UNIVERSITY OF BERGEN
Faculty of Mathematics and Natural Sciences

Master's Thesis in Topology

Log Hochschild Homology of Categorical Products and Coproducts

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Abstract

As the front page implies, this is a thesis about log Hochschild homology. Our primary goal will be to understand how the log Hochschild homology groups acts on products and coproducts in the category of commutative pre-log algebras. We will prove that the log Hochschild homology commutes with products, and we shall formulate and prove a Künneth Theorem for log Hochschild homology. These results allows for the computation of the log Hochschild homology of complicated pre-log algebras that have been constructed from components we already understand by the process of taking products and coproducts.

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Introduction

On the Structure and Style of the Thesis

We have attempted to write a paper that is accessible for everyone with a solid background in the basics of commutative algebra (where we have listed the more specific prerequisites at the beginning of Chapter 1). As a result of a having a broad audience, the preliminary chapter has become rather extensive and the more experienced reader might find herself bored long before the "actual" thesis begins. It might therefore be of interest to those who consider themselves members of the more advanced section of the readership to start reading from Chapter 2 and let Chapter 1 serve as an easily available reference for some of the more obscure details. The thesis is structured as follows:

In Chapter 1 we deliberate on the notation and terminology that we will use, and present some technical results that would otherwise have interrupted the flow of the subsequent chapters.

In Chapter 2 we will introduce Hochschild homology groups $\mathrm{HH}_*(A)$ of an associative and unital algebra A. These are defined as the homology groups associated to what is called the Hochschild complex of A. We will then delve deeper into how the Hochschild homology groups behaves with respect to the product, tensor products, and localizations of A. These are important results which we will generalize to the setting of log Hochschild homology in Chapter 4.

In Chapter 3 we present the definitions of pre-log and log structures on commutative algebras and introduce the category of commutative pre-log algebras. Then follows explicit descriptions of products and coproducts in the category of commutative pre-log algebras and we present a result describing how one can calculate limits and colimits generally in this category. At the end of this chapter we will do the groundwork that is required before we can start on Chapter 4.

In Chapter 4 we give the definition of the log Hochschild homology groups $\mathrm{HH}_*(A, M)$ of a commutative pre-log algebra (A, M, α). This is also the chapter where we state and prove the main results of this thesis. In particular we shall prove that the log Hochschild homology groups commutes with the categorical product and provide a Künneth Theorem for log Hochschild homology. We then proceed to use these results to obtain some corollaries, in particular a corollary that proves log Hochschild homology to commute with localizations. We will also use these new results to do some calculations of log Hochschild homology groups.

Notational Conventions

Most of the symbols and expressions we use are the conventional ones but there are also some anomalies the reader should be aware of, especially if one chooses to omit reading the Preliminaries. We specify the most frequently used notation below, and hope that the reader will find the list to be sufficiently complete:

- We will consistently use R to as the notation for a commutative, associative and unital ring throughout the entire thesis. When we want to put emphasis on the fact that we are working over a field, we will write Q.
- We have done our best to be consequent about using A to denote an associative and unital R-algebra. We will assume A to have the additional property of being commutative in both Chapter 3 and Chapter 4.
- If x is an element of an algebraic structure X, we write $\langle x \rangle$ for the substructure generated by x whenever we are in a setting where this has an obvious meaning. More generally we write $\langle x_1, x_2, \dots \rangle$ or $\langle x_i \rangle$ for the substructure of X generated by several elements $x_i \in X$. We use the same notation, $\langle y_i \rangle$, for the free "insert the algebraic structure we are working with" generated by the elements $y_1, y_2, \dots, y_i, \dots$
- As is usual, $\mathbb{Z}$ means all the integers both positive, negative and 0. We let $\mathbb{N}_0$ denote the non-negative integers including zero, while $\mathbb{N}^*$ will mean the non-negative integers excluding zero.
- We will generally write $\otimes$ rather than $\otimes_R$ and we often write $A^{\otimes n}$ instead of $A \otimes A \otimes \cdots \otimes A$. We have been careful in pointing it our in the few instances where we deviate from these conventions.
- We have been somewhat careless when it comes to making a proper distinction between the direct product and the free sum (denoted respectively as $\times$ and $\oplus$). The reader who is unaware of this runs the risk of being somewhat confused. This is regrettable, and we would have corrected every occurrence of this if time had permitted us to do so. Luckily all of the results are still correct, since we work exclusively with finite products of commutative algebraic structures, and so these concepts becomes isomorphic.
- Finally, be aware that our notation for the Cartesian product of simplicial modules, $C_\bullet \otimes D_\bullet$, is very similar to the notation used for the different concept of the tensor product of chain complexes, $C_* \otimes D_*$.
- The reader should be aware that the notation $C_\bullet(A, M)$, $C_*(A, M)$ and $\mathrm{H}_*(A, M)$ will be used for two different concepts. In Chapter 2 it will be the notation for the Hochschild simplical R-algebra, Hochschild chain complex and Hochschild homology groups respectively, while in Chapter 4 it will mean the log Hochschild simplical R-algebra, log Hochschild chain complex and log Hochschild homology groups.

Chapter 1

Preliminaries

In this preliminary, we recall concepts that will be frequently used throughout the thesis. As a result, this chapter is largely devoted to definitions and elementary results from algebra, category theory and homology. The purpose of this is to establish terminology and notation, and to ensure the paper to be fairly self-contained. The reader is expected to have some familiarity with algebraic objects and constructions, such as groups, rings, modules, algebras, tensor products, localizations and split exact sequences. In addition, some prior exposure to either homological algebra or algebraic topology would serve as source of motivation, although this is not an absolute prerequisite. To keep the text from becoming over-fragmented, there will be times when definitions appear inside the text. We will then lend ourself to the **boldface convention**, meaning that we write the expression that is being defined in thick letters.

1.1 Abstract Algebra

Throughout the thesis, R will be assumed to be a commutative ring with a unit element. All unspecified tensor products are taken to be over R, unless otherwise stated. So, whenever the reader finds "$\otimes$" written, what we we really mean is "$\otimes_R$". In addition to this, A is always assumed to be an **associative and unital R-algebra**. This means that A is both an R-module and a unital and associative ring, where we have that, for all $a, b \in A$ and for all $r \in R$:

$$r(ab) = (ra)b = a(rb)$$

Given R-algebras A and B, we call an R-linear ring homomorphism, $f\colon A \to B$, for an **algebra homomorphism** from A to B.

Example 1.1.1. Here is a selection of some elementary examples of R-algebras:

- The **polynomial ring over R in n variables**, $R[x_1, \ldots, x_n]$. This is an R-algebra by letting the scalar multiplication of an element $r \in R$ be ordinary multiplication by a constant: $r \cdot p(x_1, \ldots, x_n)$.

- The **n-fold product ring** of R, $R^n = R \times R \times \cdots \times R$. This is an R-algebra with R-module structure defined to be multiplication by $r \in R$ in all coordinates. In particular, we have that R is an R-algebra.

- An example of a non-commutative algebra is the vector space $\mathbb{R}^3$, with cross product as multiplicative ring structure. The $\mathbb{R}$-algebra structure is again obtained by scalar multiplication.

1.1.1 Bimodules

Definition 1.1.2. That M is an A-**bimodule** means that M is both a right A-module and a left A-module. These module structures need to be compatible with each other, in the sense that scalar multiplications of A on the right should commute with scalar multiplications of A on the left. To be precise, we have that for all $a, a' \in A$ and for all $m \in M$:

$$(am)a' = a(ma')$$

We make a remark of the fact that M inherits both a left and a right R-module structure. For all $r \in R$, $m \in M$ and $1 \in A$, we define left scalar multiplication as $r \cdot m = (r1) \cdot m$, and similarly for right scalar multiplication, we let $m \cdot r = m \cdot (r1)$.

We need bimodules in order to define Hochschild homology in Chapter 2, but then mainly in the form of the following alternative description. Let M be a bimodule over the R-algebra, A. Then there is a unique interpretation of M as a left module over the R-algebra, $A^e = A \otimes A^{\mathrm{op}}$, called the **enveloping algebra** of A. A^{op} is called the **opposite R-algebra**, defined as the R-algebra $\langle A, +, \cdot^{\mathrm{op}} \rangle$. The underlying additive group structure of A^{op} and A is the same, while multiplication in A^{op} is obtained by a permutation of the factors prior to multiplication: $a \cdot^{\mathrm{op}} b = b \cdot a$. It is routine to verify that A^{op} is an R-algebra and we omit this calculation. We can manoeuvre back and forth between A-bimodules and left A^e-modules by letting:

$$a \cdot m = (a \otimes 1_{A^{\mathrm{op}}}) \cdot m$$
$$m \cdot b = (1_A \otimes b) \cdot m$$

Let M and N be A-bimodules. An A-**bimodule homomorphism** is a right and left A-linear function $f \colon M \to N$. In the left A^e-module interpretation, this can be shown to be equivalent to $f \colon M \to N$ being A^e-linear.

Example 1.1.3. The product of a ring A by itself n times, $\prod^n A$, is an A-bimodule if we define scalar multiplication of A on $\prod^n A$ to be:

$$a(a_1, a_2, \dots, a_n) = (aa_1, aa_2, \dots, aa_n)$$
$$(a_1, a_2, \dots, a_n)a = (a_1 a, a_2 a, \dots, a_n a)$$

Example 1.1.4. The tensor product of a ring A by itself n times, $A^{\otimes n}$, is an A-bimodule if we define scalar multiplication of A on $A^{\otimes n}$ to be:

$$a(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = (aa_1 \otimes a_2 \otimes \cdots \otimes a_n)$$
$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n)a = (a_1 \otimes a_2 \otimes \cdots \otimes a_n a)$$

In particular, these two examples makes A into an A-bimodule over itself, by letting $n = 1$.

Proposition 1.1.5. *Let M be an R-module. Then we give $A \otimes M \otimes A$ and $A^e \otimes M$ the structure of A-bimodules by defining:*

$$(b \otimes b') \cdot (a \otimes m \otimes a') = (b \cdot a \otimes m \otimes a' \cdot b')$$
$$(b \otimes b') \cdot (a \otimes a' \otimes m) = (b \cdot a \otimes a' \cdot b' \otimes m)$$

Doing so makes $A \otimes M \otimes A$ and $A^e \otimes M$ isomorphic via the map:

$$\begin{aligned} A \otimes M \otimes A^e &\longrightarrow A^e \otimes M \\ a \otimes m \otimes a' &\longmapsto a \otimes a' \otimes m \end{aligned}$$

Proof. By the universal property of the tensor product, we always have an R-module isomorphism $A \otimes M \otimes A \cong A \otimes A \otimes M$, which is the one described. That this is an isomorphism of A^e-modules follows from definition. $\square$

1.1.2 Projective Modules

Projective modules generalizes the concept of free modules, while preserving many of their important properties. There are several equivalent definitions of projective modules floating around. We list the ones we will use here:

Definition 1.1.6. Let X be a ring which is not necessarily commutative. Let P be a a left X-module. We say that P is **X-projective**, or **projective** as a left X-module if one (and hence all) of the following statements are true:

- For all diagrams of left X-modules below there exists a **lift**, meaning the dashed arrow, making the diagram commute. The notation in the diagram is meant to indicate that f has to be surjective:

$$\begin{tikzcd} & N \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "\exists h"] \arrow[r, "g"'] & M \end{tikzcd}$$

- P is the **direct summand** of a free module. In other words, there exists a left X-module, Q, such that:

$$P \oplus Q \cong \bigoplus_{\alpha \in I} X$$

- $\operatorname{Hom}(P, -)\colon X\text{-}\mathbf{Mod} \to \mathbf{AbGrp}$ is an **exact functor**. I.e. applying $\operatorname{Hom}(P, -)$ to an exact sequence of left X-modules always yield a new exact sequence.

Proof. We refer the reader to Chapter 1 of [Mac Lane, 1967] for a proof of the equivalence of these definitions. $\square$

The definitions of right projective is obtained by changing every "left" to "right" in the definition above. We suppress the "left" (respectively "right") in left (respectively right) projective whenever it is obvious which is meant (in this paper, everything will be left projective)

Example 1.1.7. Any free R-module, F, is R-projective, since the direct sum $F \oplus 0 = F$ is free. Some further properties that can easily be deduced with the direct sum definition, is that the direct sum and direct summands of R-projective modules are R-projective. There are non-free projective modules, such as $\mathbb{Z}_3$ considered as a $\mathbb{Z}_6$-module. The reason is that $\mathbb{Z}_3$ is a summand of $\mathbb{Z}_6$ under the isomorphism $\mathbb{Z}_3 \oplus \mathbb{Z}_2 \cong \mathbb{Z}_6$ As a counter-example to the statement "all modules are projective", we have $\mathbb{Z}_2$, which is not $\mathbb{Z}_4$-projective. This is easily proven, as there is no lift in the diagram below. Here, q is the quotient map obtained when we quotient out by the ideal $\{0, 2\}$ in $\mathbb{Z}_4$.

$$\begin{tikzcd} & \mathbb{Z}_4 \arrow[d, "q"] \\ \mathbb{Z}_2 \arrow[ur, dashed] \arrow[r, "\mathrm{Id}"'] & \mathbb{Z}_2 \end{tikzcd}$$

In Chapter 2, we will be in need of Corollary 1.1.10. The next three results are claims found in [Loday, 1998]. We write out the proofs for completeness.

Proposition 1.1.8. *Let M and N be R-modules. If M and N are R-projective, then $M \otimes N$ is R-projective as an R-module. In particular, if A is R-projective, then $A^{\otimes n}$ is R-projective.*

Proof. Using the "exact functor" definition of projectivity, we try to prove that $\mathrm{Hom}_R(M \otimes N, -)$ is an exact functor. We start by rewriting this via the well known natural isomorphism:

$$\mathrm{Hom}_{R\text{-Mod}}(M \otimes N, -) \cong \mathrm{Hom}_{R\text{-Mod}}(M, (\mathrm{Hom}_{R\text{-Mod}}(N, -))$$

See for instance [Atiyah and Macdonald, 2016] for a proof. Since both M and N are by assumption R-projective, then the right hand side of the isomorphism is the composition of two exact functors. The composition of two exact functors is an exact functor, so $\mathrm{Hom}_{R\text{-Mod}}(M, (\mathrm{Hom}_{R\text{-Mod}}(N, -))$ is an exact functor. Since $\mathrm{Hom}_{R\text{-Mod}}(M \otimes N, -)$ is naturally isomorphic to an exact functor, it is itself an exact functor. Thus we have that $M \otimes N$ is projective as an R-module. □

Proposition 1.1.9. *If P is an R-projective module, then $A \otimes P \otimes A$ is A^e-projective, with the A^e-module structure described in Proposition 1.1.5.*

Proof. We give two proofs for this Proposition. This proof uses the lifting property, and so we want to construct a lift in the diagram of A^e-modules below:

$$\begin{tikzcd} & M \arrow[d, "g"] \\ A \otimes P \otimes A \arrow[r, "f"] & N \end{tikzcd}$$

We start by temporarily thinking of this as a diagram of R-modules rather than of A^e-modules. We can then extend this diagram by the R-linear homomorphism $\iota\colon P \to A \otimes P \otimes A$, given by $\iota\colon p \mapsto 1 \otimes p \otimes 1$. Because we have assumed that P is R-projective, we know that there exists an R-linear lift, $g\colon P \to M$, such that the diagram

$$\begin{tikzcd} & & M \arrow[d, "h"] \\ P \arrow[urr, dashed, "g"] \arrow[r, "\iota"'] & A \otimes P \otimes A \arrow[r, "f"'] & N \end{tikzcd}$$

commute. We will use the A^e-module $A \otimes M \otimes A$ as a stepping stone in the construction of our lift, giving it the A^e-module structure where $(\alpha \otimes \beta) \cdot (a \otimes m \otimes b) := \alpha \cdot a \otimes m \otimes b \cdot \beta$ (this is the same structure as in Proposition 1.1.5). The lift is then defined as the composition of the A^e-linear functions ψ and ϕ, defined on generators as:

$$\begin{aligned} \psi \colon A \otimes P \otimes A &\longrightarrow A \otimes M \otimes A \\ a \otimes p \otimes a' &\longmapsto a \otimes g(p) \otimes a' \end{aligned} \qquad\qquad \begin{aligned} \phi \colon A \otimes M \otimes A &\longrightarrow N \\ a \otimes m \otimes a' &\longmapsto a \cdot m \cdot a' \end{aligned}$$

Notice that ψ is well-defined since it defines an R-multilinear function on $\prod^n A$. It is easy to verify that this function is A^e-linear, by writing out the definitions. A similar argument can be applied to ϕ. Finally, it remains to show that the composition $\phi \circ \psi$ provides a lift in the diagram:

$$\begin{tikzcd} A \otimes M \otimes A \arrow[r, "\phi"] & M \arrow[d, "h"] \\ A \otimes P \otimes A \arrow[u, "\psi"] \arrow[ur, "\phi\circ\psi"'] \arrow[r, "f"'] & N \end{tikzcd}$$

Equivalently, this is to ask if we have the equality $h \circ \psi \circ \phi = f$. It is enough to verify this on the generators, and we see that:

$$\begin{aligned} h(\psi(\phi(a \otimes p \otimes a'))) &= h(a \cdot g(p) \cdot a') = a \cdot h(g(p)) \cdot a' = a \cdot f(\iota(p)) \cdot a' \\ &= a \cdot f(1 \otimes p \otimes 1) \cdot a' = f(a \otimes p \otimes a') \end{aligned}$$

Here, the first equality is by definition, the second equality is by A^e-linearity, the third equality is by commutativity of the second diagram, the fourth equality is by definition and the final equality is again by A^e-linearity. □

Proof. We present a second proof, where we use the direct sum definition. Assume that there exists an R-module S such that $P \oplus S \cong \bigoplus_{i \in I} R_i$. By the following isomorphisms:

$$(A \otimes P \otimes A) \oplus (A \otimes S \otimes A) \cong A^e \otimes P \oplus A^e \otimes S \cong A^e \otimes (P \oplus S) \cong A^e \otimes (\bigoplus_{i\in I} R_i) \cong \bigoplus_{i \in I} R_i \otimes A^e \cong \bigoplus_{i\in I} A^e$$

$A \otimes P \otimes A$ is the direct summand of a free A^e-module, and so $A \otimes P \otimes A$ is A^e-projective per definition. □

Proposition 1.1.8 and Proposition 1.1.9 yield the corollary below.

Corollary 1.1.10. *Let A be an R-algebra with an R-projective underlying module structure. Then we have that $A^{\otimes n+2}$ is projective as an A^e-module for all $n \in \mathbb{N}_0$, where the A^e-module structure on $A^{\otimes n+2}$ is the one given in both Example 1.1.4 and Proposition 1.1.5.*

Proof. By induction on n, we can show that $A^{\otimes n}$ is R-projective. The induction step is to apply Proposition 1.1.8 to $A \otimes (A^{\otimes k-1})$. The base cases of $n = 0, 1$ follow by the standard convention that $A^{\otimes 0} = R$, which is R-projective and from the fact that $A^{\otimes 1} = A$, which is also R-projective. Combining this with Proposition 1.1.9, where we let $P = A^{\otimes n}$, yields the desired result.

□

1.1.3 Group Completion

Group completion will be used in our study of log Hochschild homology in the subsequent chapters. In brief, to group complete is to associate a group M^{gp} to any monoid, M (see the definition immediately below). This group should be optimal with respect to other groups, in a sense made precise by Proposition 1.1.15.

Definition 1.1.11. A **monoid** is a set, M, together with a binary operation, which is associative and has a unit element, 1, such that for all $m \in M$ we have that $1 \cdot m = m = m \cdot 1$. A **commutative monoid** is a monoid where the binary operation is commutative. A **monoid homomorphism** from the monoid M to the monoid N is a function, $f\colon M \to N$, such that f commutes with the binary operation and f sends the unit of M to the unit of N

We write *the* unit since a unit element is always unique by the usual proof. Assuming both 1 and $1'$ to be different units of the monoid, M, then we get a contradiction, since by the property of units we have that:

$$1 = 1 \cdot 1' = 1'$$

Example 1.1.12. Clearly all groups are monoids, since groups are just monoids with an additional axiom of inverses. All group homomorphism are monoid homomorphisms and all monoid homomorphisms between groups are group homomorphism. Similarly, all (unital and associative) rings, $\langle X, +, \cdot \rangle$, are monoids. Both in the sense that the group $\langle X, + \rangle$ is a monoid and in the sense that $\langle X, \cdot \rangle$ satisfies the axiom of a monoid. Whenever we refer to the **underlying monoid** of the ring X we mean the monoid $\langle X, \cdot \rangle$. It is easy to verify that ring homomorphism induces monoid homomorphism on the underlying monoids.

Definition 1.1.13. The **group completion functor** associates to every commutative monoid M the quotient $M^{\text{gp}} := (M \times M)/\sim$, where we identify elements (m_1, m_2) and (n_1, n_2) of $M \times M$ if there exists an element $k \in M$ such that $m_1 \cdot n_2 \cdot k = n_1 \cdot m_2 \cdot k$

Proposition 1.1.14. *Let M be a commutative monoid. Then M^{gp} inherits a binary operation making M^{gp} into an abelian group.*

Proof. First, we prove that the inherited binary operation is well-defined. This means that we need to verify that $(m_1, m_2) \sim (m_1', m_2')$ and $(n_1, n_2) \sim (n_1', n_2')$ implies that $(m_1, m_2)(n_1, n_2) \sim (m_1', m_2')(n_1', n_2')$. By definition, this is the same as asking if $(m_1 n_1, m_2 n_2) \sim (m_1' n_1', m_2' n_2')$. Thus we need to find an element $k \in M$, which such that $m_1 n_1 m_2' n_2' k = m_2 n_2 m_1' n_1' k$. The trick here is to write k as the product of two elements to be chosen later: $k = x \cdot y$. We use commutativity of M to rearrange the left side $m_1 n_1 m_2' n_2' k = m_1 n_1 m_2' n_2' xy = (m_1 m_2' x)(n_1 n_2' y)$, and similarly, we for the right hand side we have $m_2 n_2 m_1' n_1' k = m_1 n_1 m_2' n_2' xy = (m_2 m_1' x)(n_2 n_1' y)$.

By the assumption that $(m_1, m_2) \sim (m_1', m_2')$, we know that there exists an $x \in M$ such that $m_1 m_2' x = m_2 m_1' x$. Similarly, by the assumption that $(n_1, n_2) \sim (n_1', n_2')$ we know that there exists an $y \in M$ such that $n_1 n_2' y = n_2 n_1' y$. We are now finished, since we have:

$$m_1 n_1 m_2' n_2' k = (m_1 m_2' x)(n_1 n_2' y) = (m_2 m_1' x)(n_2 n_1' y) = m_2 n_2 m_1' n_1' k$$

Secondly, we need to find a unit element in M^{gp}. This is not hard, the equivalence class of $(1, 1) \in M \times M$ clearly does this job.

Thirdly, we need to find an inverse for the equivalence class of any element: (m_1, m_2). Notice that, $(1,1) \sim (m_1,m_2)(m_2,m_1) = (m_1m_2, m_2m_1)$, since $1(m_1m_2)k = (m_2m_1)1k$ for any choice of $k \in M$. We omit the final step of verifying commutativity of the group, since this simply involves writing out a general product and comparing it to the commuted product. $\square$

Lemma 1.1.15. *Let M be a commutative monoid. Then there is a monoid homomoprhism denoted by $\gamma\colon M \to M^{\mathrm{gp}}$ such that for all abelian groups G and monoid homomorphism $f\colon M \to G$, there exists a unique monoid homomorphism $f'\colon M^{\mathrm{gp}} \to G$ making the diagram*

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[dr, "\forall f"'] & M^{\mathrm{gp}} \arrow[d, dashed, "\exists! f'"] \\ & G \end{tikzcd}$$

commute.

Proof. First, let $\gamma\colon M \to M^{\mathrm{gp}}$ be given by $\gamma\colon m \mapsto (m,1)$. Then, for any monoid homomorphism $f\colon M \to G$, we define $f'\colon M^{\mathrm{gp}} \to G$ to be the group homomorphism $f'\colon (m_1,m_2) \mapsto f(m_1)\big(f(m_2)\big)^{-1}$. To see that this function is well-defined, let $(m_1,m_2) \sim (n_1,n_2)$, i.e. $m_1n_2k = m_2n_1k$. Then we see that:

$$\begin{aligned} f((m_1,m_2)) = f(m_1)\big(f(m_2)\big)^{-1} &= f(m_1)\big(f(m_2)\big)^{-1} f(n_1)\big(f(n_1)\big)^{-1} f(n_2)\big(f(n_2)\big)^{-1} f(k)\big(f(k)\big)^{-1} \\ &= \big(f(m_1)f(n_2)f(k)\big)\big(f(m_2)f(n_1)f(k)\big)^{-1} f(n_1)\big(f(n_2)\big)^{-1} \\ &= f(n_1)\big(f(n_2)\big)^{-1} = f((n_1,n_2)) \end{aligned}$$

To see that the function f' is the only possible group homomorphism, notice that the equivalence classes of the elements of the form $(m,1)$ and $(1,m)$ generate M^{gp}. Since the diagram above should commute, f must equal $f' \circ \gamma$. Hence $f'(m,1)$ is forced to equal $f(m)$ and since $(1,m)$ is the inverse of $(m,1)$, we get that $f'(1,m) = f'((m,1)^{-1}) = f'(m,1)^{-1} = \big(f(m)\big)^{-1}$. This completes the proof. $\square$

Proposition 1.1.16. *For the commutative monoids, M and N, we have $(M \times N)^{\mathrm{gp}} \cong M^{\mathrm{gp}} \times N^{\mathrm{gp}}$.*

Proof. We prove this using the universal property of the group completion. Let $\gamma' : M \to M^{\mathrm{gp}}$, $\gamma'' : N \to N^{\mathrm{gp}}$ and $\gamma : M \times N \to (M \times N)^{\mathrm{gp}}$ be maps corresponding to the γ in Lemma 1.1.15. By the universal property of the group completion, there exists a unique monoid homomorphism $\phi\colon (M \times N)^{\mathrm{gp}} \to M^{\mathrm{gp}} \times N^{\mathrm{gp}}$ making the diagram

$$\begin{tikzcd} M \times N \arrow[r, "\gamma"] \arrow[dr, "(\gamma',\gamma'')"'] & (M \times N)^{\mathrm{gp}} \arrow[d, dashed, "\exists! \phi"] \\ & M^{\mathrm{gp}} \times N^{\mathrm{gp}} \end{tikzcd}$$

commute. We need there to be a unique monoid homomorphism $\psi\colon M^{\mathrm{gp}} \times N^{\mathrm{gp}} \to (M \times N)^{\mathrm{gp}}$, making the diagram

$$\begin{tikzcd} M \times N \arrow[r, "(\gamma',\gamma'')"] \arrow[dr, "\gamma"'] & M^{\mathrm{gp}} \times N^{\mathrm{gp}} \arrow[d, dashed, "\exists! \psi"] \\ & (M \times N)^{\mathrm{gp}} \end{tikzcd}$$

commute. We know that $M^{\mathrm{gp}} \times N^{\mathrm{gp}}$ is both the product and the coproduct in the category of commutative monoids. Let i_1 be the inclusion of M^{gp} into $M^{\mathrm{gp}} \times N^{\mathrm{gp}}$ with the identity in the second coordinate, and let i_2 be the inclusion of N^{gp} into $M^{\mathrm{gp}} \times N^{\mathrm{gp}}$ with the identity in the first coordinate, and let p_1 be the projection $M \times N \to M$ and p_2 be the projection $M \times N \to N$. The diagram below shows the existence of a unique monoid homomorphism $M^{\mathrm{gp}} \times N^{\mathrm{gp}} \to (M \times N)^{\mathrm{gp}}$, that makes the above diagram commute. Therefore there is a unique monoid homomorphism, $\psi\colon M^{\mathrm{gp}} \times N^{\mathrm{gp}} \to (M \times N)^{\mathrm{gp}}$. In the diagram, the morphisms ψ' and ψ'' follows from the universal property of the group completion, while ψ is from the universal property of the coproduct (see the next section about category theory).

$$
\begin{tikzcd}
M \arrow[dd, "\gamma'"] & M \times N \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\gamma"] & N \arrow[dd, "\gamma''"] \\
 & (M \times N)^{\mathrm{gp}} & \\
M^{\mathrm{gp}} \arrow[r, "i_1"'] \arrow[ur, dashed, "\psi'", "\exists!"'] & M^{\mathrm{gp}} \oplus N^{\mathrm{gp}} \arrow[u, dotted, "\exists!\psi"] & N^{\mathrm{gp}} \arrow[l, "i_2"] \arrow[ul, dashed, "\psi''"', "\exists!"]
\end{tikzcd}
$$

We have proved that there are unique monoid homomorphisms, $\phi\colon (M \times N)^{\mathrm{gp}} \to M^{\mathrm{gp}} \times N^{\mathrm{gp}}$ and $\psi\colon M^{\mathrm{gp}} \times N^{\mathrm{gp}} \to (M \times N)^{\mathrm{gp}}$ making both the diagrams

$$
\begin{tikzcd}
 & (M \times N)^{\mathrm{gp}} \arrow[d, dashed, "\exists!\phi"] \\
M \times N \arrow[ur, "\gamma"] \arrow[r, "(\gamma',\gamma'')"] \arrow[dr, "\gamma"'] & M^{\mathrm{gp}} \times N^{\mathrm{gp}} \arrow[d, dashed, "\exists!\psi"] \\
 & (M \times N)^{\mathrm{gp}}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M^{\mathrm{gp}} \times N^{\mathrm{gp}} \arrow[d, dashed, "\exists!\psi"] \\
M \times N \arrow[ur, "(\gamma',\gamma'')"] \arrow[r, "\gamma"] \arrow[dr, "(\gamma',\gamma'')"'] & (M \times N)^{\mathrm{gp}} \arrow[d, dashed, "\exists!\phi"] \\
 & M^{\mathrm{gp}} \times N^{\mathrm{gp}}
\end{tikzcd}
$$

commute. By the universal property of the group completion, there is also a unique monoid homomorphism equal to the composite of the vertical maps making the leftmost diagram commute. There is also at most a unique monoid homomorphism in the rightmost diagram. To see this, notice that by the universal property of the group completion, there is precisely one unique monoid homomorphism in each coordinate that can make the outer diagram to commute (the unique map from M^{gp} to itself induced by γ' and the unique map from N^{gp} to itself induced by γ''). Thus the product of these functions is the only possible monoid homomorphism that makes the diagram commute.

Clearly, the appropriate identity homomorphisms are monoid homomorphisms making both outer triangle commute in the diagram above. Since we have proven the uniqueness of these monoid homomorphisms, the compositions along the vertical maps in the above diagrams must equal the identity morphism. Hence we have that ψ and ϕ are inverses to each other.

□

There are other ways to prove the previous proposition. For instance, Lemma 1.2.15 in the next section states that all right adjoint functors preserve limits. Products and coproducts are isomorphic for commutative monoids and abelian groups, and coproducts are a kind of colimit. The group completion functor can be shown to be left adjoint functor (see [Mac Lane, 1971]), and so the product should commute with group completion. A third approach would be to prove directly that we have inverse homomorphisms

$$\begin{aligned} \psi\colon M^{\mathrm{gp}} \times N^{\mathrm{gp}} &\longrightarrow (M \times N)^{\mathrm{gp}} \\ \big((m_1, m_2), (n_1, n_2)\big) &\longmapsto \big((m_1, n_1), (m_2, n_2)\big) \end{aligned} \qquad \begin{aligned} \phi\colon (M \times N)^{\mathrm{gp}} &\longrightarrow M^{\mathrm{gp}} \times N^{\mathrm{gp}} \\ \big((m_1, n_1), (m_2, n_2)\big) &\longmapsto \big((m_1, m_2), (n_1, n_2)\big) \end{aligned}$$

1.2 Basic Category Theory

This is a short introduction which covers the category theory that will be used in this thesis. There are numerous books on the topic, and the interested reader can consult any one of these for a more detailed exposition of the subject (see for instance the standard source on the topic [Mac Lane, 1971], or the more leisurely written [Adámek et al., 2006]). We shall use [Mac Lane, 1971] as a general reference for the definitions given in this section.

1.2.1 Categories & Functors

Category theory provides both a useful language and an efficient organizing tool, permeating many areas of modern pure mathematics. If a construction in one part of mathematics can be expressed categorically, it is often easy to see how its analogue may be defined in completely different fields. The reader who is unfamiliar with the subject might want to mentally replace the word object by set and the word morphism by function in the definition below.

Definition 1.2.1. We say that $\mathcal{C}$ is a **category** if $\mathcal{C}$ consists of a class of **objects**, $\mathcal{O}_{\mathcal{C}}$, and a class of **morphisms/arrows**, $\mathcal{A}_{\mathcal{C}}$, which we describe shortly. $\mathcal{C}$ should also be equipped with four operations relating these two classes. These operations must be subject to a unit law and an associativity axiom. Explicitly, we have:

1. The **domain** operation, $\mathrm{dom}(-)$, assigns an object to every morphism.

2. The **codomain** operation, $\mathrm{cod}(-)$, assigns an object to every morphism.

3. The **identity** operation, $\mathrm{Id}_{(-)}$, assigns a morphism to every object, called the identity morphism of the object. It relates to the previous operations in that for all objects $a \in \mathcal{C}$, $\mathrm{dom}(\mathrm{Id}_a)$=$\mathrm{cod}(\mathrm{Id}_a) = a$.

4. The **composition** operation, $\circ$, assigns to every pair of morphisms (g, f) where $\mathrm{cod}(f) = \mathrm{dom}(g)$ a composite morphism $g \circ f$. The domain and codomain is given to be $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$.

Finally we have the two axioms that needs to be fulfilled. These axioms should hold for all objects a and for all morphism f, g, h, i and j in any category, where the compositions given below are defined. First we have the **axiom of associativity**, which says that composition of morphism is associative: $h \circ (g \circ f) = (h \circ g) \circ f$. Secondly, the **unit laws** says that composing a morphism with the identity is the same as doing nothing, i.e. that $i \circ \mathrm{Id}_a = i$ and $\mathrm{Id}_a \circ j = j$.

We frequently write $f\colon a \to b$ for a morphism f with domain a and codomain b. This is useful, since it is both compact and it helps visualization. For example, the identity operation can be described in the following way: For every object a there is a morphism $\mathrm{Id}_a\colon a \to a$. Similarly for the composition operation: For all objects a, b and c and for all morphism $f\colon a \to b$ and $g\colon b \to c$ there is a morphism $g \circ f\colon a \to c$.

Remark 1.2.2. The definition of a category is symmetric in the sense that if we swap every domain and codomain and reverse the order of every composition so that $f \circ g$ becomes $g \circ f$ we obtain a new category. This category is often referred to as the **dual** or **opposite** category. We write C^{op} for the opposite category of C

Definition 1.2.3. We say that $\mathcal{C}'$ is a **subcategory** of $\mathcal{C}$ if $\mathcal{C}'$ is a category where all objects and all morphisms of $\mathcal{C}'$ are respectively objects and morphisms of $\mathcal{C}$. We write $\mathcal{C}' \subset \mathcal{C}$ for this. A subcategory, $\mathcal{C}' \subset \mathcal{C}$, is said to be a **full subcategory** if the collection, $\mathrm{Hom}_{\mathcal{C}'}(a, b)$, of morphisms between any two objects $a, b \in \mathcal{C}'$, is equal to the collection, $\mathrm{Hom}_{\mathcal{C}}(a, b)$, of all morphisms between any two objects $a, b \in \mathcal{C}$.

Example 1.2.4. The following categories will all crop up at some point in the thesis.

- The category of sets, denoted as **Set**. The objects are sets and the morphisms are functions.
- The category of monoids, for which we write **Mon**. The objects are monoids and the morphisms are monoid homomorphisms, i.e. functions $f\colon M \to N$ where for all $m, n \in M$, $f(m \cdot n) = f(m) \cdot f(n)$ and $f(1) = 1$. As a subcategory of **Mon** we have another category: The category of commutative monoids, denoted **CMon**. The objects are commutative monoids and the morphisms are monoid homomorphisms.
- The category of groups, denoted as **Grp**. The objects are groups and the morphism are group homomorphisms. Similarly to above, we have a subcategory inside the category of groups called the category of abelian groups, **AbGrp**. **AbGrp** have as objects all abelian groups and as morphisms the group homomorphisms between them.
- We also have a category of rings, **Ring**, and a category commutative rings **CRing**. These are defined analogously to the above examples of **Mon** and **CMon**, and of **Grp** and **AbGrp**.
- The category of R-modules, denoted as R-**Mod**: The objects are R-modules, and the morphisms are R-linear homomorphisms.
- The category of A-bimodules, with notation A-**BiMod**: The objects are A-bimodules, and the morphisms are homomorphisms that are right A-linear and left A-linear.

These examples are all categories where the objects are sets with some additional structure and the morphisms are functions that preserve this structure. Far from every category appear in this form however, and concepts such as commutative diagrams and order relations can be interpreted as categories. There is also a category of categories, where the objects are categories and the morphisms are called functors:

Definition 1.2.5. A **functor**, F, from the category $\mathcal{C}$ to the category $\mathcal{D}$, usually written as $F\colon \mathcal{C} \to \mathcal{D}$, consists of two operations, both of which are called F. One that sends objects to objects, $F\colon \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{D}}$, and one that sends morphisms to morphisms, $F\colon \mathcal{A}_{\mathcal{C}} \to \mathcal{A}_{\mathcal{D}}$, such that for any object, a, and any pair of composable morphisms, f and g, in $\mathcal{C}$, then:

$$F(\mathrm{Id}_a) = \mathrm{Id}_{F(a)}$$
$$F(f \circ g) = F(f) \circ F(g)$$

A functor that is defined to have domain in the opposite category C^{op} of some category C is called a **contravariant** functor.

We make the last statement prior to the definition precise: Notice that the object and morphism wise composition of two functors is a functor and that there exists identity functors for any category $\mathrm{Id}\colon \mathcal{C} \to \mathcal{C}$, sending objects $\mathrm{Id}\colon a \mapsto a$ and morphisms $\mathrm{Id}\colon f \mapsto f$. Thus we have justified the claim above the definition, stating that the functors can be morphisms between categories.

Example 1.2.6. Here is a list of some relevant functors the reader might recognize. Several of the examples are called "the forgetful" functor. This prefix just imply that the functor is defined to "forget" some of the structure of the objects in the domain category, like a binary operation, an order, the topology etc.

- The forgetful functor **Mon** $\to$ **Set**. It sends a monoid $\langle M, \cdot \rangle$ to the underlying set M and a monoid homomorphism $f\colon \langle M, \cdot \rangle \to \langle N, \cdot \rangle$ to the underlying function $f\colon M \to N$.

- The forgetful functor **AbGrp** $\to$ **Grp**. It sends an abelian group to itself, but as an object in **Grp**. Similarly for the morphism, $f\colon G \to H \in$ **AbGrp** is simply sent to $f\colon G \to H \in$ **Grp**

- The forgetful functor **CRing** $\to$ **CMon**. It sends a commutative ring $\langle A, +, \cdot \rangle$ to the underlying monoid $\langle A, \cdot \rangle$. A ring homomorphism $f\colon A \to B \in$ **AbGrp** is simply sent to $\langle f, \cdot \rangle\colon \langle A, \cdot \rangle \to \langle B, \cdot \rangle \in$ **Grp**

- The free monoid functor $(-)^*\colon$ **Set** $\to$ **Mon**. It sends a set X to the **free monoid generated on the set** X. As a set, this is all the different finite "words" with "letters" in X: $X^* := \{x_1 x_2 \ldots . x_n \mid x_i \in X, n \in \mathbb{N}\}$. This is a monoid, with monoid multiplication defined to be the conjoining of two words. The unit element is then the "empty word" or the "word with no letters". This is often given some appropriate notation, like $*$ or 1. Functions $f\colon X \to Y$ are sent to the unique monoid morphism $f^*\colon X^* \to Y^*$ induced by sending $x \mapsto f(x)$

- The abelianization functor **Grp** $\to$ **AbGrp**. It sends a group G to the abelianized group $\frac{G}{[G,G]}$, and morphisms to the morphisms induced on those quotients.

- The free R-algebra functor on a monoid M, $R[-]\colon$ **CMon** $\to$ **CR-alg**. Let $\mathbb{N}^*$ denote the set of positive integers, excluding 0. The functor $R[-]$ sends a commutative monoid, M, to the **free R-algebra** generated on M, $R[M]$. This R-algebra has as its underlying set:

$$\Big\{ \sum_{i=1}^{i=n} r_i \cdot m_i \mid r_i \in R, m_i \in M, n \in \mathbb{N}^* \Big\}$$

  The addition of two elements in this set is defined as it is for addition of free group on elements in M, while multiplication is defined to be:

$$\Big(\sum_{i=1}^{i=n} r_i \cdot m_i\Big)\Big(\sum_{j=1}^{j=m} r_j m_j\Big) = \sum_{i=1}^{i=n}\sum_{j=1}^{j=m} (r_i \cdot r_j) \cdot (m_i \cdot m_j)$$

Multiplication of the scalar r in R with an element x in $R[M]$ is given by letting $r \cdot x = (r \cdot 1) \cdot (x)$. A monoid morphism $\phi\colon M \to N$ induces the morphism

$$\overline{\phi}\colon R[M] \longrightarrow R[N]$$
$$\overline{\phi}\colon \sum_{i \in I} r_i \cdot m_i \longmapsto \sum_{i \in I} r_i \cdot \phi(m_i)$$

1.2.2 Natural Transformations & Adjoint Functors

Definition 1.2.7. Let F and G be two functors $F, G\colon \mathcal{C} \to \mathcal{D}$. A **natural transformation** from F to G, written as $\alpha\colon F \to G$, is a function (operation) assigning to each object, $a \in \mathcal{C}$, a morphism in $\mathcal{D}$, $\alpha_a\colon F(a) \longrightarrow G(a)$, in such a way that for all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ we always have commutativity $\alpha_a \circ F(f) = G(f) \circ \alpha_b$. More intuitively, this condition means that the diagram beneath commutes for every morphism $f \in \mathcal{C}$:

$$\begin{tikzcd}
F(a) \arrow[r, "F(f)"] \arrow[d, "\alpha_a"] & F(b) \arrow[d, "\alpha_b"] \\
G(a) \arrow[r, "G(f)"'] & G(b)
\end{tikzcd}$$

We have a category of functors from any category $\mathcal{C}$ to any category $\mathcal{D}$, where the natural transformations are the morphisms. For the existence of an identity morphism, we let F be any functor $F\colon \mathcal{C} \to \mathcal{D}$. The natural transformation $\mathrm{Id}\colon F \to F$ defined by assigning the identity morphism of $F(a)$ to every object a in the category $\mathcal{C}$ is a natural transformation. For the existence of compositions, we let F, G, $H\colon \mathcal{C} \to \mathcal{D}$ be three functors, where there are natural transformations: $\alpha\colon F \to G$ and $\beta\colon G \to H$. Then we can define the composition $\beta \circ \alpha$ object wise, by sending each object to the composition of the two original morphisms. We write $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ for this category.

Adjoint functors occur frequently in mathematics, and there are several different definitions of these, with the different definitions illustrating different aspects. The definition we give below is referred to as the Hom-set definition of adjoints.

Definition 1.2.8. A pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are said to be **adjoint** to each other if for every object $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ there is an isomorphism:

$$\iota\colon\ \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

These isomorphisms should be "natural" for all objects in $\mathcal{C}$ and for all objects in $\mathcal{D}$. By natural we mean that for any morphism $f\colon X' \to X$ or $g\colon Y \to Y'$ respectively, the diagrams below should commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(F(X), Y) \arrow[r, "\cong", "\iota"'] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(F(f), \mathrm{Id}_Y)"] & \mathrm{Hom}_{\mathcal{D}}(X, G(Y)) \arrow[d, "\mathrm{Hom}_{\mathcal{D}}(f, \mathrm{Id}_{G(Y)})"] \\
\mathrm{Hom}_{\mathcal{C}}(F(X'), Y) \arrow[r, "\cong", "\iota"'] & \mathrm{Hom}_{\mathcal{D}}(X', G(Y))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(F(X), Y) \arrow[r, "\cong", "\iota"'] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(\mathrm{Id}_{F(X)}, g)"] & \mathrm{Hom}_{\mathcal{D}}(X, G(Y)) \arrow[d, "\mathrm{Hom}_{\mathcal{D}}(\mathrm{Id}_X, G(g))"] \\
\mathrm{Hom}_{\mathcal{C}}(F(X), Y') \arrow[r, "\cong", "\iota"'] & \mathrm{Hom}_{\mathcal{D}}(X, G(Y'))
\end{tikzcd}$$

F is then called the **left adjoint** functor to G, and G is called the **right adjoint** functor to F

This is a slightly technical formulation, and the best way to learn what it means is to have a look at a few examples. In fact, we have already defined three pairs of adjoint functors. We go through the first one in detail:

Example 1.2.9. The forgetful functor **Mon** $\to$ **Set** is right adjoint to the free monoid functor **Set** $\to$ **Mon**. A function $f\colon X \to M$ induces a unique monoid homomorphism $\overline{f}\colon X^* \to M$ by sending $x_1x_2\dots x_n \mapsto f(x_1)f(x_2)\dots f(x_n)$, similarly a monoid morphism $g\colon X^* \to M$ induces a unique function from $\underline{g}\colon X \to M$ by $x_i \mapsto g(x_i)$. Notice that $\overline{(\underline{g})} = g$ and $\underline{(\overline{f})} = f$, so there is a bijection. The naturality requirement for objects in **Mon** and **Set** is by noting that composing with a function $\phi\colon Y \to X$ or a monoid homomorphism $\psi\colon M \to N$ induces commutativity as required in the definition:

Example 1.2.10. . The forgetful functor **AbGrp** $\to$ **Grp** is right adjoint to the abelianization functor **Grp** $\to$ **AbGrp**.

Example 1.2.11. The forgetful functor **C**R**-alg** $\to$ **CMon** is right adjoint to the free R-algebra functor $R[-]\colon$ **CMon** $\to$ **C**R**-alg**.

1.2.3 Limits & Colimits

The definition of a (co)limit is rather technical, so we start with a short example: If we have groups X, G and H and a pair of morphisms $f_1\colon X \to G$ and $f_2\colon X \to H$, then we know that we can define a function $f_1 \times f_2\colon X \to G \times H$. Conversely, if we define a function $f\colon X \to G \times H$ we can obtain a pair of functions $f_1\colon X \to G$ and $f_2\colon X \to H$. There is no choice involved when we jump between these interpretations. Also, when we do translate twice, we end up with the same (two) morphism(s) we started with. In other words, we have found a construction that compresses information about a pair of morphism from a fixed object into one morphism from that object. This is a special example of what a limit is, namely a single object that "contains the information" of a collection of related or unrelated objects. The following discussion is about making this precise, culminating in the definition of a limit. We need some preliminary definitions to develop a sufficient language:

An **initial object** is an object, $\iota \in \mathcal{C}$, such that for any object, $x \in \mathcal{C}$, there exists a unique morphism $f_x\colon \iota \to x$. A **terminal object** is an object, $\eta \in \mathcal{C}$, such that for any object, $y \in \mathcal{C}$, there exists a unique morphism $g_y\colon y \to \eta$.

Let $\mathcal{J}$ be a (small) category, meaning that both $\mathcal{O}_\mathcal{J}$ and $\mathcal{A}_\mathcal{J}$ should be sets (as opposed to something "bigger"). We define a **diagram** of shape $\mathcal{J}$ in the category $\mathcal{C}$ to be a functor $F\colon \mathcal{J} \to \mathcal{C}$. Diagrams are often visualized as actual diagrams whenever the shape of the diagram, $\mathcal{J}$, consists of a small number of objects/morphisms. One of the most common examples, a commuting square, is a diagram where the shape is given by four objects and five morphism (the sides of the rectangle and an "invisible" diagonal morphism, forcing commutativity).

The **diagonal functor**, $\Delta(-)\colon \mathcal{C} \to \mathcal{C}^\mathcal{J}$ sends:

- An object $a \in \mathcal{C}$ to the **constant functor** $\Delta(a)\colon \mathcal{J} \to \mathcal{C}$. The constant functor a sends, as its name suggest, every object of $\mathcal{J}$ to a and every morphism in $\mathcal{J}$ to the identity of a (that this is a functor is clear from the definition).

- A morphism $f\colon a \to b$ in $\mathcal{C}$ is sent to the natural transformation that assigns to every object $X \in \mathcal{J}$ the same morphism: $f\colon a \to b$.

Finally, a **cone** over a diagram, $F \in \mathcal{C}^{\mathcal{J}}$, is a natural transformation from $\Delta(a)$ to the functor F. It is not hard to construct a finite category in which a specific diagram does not have any cone.

Definition 1.2.12. Let F be a fixed diagram, $F \in \mathcal{C}^{\mathcal{J}}$. Then the limit of F, $\varprojlim F$ is defined to be the terminal object in the category of cones over F. Written out, $\varprojlim F$ is a cone over F, $\Delta(t)\colon \mathcal{J} \to F$, in the category $\mathrm{Hom}_{\mathbf{Cat}}(\Delta(-), \mathcal{C}^{\mathcal{J}})$, where for all other cones, $\Delta(s)\colon \mathcal{J} \to \mathcal{C}$ there exists an unique natural transformation $\xi\colon \Delta(s) \to \Delta(t)$.

Colimits are defined similarly to limits. We define a **cocone** under a diagram, $F \in \mathcal{C}^{\mathcal{J}}$, to be a natural transformation to $\Delta(a)$ from the functor F.

Definition 1.2.13. Let F be a fixed diagram, $F \in \mathcal{C}^{\mathcal{J}}$. Then the colimit of F, $\varinjlim F$ is defined to be the initial object in the category of cocones under F. Written out, $\varinjlim F$ is a cocone over F, $\Delta(i)\colon \mathcal{J} \to F$, in the category $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}^{\mathcal{J}}, \Delta(-))$, where for all other cocones, $\Delta(j)\colon \mathcal{J} \to \mathcal{C}$ there exists an unique natural transformation $\xi'\colon \Delta(i) \to \Delta(j)$.

If the reader has never heard of (co)limits before, an internet search for examples and graphical illustration might be helpful at this point. There is an abundance of examples available, and these are important to get the feel of the subject.

Limits and colimits need not exist (again, it is easy to construct categories in which there are counter-examples). A (co)limit is said to be small if the shape $\mathcal{J}$ consists of a set of objects and a set of morphisms (as opposed to classes or something larger). A category in which all small limits exists is called a **complete category**, a category where all small colimits exists is called a **cocomplete category** and a complete cocomplete category is called a **bicomplete** category.

Example 1.2.14. Limits and colimits are common in mathematics. For instance, whenever some construction is called a "product", it is likely a limit of a diagram in which $\mathcal{J}$ is the category of two objects and two (identity) morphisms. The limit of this diagram in the category of sets, rings, groups, monoids, R-algebras and topological spaces can all be shown to be the product (see [Mac Lane, 1971]). The colimits of this kind of diagram have varying names, i.e. it is the direct sum in the category of groups and monoids, it is the disjoint union in the case of sets and in the case of commutative R-algebras it is the tensor product of the two objects over R. For proofs and more examples, see [Mac Lane, 1971].

Initial/terminal objects (if they exist in a given category) can be expressed respectively as the colimit/limit of the diagram in the shape of the empty category (no objects, no morphisms). This follows from the definitions above, though it might be challenging to see this immediately

Lemma 1.2.15. *Right adjoint functors preserve limits and left adjoint functors preserve colimits. In detail, if we let F be a diagram of shape $\mathcal{J}$ in the category $\mathcal{C}$, and let X be a right adjoint functor $X\colon \mathcal{C} \to \mathcal{D}$ and Y be a left adjoint functor $Y\colon \mathcal{C} \to \mathcal{D}$. Then if $\varprojlim F$ exists, there is a natural isomorphism:*

$$X(\varprojlim F) \cong \varprojlim (X \circ F)$$

Correspondingly, if $\varinjlim F$ exists, there is a natural isomorphism:

$$Y(\varinjlim F) \cong \varinjlim (Y \circ F)$$

Proof. For a proof of this lemma, see [Mac Lane, 1971]. $\square$

1.3 Homological Algebra

In this section we present a short review of the homological algebra that we will need later on. The main topics to be covered are chain complexes, homology, resolutions and the Tor functor. We use [Mac Lane, 1967] as a general reference for this section.

1.3.1 Chain Complexes & Homology

Definition 1.3.1. A **chain complex** of R-modules is a sequence of R-modules, C_n, and of R-linear maps, $d\colon C_n \to C_{n-1}$, with $n \in \mathbb{Z}$. The d's are called the **boundary maps** of the chain complex, and they all have to have the property that $d^2 = d \circ d = 0$. We often use the notation C_* or D_* when referring to arbitrary chain complexes.

A chain complex, C_*, is commonly visualized by writing it in the following way:

$$\dots \xrightarrow{d} C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \dots \xrightarrow{d} C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{d} C_{-1} \xrightarrow{d} \dots$$

We call the elements of the R-module C_n of a chain complex C_* for the n**-chains** of the complex. An element of C_n is said to have **dimension** n. The chain complexes we use will always be **non-negative**, by which we mean that all the $C_{-n} = 0$ when n is a positive integer different from zero. We visualize non-negative chain complexes by writing

$$\dots \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \dots \xrightarrow{d} C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{d} 0$$

Here, we have omitted the infinite trail of zero modules following C_0. From this point onwards, we suppress the "non-negative" in "non-negative chain complex", and refer to them simply as chain complexes.

Definition 1.3.2. The morphisms in the category of chain complex $f_*\colon C_* \to D_*$ are called **chain homomorphisms**. A chain homomorphism is a collection of R-module homomorphisms, $\{f_n\colon C_n \to D_n\}$, such that $d \circ f_n = f_{n-1} \circ d$ for all $n \in \mathbb{N}^*$. In other words, all the squares in the diagram below should commute:

$$\begin{tikzcd}
\dots \arrow[r, "d"] & C_{n+1} \arrow[r, "d"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d"] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r, "d"'] & D_{n+1} \arrow[r, "d"'] & D_n \arrow[r, "d"'] & D_{n-1} \arrow[r, "d"'] & \dots
\end{tikzcd}$$

The composition of two chain homomorphisms is defined by composing the homomorphisms in each dimension. We verify that the result is a chain homomorphism by writing out the composition $g_* \circ f_*\colon C_* \to E_*$, where $f_*\colon C_* \to D_*$ and $g_*\colon D_* \to E_*$.

$$\begin{tikzcd}
\dots \arrow[r, "d"] & C_{n+1} \arrow[r, "d"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d"] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r, "d'"] & D_{n+1} \arrow[r, "d'"] \arrow[d, "g_{n+1}"] & D_n \arrow[r, "d'"] \arrow[d, "g_n"] & D_{n-1} \arrow[r, "d'"] \arrow[d, "g_{n-1}"] & \dots \\
\dots \arrow[r, "d''"] & E_{n+1} \arrow[r, "d''"] & E_n \arrow[r, "d''"'] & E_{n-1} \arrow[r, "d''"] & \dots
\end{tikzcd}$$

We know that all the smallest squares of this diagram commutes by definition, and so all the rectangles in the diagrams with edges d, d'', $g_n \circ f_n$ and $g_{n-1} \circ f_{n-1}$ commutes. The collection of identity homomorphisms $\{\mathrm{Id}_{C_n} : C_n \to C_n\}$ defines a chain homomorphism. We write $\mathrm{Id}_{\mathbf{C}_*} : \mathbf{C}_* \to \mathbf{C}_*$ for this chain homomorphism. It is now easy to see that we have the **category of chain complexes** of R-modules, for which we use the notation **Ch(*R*-Mod)**.

Definition 1.3.3. The ***n*-th homology group** of the chain complex C_*, for which we write $\mathrm{H}_n(C_*)$, is defined as the quotient:

$$\mathrm{H}_n(C_*) = \frac{\ker(d\colon C_n \to C_{n-1})}{\mathrm{im}(d\colon C_{n+1} \to C_n)}$$

Homology groups are well-defined, since d^2 is always the zero map, and so $im(d\colon C_{n+1} \to C_n) \subset ker(d\colon C_n \to C_{n-1})$. We usually refer to elements of the kernel of $d\colon C_n \to C_{n-1}$ as ***n*-cycles** and the elements of the image of $d\colon C_{n+1} \to C_n$ as ***n*-boundaries**. The elements $[x] \in \mathrm{H}_n(C_*)$ will be called **homology classes of degree** n. Notice that if the homology groups $\mathrm{H}_n(C_*)$ are all zero, we get that $im(d\colon C_{n+1} \to C_n) = ker(d\colon C_n \to C_{n-1})$ for all n. This is the definition of an exact sequence, so the homology groups can be thought of as a measure of how close a chain complex is to be an exact sequence.

Proposition 1.3.4. *Homology is a functor from the category of chain complexes to the category of R-modules,* $\mathrm{H}_n(-)\colon \mathbf{Ch}(R\text{-}\mathbf{Mod}) \longrightarrow R\text{-}\mathbf{Mod}$. *A chain homomorphism* $f_*\colon C_* \to D_*$ *induces the R-module homomorphism given by:*

$$\begin{aligned} \overline{f_n}\colon \mathrm{H}_n(C_*) &\longrightarrow \mathrm{H}_n(D_*) \\ [x] &\longmapsto [f_n(x)] \end{aligned}$$

Proof. Let x be an n-cycle of C_n, so that $dx = 0$. We then have that $df(x) = f(dx) = f(0) = 0$, so every n-cycle C_* is sent to an n-cycle of D_*. To see that this homomorphism induces a function on the quotient group, let y be an n-boundary of C_n, meaning that $y = dz$ for some some $z \in C_{n+1}$. Then we have that $f(y) = f(dz) = d(fz)$. Hence f sends boundaries to boundaries, and so the morphism f_* is well-defined. □

Definition 1.3.5. Let C_* and D_* be chain complexes. We say that the two chain homomorphisms f_*, $g_*\colon C_* \to D_*$ are **chain homotopic** if there exists a **chain homotopy** from f_* to g_*, meaning a collection of maps $\{h_n\colon C_n \to D_{n+1} \mid n \in \mathbb{N}_0\}$ such that $f_n - g_n = dh_n + h_{n-1}d$ for all n. We write $f_* \simeq g_*$ to indicate that f_* and g_* are chain homotopic.

Lemma 1.3.6. *Let* C_* *and* D_* *be chain complexes and let* f_*, $g_*\colon C_* \to D_*$ *be chain homomorphisms that are chain homotopic. Then* $\overline{f_n} = \overline{g_n}\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ *for all* $n \in \mathbb{N}_0$

Proof. Let $h_n\colon f_* \to g_*$ be the chain homotopy. Then we now that $f_n - g_n = dh_n + h_{n-1}$ or equivalently that $f_n = dh_n + h_{n-1}d + g_n$. By definition, $\overline{f_n}\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ sends $[x]$ to $[f_n(x)]$. We see that this means that

$$\begin{aligned} \overline{f_n}([x]) &= [f_n(x)] = [(dh_n + h_{n-1}d + g_n)(x)] \\ &= [dh_n(x) + h_{n-1}d(x) + g_n(x)] \\ &= [dh_n(x)] + [h_{n-1}d(x)] + [g_n(x)] \\ &= [0] + [h_{n-1}(0)] + [g_n(x)] = [0] + [0] + [g_n(x)] = [g_n(x)] = \overline{g_n}([x]) \end{aligned}$$

where $dx = 0$ since $x \in \ker(d\colon C_n \to C_{n-1})$ and $dh_n(x)$ is in the same equivalence class as 0 since, $dh_n(x) \in \operatorname{im}(d\colon C_{n+1} \to C_n)$ □

Let C_* and D_* be chain complexes. We say that C_* and D_* are **homotopy equivalent** if there are chain homomorphisms $f_*\colon C_* \to D_*$ and $g_*\colon D_* \to C_*$ such that $g_* \circ f_* \simeq \mathrm{Id}_{C_*}$ and $f_* \circ g_* \simeq \mathrm{Id}_{D_*}$. We call the f_* and g_* for **homotopy equivalences**.

Corollary 1.3.7. *Let C_* and D_* be homotopy equivalent chain complexes with homotopy equivalences as in the text above. Then the induced $\overline{f_n}\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ is an isomorphism with inverse $\overline{g_n}\colon \mathrm{H}_n(D_*) \to \mathrm{H}_n(C_*)$.*

Proof. The previous Lemma implies $\overline{g_n} \circ \overline{f_n} = \overline{g_n \circ f_n} = \mathrm{Id}_{\mathrm{H}_n(C_*)}$ and $\overline{f_n} \circ \overline{g_n} = \overline{f_n \circ g_n} = \mathrm{Id}_{\mathrm{H}_n(D_*)}$ for all n, and so $\overline{f_n}$ and $\overline{g_n}$ are inverses to each other as claimed. □

1.3.2 The Tor Functor

Definition 1.3.8. Let M be an R-module. A **resolution** of M is an exact sequence of R-modules

$$C_*\colon \dots \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \dots \xrightarrow{d} C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0$$

together with a morphism $\epsilon\colon C_0 \to M$, called the **augmentation map**. This augmentation map should cause the sequence of morphisms below to become a chain complex where the zeroth homology group is equal to zero, which is equivalent to requiring that the same sequence is exact.

$$\dots \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \dots \xrightarrow{d} C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{\epsilon} M$$

We will usually write $\epsilon\colon C_* \to M$ to indicate that C_* is a resolution of M. In the next proposition we will give the product of the A^e-module X by the B^e-module Y the $(A \times B)^e$-module structure induced by the ring homomorphism:

$$\begin{aligned} p\colon (A \times B) \otimes (A \times B)^{op} &\longrightarrow A \otimes A^{op} \times B \otimes B^{op} \\ (a_1, b_1) \otimes (a_2, b_2) &\longmapsto (a_1 \otimes a_2, b_1 \otimes b_2) \end{aligned}$$

This implies that that for $(a, b) \otimes (a', b') \in (A \times B)^e$ and $(x, y) \in X \times Y$ we get a scalar multiplication defined as:

$$((a, b) \otimes (a', b')) \cdot (x, y) = (a \cdot x \cdot a', b \cdot y \cdot b')$$

Proposition 1.3.9. *Assume A and B to be R-algebras and assume there to be resolutions of A^e-modules and B^e-modules respectively, given as below:*

$$\begin{aligned} \epsilon\colon C_* &\longmapsto M \\ \delta\colon D_* &\longmapsto N \end{aligned}$$

Then we have that the degreewise product of these resolutions, $\epsilon \times \delta\colon C_* \times D_* \to M \times N$, is an $(A \times B)^e$-module resolution of $M \times N$. We use the $(A \times B)^e$-module structure as described above on each $C_n \times D_n$ and we let the boundary maps be $d \times d'\colon C_n \times D_n \to C_{n-1} \times D_{n-1}$.

Proof. It only requires some simple calculations to check that we have well-defined module structures and that the maps are $(A \times B)^e$-linear with respect to these structures. We have therefore only to show that the sequence $\epsilon \times \delta \colon C_* \times D_* \to M \times N$ is exact. This follows immediately, since we know that $ker(d \times d') = ker(d) \times ker(d') = im(d) \times im(d') = im(d \times d')$ for all the boundary maps d and d'. $\square$

The Tor functor is a construction that produces a sequence of homology groups to any R-module M. We give its construction and state some of its properties.

Definition 1.3.10. Let X be a ring, not necessarily commutative, and let A be a right X-module and B be a left X-module. $\mathrm{Tor}_n^X(A, B)$ is the group constructed by the following three steps: First, choose an **projective resolution** of A, meaning a resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to A$$

where all the X-modules, P_i, are projective. Then remove the "A" term, and tensor the above exact sequence by B to get the chain complex:

$$\cdots \to P_2 \otimes_X B \to P_1 \otimes_X B \to P_0 \otimes_X B$$

Finally, take n-th homology of this chain complex. This is by definition $\mathrm{Tor}_n^X(A, B)$.

We now paraphrase [Mac Lane, 1967, p. 160] Theorem 8.1, which states that the Tor groups above does not depend on the projective resolution chosen. We have changed the name of the variables in the original theorem.

Theorem 1.3.11. *For a resolution $\epsilon \colon P_* \to A$ of the module A_X and a module ${}_XB$, there is a homomorphism*

$$\omega \colon \mathrm{Tor}_n^X(A, B) \longrightarrow \mathrm{H}_n(P_* \otimes_X A), \quad n = 0, 1 \ldots,$$

natural in B. If P_* is a projective resolution, ω is an isomorphism natural in A and B

1.3.3 Tensor Products of Chain Complexes

In this last section about homology we introduce the definition of the tensor product of chain complex. We also state the Künneth theorem for general chain complex, which explain how the homology groups of the tensor product of chain complex relates to the tensor product of the homology groups of the same chain complexes.

Definition 1.3.12. Let C_* and D_* be two chain complexes of R-modules. Then their **tensor product**, denoted sometimes as $C_* \otimes D_*$ and sometimes as $(C \otimes D)_*$, is defined to be the chain complex where:

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

The **boundary map** of the tensor product of chain complexes is the homomorphism

$$\delta \coloneqq \bigoplus_{p+q=n} \delta_{p,q} \colon \bigoplus_{p+q=n} C_p \otimes D_q \to \bigoplus_{p+q=n-1} C_p \otimes D_q$$

where the maps $\delta_{p,q}$ are given as:

$$\delta_{p,q} \colon C_p \otimes D_q \longrightarrow C_{p-1} \otimes D_q \oplus C_p \otimes D_{q-1}$$
$$\delta_{p,q} \colon c \otimes d \longmapsto \delta(c) \otimes d + (-1)^p c \otimes \delta(d)$$

The Künneth formula gives us a relationship between the homology of a pair of chain complexes and the homology of their tensor product.

Theorem 1.3.13 (The Künneth Theorem). *Let C_* and D_* be chain complexes of R-modules. If both the module of cycles and the module of boundaries of C_n are flat for all n, we have the following natural short exact sequence:*

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(C_*) \otimes \mathrm{H}_q(D_*) \to \mathrm{H}_n((C \otimes D)_*) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(\mathrm{H}_p(C_*), \mathrm{H}_q(D_*)) \to 0$$

Under the stronger assumption that C_n and $\mathrm{H}_n(C_*)$ are flat for all n, there is an isomorphism:

$$\bigoplus_{p+q=n} \mathrm{H}_p(C_*) \otimes \mathrm{H}_q(D_*) \cong \mathrm{H}_n((C \otimes D)_*)$$

In particular, if R is a field, we will always have the isomorphism above, since all modules over fields are free [Lang, 2002].

Proof. See Theorem 10.2. on page 166 of [Mac Lane, 1967]. □

Remark 1.3.14. We will at times be in a situation where we wish calculate the tensor product of two chain complexes, C_* and D_*, where C_* itself does not satisfy the conditions of the Künneth Theorem in any obvious way. A trick that sometimes work in these cases is to search for another chain complex, C'_*, which does satisfy the criteria and is homotopy equivalent to C_*. If there exists such a C'_*, we can use the Künneth theorem on $C'_* \otimes D_*$. Since we have that $C'_* \simeq C_*$, we have natural isomorphisms $\mathrm{H}_n(C'_*) \cong \mathrm{H}_n(C_*)$ and $\mathrm{H}_n(C'_* \otimes D_*) \cong \mathrm{H}_n(C_* \otimes D_*)$, where the latter isomorphism is due to the fact that we can construct a homotopy equivalence $C_* \otimes D_* \simeq C'_* \otimes D_*$. To be precise, if $f_* \colon C_* \simeq C'_*$ is the homotopy equivalence then $(f \otimes \mathrm{Id})_n \colon (C \otimes D)_n \simeq (C' \otimes D)_n$ where $(f \otimes \mathrm{Id})_n$ is defined as the map:

$$\bigoplus_{p+q=n} f_p \otimes \mathrm{Id}_q \colon \bigoplus_{p+q=n} C_p \otimes D_q \longrightarrow \bigoplus_{p+q=n} C'_p \otimes D_q$$

The homotopy equivalence the other way is defined in a similar fashion. This means that if $g_* \colon C'_* \simeq C_*$ is the other homotopy equivalence, we have that

$$\bigoplus_{p+q=n} g_p \otimes \mathrm{Id}_q \colon \bigoplus_{p+q=n} C'_p \otimes D_q \longrightarrow \bigoplus_{p+q=n} C_p \otimes D_q$$

is the "inverse" homotopy equivalence $(g \otimes \mathrm{Id})_n \colon (C' \otimes D)_n \simeq (C \otimes D)_n$. Now, we let the first chain homotopy be denoted $h_* \colon g_* f_* \simeq \mathrm{Id}_{C_*}$. Then we can see that $(h \otimes \mathrm{Id}_{D_*})_*$ given by the homomorphisms given degreewise by:

$$\bigoplus_{p+q=n} h_p \otimes \mathrm{Id}_q \colon \bigoplus_{p+q=n} C_p \otimes D_q \longrightarrow \bigoplus_{p+q=n+1} C_{p+1} \otimes D_q$$

We can verify that $(h \otimes \mathrm{Id}_{D_*})_* \colon (g \otimes \mathrm{Id}_{D_*})_* \circ (f \otimes \mathrm{Id}_{D_*})_* \simeq (\mathrm{Id}_{C_*} \otimes \mathrm{Id}_{D_*})_*$ by writing out the definitions. The equivalent procedure can clearly be applied to the other homotopy, $h'_* \colon f_* g_* \simeq \mathrm{Id}_*$. Thus we have the Künneth Theorem holds for C_* as well under the circumstances described.

1.4 Simplicial Methods

We present here the basics of simplicial methods. Rather than taking a more minimalistic approach, we have chosen to start our exposition from scratch and include some material on the simplex category. The three main notions that we want to take with us from this section is what a simplicial module is, how a simplicial module gives rise to a chain complex and what the Eilenberg-Zilber Theorem (Theorem 1.4.9) is. The following exposition is largely based on Chapter 8.5 in [Mac Lane, 1967], and have an algebraic flavour. For a more geometrically flavoured (and very intuitive) introduction to simplicial objects, the reader might like to have look at [Friedman, 2012].

1.4.1 The Simplex Category

The "engine" in the theory of simplicial objects is the category Δ, varyingly referred to as the **simplex category**, the **simplicial category** or the **category of non-negative ordinal numbers**. The category of non-negative ordinal numbers is the name that best describes our definition of Δ:

Definition 1.4.1. Let p be a non-negative integer. The category Δ has objects of the form

$$[p] \coloneqq \{0, 1, 2, \dots, p\}$$

as objects. We give $[p]$ the usual ordering (i.e. $0 < 1 < 2 < \cdots < p$) and define the morphisms in the category to be all (non-strict) order-preserving maps, i.e. functions $\psi \colon [p] \to [q]$ such that $i \leq j \Rightarrow \psi(i) \leq \psi(j)$.

The morphisms in the category of ordinal numbers turns out to be generated by two important classes of morphisms.

- The first type of morphism, for which we will usually write $\delta_i \colon [p] \to [p+1]$, is defined for all non-negative integers p and all $0 \leq i \leq p+1$. $\delta_i \colon [p] \to [p+1]$ is defined to be the (weakly) order-preserving injective function that fails to "hit" the i-th coordinate. In other words, if we have j and k such that $0 \leq j < i \leq k \leq p$ then $\delta^i(j) = j$ and $\delta^i(k) = k+1$.

- The second type of morphism, for which we will write $\sigma^i \colon [p+1] \to [p]$, is defined for all non-negative integers p and all $0 \leq i \leq p$. $\sigma^i \colon [p+1] \to [p]$ is defined to be the (weakly) order-preserving surjective function that "hits" the i-th coordinate twice. In other words, if we have j and k such that $0 \leq j \leq i < k \leq p+1$ then $\delta^i(j) = j$ and $\delta^i(k) = k-1$.

As a short example, $\delta^2 : [4] \to [5]$ and $\sigma^1 \colon [4] \to [3]$ is defined on elements as:

$$\begin{array}{ccccc} \delta^2 \colon 0 \mapsto 0 & \delta^2 \colon 1 \mapsto 1 & \delta^2 \colon 2 \mapsto 3 & \delta^2 \colon 3 \mapsto 4 & \delta^2 \colon 4 \mapsto 5 \\ \sigma^1 \colon 0 \mapsto 0 & \sigma^1 \colon 1 \mapsto 1 & \sigma^1 \colon 2 \mapsto 1 & \sigma^1 \colon 3 \mapsto 2 & \sigma^1 \colon 4 \mapsto 3 \end{array}$$

There are "commutativity relations" that holds for every allowed composition of pairs δ^i's and σ^j's. We state them here for easy reference, since these are going to be used extensively later on in the thesis:

Proposition 1.4.2. *Let Δ^{op} be the opposite category of Δ. We write d for δ^{op} and s for σ^{op}. In the category Δ, the δ's and σ's satisfy the relations to the left whenever the compositions are* *defined. By duality, the morphisms d and s in Δ^{op} satisfy the relations to the right whenever those compositions are defined.*

$\delta^j\delta^i = \delta^i\delta^{j-1}$	$d_id_j = d_{j-1}d_i$	*if* $i < j$
$\sigma^j\sigma^i = \sigma^i\sigma^{j+1}$	$s_is_j = s_{j+1}s_i$	*if* $i \leq j$
$\sigma^j\delta^i = \delta^i\sigma^{j-1}$	$d_is_j = s_{j-1}d_i$	*if* $i < j$
$\sigma^j\delta^i = 1$	$d_is_j = 1$	*if* $i = j$, *or if* $i = j+1$
$\sigma^j\delta^i = \delta^{i-1}\sigma^j$	$d_is_j = s_jd_{i-1}$	*if* $i > j+1$

Proof. We omit the proof, since it consists of some rather tedious case-checking. □

Lemma 1.4.3. *Every morphism $f\colon [p] \to [q]$ in the category Δ of non-negative ordinal numbers has a unique factorization in terms of δ^i's and σ^j's:*

$$f = \delta^{i_1}\delta^{i_2}\delta^{i_3}\dots\delta^{i_s}\sigma^{j_1}\sigma^{j_2}\sigma^{j_3}\dots\sigma^{j_t}$$

where we have $p \geq i_1 \geq i_2 \geq i_3 \geq \cdots \geq i_s \geq 0$ and $0 \leq j_1 \leq j_2 \leq j_3 \leq \cdots \leq j_t \leq q$ and $q+s-t = p$

Proof. A different (shorter) proof of the statement is given for Lemma 5.1 of [Mac Lane, 1967] on page 234. Our proof is constructive, hence for any monotonically increasing function $f : [p] \to [q]$ we want to compose δ's and σ's so that we obtain f. We do this by induction on the set, $[p]$. First consider the base case: If f sends 0 to 0 we let $f = \mathrm{Id}$, which is the empty composition. If f sends 0 to n, we start with the function $\sigma^0\sigma^1\cdots\sigma^{n-1}$. Then we have the inductive step: Assume that we have constructed a function ψ_{j-1} that sends i to $f(i)$ for all $i < j$. If $f\colon j \mapsto f(i)$ we can compose on the left by d_{j-1}. If $f\colon j \mapsto f(j-1)+1$ we compose with the identity and if $f\colon j \mapsto f(j-1)+1+n$ then we compose by $\sigma^j\sigma^{j+1}\sigma^{j+n-1}$. After we have done this for all $j \in [p]$, we have a function $\psi_p : [p] \to [q-s]$ for some $s = q - f(p)$. We compose with $\sigma_{(p+1)}\sigma_{(p+2)}\cdots\sigma_{(q)}$ on the right of ψ_q. We call the new function ψ, and it is equal to f by its construction.

We have showed that δ' and σ' are generators for Δ, but it still remains to show that there is a unique factorization of the form $f = \delta^{i_1}\delta^{i_2}\delta^{i_3}\dots\delta^{i_s}\sigma^{j_1}\sigma^{j_2}\sigma^{j_3}\dots\sigma^{j_t}$ where $p \geq i_1 \geq i_2 \geq i_3 \geq \cdots \geq i_s \geq 0$ and $0 \leq j_1 \leq j_2 \leq j_3 \leq \cdots \leq j_t \leq q$. To see this, notice that any random composition of δ's σ's that is well-defined can be rearranged so that it is of the required form using the properties listed in Proposition 1.4.2. Two different compositions of this form can not result in the same function, as can be seen by the definition of δ and σ, and so we have completed our proof. □

1.4.2 Simplicial Objects

We are now going to demonstrate the usefulness of simplicial methods through studying the case of simplicial R-modules. These will play an important role throughout the thesis.

Definition 1.4.4. A **simplicial object** in the category $\mathcal{C}$ is a functor $F\colon \Delta^{\mathrm{op}} \to \mathcal{C}$. A simplicial map from $F\colon \Delta \to \mathcal{C}$ to $G\colon \Delta \to \mathcal{C}$ is natural transformation $\eta\colon F \to G$.

Our focus of interest will be in the cases where the objects of $\mathcal{C}$ have some algebraic structure. To be precise, $\mathcal{C}$ will be one of the following categories: the category of commutative monoids, commutative rings, R-modules or A-algebras. The following proposition gives an easier way of defining simplicial objects.

Proposition 1.4.5. *Assume there to be an object $C_q \in \mathcal{C}$ together with collections of two families of morphisms, the class of **face maps**, $d_i\colon C_q \to C_{q-1} \in \mathcal{C}$, and the class of **degeneracy maps**, $s_j\colon C_q \to C_{q+1}$, for every $q \in \mathbb{N}_0$. Further more, assume the maps d_i and s_j satisfy the relations (to the right) in Proposition 1.4.2. Then there is a simplicial objects given by the contravariant functor, $F\colon \Delta \to \mathcal{C}$ which sends objects $[q]$ to C_q and morphisms:*

$$F\colon \Big[\sigma^i\colon [q+1] \to [q]\Big] \mapsto \Big[s_i\colon C_q \to C_{q+1}\Big]$$
$$F\colon \Big[\delta^i\colon [q-1] \to [q]\Big] \mapsto \Big[d_i\colon C_q \to C_{q-1}\Big]$$

Proof. We need to show that F is a contravariant functor, but this is nearly a corollary of Lemma 1.4.3. Recall that every map $f \in \Delta$ has a unique decomposition into generators δ^i's and σ^j's. These maps are sent contravariantly to d_i and s_j, and these are maps that by assumption must satisfy the dual relations on the δ^i's and σ^j's which makes the functor well-defined. □

For the rest of the chapter, let us take the view that $\mathcal{C}$ is the category of R-modules. These simplicial objects will be called **simplicial R-modules**. To make things more tangible and to link this material to chain complexes, we shall begin to use notation like $C_\bullet$ rather than F when something is a simplicial module. We will also write C_q for the object $F([q])$, $d_i\colon C_{q+1} \to C_q$ for the map $F(\delta^i)\colon C_q \to C_{q+1}$ and $s_j\colon C_{q+1} \to C_q$ for the map $F(\sigma^j)\colon C_{q+1} \to Cq$.

Lemma 1.4.6. *There is a functorial way of associating a chain complex to every simplicial R-module, $C_\bullet$. This chain complex is given the notation $K(C_\bullet)$ or C_*, depending on which of these is the least confusing in a given setting. The chain complex is defined by letting the n-chains of C_* be the module in the n-th simplicial degree of $C_\bullet$ and by letting the boundary maps, b, be the alternating sum of face maps:*

$$b \coloneqq \sum_{i=0}^{n} (-1)^i d_i$$

Proof. We need only show that for the map b above, $b \circ b$ is the zero map. To see this, we use the linearity of the face maps together with the relations of Proposition 1.4.2 and get:

$$\begin{aligned}
b \circ b = b \circ \Big(\sum_{j=0}^{n} (-1)^j d_j\Big) &= \sum_{j=0}^{n} (-1)^j b \circ d_j = \sum_{j=0}^{n} (-1)^j \sum_{i=0}^{n-1} (-1)^i d_i \circ d_j = \sum_{j=0}^{n}\sum_{i=0}^{n-1} (-1)^i (-1)^j d_i \circ d_j \\
&= \sum_{j=0}^{n}\sum_{i=0}^{n-1} (-1)^{i+j} d_i \circ d_j = \sum_{i=0}^{i<j=n} (-1)^{i+j} d_i \circ d_j + \sum_{j=0}^{j\leq i=n-1} (-1)^{i+j} d_i \circ d_j \\
&= \sum_{i=0,j=1}^{i<j=n} (-1)^{i+j} d_{j-1} \circ d_i + \sum_{i,j=0}^{j\leq i=n-1} (-1)^{i+j} d_i \circ d_j \\
&= \sum_{i,j'=0}^{i\leq j'=n-1} (-1)^{i+j+1} d_{j'} \circ d_i + \sum_{j=0}^{j\leq i=n-1} (-1)^{i+j} d_i \circ d_j = 0
\end{aligned}$$

□

One can often use properties of the simplicial maps in a simplicial module to make deductions about the associated chain complex. Here is one example of this:

Lemma 1.4.7. *Let $C_\bullet$ be a simplicial module with facemaps d_j and let there be homomorphisms $c\colon C_q \to C_{q+1}$ for each $q \in \mathbb{N}_0$, such that $d_0 c = \mathrm{Id}_{C_q}$ and $d_i c = c d_{i-1}$ for all $1 \le i \le q-1$. Then $\mathrm{H}_n(C_*) = 0$ for all $n \in \mathbb{N}_0$*

Proof. The conditions in the lemma means that there is a chain homotopy $\mathrm{Id}_{C_q} \simeq 0$ since we have

$$bc + cb = (\sum_{i=0}^{q-1} d_i) \circ c + c \circ (\sum_{i=0}^{q-1} d_{i-1}) = \sum_{i=0}^{q-1} (d_i c + c d_{i-1}) = \mathrm{Id}_{C_q} = \mathrm{Id}_{C_q} - 0$$

Hence we must have that the identity map $\mathrm{Id}_{\mathrm{H}_q(C_*)}\colon \mathrm{H}_q(C_*) \to \mathrm{H}_q(C_*)$ is the zero map, since the above calculation gives $\mathrm{Id}_{\mathrm{H}_q(C_*)} = \overline{\mathrm{Id}_{C_q}} = \overline{0} = 0$. This completes the proof. □

1.4.3 The Eilenberg-Zilber's Theorem

We are now going to discuss how "tensor products" of simplicial modules (called Cartesian products) correlates to the tensor product of their associated chain complexes.

Definition 1.4.8. The **Cartesian product** of two simplicial R-modules, $C_\bullet$ and $D_\bullet$ is denoted by $(C \otimes D)_\bullet$ or $C_\bullet \otimes D_\bullet$. Here we let $(C \otimes D)_n$ be is defined to be $C_n \otimes D_n$ while the face and degeneracy maps are given respectively by:

$$\begin{aligned} s_i &\colon (C \times D)_n \longrightarrow (C \times D)_{n+1} & \qquad d_i &\colon (C \times D)_n \longrightarrow (C \times D)_{n-1} \\ s_i &\colon \quad c \otimes d \longmapsto s_i(c) \otimes s_i(d) & \qquad d_i &\colon \quad c \otimes d \longmapsto d_i(c) \otimes d_i(d) \end{aligned}$$

It is important to be aware of the similarity in notation of Cartesian product, $(C \otimes D)_\bullet$, and the tensor product of chain complex, $(C \otimes D)_*$. The chain complex associated to $(C \otimes D)_\bullet$, $K((C \otimes D)_\bullet)$, is seldom equal to the chain complex $(K(C_\bullet) \otimes K(D_\bullet))_*$, but these chain complexes are always chain equivalent. This is the Eilenberg-Zilber theorem:

Theorem 1.4.9 (The Eilenberg-Zilber Theorem). *Let $C_\bullet$ and $D_\bullet$ be simplicial modules. Then there exists natural chain equivalences f and g:*

$$\begin{aligned} f &\colon K((C \otimes D)_\bullet) \longrightarrow (K(C_\bullet) \otimes K(D_\bullet))_* \\ g &\colon (K(C_\bullet) \otimes K(D_\bullet)) \longrightarrow K((C \otimes D)_\bullet) \end{aligned}$$

Meaning that f and g are chain homomorphisms such that $fg \simeq \mathrm{Id}$ and $gf \simeq \mathrm{Id}$

Proof. We refer the reader to Theorem 8.1 on page 239 of [Mac Lane, 1967] □

1.4.4 Limits and Colimits of Simplicial Objects

As the final topic of the preliminaries, we will discuss general limits and colimits of simplicial objects. We need the following abstract-nonsense theorem from [Mac Lane, 1971] to do so, with the statement summarized in the remark below.

Theorem 1.4.10. *Let X and P and $\mathcal{J}$ be categories and let $F\colon \mathcal{J} \to X^P$ be a functor from $\mathcal{J}$ into the category of functors $F\colon P \to X$. For an object $p \in P$, we let $E_p\colon X^P \to X$ be the functor that sends every functor $g \in X^P$ to $g(p)$. Assume the composite functor $E_p \circ F\colon P \to X$ have a limit L_p with a limit cone $\tau_p\colon L_p \to E_p \circ F$. Then there is a unique functor $L\colon P \to X$ with object function $p \mapsto L_p$ such that $p \mapsto \tau_p$ is a natural transformation $\tau\colon \Delta(L) = \Delta_{\mathcal{J}}(L) \to S$; moreover, this τ is a limiting cone from the vertex $L \in X^P$ to the base $S\colon \mathcal{J} \to X^P$.*

Proof. See Theorem 1 on page 115 of [Mac Lane, 1971] □

Remark 1.4.11. "In a functor category, limits may be calculated pointwise, (provided the pointwise limits exists)" [Mac Lane, 1971, p. 116].

Remark 1.4.12. The dual statement of the remark above is that in a functor category, colimits may be calculated pointwise, (provided the pointwise colimits exists).

Though an interesting result, we will only use the Theorem above to obtain the following corollary. Recall that the category of simplicial objects is by definition a functor category, and so we have that:

Corollary 1.4.13. *Limits and colimits of a simplicial object can be calculated degreewise. In particular, the pushout of the diagram of simplicial commutative R-algebras below*

$$\begin{tikzcd} E_\bullet \arrow[r, "g_\bullet"] \arrow[d, "f_\bullet"'] & D_\bullet \arrow[d] \\ C_\bullet \arrow[r] & C_\bullet \otimes_{E_\bullet} D_\bullet \end{tikzcd}$$

is the simplicial commutative R-algebra given in each simplicial degree q as $C_q \otimes_{E_q} D_q$ and the face/degeneracy maps are given by

$$\begin{aligned} d_i \colon C_n \otimes_{E_n} D_n &\longrightarrow C_{n-1} \otimes_{E_{n-1}} D_{n-1} \\ x \otimes_{E_n} y_n &\longmapsto d'_i(x_n) \otimes_{E_{n-1}} d''_i(y_n) \end{aligned} \qquad \begin{aligned} s_i \colon C_n \otimes_{E_n} D_n &\longrightarrow C_{n+1} \otimes_{E_{n+1}} D_{n+1} \\ x \otimes_{E_n} y &\longmapsto s'_i(x) \otimes_{E_{n+1}} s''_i(y) \end{aligned}$$

Proof. Use the dual statement of Theorem 1.4.10, where we let the category P be the category Δ^{op} and X be the category $\mathbf{C}R\text{-}\mathbf{Alg}$ of commutative R-algebras. Then the result is that we obtain the statement of Corollary 1.4.13. □

Remark 1.4.14. Another consequence of the corollary above is that the products and coproducts can be calculated degreewise. In particular, we see that the cartesian product of simplical commutative R-algebras is the coproduct in this category. In both the category of simplicial R-modules or simplicial monoids (the latter of which will appear in Chapter 3) the coproduct of $C_\bullet$ and $D_\bullet$ is given as the free sum $C_q \oplus D_q$ in each simplicial degree. Similarily, we have that the product of $C_\bullet$ and $D_\bullet$ is $C_q \times D_q$ in each simplicial degree. Keeping in line with the convention of not distinguishing between $\oplus$ and $\times$ for finite products, we will write $C_\bullet \times D_\bullet$ for both the product and coproduct in these categories. The face and degeneracy maps of the product and coproducts are of course the same as well. These are maps of the form $d_i \times d'_i$ and $s_i \times s'_i$ where d_i is a face map and s_i a degeneracy map of $C_\bullet$ while d'_i is a face map and s'_i a degeneracy map of $D_\bullet$. Notice that the indexes has to be the same here.

1.5 A Technical Lemma and Spectral Sequences

The last subject we want to discuss in this preliminary is Lemma 1.5.1. This is a technical lemma that will play a small yet vital part in the proof of Theorem 4.2.1, which is one of the main theorems of this thesis. Unfortunately, the only proof we could come up with is very much in the area of using a sledgehammer to crack a nut, since we have to apply the quite advanced toolkit of spectral sequences. As a consequence, this last part of the preliminaries might be harder to follow than most of what we have encountered this far.

The reader who finds the following passage disheartening might appreciate knowing that the accompanying machinery will not be used elsewhere. Also, the proof does not provide any particular insight into why the lemma must be true. These two factors makes it possible to take Lemma 1.5.1 for granted or postpone reading it until it is needed. (To make up for some of this scaremongering, we would like to add that the topic of spectral sequences is quite interesting and we hope that no aversions regarding the subject has been imposed upon the reader.)

Lemma 1.5.1. *Let $C_\bullet$, $C'_\bullet$, $D_\bullet$ and $E_\bullet$ be simplicial commutative R-algebras and let $f_\bullet$, $g_\bullet$ and $f_\bullet$ as in the diagram below. Assume that we under these maps*

$$\operatorname{Tor}_q^{E_n}(C_n, D_n) = 0$$
$$\operatorname{Tor}_q^{E_n}(C'_n, D_n) = 0$$

for $q > 0$, and that there is a simplicial map of commutative R-algebras $i_\bullet \colon C_\bullet \to C'_\bullet$ such that the induced map $i_* \colon C_* \xrightarrow{\sim} C'_*$ is a quasi isomorphism. Then we have that the map induced on the tensor product by

$$\begin{tikzcd}
E_\bullet \arrow[rr, "g_\bullet"] \arrow[dr, "="] \arrow[dd, "f_\bullet"'] & & D_\bullet \arrow[dr, "="] \arrow[dd] & \\
& E_\bullet \arrow[rr, "g_\bullet"] \arrow[dd, "f'_\bullet"'] & & D_\bullet \arrow[dd] \\
C_\bullet \arrow[rr] \arrow[dr, "i_\bullet"'] & & C_\bullet \otimes_{E_\bullet} D_\bullet \arrow[dr, dotted] & \\
& C'_\bullet \arrow[rr] & & C'_\bullet \otimes_{E_\bullet} D_\bullet
\end{tikzcd}$$

induces an isomorphism on all homology groups:

$$\overline{(i_\bullet \otimes_{\mathrm{Id}_\bullet} \mathrm{Id}_\bullet)}_* \colon \mathrm{H}_*(C_\bullet \otimes_{E_\bullet} D_\bullet) \xrightarrow{\cong} \mathrm{H}_*(C'_\bullet \otimes_{E_\bullet} D_\bullet)$$

1.5.1 Spectral Sequences

We need some tools from the theory of spectral sequences before we can prove the lemma above, and so we provide a short introduction below. Here we have used Chapter 11 of [Mac Lane, 1967] as a general reference.

By definition, a **$\mathbb{Z}$-bigraded R-module** E is a family of R-modules $\{E_{p,q}\}$ indexed over integers p and q.

Definition 1.5.2. We say that E is a **spectral sequence** if E is a sequence of $\mathbb{Z}$-bigraded modules over R, $E^r_{p,q}$, where $E^r_{p,q}$ is indexed over $r \in \mathbb{N}$, with a family of homomorphisms, $d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$, for each r which are called the **differentials** of E. These differentials are subject to two conditions. First, that the differentials should play the role of a boundary map, meaning that $d \circ d = 0$. Secondly, there should be an isomorphism between E^{r+1} and the homology groups of E^r, where the homology groups are obtained from the differential d.

If this is the readers first time reading about spectral sequences, it might be good to know that it can be quite instructive to make some drawings of what spectral sequences "looks like". See page 320 of [Mac Lane, 1967] for how this may be done.

Remark 1.5.3. We write $\mathrm{H}_*(E^r)$ for the the homology groups of E^r induced by the differential d^r, thus the isomorphism in the definition above can be written more compactly as $E^{r+1} \cong \mathrm{H}_*(E^r)$. We can use this isomorphism in the following way. If we start at E^s for some index $s \in \mathbb{N}$ and take the homology of E^s, the result will be a new $\mathbb{Z}$-bigraded R-module, namely $\mathrm{H}_*(E^r)$. Normally there would be no reasonable way of taking the homology here, but since a spectral sequence is equipped with an isomorphism from these homology groups of E^r to E^{r+1}, we can use the differential maps d^{r+1} we have in E^{r+1} calculate homology of the homology of E^r. Clearly we have that this process can be repeated indefinitely.

We will be interested in a spectral sequence, E, that has the property of being a **first quadrant spectral sequences**. By this we mean that for every index r, we have that $E^r_{p,q} = 0$ whenever p or q are negative. If we draw this, we can see that this is the same as saying that only the first quadrant of each E^r is non-zero, hence the name.

Remark 1.5.4. Since we have that E^{r+1} is the homology of E^r which is in turn the homology of E^{r-1} and so on, we see that there is a tower of submodules

$$0 \subset B^s \subset B^{s+1} \subset B^{s+2} \subset \cdots \subset B^r \subset B^{r+1} \subset \cdots \subset Z^{r+1} \subset Z^r \subset \cdots \subset Z^{s+2} \subset Z^{s+1} \subset Z^s \subset E^s$$

defined inductively in the following way. First, we let Z^s and B^s be the subcomplex of cycles and the subcomplex boundaries of E^s in E^s. Since for any $r > s$, E^r is (isomorphic to) the homology of E^{r-1}, we have that $E^r \cong Z^{r-1}/B^{r-1}$. We see that under this isomorphism, the map $d^r\colon Z^{r-1}/B^{r-1} \to Z^{r-1}/B^{r-1}$ has kernel Z^r/B^{r-1} and image B^r/B^{r-1}. If we define Z^∞ to be the intersection $\bigcap Z^r$ over all r and similarly define B^∞ to be the union $\bigcup B^r$ of all r, we get that $B^\infty \subset Z^\infty$. We are therefore free to define $\{E^\infty_{p,q}\}$ as $\{Z^\infty_{p,q}/B^\infty_{p,q}\}$.

Definition 1.5.5. A **morphisms of spectral sequences** $f\colon E \to E'$ is a family of of $\mathbb{Z}$-bigraded R-module homomorphisms

$$f^r\colon E^r \longrightarrow E'^r$$

of bidegree $(0,0)$ indexed over $r \in \mathbb{N}$. The maps f^r have to satisfy two properties for all r, the first that we have commutativity with the differentials $f^r d^r = d^r f^r$ and the second that the map f^{r+1} has to be the same map as we get induced on the homology of E^r by f^r (here we use the isomorphism $\mathrm{H}_*(E^r) \cong E^{r+1}$).

Definition 1.5.6. We say that $F = \{F_pA \subset A \mid p \in \mathbb{Z}\}$ is a **filtration of the R-module** A if F is a family of submodules F_pA of A, with the property that for every $p \in \mathbb{Z}$ we have that $F_{p-1}A \subset F_pA$. More generally, we can define a **filtration of a graded R-module**, $A = A_n$ to be a family of subgraded R-modules, F_pA, that satisfy the same conditions as a filtration of A.

Remark 1.5.7. A filtration F of A has an **associated graded module**, for which we write G^FA. This is obtained by defining G^FA as $\{(G^FA)_p \mid (G^FA)_p = F_pA/F_{p-1}A\}$. The associated graded module of a filtration of a graded R-module A, G^FA, gives us a filtration $\{F_pA_n\}$ for each n.

Definition 1.5.8. To say that the spectral sequence $E = \{E^r, d^r\}$ **converges** to the graded R-module A, means that there exists a filtration F of A such that there is an isomorphism of graded modules $E^\infty_{p,q} \cong \{F_pA_{p+q}/F_{p-1}A_{p+q}\}$ for each p, with grading over q. We write $E^s_p \Rightarrow A$ to indicate that we have convergence of E^r to A.

We have now covered all relevant definitions to introduce a theorem from [Quillen, 1967]. To be precise, this is found in Chapter II.6, and it is a combination of spectral sequence (b) of Theorem 6 on page 6.5 and the Corollary on page 6.10.

Theorem 1.5.9. *Let $E_\bullet$ be a simplicial ring and let $C_\bullet$ and $D_\bullet$ be respectively right and left simplicial R-modules. If $\mathrm{Tor}_q^{E_n}(C_n, D_n) = 0$ for $q > 0$, then there is a canonical first quadrant spectral sequence*

$$E^2_{p,q} = \left[\mathrm{Tor}_p^{\mathrm{H}_*(E_\bullet)}(\mathrm{H}_*(C_\bullet), \mathrm{H}_*(D_\bullet))\right]_q \Rightarrow H_{p+q}(C_\bullet \otimes_{E_\bullet} D_\bullet)$$

Remark 1.5.10. In his proof of the previous theorem, D.G. Quillen constructs the spectral sequence from a filtration of $(C_\bullet \otimes_{E_\bullet} D_\bullet)_*$ that is convergent above and bounded below (see [Mac Lane, 1967] Chapter XI section 3 for definitions), a fact that we need in the next theorem. For a general discussion on how this may be done, we refer the reader to Section 3 on Filtered Modules in Chapter XI of [Mac Lane, 1967], starting on page 326.

We are now ready to give a proof of the Lemma appearing at the beginning of this section:

Proof of Lemma 1.5.1. By Theorem 1.5.9, we have the existence of two first quadrant spectral sequences.

$$E^2_{p,q} = \left[\mathrm{Tor}_p^{\mathrm{H}_*(E_\bullet)}(\mathrm{H}_*(C_\bullet), \mathrm{H}_*(D_\bullet))\right]_q \Rightarrow H_{p+q}(C_\bullet \otimes_{E_\bullet} D_\bullet)$$

$$E'^2_{p,q} = \left[\mathrm{Tor}_p^{\mathrm{H}_*(E_\bullet)}(\mathrm{H}_*(C'_\bullet), \mathrm{H}_*(D_\bullet))\right]_q \Rightarrow H_{p+q}(C'_\bullet \otimes_{E_\bullet} D_\bullet)$$

Further more, we know that for every combination of integers p and q we have that

$$\left[\mathrm{Tor}_p^{\mathrm{H}_*(E_\bullet)}(\overline{i_*}, \mathrm{H}_*(D_\bullet))\right]_q : \left[\mathrm{Tor}_p^{\mathrm{H}_*(E_\bullet)}(\mathrm{H}_*(C_\bullet), \mathrm{H}_*(D_\bullet))\right]_q \xrightarrow{\cong} \left[\mathrm{Tor}_p^{\mathrm{H}_*(E_\bullet)}(\mathrm{H}_*(C'_\bullet), \mathrm{H}_*(D_\bullet))\right]_q$$

is an isomorphism, since $\overline{i_*}$ is an isomorphism. We can use The mapping theorem (see Theorem 3.4 on page 331 of [Mac Lane, 1967]) to prove that this must mean that the map $i_\bullet$ induces an isomorphism on the homology groups. Paraphrasing this theorem, we have that assuming we are given differentially graded $\mathbb{Z}$-modules A_* and A'_* with filtrations that are convergent above and bounded below and a chain homomorphism, $\alpha_*\colon A_* \to A'_*$ inducing an isomorphism on the spectral sequences obtained from the filtrations. Then the induced map $\overline{\alpha_*}\colon \mathrm{H}_*(A) \to \mathrm{H}_*(A')$ has to be an isomorphism. If we let the associated chain homomorphism of $i_\bullet \otimes_{\mathrm{Id}_\bullet} \mathrm{Id}_\bullet$ play the role of α in the mapping theorem, we see that the conditions have been met due to what we wrote in Remark 1.5.10. We therefore have that the maps induced on the homology groups

$$\overline{i_* \otimes_{\mathrm{Id}_*} \mathrm{Id}_*}\colon \mathrm{H}_*(C_\bullet \otimes_{E_\bullet} D_\bullet) \longrightarrow \mathrm{H}_*(C'_\bullet \otimes_{E_\bullet} D_\bullet)$$

is an isomorphism and so we have finished our proof. $\square$

Chapter 2

Hochschild Homology

We begin this chapter by defining Hochschild homology and the bar construction. Then we move on to show that Hochschild homology groups and the Tor functor coincide under certain conditions and finally prove that the Hochschild homology of projective R-algebras sends the direct product to the direct sum. The conventions adapted in the preliminaries hold in this chapter as well. In particular, we always assume R to be a commutative ring (with unity). A is always taken to be a unital and associative R-algebra and whenever we write an unspecified tensor product, $\otimes$, it is a shorthand notation for $\otimes_R$. In addition to this, we reserve M as the standard notation for an A-bimodule this chapter (and in this chapter only). The definitions found in this section comes from the first chapter of [Loday, 1998], where the mentioned direct sum result occurs as an exercise.

2.1 The Hochschild Complex

Hochschild homology is a homology theory for associative algebras over rings. Its definition involves the construction of simplicial R-modules from A-bimodules. The chain complex associated to this simplicial module is called the Hochschild chain complex.

Let M be an A-bimodule. The **Hochschild simplicial R-module** of M, $C_\bullet(A, M)$, is defined to be a simplicial R-module where $C_n(A, M) = M \otimes A^{\otimes n}$. The **face maps of $C_\bullet(A, M)$**, $d_j \colon M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$, are given by

$$\begin{aligned}
d_0 \colon (m \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_{n-1} \otimes a_n) &\longmapsto (ma_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_{n-1} \otimes a_n) \\
d_i \colon (m \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_{n-1} \otimes a_n) &\longmapsto (m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n-1} \otimes a_n) \\
d_n \colon (m \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_{n-1} \otimes a_n) &\longmapsto (a_n m \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_{n-1})
\end{aligned}$$

where $0 < i < n$. We let the **degeneracy maps of $C_\bullet(A, M)$**, $s_j \colon M \otimes A^{\otimes n} \to M \otimes A^{\otimes n+1}$ be:

$$\begin{aligned}
s_0 \colon (m \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n) &\longmapsto (m \otimes 1 \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n) \\
s_i \colon (m \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n) &\longmapsto (m \otimes a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n) \\
s_n \colon (m \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n) &\longmapsto (m \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes 1)
\end{aligned}$$

where $0 < i < n$.

The corresponding **Hochschild boundary map**, $b\colon M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$, is then defined as the alternating sum of these maps:

$$b = \sum_{j=0}^{n} (-1)^j d_j$$

The face and degeneracy maps of a Hochschild simplicial R-module can easily be shown to satisfy the relations required of a simplicial object, as seen in Propsition 1.4.2, and so there is a justification for the term Hochschild simplicial R-modules. Further more, by Lemma 1.4.6 in the preliminaries, $b \circ b = 0$, and we obtain the following two definitions:

Definition 2.1.1. The Hochschild complex of the bimodule M over the R-algebra A is defined as the chain complex:

$$\dots \xrightarrow{b} M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \dots \xrightarrow{b} M \otimes A^{\otimes 2} \xrightarrow{b} M \otimes A \xrightarrow{b} M \xrightarrow{b} 0$$

Where the b's are the Hochschild boundary maps as described in the text above. We use the notation $C_*(A, M)$ in referring to this complex. After this chapter, we will only be interested in the Hochschild complex created by considering A as a bimodule over itself. In this case, we use the notation $C_*(A)$, rather than $C_*(A, A)$.

Definition 2.1.2. The Hochschild homology groups of the bimodule M over the R-algebra A is defined as the homology of the Hochschild chain complex of M over A. Rather than writing $\mathrm{H}_*(C_*(A, M))$ for the Hochschild homology of M over A, we suppress the "C_*", and write only $\mathrm{H}_*(A, M)$ instead. We write $\mathrm{HH}_*(A)$ rather than $\mathrm{H}_*(A, A)$ for the Hochschild homology of A with its usual A-bimodule structure.

Remark 2.1.3. The reader should be aware that the concepts of $C_\bullet(A, M)$, $C_*(A, M)$ and $\mathrm{H}_*(A, M)$ will only be used in the first part of this chapter. Later, we will only care about the cases where $M = A$ and so we will use the notation $C_\bullet(A)$, $C_*(A)$ and $\mathrm{HH}_*(A)$. The reader should also be aware that we will use $C_\bullet(A, M)$, $C_*(A, M)$ and $\mathrm{H}_*(A, M)$ to denote the log Hochschild simplical R-algebra, log Hochschild chain complex and log Hochschild homology groups respectively in Chapter 4.

The zeroth Hochschild homology group is easy to calculate and we have its explicit description below. This is not the case for the higher homology groups, which are usually much harder to find.

Proposition 2.1.4. *Let M be an A-bimodule. Then we have an isomorphism of R-modules:*

$$\mathrm{H}_0(M, A) = \frac{M}{\{ma - am \mid a \in A, m \in M\}}$$

In particular,this means that for an R-algebra A we have that $\mathrm{HH}_0(A) \cong \frac{A}{[A,A]}$ where $[A, A]$ is the commutator/center subalgebra of A.

Proof. It is perhaps a bit grand to dignify this as a proposition, since it just involves writing out definitions. To be painstakingly pedantic about it, we see that the face maps $d_i\colon C_1(M, A) \to C_0(M, A)$ are:

$$d_0\colon (m, a) \longmapsto m \cdot a \qquad\qquad d_1\colon (m, a) \longmapsto a \cdot m$$

Hence the boundary map $b(m,a) = (d_0 - d_1)(m,a) = d_0(m,a) - d_1(m,a) = ma - am$ and so, by the definition of homology we have that

$$\mathrm{H}_0(M,A) = \frac{ker(b\colon M \longrightarrow 0)}{im(b\colon M \otimes A \to M)} = \frac{M}{\{ma - am \mid a \in A, m \in M\}}$$

□

An immediate consequence of this is that $\mathrm{HH}_0(A)$ is zero if and only if A is a commutative R-algebra.

Example 2.1.5. We let R be a commutative ring, let $\mathfrak{p} \subset R$ be a prime ideal in R and let $R_{(\mathfrak{p})}$ be R localized at $\mathfrak{p}$ considered as an R-algebra. Then we we can describe $C_\bullet(R_{(\mathfrak{p})})$:

$$\begin{aligned} C_n(R_{(\mathfrak{p})}) &= R_{(\mathfrak{p})} \otimes_R R_{(\mathfrak{p})} \otimes_R R_{(\mathfrak{p})} \otimes_R \cdots \otimes_R R_{(\mathfrak{p})} \\ &\cong R_{(\mathfrak{p})} \otimes_{R_{(\mathfrak{p})}} R_{(\mathfrak{p})} \otimes_{R_{(\mathfrak{p})}} R_{(\mathfrak{p})} \otimes_{R_{(\mathfrak{p})}} \cdots \otimes_{R_{(\mathfrak{p})}} R_{(\mathfrak{p})} \\ &\cong R_{(\mathfrak{p})} \end{aligned}$$

We write $\iota_n\colon R_{(\mathfrak{p})} \otimes_R R_{(\mathfrak{p})} \otimes_R R_{(\mathfrak{p})} \otimes_R \cdots \otimes_R R_{(\mathfrak{p})} \to R_{(\mathfrak{p})}$ for the composite isomorphism above, and it is given by $\iota_n\colon (\frac{r_0}{s_0} \otimes \frac{r_1}{s_1} \otimes \cdots \otimes \frac{r_n}{s_n}) \mapsto \frac{r_0 r_1 \cdots r_n}{s_0 s_1 \cdots s_n}$. Under this isomorphism we see that the face and degeneracy maps become "trivial" in that the diagrams below commute.

$$\begin{tikzcd} C_n(R_{(\mathfrak{p})}) \arrow[r, "\iota_n", "\cong"'] \arrow[d, "d_j"] & R_{(\mathfrak{p})} \arrow[d, "\mathrm{Id}"] \\ C_{n-1}(R_{(\mathfrak{p})}) \arrow[r, "\iota_{n-1}", "\cong"'] & R_{(\mathfrak{p})} \end{tikzcd} \qquad \begin{tikzcd} C_n(R_{(\mathfrak{p})}) \arrow[r, "\iota_n", "\cong"'] & R_{(\mathfrak{p})} \\ C_{n-1}(R_{(\mathfrak{p})}) \arrow[r, "\iota_{n-1}", "\cong"'] \arrow[u, "s_j"] & R_{(\mathfrak{p})} \arrow[u, "\mathrm{Id}"'] \end{tikzcd}$$

We have by this proved that $C_\bullet(R_{(\mathfrak{p})})$ is isomorphic to a simplicial module that we will denote by $(R_{(\mathfrak{p})})_\bullet$. We let $(R_{(\mathfrak{p})})_n = R_{(\mathfrak{p})}$ for all $n \geq 0$, and let all the face and degeneracy maps be the identity. We see that the associated chain complex of $(R_{(\mathfrak{p})})_\bullet$, denoted by $(R_{(\mathfrak{p})})_*$, can explicitly described as

$$\cdots \xrightarrow{\mathrm{Id}} R_{(\mathfrak{p})} \xrightarrow{0} R_{(\mathfrak{p})} \xrightarrow{\mathrm{Id}} R_{(\mathfrak{p})} \xrightarrow{0} R_{(\mathfrak{p})} \xrightarrow{\mathrm{Id}} R_{(\mathfrak{p})} \xrightarrow{0} R_{(\mathfrak{p})} \xrightarrow{0} 0$$

since the alternating sum of $n+1$ identity morphism is 0 or Id depending on if n is odd or even. It is not hard to calculate the homology groups of $(R_{(\mathfrak{p})})_*$, as it is zero in every degree apart from in dimension 0, where it is $R_{(\mathfrak{p})}$.

Generalizing this, we let S be any multiplicatively closed set and $S^{-1}R$ be the localization of R at S. We can calculate the Hochschild homology groups, $\mathrm{HH}_*(S^{-1}R)$, of the R-algebra $S^{-1}R$ with some minor modification to the previous example. We then end up with a chain complex similar to $(R_{(\mathfrak{p})})_*$, for which we will write $(S^{-1}R)_*$:

$$\cdots \xrightarrow{\mathrm{Id}} S^{-1}R \xrightarrow{0} S^{-1}R \xrightarrow{\mathrm{Id}} S^{-1}R \xrightarrow{0} S^{-1}R \xrightarrow{\mathrm{Id}} S^{-1}R \xrightarrow{0} S^{-1}R \xrightarrow{0} 0$$

We have now seen a few examples of a special kind of simplicial module, where the general definition is:

Definition 2.1.6. Let N be an R-module. The simplicial R-module equalling N in each simplicial degree and has every face and degeneracy map equal to the identity morphism is called the **constant simplicial module** of N. We denote this simplicial module by $(N)_\bullet$. We write $(N)_*$ for the associated chain complex to $(N)_\bullet$.

Proposition 2.1.7. *Let R be a commutative ring considered as an R-algebra over itself, $R_{(\mathfrak{p})}$ be the R-algebra from Example 2.1.5 and $S^{-1}R$ be as in the text above. Then there are isomorphisms of simplicial modules*

$$C_\bullet(R) \cong (R)_\bullet \qquad C_\bullet(R_{(\mathfrak{p})}) \cong (R_{(\mathfrak{p})})_\bullet \qquad C_\bullet(S^{-1}R) \cong (S^{-1}R)_\bullet$$

where the rightmost case generalizes the first two cases. Also, we have that the Hochschild homology groups of R, $R_{(\mathfrak{p})}$ and $S^{-1}R$ are:

$$\begin{array}{lll} \mathrm{HH}_0(R) \cong R & \mathrm{HH}_0(R_{(\mathfrak{p})}) \cong R_{(\mathfrak{p})} & \mathrm{HH}_0(S^{-1}R) \cong S^{-1}R \\ \mathrm{HH}_n(R) \cong 0 \quad \textit{for } n \geq 1 & \mathrm{HH}_n(R_{(\mathfrak{p})}) \cong 0 \quad \textit{for } n \geq 1 & \mathrm{HH}_n(S^{-1}R) \cong 0 \quad \textit{for } n \geq 1 \end{array}$$

Proof. For $S^{-1}R$ and $R_{(\mathfrak{p})}$, see the discussion above and Example 2.1.5. For the calculation of $\mathrm{HH}_n(R)$, recall that we have an isomorphism of $R \cong (1_R)^{-1}R$, and apply the general case. □

The Hochschild homology groups are functorial in several ways. One of these ways is as the functor $\mathrm{H}_*(A, -)\colon A\text{-}\mathbf{BiMod} \to R\text{-}\mathbf{Mod}$, which sends the bimodule homomorphism, $f\colon M \to M'$, to the chain homomorphism which is given on the generators of the n-chains as $f\colon (m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) \mapsto (f(m) \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n))$. This map in turn induces the map on the homology groups. Another example is the functor $\mathrm{HH}_*(-)\colon R\text{-}\mathbf{Alg} \longrightarrow R\text{-}\mathbf{Mod}$. This functor sends a ring homomorphism $f\colon A \to A'$ to the chain homomorphism defined on generators as $f\colon (a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) \mapsto (f(a_0) \otimes f(a_1) \otimes f(a_2) \otimes \cdots \otimes f(a_n))$. As in the first case, this chain homomorphism induces the homomorphisms on homology groups.

2.2 The Bar Complex

The bar complex/the bar construction is a tool from homological algebra that will turn out to be quite useful. A key feature of the bar construction is that it produces a concrete resolution of any R-algebra, A, as an A^e-module. We will show that the bar resolution is A^e-projective if A is R-projective, and use this to prove there to be an isomorphism $\mathrm{H}_n(A, M) \cong \mathrm{Tor}_n^{A^e}(M, A)$ whenever A is R-projective. We have decided to go slightly deeper into the theory behind bar complexes than what is strictly necessary for this chapter, as this will pay off later.

In the following discussion let A be an R-algebra, X be a right A-module and Y be a left A-module. We start by defining the **simplicial bar construction**, $B_\bullet(X, A, Y)$, from which we obtain the **bar complex**, $B_*(X, A, Y)$, as the associated chain complex. We let $B_n(X, A, Y)$ be the R-module $X \otimes A^{\otimes n} \otimes Y$. The **face maps of** $B_\bullet(X, A, Y)$, $d_j\colon X \otimes A^{\otimes n} \otimes Y \to X \otimes A^{\otimes n-1} \otimes Y$, are given given by

$$\begin{aligned} d_0\colon (x \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes y) &\longmapsto (x \cdot a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes y) \\ d_i\colon (x \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes y) &\longmapsto (x \otimes a_1 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_n \otimes y) \\ d_n\colon (x \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes y) &\longmapsto (x \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \cdot y) \end{aligned}$$

where $0 < i < n$, while the **degeneracy maps of** $B_\bullet(X, A, Y)$, $s_j\colon X \otimes A^{\otimes n} \otimes Y \to X \otimes A^{\otimes n+1} \otimes Y$, are given by

$$\begin{aligned} s_0\colon (x \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes y) &\longmapsto (x \otimes 1 \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes y) \\ s_i\colon (x \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes y) &\longmapsto (x \otimes a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes y) \\ s_n\colon (x \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes y) &\longmapsto (x \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes 1 \otimes y) \end{aligned}$$

where $0 < i < n$. These maps are easily seen to satisfy the conditions of Proposition 1.4.2.

The corresponding boundary map of the bar complex, $b'\colon X \otimes A^{\otimes n} \otimes Y \to X \otimes A^{\otimes n-1} \otimes Y$, is then defined as the alternating sum of these maps:

$$b = \sum_{j=0}^{n} (-1)^j d_j$$

where $0 \leq j \leq n$. For the rest of this chapter we will primarily be interested in the bar complex $B_*(A, A, A)$. We define an A^e-module structure on $B_n(A, A, A) = A^{\otimes(n+2)}$, by letting the scalar multiplication be the same multiplication that we used in Example 1.1.4 and in Corollary 1.1.10 where

$$(a \otimes a') \cdot (a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = (a \cdot a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \cdot a')$$

Definition 2.2.1. Let A be an R-algebra. The **bar complex of** A is the chain complex $B_*(A, A, A)$, where for each n $A^{\otimes(n+2)}$ is considered as A^e-modules. We use the notation $C_*^{bar}(A)$ for the bar complex of A, and $C_*^{bar}(A)$ is visualized as:

$$\ldots \xrightarrow{b'} A^{\otimes n+3} \xrightarrow{b'} A^{\otimes n+2} \xrightarrow{b'} A^{\otimes n+1} \xrightarrow{b'} \ldots \xrightarrow{b'} A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2}$$

The following proposition describes how we can use the bar construction to make an A^e-projective resolution of R-algebras.

Proposition 2.2.2. *Let A be an R-algebra and give A and $C_n^{bar}(A) = A^{\otimes n+1}$ the A^e-module structure from Example 1.1.4. If we augment the bar complex $C_*^{bar}(A)$ by A by the A^e-module homomorphism $\mu\colon A \otimes A \to A$ sending $\mu\colon a \otimes a' \mapsto a \cdot a'$, we get an A^e-resolution of A. We call the resolution $\mu\colon C_*^{bar}(A) \to A$ for the* ***bar resolution*** *of A. If A is a projective R-module, then the bar resolution is a projective resolution as an A^e-module.*

Proof. To see that the bar complex is a resolution, we need to show that the chain complex $C_*^{bar}(A)$ is an exact sequence of A^e-modules in degrees above zero. We also need to argue that $\overline{\mu}\colon \mathrm{H}_0(C_*^{bar}(A)) \xrightarrow{\cong} A$ is an isomorphism.

We prove the exactness of $C_*^{bar}(A)$ first. To do so, observe that the homomorphism $s\colon A^{\otimes n} \to A^{\otimes n+1}$, defined by sending generators $s\colon a_1 \otimes \cdots \otimes a_n \mapsto 1 \otimes a_1 \otimes \cdots \otimes a_n$ satisfy the conditions for being a contracting homotopy, since $d_0 s = \mathrm{Id}$ and $d_i s = s d_{i-1}$ for $1 \leq i \leq n-1$. By Lemma 1.4.7, we get that $b's + sb' = id$. Therefore we have that all the homology groups are zero, which is equivalent to that the chain complex $C_*^{bar}(A)$ is an exact sequence of A^e-modules in positive degrees.

To prove that $\overline{\mu}\colon \mathrm{H}_0(C_*^{bar}(A)) \xrightarrow{\cong} A$ is an isomorphism, it is enough to check that $\overline{\mu}$ is well-defined and that it has an inverse function. By definition we have that:

$$\mathrm{H}_0(C_*^{bar}(A)) = \frac{A \otimes A}{\langle a_0 \cdot a_1 \otimes a_2 - a_0 \otimes a_1 \cdot a_2 \rangle}$$

The representatives of the equivalence class $\overline{a \otimes b}$ is sent to $a \cdot b$. If we apply μ to any element in the generator of the quotient module we see that:

$$\mu(a_0 \cdot a_1 \otimes a_2 - a_0 \otimes a_1 \cdot a_2) = \mu(a_0 \cdot a_1 \otimes a_2) - \mu(a_0 \otimes a_1 \cdot a_2) = (a_0 \cdot a_1 \cdot a_2) - (a_0 \cdot a_1 \cdot a_2) = 0$$

Thus we have that $\overline{\mu}$ well-defined. To show that $\overline{\mu}$ is an isomorphism, we verify that the function $\overline{\mu}^{-1}\colon a \mapsto \overline{a \otimes 1}$ is an inverse function:

$$\overline{\mu}^{-1}(\overline{\mu}(\overline{a_0 \otimes a_1})) = \overline{a_0 \cdot a_1 \otimes 1} = \overline{a_0 \otimes a_1 \cdot 1} = \overline{a_0 \otimes a_1}$$

To prove the final statement of the proposition, note that Corollary 1.1.10 in the preliminaries implies that if A is R-projective, then $A \otimes A^{\otimes n} \otimes A$ is A^e-projective. Hence we have that all the modules in the resolution are A^e-projective. By definition this means that $\mu\colon C_*^{bar}(A) \to A$ is an A^e-projective resolution. □

2.2.1 The Tor Functor and Hochschild Homology

We will now show that when A is projective as an R-module, the n-th Hochschild homology groups of the A-bimodule M becomes isomorphic to $\mathrm{Tor}_n^{A^e}(M, A)$. In this sense, the Hochschild homology groups of projective R-algebras is a special case of the homology theory of rings. The proposition is Proposition 1.1.13 on page 12 of [Loday, 1998], where we have written out the details of the proof.

Proposition 2.2.3. *Let M be an A-bimodule and A be a projective R-module. Then there is an isomorphism of R-modules:*

$$\mathrm{H}_n(A, M) \cong \mathrm{Tor}_n^{A^e}(M, A)$$

Proof. By Proposition 2.2.2, the bar resolution is a projective resolution of A as an A^e-module. Recalling the definition of the Tor functor (Definition 1.3.10), we choose the bar resolution of A as a projective resolution of A. We now argue that the chain complex $M \otimes_{A^e} C_*^{bar}(A)$ is isomorphic to the chain complex $C_*(A, M)$:

$$\begin{tikzcd}
\cdots \arrow[r, "\mathrm{Id}_{A_e}\otimes b'"] & M \otimes_{A^e} A^{\otimes(n+3)} \arrow[r, "\mathrm{Id}_{A_e}\otimes b'"] \arrow[d, "\psi_{n+1}"] & M \otimes_{A^e} A^{(\otimes n+2)} \arrow[r, "\mathrm{Id}_{A_e}\otimes b'"] \arrow[d, "\psi_n"] & M \otimes_{A^e} A^{(\otimes n+1)} \arrow[r, "\mathrm{Id}_{A_e}\otimes b'"] \arrow[d, "\psi_{n-1}"] & \cdots \\
\cdots \arrow[r, "b"'] & M \otimes A^{\otimes(n+1)} \arrow[r, "b"'] & M \otimes A^{\otimes n} \arrow[r, "b"'] & M \otimes A^{\otimes(n-1)} \arrow[r, "b"'] & \cdots
\end{tikzcd}$$

The degreewise isomorphisms, $\psi_n\colon M \otimes_{A^e} A^{\otimes(n+2)} \xrightarrow{\cong} M \otimes A^{\otimes n}$, for all $n \in \mathbb{N}_0$ are defined as the composition map:

$$\psi_n\colon M \otimes_{A^e} A^{\otimes(n+2)} \xrightarrow{=} M \otimes_{A^e} A \otimes A^{\otimes n} \otimes A \xrightarrow{\cong} M \otimes_{A^e} A^e \otimes A^{\otimes n} \xrightarrow{\cong} M \otimes A^{\otimes n}$$

Here, the middle morphism is an isomorphism by Proposition 1.1.5, and the last morphism is an isomorphism by elementary properties of the tensor product. Explicitly, the isomorphism ψ_n acts on generators by:

$$\psi_n\colon m \otimes a \otimes a_1 \otimes a_2 \cdots \otimes a_n \otimes a' \longmapsto ama' \otimes a_1 \otimes a_2 \cdots \otimes a_n$$

The collection of R-algebra isomorphisms, $\psi_n\colon M \otimes_{A^e} A^{\otimes(n+2)} \xrightarrow{\cong} M \otimes A^{\otimes n}$, for $n \in \mathbb{N}_0$, defines an isomorphism of the simplicial modules $M \otimes_{A^e} C_\bullet^{bar}(A)$ and $C_\bullet(A, M)$. It is easy (although a bit tedious) to verify that $\psi_{n-1} \circ (\mathrm{Id}_M \otimes d_i) = d_i \circ \psi_n$ for all $0 \leq i \leq n$, where the d_i's on the left side of the equality are the face maps of the bar complex, and the d_i's on the right side are the face maps of the Hochschild complex. This implies that that $\psi_{n-1} \circ (\mathrm{Id}_M \otimes b') = b \circ \psi_n$ and that the chain complexes are isomorphic. By the functoriality of homology, the homology groups are isomorphic. □

Proposition 2.2.3 allows us to us to calculate the Hochschild homology group following important example:

Corollary 2.2.4. *Recall that R is a commutative ring. We can then calculate the Hochschild homology of the R-algebra $R[x]$ to be:*

$$\mathrm{HH}_n(R[x]) = \begin{cases} R[x], & \text{for } n = 0 \\ R[x], & \text{for } n = 1 \\ 0, & \text{for } n > 1 \end{cases}$$

Proof. We start by remarking that $R[x]$ is R-projective, since there is an isomorphism of R-modules:

$$R[X] \cong \bigoplus_{n \in \mathbb{N}_0} R_{x^n}$$

Hence by Proposition 2.2.3, $\mathrm{HH}_n(R[x]) \cong \mathrm{Tor}_n^{R[x]^e}(R[x], R[x])$. We calculate this using the following free $R[x]^e$-resolution of $R[x]$:

$$0 \longrightarrow R[x] \otimes R[y] \xrightarrow{q} R[x] \otimes R[y] \xrightarrow{p} R[x] \longrightarrow 0$$

In this resolution, $q\colon r \cdot (x \otimes y) \mapsto r \cdot (x \otimes 1 - 1 \otimes y)$ and $p\colon r \cdot (x \otimes y) \mapsto r \cdot (x \cdot y)$. This resolution results in the following chain complex for calculating torsion groups:

$$0 \longrightarrow R[z] \otimes_{R[x] \otimes R[y]} R[x] \otimes R[y] \xrightarrow{\mathrm{Id} \otimes q} R[z] \otimes_{R[x] \otimes R[y]} R[x] \otimes R[y] \xrightarrow{0} 0$$

But the above chain complex is clearly isomorphic to the chain complex:

$$0 \longrightarrow R[z] \xrightarrow{0} R[z] \xrightarrow{0} 0$$

This completes our proof. $\square$

2.3 Hochschild Homology of Products and Coproducts

In this section we prove the two mains theorems of this chapter. These explain how the Hochschild homology behaves with respect to products of R-algebras and coproducts of commutative R-algebras.

2.3.1 Hochschild Homology of Products

The following theorem appears as Exercise 1.1.1 in [Loday, 1998].

Theorem 2.3.1. *Let A and B be R-algebras, such that both are projective as R-modules. Then there is an isomorphism of homology groups:*

$$\mathrm{HH}_*(A \times B) \cong \mathrm{HH}_*(A) \oplus \mathrm{HH}_*(B)$$

Proof. Using Proposition 2.2.3, we see that it suffices to prove that:

$$\mathrm{Tor}_n^{(A \times B)^e}(A \times B, A \times B) \cong \mathrm{Tor}_n^{A^e}(A, A) \oplus \mathrm{Tor}_n^{B^e}(B, B)$$

Step 1: We wish to utilize that different choices of projective resolutions in the construction of the Tor functor yield the same Tor groups. Specifically, we would like to show that we can use $C_*^{bar}(A) \times C_*^{bar}(B)$ and $C_*^{bar}(A \times B)$ interchangeably as $(A \times B)^e$-projective resolutions of $A \times B$. By Proposition 1.3.9 we know that $C_*^{bar}(A) \times C_*^{bar}(B)$ is an $(A \times B)^e$-resolution of $A \times B$ as an $(A \times B)^e$-module, and it only remains to show that $C_n^{bar}(A) \times C_n^{bar}(B)$ is $(A \times B)^e$-projective.

By assumption we know that A and B are R-projective. We have shown in Corollary 1.1.10 that this means that $A^{\otimes n}$ is A^e-projective and that $B^{\otimes n}$ is B^e-projective when $n \geq 2$. Hence there exist an A^e-module, S_A, and a B^e-module, S_B, such that:

$$A^{\otimes n} \oplus S_A \cong \bigoplus_{i \in I} A^e$$

$$B^{\otimes n} \oplus S_B \cong \bigoplus_{j \in J} B^e$$

We shall now prove that $A^{\otimes n} \times B^{\otimes n}$ is the direct summand of a free $(A \times B)^e$-module. The strategy we will use is to take the direct sum of left $(A \times B)^e$-modules three times and see that the the result is a free $(A \times B)^e$-module. We begin by adding the $(A \times B)^e$-module $(S_A \times S_B)$:

$$A^{\otimes n} \times B^{\otimes n} \oplus (S_A \times S_B) \cong (A^{\otimes n} \oplus S_A \times B^{\otimes n} \oplus S_B) \cong (\bigoplus_{i \in I} A^e) \times (\bigoplus_{j \in J} B^e)$$

The second left $(A \times B)^e$-module we want to add is

$$((\bigoplus_{j \in J} A_j^e) \times (\bigoplus_{i \in I} B_i^e))$$

We define X to be the disjoint union $J \sqcup I$ we see that we get the isomorphisms as described below.

$$\begin{aligned}
((\bigoplus_{i \in I} A^e) \times (\bigoplus_{j \in J} B^e)) \oplus ((\bigoplus_{j \in J} A^e) \times (\bigoplus_{i \in I} B^e)) &\cong (\bigoplus_{i \in I} A^e) \oplus (\bigoplus_{j \in J} A^e) \times ((\bigoplus_{i \in I} B^e)) \oplus (\bigoplus_{j \in J} B^e)) \\
&\cong (\bigoplus_{x \in I \sqcup J} A^e) \times (\bigoplus_{x \in J \sqcup I} B^e) \\
&= (\bigoplus_{x \in X} A^e) \times (\bigoplus_{x \in X} B^e)
\end{aligned}$$

The third and final left $(A \times B)^e$-module we want to add is the direct sum of X copies of the $(A \times B)^e$-module $(A \otimes B^{\mathrm{op}} \times B \otimes A^{\mathrm{op}})$. Scalar multiplication is defined on $(A \otimes B^{\mathrm{op}} \times B \otimes A^{\mathrm{op}})$ by

$$(a, b \otimes a', b') \otimes (a_1 \otimes b_1, b_2 \otimes a_2) = (a \cdot a_1 \otimes b_1 \cdot b', b \cdot b_2 \otimes a_2 \cdot a')$$

The resulting direct sum is isomorphic to the free $(A_x \times B_x)^e$-module. The last of the isomorphism below is explained immediately beneath:

$$\begin{aligned}
((\bigoplus_{x \in X} A^e) \times (\bigoplus_{x \in X} B^e)) \oplus (\bigoplus_{x \in X} A \otimes B^{\mathrm{op}} \times B \otimes A^{\mathrm{op}}) &\cong ((\bigoplus_{x \in X} A^e) \times (\bigoplus_{x \in X} B^e)) \times (\bigoplus_{x \in X} (A \otimes B^{\mathrm{op}} \times B \otimes A^{\mathrm{op}})) \\
&\cong (\bigoplus_{x \in X} A^e) \times \bigoplus_{x \in X} (A \otimes B^{\mathrm{op}} \times B \otimes A^{\mathrm{op}}) \times (\bigoplus_{x \in X} B^e) \\
&\cong \bigoplus_{x \in X} (A^e \times (A \otimes B^{\mathrm{op}} \times B \otimes A^{\mathrm{op}}) \times B^e) \\
&\cong \bigoplus_{x \in X} (A \times B)^e
\end{aligned}$$

The last isomorphism is perhaps best understood backwards. Under the isomorphisms

$$\begin{aligned}(A\times B)^e &= (A\times B)\otimes (A\times B)^{\mathrm{op}}\\ &\cong (A\times B)\otimes (A^{\mathrm{op}}\times B^{\mathrm{op}})\\ &\cong (A\otimes A^{\mathrm{op}}\times A\otimes B^{\mathrm{op}}\times B\otimes A^{\mathrm{op}}\times B\otimes B^{\mathrm{op}})\\ &\cong (A^e\times (A\otimes B^{\mathrm{op}}\times B\otimes A^{\mathrm{op}})\times B^e)\end{aligned}$$

we see that the middle term, $(A\otimes B^{\mathrm{op}}\times B\otimes A^{\mathrm{op}})$, and the remaining part, $A^e\times B^e$, both inherits the $(A\times B)^e$-module structure they were given. Hence we have an $(A\times B)^e$-module isomorphism is claimed.

Step 2: We need to show that the remaining part of the construction of $\mathrm{Tor}_n^{(A\times B)^e}(A\times B, A\times B)$ is equivalent to the remaining part of the construction of $\mathrm{Tor}_n^{A^e}(A,A)\oplus \mathrm{Tor}_n^{B^e}(B,B)$. In other words, we need to prove that there is an isomorphism:

$$A^{\otimes n+2}\otimes_{A^e} A\times B^{\otimes n+2}\otimes_{B^e} B\cong (A^{\otimes n+2}\times B^{\otimes n+2})\otimes_{(A\times B)^e}(A\times B)$$

To prove this, we show the more general isomorphism

$$M\otimes_{A^e} A\times N\otimes_{B^e} B\cong (M\times N)\otimes_{(A\times B)^e}(A\times B)$$

where M is an A^e-module and N is a B^e-module. From left to the right, we define the isomorphism $(f\times g)$ in each coordinate to be:

$$\begin{aligned} f\colon M\otimes_{A^e} A &\longrightarrow (M\times N)\otimes_{(A\times B)^e}(A\times B) \qquad & g\colon N\otimes_{B^e} B &\longrightarrow (M\times N)\otimes_{(A\times B)^e}(A\times B)\\ f\colon m\otimes_{A^e} a &\longmapsto (m,0)\otimes_{(A\times B)^e}(a,0) \qquad & g\colon n\otimes_{B^e} b &\longmapsto (0,n)\otimes_{(A\times B)^e}(0,b)\end{aligned}$$

This is an invertible function, since we have an inverse, $(f\times g)^{-1}$. This is given as the function induced by the $(A\times B)^e$-bilinear map:

$$\begin{aligned}(M\times N)\times (A\times B) &\longrightarrow M\otimes_{A^e} A\times N\otimes_{B^e} B\\ (m,n),(a,b) &\longmapsto (m\otimes_{A^e} a, n\otimes_{B^e} b)\end{aligned}$$

This can be verified to be an inverse, since we have on the generators that:

$$\begin{aligned}(f\times g)^{-1}\big((f\times g)(m\otimes_{A^e} a, n\otimes_{B^e} b)\big) &= (f\times g)^{-1}\big((m,0)\otimes_{(A\times B)^e}(a,0)+(0,n)\otimes_{(A\times B)^e}(0,b)\big)\\ &= (f\times g)^{-1}\big((m,n)\otimes_{(A\times B)^e}(a,b)\big) = \big(m\otimes_{A^e} a, n\otimes_{B^e} b\big)\end{aligned}$$

Step 3: We conclude from the two previous steps that

$$C_*^{bar}(A)\otimes_{A^e} A\times C_*^{bar}(B)\otimes_{B^e} B\cong (C_*^{bar}(A)\times C_*^{bar}(B))\otimes_{A^e\times B^e}(A\times B)\simeq \big(C_*^{bar}(A\times B)\big)\otimes_{A^e\times B^e}(A\times B)$$

where the first isomorphism is by Step 2 and the second homotopy equivalence follows from Step 1, combined with the fact that different choice of projective resolution gives isomorphic Tor groups. Hence, by the chain equivalence above and the definition of Tor, we have proven that under the conditions in the theorem:

$$\mathrm{Tor}_n^{(A\times B)^e}(A\times B, A\times B)\cong \mathrm{Tor}_n^{A^e}(A,A)\oplus \mathrm{Tor}_n^{B^e}(B,B)$$

Which means that $\mathrm{HH}_*(A\times B)\cong \mathrm{HH}_*(A)\times \mathrm{HH}_*(B)$. $\square$

According to [Loday, 1998], the previous theorem can be generalized for the case where A and B non-projective R-algebras as well, but then with a much modified proof. To be precise, we have the following;

Corollary 2.3.2. *Let A and B be as in Theorem 2.3.1 and let p_1 and p_2 be the projections*

$$\begin{aligned} p_1 &\colon A \times B \longrightarrow A \\ p_2 &\colon A \times B \longrightarrow B \end{aligned}$$

The isomorphism of Theorem 2.3.1 is then equal to the product map:

$$p_{1*} \times p_{2*} \colon \mathrm{HH}_*(A \times B) \xrightarrow{\simeq} \mathrm{HH}_*(A) \times \mathrm{HH}_*(B)$$

Proof. We show this by proving that the product of induced maps

$$p_{1*} \times p_{2*} \colon \mathrm{Tor}_n^{(A\times B)^e}(A \times B, A \times B) \longrightarrow \mathrm{Tor}_n^{A^e}(A, A) \times \mathrm{Tor}_n^{B^e}(B, B)$$

induces an isomorphism. In the proof of the previous theorem we proved in the first step that the product of the complexes $C_*^{bar}(A)$ and $C_*^{bar}(B)$ could be chosen as a projective resolution of $A \times B$. If we now let $C_*^{bar}(A)$ and $C_*^{bar}(B)$ be projective resolutions of A and B respectively. There is by definition an isomorphism of $\mathrm{Tor}_n^{(A\times B)^e}(A \times B, A \times B)$ and the n-th homology group of the chain complex

$$(C_*^{bar}(A) \times C_*^{bar}(B)) \otimes_{(A\times B)^e} (A \times B)$$

and there is an isomorphism of $\mathrm{Tor}_n^{A^e}(A, A) \times \mathrm{Tor}_n^{B^e}(B, B)$ and the n-th homology group of the chain complex:

$$(C_*^{bar}(A) \otimes_{A^e} A) \times (C_*^{bar}(B) \otimes_{B^e} B)$$

we see that the product of the maps p_1 and p_2 induces chain homomorphisms given in each n by:

$$\begin{aligned} &p_{1*} \times p_{2*} \colon (C_*^{bar}(A) \times C_*^{bar}(B)) \otimes_{(A\times B)^e} (A \times B) \longrightarrow (C_*^{bar}(A) \otimes_{A^e} A) \times (C_*^{bar}(B) \otimes_{B^e} B) \\ &(a_0 \otimes \cdots \otimes a_{n+1}, b_0 \otimes \cdots \otimes b_{n+1}) \otimes_{(A\times B)^e} (a \times b) \longmapsto (a_0 \otimes \cdots \otimes a_{n+1} \otimes_{A^e} a, b_0 \otimes \cdots \otimes b_{n+1} \otimes_{B^e} b) \end{aligned}$$

which is an isomorphism by the argument used in Step 2 of the previous theorem. Since homology is functorial, we have that the map also induces an isomorphism as promised on

$$p_{1*} \times p_{2*} \colon \mathrm{Tor}_n^{(A\times B)^e}(A \times B, A \times B) \longrightarrow \mathrm{Tor}_n^{A^e}(A, A) \times \mathrm{Tor}_n^{B^e}(B, B)$$

which means that it induces an isomorphism as we promised:

$$p_{1*} \times p_{2*} \colon \mathrm{HH}_*(A \times B) \xrightarrow{\simeq} \mathrm{HH}_*(A) \times \mathrm{HH}_*(B)$$

□

2.3.2 Hochschild Homology of Coproducts

We have seen what happens to the product of R-algebras when applying Hochschild homology. We will now investigate how the Hochschild homology behaves with respect to the coproducts in the category of commutative R-algebras. As is well known, the coproduct of A and B in this category is given as $A \otimes B$. The following theorem is Theorem 4.2.5 on page 124 of [Loday, 1998]. Since Loday has given references in his book rather than a proof, we present our own proof here.

Theorem 2.3.3. *Let A and B be R-algebras. Then, if both the submodule of cycles and the submodule of boundaries of the Hochschild complex of A are flat for all n, we have the following natural short exact sequence:*

$$0 \to \bigoplus_{p+q=n} \mathrm{HH}_p(A) \otimes \mathrm{HH}_q(B) \to \mathrm{HH}_n(A \otimes B) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(\mathrm{HH}_p(A), \mathrm{HH}_q(B)) \to 0$$

If we assume both the cycles and the homology groups of $C_*(A)$ are flat over R, then there is a natural isomorphism:

$$\mathrm{HH}_*(A \otimes B) \cong \mathrm{HH}_*(A) \otimes \mathrm{HH}_*(B)$$

Proof. Let us compare the simplicial module $C_\bullet(A \otimes B)$ and the cartesian product $C_\bullet(A) \otimes C_\bullet(B)$. It is not hard to see that these are isomorphic simplicial modules, since we have an isomorphism of each degree, defined on generators as:

$$\begin{aligned} \iota \colon C_n(A \otimes B) &\longrightarrow C_n(A) \otimes C_n(B) \\ (a_0 \otimes b_0) \otimes \dots (a_n \otimes b_n) &\longmapsto (a_0 \otimes \cdots \otimes a_n) \otimes (b_0 \otimes \cdots \otimes b_n) \end{aligned}$$

Since this map commutes with the face and degeneracy maps, this is an isomorphism of simplicial R-modules. This gives us the isomorphism below, while the Eilenberg-Zilber theorem (Theorem 1.4.9) gives us the chain equivalence to the right:

$$K(C_\bullet(A \otimes B)) \cong K(C_\bullet(A) \otimes C_\bullet(B)) \simeq C_*(A) \otimes C_*(B)$$

Here, the middle chain complex is the degreewise tensor product, while the rightmost chain complex is the tensor product of chain complex from Definition 1.3.12. Applying the ordinary Künneth theorem (Theorem 1.3.13), we obtain the desired result. □

Corollary 2.3.4. *Let Q be a field, and let A and B be Q-algebras. Then there is a natural isomorphism:*

$$\mathrm{HH}_*(A \otimes_Q B) \cong \mathrm{HH}_*(A) \otimes_Q \mathrm{HH}_*(B)$$

Proof. Every module over a field Q is free, and is hence flat. □

Remark 2.3.5. We will refer to Theorem 2.3.3 and its accompanying Corollary above as the Künneth theorem for Hochschild homology.

Corollary 2.3.6. *Let A be an R-algebra, and S be a multiplicatively closed subset of R. Then there is an isomorphism*

$$\mathrm{HH}_*(S^{-1}A) \cong S^{-1}\,\mathrm{HH}_*(A)$$

Proof. Using the isomorphism $S^{-1}A \cong S^{-1}R \otimes_R A$ we see that $\mathrm{HH}_*(S^{-1}A) \cong \mathrm{HH}_*(S^{-1}R \otimes_R A)$. From Proposition 2.1.7, we know that $C_n(S^{-1}R) \cong S^{-1}R$, which is flat. From the same proposition we know that $\mathrm{HH}_n(S^{-1}R)$ is either isomorphic to $S^{-1}R$ or 0, both of which are flat as R-modules. Hence we can apply Theorem 2.3.3, Proposition 2.1.7 and the definition of the tensor product of graded algebras to see that we get isomorphisms:

$$\mathrm{HH}_n(S^{-1}A) \cong \bigoplus_{p+q=n} \mathrm{HH}_p(S^{-1}R) \otimes \mathrm{HH}_q(A) \cong S^{-1}R \otimes \mathrm{HH}_n(A) \cong S^{-1}\,\mathrm{HH}_n(A)$$

$\square$

Corollary 2.3.7. *Let R be a commutative ring, then the Hochschild homology groups of the R-algebra $R[x_1 \dots x_n]$ are isomorphic to:*

$$\mathrm{HH}_q(R[x_i]) = \begin{cases} R[x_i]^{\oplus\binom{q}{n}}, & \text{for } 0 \le q \le n \\ 0, & \text{for } q > n \end{cases}$$

Proof. This is simply noting that $\mathrm{HH}_*(R[x_1 \dots x_n]) \cong \mathrm{HH}_*(R[x_1] \otimes \cdots \otimes R[x_n])$, and so by the fact that $R[x]$ is R-projective (see Corollary 2.2.4), we have R-projective homology groups and cycles. We can therefore repeatedly apply Theorem 2.3.3:

$$\mathrm{HH}_*(R[x_1] \otimes \cdots \otimes R[x_n]) \cong \mathrm{HH}_*(R[x_1]) \otimes \cdots \otimes \mathrm{HH}_*(R[x_n]) = \bigotimes_{i=1}^{n} \mathrm{HH}_*(R[x_i])$$

The statement now follows directly from the definition of the tensor products of graded R-algebras (see Definition 1.3.12), Corollary 2.2.4, and some easy combinatorics. $\square$

Chapter 3

Logarithmic R-Algebras

In this chapter we present the relevant definitions and theory necessary to define and work with logarithmic Hochschild homology. In addition to presenting the necessary definitions and results from John Rogne's article, [Rognes, 2009], we present some of our original work. In particular, we give an exposition on the limits and colimits in the category of pre-log algebras and give explicit constructions of products and coproducts and prove the bi-completeness of this category. We will also do the technical groundwork required before the next chapter, in which the two main theorems of this thesis are stated and proven.

Throughout this chapter, we let A and B be the standard notation for associative, unital and commutative R-algebras, while M and N will stand for commutative monoids.

3.1 Commutative Logarithmic Structures

In this first section of the chapter, we define pre-log R-algebras and log R-algebras and make some elementary observations. We use [Rognes, 2009] as a general reference for definitions and results in this part of the chapter.

3.1.1 The Generalization from Rings to R-Algebras

In [Rognes, 2009], the theory is phrased so that it concerns itself with (pre-) logarithmic structures on commutative rings. We wish to work in the more general setting, where we place (pre-) logarithmic structures on commutative R-algebras instead. Therefore, before we start defining (pre-) log structures, we would like to explain how this generalization works, and how one can translate back to the special case.

Proposition 3.1.1. *The category of commutative R-algebras is isomorphic to the category of morphisms from R into the category of commutative rings:*

$$\mathbf{C}R\text{-}\mathbf{Alg} \cong R/\mathbf{CRing}$$

Proof. It is easily verified that given an R-algebra structure on A, we get an induced ring homomorphism $\theta\colon R \to A$, defined by $\theta(r) = r \cdot 1$. Conversely, all ring homomorphisms $\psi\colon R \to A$ define an R-algebra by letting $r \cdot a \coloneqq \psi(r) \cdot a$. This correspondence can be shown to be a bijection, and so we get a 1-1 correspondence between R-algebras and ring homomorphisms from R to A. We also want to have a bijection of morphisms. An R-algebra homomorphism, $f\colon A \to B$, is a ring homomorphism such that $f(ar) = f(a)r$ for all $a \in A$ and all $r \in R$. This is exactly the same as to require that the diagram

$$\begin{tikzcd} & R \arrow[dl, "\theta"'] \arrow[dr, "\theta'"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

commute. These are precisely the morphisms in the category $R/\mathbf{CRing}$. Conversely, a commutative diagram of commutative rings like the one above induces an R-algebra homomorphisms on its associated R-algebra, since it means that $f(ar) = f(a)r$. This finishes the proof. $\square$

Proposition 3.1.2. *The category of commutative rings is isomorphic to the category of commutative $\mathbb{Z}$-algebras. To be concise:* $\mathbf{CRing} \cong \mathbf{C}\mathbb{Z}\text{-}\mathbf{Alg}$

Proof. The isomorphism factors as $\mathbf{CRing} \cong \mathbb{Z}/\mathbf{CRing} \cong \mathbf{C}\mathbb{Z}\text{-}\mathbf{Alg}$. The first isomorphism follows from the fact that $\mathbb{Z}$ is the initial object in the category of (commutative) rings, combined with the fact that $i/\mathcal{C} \cong \mathcal{C}$ for any category $\mathcal{C}$, where i is the initial object of $\mathcal{C}$. The second isomorphism is by Proposition 3.1.1. $\square$

3.1.2 Pre-Log R-Algebras

We start by recalling some definitions introduced in [Rognes, 2009], but reformulated to fit in the new setting of R-algebras.

Definition 3.1.3. Let A be a commutative R-algebra, and let us denote the underlying multiplicative monoid of A by $(A, \cdot)$. We define a **pre-log structure** on A to be a pair (M, α) consisting of a commutative monoid M and a monoid homomorphism

$$\alpha\colon M \to \langle A, \cdot\rangle$$

A **pre-log R-algebra** is an algebra with a chosen pre-log structure. We will use the notation (A, M, α) for the R-algebra A with the pre-log structure (M, α). We will sometimes shorten this to (A, M) if the map α is known.

Definition 3.1.4. A pair, (f, f^b), where $f : A \to B$ is an R-algebra homomorphism and $f^b : M \to N$ is a monoid homomorphism is a **pre-log homomorphism** if the diagram

$$\begin{tikzcd} M \arrow[r, "f^b"] \arrow[d, "\alpha"] & N \arrow[d, "\beta"] \\ (A, \cdot) \arrow[r, "{(f,\cdot)}"] & (B, \cdot) \end{tikzcd}$$

commutes. A pre-log homomorphism is often written as a pair $(f, f^b)\colon (A, M, \alpha) \to (B, N)$. The pre-log homomorphisms are the morphisms in the category **PreLog** of pre-log R-algebras.

The pre-log homomorphisms have an alternative description which we obtain from using the fact that we have adjoint functors, $R[-]\colon \mathbf{CMon} \to \mathbf{C}R\text{-}\mathbf{Alg}$ and $(-,\cdot)\colon \mathbf{C}R\text{-}\mathbf{Alg} \to \mathbf{CMon}$. We discussed adjointness in the preliminaries, just after Example 1.2.11. Explicitly, we have that a pre log structure (M, α) can be expressed in terms of the ring homomorphism $\bar{\alpha} : R[M] \to A$, where

$\bar{\alpha}$ is the left adjoint to α. A pair of an R-algebra homomorphism and a monoid homomorphism, (f, f^b), defines a pre-log morphism if and only if the diagram underneath commutes.

$$\begin{tikzcd} R[M] \arrow[r, "R[f]"] \arrow[d, "\bar{\alpha}"] & R[N] \arrow[d, "\bar{\beta}"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Example 3.1.5. The **trivial pre-log R-algebra** of A is the pre-log algebra $(A, \{1\}, \alpha\colon 1 \mapsto 1)$. The trivial pre-log R-algebras are the images of the free functor:

$$\begin{aligned} (-, \{1\})\colon \mathbf{C}R\text{-}\mathbf{Alg} &\longrightarrow \mathbf{PreLog} \\ A &\longmapsto (A, \{1\}) \\ [f\colon A \to B] &\longmapsto [(f, \mathrm{Id}_{\{1\}})\colon (A, \{1\}) \to (B, \{1\})] \end{aligned}$$

Example 3.1.6. Given a subalgebra, $B \subseteq A$, we can choose the underlying multiplicative monoid of B as the monoid of the pre-log structure and the inclusion, $i\colon \langle B, \cdot\rangle \to \langle A, \cdot\rangle$ as monoid homomorphism. More generally, we can consider the inclusion of a submonoid $M \subseteq \langle A, \cdot\rangle$. Special cases of this includes the pre-log R-algebra $(A, \langle A, \cdot\rangle, \mathrm{Id})$, and, under the presumption that the underlying ring structure of A is an integral domain, $(A, \langle A \setminus \{0\}, \cdot\rangle, i)$.

Example 3.1.7. Let I be some indexing set, and let $\{a_i \mid i \in I\}$ be a subset of A. We can then define a monoid homomorphism on the free commutative monoid generated on I to be the monoid homomorphism induced by the function:

$$\begin{aligned} \alpha\colon I &\longrightarrow A \\ i &\longmapsto a_i \end{aligned}$$

This example will occur quite often in different guises. For instance, we will look at special cases of the pre-log R-algebra $(R[x_1, \dots x_q], \langle x_1, \dots x_{q-p}\rangle, \alpha\colon x_i \mapsto x_i)$

We will primarily be interested in studying the properties of pre-log R-algebras, and will not be too concerned about log R-algebras. Still, to give closure to those who would else wonder why we write the the prefix “pre-” constantly, we provide the definition of a log R-algebra below.

Definition 3.1.8. Let $(A, \cdot)^* \subset (A, \cdot)$ be the notation for the multiplicative group consisting of all unit elements in A. We say that the pre-log algebra (A, M, α) is a **log algebra** if there is an isomorphism as indicated in the pullback diagram:

$$\begin{tikzcd} \alpha^{-1}((A, \cdot)^*) \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{\alpha}", "\cong"'] & M \arrow[d, "\alpha"] \\ (A, \cdot)^* \arrow[r, "i"] & (A, \cdot) \end{tikzcd}$$

In the diagram, both i and $\tilde{i}$ are inclusions, while $\tilde{\alpha}$ is the restriction $\alpha|_{\alpha^{-1}((A,\cdot)^*)}$. We call a pre-log structure on A that result in a log algebra for a **log structure**. The log algebras forms a full subcategory of **PreLog**, for which we will write **Log**.

There is a **logification functor**, which sends the pre log R-algebra (A, M, α) to $(A, M, \alpha)^a = (A, M^a, \alpha^a)$. Here, M^a is the pushout of the following diagram, and α^a is the morphism to $(A, \cdot)^*$ induced by α and the inclusion i

$$\begin{tikzcd}
\alpha^{-1}((A,\cdot)^*) \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{\alpha}"] & M \arrow[d] \arrow[ddr, bend left, "\alpha"] & \\
(A,\cdot)^* \arrow[r] \arrow[drr, bend right, "i"'] & M^a \arrow[dr, "\alpha^a"] & \\
& & (A,\cdot)
\end{tikzcd}$$

The following proposition gives the universal property of the logification functor:

Proposition 3.1.9. *Let (A, M, α) be a pre-log algebra. Then for all log algebras (B, N, β) and all pre-log algebra morphisms $f = (f, f^b)\colon (A^a, M^a, \alpha^a) \to (B, N, \beta)$ there exists a unique pre-log algebra morphism $f'\colon (A, M, \alpha) \to (B, N, \beta)$ such that the diagram below commutes.*

$$\begin{tikzcd}
(A, M, \alpha) \arrow[r] \arrow[dr, "f"] & (A^a, M^a, \alpha^a) \arrow[d, dashed, "f'"] \\
& (B, N, \beta)
\end{tikzcd}$$

Proof. For a proof see [Rognes, 2009], Remark 2.7 on page 414. □

3.2 Limits and Colimits in PreLog

We make a halt in our summary [Rognes, 2009] in order to briefly investigate what the limits and colimits in **PreLog** are. It turns out that the calculation of these can be reduced to case of finding limits and colimits in **CR-Alg** and **CMon** (see Lemma 3.2.3 for details). We also give explicit description of what the products and coproducts in **PreLog** are.

3.2.1 Products and Coproducts of pre-Log Algebras

We want to investigate what the categorical product and coproduct should be in the category **PreLog**. Recall from the general discussion in the preliminaries that we want the product of two pre-log rings (A, M, α) and (B, N, β) to be an object, $(A, M, \alpha) \times (B, N, \beta)$ together with two morphisms, $p_1\colon (A, M, \alpha) \times (B, N, \beta) \to (A, M, \alpha)$ and $p_2\colon (A, M, \alpha) \times (B, N, \beta) \to (B, N, \beta)$, such that for every diagram of pre-log algebras below there exists a unique dashed arrow, $f \times g$, making the diagram commute.

$$\begin{tikzcd}
& (C, P, \gamma) \arrow[dl, "f"'] \arrow[d, dashed, "f\times g"] \arrow[dr, "g"] & \\
(A, M, \alpha) & (A, M, \alpha) \times (B, N, \beta) \arrow[l, "p_1"'] \arrow[r, "p_2"] & (B, N, \beta)
\end{tikzcd} \tag{3.1}$$

The following proposition gives an explicit construction of this product:

Proposition 3.2.1. *The categorical product in the category* **PreLog** *of* (A, M, α) *and* (B, N, β), $(A, M, \alpha) \times (B, N, \beta)$ *is isomorphic to* $(A \times B, M \times N, \alpha \times \beta \colon m \times n \mapsto \alpha(m) \times \beta(n))$. *It has the following projection morphisms:*

$$\begin{aligned} p_1 \colon (A \times B, M \times N, \alpha \times \beta) &\longrightarrow (A, M, \alpha) \\ (a \times b, m \times n) &\longmapsto (a, m) \end{aligned} \qquad \begin{aligned} p_2 \colon (A \times B, M \times N, \alpha \times \beta) &\longrightarrow (B, N, \beta) \\ (a \times b, m \times n) &\longmapsto (b, n) \end{aligned}$$

Proof. It is clear that $(A \times B, M \times N, \alpha \times \beta)$ is a pre-log algebra. The unique morphisms that exists for any g and for any f in (3.1) is constructed to be:

$$f \times g \coloneqq (f \times g, f^b \times g^b) \colon (c, p) \mapsto (f(c) \times g(c)), (f^b(p) \times g^b(p))$$

It is easy to check that this is a pre-log algebra homomorphism. It is also clear that $p_1 \circ (f \times g) = f$ and that $p_2 \circ (f \times g) = g$. The existence of this map makes this a product. $\square$

Analogously, we want the coproduct of two pre-log rings (A, M, α) and (B, N, β) to be an object, $(A, M, \alpha) \sqcup (B, N, \beta)$ together with two morphisms, $i_1 \colon (A, M, \alpha) \to (A, M, \alpha) \sqcup (B, N, \beta)$ and $i_2 \colon (B, N, \beta) \to (A, M, \alpha) \sqcup (B, N, \beta)$ such that for every diagram of pre-log algebras below there exists a unique dashed arrow, $f \sqcup g$, making the diagram commute

$$\begin{tikzcd} (A, M, \alpha) \arrow[r, "i_1"] \arrow[dr, "f"'] & (A, M, \alpha) \sqcup (B, N, \beta) \arrow[d, dashed, "f \sqcup g"] & (B, N, \beta) \arrow[l, "i_2"'] \arrow[dl, "g"] \\ & (C, P, \gamma) & \end{tikzcd} \tag{3.2}$$

The following proposition gives an explicit construction of the coproduct. Keep in mind that the notation does not reflect what would happen for an infinite coproduct of commutative monoids.

Proposition 3.2.2. *The categorical coproduct* $(A, M, \alpha) \sqcup (B, N, \beta)$ *of* (A, M, α) *and* (B, N, β) *in the category* **PreLog** *is isomorphic to* $(A \otimes B, M \times N, \alpha \sqcup \beta \colon (m, n) \mapsto \alpha(m) \otimes \beta(n))$. *It has the following coprojection morphisms:*

$$\begin{aligned} i_1 \colon (A, M, \alpha) &\longrightarrow (A \otimes B, M \times N, \alpha \sqcup \beta) \\ (a, m) &\longmapsto (a \otimes 1, m \times 1) \end{aligned} \qquad \begin{aligned} i_2 \colon (B, N, \beta) &\longrightarrow (A \otimes B, M \times N, \alpha \sqcup \beta) \\ (b, n) &\longmapsto (1 \otimes b, 1 \times n) \end{aligned}$$

Proof. It is clear that $(A \otimes B, M \oplus N, \alpha \sqcup \beta)$ is a pre-log algebra. The unique morphisms that exists for any g and for any f in (3.2) is constructed to be:

$$f \sqcup g \coloneqq (a \otimes b, m \oplus n) \mapsto (f(a) \cdot g(b), f^b(m) + g^b(n))$$

It is easy to check that this is a pre-log algebra homomorphism. It is also clear that $(f \sqcup g) \circ i_1 = f$ and $(f \sqcup g) \circ i_2 = g$. The existence of this map makes this the coproduct. $\square$

3.2.2 General Limits and Colimits of log-Algebras

We are now going to analyse limits and colimits in the category of pre-log algebras. Let $\mathcal{J}$ be a diagram and $\mathbf{PreLog}^{\mathcal{J}}$ be the category of diagrams of shape $\mathcal{J}$. Let $F \in \mathbf{PreLog}^{\mathcal{J}}$ be a fixed diagram element. Then for any object, j in $\mathcal{J}$, we give $F(j)$ the notation (A_j, M_j, α_j)

Lemma 3.2.3. *Let $F\colon \mathcal{J} \to \mathbf{PreLog}$ be an object in the functor category $\mathbf{PreLog}^{\mathcal{J}}$. Then the colimit of this diagram $\varinjlim F \cong (\varinjlim A_j, \varinjlim M_j, \alpha'\colon \varinjlim M_j \longrightarrow \langle \varinjlim A_j, \cdot\rangle)$ Similarly, the limit of the diagram, $\varprojlim F \cong (\varprojlim A_j, \varprojlim M_j, \alpha''\colon \varprojlim M_j \longrightarrow \langle \varprojlim A_j, \cdot\rangle)$ The morphism α' is described in Part 1 of the proof, and the map α'' in Part 2.*

Proof. Part 1, Colimits:
Let the maps $\phi_j\colon A_j \to \varinjlim A_j$ be the coprojection maps associated to the cocone $\varinjlim A_j$ and let the maps , $\phi_j^b\colon M_j \to \varinjlim M_j$, be the coprojection maps of $\varinjlim M_j$.

Our first step is to show that there is a monoid homomorphism, $\alpha'\colon \varinjlim M_j \to \langle \varinjlim A_j, \cdot\rangle$, which is defined in such a way that $(\varinjlim A_j, \varinjlim M_j, \alpha'\colon \varinjlim M_j \longrightarrow \langle \varinjlim A_j, \cdot\rangle)$ defines a pre-log R-algebra. We define α' to be the unique map induced on $\varinjlim M_j$ by the collection of monoid homomorphisms

$$\langle \phi_j, \cdot\rangle \circ \alpha_j\colon M_j \to \langle A_j, \cdot\rangle \to \langle \varinjlim A_j, \cdot\rangle$$

We now need to verify that this map constitutes a cone over F. To see this, notice that for any $j \in \mathcal{J}$, the diagram

$$\begin{tikzcd}
M_j \arrow[r, "\phi_j^b"] \arrow[d, "\alpha_j"] & \varinjlim M_j \arrow[d, "\alpha'"', "\exists!"] \\
\langle A_j, \cdot\rangle \arrow[r, "\langle \phi_j, \cdot\rangle"] & \langle \varinjlim A_j, \cdot\rangle
\end{tikzcd}$$

commute by the definition of α'. As a result, every pair of morphisms (ϕ_j, ϕ_j^b) are all pre-log morphisms, and we have shown that our presumed colimt is a cocone over F.

We shall now prove that $(\varinjlim A_j, \varinjlim M_j, \alpha')$ is initial among cocones over F. To do this, we assume that (B, N, β) is another cocone over F, where the maps from (A_j, M_j, α_j) to (B, N, β) are called $(\xi_j, \xi_j^b)\colon (A_j, M_j, \alpha_j) \to (B, N, \beta)$. Then we want to prove that there exists a unique morphism, $(\xi, \xi^b)\colon (\varinjlim A_j, \varinjlim M_j, \alpha') \to (B, N, \beta)$, such that for all $j \in \mathcal{J}$: $(\xi, \xi^b) \circ (\phi_j, \phi_j^b) = (\xi_j, \xi_j^b)$. There is only one possible choice of an R-algebra homomorphism, $\xi\colon \varinjlim A_j \to B$, obtained by the universal property of the colimit. Analogously, there is only one possible choice of morphism ξ^b from $\varinjlim M$ to N. It now only remains to see that these two maps constitute a pre-log morphism. In other words, does the following diagram commute?

$$\begin{tikzcd}
\varinjlim M_j \arrow[r, "\xi^b"] \arrow[d, "\alpha'"'] & N \arrow[d, "\beta"] \\
\langle \varinjlim A_j, \cdot\rangle \arrow[r, "\langle \xi, \cdot\rangle)"'] & \langle B, \cdot\rangle
\end{tikzcd}$$

To see that this is the case, note that for all objects $j \in \mathcal{J}$ there exist morphisms, $\langle \xi_j, \cdot\rangle \circ \alpha_j\colon M_j \to \langle B, \cdot\rangle$. By the universal property of $\varinjlim M_j$, this induces a unique morphism as indicated in the diagram

$$\begin{tikzcd} \varinjlim M_j \arrow[r, "\xi^b"] \arrow[d, "\alpha'"'] \arrow[dr, dashed, "\exists!"] & N \arrow[d, "\beta"] \\ \langle \varinjlim A_j, \cdot \rangle \arrow[r, "\langle \xi, \cdot \rangle)"'] & \langle B, \cdot \rangle \end{tikzcd}$$

The fact that the dashed arrow along the diagonal is unique, means that the diagram must commute, which in turn means that (ξ, ξ^b) is a pre-log morphism.

Part 2, Limits:
By an analogous argument to the one we made in Part 1, we get that there is only one possible candidate to be our R-algebra and one possible candidate for the monoid in the limit which are $\varprojlim A_j$ and $\varprojlim M_j$ respectively. To see that there exists a map $\alpha'' \colon \varprojlim M_j \to \langle \varprojlim A_j, \cdot \rangle$ we need to use Lemma 1.2.15. This gives us that there is an isomorphism $\langle \varprojlim A_j, \cdot \rangle \cong \varprojlim \langle A_j, \cdot \rangle$, since $\langle -, \cdot \rangle$ is right adjoint. We can now use the fact that for all $j \in \mathcal{J}$ there is a monoid morphism from M_j to $\varprojlim \langle A_j, \cdot \rangle$, by factoring through $\langle A_j, \cdot \rangle$. This induces a unique morphism $\alpha'' \colon \varprojlim M_j \to \varprojlim \langle A_j, \cdot \rangle$, and $(\varprojlim A_j, \varprojlim M_j, \alpha)$ can be shown to be a cone over F in a similar manner to how we did it for colimits.

Proving that this cone is universal is also done in a way reminiscent of the colimt case. That is, given any other cone (C, K, γ) over F, with pre-log morphisms $(\eta_j, \eta_j^b) \colon (C, K, \gamma) \to (A_j, M_j, \alpha_j)$, we wish to find a unique pre-log morphism $(\eta, \eta^b) \colon (C, K, \gamma) \to (\varprojlim A_j, \varprojlim M_j, \alpha'')$ such that for all $j \in \mathcal{J}$: $(\psi_j, \psi_j^b) \circ (\eta, \eta^b) = (\eta_j, \eta_j^b)$. Here (ψ_j, ψ_j^b) are the projection maps from the cone $(\varprojlim A_j, \varprojlim M_j, \alpha'')$. The argument giving unique existence of two possible candidates (η, η^b), and showing that the relation $(\psi_j, \psi_j^b) \circ (\eta, \eta^b) = (\eta_j, \eta_j^b)$ holds, is "the same" as the analogous result for colimits in Part 1. Proving that (η, η^b) is a pre-log morphism is also similar but uses $\langle \varprojlim A_j, \cdot \rangle \cong \varprojlim \langle A_j, \cdot \rangle$ in the following way when we wish to prove commutativity of the square in the diagram:

$$\begin{tikzcd} K \arrow[r, "\eta^b"] \arrow[d, "\gamma"] & \varprojlim M_j \arrow[d, "\alpha''"'] \arrow[dr] & \\ \langle C, \cdot \rangle \arrow[r, "\langle \eta, \cdot \rangle)"'] & \langle \varprojlim A_j, \cdot \rangle \arrow[r, "\cong"] & \varprojlim \langle A_j, \cdot \rangle \end{tikzcd}$$

The outer morphism commutes by the fact that there is one unique morphism $K \to \varprojlim \langle A_j, \cdot \rangle$ induced by the maps $\psi_j \colon K \to \langle A_j, \cdot \rangle$, and the triangle commutes by definition. Hence the square commutes and (η, η^b) is a pre-log morphism. This concludes our proof. $\square$

3.3 Replete Homomorphisms

We pick up on our recapitulation of [Rognes, 2009], and resume to use this article as a general reference for the material we now cover. This section will be spent on introducing and investigating three subclasses of monoid homomorphism between commutative monoids, namely the virtually surjective homomorphisms, the exact homomorphisms and the replete homomorphisms. These kinds of homomorphism feature later on in this thesis, most importantly in the definition of log Hochschild homology. In the following discussion, we let $\epsilon \colon M \to P$ be standard notation for a homomorphism between commutative monoids. The category we work in is the category of morphisms from a commutative monoid to a fixed commutative monoid, P. We use the notation $\mathbf{CMon}/P$ for this category.

Definition 3.3.1. The monoid homomorphism $\epsilon\colon M \to P$ is said to be **virtually surjective** if the map $\epsilon^{\mathrm{gp}}\colon M^{\mathrm{gp}} \to P^{\mathrm{gp}}$ induced by group completion is surjective (see Definition 1.1.13).

We write $(\mathbf{CMon}/P)^{\mathrm{vsur}}$ for the full subcategory of $\mathbf{CMon}/P$ consisting of all the virtually surjective monoid homomorphism.

Proposition 3.3.2. *All surjective morphisms $\epsilon\colon M \to P$ are virtually surjective.*

Proof. We want to show that ϵ^{gp} hits every element in P^{gp}. Let $(p_1, p_2) \in P^{gp}$ be a representative of an equivalence class in P^{gp}. Let $m_1 \in \epsilon^{-1}(p_1)$ and $m_2 \in \epsilon^{-1}(p_2)$. We claim that $(m_1, m_2) \in M^{gp}$ is in the pre-image of (p_1, p_2) under ϵ^{gp}. This is true, since $\epsilon^{gp}(m_1, m_2) = \epsilon(m_1) + (-\epsilon(m_2)) = p_1 + (-p_2)$ by definition, which in P^{gp} gives us $p_1 + (-p_2) = (p_1, 0) + (0, p_2) = (p_1, p_2)$ by definition. $\square$

There are non surjective, virtually surjective morphisms. For example, we see that the inclusion $i\colon \mathbb{N} \to \mathbb{Z}$, $n \mapsto n$, under group completion results in an isomorphism $i^{\mathrm{gp}}\colon \mathbb{N}^{\mathrm{gp}} \cong \mathbb{Z}$

Definition 3.3.3. The monoid homomorphism ϵ is said to be **exact** if the commutative diagram below is a pullback square.

$$\begin{tikzcd}
M \arrow[r, "\gamma"] \arrow[d, "\epsilon"] & M^{\mathrm{gp}} \arrow[d, "\epsilon^{\mathrm{gp}}"] \\
P \arrow[r, "\gamma"'] & P^{\mathrm{gp}}
\end{tikzcd}$$

Proposition 3.3.4. *All monoid homomorphisms $\epsilon\colon M \to P$ where M and P are groups are exact morphisms.*

Proof. By the universal property of the group completion (see Lemma 1.1.15), we get that the relevant pullback diagram becomes isomorphic to the diagram below, which is always a pullback.

$$\begin{tikzcd}
H \arrow[r, "="] \arrow[d, "\epsilon"] & H \arrow[d, "\epsilon"] \\
G \arrow[r, "="] & G
\end{tikzcd}$$

$\square$

Definition 3.3.5. We call a monoid homomorphism ϵ **replete** if it is both virtually surjective and exact. The collection of all replete monoid homomorphisms constitutes a subcategory of $(\mathbf{CMon}/P)^{\mathrm{vsur}}$. We use the notation $(\mathbf{CMon}/P)^{\mathrm{rep}}$ for the category of all replete morphisms.

By the proof of the previous proposition, one can easily see that if M and P are groups, the collection of replete morphisms from M to P are the surjections. For general monoids, this question becomes more complicated.

Example 3.3.6. The inclusion of the natural numbers $i\colon \mathbb{N} \longrightarrow \mathbb{Z}$ was shown to be virtually surjective above. This morphism is not a replete morphism since the diagram below is not a pullback.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\gamma"] \arrow[d, "i"] & \mathbb{Z} \arrow[d, "i^{\mathrm{gp}}"', "\cong"] \\ \mathbb{Z} \arrow[r, "\gamma", "\cong"'] & \mathbb{Z} \end{tikzcd}$$

The morphism $j\colon \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$ given by $n \mapsto (n, n)$ is not a virtual surjection or exact, since the map induced by group completion is isomorphic to $j^{\mathrm{gp}}\colon \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ given by $z \mapsto (z, z)$ which is clearly not surjective. If one writes up the relevant diagram, it is also clear that the result is not a pullback diagram.

Definition 3.3.7. The **repletion functor** is a functor that sends the category of virtually surjective morphisms to the category of replete morphisms. We write

$$(-)^{\mathrm{rep}}\colon (\mathbf{CMon}/P)^{\mathrm{vsur}} \to (\mathbf{CMon}/P)^{\mathrm{rep}}$$

for this functor, and it is defined by sending a virtually surjective morphism, $\epsilon\colon M \to P$, to the pullback, M^{rep}, in the diagram:

$$\begin{tikzcd} M \arrow[dr] \arrow[drr, bend left, "\gamma"] \arrow[ddr, bend right, "\epsilon"'] & & \\ & M^{\mathrm{rep}} \arrow[r] \arrow[d, "\epsilon^{\mathrm{rep}}"] & M^{\mathrm{gp}} \arrow[d, "\epsilon^{\mathrm{gp}}"] \\ & P \arrow[r, "\gamma"'] & P^{\mathrm{gp}} \end{tikzcd}$$

This functor has the following universal property:

Proposition 3.3.8. *Let $\epsilon\colon M \to P$ be a virtually surjective morphism. Then for all replete morphisms $\delta\colon N \to P$ together with a monoid homomorphism $f\colon M \to N$ such that the diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\epsilon"] & N \arrow[d, "\delta"] \\ P \arrow[r, "="] & P \end{tikzcd}$$

commutes, there is a unique morphism $f'\colon M^{\mathrm{rep}} \to N$ making the diagram below commute.

$$\begin{tikzcd} M \arrow[rr] \arrow[dr, "f"] \arrow[dd, "\epsilon"'] & & M^{\mathrm{rep}} \arrow[dl, dashed, "f'"'] \arrow[dd, "\epsilon^{\mathrm{rep}}"] \\ & N \arrow[d, "\delta"] & \\ P \arrow[r, "="] & P & P \arrow[l, "="'] \end{tikzcd}$$

Proof. The proof can be found in [Rognes, 2009]. □

3.4 Bar Constructions in the Category of Monoids

We present three constructions, all of which are examples of simplicial commutative monoids. The material used here builds up to the definition of log Hochschild homology, and the results will be used extensively in Chapter 4, in particular in proving the two main results of this thesis. We still use [Rognes, 2009] as a general reference, but interspersed with some new observations, such as Proposition 3.4.15 which we will use to prove the first main theorem of the next chapter.

3.4.1 The Bar Construction

Let M be a commutative monoid. A **right action** of M on a set, X, is then defined to be a map, $(-)\cdot(-)\colon X \times M \to X$, where $x \cdot 1 = x$ and where $(x \cdot m) \cdot n = x \cdot (mn)$. A **left action** of M on a set, Y, is similarily defined as a map, $(-)\cdot(-)\colon M \times Y \to Y$, where $1 \cdot y = y$ and where $n \cdot (m \cdot y) = (nm)y$.

Definition 3.4.1. Let M be a commutative monoid, and X and Y be sets on which a right (respectively left) M-action has been defined. We then define the **bar construction of the triple**, (X, M, Y), to be the simplical object in the category of sets, for which we write $B_\bullet(X, M, Y)$. In degree q, we define $B_q(X, M, Y)$ to be the product:

$$X \times (\prod_{i=1}^{q} M) \times Y = X \times M \times M \times \cdots \times M \times Y$$

The face maps, $d_j\colon B_q(X, M, Y) \to B_{q-1}(X, M, Y)$, and the degeneracy maps, $s_j\colon B_q(X, M, Y) \to B_{q+1}(X, M, Y)$, are:

$$\begin{aligned}
d_0\colon (x, m_1, m_2, \ldots, m_i, m_{i+1}, \ldots, m_q, y) &\longmapsto (x \cdot m_1, m_2, \ldots, m_i, m_{i+1}, \ldots, m_q, y)\\
d_i\colon (x, m_1, m_2, \ldots, m_i, m_{i+1}, \ldots, m_q, y) &\longmapsto (x, m_1, m_2, \ldots, m_i \cdot m_{i+1}, \ldots, m_q, y)\\
d_q\colon (x, m_1, m_2, \ldots, m_i, m_{i+1}, \ldots, m_q, y) &\longmapsto (x, m_1, m_2, \ldots, m_i, m_{i+1}, \ldots, m_q \cdot y)
\end{aligned}$$

$$\begin{aligned}
s_0\colon (x, m_1, m_2, \ldots, m_i, m_{i+1}, \ldots, m_q, y) &\longmapsto (x, 1, m_1, m_2, \ldots, m_i, m_{i+1}, \ldots, m_q, y)\\
s_i\colon (x, m_1, m_2, \ldots, m_i, m_{i+1}, \ldots, m_q, y) &\longmapsto (x, m_1, m_2, \ldots, m_i, 1, m_{i+1}, \ldots, m_q, y)\\
s_q\colon (x, m_1, m_2, \ldots, m_i, m_{i+1}, \ldots, m_q, y) &\longmapsto (x, m_1, m_2, \ldots, m_i, m_{i+1}, \ldots, m_q, 1, y)
\end{aligned}$$

where $0 < i < q$. We will mainly be interested in the case where X and Y are one-element sets. In this case, there is a monoid structure on the $B_\bullet(X, M, Y)$, and we write $B_\bullet M$ as a shorthand notation for the simplicial monoid $B_\bullet(\{*\}, M, \{*\})$. We call $B_\bullet M$ for the **bar construction of** M.

We cannot make $B_\bullet M$ into a chain complex directly using Lemma 1.4.6, since $B_\bullet M$ is a simplicial monoid, and not a simplicial module. On the other hand, if we apply the free R-algebra functor, $R[-]$, degreewise to $B_\bullet M$, the outcome is a simplicial R-module. We will sometimes write $R[B_\bullet M]$ for this simplicial module and $R[B_* M]$ for its associated chain complex.

Remark 3.4.2. We can think of the bar construction is a functor from the category of commutative monoids to the category **sCMon** of simplicial commutative monoids:

$$\begin{aligned}
B_\bullet(-)\colon \mathbf{CMon} &\longrightarrow \mathbf{sCMon}\\
M &\longmapsto B_\bullet(M)
\end{aligned}$$

This functor takes a homomorphism $f\colon M \to N$ to the simplicial map define degreewise as the monoid homomorphism $f_q\colon (m_1, m_2, \dots m_q) \longmapsto (f(m_1), f(m_2), \dots f(m_q))$. The f_q map clearly commutes with the face and degeneracy map.

Proposition 3.4.3. *Let X, Y and M be commutative monoids together with monoid homomorphisms, $f\colon M \to X$ and $g\colon M \to Y$. Define M-actions on X and Y via f and g. Then there is an isomorphism of simplicial R-modules:*

$$R[B_\bullet(X, M, Y)] \cong B_\bullet(R[X], R[M], R[Y])$$

Proof. The proof of this is quite simple. Notice that there is an isomorphism in every simplicial degree

$$\begin{aligned} R[X] \otimes R[M] \otimes R[M] \otimes \cdots \otimes R[M] \otimes R[Y] &\longrightarrow R[X \times M \times M \times \cdots \times M \times Y] \\ r_0 x \otimes r_1 m_1 \otimes r_2 m_2 \otimes \cdots \otimes r_q m_q \otimes r_{q+1} y &\longrightarrow (r_0 \cdot r_1 \cdot r_2 \dots r_q)[x, m_1, m_2, \dots, m_q, y] \end{aligned}$$

We see this, either because we know that $R[-]$ is left adjoint, and hence takes "coproducts to coproducts" by Lemma 1.2.15, or by noting that the inverse of this morphism is the map,

$$(r_0 \cdot r_1 \cdot r_2 \dots r_q)[x, m_1, m_2, \dots, m_q, y] \mapsto (r_0 \cdot r_1 \cdot r_2 \dots r_q)(1x \otimes 1m_1 \otimes 1m_2 \otimes \cdots \otimes 1m_q \otimes 1y)$$

where the right hand side is equal to $r_0 x \otimes r_1 m_1 \otimes r_2 m_2 \otimes \cdots \otimes r_q m_q$ $\square$

This gives us the following corollary:

Corollary 3.4.4. *Let M be a commutative monoid. Then there is an isomorphism of R-modules:*

$$\mathrm{H}_n(R[B_* M]) \cong \mathrm{Tor}_n^{R[M]}(R, R)$$

Proof. To see this, simply notice that $R[B_\bullet M] = R[B_\bullet(\{*\}, M, \{*\})] \cong B_\bullet(R, R[M], R)$. Since R and $R[M]$ are projective (even free) as R-modules, this chain complex has n-th homology groups equal to $\mathrm{Tor}_n^{R[M]}(R, R)$ by the following proof:

We can take $\mu\colon B_*(R[M], R[M], R) \to R$ as an $R[M]$-projective resolution of R, where

$$\begin{aligned} \mu\colon R[M] \otimes R &\longrightarrow R \\ rm \otimes r' &\longmapsto r \cdot r' \end{aligned}$$

We can verify this as follows: The chain complex $B_*(R[M], R[M], R)$ is exact by an argument similar to the one made in Proposition 2.2.2, and it is $R[M]$-projective (since it is the tensor product of itself in each degree). The map μ can be seen to induce an isomorphism

$$\overline{\mu}\colon \frac{R[M] \otimes R}{\langle (r_1 m_1) \cdot (r_2 m_2) \otimes r_3 - r_1 m_1 \otimes r_2 \cdot r_3 \rangle} \longmapsto \frac{R}{0}$$

by noting that an element $rm \otimes r' \in R[M] \otimes R$ can be seen to be in the same equivalence class as the element $r \cdot 1_M \otimes r'$ and noting that $1 \otimes r$ is not in the equivalence class of zero unless $r = 0$.

Then if we remove the first term and apply the tensor product $R \otimes_{R[M]} (-)$ in each degree, we see that the resulting chain complex (upper chain complex) is isomorphic to the bar construction $B_\bullet(R, R[M], R)$ (lower chain complex):

$$\begin{tikzcd}
\dots \arrow[r, "f_i"] & R \otimes_{R[M]} R[M]^{\otimes q+1} \otimes R \arrow[r, "f_j"] \arrow[d, "\iota_q"] & R \otimes_{R[M]} R[M]^{\otimes q} \otimes R \arrow[r, "f_k"] \arrow[d, "\iota_{q-1}"] & \dots \\
\dots \arrow[r, "d_i"] & R \otimes R[M]^{\otimes q} \otimes R \arrow[r, "d_j"] & R \otimes R[M]^{\otimes q-1} \otimes R \arrow[r, "d_k"] & \dots
\end{tikzcd}$$

via the isomorphism is the composition along the map

$$R \otimes_{R[M]} R[M]^{\otimes q+1} \otimes R \cong R \otimes_{R[M]} R[M] \otimes R[M]^{\otimes q} \otimes R \cong R \otimes R[M]^{\otimes q} \otimes R$$

This completes the proof.

□

3.4.2 The Cyclic Bar Construction

The cyclic bar construction of monoids is similar to the simplical module structure of the Hochschild complex (see Definition 2.1.1). This similarity is exemplified just after the definition, in Proposition 3.4.7.

Definition 3.4.5. The **cyclic bar construction** $B^{\text{cy}}_\bullet M$ of the commutative monoid M is the simplicial commutative monoid with n-simplices:

$$B^{\text{cy}}_q M := \prod_{i=0}^{q} M = M \times M \times \cdots \times M$$

The face maps $d_{i'} \colon B_q M \to B_{q-1} M$ and the degeneracy maps $s_{j'} \colon B_q M \to B_{q+1} M$ are given to be

$$\begin{aligned}
d_i &\colon (m_0, m_1, m_2, \dots, m_i, m_{i+1}, \dots, m_q) \longmapsto (m_0, m_1, m_2, \dots, m_i m_{i+1}, \dots, m_{q-1}, m_q) \\
d_q &\colon (m_0, m_1, m_2, \dots, m_i, m_{i+1}, \dots, m_q) \longmapsto (m_q m_0, m_1, m_2, \dots, m_i, m_{i+1}, \dots, m_{q-1})
\end{aligned}$$

$$\begin{aligned}
s_j &\colon (m_0, m_1, m_2, \dots, m_i, m_{i+1}, \dots, m_q) \longmapsto (m_0, m_1, m_2, \dots, m_i, 1, m_{i+1}, \dots, m_q) \\
s_q &\colon (m_0, m_1, m_2, \dots, m_i, m_{i+1}, \dots, m_q) \longmapsto (m_0, m_1, m_2, \dots, m_i, m_{i+1}, \dots, m_q, 1)
\end{aligned}$$

where $0 \leq i < q$.

As with the bar construction, we can make $B^{\text{cy}}_\bullet M$ into a chain complex by applying $R[-]$ degreewise, obtaining the chain complex $R[B^{\text{cy}} \bullet M]$. We also see that the cyclic bar construction is functorial in a manner similar to how the bar construction was functorial. To be precise:

Remark 3.4.6. The cyclic bar construction is a functor from the category of commutative monoids to the category **sCMon** of simplicial commutative monoids:

$$\begin{aligned}
B_\bullet(-) \colon \mathbf{CMon} &\longrightarrow \mathbf{sCMon} \\
M &\longmapsto B_\bullet(M)
\end{aligned}$$

This functor takes a homomorphism $f \colon M \to N$ to the simplicial map define degreewise as the monoid homomorphism $f_q \colon (m_0, m_1, m_2, \dots m_q) \longmapsto (f(m_0), f(m_1), f(m_2), \dots f(m_q))$.

Proposition 3.4.7. *Let M be a commutative monoid. Then the simplicial module $R[B_\bullet M]$ is isomorphic to the simplicial module $C_\bullet(R[M])$.*

Proof. We define the isomorphism $f\colon R[B^{\mathrm{cy}}_\bullet M] \to C_\bullet(R[M])$ in degree $q \in \mathbb{N}_0$ by:

$$\begin{aligned} f_q\colon R[M \times M \times \cdots \times M] &\longrightarrow R[M] \otimes R[M] \otimes \cdots \otimes R[M] \\ f_q\colon r(m_0, m_1, \ldots, m_q) &\longmapsto r(1_R \cdot m_0 \otimes 1_R \cdot m_1 \otimes \cdots \otimes 1_R \cdot m_q) \end{aligned}$$

That this is an isomorphism of R-modules is the same argument as the one used in Proposition 3.4.3. The only thing left is a simple verification to check that these maps commute with the relevant face and degeneracy maps, which they do. □

Proposition 3.4.8. *Let M be a monoid. Then there is an isomorphism of R-modules:*

$$\mathrm{H}_n(R[B^{\mathrm{cy}}_\bullet M]) \cong \mathrm{HH}_n(R[M]) \cong \mathrm{Tor}_n^{R[M]^e}(R[M], R[M])$$

Proof. By Proposition 3.4.7, we have that the simplicial module $R[B^{\mathrm{cy}}_\bullet M]$ is isomorphic to $C_\bullet(R[M])$. As a result, we get that there is an isomorphism of homology groups $\mathrm{H}_n(R[B^{\mathrm{cy}}_\bullet M]) \cong \mathrm{H}_n(C_\bullet(R[M]))$. Now, Proposition 2.2.3 would give us an isomorphism $\mathrm{H}_n(C_*(R[M])) \cong \mathrm{Tor}_n^{R[M]^e}(R[M], R[M])$ of R-modules, if we can prove that $R[M]$ is projective as an R-module. Recall that the R-algebra, $R[M]$, has underlying set equal to the left hand side of the isomorphism below. When we consider only the R-module structure on $R[M]$, we see that we have an isomorphism of R-modules

$$\Big\{\sum_{i=1}^{i=n} r_i \cdot m_i \mid r_i \in R, m_i \in M, n \in \mathbb{N}^*\Big\} \cong \bigoplus_{m \in M} R$$

The left hand side is clearly free, and hence $R[M]$ is R-projective as we wanted. □

Proposition 3.4.9. *The cyclic bar construction sends products to products. Explicitly, there is an isomorphism of simplicial monoids, $B^{\mathrm{cy}}_\bullet(M \times N) \cong B^{\mathrm{cy}}_\bullet M \times B^{\mathrm{cy}}_\bullet N$. This isomorphism is obtained as the product of the maps induced by the projections*

$$\begin{aligned} p_1\colon M \times N &\longrightarrow M \\ (m, n) &\longmapsto m \end{aligned} \qquad\qquad \begin{aligned} p_2\colon M \times N &\longrightarrow N \\ (m, n) &\longmapsto n \end{aligned}$$

Proof. We see that the product of the two projections does indeed define an isomorphism on q-simplices by

$$\big((m_0, n_0), \ldots, (m_i, n_i), \ldots, (n_q, m_q)\big) \longmapsto \big((m_0, \ldots, m_i, \ldots, m_q), (n_0, \ldots, n_i, \ldots, n_q)\big)$$

Further more, this map can easily seen to commute with the face and degeneracy maps of the simplicial monoids. The inverse map

$$\big((m_0, \ldots, m_i, \ldots, m_q), (n_0, \ldots, n_i, \ldots, n_q)\big) \longmapsto \big((m_0, n_0), \ldots, (m_i, n_i), \ldots, (n_q, m_q)\big)$$

can be seen to commute with the face and degeneracy maps, and so we have that there is an isomorphism above as claimed. □

Lemma 3.4.10. *There is a functor from the category of pre-log R-algebras to the category of simplicial pre-log R-algebras*

$$(C(-), B^{\mathrm{cy}}(-))\colon \mathbf{PreLog} \longrightarrow \mathbf{sPreLog}$$

defined in the following manner. Given a pre-log algebra (A, M, α) then there exists a natural pre-log structure on $(C_\bullet(A), B^{\mathrm{cy}}_\bullet M)$, for which we will write $(C(A), B^{\mathrm{cy}}M, \alpha')$. The map α' is defined by

$$\begin{aligned} \alpha'_q\colon B^{\mathrm{cy}}_q M &\longrightarrow \langle C_q(A), \cdot\rangle \\ (m_0, m_1, \ldots, m_q) &\longmapsto (\alpha(m_0) \otimes \alpha(m_1)\otimes, \ldots, \otimes\alpha(m_q)) \end{aligned}$$

where the maps α' defines as a morphism of simplicial commutative monoids. The simplicial structure is given by face and degeneracy maps

$$\begin{aligned} (d_i, d_i^b)\colon (C_q(A), B^{\mathrm{cy}}_q M) &\longrightarrow (C_{q-1}(A), B^{\mathrm{cy}}_{q-1} M) \\ (s_i, s_i^b)\colon (C_q(A), B^{\mathrm{cy}}_q M) &\longrightarrow (C_{q+1}(A), B^{\mathrm{cy}}_{q+1} M) \end{aligned}$$

where d_i and s_i are the face/degeneracy maps of $C_\bullet(A)$ and d_i^b and s_i^b are the face/degeneracy maps of $B^{\mathrm{cy}}_\bullet M$

Proof. There is a lot of trivial case checking to be done in order to prove this in detail, but we hope that the reader understand that the lemma is sound. The functoriality we mentioned is arises from sending a pre-log R-algebra homomorphism $(f, f^b)\colon (A, M) \to (B, N)$ to the map

$$f_q = (\prod_{i=0}^{q} f, \prod_{i=0}^{q} f^b)\colon (C_q(A), B^{\mathrm{cy}}_q M) \longrightarrow (C_q(B), B^{\mathrm{cy}}_q N)$$

where the notation $\prod_{i=0}^{q} g_i$ is meant to indicate the map that is defined as g_i in the i-th coordinate. $\square$

3.4.3 The Replete Bar Construction

Definition 3.4.11. The **replete bar construction** of a commutative monoid M is the simplicial object obtained as the pullback of the diagram,

$$\begin{tikzcd} B^{\mathrm{rep}}_\bullet M \arrow[r] \arrow[d, "\epsilon^{\mathrm{rep}}"] & B^{\mathrm{cy}}_\bullet M^{\mathrm{gp}} \arrow[d, "\epsilon^{\mathrm{gp}}"] \\ M \arrow[r, "\gamma"] & M^{\mathrm{gp}} \end{tikzcd}$$

where the map ϵ^{rep} is defined in simplicial degree q as the repletion of $\epsilon\colon \prod_{i=0}^{q} M \longrightarrow M$, given by

$$\epsilon_q\colon (m_0, m_1, \ldots, m_q) \mapsto m_0 \cdot m_1 \cdot \cdots \cdot m_q$$

There is a unique morphism $\mu\colon B^{\mathrm{cy}}_\bullet M \to B^{\mathrm{rep}}_\bullet$ obtained by the universal property of the pullback. It is induced on each $B^{\mathrm{cy}}_q M$ by the map ϵ and the group completion map γ, given by group completing in each simplicial degree. The diagram is the following:

$$\begin{tikzcd}
B^{\mathrm{cy}}_{\bullet} M \arrow[dr, dashed, "\exists ! \mu"] \arrow[drr, bend left, "\gamma"] \arrow[ddr, bend right, "\epsilon"'] & & \\
& B^{\mathrm{rep}}_{\bullet} M \arrow[r] \arrow[d, "\epsilon^{\mathrm{rep}}"] & B^{\mathrm{cy}}_{\bullet} M^{\mathrm{gp}} \arrow[d, "\epsilon^{\mathrm{gp}}"] \\
& M \arrow[r, "\gamma"] & M^{\mathrm{gp}}
\end{tikzcd}$$

Remark 3.4.12. As with all the other bar constructions, this last one is also a functor. To see what a homomorphisms of commutative monoids $f\colon M \to N$ should be sent to, we see that this map already induces morphisms of simplicial monoids

$$\begin{tikzcd}
B^{\mathrm{rep}}_{\bullet} M \arrow[r] \arrow[dd, "\epsilon^{\mathrm{rep}}_M"] \arrow[drr, dotted] & B^{\mathrm{cy}}_{\bullet} M^{\mathrm{gp}} \arrow[drr] \arrow[dd, "\epsilon^{\mathrm{gp}}_M"] & & \\
& & B^{\mathrm{rep}}_{\bullet} N \arrow[r] \arrow[dd, "\epsilon^{\mathrm{rep}}_N"] & B^{\mathrm{cy}}_{\bullet} N^{\mathrm{gp}} \arrow[dd, "\epsilon^{\mathrm{gp}}_N"] \\
M \arrow[r, "\gamma_M"] \arrow[drr] & M^{\mathrm{gp}} \arrow[drr] & & \\
& & N \arrow[r, "\gamma_N"'] & N^{\mathrm{gp}}
\end{tikzcd}$$

and so by the universal property of pullbacks there exists a unique morphism making the diagram commute, as indicated by the dotted line.

Proposition 3.4.13. *The replete bar construction commutes with products. Explicitly, there is an isomorphism of simplicial monoids* $B^{\mathrm{rep}}_{\bullet}(M \times N) \cong B^{\mathrm{rep}}_{\bullet} M \times B^{\mathrm{rep}}_{\bullet} N$.

Proof. By Proposition 1.1.16 we get the two isomorphisms, $(M \times N)^{gp} \cong (M)^{gp} \times (N)^{gp}$ and $B^{\mathrm{cy}}_{\bullet}((M \times N)^{gp}) \cong B^{\mathrm{cy}}_{\bullet}(M^{gp} \times N^{gp})$. By Proposition 3.4.9 we have that $B^{\mathrm{cy}}_{\bullet}(M^{gp} \times N^{gp}) \cong B^{\mathrm{cy}}_{\bullet}(M^{gp}) \times B^{\mathrm{cy}}_{\bullet}(N^{gp})$. Combining all of this, we see that the right and left pullback diagrams below are isomorphic:

$$\begin{tikzcd}
B^{\mathrm{rep}}_{\bullet}(M \times N) \arrow[r] \arrow[d, "\epsilon^{\mathrm{rep}}"] & B^{\mathrm{cy}}_{\bullet}(M \times N) \arrow[d, "\epsilon^{gp}"] \\
M \times N \arrow[r, "\gamma"] & (M \times N)^{gp}
\end{tikzcd}
\qquad
\begin{tikzcd}
B^{\mathrm{rep}}_{\bullet} M \times B^{\mathrm{rep}}_{\bullet} N \arrow[r] \arrow[d, "\epsilon^{\mathrm{rep}}_M \times \epsilon^{\mathrm{rep}}_N"] & B^{\mathrm{cy}}_{\bullet} M \times B^{\mathrm{cy}}_{\bullet} N \arrow[d, "\epsilon^{gp}_M \times \epsilon^{gp}_N"] \\
M \times N \arrow[r, "\gamma_M \times \gamma_N"] & M^{gp} \times N^{gp}
\end{tikzcd}$$

This finishes the proof. ☐

We now paraphrase half of Lemma 3.17 on page 427 of [Rognes, 2009]. To be precise, this is Lemma 3.17 of that article. The term “weak equivalence” (which we have not explicitly defined at this point) can be taken to mean “a map which induces an isomorphism on the associated chain complexes”.

Lemma 3.4.14. *There is a natural isomorphism* $(\epsilon^{\mathrm{rep}}, \pi^{\mathrm{rep}})\colon B^{\mathrm{rep}}_{\bullet} M \cong M \times B_{\bullet} M^{gp}$ *of simplicial commutative monoids, where* $\pi^{\mathrm{rep}}\colon B^{\mathrm{rep}}_{\bullet} M \to B_{\bullet} M^{gp}$ *is the degreewise repletion of the projection* *map, $\pi\colon B_qM \to B_qM^{gp}$, sending $\pi\colon (m_0, m_1, \ldots, m_q) \mapsto (\gamma(m_1), \ldots, \gamma(m_q))$.*

There is a weak equivalence $\gamma\colon B_\bullet M \to B_\bullet M^{gp}$, which implies that there is a weak equivalence $(\epsilon^{\mathrm{rep}}, \pi^{\mathrm{rep}})^{-1} \circ (\mathrm{Id}, \gamma)\colon M \times B_\bullet M \to M \times B_\bullet M^{gp} \cong B_\bullet^{\mathrm{rep}} M$.

The repletion map $B_\bullet^{\mathrm{cy}} M \to B_\bullet^{\mathrm{rep}} M$ factors as the composition of $(\epsilon, \pi^{\mathrm{gp}})\colon B_\bullet^{\mathrm{cy}} M \to M \times B_\bullet M^{gp}$ and then the isomorphism above.

Proof. See Lemma 3.17 on page 427 of [Rognes, 2009]. $\square$

We will need the next proposition in the proof of Theorem 4.2.1.

Proposition 3.4.15. *Let M be a commutative monoid. Then the pushout, $\{1\} \sqcup_{B_q^{\mathrm{cy}} M} B_q^{\mathrm{rep}} M$ is isomorphic to the trivial monoid for all $q \in \mathbb{N}_0$.*

Proof. The pushout, $X = \{1\} \sqcup_{B_q^{\mathrm{cy}} M} B_q^{\mathrm{rep}} M \cong \{1\} \sqcup_{B_q^{\mathrm{cy}} M} (M \times B_q M^{gp})$, should have the property that the dashed arrow exists and is unique for any commutative diagram

$$\begin{tikzcd}
B_q^{\mathrm{cy}} M \arrow[r] \arrow[d] & M \times B_q M^{gp} \arrow[d, "j_1"] \arrow[ddr, bend left] & \\
\{1\} \arrow[r] \arrow[drr, bend right] & X \arrow[dr, dashed] & \\
& & N
\end{tikzcd}$$

In particular, any generating element $(1, \ldots, 1, m_i, 1, \ldots 1)$ of $B_q^{\mathrm{cy}} M$ has to be sent to $1 \in X$, since it factorise through the trivial monoid along the lower composition map. This means that the image of $(1, \ldots, 1, m_i, 1, \ldots 1)$ in $M \times B_q M^{gp}$ also has to be sent to the identity in X. If not, the diagram

$$\begin{tikzcd}
B_q^{\mathrm{cy}} M \arrow[r] \arrow[d] & M \times B_q M^{gp} \arrow[d, "j_1"] \arrow[ddr, bend left, "\mathrm{Id}"] & \\
\{1\} \arrow[r] \arrow[drr, bend right] & X \arrow[dr, dashed] & \\
& & M \times B_q M^{\mathrm{gp}}
\end{tikzcd}$$

would not commute. The following elements in $X \cong \{1\} \sqcup_{B_q^{\mathrm{cy}} M} (M \times B_q M^{gp})$ where $m, m_j \in M$

$$(1 \sqcup m_0, 1, 1, \ldots, 1), (1 \sqcup m_1, \gamma(m_1), 1, \ldots, 1), \ldots, (1 \sqcup m_i, 1, \ldots 1, \gamma(m_i), 1, \ldots, 1), \ldots$$

are therefore in the same equivalence class as the identity element. This implies in turn that that every element $(1 \sqcup 1, 1, \ldots 1, m_i, 1, \ldots, 1)$ has to be equivalent to the identity element as well, since an element $(1, 1, \ldots 1, m_i, 1, \ldots, 1)$ in $M \times B_q M^{gp}$

$$\begin{aligned}
j_1\big((1, 1, \ldots 1, m_i, 1, \ldots, 1)\big) &= 1 \cdot j_1\big((1, 1, \ldots 1, m_i, 1, \ldots, 1)\big) \\
&= j_1\big((m_i, 1, \ldots 1, 1, 1, \ldots, 1)\big) \cdot j_1\big((1, 1, \ldots 1, m_i, 1, \ldots, 1)\big) \\
&= j_1\big((m_i, 1, \ldots 1, m_i, 1, \ldots, 1)\big) = 1
\end{aligned}$$

Having proved that $j_1\big((1,1,\dots 1,m_1,1,\dots,1)\big) = 1$, one can use the usual argument to prove that $j_1\big((1,1,\dots 1,m_1^{-1},1,\dots,1)\big) = j_1\big((1,1,\dots 1,m_1,1,\dots,1)^{-1}\big) = 1$. Thus the pre-image of the identity under j_1, $j_1^{-1}(1)$, contains the set

$$\{(m,1,\dots,1),(1,m_1,1,\dots,1)\dots(1,1,\dots 1,m_i,1,\dots,1)\dots(1,1,\dots,m_q) \mid m \in M, m_j \in M^{gp}\}$$

which is a generating set for $M \times B_q M^{gp}$. Hence j_1 is the trivial map, sending $x \to 1$ for all elements of $M \times B_q M^{gp}$. From this we conclude that $X = \{1\}$. □

The following Lemma is a variant of Lemma 4.14, Lemma 4.15 and Proposition 4.16 of Stefano Piceghello's master thesis [Piceghello, 2015], in which the usefulness of Lemma 3.4.14 is made even clearer.

Lemma 3.4.16. *Let M be a commutative monoid. Give $R[B_\bullet^{\mathrm{rep}}M]$ the induced simplicial structure of face maps $R[d_i]$ and degeneracy maps $R[s_i]$, where d_i and s_i are face and degeneracy maps of $B_\bullet^{\mathrm{rep}}M$. Then we have a description of the homology groups of the chain complex associated to $R[B_\bullet^{\mathrm{rep}}M]$ as:*

$$\mathrm{H}_n(R[B_\bullet^{\mathrm{rep}}M]) \cong R[M] \otimes \mathrm{Tor}_n^{R[M^{gp}]}(R,R)$$

Proof. By Lemma 3.4.14 there is an isomorphism $(\epsilon^{\mathrm{rep}}, \pi^{\mathrm{rep}})\colon B_\bullet^{\mathrm{rep}}M \cong M \times B_\bullet M^{gp}$ of simplicial monoids. This induces isomorphisms of simplicial modules:

$$\mathrm{H}_n(R[B_\bullet^{\mathrm{rep}}M]) \cong \mathrm{H}_n(R[M \times BM^{gp}]) \cong \mathrm{H}_n(R[M] \otimes R[BM^{gp}])$$

Next we want to prove that $\mathrm{H}_n(R[M] \otimes R[BM_\bullet^{gp}]) \cong R[M] \otimes \mathrm{H}_n(R[BM^{gp}])$. The simplicial module structure of $R[M]$ in $R[M] \otimes R[B_\bullet M^{gp}]$, have the associated chain complex:

$$\dots \xrightarrow{b_{n+1}} R[M] \xrightarrow{b_n} R[M] \xrightarrow{b_{n-1}} \dots \xrightarrow{b_2} R[M] \xrightarrow{b_1} R[M] \xrightarrow{b_0} 0$$

The boundary maps b_i of this complex are induced by the face maps, $R[\pi^{\mathrm{rep}} \circ d_i] = R[\mathrm{Id}_M] = \mathrm{Id}_{R[M]}$. Hence we have that the boundary maps b_i is the identity when i is odd, and zero when i is even. By the definition of the tensor product of chain complexes, and by the Künneth theorem (Theorem 1.3.13), this means that $\mathrm{H}_n(R[M] \otimes R[B_\bullet M^{gp}]) \cong R[M] \otimes \mathrm{H}_n(R[B_\bullet M^{gp}])$ as we wanted. Finally, Proposition 3.4.4 gives us an isomorphism $\mathrm{H}_n(R[B_\bullet M^{gp}]) \cong \mathrm{Tor}_n^{R[M^{gp}]}(R,R)$, which finishes the proof.

□

Chapter 4

Logarithmic Hochschild Homology

This is in many ways the most important chapter of this thesis. We start by presenting the definition of log Hochschild homology as given in [Rognes, 2009] and make some elementary observations. We then proceed to prove two of the main theorems of this thesis. The first result is that the log Hochschild homology commutes with the product in the category of log-algebras. The second result is that the log Hochschild homology of coproducts in the category of log-algebras.

4.1 Log Hochschild Homology

When defining log Hochschild homology we keep in mind the remark from [Rognes, 2009]: If A is flat over $R[M]$, then $C_\bullet(A)$ is flat over $C_\bullet(R[M]) = R[B^{\mathrm{cy}}_\bullet M]$ in every simplicial degree.

Definition 4.1.1 ([Rognes, 2009]). Let (A, M, α) be a pre-log R-algebra such that A is flat over $R[M]$. Then we define the **Hochschild simplicial pre-log R-algebra** of (A, M, α) as the simplicial pre-log R-algebra, $(C_\bullet(A, M), B^{\mathrm{rep}}_\bullet M, \xi)$, where $C_\bullet(A, M)$ is defined to be the pushout of simplicial R-modules:

$$\begin{tikzcd} R[B^{\mathrm{cy}}_\bullet M] \arrow[r, "R[\mu]"] \arrow[d, "\overline{\alpha'}"] & R[B^{\mathrm{rep}}_\bullet M] \arrow[d] \\ C_\bullet(A) \arrow[r] & C_\bullet(A, M) \end{tikzcd}$$

Here, μ is the unique map described in the text after Definition 3.4.11, while $\overline{\alpha'}\colon R[B^{\mathrm{cy}}_\bullet M] \to C_\bullet(A)$ is the adjoint morphism to the pre-log structure induced by (A, M, α) as in Lemma 3.4.10. We will write $C_*(A, M)$ for the chain complex associated to $C_\bullet(A, M)$ and $\mathrm{HH}_n(A, M)$ for the n-th homology groups of $C_\bullet(A, M)$. We call $\mathrm{HH}_n(A, M)$ for the n-th **log Hochschild Homology group** of (A, M).

Proposition 4.1.2. *Let $(A, \{1\})$ be the pre-log R-algebra obtained by giving A the trivial pre-log structure on A, and let $(R[M], M)$ be the pre-log R-algebra with pre-log structure map sending $m \in M$ to $1m \in R[M]$. Then there are isomorphisms of simplicial R-algebras:*

$$C_\bullet(A, \{1\}) \cong C_\bullet(A) \qquad\qquad C_\bullet(R[M], M) \cong R[B^{\mathrm{rep}}_\bullet M]$$

Proof. These results are easily proven. First we see that

$$C_n(A, \{1\}) = C_n(A) \otimes_{R[B^{\mathrm{cy}}_n\{1\}]} R[B^{\mathrm{rep}}_n\{1\}] \cong C_n(A) \otimes_R R \cong C_n(A)$$

proving the leftmost isomorphism. For the second result, recall that from Proposition 3.4.7 that there is an isomorphism of simplicial R-modules $C_\bullet(R[M])$ and $R[B^{\mathrm{cy}}_\bullet M]$, and we obtain the rightmost isomorphism by

$$C_\bullet(R[M], M) = C_\bullet(R[M]) \otimes_{R[B^{\mathrm{cy}}_\bullet M]} R[B^{\mathrm{rep}}_\bullet M] \cong C_\bullet(R[M]) \otimes_{C_\bullet(R[M])} R[B^{\mathrm{rep}}_\bullet M] \cong R[B^{\mathrm{rep}}_\bullet M]$$

which concludes our proof. $\square$

We generalize Proposition 2.1.4 to log Hochschild homology.

Proposition 4.1.3. *Let (A, M, α) be a pre-log R-algebra. Then we have an isomorphism of R-modules:*

$$\mathrm{HH}_0(A, M) \cong A$$

Proof. By the relevant definitions we have the following R-module isomorphisms:

$$C_0(A, M) \cong C_0(A) \otimes_{R[B^{\mathrm{cy}}_0 M]} R[B^{\mathrm{rep}}_0 M] \cong A \otimes_{R[M]} R[M] \cong A$$

Similarily, by definition and by Lemma 3.4.14, we have the R-module isomorphism:

$$C_1(A, M) \cong C_1(A) \otimes_{R[B^{\mathrm{cy}}_1 M]} R[B^{\mathrm{rep}}_1 M] \cong (A \otimes A) \otimes_{R[M] \otimes R[M]} (R[M] \otimes R[M^{gp}])$$

Under these isomorphisms, the face maps $d_i \colon C_0(A, M) \to C_1(A, M)$ sends:

$$\begin{aligned} d_0 \colon (a \otimes a') \otimes_{R[M] \otimes R[M]} (rm \otimes r'm') &\longmapsto (a \cdot a') \cdot (r\alpha(m) \cdot 1) = a \cdot a' \cdot r \cdot \alpha(m) \\ d_1 \colon (a \otimes a') \otimes_{R[M] \otimes R[M]} (rm \otimes r'm') &\longmapsto (a' \cdot a) \cdot (r\alpha(m) \cdot 1) = a \cdot a' \cdot r \cdot \alpha(m) \end{aligned}$$

We see that $d_0 = d_1$, and so the boundary map of the associated chain complex, $b = d_0 - d_1 = d_0 - d_0 = 0$, is the zero map. Using this, we have that the zeroth log Hochschild homology group of (A, M, α) is

$$\mathrm{HH}_0(A, M) = \frac{ker(b \colon C_0(A, M) \to 0)}{im(b \colon C_1(A, M) \to C_0(A, M))} = \frac{A}{0} = A$$

as claimed. $\square$

Remark 4.1.4. The log Hochschild simplicial R-module is a functorial in that we can consider it as a functor from the category of pre-log R-algebras to the category of simplicial R-algebras

$$\mathrm{C}_\bullet(-) \colon \mathbf{PreLog} \longrightarrow \mathbf{s}R\text{-}\mathbf{Alg}$$

We define this in a manner similar to how we defined the replete bar construction to be a functor. As we have seen in the previous chapter, the pre-log homomorphism $(f, f^b) \colon (A, M) \to (B, N)$ induces functoriality a homomorphism on each of the components of the pushout as indicated below. In this way we obtain the diagram

$$\begin{tikzcd}
R[B^{\mathrm{cy}}_\bullet M] \arrow[r] \arrow[drr, "R[f^b_\bullet]"] \arrow[dd] & R[B^{\mathrm{rep}}_\bullet M] \arrow[drr, "R[f^b_\bullet]"] \arrow[dd] & & \\
& & R[B^{\mathrm{cy}}_\bullet N] \arrow[r] \arrow[dd] & R[B^{\mathrm{rep}}_\bullet N] \arrow[dd] \\
C_\bullet(A) \arrow[r] \arrow[drr, "f_\bullet"] & C(A, M) \arrow[drr, dotted, "(f, f^b)_\bullet"] & & \\
& & C_\bullet(B) \arrow[r] & C(B, N)
\end{tikzcd}$$

where the dotted arrow is defined by the universal property of the pushout, and let this be the morphism that $(f, f^b): (A, M) \to (B, N)$ is sent to.

4.2 Log Hochschild Homology of Products

Theorem 4.2.1. *Let (A, M, α) and (B, N, β) be pre-log R-algebras and let*

$$\begin{aligned}(p_1, p_1^b)&: (A, M) \times (B, N) \longrightarrow (A, M)\\ (p_2, p_2^b)&: (A, M) \times (B, N) \longrightarrow (B, N)\end{aligned}$$

be the projection maps of the categorical product of pre-log R-algebras. Then there is an quasi isomorphism of chain complexes given by the product of maps induced by the projections.

$$(p_1, p_1^b)_* \times (p_2, p_2^b)_* : C_*((A, M) \times (B, N)) \xrightarrow{\sim} C_*(A, M) \times C_*(B, N)$$

Proof. By Proposition 3.2.1 we know that there is an isomorphism between the categorical product $(A, M, \alpha) \times (B, N, \beta)$ of the two pre-log R-algebras, (A, M, α) and (B, N, β), and the pre-log R-algebra defined as $\big(A \times B, M \times N, \alpha \times \beta : (m, n) \mapsto (\alpha(m), \beta(n))\big)$. As a consequence of this isomorphism and the definition of the log Hocschild simplicial module associated to a pre-log R-algebra, we see that:

$$C_\bullet\big((A, M) \times (B, N)\big) \cong C_\bullet(A \times B, M \times N) = C_\bullet(A \times B) \otimes_{R[B_\bullet^{\text{cy}}(M \times N)]} R[B_\bullet^{\text{rep}}(M \times N)]$$

By Theorem 2.3.1 and Corollary 2.3.2, we know that the projections

$$\begin{aligned}p_1&: A \times B \longrightarrow A\\ p_2&: A \times B \longrightarrow B\end{aligned}$$

induce a quasi isomorphism of the Hochschild chain complexes by the products of the maps induced by the projections:

$$p_{1*} \times p_{2*} : C_*(A \times B) \xrightarrow{\sim} C_*(A) \times C_*(B)$$

Let us use the notation $f_\bullet$ for the simplicial map $p_{1\bullet} \times p_{2\bullet} : C_\bullet(A \times B) \xrightarrow{\sim} C_\bullet(A) \times C_\bullet(B)$. We want to prove that the simplicial homomorphism $f_\bullet \otimes_{R[\text{Id}_\bullet]} R[\text{Id}_\bullet]$:

$$C_\bullet(A \times B) \otimes_{R[B_\bullet^{\text{cy}}(M \times N)]} R[B_\bullet^{\text{rep}}(M \times N)] \longrightarrow \big(C_\bullet(A) \times C_\bullet(B)\big) \otimes_{R[B_\bullet^{\text{cy}}(M \times N)]} R[B_\bullet^{\text{rep}}(M \times N)]$$

induces an isomorphism on the homology groups. Before we can tackle this problem, we have to be precise about how the codomain is meant to be a simplicial module. To define this in a precise way, we have to choose a simplicial ring homomorphism, ψ, from $R[B_\bullet^{\text{cy}}(M \times N)]$ to $C_\bullet(A) \times C_\bullet(B)$ so that we have a simplical $R[B_\bullet^{\text{cy}}(M \times N)]$-module structure on $C_\bullet(A) \times C_\bullet(B)$ over which we can take the tensor product. We keep in mind what the end result we want to have is, and so we define ψ so that we have commutativity of the digram:

$$\begin{tikzcd} R[B_\bullet^{\text{cy}}(M \times N)] \arrow[r, "R[\text{Id}]"] \arrow[d, "\overline{\alpha' \times \beta'}"'] & R[B_\bullet^{\text{cy}}(M \times N) \arrow[d, "\psi"] \\ C_\bullet(A \times B) \arrow[r, "p_{1\bullet} \times p_{2\bullet}"] & C_\bullet(A) \times C_\bullet(B) \end{tikzcd}$$

There is only one obvious candidate for this map, which is to let $\psi_q = p_{1q} \circ (\overline{\alpha'_q \times \beta'_q}) \times p_{2q} \circ (\overline{\alpha'_q \times \beta'_q})$. We see that this map is defined on elements as

$$r\big((m_0, n_0) \otimes \cdots \otimes (m_q, n_q)\big) \longmapsto \big(r(\alpha(m_0) \otimes \cdots \otimes \alpha(m_q)), r(\beta(n_0) \otimes \ldots, \otimes \beta(n_q))\big)$$

Now that the codomain of $f_\bullet \otimes_{\mathrm{Id}_\bullet} \mathrm{Id}_\bullet$ has been properly defined, we proceed to the proof that this map is an isomorphism. We shall be using Lemma 1.5.1 for this. We see that all the conditions in the lemma is satisfied. In particular, the fact that

$$\mathrm{Tor}^{R[B^{\mathrm{cy}}_q(M \times N)]}(C_q(A \times B), R[B^{\mathrm{rep}}_q(M \times N)]) = 0$$

$$\mathrm{Tor}^{R[B^{\mathrm{cy}}_q(M \times N)]}(C_q(A) \times C_q(B), R[B^{\mathrm{rep}}_q(M \times N)]) = 0$$

is due to the flatness condition in Definition 4.1.1, and the commutativity of the diagram in the lemma is by the definition of the map ψ given above. The end result is that the homomorphism below induces an isomorphism on the homology groups.

$$f_\bullet \otimes_{R[\mathrm{Id}_\bullet]} R[\mathrm{Id}_\bullet] \colon C_\bullet\big((A, M) \times (B, N)\big) \xrightarrow{\sim} C_\bullet(A) \times C_\bullet(B) \otimes_{R[B^{\mathrm{cy}}_\bullet(M \times N)]} R[B^{\mathrm{rep}}_\bullet(M \times N)]$$

We shall now analyse the right hand side further. By Proposition 3.4.9 we have that there is a simplicial isomorphism, $R[B^{\mathrm{cy}}(M \times N)] \cong R[B^{\mathrm{cy}}M] \otimes R[B^{\mathrm{cy}}N]$, and by Proposition 3.4.13 we have a simplicial isomorphism $R[B^{\mathrm{rep}}(M \times N)] \cong R[B^{\mathrm{rep}}M] \otimes R[B^{\mathrm{rep}}N]$. Put together, this gives us that:

$$\begin{aligned}
&\big(C_\bullet(A) \times C_\bullet(B)\big) \otimes_{R[B^{\mathrm{cy}}_\bullet(M \times N)]} R[B^{\mathrm{rep}}_\bullet(M \times N)] \cong \\
&\big(C_\bullet(A) \times C_\bullet(B)\big) \otimes_{R[B^{\mathrm{cy}}_\bullet M] \otimes R[B^{\mathrm{cy}}_\bullet N]} R[B^{\mathrm{rep}}_\bullet(M \times N)] \cong \\
&\big(C_\bullet(A) \times C_\bullet(B)\big) \otimes_{R[B^{\mathrm{cy}}_\bullet M] \otimes R[B^{\mathrm{cy}}_\bullet N]} \big(R[B^{\mathrm{rep}}_\bullet M] \otimes R[B^{\mathrm{rep}}_\bullet N]\big)
\end{aligned}$$

Recall that the isomorphisms above where both obtained as the product of the map induced by the two projections. Summarizing what we have done this far, we have that there the simplicial map of simplicial R-algebras, given by the map

$$\begin{aligned}
&(p_{1\bullet} \times p_{2\bullet}) \otimes_{R[p^b_{1\bullet} \times p^b_{2\bullet}]} R[p^b_{1\bullet} \times p^b_{2\bullet}] \colon \\
&\qquad C_\bullet\big((A, M) \times (B, N)\big) \longrightarrow \big(C_\bullet(A) \times C_\bullet(B)\big) \otimes_{R[B^{\mathrm{cy}}_\bullet M] \otimes R[B^{\mathrm{cy}}_\bullet N]} \big(R[B^{\mathrm{rep}}_\bullet M] \otimes R[B^{\mathrm{rep}}_\bullet N]\big)
\end{aligned}$$

induces an isomorphism on homology groups. The rest of the proof focuses on finding a simplicial isomorphism

$$\big(C_\bullet(A) \times C_\bullet(B)\big) \otimes_{R[B^{\mathrm{cy}}_\bullet M] \otimes R[B^{\mathrm{cy}}_\bullet N]} \big(R[B^{\mathrm{rep}}_\bullet M] \otimes R[B^{\mathrm{rep}}_\bullet N]\big) \xrightarrow{\cong} C_\bullet(A, M) \times C_\bullet(B, N)$$

Our approach will be to show that these simplicial modules are isomorphic in each simplicial degree. We start by defining an isomorphism

$$\begin{aligned}
&\big(C_\bullet(A) \times C_\bullet(B)\big) \otimes_{R[B^{\mathrm{cy}}_\bullet M] \otimes R[B^{\mathrm{cy}}_\bullet N]} \big(R[B^{\mathrm{rep}}_\bullet M] \otimes R[B^{\mathrm{rep}}_\bullet N]\big) \cong \\
&\big(C_\bullet(A) \otimes R \times R \otimes C_\bullet(B)\big) \otimes_{R[B^{\mathrm{cy}}_\bullet M] \otimes R[B^{\mathrm{cy}}_\bullet N]} \big(R[B^{\mathrm{rep}}_\bullet M] \otimes R[B^{\mathrm{rep}}_\bullet N]\big)
\end{aligned}$$

We will use this isomorphism as a device for keeping track of how the R-modules $C_\bullet(A)$ and $C_\bullet(B)$ are considered to be $R[B^{\mathrm{cy}}_\bullet M] \otimes R[B^{\mathrm{cy}}_\bullet N]$-algebras. Concretely, we define the scalar multiplication by generating element $(r{\cdot}m{\otimes}sn)$ of $R[B_q^{\mathrm{cy}}M] \otimes R[B_q^{\mathrm{cy}}N]$ by a generating element $(a{\otimes}t) \in C_q(A){\otimes}R$ and $(u \otimes b) \in R \otimes C_q(B)$ respectively as:

$$(rm \otimes sn) \cdot (a \otimes t) = r\alpha(m) \cdot a \otimes s \cdot t$$
$$(rm \otimes sn) \cdot (u \otimes b) = r \cdot u \otimes s\beta(n) \cdot b$$

Observe that this action is compatible with the action we defined as the one we defined by the map ψ, and also that we can use the distributivity of the tensor product to obtain the isomorphism (that we have the distributivity of the simplicial module follows from the fact that we have distributivity in every degree):

$$\big(C_\bullet(A) \otimes R \times R \otimes C_\bullet(B)\big) \otimes_{R[B_\bullet^{\mathrm{cy}}M] \otimes R[B_\bullet^{\mathrm{cy}}N]} \big(R[B_\bullet^{\mathrm{rep}}M] \otimes R[B_\bullet^{\mathrm{rep}}N]\big)$$
$$\cong$$
$$\big(C_\bullet(A) \otimes R\big) \otimes_{R[B_\bullet^{\mathrm{cy}}M] \otimes R[B_\bullet^{\mathrm{cy}}N]} \big(R[B_\bullet^{\mathrm{rep}}M] \otimes R[B_\bullet^{\mathrm{rep}}N]\big)$$
$$\times$$
$$\big(R \otimes C_\bullet(B)\big) \otimes_{R[B_\bullet^{\mathrm{cy}}M] \otimes R[B_\bullet^{\mathrm{cy}}N]} \big(R[B_\bullet^{\mathrm{rep}}M] \otimes R[B_\bullet^{\mathrm{rep}}N]\big)$$

As the reader can guess from how we wrote the equations above, the length of our expression has exceeds the width of the paper. To simplify the notation, we will work exclusively with the first of factor of the direct product until we have obtained something less space consuming. The obvious analogue to all the algebraic operations that we perform on the first factor of the simplicial complex should be performed mentally on the latter factor simultaneously.

We will now do an algebraic trick, where the idea is the following: Assume that we have R-algebras X, Y, Z, X', Y' and Z' such that we have a Z-module structure on X and Y and a Z'-module structure on X' and Y'. Then there is an isomorphism

$$\iota \colon (X \otimes X') \otimes_{Z \otimes Z'} (Y \otimes Y') \longrightarrow X \otimes_Z Y \otimes X' \otimes_{Z'} Y'$$
$$x \otimes x' \otimes_{Z \otimes Z'} y \otimes y' \longmapsto x \otimes_Z y \otimes x' \otimes_{Z'} y'$$

To see that this is an isomorphis, observe that the map $x \otimes x' \times y \otimes y' \mapsto x \otimes_Z y \otimes x' \otimes_{Z'} y'$ is $Z \otimes Z'$-bilinear and that the map $x \otimes_Z y \times x' \otimes_{Z'} y' \mapsto x \otimes x' \otimes_{Z \otimes Z'} y \otimes y'$ is R-bilinear. By this argument we obtain:

$$\iota_q \colon \big(C_q(A) \otimes R\big) \otimes_{R[B_q^{\mathrm{cy}}M] \otimes R[B_q^{\mathrm{cy}}N]} \big(R[B_q^{\mathrm{rep}}M] \otimes R[B_q^{\mathrm{rep}}N]\big)$$
$$\cong$$
$$\big(C_q(A) \otimes_{R[B_q^{\mathrm{cy}}M]} R[B_q^{\mathrm{rep}}M]\big) \otimes \big(R_q \otimes_{R[B_q^{\mathrm{cy}}N]} R[B_q^{\mathrm{rep}}N]\big)$$

it is clear that this isomorphism commutes with the face and degeneracy maps, and so it induces an isomorphism in each simplicial degree:

$$\begin{aligned}
&\big(C_\bullet(A) \otimes R\big) \otimes_{R[B_\bullet^{\mathrm{cy}}M] \otimes R[B_\bullet^{\mathrm{cy}}N]} \big(R[B_\bullet^{\mathrm{rep}}M] \otimes R[B_\bullet^{\mathrm{rep}}N]\big) \cong \\
&\big(C_\bullet(A) \otimes_{R[B_\bullet^{\mathrm{cy}}M]} R[B_\bullet^{\mathrm{rep}}M]\big) \otimes \big(R_\bullet \otimes_{R[B_\bullet^{\mathrm{cy}}N]} R[B_\bullet^{\mathrm{rep}}N]\big) = \\
&C_\bullet(A, M) \otimes \big(R_\bullet \otimes_{R[B_\bullet^{\mathrm{cy}}N]} R[B_\bullet^{\mathrm{rep}}N]\big)
\end{aligned}$$

The last line follows by the definition of $C_\bullet(A,M)$. Notice that we have endowed the ring R with a simplicial structure that we have not described yet. To clarify, the structure we give $R_\bullet$ is the constant simplicial R-algebra structure, since this is the simplicial structure that makes the above map into a simplicial morphism. In fact one can easily check that if A is an R-algebra and $C_\bullet$ is a simplical R-algebra, then there is a simplicial isomorphism $A \otimes C_\bullet \cong A_\bullet \otimes C_\bullet$ where $A_\bullet$ has been given the constant simplicial structure. Since the expression we work with is more compact now, we can write the entire expression again. To be precise, we have the simplicial R-algebra:

$$C_\bullet(A,M) \otimes \big(R_\bullet \otimes_{R[B_\bullet^{\mathrm{cy}}N]} R[B_\bullet^{\mathrm{rep}}N]\big) \times C_\bullet(B,N) \otimes \big(R_\bullet \otimes_{R[B_\bullet^{\mathrm{cy}}M]} R[B_\bullet^{\mathrm{rep}}M]\big)$$

Recall that $R[-]$ is a left adjoint functor which therefore preserves colimits (Lemma 1.2.15). Notice that both of the "rubbish" terms in the expression above have been defined so that they can be thought of as pushouts in each simplicial degree as indicated below

$$\begin{tikzcd}
R[B_q^{\mathrm{cy}}N] \arrow[r, "R[g]"] \arrow[d, "R[f]"] & R[B_q^{\mathrm{rep}}N] \arrow[d] \\
R[\{*\}] \arrow[r] & R_\bullet \otimes_{R[B_\bullet^{\mathrm{cy}}N]} R[B_\bullet^{\mathrm{rep}}N]
\end{tikzcd}
\qquad
\begin{tikzcd}
R[B_q^{\mathrm{cy}}N] \arrow[r, "R[g']"] \arrow[d, "R[f']"] & R[B_q^{\mathrm{rep}}N] \arrow[d] \\
R[\{*\}] \arrow[r] & R_\bullet \otimes_{R[B_\bullet^{\mathrm{cy}}M]} R[B_\bullet^{\mathrm{rep}}M]
\end{tikzcd}$$

where the map f (respectively f') is the map sending every element of M (respectively N) to the identity element of the trivial monoid. If we use this and Proposition 3.4.15, which says that $\{1\} \sqcup_{B_q^{\mathrm{cy}}N} B^{\mathrm{rep}}N \cong \{1\}$ we obtain the following isomorphisms in each simplicial degree:

$$\begin{aligned}
& C_q(A,M) \otimes (R \otimes_{R[B_q^{\mathrm{cy}}N]} R[B_q^{\mathrm{rep}}N]) \bigtimes C_q(B,N) \otimes (R \otimes_{R[B_q^{\mathrm{cy}}M]} R[B_q^{\mathrm{rep}}M]) \\
\cong\ & C_q(A,M) \otimes R[\{1\}_q \sqcup_{B_q^{\mathrm{cy}}N} B_q^{\mathrm{rep}}N] \bigtimes C_q(B,N) \otimes R[\{1\}_q \sqcup_{B_q^{\mathrm{cy}}M} B_q^{\mathrm{rep}}M] \\
\cong\ & C_q(A,M) \otimes R[\{1\}_q] \bigtimes C_q(B,N) \otimes R[\{1\}_q] \\
\cong\ & C(A,M) \otimes R_q \bigtimes C(B,N) \otimes R_q \\
\cong\ & C(A,M) \bigtimes C(B,N)
\end{aligned}$$

We have have now proven the there exists a simplicial map

$$h\colon C_\bullet\big((A,M) \times (B,N)\big) \longrightarrow C_\bullet(A,M) \times C_\bullet(B,N)$$

that induces an isomorphism on homology groups. The part of the proof that remains is to check that this is the map obtained as the product of the maps induced by the projections. We combine the first and last part of this isomorphism in a diagram:

$$\begin{tikzcd}
C_\bullet\big((A,M) \times (B,N)\big) \arrow[r, "F"] \arrow[dr, "h"'] & \big(C_\bullet(A) \times C_\bullet(B)\big) \otimes_{R[B_\bullet^{\mathrm{cy}}M] \otimes R[B_\bullet^{\mathrm{cy}}N]} \big(R[B_\bullet^{\mathrm{rep}}M] \otimes R[B_\bullet^{\mathrm{rep}}N]\big) \arrow[d, "G"] \\
& C_\bullet(A,M) \times C_\bullet(B,N)
\end{tikzcd}$$

We have some idea of what the maps F and G are, and so we write them out carefully. The map F is defined degreewise as the following R-algebra homomorphism:

$$\begin{aligned}F\colon C_q(A\times B)\otimes_{R[B_q^{\mathrm{cy}}(M\times N)]} R[B_q^{\mathrm{rep}}(M\times N)] \longrightarrow \\ \longrightarrow \big(C_q(A)\times C_q(B)\big)\otimes_{R[B_q^{\mathrm{cy}}M]\otimes R[B_q^{\mathrm{cy}}N]} \big(R[B_q^{\mathrm{rep}}M]\otimes R[B_q^{\mathrm{rep}}N]\big)\end{aligned}$$

$$\begin{aligned}\big((a_0,b_0)\otimes\cdots\otimes(b_q,a_q)\big)\otimes_{R[B_q^{\mathrm{cy}}(M\times N)]} r\cdot z^{\mathrm{rep}} \longmapsto \\ \longmapsto \big(a_0\otimes\cdots\otimes a_q, b_0\otimes\cdots\otimes b_q\big)\otimes_{R[B_q^{\mathrm{cy}}M]\otimes R[B_q^{\mathrm{cy}}N]} r\big(p_{1q}(z^{\mathrm{rep}})\otimes p_{2q}(z^{\mathrm{rep}})\big)\end{aligned}$$

The map G is defined degreewise as the following R-algebra homomorphism:

$$\begin{aligned}G\colon \big(C_q(A)\times C_q(B)\big)\otimes_{R[B_q^{\mathrm{cy}}M]\otimes R[B_q^{\mathrm{cy}}N]} \big(R[B_q^{\mathrm{rep}}M]\otimes R[B_q^{\mathrm{rep}}N]\big) \longrightarrow \\ \longrightarrow \big(C_q(A)\otimes_{R[B_q^{\mathrm{cy}}M]} R[B_q^{\mathrm{rep}}M]\big)\times\big(C_q(A)\otimes_{R[B_q^{\mathrm{cy}}M]} R[B_q^{\mathrm{rep}}N]\big)\end{aligned}$$

$$\begin{aligned}\big(a_0\otimes\cdots\otimes a_q, b_0\otimes\cdots\otimes b_q\big)\otimes_{R[B_q^{\mathrm{cy}}M]\otimes R[B_q^{\mathrm{cy}}N]} \big(r_1x^{\mathrm{rep}}\otimes r_2y^{\mathrm{rep}}\big) \longmapsto \\ \longmapsto \Big(\big(a_0\otimes\cdots\otimes a_q\big)\otimes_{R[B_q^{\mathrm{cy}}M]}\big(r_1r_2x^{\mathrm{rep}}\big), \big(b_0\otimes\cdots\otimes b_q\big)\otimes_{R[B_q^{\mathrm{cy}}N]}\big(r_1r_2y^{\mathrm{rep}}\big)\Big)\end{aligned}$$

we see that the composition of these two maps gives us that the function is given in the each coordinate as the maps:

$$\begin{aligned}(p_{1q},p_{1q}^b)\colon C_q(A\times B)\otimes_{R[B_q^{\mathrm{cy}}(M\times N)]} R[B_q^{\mathrm{rep}}(M\times N)] \longrightarrow C_q(A)\otimes_{R[B_q^{\mathrm{cy}}M]} R[B_q^{\mathrm{rep}}M] \\ \big((a_0,b_0)\otimes\cdots\otimes(b_q,a_q)\big)\otimes_{R[B_q^{\mathrm{cy}}(M\times N)]} r\cdot z^{\mathrm{rep}} \longmapsto \big(a_0\otimes\cdots\otimes a_q\big)\otimes_{R[B_q^{\mathrm{cy}}M]}\big(r\cdot p_{1q}z^{\mathrm{rep}}\big)\end{aligned}$$

$$\begin{aligned}(p_{2q},p_{2q}^b)\colon C_q(A\times B)\otimes_{R[B_q^{\mathrm{cy}}(M\times N)]} R[B_q^{\mathrm{rep}}(M\times N)] \longrightarrow \big(C_q(A)\otimes_{R[B_q^{\mathrm{cy}}M]} R[B_q^{\mathrm{rep}}N]\big) \\ \big((a_0,b_0)\otimes\cdots\otimes(b_q,a_q)\big)\otimes_{R[B_q^{\mathrm{cy}}(M\times N)]} r\cdot z^{\mathrm{rep}} \longmapsto \big(b_0\otimes\cdots\otimes b_q\big)\otimes_{R[B_q^{\mathrm{cy}}N]}\big(r\cdot p_{2q}z^{\mathrm{rep}}\big)\end{aligned}$$

equal to the function (p_{1q},p_{1q}^b) in the first coordinate and (p_{2q},p_{2q}^b) as we stated in the theorem. This concludes our proof. □

4.3 Log Hochschild Homology of Coproducts

Now that we know how the log Hochschild Homology of products of pre-log R-algebras can be calculated, we would like to present a similar theorem for the coproduct of pre-log R-algebras. Recall that we showed in Proposition 3.2.2 that the coproduct of two pre-log algebras (A,M,α) and (B,N,α), denoted with $(A,M,\alpha)\sqcup(B,N,\beta)$, is isomorphic to the pre-log R-algebra $(A\otimes B, M\oplus N, \alpha\sqcup\beta\colon m\oplus n\mapsto \alpha(m)\oplus\beta(n))$. We adopt the notation $(A,M,\alpha)\otimes(B,N,\beta)$ for this coproduct for aesthetic purposes, and refer to it as the tensor product of pre-log R-algebras. We justify this in that the tensor product of pre-log R-algebras can even be thought of as a generalization of the tensor product of R-algebras.

By Theorem 2.3.3 we already know that the Hochschild homology of the tensor products of Q-modules (where Q is a field) is isomorphic to the tensor product of the Hochschild homology of its factors. The corresponding result, generalized to the setting of log Hochschild homology, consists in the next theorems.

Theorem 4.3.1. *Let (A, M, α) and (B, N, β) be pre-log R-algebras and let the tensor product, $\otimes$, denote the coproduct in the category of pre-log R-algebras. Then there is a simplicial isomorphism of R-algebras:*

$$C_\bullet((A, M) \otimes (B, N)) \cong C_\bullet(A, M) \otimes C_\bullet(B, N)$$

Proof. By definition, $C_\bullet((A, M) \otimes (B, N))$ is equal to $C_\bullet(A \otimes B) \otimes_{R[B_\bullet^{\mathrm{cy}}(M\times N)]} R[B_\bullet^{\mathrm{rep}}(M \times N)]$. Using Proposition 3.4.9 and Proposition 3.4.13 as we did in the proof of Theorem 4.2.1, we see that we have isomorphisms:

$$\begin{aligned} C_\bullet((A, M) \otimes (B, N)) &= C_\bullet(A \otimes B) \otimes_{R[B_\bullet^{\mathrm{cy}}(M\times N)]} R[B_\bullet^{\mathrm{rep}}(M \times N)] \\ &\cong C_\bullet(A \otimes B) \otimes_{R[B_\bullet^{\mathrm{cy}}(M\times N)]} R[B_\bullet^{\mathrm{rep}} M] \otimes R[B_\bullet^{\mathrm{rep}} N] \\ &\cong C_\bullet(A \otimes B) \otimes_{R[B_\bullet^{\mathrm{cy}} M]\otimes R[B_\bullet^{\mathrm{cy}} N]} R[B_\bullet^{\mathrm{rep}} M] \otimes R[B_\bullet^{\mathrm{rep}} N] \end{aligned}$$

Recall from the proof of Theorem 4.3.1 that the simplicial module $C_\bullet(A \otimes B)$ is isomorphic to the simplicial tensor product/cartesian product of simplicial R-modules, $C_\bullet(A) \otimes C_\bullet(B)$ by rearranging the terms. Explicitly, this isomorphism is given in each simplicial degree by the isomorphism:

$$\mathfrak{sh}\colon (a_0 \otimes b_0) \otimes (a_1 \otimes b_1) \otimes \cdots \otimes (a_q \otimes b_q) \mapsto (a_0 \otimes a_1 \otimes \cdots \otimes a_q) \otimes (b_0 \otimes b_1 \otimes \cdots \otimes b_q)$$

We also want to use the same trick that we used in the proof in Theorem 4.2.1, that R-algebras S, S', T, T', U and U' where S and T are U-algebras and S' and T' are U'-algebras, we have an isomorphism $(S \otimes S') \otimes_{U\otimes U'} (T \otimes T') \cong (S \otimes_U T) \otimes (S' \otimes_{U'} T')$. Together this gives us the simplicial isomorphisms:

$$\begin{aligned} C_\bullet((A, M) \otimes (B, N)) &\cong C_\bullet(A \otimes B) \otimes_{R[B_\bullet^{\mathrm{cy}} M]\otimes R[B_\bullet^{\mathrm{cy}} N]} R[B_\bullet^{\mathrm{rep}} M] \otimes R[B_\bullet^{\mathrm{rep}} N] \\ &\cong C_\bullet(A) \otimes C_\bullet(B) \otimes_{R[B_\bullet^{\mathrm{cy}} M]\otimes R[B_\bullet^{\mathrm{cy}} N]} R[B_\bullet^{\mathrm{rep}} M] \otimes R[B_\bullet^{\mathrm{rep}} N] \\ &\cong \big(C_\bullet(A) \otimes_{R[B_\bullet^{\mathrm{cy}} M]} R[B_\bullet^{\mathrm{rep}} M]\big) \otimes \big(C_\bullet(B) \otimes_{R[B_\bullet^{\mathrm{cy}} N]} R[B_\bullet^{\mathrm{rep}} N]\big) \\ &\cong C_\bullet(A, M) \otimes C_\bullet(B, N) \end{aligned}$$

We have now constructed a sequence of isomorphisms of simplicial R-modules from $C_\bullet((A, M) \otimes (B, N))$ to $C_\bullet(A, M) \otimes C_\bullet(B, N)$, which yields the desired result.

□

Corollary 4.3.2 (The Eilenberg-Zilber Theorem for Log Hochschild Homology). *Let (A, M, α) and (B, N, β) be pre-log R-algebras. Then there is a chain equivalence of R-algebras:*

$$C_*((A, M) \otimes (B, N)) \cong C_*(A, M) \otimes C_*(B, N)$$

Above, we let $(A, M) \otimes (B, N)$ denote the coproduct in the category of pre-log R-algebras, while $C_*(A, M) \otimes C_*(B, N)$ denotes the tensor product of chain complexes.

Proof. This is immediate by applying Theorem 1.4.9, (the Eilenberg-Zilber Theorem) to the simplicial isomorphism from Theorem 4.3.1. □

In the next corollary, we let $B_n(A, N)$ denote the n-bounadries of the chain complex $C_*(A, M)$ and let $Z_n(A, M)$ denote the n-cycles of the chain complex $C_*(A, M)$.

Corollary 4.3.3 (A Künneth Formula for Log Hochschild Homology). *Let (A, M, α) and (B, N, β) be pre-log R-algebras and assume that both $Z_n(A, M)$ and $B_n(A, N)$ are flat over R for all $n \in \mathbb{N}_0$. Then there is a short exact sequence:*

$$0 \to \bigoplus_{q+r=n} \mathrm{HH}_q(A, M) \otimes \mathrm{HH}_r(B, N) \xrightarrow{p} \mathrm{HH}_n((A, M), (B, N)) \xrightarrow{\beta}$$

$$\xrightarrow{\beta} \bigoplus_{q+r=n-1} \mathrm{Tor}_1^R(\mathrm{HH}_q(A, M), \mathrm{HH}_r(B, N)) \to 0$$

If $Z_n(A, M)$ and $\mathrm{HH}_n(A, M)$ are flat over R, then the map p is an isomorphism:

$$\bigoplus_{q+r=n} \mathrm{HH}_q(A, M) \otimes \mathrm{HH}_r(B, N) \overset{p}{\cong} \mathrm{HH}_n((A, M) \otimes (B, N))$$

If Q is a field and both (A, M) and (B, N) are pre-log Q-algebras, then we always have the isomorphism above, since every module under a field is free.

Proof. This corollary follows immediately by applying Theorem 1.3.13 (the Künneth Formula) to Corollary 4.3.2. ☐

4.3.1 Log Hochschild Homology of Localizations

Definition 4.3.4. Let S be a multiplicatively closed subset of R and let $(S^{-1}R, \{1\})$ be the R-algebra $S^{-1}R$ with the trivial pre-log structure. Then we define the **localization** of the log R-algebra (A, M, α) at S to be the pre-log algebra:

$$S^{-1}(A, M) := (S^{-1}R, \{1\}) \otimes (A, M)$$

Proposition 4.3.5. *Let S be a multiplicatively closed subset of R. A pre-log R-algebra, $(S^{-1}A, M, \alpha)$ is isomorphic to the localization at S of some pre-log algebra (A, M, α') if the image of α is contained in the image of the localization map, $\phi\colon A \to S^{-1}A$, that sends a to $\frac{a}{1}$.*

Proof. ☐

Theorem 4.3.6. *Let S be a multiplicatively closed subset of R and let $S^{-1}(A, M)$ be the localization of the pre-log R-algebra (A, M, α). Then there is an isomorphism for all $n \in \mathbb{N}_0$:*

$$\mathrm{HH}_n(S^{-1}(A, M)) \cong S^{-1}\,\mathrm{HH}_n(A, M)$$

Proof. By Proposition 4.1.2 we have a simplicial isomorphism $C_\bullet(A, \{1\}) \cong C_\bullet(A)$. Using this together with Theorem 1.4.9 (the Eilenberg-Zilber Theorem) for log Hochschild homology, we have a homotopy equivalence:

$$\begin{aligned} C_*(S^{-1}(A, M)) &\cong C_*((S^{-1}R, \{1\}) \otimes (A, M)) \\ &\simeq C_*(S^{-1}R, \{1\}) \otimes C_*(A, M)) \\ &\cong C_*(S^{-1}R) \otimes C_*(A, M) \end{aligned}$$

By Proposition 2.1.7 we know that $C_n(S^{-1}R)$ has an R-projective module of cycles and an R-projective module of. We can therefore use the the Künneth formula for log Hochschild homology, to obtain the isomorphism we wanted, since

$$\mathrm{HH}_*(S^{-1}A, M) \cong \mathrm{HH}_*(S^{-1}R, \{1\}) \otimes \mathrm{HH}_*(A, M) \cong \mathrm{HH}_*(S^{-1}R) \otimes \mathrm{HH}_*(A, M) \cong S^{-1}\,\mathrm{HH}_*(A, M)$$

The last isomorphism in the sequence above is essentially the same as in the proof we gave of Corollary 2.3.6. ☐

4.4 Polynomial Algebras

We will now use the Künneth theorem of the previous section to calculate the log Hochschild homology groups of the pre-log R-algebra $(R[x_i], \langle x_i, y_i \rangle)$. We begin by making the following rather elementary remark:

Remark 4.4.1. There is an isomorphism of R-algebras $R[\langle x \rangle^{\mathrm{gp}}] \cong R[x, x^{-1}]$. To see this, define a map on the generators of $R[\langle x \rangle^{\mathrm{gp}}]$ by $rx^n \mapsto rx^n$ and $r(x^n)^{-1} \mapsto rx^{-n}$. This map is clearly an R-algebra homomorphism and it is also clear that this homomorphism is invertible.

Proposition 4.4.2. *The pre-log algebra $(R[x], \langle x \rangle, \alpha \colon x \to x)$ has log Hochschild homology groups:*

$$\mathrm{HH}_n(A, M) = \begin{cases} R[x], & \textit{for } n = 0 \\ R[x], & \textit{for } n = 1 \\ 0, & \textit{for } n \geq 2 \end{cases}$$

Proof. We begin by using Proposition 4.1.2 on $C_\bullet(R[x], \langle x \rangle)$ to see that we can instead calculate the log Hochschild homology groups of $R[B_\bullet^{\mathrm{rep}} \langle x \rangle]$. By Lemma 3.4.16 we have an isomorphism of R-modules:

$$\mathrm{H}_n(R[B_\bullet^{\mathrm{rep}} \langle x \rangle]) \cong R[\langle x \rangle] \otimes \mathrm{Tor}_n^{R[\langle x \rangle^{gp}]}(R, R)$$

We use the following free resolution of $R[x, x^{-1}]$-modules of R:

$$0 \longrightarrow R[x, x^{-1}] \xrightarrow{\rho} R[x, x^{-1}] \xrightarrow{\sigma} R \longrightarrow 0$$

In this resolution, the homomorphisms ρ and σ are defined as:

$$\begin{aligned} \rho \colon R[x, x^{-1}] &\longrightarrow R[x, x^{-1}] \\ p(x) &\longmapsto (x-1)p(x) \end{aligned} \qquad\qquad \begin{aligned} \sigma \colon R[x, x^{-1}] &\longrightarrow R \\ x &\longmapsto 1 \end{aligned}$$

To see that this is a resolution, first notice that σ is surjective, and ρ injective. Furthermore, the kernel of σ definitely contains the image of ρ, since we have that:

$$\sigma(\rho(p(x)) = \sigma((x-1)p(x)) = \sigma(xp(x) - p(x)) = \sigma(xp(x)) - \sigma(p(x)) = 0$$

Conversely, the kernel of σ is contained in the image of ρ, since if for

$$g(x) = r_n x^n + \cdots + r_1 x^1 + r_0 x^0 + r_{-1} x^{-1} + \cdots + r_{-m} x^{-m}$$

we have that $\sigma(g(x)) = 0$, then we see that

$$g(x) = (x-1)\Big(r_n x^{n-1} + (r_n + r_{n-1})x^{n-2} + (r_n + r_{n-1} + r_{n-2})x^{n-3} \cdots + \big(\sum_{i=-m}^{i=n} r_i\big)x^{-(m+1)}\Big)$$

The next step is tensoring this sequence by $R \otimes_{R[x,x^{-1}]} (-)$ and removing the first term. The result is the sequence:

$$0 \longrightarrow R \otimes_{R[x,x^{-1}]} R[x, x^{-1}] \xrightarrow{\mathrm{Id} \otimes \rho} R \otimes_{R[x,x^{-1}]} R[x, x^{-1}] \longrightarrow 0$$

That becomes isomorphic to the chain complex

$$0 \longrightarrow R \xrightarrow{0} R \longrightarrow 0$$

under the isomorphism

$$\begin{aligned} \iota\colon R \otimes_{R[x,x^{-1}]} R[x,x^{-1}] &\longrightarrow R \\ r \otimes_{R[x,x^{-1}]} p(x) &\longmapsto r \cdot p(1) \end{aligned}$$

The map above is the zero map, since for a generator, $r \otimes_{R[x,x^{-1}]} p(x)$ of $R \otimes_{R[x,x^{-1}]} R[x,x^{-1}]$, we have that:

$$\iota(\mathrm{Id} \otimes_{R[x,x^{-1}]} \rho((r \otimes_{R[x,x^{-1}]} p(x))) = \iota(r \otimes_{R[x,x^{-1}]} (x-1)p(x)) = r \cdot (1p(1) - 1p(1)) = 0$$

We have therefore calculated $\mathrm{Tor}_n^{R[\langle x\rangle^{gp}]}(R,R)$ to be R if $n = 0, 1$ and the 0 module if $n \geq 2$. The upshot of this is that by the isomorphism $\mathrm{H}_n(R[B_\bullet^{\mathrm{rep}}\langle x\rangle]) \cong R[\langle x\rangle] \otimes \mathrm{Tor}_n^{R[\langle x\rangle^{gp}]}(R,R)$ we see that we have

$$\mathrm{HH}_n(A,M) = \begin{cases} R[x], & \text{for } n = 0 \\ R[x], & \text{for } n = 1 \\ 0, & \text{for } n \geq 2 \end{cases}$$

as claimed. □

Proposition 4.4.3. *The pre-log R-algebra*

$$(R[x_1, x_2, x_3, \ldots, x_n], \langle x_1, x_2, x_3, \ldots, x_n\rangle, \alpha_i\colon x_i \mapsto x_i),$$

for which we write $(R[x_i], \langle x_i\rangle)$, has log Hochschild homology groups isomorphic to:

$$\mathrm{HH}_q(R[x_i], \langle x_i\rangle) = \begin{cases} R[x_i]^{\oplus\binom{q}{n}}, & \textit{for } 0 \leq q \leq n \\ 0, & \textit{for } q > n \end{cases}$$

Proof. This result follows easily from the fact that

$$(R[x_1, x_2, \ldots, x_n], \langle x_1, x_2, \ldots, x_n\rangle) \cong (R[x_1, x_2, \ldots, x_{n-1}], \langle x_1, x_2, \ldots, x_{n-1}\rangle) \otimes (R[x_n], \langle x_n\rangle)$$

together with some elementary algebra and repeated use of Corollary 4.3.3 to Proposition 4.4.2. □

We can generalize the above Proposition even further. If we let $(R[x_1, \ldots, x_n, y_1, \ldots, y_m], \langle x_1, \ldots, x_n\rangle, \alpha)$ be the pre-log R-algebra where α is the morphism defined on generators by:

$$\begin{aligned} \alpha\colon \langle x_1, \ldots, x_n\rangle &\longrightarrow \langle R[x_1, \ldots, x_n, y_1, \ldots, y_m], \cdot\rangle \\ x_i &\longmapsto 1 \cdot x_i \end{aligned}$$

We will use the notation $(R[x_i, y_j], \langle x_i\rangle, \alpha)$ as a more compact way of denoting the pre-log R-algebra $(R[x_1, \ldots, x_n, y_1, \ldots, y_m], \langle x_1, \ldots, x_n\rangle, \alpha)$.

Theorem 4.4.4. *The log Hochschild homology groups of the pre-log R-algebra $(R[x_i, y_j], \langle x_i\rangle, \alpha)$ are isomorphic to:*

$$\mathrm{HH}_q(R[x_i, y_j], \langle x_i\rangle) = \begin{cases} R[x_i, y_j]^{\oplus\binom{q}{n+m}}, & \textit{for } 0 \leq q \leq n+m \\ 0, & \textit{for } q > n+m \end{cases}$$

Observe that we have isomorphisms of pre-log R algebras as below:

$$\begin{aligned} C_\bullet(R[x_1,\dots,x_n,y_1,\dots,y_m],\langle x_1,\dots,x_n\rangle) &\cong C_\bullet(R[x_1,\dots,x_n]\otimes R[y_1,\dots,y_m],\langle x_1,\dots,x_n\rangle\times\{1\}) \\ &\cong C_\bullet(R[x_1,\dots,x_n],\langle x_1,\dots,x_n\rangle)\otimes C_\bullet(R[y_1,\dots,y_m],\{1\}) \end{aligned}$$

The last isomorphism is by Theorem 2.3.3. By Proposition 4.1.2, the log Hochschild chain complex of $(R[y_1,\dots,y_m],\{1\})$ is isomorphic to the Hochschild chain complex of the R-algebra of $R[y_1,\dots,y_m]$ which by the proof of Proposition 4.1.2 was again shown to be isomorphic to the log Hochschild chain complex of the pre-log R-algebra $(R[y_1,\dots,y_m],\langle y_1,\dots,y_m\rangle)$. Summarizing, this as simplicial isomorphisms, we get three first lines of isomorphisms below. The fourth isomorphisms is by Theorem 2.3.3, the fifth is by the definition of the tensor product of pre-log R-algebras and the last isomorphism is obvious.

$$\begin{aligned} C_\bullet(R[x_1,\dots,x_n,y_1,\dots,y_m],&\langle x_1,\dots,x_n\rangle) \\ &\cong C_\bullet(R[x_1,\dots,x_n],\langle x_1,\dots,x_n\rangle)\otimes C_\bullet(R[y_1,\dots,y_m],\{1\}) \\ &\cong C_\bullet(R[x_1,\dots,x_n],\langle x_1,\dots,x_n\rangle)\otimes C_\bullet(R[y_1,\dots,y_m]) \\ &\cong C_\bullet(R[x_1,\dots,x_n],\langle x_1,\dots,x_n\rangle)\otimes C_\bullet(R[y_1,\dots,y_m],\langle y_1,\dots,y_m\rangle) \\ &\cong C_\bullet\big((R[x_1,\dots,x_n],\langle x_1,\dots,x_n\rangle)\otimes(R[y_1,\dots,y_m],\langle y_1,\dots,y_m\rangle)\big) \\ &\cong C_\bullet\big(R[x_1,\dots,x_n]\otimes R[y_1,\dots,y_m],\langle x_1,\dots,x_n\rangle\times\langle y_1,\dots,y_m\rangle\big) \\ &\cong C_\bullet\big(R[x_1,\dots,x_n,y_1,\dots,y_m],\langle x_1,\dots,x_n,y_1,\dots,y_m\rangle\big) \end{aligned}$$

Applying Proposition 4.4.3 to the last of these lines gives the statement of the theorem.

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