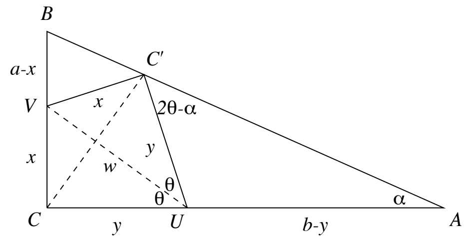

Figure 3: Setting for Sine rule

But $\angle CUC'$, as an exterior angle of $\triangle AC'U$, is the sum of the two opposite angles $\angle C'AU = \angle BAC = \alpha$ and $\angle AC'U$. Turning this equality around, we deduce that

$$\angle AC'U = \angle CUC' - \angle C'AU = 2\theta - \alpha.$$

This means that, with reference to Figure 3, application of the Sine rule yields

$$\frac{y}{\sin\alpha} = \frac{C'U}{\sin\angle C'AU} = \frac{AU}{\sin\angle AC'U} = \frac{b-y}{\sin(2\theta-\alpha)}.$$

Collecting terms in y gives

$$b\sin\alpha = y(\sin\alpha + \sin(2\theta-\alpha)) = y(\sin\alpha + \sin 2\theta\cos\alpha - \cos 2\theta\sin\alpha).$$

Hence, on recalling that $\cot\alpha = b/a$ and deploying the double angle formulae, we find that

$$y = \frac{ab}{2\sin\theta(a\sin\theta + b\cos\theta)}. \tag{4}$$

Turning now to the analogous triangle $\triangle BC'V$, we see that

$$\angle C'VB = \pi - \angle C'VC = \angle C'UC = 2\theta,$$

because $\angle UC'V = \angle UCV = \pi/2$, while

$$\angle BC'V = \pi - \angle UC'V - \angle AC'U = \frac{\pi}{2} + \alpha - 2\theta.$$

So, we are also in a position to apply the Sine rule in this second triangle, deriving thereby an expression for x matching (4):

$$x = \frac{ab}{2\cos\theta(a\sin\theta + b\cos\theta)}. \tag{5}$$

But, in view of Figure 1(i), $\Delta = xy/2$, so (4) and (5) delivers (2). Thus we are able to dispense with Nahin's much longer derivation, if at the price of knowing the double angle formulae. However, these are also suggestive on turning to the minimisation problem.

Now, as ab is a constant, (2) is really just a reciprocal. So, minimising $\Delta(\theta)$ is equivalent to maximising a multiple of the denominator, say,

$$f(\theta) = 2\sin\theta\cos\theta(a\sin\theta + b\cos\theta)^2,$$

taking care to avoid $f(\theta)$ vanishing. Differentiation will be simplified if we revert to double angles and write

$$f(\theta) = \sin 2\theta(a\sin\theta + b\cos\theta)^2.$$

It follows that

$$\begin{aligned} f'(\theta) &= 2\cos 2\theta(a\sin\theta + b\cos\theta)^2 + 2\sin 2\theta(a\sin\theta + b\cos\theta)(a\cos\theta - b\sin\theta) \\ &= 2[a(\sin\theta\cos 2\theta + \cos\theta\sin 2\theta) + b(\cos\theta\cos 2\theta - \sin\theta\sin 2\theta)](a\sin\theta + b\cos\theta) \\ &= 2(a\sin 3\theta + b\cos 3\theta)(a\sin\theta + b\cos\theta), \end{aligned} \tag{6}$$

making use of the treble angle formulae to achieve the final simplification in (6). Consequently, the only turning points of $f(\theta)$, with $f(\theta)$ non-zero, are given by

$$\tan 3\theta = -\frac{b}{a}. \tag{7}$$

But since $\tan\alpha = a/b$, we also know that

$$\tan(\pi/2 + \alpha) = -\frac{b}{a}.$$

Hence, the solution to (7) in the range (1) is given by (3); and this solution does give a maximun for $f(\theta)$, as expected.

What if we have (2), but do not (think to) make use of double and treble angles? Then there is some choice as to what might stand in place of the term in 3θ in the product in (6). But if we leave it as

$$a(3\cos^2\theta - \sin^2\theta)\sin\theta + b(\cos^2\theta - 3\sin^2\theta)\cos\theta,$$

then instead of (7), the location of the minimum of $\Delta(\theta)$ is given by a cubic in $t = \tan\theta$:

$$at^3 + 3bt^2 - 3at - b = 0. \tag{8}$$

Clearly, (8) disguises the angle trisection in (3). Nevertheless, it turns out to be tractable.

In order to recast (1) in terms of t, it is helpful to recall that the tangents of the half-angles of a right triangle can be expressed by means of the sides together with the radius r of the inscribed circle:

$$\tan\frac{\alpha}{2} = \frac{r}{b-r}, \quad \tan(\frac{\pi}{4} - \frac{\alpha}{2}) = \frac{r}{a-r},$$

where $r = (a+b-c)/2$. It follows that (1) translates to

$$\frac{a+b-c}{b+c-a} \leq t \leq \frac{a+c-b}{a+b-c}. \tag{9}$$

On the other hand, since $a \leq b$, it is straightforward to check that (8) has a root, $t = t_\Delta$, say, in the interval $[\frac{a}{b}, 1]$, so satisfying (9). The sign of the constant term in (8) implies there are either one or three positive real roots. But with three positive real roots, the cubic would have both its own turning points positive, which is not the case. Hence the root t_Δ in $[\frac{a}{b}, 1]$ is, in fact, the unique positive root of (8). Thus, comparing (8) with (6), at this turning point for $f(\theta)$, $f'(\theta)$ changes from positive to negative, ensuring that $f(\theta)$ assumes a maximum.

3 Minimising the crease

In obtaining (2) in the previous section, we made use of the fact that the area Δ of the folded triangle $\triangle UVC$ is given by $\Delta = xy/2$. Of course, this depends on the angle at C being right. But, whenever we fold, the displacement $l = CC'$ and the width $w = UV$ of the crease are orthogonal. Thus, with reference to Figure 1(i), we have an alternative expression for Δ:

$$\Delta = \frac{lw}{4}. \tag{10}$$

Now, since the angle at C is right, $w^2 = x^2 + y^2$. So, from (4) and (5), we find that

$$w = \frac{ab}{2\sin\theta\cos\theta(a\sin\theta + b\cos\theta)}, \tag{11}$$

and then comparison of (2), (10) and (11) gives

$$l = \frac{ab}{a\sin\theta + b\cos\theta}. \tag{12}$$

In (11) and (12), we have two further examples where, as with (2), minimisation can be effected through maximising a denominator. Of these, the minimisation of the displacement l in (12) is the simpler and neater. Indeed, differentiation of the denominator $a\sin\theta + b\cos\theta$ quickly reveals that l attains a minimum when $\theta = \alpha$. that is, when the crease is parallel to the hypotenuse. Consequently, in this case, C' is the foot of the altitude of $\triangle ABC$ from C, while U and V are the midpoints of AC and BC (compare Figure 1(ii)). As $\sin\alpha = a/c$ and $\cos\alpha = b/c$, with $c^2 = a^2 + b^2$, we infer from (11) and (12) that $w = ab/c$ and $l = c/2$.

On the other hand, minimisation of w itself proceeds along the lines of our discussion in the final paragraph of the previous section. The upshot is that the minimum is located by another cubic in $t = \tan\theta$:

$$at^3 + 2bt^2 - 2at - b = 0. \tag{13}$$

Without being alert to the quirk of angle trisection, there would seem little to tell (8) and (13) apart. In particular, arguing as for (8), (13) has a unique positive root $t = t_w$, say, which is located in the interval $[\frac{a}{b},1]$, so satisfying (9), and gives a minimum for w in (11). Moreover, these unique positive roots of (8) and (13) are related by

$$\frac{a}{b} \leq t_\Delta \leq t_w \leq 1.$$

Any one of these inequalities holds with equality only in the case $a = b$. that is, the right triangle $\triangle ABC$ is isosceles, and equality holds throughout. But Nahin's computer-aided study might come in more handy with (13), where we miss out on such a neat relation as (3).

4 Minimising another area

Envelopes make a handy source of right triangles ready for folding. But there is nothing in the nature of folding to restrict attention to right triangles and readers might be encouraged to experiment with folding vertices of triangles onto opposite sides for triangles and vertices of their choice. Entering into this spirit, we stay with Nahin's right triangle $\triangle ABC$, but now consider folding A onto BC at, say, A', by a crease UV with U on AC and V on AB.

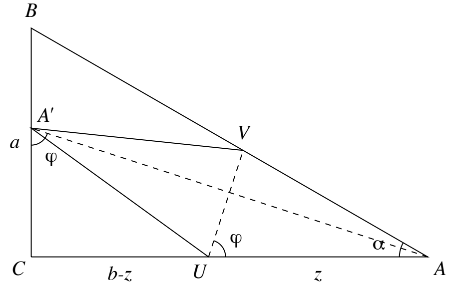

Figure 4: Another folding

If A' is on the line segment BC, rather than the line BC produced, the internal angle bisectors at B and C are inaccessible as folds for $a < b$, and the crease varies instead from the perpendicular bisector of AB, when A is folded onto B, to the perpendicular bisector of AC, when A is folded onto C. Hence, if we let $\phi = \angle AUV$ be the angle that the crease UV makes with CA, as shown in Figure 4, then

$$\frac{\pi}{2} - \alpha \leq \phi \leq \frac{\pi}{2}. \tag{14}$$

Further, let O be the point of intersection of the crease UV with the displacement AA'. Since the crease and the displacement intersect at right angles, the right triangles $\triangle AA'C$ and $\triangle AUO$ are similar, so that

$$\angle AA'C = \angle AUO = \angle AUV = \phi. \tag{15}$$

We convert our notation to this new setting, writing Δ for the area of the folded triangle $\triangle AUV$, l for the displacement AA', w for the width of the crease UV and $z = AU = A'U$. With this understanding, Δ is given by (10), as before. From Figure 4, we see that

$$l \sin\phi = b, \quad 2z \sin\phi = l.$$

Thus, z comes out as:

$$z = \frac{b}{2\sin^2\phi}. \tag{16}$$

Now, arguing on the lines of our derivation of (4) in Section 2, application of the Sine rule to $\triangle AUV$ yields, with reference to Figure 4,

$$\frac{w}{\sin\alpha} = \frac{UV}{\sin\angle UAV} = \frac{AU}{\sin\angle AVU} = \frac{z}{\sin(\pi-\alpha-\phi)},$$

that is,

$$z\sin\alpha = w\sin(\alpha+\phi) = w(\sin\alpha\cos\phi + \cos\alpha\sin\phi).$$

Noting once more that $\cot\alpha = b/a$ and eliminating z by means of (16) leads to

$$w = \frac{az}{a\cos\phi + b\sin\phi} = \frac{ab}{2\sin^2\phi(a\cos\phi + b\sin\phi)}. \tag{17}$$

Hence,

$$\Delta = \frac{lw}{4} = \frac{ab^2}{8\sin^3\phi(a\cos\phi + b\sin\phi)}. \tag{18}$$

It is immediate that l is minimised by folding A onto C. But the functions in (17) and (18) provide further candidates for computer-aided study of the sort favoured by Nahin. However, as with the counterpart expressions in the previous two sections, there is really no problem in locating minimums in (17) and (18) by maximising the denominators. Perhaps what is most interesting here is that, in contrast with the cubic equations (12) and (8), these new minimums are located as the positive roots of a *quadratic*, meaning that, in some sense, the problems in this section are simpler — for example, we can construct their solutions by means of straightedge and compasses, which is not possible for the solution of the cubics. Thus the minimum of w in (17) is located at the positive root of

$$at^2 - 4bt - 3a = 0,$$

while that of Δ in (18) is located at the positive root of

$$at^2 - 3bt - 2a = 0,$$

where now $t = \tan\phi$. It is easy the check that these positive roots are greater than b/a, in keeping with (14), although it is less easy to recognise the angles that have these tangents, even in the isosceles case.

There may be somewhat more transparency here in using as the parameter $h = A'C$. From (15), $ht = h\tan\phi = b$. Moreover, if L is the foot of the perpendicular from O onto AC, then L is the midpoint of AC while $OL = \frac{1}{2}h$. Thus, the discussion in this section can be reformulated in the language of coordinate geometry by means of h, if so wished. At all events, in terms of h, we find that

$$l = \sqrt{b^2 + h^2}, \quad w = \frac{a(b^2 + h^2)\sqrt{b^2 + h^2}}{2b(ah + b^2)}, \quad \Delta = \frac{a(b^2 + h^2)^2}{8b(ah + b^2)}.$$

Of course, the minimum of l is b. The minimum of w occurs for

$$h_w = b(\sqrt{8a^2 + 9b^2} - 3b)/4a;$$

that for Δ occurs for

$$h_\Delta = b(\sqrt{3a^2 + 4b^2} - 2b)/3a.$$

For what it is, worth $0 < h_\Delta < h_w < a$.

5 A stripped-down minimisation

Nahin has not been alone in combining folding with minimisation. For example, the questions of minimising the width of the crease when a corner C of a strip of width a is folded onto the opposite edge, as in Figure 5, has already achieved a certain veneration, perhaps because the answer is surprisingly neat: the minimum occurs when $x = 3a/4$, that is when $\tan\theta = \sqrt{2}/2$ (compare (13)). For the folded triangle in Figure 5, the minimum area occurs when $x = 2a/3$, that is, when $\theta = \pi/6$, the limiting case of (3) as α tends to zero. Clearly, Nahin's envelope-folding problem can be seen as extending this setting to allow the line through B onto which C is folded to be at an acute angle to BC, rather than a right angle.

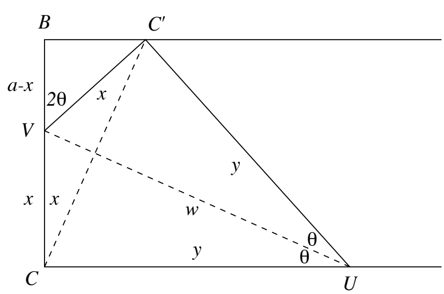

Figure 5: The folded strip

The well-known puzzlist Henry Ernest Dudeney (1857–1930) posed this minimum crease problem in the issue of *The Strand Magazine* for August, 1919, as part of his long-running series of *Perplexities* [3, (a), Prob. 469] that had appeared there since

May 1910 (and was to run until his death in 1930). He included it in *Modern Puzzles and How to Solve Them* [3, (b) Prob. 139, pp. 55, 144], a collection drawn mainly from this series that came out in 1926 (for a contemporary review, see [3, (c)]). However, in this instance no justification is given for the answer — the pages from Dudeney's book are reproduced in Figure 6 on p. 89. Similarly, Martin Gardner (1914–) gave the problem a further airing as an aside when discussing paper-folding in an early column on *Mathematical Games* [6] in the issue of *Scientific American* [6] for July, 1959. Although the problem has done long service as a staple in calculus textbooks, a fresh look has been taken recently in [4, 9].

But the questions of minimising the folded triangle and the width of the crease in Figure 5 are also among the examples in an article [5, Ex. 17 and Fig. 7] by Trevor James Fletcher (1921–) in 1971. Fletcher's intent is to show how to *dispense* with calculus in working a variety of maximisation and minimisation problems — an aim of which Dudeney and Gardner would no doubt approve. The "fundamental principle", as he puts it, running through his solutions is the inequality between the arithmetic and geometric means. But he concedes that the folding problems are "a little tricky" managed this way, and opts to leave them for "enthusiasts to solve for themselves". We are happy to pass on this encouragement to readers of *Normat*, now with Nahin's generalisation.

Indeed, there is a further puzzle we should like to include. Folding the corner of a paper strip over onto the opposite edge has been developed for classroom activity, not only as a computer-aided exercise in optimisation, for example in [1], but also as a construction for Pythagorean triples, for example in [7]. However, the triangle of interest in [1, 7] is not the folded triangle $\triangle UVC$, but rather the right triangle $\triangle C'VB$. Now, the maximum area of this triangle can be obtained by a simple direct argument. Let V' be the reflection of V in the line through B perpendicular to BC. Then $\triangle C'V'B$ is congruent to $\triangle C'VB$. Hence, $\triangle C'VV'$ is isosceles, with equal sides x and base $2(a-x)$, and so with constant perimeter $2a$. But among all (isosceles) triangles with given perimeter, the equilateral triangle has largest area. Thus, $\triangle C'VB$ attains its maximum area when $\triangle C'VV'$ is equilateral, that is, when $x = 2(a-x)$ or $x = 2a/3$. So, our further puzzle is to explain why this maximum area is achieved where the folded triangle $\triangle UVC$ attains its minimum area.

Moreover, this agreement is not peculiar to the folded strip, but holds more generally in Nahin's envelope-folding problem. For, let $\Gamma = \Gamma(\theta)$ denote the area of triangle $\triangle C'VB$ in Figure 3, so that

$$\Gamma = \Gamma(\theta) = \frac{1}{2}(a-x)x\sin 2\theta.$$

From (5), after a little algebra with trigonometric expansions, we find that

$$\Gamma(\theta) = \frac{a^2 b\sin\theta(a\sin 2\theta + b\cos 2\theta)}{4\cos\theta(a\sin\theta + b\cos\theta)^2}.$$

Now, if Nahin thought $\Delta(\theta)$ in (2) was messy to differentiate, then $\Gamma(\theta)$ looks even more formidable. Yet, the derivative comes out comparatively cleanly:

$$\Gamma'(\theta) = \frac{(ab)^2(a\sin 3\theta + b\cos 3\theta)}{4\cos^2\theta(a\sin\theta + b\cos\theta)^3}.$$

Hence, we see that turning points for $\Gamma(\theta)$ are also given by (7), as for $\Delta(\theta)$, and we can then verify that $\Gamma(\theta)$ attains a maximum at (3). But this working only quickens our interest in having some more transparent explanation.

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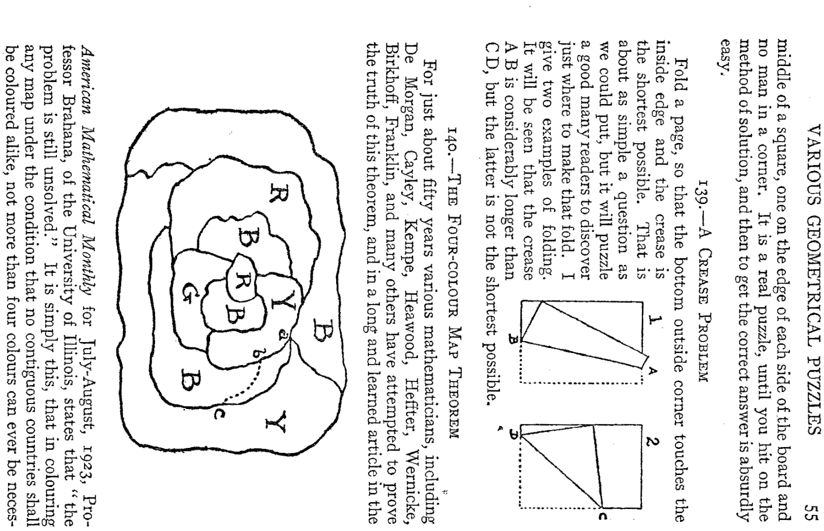

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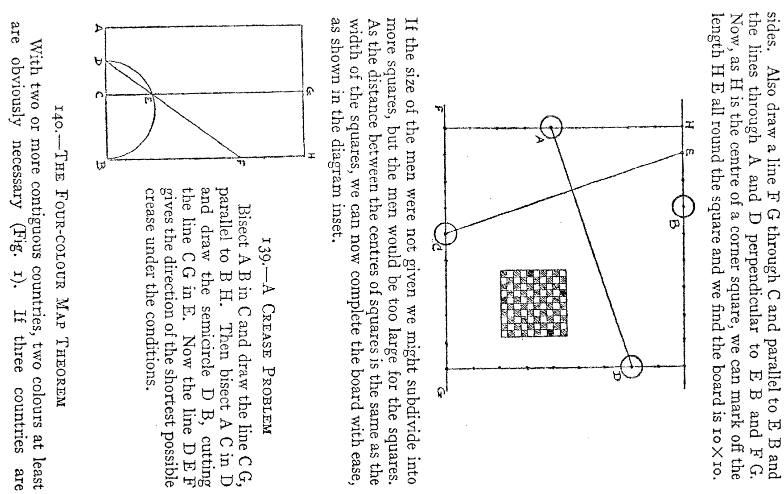
middle of a square, one on the edge of each side of the board and no man in a corner. It is a real puzzle, until you hit on the method of solution, and then to get the correct answer is absurdly easy.

139.—A Crease Problem

Fold a page, so that the bottom outside corner touches the inside edge and the crease is the shortest possible. That is about as simple a question as we could put, but it will puzzle a good many readers to discover just where to make that fold. I give two examples of folding. It will be seen that the crease A B is considerably longer than C D, but the latter is not the shortest possible.

140.—The Four-colour Map Theorem

For just about fifty years various mathematicians, including De Morgan, Cayley, Kempe, Heawood, Heffter, Wernicke, Birkhoff, Franklin, and many others have attempted to prove the truth of this theorem, and in a long and learned article in the *American Mathematical Monthly* for July-August, 1923, Professor Brahana, of the University of Illinois, states that "the problem is still unsolved." It is simply this, that in colouring any map under the condition that no contiguous countries shall be coloured alike, not more than four colours can ever be neces-



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sides. Also draw a line F G through C and parallel to E B and the lines through A and D perpendicular to E B and F G. Now, as H is the centre of a corner square, we can mark off the length H E all round the square and we find the board is 10×10.

If the size of the men were not given we might subdivide into more squares, but the men would be too large for the squares. As the distance between the centres of squares is the same as the width of the squares, we can now complete the board with ease, as shown in the diagram inset.

139.—A Crease Problem

Bisect A B in C and draw the line C G, parallel to B H. Then bisect A C in D and draw the semicircle D B, cutting the line C G in E. Now the line D E F gives the direction of the shortest possible crease under the conditions.

140.—The Four-colour Map Theorem

With two or more contiguous countries, two colours at least are obviously necessary (Fig. 1). If three countries are

Figure 6: Facscimiles from Dudeney's 1926 book *Modern Puzzles and How to Solve Them.*