UNIVERSITY OF BERGEN
Faculty of Mathematics and Natural Sciences

Master's Thesis in Topology

Higher Hochschild homology is not a stable invariant

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Spring 2016

difficilesque symmetriarum
quaestiones geometricis rationibus
et methodis inveniuntur.

Marcus Vitruvius Pollio,
De architectura,
Liber Primum, Caput Primum

Acknowledgements

First of all, I want to thank my supervisor, Bjørn Ian Dundas for his indefatigable help and support during the whole project. I am also grateful to him for proposing me the topic of this thesis.

I heartily thank the topology research group for the positive and stimulating environment they created. I am particularly grateful to Anders Husebø for sharing with me his early results and to Stefano Piceghello for providing me helpful tips and suggestions in the writing of the thesis.

I am most thankful to my family, who always supported me, despite the distance. And to Francesca Azzolini, who always supported me, despite the closeness.

Contents

Foreword

One way we can interpret the Hochschild homology $HH_*(A)$ of a commutative k-algebra A is as the homology of the composition of the simplicial circle $S^1 : \Delta^{\mathrm{op}} \to \mathbf{Fin}$ with the non-pointed Loday functor $X \to \bigotimes_X A$. A motivation for studying the Hochschild homology comes from the relation it has with K-theory. In particular, there is a map $\tau : K_*(A) \to HH_*(A)$, called the Dennis trace map, that can detect some of the properties of the K-theory of A ([Goo86]).

Pirashvili, in [Pir00], studies the particular case for $X = S^n$, which he calls the n-th order Hochschild homology. The case for the simplicial torus, $X = T^n$ is of particular interest, as we will see below. We we will refer to it as iterated Hochschild homology. In this thesis we will study how the n-th order Hochschild homology is related to the n-th iterated Hochschild homology.

While the n-th order Hochschild homology is understood, we cannot say the same for the n-th iterated Hochschild homology. For some commutative algebras A, the higher Hochschild homology factors through the stable homotopy category. For such algebras, we can write the n-th iterated Hochschild homology as a tensor product of some i-th order Hochschild homology. As an example the homology over T^2 is isomorphic the the tensor of the homology over $S^1 \vee S^1 \vee S^2$.

There is a topological version, THH, of the Hochschild homology, which is also related via a Dennis trace map to K-theory

$$D : K_*(A) \to THH(A).$$

Rognes' Red-shift conjecture ([Gui08]) claims that K-theory transforms some n-chromatic type algebras to $(n+1)$-chromatic type algebras. So, it comes natural to study $K^n(A)$, the iterated K-theory of an algebra and, consequently, $THH^n(A)$, the iterated topological Hochschild homology. Even though the topological Hochschild homology does not capture the chromatic shift, every such shift detected for K has been actually detected also in the fixed points of the action of the circle on the topological Hochschild homology ([DGM13, Section 7.3]).

We will, therefore, study the iterated Hochschild homology and its equivariant structure as a toy model, in order to shed some lights on the limits of the iterated topological Hochschild homology.

The thesis is structured as follows:

In Chapter 1 we will recall the basic notions in algebraic topology, homological algebra and simplicial methods required further on in the thesis.

In Chapter 2 we will give the construction for the higher Hochschild homology with special consideration to the iterated Hochschild homology and to some of its properties.

In Chapter 3 we will study the higher Hochschild homology over the spheres. We will start by recalling some results from Pirashvili concerning smooth finite type algebras A over a field of characteristic 0. The problem has already been analysed for n an odd number; we will improve these results, giving a description of the n-th order Hochschild homology of A for n even, using the Greenlees spectral sequence.

In Chapter 4 we will see why the higher Hochschild homology of a symmetric algebra is stably invariant. Moreover, we will adapt Lundervold's results ([Lun07]) to study the case of the dual numbers, in which case the higher Hochschild homology is not a stable invariant. We will discuss in detail the problem for the homology up to degree two. For higher degrees, the lack of flatness conditions makes it difficult to get explicit formulas. However, the unstability features will be made evident in every homology group when working with homology with coefficients.

In Chapter 5, after recollecting some notions about group homology, we will present the description of the homotopy orbits of some examples of groups acting on T^2. In particular, we will focus on $GL_2(\mathbb{Z})$. We will also be explicit about the infinite order cyclic subgroup of $GL_2(\mathbb{Z})$, generated by

$$B := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$$

since it is troublesome to study in positive characteristic.

Chapter 1

Basic notions

In this chapter we give the basic notations, definitions and properties of the tools that we are going to use in the rest of the thesis.

1.1 Some notions in algebra

Definition 1.1. Let k be a ring with unity. A (left) **k-module** $(A, +)$ is an abelian group together with an operation $\cdot : k \times A \to A$ satisfying the following properties:

1. $1 \cdot a = a$,

2. $(x + y) \cdot a = x \cdot a + y \cdot a$,

3. $(xy) \cdot a = x \cdot (y \cdot a)$,

4. $x \cdot (a + b) = x \cdot a + x \cdot b$

for all $x, y \in k$ and $a, b \in A$. A **morphism of k-modules** (also called k-linear maps) is a group homomorphism $f : A \to B$ such that $f(x \cdot a) = x \cdot f(a)$.

A right module is defined similarly, except that the ring k acts on the right instead of acting on the left.

If a k-module is equipped with a suitable multiplication, we call it a k-algebra.

Definition 1.2. Let k be a ring. A **k-algebra** A is a k-module together with a product $\cdot : A \times A \to A$, satisfying:

1. $(a+b)\cdot c = a\cdot c + b\cdot c$,
2. $a\cdot(b+c) = a\cdot b + a\cdot c$,
3. $(xa)\cdot(yb) = (xy)(a\cdot b)$

for all $a, b, c \in A$ and $x, y \in k$.

We say that a k-algebra A is **commutative** if $a\cdot b = b\cdot a$ for all a and b. We say that A has a unity if there exists $e \in A$ such that $a\cdot e = e\cdot a = a$.

From now on, every time we introduce a ring k we mean a commutative ring with unity; every time we introduce a k-algebra we mean a commutative k-algebra with unity, except when explicitly stated otherwise.

Definition 1.3. Given two k-modules A, B we can construct their **tensor product** $A \otimes_k B$ over k, defined as the unique (up to isomorphisms) $k-$module with the following universal property: for each k-bilinear map $f : A \times B \to P$, where P is a k-module, there exists a unique k-linear map $\tilde{f} : A \otimes_k B \to P$ such that the diagram

$$\begin{tikzcd} A \times B \arrow[r] \arrow[dr, "f"'] & A \otimes_k B \arrow[d, dashed, "\tilde{f}"] \\ & P \end{tikzcd}$$

commutes.

The existence of a tensor product between modules, together with its basic properties (such as Proposition 1.4) can be found in [AM69].

Proposition 1.4. Let A, B be two k-modules. Then every element $x \in A \otimes_k B$ has the following form:

$$x = \sum_{\text{finite}} a_i \otimes b_j$$

for some $a_i \in A, b_j \in B$. In particular, the set $\{a \otimes b \mid a \in A, b \in B\}$ generates $A \otimes_k B$ as a k-module.

Notation

In general we will avoid to write the ring under the symbol $\otimes$ when it is clear which ring we are tensoring over.

It will often happen that we have a module tensored with itself several times. In such situations we will write $A^{\otimes n} := A \otimes \cdots \otimes A$.

Several times we are going to deal with k-linear maps $\psi : A^{\otimes n} \to A^{\otimes m}$. Such maps are determined by their behaviour on the generators of $A^{\otimes n}$ and $A^{\otimes m}$. In this situation we will write $(a_1, \ldots, a_n)$ in place for $a_1 \otimes \cdots \otimes a_n$ in order to maintain a light notation.

Given a k-module M, there are several important k-algebras we can build from it. The following definitions take into consideration some of them.

Definition 1.5. We define the **tensor algebra** of M as the k-algebra, whose underlying module is $T(M) := k \oplus M \oplus M^{\otimes 2} \oplus M^{\otimes 3} \oplus \ldots$. We write a generator of $M^{\otimes n}$, $(m_1, \ldots, m_n)$ as $m_1 \ldots m_n$. The multiplicative structure is the multilinear extension of $- \cdot - : V^{\otimes n} \times V^{\otimes s} \to V^{\otimes n+s}$, given by concatenation:

$$(m_1 \ldots m_n) \cdot (m'_1 \ldots m'_s) := (m_1 \ldots m_n m'_1 \ldots m'_s).$$

The **symmetric algebra** of M is the k-algebra whose underlying k-module is $S(M) := k \oplus (M/\sim) \oplus (M^{\otimes 2}/\sim) \oplus \ldots$, where the relations are generated by $(m_1, \ldots, m_n) \sim (m'_1, \ldots, m'_n)$ if and only if there is a permutation $\sigma \in \Sigma_n$ such that $m_{\sigma(i)} = m'_i$ for all i. The product is, again, given the multilinear extension of the concatenation.

The **exterior algebra** of M is the k-algebra whose underlying k-module is $E(M) := k \oplus (M/\sim) \oplus (M^{\otimes 2}/\sim) \oplus \ldots$, where the relations are generated by $(m_1, \ldots, m_n) \sim \operatorname{sgn}(\sigma)(m_{\sigma(1)}, \ldots, m_{\sigma(n)})$ for a permutation $\sigma \in \Sigma_n$. Once again, the product is given by multilinear extension of the concatenation.

If we add the hypothesis that the k-module M is graded, we can perform another important construction on it.

Definition 1.6. We say that a ring k is **graded** if it is possible to write it as direct sum of rings $k = k_0 \oplus k_1 \oplus \ldots$ such that $k_i \cdot k_j \subset k_{i+j}$ for all i and j.

Similarly, for a graded ring k, a **graded module** over k is a k-module M such that $M = M_0 \oplus M_1 \oplus \dots$, where all M_i are k-modules and the following inclusion must hold: for all i and j, $k_i \cdot M_j \subset M_{i+j}$.

Let M be a graded module, we define $M^{\mathrm{ev}} := \bigoplus_{n \geq 0} M_{2n}$ and $M^{\mathrm{odd}} := \bigoplus_{n \geq 0} M_{2n+1}$. The **graded symmetric algebra** of M is the k-algebra given by the tensor product:

$$\Lambda(M) := S(M^{\mathrm{ev}}) \otimes_k E(M^{\mathrm{odd}}).$$

1.2 Chain complexes, chain maps and chain homotopies

Definition 1.7. Let k be a commutative ring. A **chain complex** of k-modules C_* is a sequence of k-modules and k-linear maps of the form:

$$... \to C_n \overset{\partial_n}{\to} C_{n-1} \overset{\partial_{n-1}}{\to} \dots \overset{\partial_1}{\to} C_0 \overset{\partial_0}{\to} C_{-1} \overset{\partial_{-1}}{\to} ...$$

such that $\partial_n \circ \partial_{n-1} = 0$, for every n. The maps ∂_n are called **boundary maps**. If it is clear which ones of them we are referring to, we will omit the indices.

Proposition 1.8. Let C_* be a chain complex such that ∂_n is an isomorphism. Then ∂_{n+1} is the 0 map.

Proof. If ∂_n is an isomorphism then its kernel is trivial. By definition, we have that $\operatorname{Im} \partial_{n+1} \subset \ker \partial_n$. Hence, the image of ∂_{n+1} is 0. □

On chain complexes we can perform several constructions. In particular we have morphisms and equivalences.

Definition 1.9. Let C_*, D_* be chain complexes of k-modules. A **chain map** from C_* to D_* is a collection of k-linear maps $f := \{f_n : C_n \to D_n\}$ so that the following diagrams commute for each n:

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "\partial"'] & D_n \arrow[d, "\partial"] \\ C_{n-1} \arrow[r, "f_{n-1}"'] & D_{n-1} \end{tikzcd}$$

We say that two chain maps $f, f' : C_* \to D_*$ are **chain homotopic** if there is a collection of k-linear maps $T := \{T_n : C_n \to D_n\}$ so that:

$$\partial^D_{n+1} T_n + T_{n-1}\partial^C_n = f_n - f'_n$$

where ∂^X is the boundary map of the chain complex X_*. We will call the map T a **chain homotopy** between f and f'.

This is an equivalence relation (see, e.g. [Hat02]), which we will denote with $f \sim f'$.

We say that two chain complexes C_*, D_* are **chain homotopic** if there are chain maps $C_* \underset{g}{\overset{f}{\rightleftarrows}} D_*$ such that $g \circ f$ is chain homotopic to Id_{C_*} and $f \circ g$ is chain homotopic to Id_{D_*}.

Definition 1.10. Given a commutative ring k and two k-chain complexes C_* and D_*, we define the **tensor product of chain complexes** over the ground ring k as

$$(C_* \otimes D_*)_n := \bigoplus_{i+j=n} C_i \otimes D_j$$

with boundary maps

$$\partial^{C\otimes D}(c_i \otimes d_j) := \partial^C(c_i) \otimes d_j + (-1)^i c_i \otimes \partial^D d_j.$$

This construction behaves well with respect to the relation of being chain homotopic, as shown in the following proposition.

Proposition 1.11. Consider two chain maps $C_* \underset{f'}{\overset{f}{\rightrightarrows}} C'_*$ and a chain complex D. If $f \sim f'$, then $f \otimes \mathrm{Id}_D \sim f' \otimes \mathrm{Id}_D$.

Proof. We need to find a set of maps $\{\tilde{T}_n\}$ such that $\partial^{C'\otimes D}\tilde{T} + \tilde{T}\partial^{C\otimes D} = f \otimes \mathrm{Id}_D - f' \otimes \mathrm{Id}_D$.

Call $\{T_n\}$ the maps realizing the chain homotopy between f and f'. Define $\tilde{T}_n(c_i \otimes d_j) := T(c_i) \otimes d_j$. With this definition, we get:

$$\begin{aligned}
&\tilde{T}\partial(c_i \otimes d_j) + \partial\tilde{T}(c_i \otimes d_j) = \\
&= (\partial T c_i) \otimes d_j + (-1)^{i+1} T(c_i) \otimes (\partial d_j) + (T\partial c_i) \otimes d_j + (-1)^i T c_i \otimes (\partial d_j) = \\
&= ((\partial T + T\partial)(c_i)) \otimes d_j = (f - f')(c_i) \otimes \mathrm{Id}(d_j),
\end{aligned}$$

as we wanted to prove. □

Theorem 1.12 (Künneth formula)**.** *Let k be a commutative ring and let C_* and D_* be chain complexes of k-modules. If C_n and D_n are flat for all n over k and if the cycles $\ker(\partial^C)$ and $\ker(\partial^D)$ are flat over k as well, then there is a short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(D_*) \to H_n(C_* \otimes D_*) \to$$
$$\to \bigoplus_{p+q=n-1} \operatorname{Tor}_1^k(H_p(C_*), H_q(D_*)) \to 0.$$

Corollary 1.13. Under the additional hypothesis that $H_n(C_*)$ or $H_n(C_*)$ are projective as k module for each n, there is an isomorphism

$$\bigoplus_{p+q=n} H_p(C_*) \otimes H_q(D_*) \cong H_n(C_* \otimes D_*).$$

Definition 1.14. We say that a chain complex C_* is **exact** at C_n if $H_n(C_*) = 0$.

1.3 Simplicial Sets

Consider the category Δ with, as objects, the finite ordered sets $[n] := \{0 < \cdots < n\}$ and, as morphisms, the order-preserving functions between them.

Definition 1.15. The category **SSet** is the category with, as objects, the covariant functors from $\Delta^{\mathrm{op}} \to$ **Set** and, as morphisms, the natural transformations between them.

Another way of defining a simplicial set is the following. A simplicial set is a family of sets $\{X_n\}_{n\in\mathbb{N}}$ together with functions of sets: $d_i : X_n \to X_{n-1}$, $s_j : X_n \to X_{n+1}$ for $i = 0, \dots, n$, satisfying the simplicial identities:

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j, \\ \mathrm{Id} & \text{for } i = j, j+1, \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases}$$
$$d_i d_j = d_{i-1} d_i \qquad \text{for } i < j,$$
$$s_i s_j = s_j s_{i-1} \qquad \text{for } i > j.$$

We call **face maps** the maps d_i and **degeneracy maps** the maps s_j.

More generally, a **simplicial object** in a category C is a functor $\Delta^{\mathrm{op}} \to C$; a morphism between two such is a natural transformation between them.

Example 1.16 (Standard simplex)**.** The **standard n-simplex** $\Delta[n]$ has as set of m-simplices:

$$\Delta[n]_m := \Delta([m],[n]) = \{[m] \to [n] \in \Delta\}$$

Face and degeneracy maps are defined by precomposition with the maps of finite ordered sets d^i and s^j, where $d^i : [n-1] \to [n]$ omits the i-th coordinate and $s^j : [n+1] \to [n]$ takes the value i twice. It will happen that we will refer to simplicial sets as **spaces**.

A **subspace** A of a simplicial set X is a simplicial set together with a map of simplicial sets $i : A \to X$, such that $i_n : A_n \to X_n$ is an inclusion of sets.

Example 1.17 (Boundary)**.** The boundary $\partial\Delta[n]$ of $\Delta[n]$ is the greatest subspace of $\Delta[n]$ not containing the simplex $\mathrm{Id}_{[n]}$.

Example 1.18 (Horns)**.** The k-th horn $\Lambda^k[n]$ is the greatest subspace of $\Delta[n]$ not containing $d^k \in \Delta([n-1],[n])$.

Example 1.19 (Products)**.** Let X, Y be simplicial sets. The product $X \times Y$ is the simplicial set whose set of n-simplices is the cartesian product $X_n \times Y_n$ and whose structure maps are built componentwise:

$$d_i(x,y) = (d_i(x), d_i(y)), \qquad s_j(x,y) = (s_j(x), s_j(y))$$

for $x \in X_n$ and $y \in Y_n$.

Remark 1.20. More generally, if I is a small category and $X : I \to \mathbf{SSet}$ is a functor, then the limit $\lim_I X$ is the space whose set of n-simplices is the set $\lim_I X_n$ and whose structure maps come from functoriality.

Similarly the colimit $\operatorname{colim}_I X$ is the space whose n-simplices are the elements of the set $\operatorname{colim}_I X_n$ and with structure maps coming from functoriality.

Example 1.21 (Simplicial spheres)**.** The simplicial model for the n-sphere in **SSet** is given by the quotient $\Delta[n]/\partial\Delta[n]$. Notice that an n-sphere has one 0-simplex, no non-degenerate m-simplices for $n \neq m > 0$ and one non-degenerate n-simplex.

Example 1.22 (Singular complex)**.** Let X be a topological space. Let Δ_n be the topological n-simplex given by

$$\Delta_n := \{(t_0, \dots, t_n) \in \mathbb{R}^n \mid t_0 + \dots + t_n = 1, t_i \geq 0 \ \forall i\}.$$

The **singular complex** $\operatorname{sing} X$ is the simplicial set whose n-simplices are $(\operatorname{sing} X)_n := \{f : \Delta_n \to X \mid f \text{ is continuous}\}$.

Example 1.23 (Classifying space)**.** Let G be a group. The **classifying space** of G is the simplicial set BG whose set of n-simplices is $BG_n := G^n$ and whose structure maps are:

$$d_i(a_1, \dots, a_n) := \begin{cases} (a_2, \dots, a_n) & \text{for } i = 0, \\ (a_1, \dots, a_i a_{i+1}, a_{i+2}, \dots, a_n) & \text{for } 0 < i < n, \\ (a_1, \dots, a_{n-1}) & \text{for } i = n \end{cases}$$
$$s_j(a_1, \dots, a_n) := (a_1, \dots, 1, a_j, \dots, a_n).$$

Definition 1.24. A **pointed simplicial set** (X, x) is a simplicial set X, together with a 0-simplex $x \in X_0$. Let Y be another pointed simplicial set with basepoint y. A **morphism of pointed simplicial sets** is a simplicial map $f : X \to Y$ such that $f(x) = y$.

In order to keep the notation light, if it is not important which zero-simplex of a simplicial set is its base point, we will omit it in the definition of the pointed simplicial set.

Example 1.25 (Wedge sum)**.** Let (X, x), (Y, y) be pointed simplicial sets. The **wedge sum** $X \vee Y$ is the pushout of the following solid diagram:

$$\begin{tikzcd} \text{point} \arrow[r, "x"] \arrow[d, "y"'] & X \arrow[d, dashed] \\ Y \arrow[r, dashed] & X \vee Y \end{tikzcd}$$

Example 1.26 (Suspension of a space)**.** Let X be a pointed simplicial set. The **suspension** of X is the quotient:

$$\Sigma X := \frac{X \times S^1}{X \vee S^1}$$

Definition 1.27. We say that a map of simplicial sets $f : X \to Y$ is a **Kan fibration** if for every commutative square of the form

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] & Y \end{tikzcd}$$

there is a lifting, i.e. a map $s : \Delta[n] \to X$ such that the resulting diagrams commute. A space X is called **fibrant** if the unique map $X \to *$ is a Kan fibration.

Proposition 1.28. The underlying simplicial set of a simplicial abelian group is fibrant.

Remark 1.29. The functor sing: **Top** $\to$ **SSet**, defined in Example 1.22 has an adjoint, called the **geometric realization**. The geometric realization gives a recipe to build a CW-complex from a simplicial set.

Model Categories

SSet is a useful category for our purposes. We saw that we can realize them as CW-complexes. We need, though, a way to describe homotopies.

Definition 1.30. Let X and Y be simplicial sets. We say that $H : X \times \Delta[1] \to Y$ is a **simplicial homotopy** from f to g, where $f : X \cong X \times \Delta[0] \xrightarrow{d^0} X \times \Delta[1] \xrightarrow{H} Y$ and $g : X \cong X \times \Delta[0] \xrightarrow{d^1} X \times \Delta[1] \xrightarrow{H} Y$.

We say that a map $f : X \to Y$ is a **homotopy equivalence** if there exists $g : Y \to X$ such that there is a simplicial homotopy from Id_X to gf and one from Id_Y to fg.

Simplicial homotopies do not form, a priori, equivalence relations. In particular they do not satisfy the symmetry axiom. Moreover, for a general map of simplicial sets, it is unlikely to be a homotopy equivalence, even if its geometric realization is. In order to solve this problem, we need to introduce the notion of *weak equivalence*, that will coincide with the notion of homotopy equivalence when we restrict to a certain class of spaces.

We will then give to **SSet** the structure of a model category.

Definition 1.31. Let $\mathcal{C}$ be a category with all small limits and colimits. We say that $\mathcal{C}$ is a **model category** if it has three subcategories: cof$\mathcal{C}$ (called **cofibrations**), fib$\mathcal{C}$ (called **fibrations**) and w$\mathcal{C}$ (called **weak equivalences**), satisfying the following properties:

1. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{C}$. If at least two among f, g, gf are weak equivalences, so is the third.

2. The subcategories cof$\mathcal{C}$, fib$\mathcal{C}$ and w$\mathcal{C}$ are closed under retraction.

3. Consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ Z \arrow[r, "g"'] & W \end{tikzcd}$$

   If p is a weak equivalence and a cofibration (usually called **trivial cofibration**) and q is a fibration, then there is a lifting $s : Z \to Y$ making the two triangles commute. If p is a cofibration and q is a fibration and a weak equivalence (usually called **trivial fibration**), then there is a lifting $t : Z \to Y$ making the two triangles commute.

4. For each morphism in $\mathcal{C}$ there are two functorial factorizations:

$$f = q(f)i(f),$$
$$f = p(f)j(f)$$

   where $q(f)$ is a trivial fibration and $i(f)$ is a cofibration, and $p(f)$ is a fibration and $j(f)$ is a trivial cofibration.

We say that an object c in $\mathcal{C}$ is **fibrant** if the map from c to the final object in $\mathcal{C}$ is a fibration.

The following result, stated and proven in [GJ09], gives a model structure to **SSet** and underline the key role played by fibrant spaces for homotopical purposes.

Proposition 1.32. The category **SSet** is a model category with the following subcategories:

- $f : X \to Y$ is a fibration if it is a Kan fibration.
- $f : X \to Y$ is a cofibration if for each n, the underlying map of sets $f_n : X_n \to Y_n$ is an injection.
- $f : X \to Y$ is a weak equivalence if for all fibrant simplicial sets K, the induced function $\mathrm{Hom}_{\mathbf{SSet}}(Y, K) \to \mathrm{Hom}_{\mathbf{SSet}}(X, K)$ is a bijection.

Proposition 1.33. Let $f : X \to Y$ be a map of fibrant spaces. f is a weak equivalence if and only if it is a homotopy equivalence.

Definition 1.34. Let X be a fibrant simplicial set with a vertex $x \in X_0$. The n-th **homotopy group** of X with base point in x is the set of equivalence classes $\pi_n(X, x) := \{t : S^n \to X \mid t_n(\Delta[n]) = s^n(x)\}/\sim$, where $t \sim t'$ if they are homotopic relatively to the boundary.

The Moore complex

Given a simplicial set X, we can form the **Moore chain complex** $\overline{Ch(X)}$ in the following way:

$$\overline{Ch(X)} := \{... \to X_n \overset{\partial_n}{\to} X_{n-1} \to ... \to X_0\}$$

with the maps ∂_n given by $\partial_n = \sum_{i=0}^{n}(-1)^i d_i$, where the d_i's are the face maps of the simplicial set.

There is another way of building a chain complex from a particular simplicial set X.

Definition 1.35. Let X be a simplicial object in $A^{\Delta^{\mathrm{op}}}$, where A is an abelian category. We define the **normalized complex** of X as the chain complex $Ch(X)$, given by

$$Ch(X) : \dots \{... \to Y_n \overset{\partial^n}{\to} Y_{n-1} \to ... \to Y_0\}$$

where $Y_n := \bigcap_{i=1}^{n} \ker d_i$, the intersection of the kernels of the last n face maps. The maps ∂_n are still given by the alternating sums of the face maps.

Notice that:

- The constructions we performed are always chain complexes, since $\partial_{n-1}\partial_n = 0$ is a formal property of the alternating sums of the face maps (indeed they satisfy the simplicial identities).

- The converse is, in general, not true: given a positive chain complex C, it is not possible to determine a simplicial set whose chain complex is C.

From the last remark we see that we cannot always reverse the process that brings us from a simplicial set to the chain complex associated to it. There are, however classes of simplicial sets with this property: for example, the underlying simplicial set of a simplicial abelian group.

The following theorem, stated and proved in [DP61] describes how the notions of chain complex and of simplicial abelian group are related to each other.

Theorem 1.36 (Dold-Puppe Theorem)**.** *Let A be an abelian category. There is an equivalence of categories $A^{\Delta^{\mathrm{op}}} \leftrightarrow Ch^+(A)$, where $Ch^+(A)$ is the category of the chain complex with zeros in the negative degree part and A-chain morphisms.*

In particular the functor $A^{\Delta^{\mathrm{op}}} \to Ch^+(A)$ takes a simplicial object of A to its normalized chain complex.

Set is not an abelian category, but, for example, **Ab** is. So, if we consider simplicial abelian groups, the previous theorem holds and it will be the same for us to consider either the simplicial structure or the chain structure.

Notice that the Moore chain complex and the normalized chain complex differ as chain complexes, but they are quasi-isomorphic, as described in the following remarks. Let us consider the case of **Ab**.

Remark 1.37. As shown in [GJ09, Theorem III 2.4], the inclusion of the normalized chain complex in the Moore complex for a simplicial abelian group G,

$$Ch(G) \hookrightarrow \overline{Ch}(G)$$

is a natural chain homotopy equivalence and therefore the homology of the two chain complexes is the same.

Remark 1.38. Again in [GJ09, Corollary III 2.7] it is stated that, given a simplicial abelian group G, there are isomorphisms:

$$H_n(Ch(G)) \cong \pi_n(G, 0).$$

These isomorphisms are natural in G.

1.4 The Hochschild homology and the Kähler differentials

Let k be a commutative ring with unity and let A be a unitary, associative and commutative k-algebra. Consider the following chain complex, called the **Hochschild complex**:

$$\overline{Ch}(A) := \cdots \to A \otimes A^{\otimes n} \xrightarrow{b} A \otimes A^{\otimes n-1} \to \ldots A \to 0$$

where the map b does the following:

$$b(a_0, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (a_n a_0, \ldots, a_{n-1}).$$

The homology of this chain complex is called the **Hochschild homology** of the algebra A. Several results on the Hochschild homology are listed in [Lod98]; we will state the results we are going to use further on in the thesis regarding this homology theory for algebras.

The Hochschild chain complex presents a wide acyclic subcomplex, namely the one whose homology is trivial. This subcomplex is generated by the so-called **degenerate elements**, the ones with $a_i = 1$ for at least one $i > 0$. We denote this subcomplex by D_*. The chain complex given, in degree n, by $A \otimes A^{\otimes n}/D_n$ is called **normalized Hochschild complex** and is denoted by $Ch(A)$.

Proposition 1.39. ([Lod98, Proposition 1.6.5]) The complex D_n is acyclic and the projection $\overline{Ch}(A) \to Ch(A)$ is a quasi-isomorphism, i.e. it induces an isomorphism in homology.

Remark 1.40. Proposition 1.39 does not hold only for the Hochschild homology. In Theorem 1.36 we saw that every positive chain complex C_* over an abelian category A can be thought as a simplicial object in A. Hence in C_* there will be degenerate simplices which will be acyclic.

Example 1.41. Consider a field k, whose characteristic in not 2, and define $A := k[x]/x^2$. We want to compute the Hochschild homology of this algebra. First of all, consider the normalized Hochschild complex. The module $Ch(A)_n$ is generated by elements of the form $(a + bx, x, \dots, x)$. We identify different cases:

- if n is even,

$$\begin{aligned} b(a + dx, \dots, x) = (ax, x, \dots, x) + (ax, x, \dots, x) = \\ = 2(ax, x, \dots, x); \end{aligned}$$

- if n is odd,

$$b(a + dx, \dots, x) = (ax, x, \dots, x) - (ax, x, \dots, x) = 0.$$

Hence the Hochschild homology of A, denoted by $HH_*(A)$ is given by:

$$HH_n(A) = \begin{cases} A & \text{for } n = 0 \\ k\{(1, x, \dots, x)\} & \text{for } n \text{ odd,} \\ k\{(x, x, \dots, x)\} & \text{for } n \text{ even} \end{cases}$$

Definition 1.42. Let A be a k-algebra. The module of the **Kähler differentials** of A is the left A-module $\Omega^1_{A|k}$, that is the A-module generated by the k-linear symbols da, satisfying the following relations:

- $d(\lambda a + \mu b) = \lambda da + \mu db$
- $d(ab) = a(db) + b(da)$

for all $a, b \in A$ and $\lambda, \mu \in k$.

The Kähler differentials are useful items since they give a concrete description of the first group of the Hochschild homology as stated in the following proposition.

Proposition 1.43. ([Lod98, Proposition 1.1.10]) Let A be a unital and commutative k-algebra. There is an isomorphism $\Omega^1_{A|k} \cong HH_1(A)$.

Proof. Notice that, by commutativity, the map $b : A \otimes A \to A$ is the trivial map. Hence $HH_1(A) = A/\operatorname{Im}(b)$, which is, A divided out by the relation

$$xy \otimes z - x \otimes yz + xz \otimes y = 0.$$

Consider the map $HH_1(A) \to \Omega^1_{A|k}$ sending $(x \otimes y) \mapsto xdy$. This is well-defined by the construction of the Kähler differentials. Its inverse is the map $xdy \mapsto x \otimes y$, which is a cycle. This is also well-defined, since $xd(yz) \mapsto x \otimes yz$ but $xy(dz) + xz(dy) \mapsto xy \otimes z + xz \otimes y$. Therefore, the difference between the two resulting terms is:

$$xy \otimes z - x \otimes xy + xz \otimes y,$$

which is a boundary, hence 0 in the quotient. □

Under certain conditions we can improve this result.

Definition 1.44. Let k be a noetherian ring and A a commutative algebra over k such that A is flat as k-module. We say that A is **smooth** over k if it satisfies the following property: given a k-algebra C, an ideal $I \subset C$ such that $I^2 = 0$ and a k-algebra morphism $\varphi : A \to C/I$, there is a lifting $s : A \to C$ such that

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "s"'] & C/I \\ & C \arrow[u, two heads, "p"'] \end{tikzcd}$$

commutes.

Example 1.45. The polynomial algebra $k[t]$ over a field k is a smooth k-algebra.

The following theorem is stated and proved in [Lod98, Section 3.4].

Theorem 1.46 (Hochschild-Kostant-Rosenberg)**.** *Let k be a field and A a smooth k-algebra which is finitely presented. There is an isomorphism of graded k-algebras:*

$$\psi : \Omega^n_{k|A} \to HH_n(A)$$

where $\Omega^n_{k|A} := \wedge^n_A(\Omega^1_{k|A})$.

The isomorphism is called the antisymmetrization map and is denoted by ε_n.

1.5 Spectral sequences

Spectral sequences are an important tool in algebraic topology. The main source for this section is [McC01, Chapter 2].

We define **differential bigraded module** over a ring k as a collection of k-modules $\{E_{p,q}\}$, for $p, q \in \mathbb{Z}$, and differentials $d_r : E_{p,q} \to E_{p-r,q+r-1}$, for a fixed $r \in \mathbb{Z}$ such that $d_r^2 = 0$. By this property, we are allowed to take the homology:

$$H_{p,q}(E_{*,*}, d_r) := \ker d / \operatorname{Im} d.$$

Definition 1.47. A **spectral sequence** is a collection of differential bigraded k-modules $\{E^r_{p,q}, d^r\}_{r\in\mathbb{N}}$ such that for all p, q, r there is an isomorphism $E^{r+1}_{p,q} \cong H_{p,q}(E^r_{*,*}, d_r)$. The pair $\{E^r_{*,*}, d^r\}$ is called the r-th **page** of the spectral sequence.

We define a **filtration** F_* of a graded k-module A_* to be a family of submodules $\{F_pA_*\}$ with $p \in \mathbb{Z}$ such that

$$A \supset \cdots \supset F_pA_* \supset F_{p-1}A_* \supset \dots .$$

We can obtain a filtration in each degree by considering $F_pA_q := F_pA_* \cap A_q$. Collapsing the filtration, we get a filtered bimodule:

$$E^0_{p,q}(A_*, F_*) := F_pA_{p+q}/F_{p-1}A_{p+q}.$$

Definition 1.48. We say that a spectral sequence $\{E^r_{p,q}, d^r\}$ **converges** to a graded k-module A_*, if there is a filtration F_* on A_* such that

$$E^\infty_{p,q} \cong E^0_{p,q}(A_*, F_*).$$

Chapter 2

The higher Hochschild homology

In this chapter we will give the definition of the higher Hochschild homology of an algebra over a simplicial set. We will explore its properties and look at the case of the simplicial set given by the n-fold product $S^1 \times \cdots \times S^1$. The idea comes from [Pir00]. He studies the pointed version of the Loday functor, while we consider its non-pointed version.

2.1 The construction of the higher Hochschild homology

Let **Fin** be the category with, as objects, the finite sets $\underline{n} := \{1, \ldots, n\} \subset \mathbb{N}$ for all $n \in \mathbb{N}$ and, as morphisms, the functions of sets between them. Call **Fin'** the category of all finite sets. For each object S in **Fin'** we choose an isomorphism $\alpha_S : S \to \underline{|S|}$, whose restriction to **Fin** is the identity. We have therefore built a skeleton for **Fin'**, that makes the following triangle commute.

$$\begin{tikzcd}
\mathbf{Fin} \arrow[r, hook] \arrow[dr, "\mathrm{Id}"'] & \mathbf{Fin'} \arrow[d] \\
& \mathbf{Fin}
\end{tikzcd} \tag{2.1}$$

Let A be a commutative and associative algebra with unity over a commutative ring k. Consider the functor: $\bigotimes_{\bullet} A$: **Fin** $\to$ **k-Alg**, defined as follows. On objects,

$$\bigotimes_{\underline{n}} A := A^{\otimes n}.$$

On the morphisms,

$$\bigotimes_{\varphi} A : \bigotimes_{i\in\underline{n}} a_i \mapsto \bigotimes_{j\in\underline{m}} b_j$$

for $\varphi : \underline{n} \to \underline{m}$, where

$$b_j := \begin{cases} \prod_{i\in\varphi^{-1}(j)} a_i & \text{for } \varphi^{-1}(j) \neq \emptyset, \\ 1 & \text{for } \varphi^{-1}(j) = \emptyset \end{cases}$$

This can be extended by linearity on all $A^{\otimes n}$.

Proposition 2.1. $\bigotimes_S A$: **Fin** $\to$ **k-Alg** is a functor.

Proof. We only need to check that the induced maps are k-linear and that the product and the composition are preserved. Assume that X, Y ,Z are objects in **Fin**.

Consider $\varphi : X \to Y$. Then $\bigotimes_{\varphi} A$ is k-linear by the universal property of the tensor product. As for the product, we have:

$$\bigotimes_{\varphi} A : \bigotimes_{x\in X}(a_x b_x) \mapsto \bigotimes_{y\in Y} \prod_{x\in f^{-1}(y)} a_x b_x = \bigotimes_{y\in Y} \prod_{x\in f^{-1}(y)} a_x \cdot \bigotimes_{y\in Y} \prod_{x\in f^{-1}(y)} b_x.$$

For the last equality we used commutativity of A and the fact that the product is performed componentwise.

About functoriality: let Id : $X \to X$ be the identity in **Fin**. The induced map is:

$$\bigotimes_{Id} A : \bigotimes_{x\in X} a_s \mapsto \bigotimes_{y\in X} \prod_{x=y} a_x = \bigotimes_{y\in X} a_y.$$

Consider two morphisms $\varphi : X \to Y$, $\psi : Y \to Z$ in **Fin**. We get:

$$\bigotimes_{\psi\varphi} A : \bigotimes_{x\in X} a_x \mapsto \bigotimes_{z\in Z} \prod_{x\in(\psi\varphi)^{-1}(z)} a_x.$$

On the other hand,

$$\bigotimes_{\psi} A \bigotimes_{\varphi} A : \bigotimes_{x \in X} a_x \mapsto \bigotimes_{\psi} A \left(\bigotimes_{y \in Y} \prod_{x \in (\varphi)^{-1}(y)} a_x \right) =$$
$$= \bigotimes_{z \in Z} \prod_{y \in \psi^{-1}(z)} \left(\prod_{x \in \varphi^{-1}(y)} a_x \right).$$

Since the product is commutative, we can reorder the terms, obtaining that the product rearranges exactly over $x \in (\psi\varphi)^{-1}(z)$, as we desired. $\square$

We have, therefore, a functor from **Fin** to **k-Alg**. We would like to extend it to **Fin'**. In order to do this, we use the vertical arrow in (2.1) and compose it with $\bigotimes_{\bullet} A$. Since the triangle in (2.1) commutes, we can call without ambiguity the composite functor $\bigotimes_{\bullet} A : \textbf{Fin'} \to \textbf{k-Alg}$.

Remark 2.2. If we equip **Fin** with the unital coproduct $\coprod$, we see that $\bigotimes_{\bullet} A$ behaves well with respect to it. Namely, there is a natural isomorphism $\bigotimes_{\underline{n} \coprod \underline{m}} A \cong \bigotimes_{\underline{n+m}} A$.

Moreover, given a bijection $\varphi : \underline{n} \to \underline{n}$, the map $\bigotimes_{\varphi} A$ is a natural isomorphism of k-algebras.

This construction behaves well also with respect to products, in the sense of the following proposition.

Proposition 2.3. Let X and Y be finite sets. There is an isomorphism

$$\bigotimes_{X \times Y} A \cong \left(\bigotimes_{X} \left(\bigotimes_{Y} A \right) \right),$$

natural in A, X and Y.

Proof. We have:

$$\bigotimes_{X \times Y} A = \bigotimes_{(a,b) \in X \times Y} A \cong \bigotimes_{a \in X} \bigotimes_{b \in Y} A = \left(\bigotimes_{X} \left(\bigotimes_{Y} A \right) \right)$$

where the isomorphism is given by $\bigotimes_{(x,y)} a_{(x,y)} \mapsto \bigotimes_x \bigotimes_y (a_{(x,y)})$. One can easily check naturality. $\square$

Remark 2.4. The functor $\bigotimes_\bullet A$ preserves colimits. Namely, given a functor $F : J \to \mathbf{Set}$, then

$$\bigotimes_{\underset{\to}{\lim} F} A \cong \underset{\to}{\lim} \left(\bigotimes_F A \right).$$

Indeed, in the category of commutative k-algebras, the tensor product is given by the coproduct.

Remark 2.5. In **Set** it is possible to write each infinite set as as a colimit in the following way. Let X be an object in **Set**; then

$$X = \bigcup_{S \subset X,\ S \text{ finite}} S$$

that can also be described as the colimit of the diagram given by the finite subsets of X, with morphisms given by the partial order induced by inclusion. In this way it is possible to extend $\bigotimes_\bullet A : \mathbf{Set} \to \mathbf{k\text{-}Alg}$. The elements of the k-algebra $\bigotimes_X A$ for an infinite set X are strings $(a_1, a_2, \dots)$ of elements $a_i \in A$, where only finitely many of them are different from the unit in A.

We can compose $\bigotimes_\bullet A$ with a simplicial set $X : \Delta^{\text{op}} \to \mathbf{Set}$ and get a simplicial k-algebra $\bigotimes_X A : \Delta^{\text{op}} \to \mathbf{k\text{-}Alg}$. Hence, we can look at $\bigotimes_\bullet A$ as a functor from **SSet** to simplicial k-algebras.

Remark 2.6. Proposition 2.3 and Remarks 2.2 and 2.4 apply also for $\bigotimes_A : \mathbf{SSet} \to \mathbf{Sk\text{-}Alg}$.

A simplicial k-algebras B is, in particular, a simplicial abelian group. Hence, in order to compute the homotopy groups of the underlying simplicial set, we can consider the homology of the normalized complex of B.

Definition 2.7 (Higher Hochschild Homology)**.** Let X be a simplicial set and A a unitary, commutative, associative k-algebra. The **higher Hochschild homology** of A over X is the homology of the normalized chain complex

$$Ch_* \left(\bigotimes_X A \right).$$

The following proposition is a reformulation of [DGM13, Lemma 2.2.1.3].

Proposition 2.8. Let Y, Y' be simplicial sets and A a commutative k-algebra. If $Y \overset{\simeq}{\to} Y'$ is a weak equivalence, then the induced map $\bigotimes_Y A \to \bigotimes_{Y'} A$ is a weak equivalence too.

Hence we have the following.

Corollary 2.9. Let Y, Y' be two weakly equivalent simplicial sets. Then there is an isomorphism

$$H_*\left(\bigotimes_Y A\right) \cong H_*\left(\bigotimes_{Y'} A\right).$$

Proof. Since $\bigotimes_Y A$ and $\bigotimes_{Y'} A$ are weakly equivalent fibrant spaces (because they are simplicial abelian groups), the induced map in homotopy is an isomorphism. By Remark 1.38, we have the stated isomorphism. $\square$

Example 2.10. As first example, we consider the case for $X = S^1$. S^1 has $n+1$ n-simplices, so we can easily write $(\bigotimes_{S^1} A)_n = A \otimes \cdots \otimes A$ ($n+1$ factors).

In order to understand the chain complex structure, we need to study the behaviour of the face maps. These are defined as

$$d_i(a_0, \dots, a_n) = \begin{cases} (a_0 a_1, \dots, a_n) & \text{for } i = 0, \\ (a_0, \dots, a_i a_j, \dots, a_n) & \text{for } 0 < i < n, \\ (a_n a_0, \dots, a_{n-1}) & \text{for } i = n \end{cases}$$

The complex we get in this case is exactly the Hochschild complex, whose homology $H_n(Ch(\bigotimes_{S^1} A)) =: HH_n(A)$ is the usual Hochschild homology.

Example 2.11. Another easy example is given when X is the trivial space $*$. The space $*$ has one n-simplex for each $n \in \mathbb{N}$, so $C_n(\bigotimes_* A) = A$. The boundary maps behave as follows:

$$\partial^n = \begin{cases} \mathrm{Id} & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ even} \end{cases}$$

Hence the homology of the complex is:

$$H_n\left(\bigotimes_* A\right) = \begin{cases} 0 & \text{for } n > 0, \\ A & \text{for } n = 0 \end{cases}$$

As for the Hochschild homology, also for the higher Hochschild homology it is possible to consider coefficients in an A-bimodule M.

We say that M is an A-**bimodule** if it is a left and right A-module with maps $\mu : A \times M \to M$ and $\nu : M \times A \to M$, such that $\mu(a, xa') = \nu(ax, a')$ for all $a \in A$ and $x \in M$.

Definition 2.12. Let k be a commutative ring and let A be a commutative k-algebra. Given an A-bimodule M, we can define the **Loday functor** $\mathcal{L}(A, M) : \mathbf{Fin}_* \to \mathbf{k\text{-}mod}$ as follows:

$$\mathcal{L}(A, M)(\underline{n} \cup \{0\}) := M \otimes A^{\otimes n}.$$

On a morphism $\varphi : \underline{n} \cup \{0\} \to \underline{m} \cup \{0\}$, the functor $\mathcal{L}(A, M)$ acts as follows:

$$\mathcal{L}(A, M)(\varphi) : (a_0, a_1, \dots, a_n) \mapsto (b_0, b_1, \dots, b_m)$$

where $a_0 \in M$, $a_i \in A$ for $i > 0$ and

$$b_j := \begin{cases} \prod_{i \in \varphi^{-1}(j)} a_i & \text{for } \varphi^{-1}(j) \neq \emptyset, \\ 1 & \text{for } \varphi^{-1}(j) = \emptyset \end{cases}$$

Remark 2.13. As we did for the functor $\bigotimes_\bullet A$, we can extend the input of the Loday functor to simplicial sets.

Let X be a pointed simplicial set; we will denote the homology of the simplicial k-module $\mathcal{L}(A, M)(X)$ as $H_*(Ch(\bigotimes_X A); M)$. The reason is explained in the following remark.

Remark 2.14. Choosing $M := A$, we get

$$H_*(Ch(\bigotimes_X A); A) \cong H_*(Ch(\bigotimes_X A)).$$

Example 2.15. Let $X := S^1$. The higher Hochschild homology of an algebra A over X with coefficients in the A-bimodule M is the standard Hochschild homology of A with coefficients in M, i.e. $H_*(A, M)$, as described in [Lod98].

By abuse of notation, we will often refer to $\mathcal{L}(A, A)(X)$ as $\bigotimes_X A$ for a pointed simplicial set X.

Proposition 2.16. Let X and Y be finite pointed simplicial sets. Let A be a k-algebra. There is an isomoprhism of A-algebras

$$\bigotimes_{X \vee Y} A \cong \left(\bigotimes_{X} A \right) \otimes_A \left(\bigotimes_{Y} A \right).$$

The proof follows from Remark 2.4, since pushouts are colimits.

2.2 Iterated Hochschild homology

One of the spaces we are interested in is the n-dimensional torus, T^n. There are several models for T^n in the category of simplicial sets, which are all weakly equivalent. We will describe the two models that we are going to use more often in this thesis.

Definition 2.17. Consider S^1 as the simplicial set given by the quotient $\Delta[1]/\partial\Delta[1]$. We define the **simplicial torus** T^n as the n-fold product $S^1 \times \cdots \times S^1$.

Definition 2.18. Consider the simplicial set $\mathbb{T}^n := B(\mathbb{Z}^n)$, namely the classifying space of the group $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (n copies of the integers). This is another model for the n-dimensional torus.

Remark 2.19. We notice that $\mathbb{T}^n$ is fibrant. Indeed it is the classifying space of an abelian group and therefore it is a simplicial group. On the other hand, T^n is not a fibrant space for $n > 0$.

Lemma 2.20. T^n is weakly equivalent to $\mathbb{T}^n$.

Proof. Since the geometric realization of $B\mathbb{Z}$ is homotopy equivalent to the topological 1-sphere, $B\mathbb{Z}$ is weakly equivalent to S^1. So we only need to show that $B(\mathbb{Z}^n)$ is weakly equivalent (hence, homotopy equivalent) to $B(\mathbb{Z})^n$. By definition of the classifying space BG of a group G, is the nerve of the grupoid with one object and whose set of morphisms is G itself, with the composition given by the product in G. We know that, given two groups G and H, there is an isomorphism $B(G \times H) \cong B(G) \times B(H)$. Choosing $G = H = \mathbb{Z}$ and iterating the process, we get the result. $\square$

In the last part of the proof we saw that we can write $B(\mathbb{Z})\times\cdots\times B(\mathbb{Z})$ for $\mathbb{T}^n$, which is, like T^n, a product. This suggests us that we can use Proposition 2.3 to compute the higher Hochschild homology for $X = T^n$ (or $X = \mathbb{T}^n$; the two models for the simplicial torus are weakly equivalent, hence their homology is the same).

Corollary 2.21. There is an isomorphism of algebras

$$\pi_*\left(\bigotimes_{T^n} A\right) \cong \pi_*\left(\bigotimes_{S^1}\cdots\bigotimes_{S^1} A\right).$$

given by the iteration of the isomorphism in Proposition 2.3.

This is why it is common practice to call the homology of an algebra over the n-torus as the n-th **iterated Hochschild homology**. Indeed $Ch(\bigotimes_{S^1} A)$ is just the Hochschild complex of A.

For this reason, it would fit nicely if, taking the Hochschild complex were an operation closed in some classes of algebras. This is the case for commutative differential graded algebras.

Definition 2.22. A **differential graded Algebra** (or DG algebra) is an algebra A over a field k together with a decomposition $A = \bigoplus_{i\geq 0} A_i$ in k-modules A_i, and a differential $\delta : A_i \to A_{i-1}$. We say that the degree of $|a|$ of a is equal to i if $a \in A_i$. The product μ in A needs to be such that the image of $\mu_{|A_i\times A_j}$ is in A_{i+j}. The differential needs to satisfy two properties

1. $\delta(a\cdot b) = \delta(a)b + (-1)^{|a|}a\delta(b)$, the so-called Leibniz rule;
2. $\delta^2 : A_i \to A_{i-2}$ factors through the 0-module of k.

A DG algebra (A, δ) is said to be **commutative** (we will refer to it as CDGA) if for every $a, b \in A$, we have: $ab = (-1)^{|a||b|}ba$.

We need to equip the Hochschild complex of A, $Ch(A)$ with a graded product. This will be the **shuffle product**, $sh : Ch(A)_n \times Ch(A)_m \to Ch(A)_{m+n}$, given by

$$(a_0, a_1, \ldots, a_n)\times(b_0, b_{n+1}, \ldots, b_{n+m}) \mapsto \sum_\sigma \mathrm{sgn}(\sigma)(a_0b_0, a_{\sigma^{-1}(1)}, \ldots a_{\sigma^{-1}(m+n)})$$

where the sum is performed over all the possible (n, m)-shuffles, namely a permutation in the symmetric group over $m+n$ letters such that $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$ and $\sigma(n+1) < \sigma(n+2) < \cdots < \sigma(n+m)$.

The following statement is [Lod98, Lemma 4.2.2].

Proposition 2.23. The Hochschild boundary b is a graded derivation for the shuffle product, so the Hochschild complex is a differential graded algebra.

We say that a CDGA of the form $(\Lambda V, \delta)$ is called a **free CDGA**. In [Lod98] it is shown that, in characteristic zero, every CDGA (A, δ) is equivalent to a free CDGA $(\Lambda V, \delta)$. This is called a **free model** for (A, δ).

Remark 2.24. In [Lod98, Remark 5.4.14], it is given a free model for the dual numbers, where $V := kx \oplus ky$ where $|x| = 0, |y| = 1$ with differential $\delta(x) = 0, \delta(y) = x^2$.

Chapter 3

The homology over the spheres

In this chapter we will try to extend the result of Pirashvili in [Pir00], giving a description of the algebra structure of $H_*(\bigotimes_{S^d} A)$ for certain commutative algebras A over a field k of characteristic zero. The main tool we are going to use is the Greenlees spectral sequence.

3.1 The Hodge decomposition for the higher Hochschild homology

We consider the functor $A \mapsto \bigotimes_{S^1} A$. Its homotopy groups have a decomposition, called **Hodge decomposition**:

$$\pi_n \left(\bigotimes_{S^1} A \right) \cong \bigoplus_{i=0}^{n} HH_n^{(i)}(A).$$

This was obtained by Gerstenhaber and Schack in [GS87] and independently by Loday in [Lod89]. The idea is to give the Hochschild complex the structure of a Hopf algebra. With such a structure it is possible to build some orthogonal idempotents $e_n^{(i)}$, called **eulerian idempotents**. Using these it is possible to decompose the Hochschild complex as a direct sum of complexes $Ch_n(A) = \bigoplus_i Ch_n^{(i)}$, with

$$Ch_n^{(i)}(A) := e_n^{(i)} Ch_n(A)$$

where $Ch_*(A)$ is the Hochschild complex of A. Taking homology yields the desired decomposition.

In [Pir00] the first definition of higher Hochschild homology is provided. Pirashvili developed a generalization of the Hodge decomposition for the homology over the spheres S^d. Although he used a different approach, for the case $d = 1$ the decompositions are isomorphic.

Pirashvili provided also some calculations for the case of smooth algebras and for the case of truncated polynomial algebras, which we will now report.

Proposition 3.1. Let A be a smooth algebra of finite type over a field k of characteristic zero. Then, for d an odd natural number, we get:

$$\pi_n\left(\bigotimes_{S^d} A\right) \cong \begin{cases} A & \text{for } n = 0, \\ 0 & \text{for } n \neq dj, \\ \Omega^j_{A|k} & \text{for } n = dj \end{cases}$$

where $\Omega^j_{A|k}$ are the Kähler differentials defined in Chapter 1.

Proof. First of all we recall Pirashvili's decomposition of the homology over odd spheres. By [Pir00, Proposition 5.3], we have that

$$\pi_n\left(\bigotimes_{S^d} A\right) \cong \bigoplus_{i+dj=n} HH^{(j)}_{i+j}(A). \tag{3.1}$$

By [Lod98, Theorem 4.5.12], under these hypotheses on A, there is a canonical isomorphism $\Omega^n_{A|k} \cong HH^{(n)}_n$. The Hochschild-Kostant-Rosenberg Theorem (Theorem 1.46) guarantees that $HH^{(j)}_n$ for $0 < j < n$ is the trivial group, since A is smooth.

Therefore if $n = qd$ for some q, then $\pi_n\left(\bigotimes_{S^d} A\right) \cong \bigoplus_{i+j=q} HH^{(j)}_{di+j}(A)$. For $i \neq 0$ the summands vanish, while for $i = 0$ we get $HH^{(j)}_j(A) \cong \Omega^j_{A|k}$.

If n is not divided by d, then in (3.1) all the summands present a positive factor i and, by what we noticed above, all the groups vanish. □

Remark 3.2. The isomorphism in (3.1) applies also when A is not smooth. The only needed assumption is that k has characteristic zero ([Pir00, Proposition 5.2]).

Proposition 3.3. Let A be a smooth algebra of finite type over a field k of characteristic zero. Then, for d an even natural number, we get:

$$\pi_n\left(\bigotimes_{S^d} A\right) \cong \begin{cases} A & \text{for } n = 0, \\ 0 & \text{for } n \neq dj \end{cases}$$

The case $n = dj$ for some j is not concretely given and we will try later to relate it to $HH_*(A)$ using a Greenlees spectral sequences, described in [Gre16, Section 3].

We now consider the case of a truncated polynomial algebra. Let k be a field of characteristic zero. We define $A := k[x]/x^{r+1}$ for a positive integer r. Consider a sphere S^d with d odd; since we are in the odd case, we want to exploit (3.1). By [Lod98, Proposition 5.4.15] we know that $HH_{2n}A = HH_{2n}^{(n)}A$ and that $HH_{2n-1}A = HH_{2n-1}^{(n)}A$. By the Hodge decomposition as in [Lod98], this implies that all $HH_n^{(m)}$ are trivial, except for $HH_{2n}^{(n)}A$ and $HH_{2n-1}^{(n)}A$, which are isomorphic to k.

Moreover, in the case of characteristic zero, $HH_n(A) \cong k^r$. Using the decomposition in (3.1) we get that:

$$\pi_n\left(\bigotimes_{S^d} A\right) = \begin{cases} A & \text{for } n = 0, \\ k^r & \text{for } n \equiv 0, 1 \pmod{d+1}, \\ 0 & \text{otherwise} \end{cases}$$

Again we have a concrete description of the homotopy groups only for odd-dimensional spheres. Unfortunately, it is not possible to describe explicitly the homology groups of the even-dimensional spheres using a Greenlees spectral sequence. It is possible, though, to consider an approximation of $\bigotimes_{S^d} A$ and to look at its homology.

3.2 Greenlees spectral sequence

Definition 3.4. Let k be a commutative ring with unity. A **cofibre sequence** of commutative k-algebras is a sequence $S \to R \to Q$ of commutative k-algebras augmented over k, such that $Q \cong R \otimes_S k$.

We now state [Gre16, Lemma 3.1].

Lemma 3.5. Let $S \to R \to Q$ be a cofibre sequence of connective commutative algebras augmented over k such that:

- $\pi_0(S) = k$,
- R is upward finite type as an S module,
- either $\pi_n(S)$ is flat over k for all n or $\pi_n(Q)$ is flat over k for all n.

Then there is a multiplicative spectral sequence

$$E^2_{p,q} = \pi_p(Q) \otimes_k \pi_q(S) \Rightarrow \pi_{p+q}(R).$$

Remark 3.6. As Greenlees himself points out in his paper (referring to [DGI06, Proposition 3.13]), the second condition is satisfied whenever $\pi_n(R)$ is finite-dimensional for each n.

We notice that it is possible to use this spectral sequence to iteratively compute the homology $H_*(Ch(\bigotimes_{S^d} A))$ over spheres of an algebra A satisfying the hypothesis, once we know the standard Hochschild homology of the algebra A. The following example will clarify how we want to proceed.

Example 3.7. We consider the case of a k-algebra A with k a field of characteristic zero. The cofibre sequence we choose is $\bigotimes_{S^1} A \to \bigotimes_{D^2} A \to \bigotimes_{S^2} A$. We notice that S^2 is the pushout of the solid diagram

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \arrow[d, dashed] \\ * \arrow[r, dashed] & S^2 \end{tikzcd} \tag{3.2}$$

By Remark 2.4, the diagram in (3.2) induces another pushout square:

$$\begin{tikzcd} \bigotimes_{S^1} A \arrow[r] \arrow[d] & \bigotimes_{D^2} A \arrow[d] \\ \bigotimes_* A \arrow[r] & \bigotimes_{S^2} A \end{tikzcd}$$

Hence $\bigotimes_{S^2} A \cong \bigotimes_{D^2} A \otimes_{(\bigotimes_{S^1} A)} \bigotimes_* A$. Connectiveness is not a problem in this case, since we are dealing with simplicial rings, that can only have positive nontrivial homotopy groups (this is, exactly, what it means to be connective). The condition about $\pi_0(S)$ is met by the properties of higher Hochschild homology, as described in [Pir00, Section 5] and it holds for any sphere S^d and not just in the case of $d = 1$. We use Remark 3.6: since the disk (of any dimension) is contractible, we get $\pi_i(\bigotimes_{D^n} A) \simeq \pi_i(\bigotimes_* A)$. Moreover $\pi_i(\bigotimes_* A)$ is trivial for $i > 0$ and it is isomorphic to A for $i = 0$, so the finite-dimension condition of Remark 3.6 is satisfied and so is the second hypothesis of the Lemma. So, the only things we will assume are the flatness conditions: $\pi_i(\bigotimes_{S^1} A)$ is flat over A and so are all the homotopy groups of $\bigotimes_{S^d} A$.

The important advantage we got is that we have a spectral sequence whose (p, q)-term is the tensor product of two k-modules. We already know one of them (either from the standard Hochschild homology or from the inductive hypothesis); as for the other ones, it may be constrained by the fact that the total complex should have trivial reduced homology, since the spectral sequence converges to the homotopy of $\bigotimes_* A$.

Now we look specifically at $A = k[t]$ to deduce concretely the homology over S^2. Indeed for this algebra, the flatness conditions are satisfied (in particular all the homotopy groups are free over the ground ring A). We start recalling that $HH_*(A)$ is the free graded commutative algebra over k with generators t and x with $|t| = 0, |x| = 1$. We insert them in the 0-th column in the spectral sequence $E^2_{p,q}$

$$
\begin{array}{c|cccccc}
q & \vdots & & & & & \\
3 & 0 & & & & & \\
2 & 0 & & & & & \\
1 & x & & & & & \\
0 & t & & & & & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & p
\end{array}
$$

Since all the groups in $E^2_{0,q}$ are trivial for $q > 1$, then all the groups in $E^2_{p,q}$ for $q > 1$ are trivial too. Moreover, since $E^2_{1,0}$ is not reached by any differential, it needs to be trivial (and therefore the same happens to the whole column above it), otherwise the homology of the total complex would not be zero. So we get:

$$
\begin{array}{c|ccccccc}
q & \vdots & \vdots & & & & \\
2 & 0 & 0 & 0 & 0 & \dots & \\
1 & x & 0 & & & & \\
0 & t & 0 & & & & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & p
\end{array}
$$

Now x is contributing to the homology of the total complex, so we need to hit it with a differential. The only differential that can do that is the one coming from $E^2_{2,0}$. We call y a generator for that group. There is nothing more in $E^2_{2,0}$, since no differential hits it and nothing should survive. The E^2-page with the new information becomes

$$
\begin{array}{c|ccccccc}
q & \vdots & \vdots & & & & \\
2 & 0 & 0 & 0 & 0 & \dots & \\
1 & x & 0 & xy & & & \\
0 & t & 0 & y & & & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & p
\end{array}
$$

We still need to hit xy with a differential; the obvious candidate is y^2. Indeed $d^2(y^2) = 2xy$. Since our field does not have characteristic 2 all the elements of $E^2_{2,1}$ are hit by elements coming from $E^2_{4,0}$.

Again, we get xy^2, a generator we need to hit with a differential. Using the same argument as before we get that the homology over the 2-sphere is the polynomial algebra with two generators t, y where $|t| = 0, |y| = 2$.

Proposition 3.8. Let $A = k[t]$, for k a field of characteristic zero. Then

$$
\pi_* \left(\bigotimes_{S^d} A \right) = \begin{cases} S(kt) \otimes E(kx) & \text{for } d \text{ odd,} \\ S(kt \oplus kx) & \text{for } d \text{ even} \end{cases}
$$

with $|t| = 0, |x| = d$.

Proof. The case when d is odd is given by Proposition 3.1, once we note that the Kähler differentials algebra $\Omega^*_{A|k}$ has exactly one generator, whose degree is one.

The case when d is even is performed with the same argument as above. We use the Greenlees spectral sequence induced by the following cofibre sequence of simplicial sets: $S^{d-1} \to D^d \to S^d$ (for which we already noticed that all the hypotheses are satisfied). Since $d-1$ is odd, we can use the first part of the proposition to build the 0-th column column of the Greenlees spectral sequence.

$$
\begin{array}{c|cccccc}
q & \vdots & & & & & \\
d & 0 & & & & & \\
d-1 & x & & & & & \\
\vdots & \vdots & & & & & \\
2 & 0 & & & & & \\
1 & 0 & & & & & \\
0 & t & & & & & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & p
\end{array}
$$

The first $d-1$ columns need to be composed by trivial groups, while the d-th one has a generator y hitting x via the $(d-1)$-st differential. We then get:

$$
\begin{array}{c|cccccc}
q & \vdots & & & & & \\
d & 0 & & & & & \\
d-1 & x & 0 & 0 & \dots & xy & \\
\vdots & \vdots & & & & \vdots & \\
2 & 0 & 0 & 0 & \dots & 0 & \\
1 & 0 & 0 & 0 & \dots & 0 & \\
0 & t & 0 & 0 & \dots & y & \\
\hline
 & 0 & 1 & 2 & \dots & d & p
\end{array}
$$

The term xy needs to be hit and $d^2(y^2) = 2xy$, exactly as for the case $d = 2$. $\square$

In case the flatness conditions are not satisfied, we can look at approximations of $\pi_*(\bigotimes_{S^d} A)$, considering $\pi_*((\bigotimes_{S^d} A) \otimes_A k)$ instead, whose homological version is the higher Hochschild homology with coefficients. In this case, the pushout diagram is given by

$$\begin{tikzcd}
\bigotimes_{S^1} A \otimes_A k \arrow[r] \arrow[d] & \bigotimes_{D^2} A \otimes_A k \arrow[d] \\
\bigotimes_{*} A \otimes_A k \arrow[r] \arrow[d, "\cong"] & \bigotimes_{S^2} A \otimes_A k \\
k &
\end{tikzcd}$$

and since k is a field, every k-module is flat.

As an example, we analyse the case $A = k[\varepsilon]/\varepsilon^2$ with k a field of characteristic 0. First of all we consider $Ch(\bigotimes_{S^1} A \otimes_A k)_*$, the normalized complex with coefficients. The n-th part of the chain complex has generators of the form:

$$x = (a\varepsilon + b, \varepsilon, \dots, \varepsilon) \otimes_A (c) = (1, \varepsilon, \dots, \varepsilon) \otimes (bc) = (bc, \varepsilon, \dots, \varepsilon) \otimes_A (1)$$

for a, b, $c \in k$. We also notice that

$$\varphi : Ch(\bigotimes_{S^1} A \otimes_A k)_n \to k,$$

sending $x \mapsto bc \in k$ is an isomorphism of A-modules. The boundary map, applied to x, yields:

$$\partial(x) = (bc\varepsilon, \varepsilon, \dots, \varepsilon) \otimes_A (1) + (-1)^n (bc\varepsilon, \varepsilon, \dots, \varepsilon) \otimes_A (1) = 0.$$

Therefore, we get that the chain morphism, induced by φ at each level of the chain

$$\begin{tikzcd}
\dots \arrow[r, "\partial"] & Ch\left(\bigotimes_{S^1} A \otimes_A k\right)_n \arrow[r, "\partial"] \arrow[d, "\varphi"] & \dots \arrow[r, "\partial"] & Ch\left(\bigotimes_{S^1} A \otimes_A k\right)_0 \arrow[r] \arrow[d, "\varphi"] & 0 \\
\dots \arrow[r, "0"] & k \arrow[r, "0"] & \dots \arrow[r, "0"] & k \arrow[r, "0"] & 0
\end{tikzcd}$$

is an isomorphism of chain complexes.

In each degree n, the homology of the second complex is freely generated as a k-module by the single element $1 \in k$, corresponding via φ^{-1} to $(1, \varepsilon, \dots, \varepsilon) \otimes_A (1) \in (\bigotimes_{S^1} A \otimes_A k)_n$.

The algebra structure of $\pi_*(\bigotimes_{S^1} A \otimes_A k)$ is inherited from the one of $\pi_*(\bigotimes_{S^1} A)$, given by the shuffle product. Precisely, as graded k-algebra, $\pi_*(\bigotimes_{S^1} A \otimes_A k) \cong E(kx) \otimes S(ky)$, where $|x| = 1$ and $|y| = 2$.

So we can formulate an argument, which is similar to the inductive one for the case $A = k[t]$ to get $\pi_*(\bigotimes_{S^d} A \otimes_A k)$, described in the following proposition.

Proposition 3.9. Let k be a field of characteristic 0 and let $A = k[\varepsilon]/\varepsilon^2$. There is an isomorphism of graded k-algebras:

$$\pi_* \left(\bigotimes_{S^d} A \otimes_A k \right) \cong \Lambda(xk \oplus yk)$$

where $|x| = d$ and $|y| = d + 1$ and $\Lambda(V)$ is the graded symmetric algebra of the graded module V.

Proof. We are going to use an inductive argument over the dimension of the sphere d. The case $d = 1$ has been explained before. Assume that the proposition holds for $n - 1$.

The first columns of the E^2-page of the Greenlees spectral sequence are:

$$\begin{array}{c|cccccccc}
q & \vdots & & & & & & & \\
\vdots & \vdots & & & & & & & \\
d(d-1) & xy & & & & & & & \\
\vdots & \vdots & & & & & & & \\
d & y & 0 & 0 & \dots & \varphi y & \psi y & & \\
d-1 & x & 0 & 0 & \dots & \varphi x & \psi x & & \\
\vdots & \vdots & & & & \vdots & \vdots & & \\
2 & 0 & 0 & 0 & \dots & 0 & 0 & & \\
1 & 0 & 0 & 0 & \dots & 0 & 0 & & \\
0 & 1 & 0 & 0 & \dots & \varphi & \psi & & \\
\hline
 & 0 & 1 & 2 & \dots & d & d+1 & \dots & p
\end{array}$$

where φ is the generator of $\bigotimes_{S^d} A \otimes_A k$ that hits x, as ψ is the one hitting y. In particular, $d^d(\varphi) = x$ and $d^{d+1}(\psi) = y$. Notice that all the differentials vanish, up to degree d, so the E^2-page presents the same groups of the E^d-page.

Assume, now, that d is even. At the E^d-page the differentials behave as follows:

$$
\begin{aligned}
d^d(y^k) &= 0,\\
d^d(xy^k) &= 0,\\
d^d(\varphi^n y^k) &= nx\varphi^{n-1}y^k,\\
d^d(y^k\psi) &= 0,\\
d^d(xy^k\psi) &= 0,\\
d^d(x\varphi^n y^k) &= 0,\\
d^d(\varphi^n y^k\psi) &= nx\varphi^{n-1}y^k\psi,\\
d^d(x\varphi^n y^k\psi) &= 0,
\end{aligned}
$$

for $n > 0$ and $k \geq 0$. Notice that

$$
\begin{aligned}
xy^k &= d^d(\varphi y^k),\\
x\varphi^n y^k &= d^d(\varphi^{n+1}y^k),\\
xy^k\varphi &= d^d(\varphi y^k\psi),\\
x\varphi^n y^k\psi &= d^d(\varphi^{n+1}y^k\psi).
\end{aligned}
$$

So, at the E^{d+1}-page appear just the generators y^k and $y^k\psi$.

The $d+1$ differentials act on them as follows:

$$
\begin{aligned}
d^{d+1}(y^k) &= 0,\\
d^{d+1}(y^k\psi) &= y^k + 1.
\end{aligned}
$$

Notice that $y^k = d^{d+1}(y^{k-1}\psi)$ and hence no generators survive the E^{d+1}-page.

For d odd, the roles of x and y are exchanged, as the roles of φ and ψ. $\square$

3.3 The homotopy of finite type smooth algebra over even spheres

In the previous section we saw that there is a partial result regarding the homotopy over even-dimensional spheres. Through Greenlees spectral sequences we can extend the result of Pirashvili given in Proposition 3.1 and compute the homotopy groups $\pi_n(\bigotimes_{S^d} A)$ when d divides n for a finite type smooth algebra A.

Lemma 3.10. Let k be a commutative ring and let $F, G : \textbf{k-mod} \to \textbf{k-mod}$ be functors such that there is a natural transformation

$$\varphi : F \to G.$$

Let P be a projective k-module, which is a retract of the free k-module V. If $\varphi(V) : F(V) \to G(V)$ is an isomorphism, then $\varphi(V) : F(V) \to G(V)$ is an isomorphism.

Proof. Since P is a retract of V, there are maps $i : P \hookrightarrow V$ and $p : V \twoheadrightarrow P$ such that $p \circ i = \mathrm{Id}_P$. By naturality we get a commutative diagram

$$\begin{tikzcd} F(P) \arrow[r, "F(i)"] \arrow[d, "\varphi(P)"'] & F(V) \arrow[r, "F(p)"] \arrow[d, "\varphi(V)"', "\cong"] & F(P) \arrow[d, "\varphi(P)"] \\ G(P) \arrow[r, "G(i)"'] & G(V) \arrow[r, "G(p)"'] & G(P) \end{tikzcd}$$

and, by functoriality, $F(p)F(i) : F(P) \to F(P)$ is the identity, as $G(p)G(i)$. Hence $F(i)$ is injective and so needs to be $\varphi(P)$. On the other hand, $G(p)$ is surjective and so needs to be $\varphi(P)$. Therefore, $\varphi(P)$ is an isomorphism. $\square$

Lemma 3.11. Let A be a commutative ring with unity and P a projective A-module. Then for every positive integer n, the n-th degree part $E^n(P)$ of the exterior algebra is projective over A.

Proof. For P to be projective means that there is a free A-module V such that $V = P \oplus R$ for some A-module R. Hence there is a commutative triangle

$$\begin{tikzcd} P \arrow[r, "\mathrm{inc}"] \arrow[dr, "="'] & V \arrow[d, "\mathrm{pr}_P"] \\ & P. \end{tikzcd}$$

Since $E^n(-) : \textbf{A-mod} \to \textbf{A-mod}$ is a functor, the commutativity of the induced diagram of exterior algebras is preserved. Moreover $E^n(-)$ preserves freeness (see e.g. [Eis95, Corollary A.2.3]). Therefore, $E^n(P)$ is a direct summand of the free A-module $E^n(V)$. $\square$

The following lemma is a particular case of [Lod98, Proposition E.2(b)].

Lemma 3.12. Let k be a field and let A be a commutative smooth k-algebra of finite type. Then the A-module of the Kähler differentials $\Omega^1_{A|k}$ is projective over A.

This result extends to the higher-dimensional differentials.

Corollary 3.13. Under the same hypotheses of Lemma 3.12, $\Omega^n_{A|k}$ is a projective A-module for all n.

Proof. By definition $\Omega^n_{A|k} = E^n(\Omega^1_{A|k})$. We apply Lemma 3.11 to $\Omega^1_{A|k}$ for A as in Lemma 3.12, which is projective by Lemma 3.12. □

We notice that the hypotheses of Lemma 3.5 are satisfied. Hence, we can use a Greenlees spectral sequence to compute the homotopy of the even-dimensional spheres.

In the proof we will witness an occurrence of the chain complex

$$\cdots \to E^m(P) \otimes S^{n-m}(P) \to E^{m+1}(P) \otimes S^{n-m-1}(P) \to \dots$$

. It is the dual version of the tautological Koszul complex, as described in [Buc13]. The divided power algebra in Buchsbaum's example is isomorphic to the symmetric algebra, since we are in characteristic 0.

Proposition 3.14. Under the same hypotheses of Lemma 3.12, for d even there is an isomorphism:

$$\pi_* \left(\bigotimes_{S^d} A \right) \cong S_A(\Omega^1_{A|k})$$

where all the elements of $\Omega^1_{A|k}$ are considered of degree d.

Proof. We do the case $d = 2$; the general case holds with the same argument. Let $a_1, \dots, a_r$ be the generators of A as k-algebra. Then $da_1, \dots, da_r$ generate $\Omega^1_{A|k}$ as A-algebra.

Consider the Greenlees spectral sequence induced by the cofibre sequence $S^1 \to D^2 \to S^2$. Using the information we got in Proposition 3.1, we know that $E^2_{p,q} = 0$ for p even. Therefore, there is an isomorphism $E^2_{2,0} \xrightarrow{d^2} E^2_{0,1} =$

$\Omega^1_{A|k}$ since the homology of the E^∞-page needs to be trivial. For convenience, we replace $E^2_{2,0}$ with $\Omega^1_{A|k}$, so that the map d^2 is the identity. Hence, the first three columns of the spectral sequence are:

$$\begin{array}{c|ccccccc}
q & \vdots & & & & & & \\
\vdots & \vdots & & & & & & \\
n & \Omega^n_{A|k} & & & & & & \\
\vdots & \vdots & & \vdots & & & & \\
2 & \Omega^2_{A|k} & \vdots & \Omega^2_{A|k}\otimes\Omega^1_{A|k} & \vdots & & & \\
1 & \Omega^1_{A|k} & 0 & \Omega^1_{A|k}\otimes\Omega^1_{A|k} & 0 & & & \\
0 & \mathrm{A} & 0 & \Omega^1_{A|k} & 0 & & & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \dots & p
\end{array}$$

We define inductively K_n

$$K_n := \begin{cases} A & \text{for } n = 0, \\ \Omega^1_{A|k} & \text{for } n = 1, \\ \ker(\Omega^1_{A|k} \otimes K_{n-1} \xrightarrow{d^2} \Omega^2_{A|k} \otimes K_{n-2}) & \text{for } n > 1 \end{cases}$$

The 2-differentials $d^2 : \Omega^j_{A|k} \otimes K_1 \to \Omega^{j+1}_{A|k}$ are surjective. Indeed

$$d^2(da_{i_1} \dots da_{i_j} \otimes da_i) = da_{i_1} \dots da_{i_j} da_i.$$

We claim that the spectral sequence collapses at the E^3-page, i.e. there are long exact sequences

$$0 \to K_n \xrightarrow{d^2} \Omega^1_{A|k} \otimes K_{n-1} \xrightarrow{d^2} \dots \xrightarrow{d^2} \Omega^{n-1}_{A|k} \otimes K_1 \xrightarrow{d^2} \Omega^n_{A|k} \xrightarrow{d^2} 0.$$

We prove the claim by induction on n. We already did the case for $n = 1$. Assume the result holds for $n-1$ and that the image of one of the 2-differential in the sequence is not onto the kernel of the next differential. Assume that the sequence is not exact at $\Omega^{n-j}_{A|k} \otimes K_j$. Hence, there is a higher even differential (notice that the odd differentials are all trivial) $d^m : E^m_{2j-2+m,n-j+2-m} \to E^m_{2(j-1),n-j+1}$ hitting some nonzero element of the kernel of

$$d^2 : \Omega^{n-j}_{A|k} \otimes K_j \to \Omega^{n-j+1}_{A|k} \otimes K_{j-1}.$$

Notice, though, that $E^m_{2j-2+m,n-j+2-m}$ is zero, since $\Omega^{n-j+2-m}_{A|k} \otimes K_{j-1+m/2}$ belongs, by induction, to the long exact sequence

$$0 \leftarrow \Omega^{\alpha}_{A|k} \leftarrow \Omega^{\alpha-1}_{A|k} \otimes K_1 \leftarrow \dots \tag{3.3}$$

where $\alpha = n - (m/2) + 1$.

Therefore, $\Omega^{n+1-m}_{A|k} \otimes K_{m/2}$ does not survive in the E^3-page. So we have a contradiction. By the definition of K_n we have exactness also at $K_n \to \Omega^1_{A|k} \otimes K_{n-1}$ and hence the claim is proven.

Now we only need to prove that $K_n \cong S^n(\Omega^1_{A|k})$. Let X be an A-module. We can define $K_n(X)$ inductively as X for $n = 1$ and as the kernel of the natural transformation $D : \mathrm{Id}(-) \otimes_A K_{n-1}(-) \to E^2(-) \otimes_A K_{n-2}(-)$, that is, the restriction of the natural transformation

$$D' : \bullet^{\otimes_A n} \to (- \wedge -) \otimes_A -^{\otimes_A (n-2)},$$

acting as follows over X:

$$D'(X) : (x_1, \dots, x_n) \mapsto (x_1 x_2) \otimes (x_3, \dots, x_n).$$

We notice that, by construction, for all n, $K_n(\Omega^1_{A|k}) \cong K_n$.

For every n, $m > 1$, we consider the functors H_m : **A-mod** $\to$ **A-mod**, defined as

$$\begin{aligned} H_m(X) &:= E^m(X) \otimes_A K_{n-m}(X), \\ H_m(f) &: (x_1 \dots x_m) \otimes (x_{m+1} \dots x_n) \mapsto \\ &\mapsto (f(x_1) \dots f(x_m)) \otimes (f(x_{m+1}), \dots, f(x_n)) \end{aligned}$$

where $f : X \to Y$ is an A-linear map and $x_j \in X$. We define the natural transformation $\partial : H_m \to H_{m+1}$ as the restriction of the natural transformation

$$\partial' : E^m(-) \otimes_A -^{\otimes_A (n-m)} \to E^{m+1}(-) \otimes_A -^{\otimes_A (n-m-1)}$$

acting as follows over X:

$$\partial'(X) : (x_1 \dots x_m) \otimes (x_{m+1}, \dots, x_n) \mapsto (x_1 \dots x_{m+1}) \otimes (x_{m+2}, \dots, x_n)$$

where $x_j \in X$. So, for each X we can form a sequence of A-linear maps

$$F_0(X) \xrightarrow{\partial} F_1(X) \xrightarrow{\partial} \dots \xrightarrow{\partial} F_n(X)$$

with

$$F_i(-) := \begin{cases} K_n(-) & \text{for } i = 0, \\ \mathrm{Id}(-) \otimes_A K_{n-1}(-) & \text{for } i = 1, \\ H_i(-) & \text{for } i > 1 \end{cases}$$

Moreover $\partial : F_0 \to F_1$ is the inclusion; $\partial : F_1 \to F_2$ is D and $\partial : F_i \to F_{i+1}$ is $\partial : H_i \to H_{i+1}$ for every $i > 1$.

We notice that there is a natural transformation $\varphi : S^n(\bullet) \to K_n(\bullet)$, sending

$$\varphi(X) : (x_1 \dots x_n) \mapsto \sum_{\sigma \in \Sigma_n} (x_{\sigma(1)}, \dots x_{\sigma(n)}).$$

It is easy to check naturality and the fact that the output lies in $K_n(X)$.

We consider, now, the application of the F_i's to a free and finitely generated A-module V. First of all, we want to understand what $K_n(V)$ is. We do it by induction on n. By construction, $K_1(V) \cong V \cong S^1(V)$. So we start with the case for $n = 2$. We have that $K_2(V)$ is the kernel of the morphism $V \otimes V \to E^2(V)$ sending $a \otimes b$ to ab. In the free case, the kernel of this morphism is generated by $a \otimes b + b \otimes a$.

By inductive hypothesis, we have maps

$$\begin{tikzcd} K_n(V) \arrow[r, hook] & V \otimes K_{n-1}(V) \arrow[r, "\partial(V)"] & E^2(V) \otimes K_{n-2}(V) \\ & V \otimes S^{n-1}(V) \arrow[u, "\mathrm{Id} \otimes \varphi", "\cong"'] \arrow[r, "\alpha"'] & E^2(V) \otimes S^{n-2}(V) \arrow[u, "\mathrm{Id} \otimes \varphi", "\cong"'] \end{tikzcd} \tag{3.4}$$

where α is the unique map making the diagram commute. We notice that

$\partial(V)$ acts as follows:

$$\begin{aligned}\partial(V): a_1 \otimes \left(\sum_{\sigma\in\Sigma_{n-1}} (a_{\sigma(2)},\dots,a_{\sigma(n)}) \right) \mapsto & \\ \mapsto (a_1a_2) \otimes \left(\sum_{\tau\in\Sigma_{n-2}} (a_{\tau(3)},\dots,a_{\tau(n)}) \right) & + \\ +(a_1a_3) \otimes \left(\sum_{\tau\in\Sigma_{n-2}} (a_{\tau(2)},a_{\tau(4)},\dots,a_{\tau(n)}) \right) & + \dots + \\ +(a_1a_n) \otimes \left(\sum_{\tau\in\Sigma_{n-2}} (a_{\tau(2)},\dots,a_{\tau(n-1)}) \right) & .\end{aligned}$$

Therefore, the morphism α behaves as follows:

$$\alpha(a_1 \otimes (a_2 \dots a_n)) = (a_1a_2) \otimes (a_3 \dots a_n) + \dots + (a_1a_n) \otimes (a_2 \dots a_{n-1}).$$

Choosing a basis for V, we can see that the kernel is generated by the elements of the form

$$a := a_1 \otimes (a_2 \dots a_n) + \dots + a_n \otimes (a_1 \dots a_{n-1}).$$

Choosing a basis for V, it is possible to see that we have an isomorphism $S^n(V) \cong \ker\alpha$ sending the generator $(a_1 \dots a_n)$ to a.

So we can complete the diagram (3.4) as follows

$$\begin{tikzcd} & K_n(V) \arrow[r, hook] & V \otimes K_{n-1}(V) \arrow[r, "\partial(V)"] & E^2(V) \otimes K_{n-2}(V) \\ S^n(V) \arrow[r, "\cong"'] & \ker(\alpha) \arrow[r, hook] \arrow[u, "(\mathrm{Id}\otimes\varphi)_{|\ker\alpha}", "\cong"'] & V \otimes S^{n-1}(V) \arrow[r, "\alpha"'] \arrow[u, "(\mathrm{Id}\otimes\varphi)", "\cong"'] & E^2(V) \otimes S^{n-2}(V) \arrow[u, "(\mathrm{Id}\otimes\varphi)", "\cong"'] \end{tikzcd}$$

The composition of isomorphisms $S^n(V) \to \ker(\alpha) \to K_n(V)$ gives $\varphi(V)$.

Indeed, on a generator of $S^n(V)$, the composition acts as follows:

$$\begin{aligned}(a_1 \dots a_n) \mapsto & a_1 \otimes (a_2 \dots a_n) + \dots + a_n \otimes (a_1 \dots a_{n-1}) \mapsto \\ & a_1 \otimes \left(\sum_{\sigma \in \Sigma_{n-1}} (a_{\sigma(2)}, \dots, a_{\sigma(n)}) \right) + \dots + \\ & a_n \otimes \left(\sum_{\sigma \in \Sigma_{n-1}} (a_{\sigma(1)}, \dots, a_{\sigma(n-1)}) \right) = \sum_{\tau \in \Sigma_n} (a_{\tau(1)}, \dots, a_{\tau(n)}).\end{aligned}$$

We are now in the conditions of applying Lemma 3.10, to get that

$$\varphi(\Omega^1_{A|k}) : S^n(\Omega^1_{A|k}) \to K_n(\Omega^1_{A|k})$$

is an isomorphism. $\square$

Chapter 4

Relation between T^2 and $S^1 \vee S^1 \vee S^2$

In the setting we have described - the higher Hochschild homology - we can build another construction, induced from one on simplicial sets. In particular we can study what happens when we perform an action of a group on them.

A case of particular interest is the one of the torus, which can be described through different models, as we have seen in Section 2.2.

We are interested in studying the homotopy fixed points and the homotopy orbits of a group action on the torus. There are many groups acting on the Torus, as we will see in Chapter 5.

The main tool we can use is the convergence of the following spectral sequences, described in [GM95].

Theorem 4.1. *Let X be a space and G a group acting on it. There are a homotopy fixed point spectral sequence:*

$$H^{-p}(G, \pi_q(X)) \Rightarrow \pi_{p+q}(X^{h(G)}),$$

and a homotopy orbit spectral sequence:

$$H_p(G, \pi_q(X)) \Rightarrow \pi_{p+q}(X_{h(G)}).$$

In order to use this theorem we need to know the homotopy groups of the space X, which, in our case, is $\bigotimes_{T^n} A$ for a k-algebra A. As we will see, the case for $n = 2$ is particularly interesting for certain properties of the groups acting on $\mathbb{T}^2$.

Lundervold, in his master's thesis ([Lun07]), provided a tool to compute the iterated Hochschild homology for CGDAs. In particular, for symmetric algebras the homology over the torus splits into the homology of the cells, as we will see in Proposition 4.4. Namely:

$$\pi_* \left(\bigotimes_{T^2} A \right) \cong \pi_* \left(\bigotimes_{S^1 \vee S^1 \vee S^2} A \right). \tag{4.1}$$

This generalizes to T^n.

Via the Dold-Kan correspondence we can translate the simplicial A-algebras to chain complexes without losing information about the homotopy. So we get that

$$Ch \left(\bigotimes_{T^2} A \right)$$

is chain homotopic to

$$Ch \left(\bigotimes_{S^1} A \right) \otimes_A Ch \left(\bigotimes_{S^1} A \right) \otimes_A Ch \left(\bigotimes_{S^2} A \right).$$

Notice that the first two terms are Hochschild complexes. Hence, the homotopy groups of $\bigotimes_{T^2} A$ are given by

$$\pi_n \left(\bigotimes_{T^2} A \right) \cong H_n \left(Ch \left(\bigotimes_{S^1} A \right) \otimes_A Ch \left(\bigotimes_{S^1} A \right) \otimes_A Ch \left(\bigotimes_{S^2} A \right) \right)$$

and, assuming projectiveness of $H_*(\bigotimes_{S^i} A)$ over A for $i = 1, 2$, we can use the strong version of the Künneth formula (Corollary 1.13) to get:

$$\pi_n \left(\bigotimes_{T^2} A \right) \cong \bigoplus_{i+j+k=n} HH_i(A) \otimes_A HH_j(A) \otimes_A H_k \left(\bigotimes_{S^2} A \right).$$

This formulation is particularly interesting, since we have a good knowledge of the homology of an algebra over a sphere.

There is, actually, a stronger result for some particular algebras, as the following proposition shows.

Proposition 4.2. Let V be a graded module over a field k of characteristic 0. Let, moreover, X and Y be simplicial sets so that their suspensions are weakly equivalent, i.e. $\Sigma X \simeq \Sigma Y$. Then for every n, there is an isomorphism

$$\pi_n\left(\bigotimes_X S(V)\right) \cong \pi_n\left(\bigotimes_Y S(V)\right)$$

where $S(V)$ is the symmetric k-algebra over V.

Proof. We notice that the symmetric functor S from the category of k-modules to the category of commutative k-algebras is left adjoint to the forgetful functor. In particular it preserves colimits. Hence,

$$\bigotimes_X S(V) \cong S\left(k[X] \otimes_k V\right)$$

where $k[X]$ is the free simplicial k-module over X. As described in [GJ09, Section V.5], there is a weak equivalence $k[X] \simeq \Omega(k[\Sigma X])$, where $\Omega(k[\Sigma X])$ is the loop space of $k[\Sigma X]$. By hypothesis, $\Sigma X \simeq \Sigma Y$. Hence, $k[\Sigma X] \simeq k[\Sigma Y]$. This gives the desired isomorphism in homotopy, since taking the loop space of fibrant simplicial sets preserves weak equivalences. $\square$

Proposition 4.3. ([Hat02, Proposition 4I.1]) Let X, Y be CW-complexes. There is a homeomorphism

$$\Sigma(X \times Y) \cong \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y).$$

In particular, the suspension of the torus ΣT^n is homeomorphic to the suspension of its cells.

Corollary 4.4. Let V be a graded k-module. Then there is an isomorphism

$$\pi_*\left(\bigotimes_{T^n} S(V)\right) \cong \pi_*\left(\bigotimes_{\bigvee_i S_i^{d_i}} S(V)\right)$$

where the $S_i^{d_i}$'s are the cells of T^n.

Proof. By Proposition 4.3 we know that $\Sigma T^n \cong \Sigma(\bigvee_i S_i^{d_i})$. Applying Proposition 4.2 we get the result. $\square$

4.1 A counterexample

We have seen that for symmetric algebras the higher Hochschild homology factors through the stable homotopy. We can ask ourselves if this holds for any commutative k-algebra A. Unfortunately, this is not true in the most general sense. For example, considering the dual numbers $k[\varepsilon]/\varepsilon^2$ over a field of characteristic zero, the second homology group over T^2 and the one over $S^1 \vee S^1 \vee S^2$ do not agree.

We are going to use the free model for the dual numbers, described in Remark 2.24.

Lundervold, in [Lun07] computed the second iteration of the Hochschild homology for the dual numbers. His computation does not provide a deep insight, so we will develop it in more detail. First of all, we need a general result of Lundervold.

Proposition 4.5. ([Lun07, Corollary 1.3.9]) Let V be a graded k-module. There is an isomorphism of graded k-algebras

$$H_*(Ch(Ch(\Lambda V, \delta))) \to H_*(\Lambda(V \oplus d_1V \oplus d_2V \oplus d_2d_1V), \delta'')$$

where the differential δ'' is given by:

$$\begin{aligned} \delta''(a) &= \delta(a), \\ \delta''(d_1a) &= -d_1\delta(a), \\ \delta''(d_2a) &= -d_2\delta(a), \\ \delta''(d_2d_1a) &= d_2d_1\delta(a). \end{aligned}$$

The generators for $\Lambda(V \oplus d_1V \oplus d_2V \oplus d_2d_1V)$ are x, d_1x, d_2x, y, d_2d_1x, d_1y, d_2y, d_2d_1y. The differential d_i raises the degree by one, while d_2d_1 raises it by two. By the graded commutativity, those generators satisfy $(d_1x)^2 = (d_2x)^2 = y^2 = (d_2d_1y)^2 = 0$.

For convenience we will compute the following differential.

Lemma 4.6. We have: $\delta''((d_iy)^j) = -2jx(d_ix)(d_iy)^{j-1}$.

Proof. By induction on j.

For $j = 1$ we have $\delta''(d_i y) = -d_i(\delta'' y) = -d_i(x^2) = -2x d_i x$. In the general case:

$$\delta''((d_i y)^j) = \delta''(d_i y (d_i y)^{j-1}) = -2x d_i x (d_i y)^{j-1} + d_i y \delta''((d_i y)^{j-1}).$$

The result follows by induction. $\square$

We will now explicitly analyse the application of δ'' to the generators of $\Lambda(V \oplus d_1 V \oplus d_2 V \oplus d_2 d_1 V)$.

$$\begin{aligned}
(1)_{j,k} := (d_1 y)^j (d_2 y)^k \mapsto & -2jx(d_1 x)(d_1 y)^{j-1}(d_2 y)^k + \\
& -2kx(d_2 x)(d_1 y)^j (d_2 y)^{k-1}, \\
(2)_{j,k} := y(d_1 y)^j (d_2 y)^k \mapsto & x^2 (d_1 y)^j (d_2 y)^k + 2jxy(d_1 x)(d_1 y)^{j-1}(d_2 y)^k + \\
& + 2kxy(d_2 x)(d_1 y)^j (d_2 y)^{k-1}, \\
(3)_{j,k} := (d_1 y)^j (d_2 y)^k (d_2 d_1 y) \mapsto & -2jx(d_1 x)(d_1 y)^{j-1}(d_2 y)^k (d_2 d_1 y) + \\
& -2kx(d_2 x)(d_1 y)^j (d_2 y)^{k-1}(d_2 d_1 y) + \\
& + 2 d_2 x d_1 x (d_1 y)^j (d_2 y)^k + \\
& + 2x(d_2 d_1 x)(d_1 y)^j (d_2 y)^k, \\
(4)_{j,k} := y(d_1 y)^j (d_2 y)^k (d_2 d_1 y) \mapsto & x^2 (d_1 y)^j (d_2 y)^k (d_2 d_1 y) + \\
& -2jxy(d_1 x)(d_1 y)^{j-1}(d_2 y)^k (d_2 d_1 y) + \\
& -2kxy(d_2 x)(d_1 y)^j (d_2 y)^{k-1}(d_2 d_1 y) + \\
& -2y(d_2 x d_1 x)(d_1 y)^j (d_2 y)^k + \\
& -2xy(d_2 d_1 x)(d_1 y)^j (d_2 y)^k.
\end{aligned}$$

When we multiply the generators by $d_1 x$ or $d_2 x$ and then apply δ'', we get:

$$
\begin{aligned}
(d_1x)(1)_{j,k} \mapsto & -2kx(d_1xd_2x)(d_1y)^j(d_2y)^{k-1},\\
(d_2x)(1)_{j,k} \mapsto & -2jx(d_2xd_1x)(d_1y)^{j-1}(d_2y)^k,\\
(d_1x)(2)_{j,k} \mapsto & x^2(d_1x)(d_1y)^j(d_2y)^k - 2kxy(d_1xd_2x)(d_1y)^j(d_2y)^{k-1},\\
(d_2x)(2)_{j,k} \mapsto & x^2(d_2x)(d_1y)^j(d_2y)^k + 2jxy(d_2xd_1x)(d_1y)^{j-1}(d_2y)^k,\\
(d_1x)(3)_{j,k} \mapsto & 2kx(d_1xd_2x)(d_1y)^j(d_2y)^{k-1}(d_2d_1y)+\\
& + 2x(d_1x)(d_2d_1x)(d_1y)^j(d_2y)^k,\\
(d_2x)(3)_{j,k} \mapsto & -2jx(d_1xd_2x)(d_2y)^{j-1}(d_2y)^k(d_2d_1y)+\\
& + 2x(d_2x)(d_2d_1x)(d_1y)^j(d_2y)^k,\\
(d_1x)(4)_{j,k} \mapsto & x^2(d_1x)(d_1y)^j(d_2y)^k(d_2d_1y)+\\
& - 2kxy(d_1xd_2x)(d_1y)^j(d_2y)^{k-1}(d_2d_1y)+\\
& - 2(d_1x)y(d_1y)^j(d_2y)^kx(d_2d_1x),\\
(d_2x)(4)_{j,k} \mapsto & x^2(d_2x)(d_1y)^j(d_2y)^k(d_2d_1y)+\\
& + 2jxy(d_1xd_2x)(d_1y)^{j-1}(d_2y)^k(d_2d_1y)+\\
& - 2(d_2x)y(d_1y)^j(d_2y)^kx(d_2d_1x).
\end{aligned}
$$

One can easily check that if $j,\ k > 1$ all the terms do not belong to the kernel of δ'' and there is no way of summing up generators to get an element of $\ker\delta''$. In particular one can start showing that all but the first of the summands in $\delta''(4)_{j,k}$ do not appear anywhere else. Then, it results impossible to replicate summands in the other images without using $(4)_{j,k}$.

Something similar happens for $j = 1$ or $k = 1$. So what remains is just the case for $j = 0$ or $k = 0$. By the symmetry between the expressions for d_1y and d_2y it is enough to examine the case for $j = 0$.

$$
\begin{aligned}
(1)_{0,k} \mapsto & -2kxd_2x(d_2)^{j-1},\\
(2)_{0,k} \mapsto & x^2(d_2y)^k + 2kxyd_2x(d_2y)^{k-1},\\
(3)_{0,k} \mapsto & -2kx(d_2x)(d_2y)^{k-1}(d_2d_1y) + 2(d_2xd_1x)(d_2y)^k + 2x(d_2d_1x)(d_2y)^k,\\
(4)_{0,k} \mapsto & x^2(d_2y)^k(d_2d_1y) - 2kxy(d_2x)(d_2y)^{k-1}(d_2d_1y) + 2y(d_2xd_1x)(d_2y)^k+\\
& - 2xy(d_2d_1x)(d_2y)^k.
\end{aligned}
$$

Again, multiplying by d_1x and by d_2x, we get:

$$\begin{aligned}
d_1x(1)_{0,k} \mapsto & -2kxd_1xd_2x(d_2)^{j-1}, \\
d_2x(1)_{0,k} \mapsto & 0, \\
d_1x(2)_{0,k} \mapsto & x^2d_1x(d_2y)^k + 2kxy(d_1xd_2x)(d_2y)^{k-1}, \\
d_2x(2)_{0,k} \mapsto & x^2d_2x(d_2y)^k, \\
d_1x(3)_{0,k} \mapsto & -2kx(d_1xd_2x)(d_2y)^{k-1}(d_2d_1y) + 2x(d_1x)(d_2d_1x)(d_2y)^k, \\
d_2x(3)_{0,k} \mapsto & 2x(d_2x)(d_2d_1x)(d_2y)^k, \\
d_1x(4)_{0,k} \mapsto & x^2(d_1x)(d_2y)^k(d_2d_1y) - 2kxy(d_1xd_2x)(d_2y)^{k-1}(d_2d_1y)+ \\
& -2xy(d_1x)(d_2d_1x)(d_2y)^k, \\
d_1x(4)_{0,k} \mapsto & x^2(d_2x)(d_2y)^k(d_2d_1y) - 2xy(d_2x)(d_2d_1x)(d_2y)^k.
\end{aligned}$$

In this situation we get some cycles. Specifically:

$$\begin{aligned}
(\alpha)_1^j &:= d_1x(d_1y)^j, \\
(\alpha)_2^k &:= d_2x(d_2y)^k, \\
(\beta)_1^j &:= 2(j+1)d_1x(y(d_1y)^j) - x(d_1y^{j+1}), \\
(\beta)_2^k &:= 2(k+1)d_2x(y(d_2y)^k) - x(d_2y^{k+1}), \\
(\gamma)_1^j &:= (j+1)d_1x((d_1y)^jd_2d_1y) - d_2d_1x(d_1y^{j+1}), \\
(\gamma)_2^k &:= (k+1)d_2x((d_2y)^kd_2d_1y) - d_2d_1x(d_2y^{k+1}),
\end{aligned}$$

where $|(\alpha)_i^j| = 2j+1, |(\beta)_i^j| = 2+2j, |(\gamma)_i^j| = 4+2j$. Together with them, among the cycles, we have their product with $x^n(d_2d_1x^m)$. Notice that such a multiplication raises the degree by $2m$. Since we are particularly interested in $H_2(\Lambda(V \oplus d_1V \oplus d_2V \oplus d_2d_1V), \delta''))$, it is enough to consider the multiplication by x^n.

Notice that $x(\alpha)_1^j = \delta''((1)_{j,0})$ and, similarly $x(\alpha)_2^k = \delta''((1)_{0,k})$. Moreover, $x(\beta)_1^j = -\delta''((2)_{j+1,0})$ and $x(\beta)_2^k = -\delta''((2)_{0,k+1})$. Hence, $x(\alpha)_i^j$ and $x(\beta)_i^j$ are in the image of δ''. Since the multiplication by x commutes with the application of δ'', also $x^n(\alpha)_i^j$ and $x^n(\beta)_i^j$ are in the image of δ'' for every positive integer n.

Proposition 4.7. $H_2(Ch(Ch(A, \delta))$ is isomorphic to $k \oplus k \oplus k$, with generators

$$
\begin{aligned}
[(\beta)_1^0] &= [2(d_1x)y], \\
[(\beta)_2^0] &= [2(d_2x)y], \\
[(\alpha)_1^0(\alpha)_2^0] &= [d_1xd_2x].
\end{aligned}
$$

Proof. By the calculations above, in $H_*(Ch(Ch(A,\delta))$ the only elements of degree 2 are just $(\beta)_i^0$ and $(\alpha)_i^0 \cdot (\alpha)_l^0$. Notice that if $i = l$ then

$$
(\alpha)_i^0 \cdot (\alpha)_i^0 = (d_ix)^2 = 0.
$$

By definition, $(\beta)_i^0 = 2(d_1x)y - x(d_1y)$, where the second summand is a boundary, so $[(\beta)_i^0] = [2(d_ix)y]$, as stated. $\square$

We now aim to study $\pi_2(\bigotimes_{S^1\vee S^1\vee S^2} A)$. It is not possible to write the second homotopy group $\pi_2(\bigotimes_{S^1\vee S^1\vee S^2} A)$ as a direct sum of the tensor product of its components since $\bigotimes_{S^1\vee S^1\vee S^2} A$ is not flat over A.

First of all, we will examine $\pi_*(\bigotimes_{S^1\vee S^1} A)$. The Tor spectral sequence converges to it:

$$
E^2_{p,q} = \bigoplus_{q_1+q_2=q} \mathrm{Tor}_p^A(HH_{q_1}A, HH_{q_2}A) \Rightarrow \pi_{p+q}\left(\bigotimes_{S^1\vee S^1} A\right).
$$

Remark 4.8. We immediately notice that since the Tor functor is computed over the commutative ring A, for any A-bimodule M we get

$$
\mathrm{Tor}_*^A(A, M) \cong \mathrm{Tor}_*^A(M, A) \cong A
$$

as a graded algebra concentrated in degree 0.

It is easy, therefore, to compute the first two rows of the Tor spectral sequence. Let $q = q_1 + q_2$. Then

$$
E^2_{p,q} = \begin{cases} A & \text{for } p = 0, q = 0, \\ k \oplus k & \text{for } p = 0, q = 1, \\ 0 & \text{for } p > 0, q < 2 \end{cases}
$$

For $q = 2$ there are three summands. Two of them are $\mathrm{Tor}_p^A(k, A)$, while the other one is $\mathrm{Tor}_p^A(k, k)$, where k in both entries has the A-module structure described before. For k we have the following resolution:

$$\ldots \xrightarrow{\cdot\varepsilon} A \xrightarrow{\cdot\varepsilon} A \xrightarrow{\cdot\varepsilon} \ldots \xrightarrow{\cdot\varepsilon} A \xrightarrow{\cdot\varepsilon} k \to 0$$

where the maps are given by multiplication by ε. As we tensor this with k, we get the following chain complex, where the boundary maps are 0:

$$\ldots \to k \to k \to \cdots \to k \to k \to 0.$$

Hence the homology of the chain is just the chain itself. Therefore the second row of the E^2-page is given by

$$E^2_{*,2} = \begin{cases} k^{\oplus 3} & \text{for } p = 0, \\ k & \text{for } p > 0 \end{cases}$$

where each k_i is a copy of k with degree i.

One can easily go on using the same resolutions in order to get the E^2-page of the spectral sequence:

q						
4	$k^{\oplus 5}$	$k^{\oplus 3}$	$k^{\oplus 3}$	$k^{\oplus 3}$	$\ldots$	
3	$k^{\oplus 4}$	$k^{\oplus 2}$	$k^{\oplus 2}$	$k^{\oplus 2}$	$\ldots$	
2	$k^{\oplus 3}$	k	k	k	$\ldots$	
1	$k^{\oplus 2}$	0	0	0	$\ldots$	
0	A	0	0	0	$\ldots$	
	0	1	2	3	4	p

We are now ready to compute some of the homotopy groups of $(\bigotimes_{S^1 \vee S^1} A)$. $\pi_1(\bigotimes_{S^1 \vee S^1} A)$ is isomorphic to $k \oplus k$ since it is the sum of the groups in the second antidiagonal, and the only differential that can hit one of them is zero. Similarly, the elements in the third antidiagonal sum up to $k^{\oplus 3}$ and all the differentials hitting it are 0. Hence, $\pi_2(\bigotimes_{S^1 \vee S^1} A) \cong k^{\oplus 3}$.

We will use this result to compute $\pi_2(\bigotimes_{S^1 \vee S^1 \vee S^2} A)$.

Proposition 4.9. Let k be a field of characteristic 0. For $A := k[\varepsilon]/\varepsilon^2$ we have that $\pi_2(\bigotimes_{S^1 \vee S^1 \vee S^2} A)$ is isomorphic to $k^{\oplus 4}$.

Proof. Again, we use the Tor spectral sequence:

$$E^2_{p,q} = \bigoplus_{q_1+q_2=q} \mathrm{Tor}^A_p \left(H_{q_1} \left(\bigotimes_{S^1 \vee S^1} A \right), H_{q_2} \left(\bigotimes_{S^2} A \right) \right) \Rightarrow \pi_{p+q} \left(\bigotimes_{S^1 \vee S^1 \vee S^2} A \right).$$

We are only interested in the first three rows.

Using Remark 4.8 we can easily compute the first two rows of the spectral sequence:

$$E^2_{p,q} = \begin{cases} A & \text{for } p = 0, q = 0, \\ k \oplus k & \text{for } p = 0, q = 1, \\ 0 & \text{for } p > 0, q < 2 \end{cases}$$

The third row is:

$$E^2_{*,2} = \mathrm{Tor}^A_*(A, k) \oplus \mathrm{Tor}^A_*(k \oplus k, 0) \oplus \mathrm{Tor}^A_*(k^{\oplus 3}, A) \cong k \oplus 0 \oplus k^{\oplus 3} \cong k^{\oplus 4}$$

as a graded k-module concentrated in degree 0.

Therefore, the first rows of the spectral sequence are given by:

$$\begin{array}{c|cccccc} q & & & & & & \\ 2 & k^{\oplus 4} & 0 & 0 & 0 & \dots & \\ 1 & k^{\oplus 2} & 0 & 0 & 0 & \dots & \\ 0 & A & 0 & 0 & 0 & \dots & \\ \hline & 0 & 1 & 2 & 3 & 4 & p \end{array}$$

The elements in the second antidiagonal sum up to $k^{\oplus 4}$ and no differential hits them. Therefore, $\pi_2(\bigotimes_{S^1 \vee S^1 \vee S^2} A)$ is isomorphic to $k^{\oplus 4}$. $\square$

The difference between $H_*(Ch(\bigotimes_{S^1 \vee S^1 \vee S^2} A)$ and $H_*(Ch(\bigotimes_{T^2} A)$ for the algebra $A := k[\varepsilon]/\varepsilon^2$ is more evident when we consider the higher Hochschild homology with coefficients in the field k.

As we saw in Proposition 3.9, the homology groups with coefficients in k over the sphere S^d of A are projective k-modules. Hence, we can apply the stronger version of the Künneth Formula to get:

$$H_n \left(\bigotimes_{S^1 \vee S^1 \vee S^2} A; k \right) \cong \bigoplus_{i+j+l=n} HH_i(A; k) \otimes HH_j(A; k) \otimes H_l \left(\bigotimes_{S^2} A; k \right).$$

To compute the homology of A over the torus with coefficients in k, we can use Lundervold's argument again. We get:

$$H_*(Ch(Ch(\Lambda V,\delta));k) \cong H_*((\Lambda(V\oplus d_1V\oplus d_2V\oplus d_2d_1V),\delta'');k).$$

We notice that the differential structure of $\Lambda(V\oplus d_1V\oplus d_2V\oplus d_2d_1V)\otimes_{\Lambda(V)} k$ is particularly simple. Indeed, the $\Lambda(V)$-algebra structure of k is given by the following composition:

$$\Lambda(V) \longrightarrow k[\varepsilon]/\varepsilon^2 \xrightarrow{\cdot\varepsilon} k\ ,$$

sending $ax+by\mapsto a\varepsilon\mapsto 0$.

Therefore, the chain complex considered with coefficients is

$$(\Lambda(d_1V\oplus d_2V\oplus d_2d_1V),\delta''')$$

with

$$\begin{aligned}\delta'''(d_ix)&=0,\\ \delta'''(d_iy)&=0,\\ \delta'''(d_2d_1x)&=0,\\ \delta'''(d_2d_1y)&=d_2xd_1x.\end{aligned}$$

Hence the homology is given by the quotient

$$H_*\left(Ch\left(\bigotimes_{T^2}A\right);k\right)\cong \Lambda(d_1V\oplus d_2V\oplus d_2d_1(kx))/\sim$$

where $\sim$ is the equivalence relation generated by $d_2xd_1x\sim 0$.

It is easy to check that for each $n>1$ the homology group

$$H_n\left(Ch\left(\bigotimes_{T^2}A\right);k\right)\ncong H_n\left(\bigotimes_{S^1\vee S^1\vee S^2}A;k\right).$$

Chapter 5

Homotopy orbits

We are now interested in the homotopy orbits of an A-algebra G obtained via the functor $\bigotimes_\bullet A$. The main tool we are going to use is the following spectral sequence.

$$H_p\left(G, \pi_q\left(\bigotimes_X A\right)\right) \Rightarrow \pi_{p+q}\left(\left(\bigotimes_X A\right)_{hG}\right). \tag{5.1}$$

First of all we need to define the objects we are working with. The main reference for the definitions and the results we are presenting is [Bro94].

5.1 Background on group homology

Let G be a group, written multiplicatively, the **group ring** $\mathbb{Z}[G]$ of G is defined as the free abelian group generated by the elements of G. $\mathbb{Z}[G]$ has a ring structure, where the product is the extension of the multiplication in G.

As an example, we can consider $G = \mathbb{Z}$. The group ring of the integers $\mathbb{Z}[\mathbb{Z}]$ can be described as the Laurent polynomials over the integers: $\mathbb{Z}[t, t^{-1}] := \{a_{-n}t^{-n} + \cdots + a_n t^n | a_j \in \mathbb{Z}, n \in \mathbb{N}\}$.

Definition 5.1. Let M be a $\mathbb{Z}[G]$-module. Let F_* be a projective resolution for G as a $\mathbb{Z}[G]$-module. We define the **homology** of G with coefficients in M, written $H_*(G; M)$, as the homology of the chain complex $F_* \otimes_{\mathbb{Z}[G]} M$.

We notice that, in the definition, the choice of the resolution appears irrelevant. It is actually the case, since any two resolutions are quasi-isomorphic, see, e.g. [Bro94, Theorem 1.7.5].

For how it is defined, the group homology can be expressed as $H_*(G; M) = Tor_*^{\mathbb{Z}[G]}(G, M)$.

Remark 5.2. Let us consider $M = \mathbb{Z}$ equipped with the trivial action by G, namely $g \cdot n = n$. In this case we have $F_q \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong F_q$ since every integer can be carried on the left-hand side of the tensor product.

It is possible to give a topological interpretation of the homology of groups. We say that a topological space X is a $K(G, n)$ **space** (or an **Eilenberg-Maclane space**) if the only nontrivial homotopy group of X is $\pi_n \cong G$. There is a standard process to build a $K(G, 1)$ space for every group G. It is possible to build also $K(G, n)$ spaces for arbitrary n and abelian G. We now present a way to construct a $K(G, 1)$ space for an arbitrary G.

Definition 5.3. Let G be a group; we define the **classifying topological space** of G, denoted by $|BG|$ to be the geometric realization of the simplicial set BG. The universal cover of $|BG|$ is called EG, the universal G-bundle.

Proposition 5.4. ([May99, Section 16.5]) The geometric realization of BG, the classifying space of G is a $K(G, 1)$ space.

The following proposition ([Bro94, Proposition II.4.1]), gives a topological interpretation of the homology of a group.

Proposition 5.5. The homology of G with coefficients in M is the homology of the complex $C_*(EG) \otimes_{\mathbb{Z}[G]} M$, where $C_*(EG)$ is the cellular homology of EG.

5.2 Groups acting on T^2

There are several groups (and simplicial groups) acting on the torus (we just need to be careful for the choice of the model). We want to focus on discrete groups acting on the model $\mathbb{T}^n := B(\mathbb{Z}) \times \cdots \times B(\mathbb{Z})$.

Consider the group of endomorphism of $\mathbb{T}^n$: $\mathrm{Hom}_{\mathbf{Sset}}(\mathbb{T}^n, \mathbb{T}^n)$. We can manipulate this expression as follows:

$$\mathrm{Hom}_{\mathbf{Sset}}(\mathbb{T}^n, \mathbb{T}^n) = \mathrm{Hom}_{\mathbf{Sset}}(B(\mathbb{Z}^n), B(\mathbb{Z})^n) \cong \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}^n, \mathbb{Z}^n) \cong M_{n \times n}\mathbb{Z}$$

where the second isomorphism is given by the following lemma.

Lemma 5.6. Let G, H be groups. Then there is an isomorphism of groups $\mathrm{Hom}_{\mathbf{Sset}}(BG, BH) \overset{\cong}{\to} \mathrm{Hom}_{\mathbf{Grp}}(G, H)$.

Proof. We recall that $(BG)_n = (g_1, \ldots, g_n)$ and that the face and degeneracy maps act as follows:

$$d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_{n-1}) & \text{for } i = 0, \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{for } 0 < i < n, \\ (g_1, \ldots, g_{n-1}) & \text{for } i = n \end{cases}$$
$$s_j(g_0, \ldots, g_n) = (g_0, \ldots, 1, g_j, \ldots, g_n).$$

We notice that the zeroth degree of a map $f : BG \to BH$ is determined uniquely by a map $\tilde{f} : G \to H$. By induction we assume that the degree $n-1$ is uniquely determined by $\tilde{f}$. We claim that the n-th degree of f is uniquely determined by the $(n-1)$-st degree of f and, hence by $\tilde{f}$.

Since f is a map of simplicial sets, it needs to commute with the face maps. In particular, we have a commutative diagram

$$\begin{tikzcd}
(g_1, \ldots, g_{n-1}) \arrow[r, mapsto, "f_{n-1}"] & (h_1, \ldots, h_{n-1}) \\
(g_1, \ldots, g_n) \arrow[r, mapsto, "f_n"] \arrow[u, mapsto, "d_n"] \arrow[d, mapsto, "d_0"'] & (h_1, \ldots, h_n) \arrow[u, mapsto, "d_n"'] \arrow[d, mapsto, "d_0"] \\
(g_2, \ldots, g_n) \arrow[r, mapsto, "f_{n-1}"] & (h_2, \ldots h_n)
\end{tikzcd}$$

that proves our claim, since by inductive hypothesis, the upper rectangle tells us that $h_i = \tilde{f}(g_i)$ for $1 \leq i < n$ and the lower rectangle that $h_i = \tilde{f}(g_i)$ for $1 < i \leq n$. $\square$

Remark 5.7. We obtained that every element in a group that acts on $\mathbb{T}^n$ can be represented by a matrix of dimension $n \times n$. Nevertheless, not all matrices represent the action of an element of a group acting on T^n. Indeed if $g \in G$, then $g \in M_{n\times n}\mathbb{Z}$. Since g is in a group, there should exist g^{-1} such that $gg^{-1} = g^{-1}g = Id$ in $M_{n\times n}\mathbb{Z}$. Therefore we can restrict to the invertible endomorphism, namely to $GL_n(\mathbb{Z}) \subset M_{n\times n}\mathbb{Z}$.

Moreover $GL_n(\mathbb{Z})$ acts on $\mathbb{T}^n$ as follows. Let $x \in (\mathbb{T}^n)_1$. It has the form $x = (x_1, \dots, x_n)$, for $x_i \in (B\mathbb{Z})_1$. Let $M = (m_{i,j}) \in GL_n(\mathbb{Z})$. We define

$$M \cdot x := \left(\sum_k m_{1,k} x_k, \dots, \sum_k m_{n,k} x_k \right).$$

This determines the action on every degree of $\mathbb{T}^2$ by Lemma 5.6.

So every discrete group acting on $\mathbb{T}^n$ is a subgroup of $GL_n(\mathbb{Z})$.

Example 5.8. By what we just saw the group $G := \langle B \rangle$, for

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

acts on T^2. We want to compute the homotopy orbits of the action of G on $\bigotimes_{T^2} A$, using (5.1). We will use the topological interpretation of the homology of a group. In order to reduce the notation, we define $M_q := \pi_q(\bigotimes_{\mathbb{T}^2} A)$.

We notice that $G \cong \mathbb{Z}$ and, therefore, the universal G-bundle is homeomorphic to the real numbers $\mathbb{R}$ with the standard topology. One can easily check that the cellular complex of $\mathbb{R}$ is

$$\cdots \to 0 \to C_1(\mathbb{R}) \to C_0(\mathbb{R})$$

where $C_1(\mathbb{R})$ is generated as $\mathbb{Z}[t, t^{-1}]$-module by one single element, which we call e_1^0. The same happens to $C_0(\mathbb{R})$ and we call its generator e_0^0. We choose the latter, so that the extremal points of e_1^0 are mapped to e_0^0 and to te_0^0. Hence the unique nontrivial boundary map sends $e_1^0 \mapsto e_0^0 - te_0^0$. Since the action of $\mathbb{Z}[t, t^{-1}]$ on $C_i(\mathbb{R})$ is free for $i = 0, 1$, then every element in $C_1(\mathbb{R}) \otimes_{\mathbb{Z}[t,t^{-1}]} M_q$ has a representative of the form $1 \otimes y$. The nontrivial map of the complex $C_*(\mathbb{R}) \otimes_{\mathbb{Z}[t,t^{-1}]} M_q$ sends it to

$$1 \otimes y \mapsto 1 \otimes y - t \otimes y = 1 \otimes y - 1 \otimes t \cdot y.$$

We can deduce that

$$H_i(G; M_q) = \begin{cases} M_q/(t-1) & \text{for } i = 0, \\ M_q^t & \text{for } i = 1, \\ 0 & \text{for } i > 1 \end{cases}$$

The spectral sequence in (5.1) collapses at the E^3-page and gives us that

$$\pi_n\left(\left(\bigotimes_{\mathbb{T}^2} A\right)_{hG}\right) = \begin{cases} M_0/(t-1) & \text{for } n=0, \\ M_n/(t-1) \oplus M^t_{n-1} & \text{for } i>0 \end{cases}$$

The case for the 2-torus is particularly nice because $GL_2(\mathbb{Z})$ is an amalgamated product of dihedral groups ([Zie81, Theorem 23.1]). In particular $GL_2(\mathbb{Z}) = D_{12} *_{D_4} D_8$, where the generators are the following

$$\begin{aligned} D_{12} &= \left\langle A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, \\ D_8 &= \left\langle B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, \\ D_4 &= \left\langle A^3 = B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle. \end{aligned}$$

This is quite helpful, since we have tools to deal with amalgamated products. One of them is the following Mayer-Vietoris exact sequence, as shown in [Bro94, Section VII.9].

$$\begin{aligned} \ldots \to H_n(D_4; M_q) \to H_n(D_{12}; M_q) \oplus H_n(D_8; M_q) \to \\ \to H_n(GL_2(\mathbb{Z}); M_q) \to H_{n-1}(D_4; M_q) \to \ldots \end{aligned} \tag{5.2}$$

Hence, the homology of the three finite subgroups will give us the spectral sequence converging to the homotopy orbits.

We notice that we can express the dihedral group D_{2n} as the semidirect product $C_n \ltimes C_2$. In particular, C_2 is a normal subgroup of D_{2n} with $D_{2n}/C_n \simeq C_2$. Hence, we have a **Lyndon–Hochschild–Serre spectral sequence** ([McC01, Theorem 8^{bis}.12]), whose E^2-page is

$$E^2_{p,q} := H_p(C_2; H_q(C_n; M)) \Rightarrow H_{p+q}(D_{2n}; M)$$

for any $\mathbb{Z}[D_{2n}]$-module M.

Using this spectral sequence we can compute the homology of the dihedral subgroups generating $GL_2(\mathbb{Z})$.

Lemma 5.9. Let $G = C_n = \langle g \rangle$. A free resolution of G as $\mathbb{Z}[G]$-module is

$$\ldots \longrightarrow \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{1} G \tag{5.3}$$

with $N := 1 + g + g^2 + \ldots g^{n-1}$.

Corollary 5.10. Let k be a field of characteristic 0 and let M be a k-module equipped with the trivial action by C_n. Then the homology of C_n with coefficient in M is given by:

$$H_p(C_n; M) \begin{cases} M & \text{for } p = 0, \\ 0 & \text{otherwise} \end{cases}$$

Proof. The homology of C_n with coefficients in M is the homology of the chain complex

$$\ldots \longrightarrow M \xrightarrow{N} M \xrightarrow{g-1} M \xrightarrow{N} M \xrightarrow{g-1} M \longrightarrow 0 \, .$$

We notice that the maps N is multiplication by n; so it is an isomorphism since we are in characteristic 0. On the other hand, $g - 1$ is the 0 map. Therefore, the homology is the one we described in the statement. $\square$

In [Lun07], Lundervold shows how $GL_2(\mathbb{Z})$ acts on $\pi_*(\bigotimes_T^2 k[t])$. In particular, a matrix

$$B := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

acts on the generators of $\pi_*(\bigotimes_T^2 k[t])$ as follows:

$$\begin{aligned} B(x) &= x, \\ B(\eta) &= a\eta + b\beta, \\ B(\beta) &= c\eta + d\beta, \\ B(z) &= \det(B)z. \end{aligned}$$

We also notice that

$$B(\eta\beta) = (a\eta + b\beta)(c\eta + d\beta) = ad\eta\beta + bc\beta\eta = \det(B)\eta\beta$$

With this information we can proceed and compute the homotopy orbits of the action of $GL_2(\mathbb{Z})$ on $\bigotimes_{\mathbb{T}^2} k[t]$

Proposition 5.11. Let $M_r := \pi_r(\bigotimes_{\mathbb{T}^2} A)$ for $A = k[t]$. The homology of D_4 with coefficients in M_r is

$$H_p(D_4; M_r) \cong \begin{cases} M_r & \text{for } p = 0, r \equiv 0 \pmod 4, \\ 0 & \text{otherwise} \end{cases}$$

Proof. We want to use the Lyndon-Hochschild-Serre spectral sequence, so we start by calculating $H_q(C_2; M_r)$. The generator for C_2 is the matrix

$$g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

which acts as follows on the generators of M_*:

$$\begin{aligned} g(x) &= x, \\ g(z) &= z, \\ g(\eta) &= -\eta, \\ g(\beta) &= -\beta, \\ g(\eta\beta) &= \eta\beta. \end{aligned}$$

We are now going to use the resolution in (5.3). If r is even, then the group acts trivially on M_r. Hence, the homology is trivial for each $p > 0$ and it is M_r for $q = 0$. On the other hand, if r is odd, then the action is the multiplication by -1, meaning that $N = 0$, while $(1 - g)$ is an isomorphism. Hence the homology is trivial for each q.

Notice that the homology

$$H_p\left(\left\langle\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\rangle; H_q\left(\left\langle\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\rangle; M_r\right)\right)$$

is trivial if r is odd or $q > 0$. So we will only focus on r even and $q = 0$. In this case we have:

$$H_p(C_2; M_r).$$

This time, C_2 does not act trivially on z, but $R(z) = -z$. Moreover $R(\eta\beta) = -\eta\beta$. We need to distinguish three scenarios.

- if $r \equiv 2 \pmod 4$, then the generators of M_r are $z^{r/2}$ and $z^{r/2-1}\eta\beta$. Applying R yields:

$$R(z^{r/2}) = -z^{r/2}, \qquad R(z^{r/2-1}\eta\beta) = z^{r/2-1} \cdot (-\eta\beta) = -z^{r/2-1}\eta\beta.$$

  So, the action is given by the multiplication by -1, meaning that the homology is trivial for each p.

- If $r \equiv 0 \pmod 4$, then then R acts trivially on the generators, meaning that the homology is trivial for positive p and it is M_r for $p = 0$.

Hence, the Lyndon-Hochschild-Serre spectral sequence has only one nontrivial column (the first one) and we get that

$$H_p(D_4; M_r) \cong \begin{cases} M_r & \text{for } p = 0, r \equiv 0 \pmod 4, \\ 0 & \text{otherwise} \end{cases}$$

As we wanted to prove.

□

Proposition 5.12. Let $M_r := \pi_r(\bigotimes_{\mathbb{T}^2} A)$ for $A = k[t]$. The homology of D_8 with coefficients in M_r is

$$H_p(D_8; M_r) \cong \begin{cases} M_r & \text{for } p = 0, r \equiv 0 \pmod 4, \\ 0 & \text{otherwise} \end{cases}$$

Proof. The proof is very similar to the one of Proposition 5.11. The generator of C_8 is the matrix

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which acts on the generators of M_* as follows:

$$\begin{aligned} g(x) &= x, \\ g(z) &= z, \\ g(\eta) &= \beta, \\ g(\beta) &= -\eta, \\ g(\eta\beta) &= \eta\beta. \end{aligned}$$

For r odd, $g-1$ is an isomorphism and hence the homology is trivial for all q. For $r=0$, g acts trivially on x, which is the only generator, so we have $H_p(\langle R\rangle; M_0) = M_0$ if $p=0$ and $H_p(\langle R\rangle; M_0) = 0$ for positive p. If r is even and positive, we have two generators: $z^{r/2}$ and $z^{r/2-1}\eta\beta$. On both of the generators, the action is trivial. Hence there is nontrivial homology only for $q=0$ and r even and $H_0(C_4; M_r) \cong M_r$.

Notice that we are in the same situation as we were in Proposition 5.11: we need to compute $H_p(\langle R\rangle; M_r)$ for r even. The matrix R acts on M_r in the same way it did in Proposition 5.11 and, therefore, we get that

$$H_p(D_8; M_r) \cong \begin{cases} M_r & \text{for } p=0, r \equiv 0 \pmod 4, \\ 0 & \text{otherwise} \end{cases}$$

as we wanted to prove. □

Proposition 5.13. Let $M_r := \pi_r(\bigotimes_{\mathbb{T}^2} A)$ for $A = k[t]$. The homology of $D_{12}4$ with coefficients in M_r is:

$$H_p(D_{12}; M_r) \cong \begin{cases} M_r & \text{for } p=0, r \equiv 0 \pmod 4, \\ 0 & \text{otherwise} \end{cases}$$

Proof. The proof is the same as the one for Proposition 5.12. The action changes but, given

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$1-g$ acts on the generators as in Proposition 5.12: it is 0 when r is even and it is an isomorphism when r is odd. □

We can use the results of Propositions 5.11, 5.12 and 5.13 to compute the homotopy orbits of $\pi_r(\bigotimes_{\mathbb{T}^2} A_{hGL_2(\mathbb{Z})})$.

Proposition 5.14. Let $M_r := \pi_r(\bigotimes_{\mathbb{T}^2} A)$ for $A = k[t]$. The homotopy groups of the homotopy orbits of the action of $GL_2(\mathbb{Z})$ on $\bigotimes_{\mathbb{T}^2} A$ are:

$$\pi_n\left(\left(\bigotimes_{\mathbb{T}^2} A\right)_{hGL_2(\mathbb{Z})}\right) \cong \begin{cases} M_n & \text{for } n \equiv 0 \pmod 4, \\ 0 & \text{otherwise} \end{cases}$$

Proof. For every r we will use the long exact sequence in (5.2). We can distinguish two cases: $r \not\equiv 0 \pmod 4$ and $r \equiv 0 \pmod 4$.

For $r \not\equiv 0 \pmod 4$ all the known homology groups in (5.2) vanish and, therefore, $H_p(GL_2(\mathbb{Z})) = 0$ for all p.

For $r \equiv 0 \pmod 4$ the only nonzero homology groups are:

$$0 \to H_1(GL_2(\mathbb{Z})) \to M_r \to M_r \oplus M_r \to H_0(GL_2(\mathbb{Z})) \to 0.$$

We notice that the map $i : H_0(D_4; M_r) \to H_0(D_8; M_r) \oplus H_0(D_12; M_r)$ is induced by the maps $C_2 \ltimes C_2 \to C_n \ltimes C_2$ for $n = 4$ or $n = 6$. Hence, i applied to x, $z^{r/2}$ and $z^{r/2-1}\eta\beta$ gives

$$\begin{aligned} i(x) &= (x, x), \\ i(z^{r/2}) &= (z^{r/2}, z^{r/2}), \\ i(z^{r/2-1}\eta\beta) &= (z^{r/2-1}\eta\beta, z^{r/2-1}\eta\beta). \end{aligned}$$

So i is injective, meaning that $H_1(GL_2(\mathbb{Z})) = 0$ and $H_0(GL_2(\mathbb{Z})) \cong M_r$.

The homotopy orbits spectral sequence is, therefore,

$$\begin{array}{c|cccccc} q & \vdots & & & & & \\ 4j & M_{4j} & 0 & 0 & \dots & & \\ \vdots & \vdots & & & & & \\ 4 & M_4 & 0 & 0 & \dots & & \\ \vdots & \vdots & \vdots & \vdots & & & \\ 1 & 0 & 0 & 0 & \dots & & \\ 0 & M_0 & 0 & 0 & \dots & & \\ \hline & 0 & 1 & 2 & 3 & 4 & p \end{array}$$

The E^2-page is the same as the E^∞-page since all the differentials are 0, yielding the isomorphisms in the statement of the proposition. □

This result can be easily generalized to any polynomial ring over a field of characteristic 0.

Lemma 5.15. Let A and B be k-algebras and let X be a space. There is an isomorphism, natural in X:

$$\bigotimes_X (A \otimes B) \cong \bigotimes_X (A) \otimes \bigotimes_X (B).$$

Proof. We consider the isomorphism at each degree n. Call

$$\varphi_n : \bigotimes_{X_n} (A \otimes B) \to \bigotimes_{X_n} (A) \otimes \bigotimes_{X_n} (B),$$

the candidate for being the isomorphism we are looking for. On generators it acts as follows:

$$\varphi_n \left(\bigotimes_{x \in X_n} (a_x \otimes b_x) \right) = \left(\bigotimes_{x \in X_n} a_x \right) \otimes \left(\bigotimes_{x \in X_n} b_x \right).$$

Consider a map of simplicial sets, $f : X \to Y$. We get a naturality square and we need to check its commutativity:

$$\begin{tikzcd}
\bigotimes_{X_n} (A \otimes B) \arrow[r, "\varphi_n"] \arrow[d, "(f_n)_*"'] & \bigotimes_{X_n} (A) \otimes \bigotimes_{X_n} (B) \arrow[d, "(f_n)_* \otimes (f_n)_*"] \\
\bigotimes_{Y_n} (A \otimes B) \arrow[r, "\varphi_n"] & \bigotimes_{Y_n} (A) \otimes \bigotimes_{Y_n} (B)
\end{tikzcd}$$

Choosing a generator $\bigotimes_{x \in X_n} (a_x \otimes b_x)$ in $\bigotimes_{X_n} (A \otimes B)$, following the diagram in the two directions gives

$$\bigotimes_{y \in Y_n} c_y \otimes \bigotimes_{y \in Y_n} d_y$$

with

$$c_y := \prod_{x \in f^{-1}(y)} a_x, \qquad d_y := \prod_{x \in f^{-1}(y)} b_x.$$

A similar diagram shows that φ preserves the structure maps and, therefore, φ is a natural isomorphism. $\square$

Corollary 5.16. Let X be a G-simplicial set. Then

$$\bigotimes_X (A \otimes B) \cong \bigotimes_X (A) \otimes \bigotimes_X (B)$$

as G-simplicial k-algebras. The action of G is diagonally.

We remark that, in general, given an action of a simplicial group G on a pointed simplicial set X, it is not possible to lift the action to $\bigotimes_X A \otimes_A M$ for an A-bimodule M. Therefore, if we want to use homology with coefficient we can only consider actions that fix the base point.

Bibliography

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[Bro94] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[Buc13] David A. Buchsbaum. Selections from the letter-place panoply. In *Commutative algebra*, pages 237–284. Springer, New York, 2013.

[DGI06] W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar. Duality in algebra and topology. *Adv. Math.*, 200(2):357–402, 2006.

[DGM13] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*, volume 18 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2013.

[DP61] Albrecht Dold and Dieter Puppe. Homologie nicht-additiver Funktoren. Anwendungen. *Ann. Inst. Fourier Grenoble*, 11:201–312, 1961.

[Eis95] David Eisenbud. *Commutative algebra: With a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[GJ09] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition [MR1711612].

[GM95] J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. *Mem. Amer. Math. Soc.*, 113(543):viii+178, 1995.

[Goo86] Thomas G. Goodwillie. Relative algebraic K-theory and cyclic homology. *Ann. of Math. (2)*, 124(2):347–402, 1986.

[Gre16] J. P. C. Greenlees. Ausoni-Bökstedt duality for topological Hochschild homology. *J. Pure Appl. Algebra*, 220(4):1382–1402, 2016.

[GS87] Murray Gerstenhaber and S. D. Schack. A Hodge-type decomposition for commutative algebra cohomology. *J. Pure Appl. Algebra*, 48(3):229–247, 1987.

[Gui08] *Guido's book of conjectures*, volume 40 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*. L'Enseignement Mathématique, Geneva, 2008. A gift to Guido Mislin on the occasion of his retirement from ETHZ June 2006, Collected by Indira Chatterji.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[Lod89] Jean-Louis Loday. Opérations sur l'homologie cyclique des algèbres commutatives. *Invent. Math.*, 96(1):205–230, 1989.

[Lod98] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1998. Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.

[Lun07] A. Lundervold. Higher order cyclic homology for rational algebras. Master's thesis, University of Bergen, 2007.

[May99] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.

[McC01] John McCleary. *A user's guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[Pir00] Teimuraz Pirashvili. Hodge decomposition for higher order Hochschild homology. *Ann. Sci. École Norm. Sup. (4)*, 33(2):151–179, 2000.

[Zie81] Heiner Zieschang. *Finite groups of mapping classes of surfaces*, volume 875 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.