Enumerating Minimal Connected Dominating Sets in Graphs of Bounded Chordality*

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Abstract

Listing, generating or enumerating objects of specified type is one of the principal tasks in algorithmics. In graph algorithms one often enumerates vertex subsets satisfying a certain property. We study the enumeration of all minimal connected dominating sets of an input graph from various graph classes of bounded chordality. We establish enumeration algorithms as well as lower and upper bounds for the maximum number of minimal connected dominating sets in such graphs. In particular, we present algorithms to enumerate all minimal connected dominating sets of chordal graphs in time $O(1.7159^n)$, of split graphs in time $O(1.3803^n)$, and of AT-free, strongly chordal, and distance-hereditary graphs in time $O^*(3^{n/3})$, where n is the number of vertices of the input graph. Our algorithms imply corresponding upper bounds for the number of minimal connected dominating sets for these graph classes.

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1 Introduction

Listing, generating or enumerating objects of specified type and properties has important applications in various domains of computer science, such as data mining, machine learning, and artificial intelligence, as well as in other sciences, especially biology. In particular, enumeration algorithms whose running time is measured in the size of the input have gained increasing interest recently. The reason for this is two-fold. Firstly, many exact exponential-time algorithms for the solution of NP-hard problems rely on such enumeration algorithms. Sometimes the fastest known algorithm to solve an optimization problem is by simply enumerating all minimal or maximal feasible solutions (e.g., for subset feedback vertex sets [15]), whereas other times the enumeration of some objects is useful for algorithms for completely different problems (e.g., enumeration of maximal independent sets in triangle-free graphs for computing graph homomorphisms [14]). Secondly, the running times of such enumeration algorithms very often imply an upper bound on the maximum number of enumerated objects a graph can have. This is a field of research that has long history within combinatorics, and enumeration algorithms provide an alternative way to prove such

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combinatorial bounds. In fact, several classical examples exist in this direction, of which one of the most famous is perhaps that of Moon and Moser [25] who showed that the maximum number of maximal independent sets in a graph on n vertices is $3^{n/3}$. Although the arguments of [25] were purely combinatorial, the same bound is also achieved by an enumeration algorithm with running time $O^*(3^{n/3})$, where the O^*-notation suppresses polynomial factors.

The mentioned result on the number of maximal independent sets is tight, as there is a graph that has exactly $3^{n/3}$ maximal independent sets, namely a disjoint union of $n/3$ triangles. However, for many upper bounds, no such matching lower bound is known, and hence for the maximum number of many objects there is a gap between the known upper and lower bounds. This motivates the study of enumeration of objects in graphs belonging to various graph classes. For example, the maximum number of minimal dominating sets in graphs is known to be at most 1.7159^n [13], however no graph having more than 1.5704^n minimal dominating sets is known. On the other hand, on many graph classes matching upper and lower bounds can be shown on the maximum number of minimal dominating sets [7, 9]. Furthermore, even if the bound on general graphs is tight, a better bound might exist for graph classes, which might be useful algorithmically and interesting combinatorially. For example, the maximum number of maximal independent sets in triangle-free graphs is at most $2^{n/2}$ and they can be listed in time $O^*(2^{n/2})$ [5], which was used in the above mentioned algorithm for homomorphisms [14]. As a consequence, there has been extensive research in this direction recently, both on general graphs and in particular on graph classes. Examples of algorithms for the enumeration and combinatorial lower and upper bounds on graph classes exist for minimal feedback vertex sets, minimal subset feedback vertex sets, minimal dominating sets, minimal separators, and potential maximal cliques [7, 8, 9, 12, 15, 17, 19, 20, 21].

In this paper we initiate the study of the enumeration and maximum number of minimal *connected* dominating sets in a given graph. Interestingly, the best known upper bound for the maximum number of minimal connected dominating sets in an arbitrary n-vertex graph is 2^n, i.e., the trivial one. The best lower bound we achieve in this paper is $3^{(n-2)/3}$, and thus the gap between the known lower and upper bounds is huge on arbitrary graphs. Furthermore, although connected dominating sets have been subject to extensive study when it comes to optimization and decision variants, their enumeration has been left completely unattended. In fact computing a minimum connected dominating set is one of the classical NP-hard problems already mentioned in the monograph of Garey and Johnson [18]. The best known running time of an algorithm solving this problem is $O(1.8619^n)$ [1], which is surprisingly larger than the best known lower bound $3^{(n-2)/3} \approx 1.4423^n$.

The results that we present in this paper are summarized in the following table, where n is the number of vertices and m is the number of edges of an input graph belonging to the given class.

Graph Class	Lower Bound	Upper Bound	Enumeration Algorithm
chordal	$3^{(n-2)/3}$	$O(1.7159^n)$	$O(1.7159^n)$
split	1.3195^n[7]	1.3803^n	$O(1.3803^n)$
cobipartite	1.3195^n[7]	1.3803^n	$O(1.3803^n)$
interval	$3^{(n-2)/3}$	$3^{(n-2)/3}$	$O^*(3^{n/3})$
AT-free	$3^{(n-2)/3}$	$O^*(3^{(n-2)/3})$	$O^*(3^{n/3})$
strongly chordal	$3^{(n-2)/3}$	$3^{n/3}$	$O^*(3^{n/3})$
distance-hereditary	$3^{(n-2)/3}$	$3^{n/3} \cdot n$	$O^*(3^{n/3})$
cograph	m	m	$O(m)$

2 Preliminaries

We consider finite undirected graphs without loops or multiple edges. For each of the graph problems considered in this paper, we let $n = |V(G)|$ and $m = |E(G)|$ denote the number of vertices and edges, respectively, of the input graph G. For a graph G and a subset $U \subseteq V(G)$ of vertices, we write $G[U]$ to denote the subgraph of G induced by U. We write $G - U$ to denote the subgraph of G induced by $V(G) \setminus U$, and $G - u$ if $U = \{u\}$. A set $U \subseteq V(G)$ is *connected* if $G[U]$ is a connected graph. For a vertex v, we denote by $N_G(v)$ the *(open) neighborhood* of v, i.e., the set of vertices that are adjacent to v in G. The *closed neighborhood* $N_G[v] = N_G(v) \cup \{v\}$. For a set of vertices $U \subseteq V(G)$, $N_G[U] = \cup_{v \in U} N_G[v]$ and $N_G(U) = N_G[U] \setminus U$. We say that P is a *(u,v)-path* if P is a path that joins u and v. The vertices of P different from u and v are the *inner* vertices of P. The *distance* $\mathrm{dist}_G(u,v)$ between vertices u and v of G is the number of edges on a shortest (u,v)-path. A path (cycle) P is *induced* if it has no *chord*, i.e., there is no edge of G incident to any two vertices of P that are not adjacent in P. The *chordality* $\mathrm{chord}(G)$ of a graph G is the length of a longest induced cycle in G; if G has no cycles, then $\mathrm{chord}(G) = 0$. A vertex v is a *cut vertex* of a connected graph G with at least two vertices if $G - v$ is disconnected.

For a non-negative integer k, a graph G is *k-chordal* if $\mathrm{chord}(G) \le k$. A graph is *chordal* if it is 3-chordal. A graph G is *strongly chordal* if G is chordal and every cycle C of even length at least 6 in G has an *odd chord*, i.e., an edge that connects two vertices of C that are an odd distance apart from each other in the cycle. A graph G is *distance-hereditary* if for any connected induced subgraph H of G, $\mathrm{dist}_H(u,v) = \mathrm{dist}_G(u,v)$ for $u, v \in V(H)$. Distance hereditary graphs are 4-chordal. An *asteroidal triple (AT)* is a set of three pairwise non-adjacent vertices such that between each pair of them there is a path that does not contain a neighbor of the third. A graph is *AT-free* if it contains no AT. Consequently, AT-free graphs are 5-chordal.

A graph is a *split* graph if its vertex set can be partitioned in an independent set and a clique. Split graphs are chordal. A graph G is an *interval graph* if it is the intersection graph of a set of closed intervals on the real line, i.e., the vertices of G correspond to the intervals and two vertices are adjacent in G if and only if their intervals have at least one point in common. Interval graphs are strongly chordal and AT-free. A graph is a *cobipartite* graph if its vertex set can be partitioned into two cliques. A graph is a *cograph* if it has no induced path on 4 vertices. Cobipartite graphs and cographs are AT-free as well as 4-chordal.

Each of the above-mentioned graph classes can be recognized in polynomial (in most cases linear) time, and they are closed under taking induced subgraphs [4, 22]. See the monographs by Brandstädt et al. [4] and Golumbic [22] for more properties and characterizations of these classes and their inclusion relationships.

A vertex v of a graph G *dominates* a vertex u if $u \in N_G[v]$; similarly v dominates a set of vertices U if $U \subseteq N_G[v]$. For two sets $D, U \subseteq V$, D dominates U if $U \subseteq N_G[D]$. A set of vertices D is a *dominating set* of G if D dominates $V(G)$. A set of vertices D is a *connected dominating set* if D is connected and D dominates $V(G)$. A (connected) dominating set is *minimal* if no proper subset of it is a (connected) dominating set. Let $D \subseteq V(G)$, and let $v \in D$. Vertex u is a *private vertex*, or simply *private*, for vertex v (with respect to D) if u is dominated by v but is not dominated by $D \setminus \{v\}$. Clearly, a dominating set D is minimal if and only if each vertex of D has a private vertex. Notice also that a connected dominating set D is a minimal if and only if for any $v \in D$, v has a private vertex or $D \setminus \{v\}$ is disconnected, i.e., v is a cut vertex of $G[D]$. Observe that a vertex can be private for itself with respect to a dominating set D, but if D is a connected dominating set of size at least two, then any private u of v is a neighbor of v.

For technical reasons, we also consider *red-blue domination*. Let $\{R, B\}$ form a bipartition of the vertex set of a graph G. We refer to the vertices of R as the *red* vertices, the vertices of B as the *blue* vertices, and we say that G is a *red-blue graph*. A set of vertices $D \subseteq R$ is a *red dominating set* if D dominates B, and D is *minimal* if no proper subset of it dominates B. It is straightforward to see that $D \subseteq R$ is a minimal red dominating set if and only if D dominates B and each vertex of D has a private blue vertex.

We conclude this section by showing a lower bound for the maximum number of minimal connected dominating sets.

▶ **Proposition 1.** *There are interval and distance-hereditary graphs with at least $3^{(n-2)/3}$ minimal connected dominating sets.*

Proof. To obtain the bound for interval and distance-hereditary graphs, consider the graph G constructed as follows for a positive integer k.

- For $i \in \{1, \ldots, k\}$, construct a triple of pairwise adjacent vertices $T_i = \{x_i, y_i, z_i\}$.
- For $i \in \{2, \ldots, k\}$, join each vertex of T_{i-1} with every vertex of T_i by an edge.
- Construct two vertices u and v and edges ux_1, uy_1, uz_1 and vx_k, vy_k, vz_k.

Clearly, G has $n = 3k+2$ vertices. Notice that $D \subseteq V(G)$ is a minimal connected dominating set of G if and only if $u, v \notin D$ and $|D \cap T_i| = 1$ for $i \in \{1, \ldots, k\}$. Therefore, G has $3^k = 3^{(n-2)/3}$ minimal connected dominating sets. It remains to observe that G is both interval and distance-hereditary. ◀

3 Chordal graphs

In this section we shall heavily rely on minimal separators of graphs and minimal transversals of hypergraphs. A vertex set $S \subseteq V$ is a *separator* of the graph $G = (V, E)$ if $G - S$ is disconnected. A component C of $G - S$ is *full* if every vertex of S has a neighbor in C. A separator S of G is a *minimal separator* of G if $G - S$ has at least two full components. Minimal separators of graphs have been studied intensively in the last twenty years. They play a crucial role in minimal triangulations, and in solving problems like treewidth and minimum fill-in. For more information we refer to [23].

Let us start with a strong relationship between the minimal connected dominating sets of a graph and its minimal separators, established by Kante, Limouzy, Mary and Nourine [24]. First note that disconnected graphs have no connected dominating set, and all minimal connected dominating sets of a complete graph G are singletons $\{v\}$ with $v \in V(G)$. Now we define the *minsep hypergraph* $H = (V(H), E(H))$ of a graph $G = (V, E)$. The vertex set of H consists of all vertices of G belonging to some minimal separator of G, hence $V(H) \subseteq V(G)$. The hyperedges of H are exactly the minimal separators of G. Hence $|E(H)|$ is the number of minimal separators of G. Recall that a transversal of a hypergraph H is a vertex set $T \subseteq V(H)$ intersecting all hyperedges of H, i.e. $T \cap A \neq \emptyset$ for all $A \in E(H)$. Furthermore a transversal is minimal if no proper subset of it is a transversal.

▶ **Theorem 2** ([24]). *Let $G = (V, E)$ be a connected and non complete graph. Then $D \subseteq V$ is a minimal connected dominating set of G if and only if D is a minimal transversal of the minsep hypergraph H of G.*

To enumerate the minimal transversals of the minsep hypergraph of chordal graphs, we will be relying on a branching algorithm and its analysis which is due to Fomin, Grandoni, Pyatkin and Stepanov [13]. The main result of their paper is that a graph has at most 1.7159^n minimal dominating sets. The crucial result of the paper for us is the branching algorithm to enumerate all minimal set covers of a set cover instance $(\mathcal{U}, \mathcal{S})$, where $\mathcal{U}$ is a universe and $\mathcal{S}$ is a collection of subsets of $\mathcal{U}$. When studying the algorithm and its analysis one observes that it can be applied to all set cover instances $(\mathcal{U}, \mathcal{S})$. Only at the very end of the analysis the obtained general bound is applied to the particular instances obtained from graphs satisfying $|\mathcal{U}| = |\mathcal{S}|$, which includes tailoring the weights to the case $|\mathcal{U}| = |\mathcal{S}|$. The interested reader may study Sections 3 and 4 of [13] to find the following implicit result.

▶ **Theorem 3** ([13]). *A set cover instance $(\mathcal{U}, \mathcal{S})$ has $O^*(\lambda^{|\mathcal{U}|+\alpha\cdot|\mathcal{S}|})$ minimal set covers, where $\lambda = 1.156154$ and $\alpha = 2.720886$. These minimal set covers can be enumerated in time $O^*(\lambda^{|\mathcal{U}|+\alpha|\mathcal{S}|})$.*

By Corollary 2 we are interested in enumerating the minimal transversals of a hypergraph $H = (V(H), E(H))$ with $V(H) = \{v_1, v_2, \dots v_s\}$ and $E(H) = \{E_1, E_2, \dots, E_t\}$ where $E_j \subseteq V(H)$ for all hyperedges E_j. It is well-known that enumerating the minimal transversals of a hypergraph H is equivalent to enumerating the minimal set covers of a set cover instance (corresponding to the dual hypergraph of H) constructed as follows. First we set $\mathcal{U} = E(H)$ and then $\mathcal{S}$ is a collection of sets $S(v_1), S(v_2), \dots S(v_s)$ such that for all $i \in \{1, 2, \dots s\}$ the set $S(v_i) \subseteq E(H)$ consists of all hyperedges E_j containing v_i. Consequently $|\mathcal{U}| = |E(H)|$ and $|\mathcal{S}| = |V(H)|$. By the construction enumerating the minimal set covers of the dual set cover instance $(\mathcal{U}, \mathcal{S})$ is equivalent to enumerating the minimal transversals of H. Consequently Theorem 3 implies

▶ **Corollary 4.** *A hypergraph $(V(H), E(H))$ has $O^*(\lambda^{|E(H)|+\alpha\cdot|V(H)|})$ minimal transversals, where $\lambda = 1.156154$ and $\alpha = 2.720886$. These minimal transversals can be enumerated in time $O^*(\lambda^{|E(H)|+\alpha|V(H)|})$.*

Now we are ready to consider the enumeration of minimal connected dominating sets on chordal graphs.

▶ **Theorem 5.** *A chordal graph has $O(1.7159^n)$ minimal connected dominating sets, and these sets can be enumerated in time $O(1.7159^n)$.*

Proof. Note that every chordal graph has a simplicial vertex, and no simplicial vertex belongs to a minimal separator. Furthermore a chordal graph has at most n minimal separators. Now let H be the minsep hypergraph of a chordal graph G. Let $s \geq 1$ be the number of maximal cliques containing a simplicial. Then $|V(H)| \leq |V(G)| - s = n - s$ and $|E(H)| \leq |V(G)| = n$. By Corollary 4, the number of minimal transversals of H is $O^*(\lambda^{|E(H)|+\alpha\cdot|V(H)|})$. Hence we can upper bound (up to a polynomial factor) the running time of the algorithm enumerating the minimal transversals and the number of minimal transversals by

$$\lambda^{|E(H)|+\alpha\cdot|V(H)|} \leq \lambda^{(1+\alpha)n} < 1.7159^n.$$

Consequently the number of minimal transversals of H is $O(1.7159^n)$, and they can be enumerated in time $O(1.7159^n)$. Finally by Corollary 2 these minimal transversals are precisely the minimal connected dominating sets of G. ◀

4 Split graphs and cobipartite graphs

We denote a split graph by $G = (C, I, E)$ to indicate that its vertex set $V(G)$ can be partitioned into a clique C and an independent set I. The following simple lemma will be crucial for our branching algorithm.

▶ **Lemma 6.** *Let $G = (C, I, E)$ be a split graph. Then D is a minimal connected dominating set of G if and only if either $D \subseteq C$ and D is minimal dominating set of G, or $|I| = 1$ and $D = I$ is a dominating set of G.*

Proof. Suppose $|I| \geq 2$. Then a minimal connected dominating set D of a split graph G cannot contain a vertex $v \in I$ since this would imply $w \in D$ for some $w \in C$ being adjacent to v. But then $D \setminus \{v\}$ would also be a connected dominating set, contradicting the minimality of D. ◀

Couturier et al. have shown that the maximum number of minimal dominating sets in a split graph is $3^{n/3}$, and that these sets can be enumerated in time $O^*(3^{n/3})$ [9]. Combined with Lemma 6, this implies that the same results hold for minimal connected dominating sets in split graphs. With a branching algorithm, given in the appendix, we are able to establish a significant improvement.

▶ **Theorem 7.** *A split graph has at most 1.3803^n minimal connected dominating sets, and these can be enumerated in time $O(1.3803^n)$.*

Theorem 7 implies an improvement on the number of minimal dominating sets for n-vertex cobipartite graphs. The previous best known bound is $O(1.4511^n)$ [9].

▶ **Corollary 8.** *A cobipartite graph has $O(1.3803^n)$ minimal (connected) dominating sets, and these sets can be enumerated in time $O(1.3803^n)$.*

Proof. Let $G = (C_1, C_2, E)$ be a cobipartite where its vertex set can be partitioned into cliques C_1 and C_2. Let D be a minimal dominating set of $G = (C_1, C_2, E)$. Then $D = \{x, y\}$ with $x \in C_1$ and $y \in C_2$, $D \subseteq C_1$ or $D \subseteq C_2$. Hence with the exception of the $O(n^2)$ minimal dominating sets $D = \{x, y\}$, all other minimal dominating sets are connected. There is a one-to-one relation of the minimal (connected) dominating sets of $G = (C_1, C_2, E)$ being a subset of C_1 and the minimal connected dominating sets of the split graph $G = (C_1, C_2, E)$ obtained by transforming C_2 into an independent set. Similarly, there is a one-to-one relation of the minimal (connected) dominating sets of $G = (C_1, C_2, E)$ being a subset of C_2 and the minimal connected dominating sets of the split graph $G = (C_2, C_1, E)$ obtained by transforming C_1 into an independent set. Hence the maximum number of minimal (connected) dominating sets of an n-vertex cobipartite graphs is equal to the maximum number of minimal connected dominating sets in an n-vertex split graph up to a polynomial factor. This together with Theorem 7 implies the corollary. ◀

The above one-to-one correspondence can also be used to obtain the best known lower bound for the maximum number of minimal connected dominating sets in an n-vertex split graph which is 1.3195^n, based on a lower bound construction for cobipartite graphs given in [7]. The following corollary will be useful in the next section.

▶ **Corollary 9.** *A red-blue graph G has $O(1.3803^n)$ minimal red dominating sets, and these can be enumerated in time $O(1.3803^n)$.*

Proof. Let $G = (R, B, E)$ be a red-blue graph. We construct a split graph $G' = (R, B, E)$ with clique R and independent set B. Then there is a one-to-one relation between the minimal red dominating sets of the red-blue graph G and the minimal connected dominating sets of the split graph G. Using this and Theorem 7 we get the result. ◀

5 AT-free graphs

We need the following folklore observation about the number of induced paths.

▶ **Lemma 10.** *For every pair of vertices u and v of a graph G, G has at most $3^{(n-2)/3}$ induced (u, v)-paths, and these paths can be enumerated in time $O^*(3^{n/3})$.*

Using Lemma 10, we can obtain the tight upper bound for the number of minimal connected dominating sets for interval graphs. Let G be an interval graph with at least two non-adjacent vertices. Consider an interval model of G, i.e., a collection of closed intervals on the real line corresponding to the vertices of G such that two vertices are adjacent in G if and only if their intervals intersect. Let u be the vertex of G corresponding to an interval with the leftmost right end-point and let v be the vertex corresponding to an interval with the rightmost left end-point. Notice that $u \neq v$ and u and v are not adjacent, because G is not a complete graph. It can be shown that $D \subseteq V(G)$ is a minimal connected dominating set of G if and only if D is the set of inner vertices of an induced (u, v)-path. This observation together with Lemma 10 immediately imply the following proposition.

▶ **Proposition 11.** *An interval graph has at most $3^{(n-2)/3}$ minimal connected dominating sets, and these sets can be enumerated in time $O^*(3^{n/3})$.*

Proposition 1 shows that the bound is tight. To extend it to AT-free graphs we need some additional terminology and auxiliary results. A path P in a graph G is a *dominating path* if $V(P)$ is a dominating set of G. A pair of vertices $\{u, v\}$ of G is a *dominating pair* if any (u, v)-path in G is a dominating path.

▶ **Lemma 12** ([6])**.** *Every connected AT-free graph has a dominating pair.*

We show the following properties of minimal connected dominating sets of AT-free graphs. Notice that if D is a connected dominating set of a graph G, then $G[D]$ is a connected AT-free graph and, therefore, $G[D]$ has a dominating pair by Lemma 12.

▶ **Lemma 13.** *Let D be a minimal connected dominating set of an AT-free graph G. Let $\{u, v\}$ be a dominating pair of $H = G[D]$ and suppose that $P = v_1 \dots v_k$, where $u = v_1$ and $v = v_k$, is a shortest (u, v)-path in H. Let $X_1 = N_G(\{v_1, v_2\}) \setminus N_G[v_4]$, $X_2 = N_G(\{v_{k-1}, v_k\}) \setminus N_G[v_{k-3}]$, $Y_1 = N_G(X_1) \setminus N_G[\{v_1, \dots, v_4\}]$ and $Y_2 = N_G(X_2) \setminus N_G[\{v_{k-3}, \dots, v_k\}]$. Then $D \subseteq N_G[V(P)]$ and if $k \geq 6$, the following holds:*

(i) *$D \subseteq V(P) \cup X_1 \cup X_2$,*

(ii) *for the (v_6, v_k)-subpath P_1 of P, $V(P_1) \cap N_G[\{v_1, \dots, v_4\} \cup Y_1] = \emptyset$, and for the (v_1, v_{k-5})-subpath P_2 of P, $V(P_2) \cap N_G[\{v_{k-3}, \dots, v_k\} \cup Y_2]) = \emptyset$.*

Proof omitted for lack of space.

Now we are ready to enumerate the minimal connected dominating sets of AT-free graphs.

▶ **Theorem 14.** *An AT-free graph has $O^*(3^{(n-2)/3})$ minimal connected dominating sets, and these sets can be enumerated in time $O^*(3^{n/3})$.*

Proof. First, we show that there are at most $3^{n/3} \cdot n^9$ minimal connected dominating sets D such that $H = G[D]$ has a dominating pair $\{u, v\}$ with $\text{dist}_G(u, v) \leq 8$ and enumerate these sets. Consider all the at most $\binom{n}{1} + \dots + \binom{n}{9} \leq n^9$ possible choices of a pair of vertices $\{u, v\}$ and an induced path $P = v_1 \dots v_k$ with $u = v_1$ and $v = v_k$ such that $k \leq 9$. For each $\{u, v\}$ and P we enumerate the minimal connected dominating sets D such that $\{u, v\}$ is a dominating pair in $H = G[D]$ and P is a shortest (u, v)-path in H.

Let P be any induced path $P = v_1 \dots v_k$ with $u = v_1$ and $v = v_k$ such that $k \leq 9$. Consider the red-blue graph $G' = G - V(P)$, where the set of red vertices is $R = N_G(V(P))$ and the set of blue vertices is $B = V(G') \setminus R$. Let D be a minimal connected dominating set of G such that $\{u, v\}$ is a dominating pair of $H = G[D]$ and P is a shortest (u, v)-path in H. By Lemma 13, $D \subseteq N_G[V(P)]$. It is straightforward to see that $D \setminus V(P)$ is a red dominating set of G' that dominates all blue vertices and by minimality, $D \setminus V(P)$ is a minimal red dominating set. By Corollary 9, there are at most $1.3803^{|V(G')|} \leq 3^{n/3}$ such sets D and they can be enumerated in time $O(3^{n/3})$. We obtain that there are at most $3^{n/3} \cdot n^9$ minimal connected dominating sets D such that $H = G[D]$ has a dominating pair $\{u, v\}$ with $\text{dist}_G(u, v) \leq 8$, and these sets can be enumerated in time $O(3^{n/3} \cdot n^9)$.

Now we enumerate minimal connected dominating sets D such that $H = G[D]$ has a dominating pair $\{u, v\}$ with $\text{dist}_G(u, v) \geq 9$. Consider all the at most $\binom{n}{1} + \dots + \binom{n}{10} \leq n^{10}$ possible choices of a pair of vertices $\{u, v\}$ and 2 disjoint induced paths $P_1 = x_1 \dots x_5$ and $P_2 = y_1 \dots y_5$ with $u = x_1$ and $v = y_5$. For each $\{u, v\}$, P_1 and P_2 we enumerate the minimal connected dominating sets D such that $\{u, v\}$ is a dominating pair in $H = G[D]$ and H has a shortest (u, v)-path $P = v_1 \dots v_k$ such that $v_i = x_i$ for $i \in \{1, \dots, 5\}$ and $v_i = y_{i+5-k}$ for $i \in \{k-4, \dots, k\}$.

Denote by $X_1 = N_G(\{x_1, x_2\}) \setminus N_G[x_4]$, $X_2 = N_G(\{y_4, y_5\}) \setminus N_G[y_2]$, $Y_1 = N_G(X_1) \setminus N_G[\{x_1, \dots, x_4\}]$ and $Y_2 = N_G(X_2) \setminus N_G[\{y_2, \dots, y_5\}]$. Consider the red-blue graph $G_1 = G[X_1 \cup X_2 \cup Y_1 \cup Y_2]$, where the set of red vertices $R = X_1 \cup X_2$ and the set of blue vertices $B = Y_1 \cup Y_2$. Let $n_1 = |V(G_1)|$. Let D be a minimal connected dominating set of G such that $\{u, v\}$ is a dominating pair of $H = G[D]$ and P is a shortest (u, v)-path in H. By Lemma 13, $D \subseteq N_G[V(P)]$ and $D' = D \setminus V(P) \subseteq X_1 \cup X_2$ is a red dominating set of G_1 that dominates all blue vertices and by minimality, D' is a minimal red dominating set. By Corollary 9, there are at most $1.3803^{n_1} \leq 3^{n_1/3}$ minimal red dominating sets in G_1, and they can be enumerated in time $O(3^{n_1/3})$.

Let $G_2 = G - (V(G_1) \cup \{x_1, \dots, x_4\} \cup \{y_2, \dots, y_5\})$ and let $n_2 = |V(G_2)|$. By Lemma 13, the (v_5, v_{k-4})-subpath of P is an induced (x_5, y_1)-path in G_2. By Lemma 10, there are at most $3^{(n_2-2)/3}$ such paths, and they can be enumerated in time $O^*(3^{n_2/3})$.

Since $D = D' \cup V(P)$, we obtain that there are at most $3^{n_1/3} \cdot 3^{(n_2-1)/3} \leq 3^{(n_1+n_2)/3} \leq 3^{n/3}$ minimal connected dominating sets D with the dominating pair $\{u, v\}$ in $H = G[D]$ and such that H has a shortest (u, v)-path $P = v_1 \dots v_k$ such that $v_i = x_i$ for $i \in \{1, \dots, 5\}$ and $v_i = y_{i+5-k}$ for $i \in \{k-4, \dots, k\}$. Moreover, these sets can be enumerated in time $O(3^{n/3})$. It follows that there are at most $3^{n/3} \cdot n^{10}$ minimal connected dominating sets D such that $H = G[D]$ has a dominating pair $\{u, v\}$ with $\text{dist}_G(u, v) \geq 9$, and these sets can be enumerated in time $O(3^{n/3} \cdot n^{10})$.

We conclude that G has at most $3^{n/3} \cdot (n^{10} + n^9)$ minimal connected dominating sets that can be enumerated in time $O^*(3^{n/3})$. ◀

Proposition 1 implies that the bound for AT-free graphs is tight up to a polynomial factor.

6 Strongly chordal graphs

A vertex u of a graph G is *simple* if for any two neighbors x and y, $N_G[x] \subseteq N_G[y]$ or $N_G[y] \subseteq N_G[x]$. In other words, the closed neighborhoods of the neighbors of u are linearly ordered by inclusion. An ordering $v_1, \dots, v_n$ of $V(G)$ is a *simple elimination ordering* if for each $i \in \{1, \dots, n\}$, v_i is a simple vertex of $G[\{x_i, \dots, x_n\}]$.

▶ **Lemma 15** ([11]). *A graph is strongly chordal if and only if it has a simple elimination ordering.*

▶ **Theorem 16.** *A strongly chordal graph has at most $3^{n/3}$ minimal connected dominating sets, and these set can be enumerated in time $O^*(3^{n/3})$.*

Proof. We consider the following EnumCDS(H, X) algorithm that for a connected induced subgraph H of G and a set of vertices $X \subseteq V(G)$ enumerates the minimal connected dominating sets D of G such that $X \subseteq D$, $D \cap (V(G) \setminus V(H)) = X \cap (V(G) \setminus V(H))$ and $D \cap V(H)$ is a connected dominating set of H. This is a branching algorithm based on the property that any strongly chordal graph has a simple vertex by Lemma 15. If G is disconnected, then G has no connected dominating set. Assume that G is connected.

EnumCDS(H, X)

1. If $X \cap V(H)$ is a connected dominating set of H, then return X if X is a minimal connected dominating set of G and stop.
2. If H is a complete graph, then for each $v \in V(H)$, return $X \cup \{v\}$, and stop.
3. Consider a simple vertex $u \in V(H)$, and for each $v \in N_H(u)$, let $H' = H - (N_H[u] \setminus \{v\})$, $X' = X \cup \{v\}$ and call EnumCDS(H', X').

We call EnumCDS$(G, \emptyset)$ to enumerate minimal connected dominating sets of G.

The correctness of the algorithm is proved via the following claim, whose proof is given in the appendix.

Claim (∗). *Suppose that D is a minimal connected dominating set of G, $X \subseteq D$ and H is a connected induced subgraph G such that*

(i) *$D \cap (V(G) \setminus V(H)) = X \cap (V(G) \setminus V(H))$ and X dominates $V(G) \setminus V(H)$,*
(ii) *any vertex w of H dominated by X in G is dominated by $X \cap V(H)$ in H and, moreover, if w is dominated by a vertex of a component F of $G[X]$, then w is dominated by $V(F) \cap V(H)$,*
(iii) *for any component F of $G[X]$, $V(H) \cap V(F) \neq \emptyset$ and $G[V(F) \cap V(H)]$ is a component of $G[X \cap V(H)]$.*

Then

(a) *if $X \cap V(H)$ is a connected dominating set of H, then $X = D$,*
(b) *otherwise, if H is a clique, then $D = X \cup \{v\}$ for some $v \in V(H)$,*
(c) *otherwise, if u is a simple vertex of H, then there is $v \in N_H(u)$ such that $X' = X \cup \{v\} \subseteq D$ and (i)–(iii) are fulfilled for $H' = H - (N_H[u] \setminus \{v\})$ and X'.*

Observe that the conditions (i)–(iii) of Claim (∗) are fulfilled for $H = G$ and $X = \emptyset$. Applying Claim (∗) recursively, we obtain that for any minimal connected dominating set D of G, EnumCDS$(G, \emptyset)$ outputs D at least once, i.e., EnumCDS$(G, \emptyset)$ enumerates the minimal connected dominating sets.

To obtain an upper bound for the number of minimal connected dominating sets, it is sufficient to upper bound the number of leaves in the search tree produced by EnumCDS. Notice that when we perform branching on Step 3 of EnumCDS for a simple vertex u with $d = d_H(u)$, we get d branches and in each branch we call EnumCDS for a graph with $|V(H)| - d$ vertices. By standard arguments (see, e.g., [16]), we obtain that the search tree for G has at most $3^{n/3}$ leaves. Hence, G has at most $3^{n/3}$ minimal connected dominating sets.

To complete the proof, notice that it is known that a simple elimination ordering of a strongly chordal graph can be found in polynomial time (see, e.g., [2, 4]). Observe also that for any induced subgraph H of G, the ordering of its vertices induced by the simple elimination ordering for G is a simple elimination ordering. As each step of EnumCDS can be done in linear time, we conclude that EnumCDS runs in time $O^*(3^{n/3})$. ◀

As the class of interval graphs is a subclass of the strongly chordal graphs, the bound $3^{n/3}$ is tight by Proposition 1.

7 Distance-hereditary graphs

First we observe that the number of minimal connected dominating sets of a cograph is polynomial. The proof of Proposition 17 is given in the appendix.

▶ **Proposition 17.** *A cograph G with at least 3 vertices has at most $m = |E(G)|$ minimal connected dominating sets, and these can be enumerated in time $O(m)$.*

Notice that this bound is tight, e.g., for complete bipartite graphs. Now we consider distance-hereditary graphs. First, we need some additional terminology.

Let G be a connected graph and $u \in V(G)$. Denote by $L_0(u), \ldots, L_{s(u)}(u)$ the levels in the breadth-first search (BFS) of G starting at u. Hence for all $i \in \{0, \ldots, s(u)\}$, $L_i(u) = \{v \in V(G) \mid \text{dist}_G(u, v) = i\}$. Clearly, the number of levels in this decomposition is $s(u) + 1$. For $i \in \{1, \ldots, s(u)\}$, we denote by $\mathcal{G}_i(u)$ the set of components of $G[L_i(u) \cup \ldots \cup L_{s(u)}(u)]$, and $\mathcal{G}(u) = \cup_{i=1}^{s(u)} \mathcal{G}_i(u)$. Let $H \in \mathcal{G}_i(u)$ and $B = N_G(V(H))$. Clearly, $B \subseteq L_{i-1}(u)$. We say that B is the *boundary of H in* $L_{i-1}(u)$. For $i \in \{0, \ldots, s(u) - 1\}$, $\mathcal{B}_i(u)$ is the set of boundaries in $L_i(u)$ of the graphs of $\mathcal{G}_{i+1}(u)$ and $\mathcal{B}(u) = \cup_{i=0}^{s(u)-1} \mathcal{B}_i(u)$.

▶ **Lemma 18** ([3, 10])**.** *A connected graph G is distance-hereditary if and only if for any vertex $u \in V(G)$, any $H \in \mathcal{G}(u)$ with the boundary B, the following holds: $N_G(u) \cap V(H) = N_G(v) \cap V(H)$ for $u, v \in B$.*

We also need the next observation that is implicit in [3, 10] but also can be easily proved directly.

▶ **Lemma 19** ([3, 10])**.** *Let G be a connected distance hereditary graph and $u \in V(G)$. Then for any $B_1, B_2 \in \mathcal{B}(u)$, either $B_1 \cap B_2 = \emptyset$ or $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.*

For a graph G and $u \in V(G)$, denote by $\mathcal{B}'(u)$ the set of inclusion minimal sets of $\mathcal{B}(u)$. Notice that by Lemma 19, the sets of $\mathcal{B}'(u)$ are pairwise disjoint if G is distance-hereditary. The enumeration of minimal connected dominating sets for distance-hereditary graphs is based on the following lemma.

▶ **Lemma 20.** *Let G be a distance hereditary graph with at least two vertices and $u \in V(G)$. Then for any minimal connected dominating set D with $u \in D$, $D \subseteq \cup_{B \in \mathcal{B}'(u)} B$ and $|B \cap D| = 1$ for $B \subseteq \mathcal{B}'(u)$.*

Proof. Let $H \in \mathcal{G}(u)$ and let $B \in \mathcal{B}(u)$ be its boundary. If $v \in V(H) \cap D \neq \emptyset$, then $B \cap D \neq \emptyset$, because $G[D]$ has an (u, v)-path and this path has a vertex of B. If $V(H) \cap D = \emptyset$, then D has a vertex v that dominates a vertex of H. Clearly, $v \in B$. We conclude that for each $B \in \mathcal{B}'(u)$, $|B \cap D| \geq 1$.

Since $n \geq 2$, $s(u) \geq 1$ and, therefore, $\mathcal{G}(u) \neq \emptyset$ and $\mathcal{B}'(u) \neq \emptyset$. In particular $\{u\} \in \mathcal{B}'(u)$. Recall that the sets of $\mathcal{B}'(u)$ are pairwise disjoint. Hence, there is a $D' \subseteq D$ such that $D \subseteq \cup_{B \in \mathcal{B}'(u)} B$ and $|B \cap D| = 1$ for $B \subseteq \mathcal{B}'(u)$. If $H \in \mathcal{G}_i(u)$ for $i \in \{1, \ldots, s(u)\}$, then any vertex of its boundary dominates $V(H) \cap L_i(u)$ by Lemma 18. Therefore, for each $H \in \mathcal{G}_i(u)$, the vertices of $V(H) \cap L_i(u)$ are dominated. Hence, D' is a dominating set. Because for each $v \in D'$, $G[D']$ has a (u, v)-path by Lemma 18, D' is a connected dominating set. By minimality, we obtain that $D = D'$. ◀

▶ **Theorem 21.** *A distance-hereditary graph has at most $3^{n/3} \cdot n$ minimal connected dominating sets, and these sets can be enumerated in time $O^*(3^{n/3})$.*

Proof. If G is disconnected, it has no connected dominating set. If $n = 1$, then G has one connected dominating set and $1 \leq 3^{n/3} \cdot n$. Suppose that G is a connected graph and $n \geq 2$. For each $u \in V(G)$, G has at most $\prod_{B \in \mathcal{B}'(u)} |B|$ sets $D \subseteq \cup_{B \in \mathcal{B}'(u)} B$ such that $|D \cap B| = 1$ for $B \in \mathcal{B}'(u)$. By Lemma 20, G has at most $\prod_{B \in \mathcal{B}'(u)} |B|$ minimal connected dominating sets containing u. Because the sets of $\mathcal{B}'(u)$ are pairwise disjoint, $\sum_{B \in \mathcal{B}'(u)} |B| \leq n$. It is well known (see, e.g., [16]) that $\prod_{B \in \mathcal{B}'(u)} |B| \leq 3^{n/3}$ in this case. We obtain that the total number of minimal connected dominating sets is at most

$$\sum_{u \in V(G)} \prod_{B \in \mathcal{B}'(u)} |B| \leq 3^{n/3} \cdot n.$$

It is trivial to enumerate minimal connected dominating sets if G is disconnected or $n = 1$. To enumerate minimal connected dominating sets of a connected graph G with $n \geq 2$, we consider all possible choices of a vertex u. For each u, we perform the breadth-first search from u that can be done in linear time, and in time $O(nm)$ construct $\mathcal{B}'(u)$. Then the sets $D \subseteq \cup_{B \in \mathcal{B}'(u)} B$ such that $|D \cap B| = 1$ for $B \in \mathcal{B}'(u)$ can be enumerated in straightforward way in time $O^*(3^{n/3})$. Hence, the total running time is $O^*(3^{n/3})$. ◀

By Proposition 1 the obtained upper bound for distance-hereditary graphs is tight up to factor n.

8 Open questions

The most challenging question concernes the maximum number of minimal connected dominating sets in an n-vertex graph. No upper bound c^n with $c < 2$ is known; neither such an enumeration algorithm was achieved nor could it be achieved by combinatorics. The large gap between lower bound $3^{n/3}$ and upper bound 2^n is astonishing. Let us mention that it seems unlikely that the maximum number of minimal connected dominating sets in an n-vertex graph is upper bounded by $3^{n/3}$. If this were the case and the upper bound could be obtained by an enumeration algorithm, then this would drastically improve the running time of the best algorithm solving the minimum connected dominating set problem from $O(1.8619^n)$ to $O(1.4423^n)$.

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