UNIVERSITY OF BERGEN
Faculty of Mathematics and Natural Sciences

Master's Thesis in Analysis

Automorphism groups of pseudo H-type Lie algebras

Francesca Azzolini

Spring 2017

Quelli che s'innamoran di pratica sanza scienzia son come 'l nocchier ch'entra in navilio senza timone o bussola, che mai ha certezza dove si vada

Leonardo da Vinci

Acknowledgments

Firstly, I would like to thank my supervisor, Irina Markina, for suggesting me a topic for my thesis and for supporting me throughout the entire process of writing it.

I am very grateful to professor Kenro Furutani, who invited me in Japan and gave me some very useful insights on the theory behind my project.

I am most thankful to the analysis group and all the students at the department for the positive and stimulating environment they created. In particular, I would like to thank Stefano Piceghello for his crucial suggestions during the writing of the final draft.

I would like to thank my parents, for their wise words and their encouragement to fulfil a scientific career, and my husband Andrea Tenti, for his endless support.

Contents

Acknowledgments **3**

Foreword **7**

1 Preliminary notions **9**
1.1 Classical Lie groups 9
1.2 Isomorphisms 12

2 Automorphism groups of pseudo H-type Lie algebras **21**
2.1 Two-step nilpotent Lie algebras 21
2.2 Pseudo H-type Lie algebras 23
2.2.1 The subgroup $C(\mathfrak{n})$ 25
2.2.2 Commutation of J_{Z_i} 27
2.3 Admissible Clifford modules 29
2.3.1 Block structure of an admissible module 30
2.3.2 The group $\mathrm{Aut}^0(\mathfrak{n}^{r,0})$ 32

3 Classification of $\mathrm{Aut}^0(\mathfrak{n}^{\mathrm{r,s}})$ **35**
3.1 One-dimensional common eigenspaces 37
3.1.1 Cases $\mathfrak{n}^{7,0}$ and $\mathfrak{n}^{3,4}$ 38
3.1.2 Cases $\mathfrak{n}^{8,0}$, $\mathfrak{n}^{0,8}$ and $\mathfrak{n}^{4,4}$ 39
3.2 Two-dimensional common eigenspaces 39
3.2.1 Cases $\mathfrak{n}^{1,0}$ and $\mathfrak{n}^{0,1}$ 39
3.2.2 Cases $\mathfrak{n}^{6,0}$ and $\mathfrak{n}^{2,4}$ 40
3.2.3 Cases $\mathfrak{n}^{0,7}$ and $\mathfrak{n}^{4,3}$ 41
3.2.4 Cases $\mathfrak{n}^{7,1}$ and $\mathfrak{n}^{3,5}$ 41
3.3 Four-dimensional common eigenspaces 42
3.3.1 Cases $\mathfrak{n}^{2,0}$ and $\mathfrak{n}^{0,2}$ 42
3.3.2 Case $\mathfrak{n}^{1,1}$ 43
3.3.3 Case $\mathfrak{n}^{3,0}$ 43
3.3.4 Cases $n^{1,2}$, $n^{3,2}$ and $n^{2,3}$ 44
3.3.5 Cases $\mathfrak{n}^{4,0}$, $\mathfrak{n}^{2,2}$, $\mathfrak{n}^{0,4}$, $\mathfrak{n}^{6,2}$ and $\mathfrak{n}^{2,6}$ 44
3.3.6 Cases $\mathfrak{n}^{5,0}$ and $\mathfrak{n}^{1,4}$ 45
3.3.7 Cases $\mathfrak{n}^{0,6}$ and $\mathfrak{n}^{4,2}$ 45
3.3.8 Cases $\mathfrak{n}^{6,1}$, $\mathfrak{n}^{1,6}$, $\mathfrak{n}^{5,2}$ and $\mathfrak{n}^{2,5}$ 46
3.3.9 Cases $\mathfrak{n}^{1,7}$ and $\mathfrak{n}^{5,3}$ 46
3.3.10 Cases $\mathfrak{n}^{3,6}$ and $\mathfrak{n}^{7,2}$ 47
3.4 Eight-dimensional common eigenspaces 47
3.4.1 Case $\mathfrak{n}^{0,3}$ 47

3.4.2 Case $\mathfrak{n}^{2,1}$ 48
3.4.3 Case $\mathfrak{n}^{1,3}$ 49
3.4.4 Cases $\mathfrak{n}^{0,5}$, $\mathfrak{n}^{4,1}$, $\mathfrak{n}^{2,7}$ and $\mathfrak{n}^{6,3}$ 49
3.4.5 Case $n^{3,1}$ 50
3.4.6 Cases $\mathfrak{n}^{5,1}$, $\mathfrak{n}^{1,5}$, $\mathfrak{n}^{7,3}$ and $\mathfrak{n}^{3,7}$ 50

A Tables for the constructions of the bases **53**

Foreword

The purpose of this thesis is to analyse and classify the automorphism groups of pseudo H-type Lie algebras, which are particular types of two-step nilpotent Lie algebras.

The paper which marks the beginning of the study of two-step nilpotent Lie algebras is [M$^+$] by Metrevier. In particular, Metrevier considers those two-step nilpotent Lie algebras which satisfy the so-called **hypothesis H**: given a Lie algebra $\mathfrak{n} = Z \oplus V$, where Z is the centre of $\mathfrak{n}$ and V is its complement, the adjoint map $\mathrm{ad}_X : \mathfrak{n} \to Z$ is surjective for any $X \in V$. These Lie algebras have also been called **fat** or **non-singular** in [KT13].

The research on two-step nilpotent Lie algebras then branched out in two directions, investigating, respectively, their associated Lie groups (see [Ebe94]) and the **Lie algebras of Heisenberg type**, also called Lie algebras of H-type. These particular algebras were first defined by A. Kaplan in [Kap80]. In particular, Kaplan used H-type Lie algebras to examine a class of hypoellyptic PDE; later, a relation was found between the H-type Lie algebras and the Clifford algebra representations over a scalar product of signature $(r, 0)$. In particular, the H-type Lie algebras inherit the periodicity specific of the Clifford algebras, and in the paper [Saa96], L. Saal classifies the group of automorphisms of H-type Lie algebras.

The starting point for our thesis is the notion of **pseudo H-type Lie algebra**, which was introduced independently by P. Ciatti [Cia00] and by I. Markina, M. Molina and A. Korolko [ref]. Such Lie algebras are correlated to Clifford algebra representations over a scalar product with a signature (r, s). In [FM17], I. Markina and K. Furutani study the isomorphism groups of pseudo H-type Lie algebras, providing the structure of a generic isomorphism $\Phi : \mathfrak{z} \oplus V \to \mathfrak{z} \oplus V$. In particular, they show that an isomorphism is possible only between certain pseudo H-type Lie algebras, namely between $\mathfrak{n}^{r,s}$ and $\mathfrak{n}^{s,r}$, where (r, s) and (s, r) represent the signatures of the which is the carrier space of the respective Clifford algebras representations.

Our goal is to describe the structure of the group of automorphisms of a generic pseudo H-type Lie algebra and to provide a classification of these groups according to the signature. Such classification will be finite because of the mentioned periodicity within pseudo H-type Lie algebras.

The thesis is composed of the following parts.

In Chapter 1 we introduce the basic definitions that we will use during our classification. We will also list a number of isomorphisms between some of the classical Lie groups constructed over different fields.

In Chapter 2 we deal with the structure of the automorphism groups. We will start from the known results for two-step nilpotent Lie algebras ([KT13] and [Saa96]), which will allow us to characterise the automorphism group $\mathrm{Aut}(\mathfrak{n})$ of a pseudo H-type Lie algebra $\mathfrak{n}$ by a particular subgroup, called $\mathrm{Aut}^0(\mathfrak{n})$. We will then define a pseudo H-type Lie algebra as a two-step nilpotent Lie algebra satisfying an additional condition; all the results for two-step nilpotent Lie algebras will then still hold in our case. We will see how the additional condition will produce an important tool for the sought classification. We will then briefly show the correlation between pseudo H-type Lie algebras and Clifford algebras, namely the one-to-one correspondence between the former and admissible modules of the latter. Lastly, we will list some of the already known results about H-type Lie algebras ([Saa96]).

Chapter 3 is dedicated to the classification of the automorphism groups of pseudo H-type Lie algebras; using the tables presented in the Appendix and the isomorphisms illustrated in Chapter 1, we will describe $\mathrm{Aut}^0(\mathfrak{n})$ for every pseudo H-type Lie algebra $\mathfrak{n} = \mathfrak{n}^{r,s}$. We will study together all the cases in which the admissible modules appear to have similar bases. All the groups $\mathrm{Aut}^0(\mathfrak{n})$ will result to be isomorphic to a classical Lie group.

Lastly, the Appendix, which constitutes an important part of this thesis, presents the tables of involutions and bases of admissible modules, which are used in Chapter 3. Once we know such involutions, we will be able to provide a basis for the minimal admissible module of each pseudo H-type Lie algebra, and hence to conclude our classification.

Chapter 1

Preliminary notions

In this chapter we will list the basic notions that will be employed throughout the exposition. We will present an account of some classical Lie groups and the definition of split-complex and split-quaternion numbers, which, despite lacking the property of being a field, still can be used to construct matrix Lie groups. We will also provide some useful isomorphisms between low-dimensional matrix Lie groups.

1.1 Classical Lie groups

We will start with the main definitions.

Definition 1.1. Let us consider a vector space $\mathfrak{v}$. We call **scalar product** a bilinear operator

$$\begin{aligned} \langle -,- \rangle \; &: \mathfrak{v} \times \mathfrak{v} \to \mathbb{R} \\ (v,w) &\mapsto \langle v,w \rangle \end{aligned}$$

such that:

- $\langle -,- \rangle$ is symmetric, i.e. $\langle v,w \rangle = \langle w,v \rangle$ for all v, $w \in \mathfrak{v}$.
- $\langle -,- \rangle$ is non-degenerate, i.e. $\langle v,w \rangle = 0$ for all $v \in \mathfrak{v}$, then $w = 0$.

We say that $\langle -,- \rangle$ is **positive definite** if for every $v \in \mathfrak{v}$ we have that $\langle v,v \rangle \geq 0$, and that $\langle v,v \rangle = 0$ if and only if $v = 0$. We say that $\langle -,- \rangle$ is **negative definite** if for every $v \in \mathfrak{v}$ we have that $\langle v,v \rangle \leq 0$, and that $\langle v,v \rangle = 0$ if and only if $v = 0$.

Definition 1.2. Given a vector space $\mathfrak{v}$ of dimension n endowed with a scalar product $\langle -,- \rangle$, we say that (r,s) is the **signature** of $\langle -,- \rangle$ if $r + s = n$ and there exists a basis $\{Z_1, \ldots, Z_n\}$ of $\mathfrak{v}$ such that

$$Z_i Z_j + Z_j Z_i = 2\varepsilon_i(r,s)\delta_{ij}, \tag{1.1}$$

where $\varepsilon_i(r,s) = \begin{cases} 1 & \text{if } i \in \{1,\ldots,r\} \\ -1 & \text{if } i \in \{r+1,\ldots,r+s\} \end{cases}$ and δ_{ij} is the Kronecker delta.

Definition 1.3. Given a matrix A, we denote with A^t its transpose, and with A^T its transpose with respect to the metric given by a scalar product, i.e. given a scalar product $\langle -,- \rangle$ over $\mathfrak{v}$,

$$\langle Ax, y \rangle = \langle x, A^T y \rangle \text{ for all } \; x, y \in \mathfrak{v}.$$

Definition 1.4. A **complex number** is a number written as $a + ib$, where a, $b \in \mathbb{R}$ and i satisfies the condition $i^2 = -1$.
A **quaternion number** is a number written in the form $a + ib + jc + kd$ where a, b, c, $d \in \mathbb{R}$ and i, j, k satisfy the relations:

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \qquad jk = -kj = i, \qquad ik = -ki = -j.$$

A triple of **quaternion units** in a group is a triple of elements satisfying the same relations as i, j and k.
Both the complex numbers and the quaternion numbers form a field.

Definition 1.5. A **split-complex number** is a number written as $a+i^*b$ where a, $b \in \mathbb{R}$ and i^* satisfies $i^{*2} = 1$. We denote the split-complex numbers with the symbol $\mathbb{SC}$.
We define the conjugation of a split-complex number $z = a + i^*b$ as $\overline{z} := a - i^*b$.
A **split-quaternion number** is a number written as $a + i^*b + j^*c + k^*d$ where a, b, c, $d \in \mathbb{R}$ and i^*, j^*, k^* satisfy:

$$i^{*2} = -1, \quad j^{*2} = k^{*2} = 1, \quad i^*j^* = -j^*i^* = k^*, \quad j^*k^* = -k^*j^* = -i^*, \quad k^*i^* = -i^*k^* = j^*.$$

The set $\{1, i^*, j^*, k^*\}$ is a basis of a four-dimensional real vector space equipped with a multiplicative operation. We denote the split-quaternion numbers with the symbol $\mathbb{SH}$.
Let $q = a + i^*b + j^*c + k^*d$ be a split-quaternion number; then we define two different types of conjugations:

$$\overline{q} := a - i^*b - j^*c - k^*d$$
$$\widetilde{q} := a - i^*b + j^*c + k^*d$$

Remark 1.6. The split-complex and the split-quaternion numbers are not fields, since they both contain zero divisors. Nevertheless, they are both associative algebras; hence, we can provide a definition for all the groups in Definition 1.7 also when using $\mathbb{SC}$ and $\mathbb{SH}$ instead of $\mathbb{F}$.

Definition 1.7. Given a field $\mathbb{F}$ and a space $M_{n,n}(\mathbb{F})$ of $(n \times n)-$matrices over $\mathbb{F}$, we give the following definitions.

- The **general linear group** $\mathrm{GL}(n, \mathbb{F})$ of degree n over $\mathbb{F}$ is

$$\mathrm{GL}(n, \mathbb{F}) := \{M \in M_{n,n}(\mathbb{F}) \mid M \text{ is invertible}\}.$$

- The **special linear group** $\mathrm{SL}(n, \mathbb{F})$ of degree n over $\mathbb{F}$ is

$$\mathrm{SL}(n, \mathbb{F}) := \{M \in \mathrm{GL}(n, \mathbb{F}) \mid \det(M) = 1\}$$

- The **general orthogonal group** $\mathrm{O}(p, q, \mathbb{F})$ over $\mathbb{F}$ is

$$\mathrm{O}(p, q, \mathbb{F}) = \mathrm{O}(p, q) := \{M \in \mathrm{GL}(p + q, \mathbb{F}) \mid M^t \eta M = \eta\}$$

  with

$$\eta := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \tag{1.2}$$

  where I_k is the $(k \times k)$ identity matrix.
  The subgroup $\mathrm{O}(p, 0, \mathbb{F}) < \mathrm{O}(p, q, \mathbb{F})$ is called **orthogonal group** of degree p and is denoted with $\mathrm{O}(p, \mathbb{F})$. In particular,

$$\begin{aligned} \mathrm{O}(p, \mathbb{F}) &:= \{M \in \mathrm{GL}(p, \mathbb{F}) \mid M^t M = M M^t = \mathrm{Id}\} \\ &= \{M \in \mathrm{GL}(p, \mathbb{F}) \mid M^{-1} = M^t\}. \end{aligned}$$

  The matrices in the orthogonal group, also called **orthogonal matrices**, have the property that $\det(M) = \pm 1$. When we consider $\mathbb{F} = \mathbb{R}$, we simply write $\mathrm{O}(p)$.

- The **special general orthogonal group** $\mathrm{SO}(p,q,\mathbb{F})$ over $\mathbb{F}$ is

$$\mathrm{SO}(p,q,\mathbb{F}) := \{M \in \mathrm{O}(p,q,\mathbb{F}) \mid \det(M) = 1\}.$$

  The subgroup $\mathrm{SO}(p,0,\mathbb{F}) < \mathrm{SO}(p,q,\mathbb{F})$ is called **special orthogonal group** of degree p and is denoted by $\mathrm{SO}(p,\mathbb{F})$. When we consider $\mathbb{F} = \mathbb{R}$, we simply write $\mathrm{SO}(n)$.
- The **general unitary group** $\mathrm{U}(p,q,\mathbb{F})$ of degree n over the field $\mathbb{F}$ is

$$\mathrm{U}(p,q,\mathbb{F}) := \{M \in \mathrm{GL}(1,\mathbb{F}) \mid \overline{M}^t \eta M = \eta\}$$

  where η is as in (1.2).
  The subgroup $\mathrm{U}(p,0,\mathbb{F}) < \mathrm{U}(p,q,\mathbb{F})$ is called **unitary group** $\mathrm{U}(p,\mathbb{F})$ of degree p. In particular,

$$\mathrm{U}(p,\mathbb{F}) := \{M \in \mathrm{GL}(p,\mathbb{F}) \mid \overline{M}^t = M^{-1}\}.$$

  In particular, if we consider $\mathbb{F} = \mathbb{R}$, then $\mathrm{U}(p,q,\mathbb{R}) = \mathrm{O}(p,q,\mathbb{R})$.
- The **symplectic group** $\mathrm{Sp}(2n,\mathbb{F})$ of degree $2n$ over $\mathbb{F}$ is

$$\mathrm{Sp}(2n,\mathbb{F}) := \{M \in \mathrm{GL}(2n,\mathbb{F}) \mid M^t \Omega_n M = \Omega_n\}$$

  where $\Omega_n := \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$.
  The **compact symplectic group** $\mathrm{Sp}(n)$ of degree $2n$ is

$$\mathrm{Sp}(n) := \mathrm{U}(2n) \cap \mathrm{Sp}(2n,\mathbb{C}).$$

- The **conjugate symplectic group** $\overline{\mathrm{Sp}}(2n,\mathbb{F})$ of degree $2n$ over $\mathbb{F}$ is

$$\overline{\mathrm{Sp}}(2n,\mathbb{F}) := \{M \in \mathrm{GL}(2n,\mathbb{F}) \mid \overline{M^t} \Omega_n M = \Omega_n\}$$

  where Ω_n is as in the definition of the symplectic group.
  Observe that $\mathrm{Sp}(2n,\mathbb{R}) = \overline{\mathrm{Sp}}(2n,\mathbb{R})$.
  If $\mathbb{F} = \mathbb{SH}$, we have two different definitions of conjugation; in particular, we denote with

$$\overline{\mathrm{Sp}}(2n,\mathbb{SH}) := \{M \in \mathrm{GL}(2n,\mathbb{SH}) \mid \overline{M^t} \Omega_n M = \Omega_n\}$$
$$\widetilde{\mathrm{Sp}}(2n,\mathbb{SH}) := \{M \in \mathrm{GL}(2n,\mathbb{SH}) \mid \widetilde{M^t} \Omega_n M = \Omega_n\}.$$

- The group $\mathrm{T}(n,\mathbb{F})$ is defined as

$$T(n,\mathbb{F}) := \{M \in \mathrm{GL}(n,\mathbb{F}) \mid \overline{M^t} \sigma_n M = \sigma_n\},$$

  where $\sigma_n := \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$.

Remark 1.8. All symplectic matrices have determinant equal to 1, so

$$\mathrm{Sp}(2n,\mathbb{F}) < \mathrm{SL}(2n,\mathbb{F}). \tag{1.3}$$

Moreover, the following isomorphism holds:

$$\mathrm{Sp}(2,\mathbb{F}) \cong \mathrm{SL}(2,\mathbb{F}).$$

Indeed, the left inclusion follows trivially from (1.3); the right inclusion follows from the fact that, given a generic $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, $d \in \mathbb{F}$, we have that

$$A^t \Omega A = \begin{pmatrix} 0 & bc - ad \\ ad - bc & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\det(A) \\ \det(A) & 0 \end{pmatrix} = \Omega,$$

since $\det(A) = 1$ by construction.

Remark 1.9. The groups $\mathrm{O}(1, 0, \mathbb{R})$, $\mathrm{O}(0, 1, \mathbb{R})$ and $\mathrm{O}(1, \mathbb{R})$ are isomorphic. In fact:

$$\begin{aligned} \mathrm{O}(1, \mathbb{R}) &= \{M \in \mathrm{GL}(1, \mathbb{R}) \mid M^{-1} = M^T\} = \{a \in \mathbb{R} \mid a = \frac{1}{a}\} = \{\pm 1\}. \\ \mathrm{O}(1, 0, \mathbb{R}) &= \{M \in \mathrm{GL}(1, \mathbb{R}) \mid M^T \operatorname{Id} M = \operatorname{Id}\} = \{a \in \mathbb{R} \mid a^T a = 1\} = \{\pm 1\}. \\ \mathrm{O}(0, 1, \mathbb{R}) &= \{M \in \mathrm{GL}(1, \mathbb{R}) \mid M^T (-\operatorname{Id}) M = -Id\} \\ &= \{a \in \mathbb{R} \mid -a^T a = -1 \Rightarrow a^T a = 1\} = \{\pm 1\}. \end{aligned}$$

Remark 1.10. The group $\mathrm{O}(1, \mathbb{C})$ is given by $\{\pm 1\}$. In fact, given $A = \begin{pmatrix} z \end{pmatrix} \in \mathrm{O}(1, \mathbb{C})$, we have that z is a complex number which satisfies the condition $A^T A = \operatorname{Id}$; since A is a number, then $A^T = A$ and $A^T A = \operatorname{Id}$, so $A^2 = \operatorname{Id}$. Hence, if $z = a + ib$, then the condition becomes $a^2 - b^2 + i2ab = 1$; this implies

$$\begin{cases} a^2 - b^2 = 1 \\ 2ab = 0. \end{cases}$$

Hence $b = 0$ and $a^2 = 1$, implying $A = \begin{pmatrix} \pm 1 \end{pmatrix}$.

1.2 Isomorphisms

In Chapter 3 we will deal with certain computations on matrices. Since such computations are easier when the matrices involved are of lower dimensions, we will make use of the isomorphisms delineated in this section, which relate some classes of four- or eight-dimensional real matrices to two-dimensional complex or quaternion matrices. The same isomorphisms are also useful for the identification of the specific groups we will work with. We start with a known remark, and we proceed with a list of isomorphisms.

Remark 1.11. Consider a 2×2 real matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ commuting with $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. By easy computation, one can see that A must be of the form

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a \cdot \operatorname{Id} - b \cdot i,$$

which that implies $A \in \mathrm{GL}(1, \mathbb{C})$.

Consider now a (4×4) real matrix A; let i, j, k be quaternion units in $\mathrm{GL}(4, \mathbb{R})$. Assume that A commutes with two of the three matrices i, j and k; then it also commutes with the third one. For example, if A commutes with i and j, we have the chain of implications:

$$A \cdot i = i \cdot A \Rightarrow A \cdot i \cdot j = i \cdot A \cdot j \Rightarrow A \cdot i \cdot j = i \cdot j \cdot A \Rightarrow A \cdot k = k \cdot A.$$

In this case, by easy computation, one can see that $A = a \cdot \operatorname{Id} + b \cdot i + c \cdot j + d \cdot k$, hence we can conclude that $A \in \mathrm{GL}(1, \mathbb{H})$.

Analogously, given a 4×4 real matrix A; let i^*, j^* and k^* be split-quaternion units written as 4×4 real matrices. Assume that A commutes with two of them; then it also commutes with the third one. For example, if A commutes with i^* and j^*, we have the chain of implications:

$$A \cdot i^* = i^* \cdot A \Rightarrow A \cdot i^* \cdot j^* = i^* \cdot A \cdot j^* \Rightarrow A \cdot i^* \cdot j^* = i^* \cdot j^* \cdot A \Rightarrow A \cdot k^* = k^* \cdot A.$$

In this case, by easy computation, one can see that $A = a \cdot \mathrm{Id} + b \cdot i^* + c \cdot j^* + d \cdot k^*$, hence we can conclude that $A \in \mathrm{GL}(1, \mathbb{SH})$.

Proposition 1.12. *A matrix $A \in \mathrm{GL}(4, \mathbb{R})$ which commutes with*

$$I = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

has the form

$$M = \begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ -b_1 & a_1 & -b_2 & a_2 \\ a_3 & b_3 & a_4 & b_4 \\ -b_3 & a_3 & -b_4 & a_4 \end{pmatrix}. \tag{1.4}$$

The matrices in the form (1.4) form a subgroup of $\mathrm{GL}(4, \mathbb{R})$ which is isomorphic to $\mathrm{GL}(2, \mathbb{C})$.

Proof. If we take a generic $A \in \mathrm{GL}(4, \mathbb{R})$ and impose the condition $A \cdot I = I \cdot A$, it follows from easy computations that A has to be in the form (1.4).
M has trivially an inverse since it belongs to $\mathrm{GL}(4, \mathbb{R})$; moreover, simple computations prove that the product of any two matrices in the form (1.4) has still the same form.
We will now construct a group homomorphism between $\mathrm{GL}(2, \mathbb{C})$ and the subgroup of the matrices in the form (1.4). Let us consider z_1, z_2, z_3, $z_4 \in \mathbb{C}$ written in the form $z_j = a_j + ib_j$ for every $j = 1, \dots, 4$. The map

$$\varphi : \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ -b_1 & a_1 & -b_2 & a_2 \\ a_3 & b_3 & a_4 & b_4 \\ -b_3 & a_3 & -b_4 & a_4 \end{pmatrix} \tag{1.5}$$

is trivially bijective and maps Id into Id. We will see that φ is a group homomorphism. Indeed, let us consider two matrices $A, B \in \mathrm{GL}(2, \mathbb{C})$ of the form

$$A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}, \qquad B = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix},$$

where $z_j = a_j + ib_j$ and $w_j = a'_j + ib'_j$ for all $j \in \{1, \dots, 4\}$. Then

$$A \cdot B = \begin{pmatrix} z_1 w_1 + z_2 w_3 & z_1 w_2 + z_2 w_4 \\ z_3 w_1 + z_4 w_3 & z_3 w_2 + z_4 w_4 \end{pmatrix}.$$

We notice that

$$\varphi(A \cdot B) = \begin{pmatrix} X_{11,23} & Y_{11,23} & X_{12,24} & Y_{12,24} \\ -Y_{11,23} & X_{11,23} & -Y_{12,24} & X_{12,24} \\ X_{31,43} & Y_{31,43} & X_{32,44} & Y_{32,44} \\ -Y_{31,43} & X_{31,43} & -Y_{32,44} & X_{32,44} \end{pmatrix}$$

where

$$X_{jk,lm} = a_j a'_k + a_l a'_m - b_j b'_k - b_l b'_m$$
$$Y_{jk,lm} = a_j b'_k + a'_k b_j + a_l b'_m + a'_m b_l.$$

On the other side,

$$\varphi(A) = \begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ -b_1 & a_1 & -b_2 & a_2 \\ a_3 & b_3 & a_4 & b_4 \\ -b_3 & a_3 & -b_4 & a_4 \end{pmatrix}$$

and $\varphi(B)$ is of a form akin to $\varphi(A)$, once substituted $a_j,\ b_j$ for $a'_j,\ b'_j$ respectively. After easy computations, it follows that $\varphi(A) \cdot \varphi(B) = \varphi(A \cdot B)$. As φ was bijective and its inverse is the inverse group homomorphism, it is an isomorphism of groups. □

Proposition 1.13. *Any matrix $A \in \mathrm{GL}(4, \mathbb{R})$ which commutes with*

$$J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is of the form

$$N = \begin{pmatrix} a_1 & a_2 & -b_1 & -b_2 \\ a_3 & a_4 & -b_3 & -b_4 \\ b_1 & b_2 & a_1 & a_2 \\ b_3 & b_4 & a_3 & a_4 \end{pmatrix}. \tag{1.6}$$

The matrices of the form (1.6) form a subroup of $\mathrm{GL}(4, \mathbb{R})$ *which is isomorphic to* $\mathrm{GL}(2, \mathbb{C})$.

Proof. Proving the first part of the statement follows from some easy computations. For the second part, we want to construct an isomorphism between $\mathrm{GL}(2, \mathbb{C})$ and the subgroup of $\mathrm{GL}(4, \mathbb{R})$ of matrices in the form (1.6).
We define the map

$$\varphi : \begin{pmatrix} a_1 + ib_1 & a_2 + ib_2 \\ a_3 + ib_3 & a_4 + ib_4 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & a_2 & -b_1 & -b_2 \\ a_3 & a_4 & -b_3 & -b_4 \\ b_1 & b_2 & a_1 & a_2 \\ b_3 & b_4 & a_3 & a_4 \end{pmatrix}. \tag{1.7}$$

This is trivially a bijection; moreover, it maps Id into Id. We want to prove that, given any two matrices A, $B \in \mathrm{GL}(2, \mathbb{C})$, then $\varphi(A) \cdot \varphi(B) = \varphi(A \cdot B)$. Write A and B as

$$A = \begin{pmatrix} a_1 + ib_1 & a_2 + ib_2 \\ a_3 + ib_3 & a_4 + ib_4 \end{pmatrix}, \qquad B = \begin{pmatrix} a'_1 + ib'_1 & a'_2 + ib'_2 \\ a'_3 + ib'_3 & a'_4 + ib'_4 \end{pmatrix}$$

Then,

$$\varphi(A \cdot B) = \begin{pmatrix} X_{11,23} & X_{12,24} & -Y_{11,23} & -Y_{12,24} \\ X_{31,43} & X_{32,44} & -Y_{31,43} & -Y_{32,44} \\ Y_{11,23} & Y_{12,24} & X_{11,23} & X_{12,24} \\ Y_{31,43} & Y_{32,44} & X_{31,43} & X_{32,44} \end{pmatrix}, \tag{1.8}$$

where

$$X_{jk,lm} = a_j a'_k + a_l a'_m - b_j b'_k - b_l b'_m$$
$$Y_{jk,lm} = a_j b'_k + b_j a'_k + a_l b'_m + b_l a'_m$$

Computing $\varphi(A) \cdot \varphi(B)$, one can see that it has the form as in (1.8). □

Remark 1.14. Let $A \in \mathrm{GL}(4,\mathbb{R})$ be a matrix of the form (1.4) or (1.6). Then

$$\varphi^{-1}(A^t) = \overline{\varphi^{-1}(A)^t},$$

where $\overline{}$ is the complex conjugation.

Proposition 1.15. *Let $A \in \mathrm{GL}(4,\mathbb{R})$ be a matrix which commutes with*

$$I = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix};$$

then A is of the form

$$O = \begin{pmatrix} a_1 & a_2 & b_2 & b_1 \\ a_3 & a_4 & b_4 & b_3 \\ b_3 & b_4 & a_4 & a_3 \\ b_1 & b_2 & a_2 & a_1 \end{pmatrix}. \tag{1.9}$$

The matrices of the form (1.9) form a subgroup of $\mathrm{GL}(4,\mathbb{R})$ which is isomorphic to $\mathrm{GL}(2,\mathbb{SC})$.

Proof. Given z_1, z_2, z_3 and z_4 split-complex numbers in the form $z_j = a_j + i^* b_j$, we can construct the bijective map

$$\varphi : \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & a_2 & b_2 & b_1 \\ a_3 & a_4 & b_4 & b_3 \\ b_3 & b_4 & a_4 & a_3 \\ b_1 & b_2 & a_2 & a_1 \end{pmatrix}. \tag{1.10}$$

The map φ maps Id to Id; moreover, it is a group homomorphism. Indeed, given two matrices $A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$ with $z_j = a_j + i^* b_j$ and $B = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$ with $w_j = a'_j + i^* b'_j$, then

$$A \cdot B = \begin{pmatrix} z_1 w_1 + z_2 w_3 & z_1 w_2 + z_2 w_4 \\ z_3 w_1 + z_4 w_3 & z_3 w_2 + z_4 w_4 \end{pmatrix}.$$

Hence,

$$\varphi(A \cdot B) = \begin{pmatrix} X_{11,23} & X_{12,24} & Y_{12,24} & Y_{11,23} \\ X_{31,43} & X_{32,44} & Y_{32,44} & Y_{31,43} \\ Y_{31,43} & Y_{32,44} & X_{32,44} & X_{31,43} \\ Y_{11,23} & Y_{12,24} & X_{12,24} & X_{11,23} \end{pmatrix}, \tag{1.11}$$

where

$$X_{jk,lm} = a_j a'_k + a_l a'_m + b_j b'_k b_l b'_m$$
$$Y_{jk,lm} = a_j b'_k + b_j a'_k + a_l b'_m + b_l a'_m$$

One can compute $\varphi(A) \cdot \varphi(B)$ and observe that it is in the form (1.11). □

Remark 1.16. Let $A \in \mathrm{GL}(4,\mathbb{R})$ be a matrix of the form (1.9). Then

$$\varphi^{-1}(A^t) = \varphi^{-1}(A)^t.$$

Proposition 1.17. *A matrix* $A \in \mathrm{GL}(8,\mathbb{R})$ *which commutes with the matrices*

$$I = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

is of the form

$$\alpha = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 & a_2 & b_2 & c_2 & d_2 \\ -b_1 & a_1 & -d_1 & c_1 & -b_2 & a_2 & -d_2 & c_2 \\ -c_1 & d_1 & a_1 & -b_1 & -c_2 & d_2 & a_2 & -b_2 \\ -d_1 & -c_1 & b_1 & a_1 & -d_2 & -c_2 & b_2 & a_2 \\ a_3 & b_3 & c_3 & d_3 & a_4 & b_4 & c_4 & d_4 \\ -b_3 & a_3 & -d_3 & c_3 & -b_4 & a_4 & -d_4 & c_4 \\ -c_3 & d_3 & a_3 & -b_3 & -c_4 & d_4 & a_4 & -b_4 \\ -d_3 & -c_3 & b_3 & a_3 & -d_4 & -c_4 & b_4 & a_4 \end{pmatrix}. \tag{1.12}$$

The matrices of the form (1.12) form a subgroup of $\mathrm{GL}(8,\mathbb{R})$ *isomorphic to* $\mathrm{GL}(2,\mathbb{H})$.

Proof. The first part of the statement can be proven via some easy computations. As for the second part, given z_1, z_2, z_3, $z_4 \in \mathbb{H}$ in the form $z_m = a_m + ib_m + jc_m + kd_m$, we can construct a bijective map

$$\varphi : \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \alpha. \tag{1.13}$$

The map φ maps Id to Id; moreoveprover, it is a group homomorphism. Indeed, if we have two matrix $A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$ and $B = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$ with $z_m = a_m + ib_m + jc_m + kd_m$ and $w_l = a'_l + ib'_l + jc'_l + kd'_l$, then

$$A \cdot B = \begin{pmatrix} z_1w_1 + z_2w_3 & z_1w_2 + z_2w_4 \\ z_3w_1 + z_4w_3 & z_3w_2 + z_4w_4 \end{pmatrix}.$$

Then

$$\varphi(A \cdot B) = \begin{pmatrix} X_{11,23} & Y_{11,23} & W_{11,23} & Z_{11,23} & X_{12,24} & Y_{12,24} & W_{12,24} & Z_{12,24} \\ -Y_{11,23} & X_{11,23} & -Z_{11,23} & W_{11,23} & -Y_{12,24} & X_{12,24} & -Z_{12,24} & W_{12,24} \\ -W_{11,23} & Z_{11,23} & X_{11,23} & -Y_{11,23} & -W_{12,24} & Z_{12,24} & X_{12,24} & -Y_{12,24} \\ -Z_{11,23} & -W_{11,23} & Y_{11,23} & X_{11,23} & -Z_{12,24} & -W_{12,24} & Y_{12,24} & X_{12,24} \\ X_{31,43} & Y_{31,43} & W_{31,43} & Z_{31,43} & X_{32,44} & Y_{32,44} & W_{32,44} & Z_{32,44} \\ -Y_{31,43} & X_{31,43} & -Z_{31,43} & W_{31,43} & -Y_{32,44} & X_{32,44} & -Z_{32,44} & W_{32,44} \\ -W_{31,43} & Z_{31,43} & X_{31,43} & -Y_{31,43} & -W_{32,44} & Z_{32,44} & X_{32,44} & -Y_{32,44} \\ -Z_{31,43} & -W_{31,43} & Y_{31,43} & X_{31,43} & -Z_{32,44} & -W_{32,44} & Y_{32,44} & X_{32,44} \end{pmatrix}$$

where

$$\begin{aligned} X_{lm,no} &= a_l a'_m - b'_m b_l - c_l c'_m - d_l d'_m + a_n a'_o - b_n b'_o - c_n c'_o - d_n d'_o \\ Y_{lm,no} &= a_l b'_m + b_l a'_m + c_l d'_m - d_l c'_m + a_n b'_o b_n a'_o + c_n d'_o - d_n c'_o \\ W_{lm,no} &= a_l c'_m - b_l d'_m + c_l a'_m + d_l b'_m + a_n c'_o - b_n d'_o + c_n a'_o + d_n b'_o \\ Z_{lm,no} &= a_l d'_m + b_l c'_m - c_l b'_m + d_l a'_m + a_n d'_o + b_n c'_o - c_n b'_o + d_n a'_o \end{aligned}$$

It follows by computation that $\varphi(A) \cdot \varphi(B) = \varphi(A \cdot B)$. □

Proposition 1.18. *A matrix $A \in \mathrm{GL}(8,\mathbb{R})$ which commutes with the matrices*

$$I = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is of the form

$$\beta = \begin{pmatrix} a_1 & a_2 & -b_2 & b_1 & c_2 & -c_1 & d_1 & d_2 \\ a_3 & a_4 & -b_4 & b_3 & c_4 & -c_3 & d_3 & d_4 \\ b_3 & b_4 & a_4 & -a_3 & d_4 & -d_3 & -c_3 & -c_4 \\ -b_1 & -b_2 & -a_2 & a_1 & -d_2 & d_1 & c_1 & c_2 \\ -c_3 & -c_4 & -d_4 & d_3 & a_4 & -a_3 & -b_3 & -b_4 \\ c_1 & c_2 & d_2 & -d_1 & -a_2 & a_1 & b_1 & b_2 \\ -d_1 & -d_2 & c_2 & -c_1 & b_2 & -b_1 & a_1 & a_2 \\ -d_3 & -d_4 & c_4 & -c_3 & b_4 & -b_3 & a_3 & a_4 \end{pmatrix}. \tag{1.14}$$

The matrices of the form (1.14) form a subgroup of $\mathrm{GL}(8,\mathbb{R})$ *isomorphic to* $\mathrm{GL}(2,\mathbb{H})$.

Proof. Again, the first part of the statement can be proven by easy computation. As for the second part, given z_1, z_2, z_3, $z_4 \in \mathbb{H}$ in the form $z_l = a_l + ib_l + jc_l + kd_l$, we can construct a map

$$\varphi : \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \beta. \tag{1.15}$$

The map φ is trivially bijective and maps Id to Id. Moreover, it is a group homomorphism: in fact, given two matrices $A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$ and $B = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$ with $z_l = a_l + ib_l + jc_l + kd_l$ and $w_m = a'_m + ib'_m + jc'_m + kd'_m$; then

$$A \cdot B = \begin{pmatrix} z_1 w_1 + z_2 w_3 & z_1 w_2 + z_2 w_4 \\ z_3 w_1 + z_4 w_3 & z_3 w_2 + z_4 w_4 \end{pmatrix}.$$

Then

$$\varphi(A \cdot B) = \begin{pmatrix} X_{11,23} & X_{12,24} & -Y_{12,24} & Y_{11,23} & W_{12,24} & -W_{11,23} & Z_{11,23} & Z_{12,24} \\ X_{31,43} & X_{32,44} & -Y_{32,44} & Y_{31,43} & W_{32,44} & -W_{31,43} & Z_{31,43} & Z_{32,44} \\ Y_{31,43} & Y_{32,44} & X_{32,44} & -X_{31,43} & Z_{32,44} & -Z_{31,43} & -W_{31,43} & -W_{32,44} \\ -Y_{11,23} & -Y_{12,24} & -X_{12,24} & X_{11,23} & -Z_{12,24} & Z_{11,23} & W_{11,23} & W_{12,24} \\ -W_{31,43} & -W_{32,44} & -Z_{32,44} & Z_{31,43} & X_{32,44} & -X_{31,43} & -Y_{31,43} & -Y_{32,44} \\ W_{11,23} & W_{12,24} & Z_{12,24} & -Z_{11,23} & -X_{12,24} & X_{11,23} & Y_{11,23} & Y_{12,24} \\ -Z_{11,23} & -Z_{12,24} & W_{12,24} & -W_{11,23} & Y_{12,24} & -Y_{11,23} & X_{11,23} & X_{12,24} \\ -Z_{31,43} & -Z_{32,44} & W_{32,44} & -W_{31,43} & Y_{32,44} & -Y_{31,43} & X_{31,43} & X_{32,44} \end{pmatrix}$$

where

$$\begin{aligned} X_{lm,no} &= a_l a'_m - b'_m b_l - c_l c'_m - d_l d'_m + a_n a'_o - b_n b'_o - c_n c'_o - d_n d'_o \\ Y_{lm,no} &= a_l b'_m + b_l a'_m + c_l d'_m - d_l c'_m + a_n b'_o b_n a'_o + c_n d'_o - d_n c'_o \\ W_{lm,no} &= a_l c'_m - b_l d'_m + c_l a'_m + d_l b'_m + a_n c'_o - b_n d'_o + c_n a'_o + d_n b'_o \\ Z_{lm,no} &= a_l d'_m + b_l c'_m - c_l b'_m + d_l a'_m + a_n d'_o + b_n c'_o - c_n b'_o + d_n a'_o \end{aligned}$$

One can compute $\varphi(A) \cdot \varphi(B)$ and prove it has the same form as $\varphi(A \cdot B)$. □

Remark 1.19. Given a matrix $A \in \mathrm{GL}(8,\mathbb{R})$ in the form (1.12) or (1.14), we have $\varphi^{-1}(A^t) = \overline{\varphi^{-1}(A)^t}$ where $\overline{}$ is the quaternion conjugation.

Proposition 1.20. *A matrix* $A \in \mathrm{GL}(8,\mathbb{R})$ *which commutes with the matrices*

$$I^* = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is of the form

$$\gamma = \begin{pmatrix} a_1 & a_2 & -b_2 & b_1 & c_2 & c_1 & d_1 & -d_2 \\ a_3 & a_4 & -b_4 & b_3 & c_4 & c_3 & d_3 & -d_4 \\ b_3 & b_4 & a_4 & -a_3 & -d_4 & -d_3 & c_3 & -c_4 \\ -b_1 & -b_2 & -a_2 & a_1 & d_2 & d_1 & -c_1 & c_2 \\ c_3 & c_4 & -d_4 & d_3 & a_4 & a_3 & b_3 & -b_4 \\ c_1 & c_2 & -d_2 & d_1 & a_2 & a_1 & b_1 & -b_2 \\ d_1 & d_2 & c_2 & -c_1 & -b_2 & -b_1 & a_1 & -a_2 \\ -d_3 & -d_4 & -c_4 & c_3 & b_4 & b_3 & -a_3 & a_4 \end{pmatrix}. \tag{1.16}$$

The matrices of the form (1.16) form a subgroup of $\mathrm{GL}(8,\mathbb{R})$ *isomorphic to* $\mathrm{GL}(2,\mathbb{SH})$.

Proof. As in the previous propositions, the first part of the statement can be proven by easy computations. As for the second part, given z_1, z_2, z_3, $z_4 \in \mathbb{SH}$ in the form $z_l = a_l + i^*b_l + j^*c_l + k^*d_l$, we can construct a map

$$\varphi : \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \gamma. \tag{1.17}$$

The map φ is trivially bijective and maps Id to Id. Moreover, it is a group homomorphism: in fact, given two matrices $A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$ and $B = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$ with $z_l = a_l + i^*b_l + j^*c_l + k^*d_l$ and $w_l = a'_l + i^*b'_l + j^*c'_l + k^*d'_l$; then

$$A \cdot B = \begin{pmatrix} z_1w_1 + z_2w_3 & z_1w_2 + z_2w_4 \\ z_3w_1 + z_4w_3 & z_3w_2 + z_4w_4 \end{pmatrix}.$$

Then

$$\varphi(A \cdot B) = \begin{pmatrix} X_{11,23} & X_{12,24} & -Y_{12,24} & Y_{11,23} & W_{12,24} & W_{11,23} & Z_{11,23} & -Z_{12,24} \\ X_{31,43} & X_{32,44} & -Y_{32,44} & Y_{31,43} & W_{32,44} & W_{31,43} & Z_{31,43} & -Z_{32,44} \\ Y_{31,43} & Y_{32,44} & X_{32,44} & -X_{31,43} & -Z_{32,44} & -Z_{31,43} & W_{31,43} & -W_{32,44} \\ -Y_{11,23} & -Y_{12,24} & -X_{12,24} & X_{11,23} & Z_{12,24} & Z_{11,23} & -W_{11,23} & W_{12,24} \\ W_{31,43} & W_{32,44} & -Z_{32,44} & Z_{31,43} & X_{32,44} & X_{31,43} & Y_{31,43} & -Y_{32,44} \\ W_{11,23} & W_{12,24} & -Z_{12,24} & Z_{11,23} & X_{12,24} & X_{11,23} & Y_{11,23} & -Y_{12,24} \\ Z_{11,23} & Z_{12,24} & W_{12,24} & -W_{11,23} & -Y_{12,24} & -Y_{11,23} & X_{11,23} & -X_{12,24} \\ Z_{31,43} & Z_{32,44} & W_{32,44} & -W_{31,43} & -Y_{32,44} & -Y_{31,43} & X_{31,43} & -X_{32,44} \end{pmatrix}$$

where

$$\begin{aligned}
X_{lm,no} &= a_l a'_m - b'_m b_l + c_l c'_m + d_l d'_m + a_n a'_o - b_n b'_o + c_n c'_o + d_n d'_o \\
Y_{lm,no} &= a_l b'_m + b_l a'_m + c_l d'_m - d_l c'_m + a_n b'_o + b_n a'_o + c_n d'_o - d_n c'_o \\
W_{lm,no} &= a_l c'_m + b_l d'_m + c_l a'_m - d_l b'_m + a_n c'_o - b_n d'_o + c_n a'_o - d_n b'_o \\
Z_{lm,no} &= a_l d'_m + b_l c'_m - c_l b'_m + d_l a'_m + a_n d'_o + b_n c'_o - c_n b'_o + d_n a'_o
\end{aligned}$$

One can compute $\varphi(A) \cdot \varphi(B)$ and prove it has the same form as $\varphi(A \cdot B)$. □

Remark 1.21. Let $A \in \mathrm{GL}(8, \mathbb{R})$ be a matrix of the form (1.16). Then

$$\varphi^{-1}(A^t) = \widetilde{\varphi^{-1}(A)}^t,$$

where $\widetilde{\ }$ is the conjugation of split-quaternion as defined in Definition 1.5.

Proposition 1.22. *The group* $\mathrm{GL}(1, \mathbb{H})$ *is isomorphic to the subgroup* $\mathrm{U}(2)$ *of* $\mathrm{GL}(2, \mathbb{C})$ *given by the matrices of the form* $\alpha = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}$.

Proof. We know by Proposition 1.12 that a matrix $A \in \mathrm{GL}(2, \mathbb{C})$, $A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$ with $z_j = a_j + ib_j$, is isomorphic to a matrix in the form

$$\begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ -b_1 & a_1 & -b_2 & a_2 \\ a_3 & b_3 & a_4 & b_4 \\ -b_3 & a_3 & -b_4 & a_4 \end{pmatrix},$$

while it is known that the matrices in $\mathrm{GL}(1, \mathbb{H})$ can be represented as matrices in the form

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}.$$

These two matrices are equal if we impose the conditions

$$a_1 = a, \quad b_1 = b, \quad a_2 = c, \quad b_2 = d, \quad a_3 = -c, \quad b_3 = d, \quad a_4 = a, \quad b_4 = -b,$$

which are equivalent to the conditions

$$z_1 = a + ib, \qquad z_2 = c + id, \qquad z_3 = -c + id, \qquad z_4 = a - ib,$$

hence $z_3 = -\overline{z_2}$ and $z_4 = \overline{z_1}$. □

Proposition 1.23. *The group* $\mathrm{GL}(2, \mathbb{H})$ *is isomorphic to the subgroup of* $\mathrm{GL}(4, \mathbb{C})$ *of the matrices in the form*

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ -\bar{z}_2 & \bar{z}_1 & -\bar{z}_4 & \bar{z}_3 \\ z_5 & z_6 & z_7 & z_8 \\ -\bar{z}_6 & \bar{z}_5 & -\bar{z}_8 & \bar{z}_7 \end{pmatrix}.$$

Proof. We know by Proposition 1.17 that a matrix

$$A = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{H}),$$

with $w_l = a_l + ib_l + jc_l + kd_l$, can be written as

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & a_2 & b_2 & c_2 & d_2 \\ -b_1 & a_1 & -d_1 & c_1 & -b_2 & a_2 & -d_2 & c_2 \\ -c_1 & d_1 & a_1 & -b_1 & -c_2 & d_2 & a_2 & -b_2 \\ -d_1 & -c_1 & b_1 & a_1 & -d_2 & -c_2 & b_2 & a_2 \\ a_3 & b_3 & c_3 & d_3 & a_4 & b_4 & c_4 & d_4 \\ -b_3 & a_3 & -d_3 & c_3 & -b_4 & a_4 & -d_4 & c_4 \\ -c_3 & d_3 & a_3 & -b_3 & -c_4 & d_4 & a_4 & -b_4 \\ -d_3 & -c_3 & b_3 & a_3 & -d_4 & -c_4 & b_4 & a_4 \end{pmatrix}.$$

On the other hand, a matrix

$$B = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ z_5 & z_6 & z_7 & z_8 \\ z_9 & z_{10} & z_{11} & z_{12} \\ z_{13} & z_{14} & z_{15} & z_{16} \end{pmatrix} \in \mathrm{GL}(4, \mathbb{C}),$$

with $z_j = x_j + iy_j$, can be written as

$$\begin{pmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & x_4 & y_4 \\ -y_1 & x_1 & -y_2 & x_2 & -y_3 & x_3 & -y_4 & x_4 \\ x_5 & y_5 & x_6 & y_6 & x_7 & y_7 & x_8 & y_8 \\ -y_5 & x_5 & -y_6 & x_6 & -y_7 & x_7 & -y_8 & x_8 \\ x_9 & y_9 & x_{10} & y_{10} & x_{11} & y_{11} & x_{12} & y_{12} \\ -y_9 & x_9 & -y_{10} & x_{10} & -y_{11} & x_{11} & -y_{12} & x_{12} \\ x_{13} & y_{13} & x_{14} & y_{14} & x_{15} & y_{15} & x_{16} & y_{16} \\ -y_{13} & x_{13} & -y_{14} & x_{14} & -y_{15} & x_{15} & -y_{16} & x_{16} \end{pmatrix}.$$

In order to have an equality between these two expressions of A and B, we need to impose the following conditions on x_j and y_j:

$$\begin{array}{llllllll} x_1 = a_1, & x_2 = c_1, & x_3 = a_2, & x_4 = c_2, & x_5 = -c_1, & x_6 = a_1, & x_7 = -c_2, & x_8 = a_2 \\ x_9 = a_3, & x_{10} = c_3, & x_{11} = a_4, & x_{12} = c_4, & x_{13} = -c_3, & x_{14} = a_3, & x_{15} = -c_4, & x_{16} = a_4 \\ y_1 = b_1, & y_2 = d_1, & y_3 = b_2, & y_4 = d_2, & y_5 = d_1, & y_6 = -b_1, & y_7 = d_2, & y_8 = -b_2 \\ y_9 = b_3, & y_{10} = d_3, & y_{11} = b_4, & y_{12} = d_4, & y_{13} = d_3, & y_{14} = -b_3, & y_{15} = d_4, & y_{16} = -b_4. \end{array}$$

These conditions imply $z_5 = -\overline{z_2}$, $z_6 = \overline{z_1}$, $z_7 = -\overline{z_2}$, $z_8 = \overline{z_2}$ and $z_{13} = -\overline{z_{10}}$, $z_{14} = \overline{z_9}$, $z_{15} = -\overline{z_{12}}$, $z_{16} = \overline{z_{11}}$; hence, the proposition is proved. □

Chapter 2

Automorphism groups of pseudo H-type Lie algebras

Our goal is to describe the structure of the automorphism groups of pseudo H-type Lie algebras. We will start by giving some insight on the theory already known for two-step nilpotent Lie algebras. Our main reference for this section is the paper by Kaplan and Tiraboschi [KT13], which describes the structure of the automorphism group of a fat Lie algebra. Some of the results appearing in that paper are also relevant to the more general class of two-step nilpotent Lie algebras. We will later see that pseudo H-type Lie algebras are two-step nilpotent, hence the same results will apply to our analysis.

2.1 Two-step nilpotent Lie algebras

We will begin with the general definitions of a Lie algebra and a nilpotent algebra.

Definition 2.1. A **Lie algebra** is a vector space $\mathfrak{n}$ over some field $\mathbb{F}$ together with a binary operation $[-,-] : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$, called **Lie bracket**, that satisfies:

- bilinearity, i.e. $[ax+by, z] = a[x,z] + b[y,z]$ and $[x, ay+bz] = a[x,y] + b[x,z]$
- anticommutativity, i.e. $[x,y] = -[y,x]$
- the Jacobi identity, i.e. $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$

for all x, y, $z \in \mathfrak{n}$ and for all a, $b \in \mathbb{F}$.
Anticommutativity implies alternativity, i.e. $[x,x] = 0$ for all $x \in \mathfrak{n}$. We define the **centre** of $\mathfrak{n}$ as the set $\mathfrak{z} = \mathfrak{z}_{\mathfrak{n}} := \{x \in \mathfrak{n} \mid [x,s] = 0 \text{ for all } s \in \mathfrak{n}\}$.

Definition 2.2. A Lie algebra is called **two-step nilpotent** if $[x,[y,z]] = 0$ for all x, y, $z \in \mathfrak{n}$. In other words, if $\mathfrak{n}$ is two-step nilpotent, then the Lie bracket is a map $[-,-] : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{z}$.

We are now interested in studying the structure of the automorphisms of a generic two-step nilpotent Lie algebra.

Let $\mathfrak{n} = (V \oplus \mathfrak{z}, [-,-])$ be a two-step nilpotent Lie algebra and $\mathrm{Aut}(\mathfrak{n})$ be a group of automorphisms of $\mathfrak{n}$. Let $n = \dim(V)$ and $m = \dim(\mathfrak{z})$. The automorphisms of two-step nilpotent Lie algebras preserve the centre; therefore, an element $\varphi \in \mathrm{Aut}(\mathfrak{n})$ has to be of the form

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}, \quad \text{with } A \in \mathrm{GL}(n),\ C \in \mathrm{GL}(m),\ B \in M_{n\times m},$$

where $C([u,v]) = [Au, Av]$.

Remark 2.3. The subgroup $\mathcal{B} < \operatorname{Aut}(\mathfrak{n})$ given by

$$\mathcal{B} = \left\{ \begin{pmatrix} t \operatorname{Id}_n & 0 \\ B & t^2 \operatorname{Id}_m \end{pmatrix},\ B \in M_{n\times m},\ t \neq 0 \right\}$$

is normal. Indeed, for $\varphi = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$, we have $\varphi^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{pmatrix}$. Then, for any $b = \begin{pmatrix} t \operatorname{Id}_n & 0 \\ B & t^2 \operatorname{Id}_m \end{pmatrix}$, we have

$$\begin{aligned} \varphi b \varphi^{-1} &= \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} t \operatorname{Id}_n & 0 \\ B & t^2 \operatorname{Id}_m \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{pmatrix} \\ &= \begin{pmatrix} t \operatorname{Id}_n & 0 \\ ((t-t^2)\operatorname{Id}_m + C)BA^{-1} & t^2 \operatorname{Id}_m \end{pmatrix} \in \mathcal{B}. \end{aligned}$$

Remark 2.4. The factor group

$$\operatorname{Aut}_{\mathrm{gr}}(\mathfrak{n}) := \operatorname{Aut}(\mathfrak{n})/\mathcal{B} = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix},\ A \in \mathrm{SL}(n), C([u,v]) = [Au, Av] \right\}$$

, for u, $v \in \mathfrak{n}$, is a subgroup complementary to the normal group $\mathcal{B}$. Indeed, if $b \in \mathcal{B}$ and $\psi \in \operatorname{Aut}_{\mathrm{gr}}(\mathfrak{n})$, then

$$b\psi = \begin{pmatrix} t \operatorname{Id}_n & 0 \\ B & t^2 \operatorname{Id}_m \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} tA & 0 \\ BA & t^2 C \end{pmatrix} \in \operatorname{Aut}(\mathfrak{n}).$$

We will now define the semi-direct product $\mathcal{B} \rtimes \operatorname{Aut}_{\mathrm{gr}}(\mathfrak{n})$ by means of the action $\mu : \operatorname{Aut}_{\mathrm{gr}}(\mathfrak{n}) \times \mathcal{B} \to \mathcal{B}$ defined by $\mu(\psi, b) = \psi(b)$, where

$$\psi(b) = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} t \operatorname{Id}_n & 0 \\ B & t^2 \operatorname{Id}_m \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix} = \begin{pmatrix} t \operatorname{Id}_n & 0 \\ CBA^{-1} & t^2 \operatorname{Id}_m \end{pmatrix} \in \mathcal{B}.$$

The product $\bullet$ on $\operatorname{Aut}(\mathfrak{n}) = \mathcal{B} \rtimes \operatorname{Aut}_{\mathrm{gr}}(\mathfrak{n}) = (\mathcal{B} \times \operatorname{Aut}_{\mathrm{gr}}(\mathfrak{n}), \bullet)$ is now defined by:

$$(n_1, \psi_1) \bullet (n_2, \psi_2) = (n_1 \psi_1(n_2), \psi_1\psi_2) \in \mathcal{B} \times \operatorname{Aut}_{\mathrm{gr}}(\mathfrak{n}).$$

The next step is to show that the group

$$\operatorname{Aut}^0(\mathfrak{n}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & \operatorname{Id}_m \end{pmatrix},\ A \in \mathrm{SL}(n) \right\}$$

is a normal subgroup of $\operatorname{Aut}_{\mathrm{gr}}(\mathfrak{n})$. Recall that $\operatorname{Aut}^0(\mathfrak{n})$ acts on $\mathfrak{n}$ by $[Ax, Ay] = [x, y]$ for any $x, y \in \mathfrak{v}$. We have that, for any $\psi \in \operatorname{Aut}_{\mathrm{gr}}(\mathfrak{n})$ and $\zeta \in \operatorname{Aut}^0(\mathfrak{n})$,

$$\psi\zeta\psi^{-1} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & \operatorname{Id}_m \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix} = \begin{pmatrix} AA_1A^{-1} & 0 \\ 0 & \operatorname{Id}_m \end{pmatrix} \in \operatorname{Aut}^0(\mathfrak{n}).$$

Thus the quotient group $\operatorname{Aut}_{\mathrm{gr}}(\mathfrak{n})/\operatorname{Aut}^0(\mathfrak{n})$ is isomorphic to the group

$$C(\mathfrak{n}) = \{C \in \mathrm{GL}(m) \mid C[x,y] = [A'x, A'y] \text{ for some } A' \in \mathrm{SL}(n)\}.$$

Knowing $C(\mathfrak{n})$ we can write $\operatorname{Aut}_{\mathrm{gr}}(\mathfrak{n}) = \operatorname{Aut}^0(\mathfrak{n}) \rtimes C(\mathfrak{n})$. We define the action of $C(\mathfrak{n})$ on $\operatorname{Aut}^0(\mathfrak{n})$ by $\nu(c, a) = c(a)$, where $c \in C(\mathfrak{n})$, $a \in \operatorname{Aut}^0(\mathfrak{n})$, and

$$c(a) = \begin{pmatrix} A' & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \operatorname{Id}_m \end{pmatrix} \begin{pmatrix} A'^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix} = \begin{pmatrix} A'AA'^{-1} & 0 \\ 0 & \operatorname{Id}_m \end{pmatrix} \in \operatorname{Aut}^0(\mathfrak{n}).$$

The product $\bullet$ on $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{n}) = \mathrm{Aut}^0(\mathfrak{n}) \rtimes C(\mathfrak{n})$ is now defined by

$$(a_1, c_1) \bullet (a_2, c_2) = (a_1 c(a_2), c_1 c_2) \in \mathrm{Aut}^0(\mathfrak{n}) \rtimes C(\mathfrak{n}).$$

In particular, we have a decomposition of $\mathrm{Aut}(\mathfrak{n})$ which focuses on the normal subgroup $\mathrm{Aut}^0(\mathfrak{n})$. This implies that if we find a way to describe $C(\mathfrak{n})$, then $\mathrm{Aut}^0(\mathfrak{n})$ will be the only unknown component of the automorphism group of a two-step nilpotent Lie algebra.

2.2 Pseudo H-type Lie algebras

We will now introduce pseudo H-type Lie algebras and study the structure of their automorphisms. These algebras are two-step nilpotent and will hence satisfy the conditions depicted in the previous section. Moreover, we will describe an ulterior property satisfied by pseudo H-type Lie algebras, which will be the tool we will use in order to classify $\mathrm{Aut}^0(\mathfrak{n})$.

Definition 2.5. Let $\mathfrak{n}$ be a (real) two-step nilpotent Lie algebra endowed with a scalar product $\langle -, - \rangle$. Assume that the restriction $\langle -, - \rangle_{\mathfrak{z}}$ of the scalar product to the centre $\mathfrak{z}$ of $\mathfrak{n}$ is not degenerate; this is equivalent to say that $\mathfrak{n} = \mathfrak{z} \oplus V$, where $V = \mathfrak{z}^{\perp}$. We define, for all $Z \in \mathfrak{z}$, a map $J_Z : V \to V$ via the condition

$$\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle \tag{2.1}$$

for all X, $Y \in V$.

Remark 2.6. J_Z is a skew-adjoint operator. In fact

$$\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle = -\langle Z, [Y, X] \rangle = -\langle J_Z Y, X \rangle = -\langle X, J_Z Y \rangle.$$

Definition 2.7. We call $\mathfrak{n}$ a **pseudo H-type Lie algebra** if

$$\langle J_Z X, J_Z X \rangle = \langle Z, Z \rangle \langle X, X \rangle \tag{2.2}$$

for all $Z \in \mathfrak{z}$, $X \in V$. In particular we say that $\mathfrak{n}$ is a (r, s)**-H-type algebra**, or a (r, s)**-algebra**, if $\langle -, - \rangle_{\mathfrak{z}}$ has signature (r, s). We reserve the notation $\mathfrak{n}^{r,s}$ for a generic (r, s)-algebra.

Remark 2.8. By polarization, if $\mathfrak{n}$ is a pseudo H-type Lie algebra, then the following equalities hold:

$$\langle J_Z X, J_{Z'} X \rangle = \langle Z, Z' \rangle \langle X, X \rangle, \qquad \langle J_Z X, J_Z X' \rangle = \langle Z, Z \rangle \langle X, X' \rangle.$$

From (2.2) and Remark 2.6 we also get

$$J_Z^2 = -\langle Z, Z \rangle \, \mathrm{Id}_V \,. \tag{2.3}$$

Proposition 2.9. *Any two of the following three statements imply the other one:*
1) $\langle J_Z X, J_Z X \rangle = \langle Z, Z \rangle \langle X, X \rangle$,
2) $\langle J_Z X, Y \rangle = -\langle X, J_Z Y \rangle$,
3) $J_Z^2 = -\langle Z, Z \rangle \, \mathrm{Id}_V$.

Proof. First, we will prove that (1) and (2) imply (3). This follows from:

$$\langle Z, Z\rangle\langle X, X\rangle = \langle J_Z X, J_Z X\rangle = -\langle J_Z J_Z X, X\rangle = -\langle J_Z^2 X, X\rangle.$$

Then, we will prove that (2) and (3) imply (1). This follows from:

$$\langle J_Z X, J_Z X\rangle = -\langle J_Z J_Z X, X\rangle = -\langle J_Z^2 X, X\rangle = -\langle -||Z||^2 X, X\rangle = \\ = ||Z||^2\langle X, X\rangle = \langle Z, Z\rangle\langle X, X\rangle.$$

Lastly, we prove that (1) and (3) imply (2). We see that

$$\langle J_Z^2 X, X\rangle = \langle J_Z X, J_Z^T X\rangle,$$

but also

$$\langle J_Z^2 X, X\rangle = \langle -||Z||^2 X, X\rangle = -\langle Z, Z\rangle\langle X, X\rangle = -\langle J_Z X, J_Z X\rangle = \langle J_Z X, -J_Z X\rangle.$$

Therefore, $J_Z^T = -J_Z$. □

We will now study the structure of the group of automorphisms of pseudo H-type Lie algebras. We already have some results for the general two-step nilpotent Lie algebras, as described in the previous section. The extra condition (2.2) on pseudo H-type Lie algebras will provide us with a tool used in the classification of $\mathrm{Aut}(\mathfrak{n})$; in particular, as we have seen in the previous section, we are mainly interested in the subgroup $\mathrm{Aut}^0(\mathfrak{n}) < \mathrm{Aut}(\mathfrak{n})$.

What follows in this section is an adaptation of [FM17]. This paper deals with the isomorphisms of pseudo H-type Lie algebras and many results can be adapted to fit the study of the groups of automorphisms.

Theorem 2.10. *Let $\Phi : \mathfrak{n}^{r,s}(V) \to \mathfrak{n}^{r,s}(V)$ be an automorphism of pseudo H-type Lie algebras. Then Φ is of the form*

$$\Phi = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} : V \oplus \mathbb{R}^{r,s} \to V \oplus \mathbb{R}^{r,s}, \tag{2.4}$$

where $A : V \to V$ and $C : \mathbb{R}^{r,s} \to \mathbb{R}^{r,s}$ are linear bijective maps satisfying the relation

$$A^T J_Z A = J_{C^T(Z)} \text{ for all } Z \in \mathbb{R}^{r,s}.$$

Moreover, there is no condition on $B : V \to \mathbb{R}^{r,s}$. Finally, if $|\det(AA^T)| = 1$, then $CC^T = \pm \mathrm{Id}$ if $r = s$ and $CC^T = \mathrm{Id}$ if $r \neq s$.

Proof. If a Lie Algebra automorphism $\Phi : \mathfrak{n}^{r,s}(V) \to \mathfrak{n}^{r,s}(V)$ exists, then it must be in the form (2.4), since it maps the centre to the centre. By the definition of Lie bracket we obtain:

$$\langle A^T J_Z A(X), Y\rangle_V = \langle J_Z A(X), A(Y)\rangle_V = \langle Z, [A(X), A(Y)]\rangle_{\mathbb{R}^{r,s}} = \langle Z, C([X, Y])\rangle_{\mathbb{R}^{r,s}} \\ = \langle C^T(Z), [X, Y]\rangle_{\mathbb{R}^{r,s}} = \langle J_{C^T(Z)} X, Y\rangle_V$$

for all X, $Y \in V$, for all $Z \in \mathbb{R}^{r,s}$. Hence, $A^T J_Z A = J_{C^T(Z)}$.
Conversely, if we know that $A^T J_Z A = J_{C^T(Z)}$ holds, by the previous calculations we can obtain $[A(X), A(Y)] = C([X, Y])$, so $\Phi = A \oplus C$ is a Lie algebra automorphism.
Let us now consider a Lie algebra automorphism $\Phi : \mathfrak{n}^{r,s}(V) \to \mathfrak{n}^{r,s}(V)$ in the form (2.4). Then

$$(A^T J_Z A)^2 = J_{C^T Z}^2 = -\langle C^T(Z), C^T(Z)\rangle_{\mathbb{R}^{r,s}}$$

Hence, assuming $\dim(V) = 2N$, we have:

$$\det((A^T J_Z A)^2) = (\det(AA^T))^2 \langle Z, Z\rangle_{\mathbb{R}^{r,s}}^{2N} = \langle C^T(Z), C^T(Z)\rangle_{\mathbb{R}^{r,s}}^{2N}$$

Now assume $r \neq s$. Then the map $C^T : \mathbb{R}^{r,s} \to \mathbb{R}^{r,s}$ preserves the sign. This means that

$$|\det(AA^T)|^{\frac{1}{N}} \langle Z, Z\rangle_{\mathbb{R}^{r,s}} = \langle C^T(Z), C^T(Z)\rangle_{\mathbb{R}^{r,s}} = \langle CC^T(Z), Z\rangle_{\mathbb{R}^{r,s}}$$

Under the assumption that $|\det(AA^T)| = 1$, we then obtain $CC^T = \mathrm{Id}$.
Assume now $r = s$. Then the map $C^T : \mathbb{R}^{r,s} \to \mathbb{R}^{r,s}$ may preserve or reverse the sign. This means that

$$|\det(AA^T)|^{\frac{1}{N}} \langle Z, Z\rangle_{\mathbb{R}^{r,s}} = \pm\langle C^T(Z), C^T(Z)\rangle_{\mathbb{R}^{r,s}} = \pm\langle CC^T(Z), Z\rangle_{\mathbb{R}^{r,s}}$$

Again assuming that $|\det(AA^T)| = 1$, we obtain, in this case, that $CC^T = \pm\,\mathrm{Id}$. □

Remark 2.11. The assumption $|\det(AA^T)| = 1$ in Theorem 2.10 is a rephrasing of the fact that $A \in \mathrm{SL}(n)$.

What emerges from this proof is the condition $A^T J_Z A = J_{C(Z)}$, which characterizes the automorphism groups of pseudo H-type Lie algebras. This condition will be heavily employed during the classification of $\mathrm{Aut}^0(\mathfrak{n})$.

Remark 2.12. It has been proven in [FM17] that, when studying these groups of automorphisms, one can assume that the condition $C^T C = \mathrm{Id}$ holds not only when $r \neq s \pmod 4$, but also when $r = s \neq 3 \pmod 4$. As we will see, $\mathfrak{n}^{3,3}$ and $\mathfrak{n}^{7,7}$ have isomorphic groups of automorphisms. This implies that the only case that we need to study for which $C^T C = -\,\mathrm{Id}$ is the case $\mathfrak{n}^{3,3}$.

2.2.1 The subgroup $C(\mathfrak{n})$

Now we want describe the group $C(\mathfrak{n})$; after doing so, we will be able to focus on the subgroup $\mathrm{Aut}^0(\mathfrak{n})$. We denote the group $C(\mathfrak{n})$ by $\mathrm{Cliff}(\mathfrak{n}_{r,s}(V))$. The map

$$\mathbb{R}^{r,s} \ni z \mapsto -z \in \mathbb{R}^{r,s} \subset \mathrm{Cl}_{r,s}$$

can be extended to the Clifford algebra automorphism $\alpha : \mathrm{Cl}_{r,s} \to \mathrm{Cl}_{r,s}$ by the universal property of Clifford algebras. We denote by $\mathrm{Cl}_{r,s}^{\times}$ the group of invertible elements in $\mathrm{Cl}_{r,s}$ and in particular $\mathbb{R}^{r,s\times} = \{v \in \mathbb{R}^{r,s} \mid \langle v, v\rangle_{r,s} \neq 0\}$. The representation

$$\widetilde{\mathrm{Ad}} \colon \mathbb{R}^{r,s\times} \to \mathrm{End}(\mathbb{R}^{r,s})$$

is defined as

$$\widetilde{\mathrm{Ad}}_v(Z) = -vZv^{-1} = \left(Z - 2\frac{\langle Z, v\rangle_{r,s}}{\langle v, v\rangle_{r,s}} v\right) \in \mathbb{R}^{r,s} \quad \text{for } Z \in \mathbb{R}^{r,s}, v \in \mathbb{R}^{r,s\times}.$$

The map $\widetilde{\mathrm{Ad}}_v : \mathbb{R}^{r,s} \to \mathbb{R}^{r,s}$ is the reflection of the vector $z \in \mathbb{R}^{r,s}$ with respect to the hyperplane orthogonal to the vector $v \in \mathbb{R}^{r,s}$. This extends to the so-called **twisted adjoint representation** $\widetilde{\mathrm{Ad}} : \mathrm{Cl}_{r,s}^{\times} \to \mathrm{GL}(\mathrm{Cl}_{r,s})$ by setting

$$\mathrm{Cl}_{r,s}^{\times} \ni \varphi \mapsto \widetilde{\mathrm{Ad}}_{\varphi}, \quad \widetilde{\mathrm{Ad}}_{\varphi}(z) = \alpha(\varphi) z \varphi^{-1}, \quad z \in \mathrm{Cl}_{r,s}. \tag{2.5}$$

The map $\widetilde{\mathrm{Ad}}_v$ for $v \in \mathbb{R}^{r,s\times}$, leaving the space $\mathbb{R}^{r,s} \subset \mathrm{Cl}_{r,s}$ invariant, is also an isometry: indeed, $\langle \widetilde{\mathrm{Ad}}_v(Z), \widetilde{\mathrm{Ad}}_v(Z)\rangle_{r,s} = \langle Z, Z\rangle_{r,s}$. Moreover, the properties to preserve the space $\mathbb{R}^{r,s}$ and the bilinear symmetric form $\langle -,-\rangle_{r,s}$ are fulfilled for the group

$$P(\mathbb{R}^{r,s}) = \{v_1 \cdots v_k \in \mathrm{Cl}^{\times}_{r,s} \mid \langle v_i, v_i\rangle_{r,s} \neq 0\}.$$

Note that $(\widetilde{\mathrm{Ad}}_{\varphi^{-1}})^T = \widetilde{\mathrm{Ad}}_{\varphi}$. The subgroups of $P(\mathbb{R}^{r,s}) \subset \mathrm{Cl}^{\times}_{r,s}$ defined by

$$\mathrm{Pin}(r,s) = \{v_1 \cdots v_k \in \mathrm{Cl}^{\times}_{r,s} \mid \langle v_i, v_i\rangle_{r,s} = \pm 1\},$$
$$\mathrm{Spin}(r,s) = \{v_1 \cdots v_k \in \mathrm{Cl}^{\times}_{r,s} \mid k \text{ is even, } \langle v_i, v_i\rangle_{r,s} = \pm 1\},$$

are called **pin** and **spin groups**, respectively. More information about the twisted adjoint representation and the groups Pin and Spin can be found in [LM89].

Proposition 2.13. *[LM89] The maps*

$$\widetilde{\mathrm{Ad}}\colon \mathrm{Pin}(r,s) \to \mathrm{O}(r,s) \quad \textit{and} \quad \widetilde{\mathrm{Ad}}\colon \mathrm{Spin}(r,s) \to \mathrm{SO}(r,s)$$

are the double covering maps.

We make the identification $\mathrm{Spin}(r) \times \mathrm{Pin}(s) \cong \mathrm{Spin}(r,0) \times \mathrm{Pin}(0,s) \subset \mathrm{Pin}(r,s)$.

Proposition 2.14. *Let $J\colon \mathrm{Cl}_{r,s} \to \mathrm{End}(U)$ be a Clifford algebra representation and $\varphi \in \mathrm{Spin}(r) \times \mathrm{Pin}(s)$. Then $J_{\varphi^{-1}} \oplus (\widetilde{\mathrm{Ad}}_{\varphi})^{\tau} \in \mathrm{Aut}_{gr}(\mathfrak{n}_{r,s}(U))$. The group homomorphism*

$$\begin{array}{rccc} \mathcal{A}\colon & \mathrm{Spin}(r) \times \mathrm{Pin}(s) & \to & \mathrm{Aut}_{gr}(\mathfrak{n}_{r,s}(U)), \\ & \varphi & \mapsto & J_{\varphi^{-1}} \oplus (\widetilde{\mathrm{Ad}}_{\varphi})^{\tau} \end{array}$$

is injective and the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Aut}^0(\mathfrak{n}_{r,s}(U)) & \longrightarrow & \mathrm{Aut}_{gr}(\mathfrak{n}_{r,s}(U)) & \xrightarrow{A\oplus C \mapsto C} & \mathrm{O}(r,s) \\ & & \uparrow & & \mathcal{A}\uparrow & & \uparrow = \\ 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathrm{Spin}(r) \times \mathrm{Pin}(s) & \xrightarrow{\widetilde{\mathrm{Ad}}} & O(r,s) \end{array} \tag{2.6}$$

is commutative. The kernel $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$ consists of automorphisms of the form $A \oplus \mathrm{Id}$.

Proof. By the definition of the twisted adjoint representation, we have

$$J_{\alpha(\varphi)} J_z J_{\varphi^{-1}} = J_{\widetilde{\mathrm{Ad}}_{\varphi}(z)}, \quad z \in \mathbb{R}^{r,s\times}.$$

If we show that $J_{\alpha(\varphi)} = J^T_{\varphi^{-1}}$, or equivalently $J_{\alpha(\varphi^{-1})} = J^T_{\varphi}$ for $\varphi \in \mathrm{Pin}(r,s)$, then it will imply that $J_{\varphi^{-1}} \oplus (\widetilde{\mathrm{Ad}}_{\varphi})^T \in \mathrm{Aut}_{gr}(\mathfrak{n}^{r,s})$ due to the relation $A^T J_z A = J_{C^T(z)}$.
If $v \in \mathbb{R}^{r,s\times}$ is such that $\langle v, v\rangle_{r,s} = -1$, then

$$J^T_{v^{-1}} = J^T_v = -J_v = J_{\alpha(v)},$$

and hence $J_{v^{-1}} \oplus (\widetilde{\mathrm{Ad}}_v)^T \in \mathrm{Aut}_{gr}(\mathfrak{n}^{r,s}(V))$. If instead v is such that $\langle v, v\rangle_{r,s} = 1$, then

$$J^T_{v^{-1}} = J^T_{-v} = J_v \neq J_{\alpha(v)},$$

and therefore the map $J_{v^{-1}} \oplus (\widetilde{\mathrm{Ad}}_v)^T$ does not belong to $\mathrm{Aut}_{gr}(\mathfrak{n}^{r,s}(V))$. If $\varphi = v_1 v_2$ with $\langle v_i, v_i\rangle_{r,s} = \pm 1$, $i = 1, 2$, then

$$J_{(v_1 v_2)^{-1}} = J_{v_2 v_1} = J^T_{\alpha(v_1 v_2)}.$$

This implies that $J_{(v_1v_2)^{-1}} \oplus (\widetilde{\mathrm{Ad}}_{v_1v_2})^T \in \mathrm{Aut}_{gr}(\mathfrak{n}^{r,s}(V))$.
In general, if $\varphi = x_1 \cdots x_{2p} \cdot y_1 \cdots y_q \in \mathrm{Pin}(r, s)$ with $\langle x_i, x_i\rangle_{r,s} = 1$, $i = 1, \ldots, 2p$, and $\langle y_j, y_j\rangle_{r,s} = -1$, $j = 1, \ldots, q$, then we obtain

$$(J_{(x_1\cdots x_{2p}\cdot y_1\cdots y_q)^{-1}})^T = (J_{y_q,\cdots y_1\cdot x_{2p}\cdots x_1})^T = (-1)^{2p+q} J_{x_1\cdots x_{2p}\cdot y_1\cdots y_q} = J_{\alpha(x_1\cdots x_{2p}\cdot y_1\cdots y_q)}. \quad \square$$

Remark 2.15. Let G be a group with a normal subgroup N and a subgroup H, such that every element $g \in G$ can be written uniquely in the form $g = nh$ where $n \in N$ and $h \in H$. Let $\varphi : H \to \mathrm{Aut}(N)$ be the homomorphism $h \mapsto \varphi_h$, defined by $\varphi_h(n) = hnh^{-1}$ for all $n \in N$, $h \in H$. Then G is isomorphic to the semidirect product $N \rtimes \varphi H$; and applying the isomorphism to the product nh gives the tuple (n, h). In G, we have

$$(n_1h_1)(n_2h_2) = n_1h_1n_2\big(h_1^{-1}h_1\big)h_2 = \big(n_1\varphi_{h_1}(n_2)\big)\big(h_1h_2\big) = (n_1, h_1) \bullet (n_2, h_2)$$

which shows that the map above is indeed an isomorphism and also explains the definition of the multiplication in $N \rtimes \varphi H$.

Recall a version of *the splitting lemma* for groups. It states that a group G is isomorphic to a semidirect product of the two groups N and H if and only if there exists a short exact sequence

$$0 \longrightarrow N \xrightarrow{\ \beta\ } G \xrightarrow{\ \alpha\ } H \tag{2.7}$$

and a group homomorphism $\gamma : H \to G$ such that $\alpha\gamma = \mathrm{Id}_H$. In this case, the map $\varphi : H \to \mathrm{Aut}(N)$ is given by $\varphi(h) = \varphi_h$, where

$$\varphi_h(n) = \beta^{-1}\big(\gamma(h)\beta(n)\gamma(h^{-1})\big).$$

2.2.2 Commutation of J_{Z_i}

In the proof of Theorem 2.10 we have seen that the groups of automorphisms of pseudo H-type Lie algebras are defined by the condition $A^T J_Z A = J_{C(Z)}$. This equation is used in the following lemma to obtain other relations between the matrix A and products of operators J_{Z_i}.

Lemma 2.16. *Let $\{Z_i\}_{i=1}^{r+s}$ be an orthogonal basis of $\mathbb{R}^{r,s}$ with $r \neq s$ or $r = s \neq 3$ (mod 4), and let*

$$\Phi = A \oplus C \,:\, V \oplus \mathbb{R}^{r,s} \to V \oplus \mathbb{R}^{r,s}$$

be an automorphism of Lie algebras (as in Theorem 2.10). The following relations hold:

- *If $p = 2m$, $m \in \mathbb{N}$, then*

$$A\prod_{j=1}^{p} J_{Z_j} = \prod_{j=1}^{p} J_{C(Z_j)}A, \qquad A^T\prod_{j=1}^{p} J_{Z_j} = \prod_{j=1}^{p} J_{C^T(Z_j)}A^T \tag{2.8}$$

$$A^TA\prod_{j=1}^{p} J_{Z_j} = \prod_{j=1}^{p} J_{Z_j}A^TA, \qquad AA^T\prod_{j=1}^{p} J_{C(Z_j)} = \prod_{j=1}^{p} J_{C(Z_j)}AA^T \tag{2.9}$$

- *If $p = 2m+1$, $m \in \mathbb{N}$, then*

$$A \prod_{j=1}^{p} J_{Z_j} A^T = \prod_{j=1}^{p} J_{C(Z_j)}, \qquad A^T \prod_{j=1}^{p} J_{Z_j} A = \prod_{j=1}^{p} J_{C^T(Z_j)} \tag{2.10}$$

$$A^T A \prod_{j=1}^{p} J_{Z_j} A^T A = \prod_{j=1}^{p} J_{Z_j}, \qquad A A^T \prod_{j=1}^{p} J_{Z_j} A A^T = \prod_{j=1}^{p} J_{Z_j} \tag{2.11}$$

Proof. We only prove the equalities on the right, since one can obtain the ones from the left by transposition. We will start by proving (2.10) and (2.8).

Firstly, observe that (2.10) for $m = 0$ is $A^T J_Z A = J_{C^T(Z)}$, which holds because of Theorem 2.10. We will prove that (2.10) for $m = 0$ implies (2.8) for $m = 1$.

$$A^T J_{Z_1} J_{Z_2} = A^T J_{Z_1} A A^{-1} J_{Z_2} = J_{C^T(Z_1)} A^{-1} J_{Z_2} =$$
$$J_{C^T(Z_1)} A^{-1} J_{Z_2} (A^T)^{-1} A^T = J_{C^T(Z_1)} J_{C^T(Z_2)} A^T,$$

where the second-last equality comes from the following observation:

$$\begin{aligned} A^{-1} J_Z (A^T)^{-1} &= (A^T J_Z^{-1} A)^{-1} = \left(-\frac{1}{\langle Z, Z\rangle} A^T J_Z A\right)^{-1} = \left(-\frac{1}{\langle Z, Z\rangle} J_{C^T(Z)}\right)^{-1} \\ &= -\langle Z, Z\rangle \left(-\frac{1}{\langle C^T(Z), C^T(Z)\rangle} J_{C^T(Z)}\right) = J_{C^T(Z)} \end{aligned}$$

for all Z, since

$$\langle C^T(Z), C^T(Z)\rangle = \langle CC^T(Z), Z\rangle = \langle Z, Z\rangle.$$

We will now prove that (2.8) for $m = 1$ and (2.10) for $m = 0$ imply (2.10) for $m = 1$.

$$A^T J_{Z_1} J_{Z_2} J_{Z_3} A = J_{C^T(Z_1)} J_{C^T(Z_2)} A^T = J_{C^T(Z_1)} J_{C^T(Z_2)} J_{C^T(Z_3)}.$$

In general, if (2.10) holds for $m \geq 1$, then (2.8) holds for $m + 1$:

$$\begin{aligned} A^T \prod_{j=1}^{2m+1} J_{Z_j} J_{Z_{2m+2}} &= A^T \prod_{j=1}^{2m+1} J_{Z_j} A A^{-1} J_{Z_{2m+2}} \\ &= \prod_{j=1}^{2m+1} J_{C^T(Z_j)} A^{-1} J_{Z_{2m+2}} (A^T)^{-1} A^T = \prod_{j=1}^{2(m+1)} J_{Z_j} A^T. \end{aligned}$$

Moreover, if (2.8) holds for $m \geq 2$, then (2.10) holds for m as well:

$$\begin{aligned} A^T \prod_{j=1}^{2m+2} J_{Z_j} J_{Z_{2m+3}} A &= \prod_{j=1}^{2m+2} J_{C^T(Z_j)} A^T J_{Z_{2m+3}} A \\ &= \prod_{j=1}^{2m+2} J_{C^T(Z_j)} J_{C^T(Z_{2m+3})} = \prod_{j=1}^{2m+3} J_{C^T(Z_j)}. \end{aligned}$$

we have hence proven (2.8) and (2.10) for every m.
We can now prove (2.9) for any m using both the equations in (2.8):

$$A^T A \prod_{j=1}^{2m} J_{Z_j} = A^T \prod_{j=1}^{2m} J_{C(Z_j)} A = \prod_{j=1}^{2m} J_{C^T C(Z_j)} A^T A = \prod_{j=1}^{2m} J_{Z_j} A^T A.$$

Lastly, we prove (2.11) for any m using both equations in (2.10):

$$A^T A \prod_{j=1}^{2m+1} J_{Z_j} A^T A = A^T \prod_{j=1}^{2m+1} J_{C(Z_j)} A = \prod_{j=1}^{2m+1} J_{C(Z_j)} = \prod_{j=1}^{2m+1} J_{C^T C(Z_j)} = \prod_{j=1}^{2m+1} J_{Z_j}.$$

□

Remark 2.17. If we restrict to the subgroup of automorphisms that act trivially on the centre, i.e. if we consider $\mathrm{Aut}^0(\mathfrak{n})$ where $C = \mathrm{Id}$, then we have three interesting results:

$$A^T J_{Z_i} A = J_{Z_i}, \qquad A^T J_{Z_i} J_{Z_k} J_{Z_l} A = J_{Z_i} J_{Z_k} J_{Z_l}, \qquad A J_{Z_i} J_{Z_j} = J_{Z_i} J_{Z_j} A.$$

2.3 Admissible Clifford modules

We will now describe a way to construct pseudo H-type Lie algebras from a Clifford algebra. We will start by giving the definition of an admissible module of a Clifford algebra; we will then follow [Cia00] and show that there is a one-to-one correspondence between pseudo H-type Lie algebras and admissible modules of Clifford algebras.

Definition 2.18. An **n-ary quadratic form** over a field $\mathbb{F}$ is a homogeneous polynomial of degree 2 in n variables with coefficients in $\mathbb{F}$:

$$q(x_1, \dots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j, \quad a_{ij} \in \mathbb{F}.$$

Remark 2.19. Any bilinear form has an associated quadratic form. In particular, if we consider the scalar product, the associated quadratic form is $q(v) = \langle v, v \rangle$.

Definition 2.20. Let $\mathfrak{v}$ be a vector space over the field $\mathbb{F}$ and let q be a quadratic form on $\mathfrak{v}$. The **Clifford algebra** $\mathrm{Cl}(\mathfrak{v}, q)$ associated to $\mathfrak{v}$ and q is an associative algebra with unit 1 defined as follows. Consider the tensor algebra $T(\mathfrak{v}) := \sum_{r=0}^{\infty} \bigotimes^r \mathfrak{v}$ of $\mathfrak{v}$. We define $I_q(\mathfrak{v})$ to be the ideal in $T(\mathfrak{v})$ generated by all the elements of the form $v \otimes v + q(v)1$ for $v \in \mathfrak{v}$. Then $\mathrm{Cl}(\mathfrak{v}, q) := T(\mathfrak{v})/I_q(\mathfrak{v})$.

Remark 2.21. We denote with $\mathrm{Cl}(r, s)$ the Clifford algebra built on $\mathfrak{v} = \mathbb{R}^{r+s}$ associated to a scalar product with signature (r, s). An orthogonal basis $\{Z_1, ..., Z_{r+s}\}$ of normalized vectors of $\mathbb{R}^{r+s}$ is called a **set of Clifford generators** of $\mathrm{Cl}(r, s)$. They satisfy the relations described in (1.1), which are called **fundamental relations of** $\mathrm{Cl}(r, s)$**.**

Remark 2.22. It is known ([LM89] and [ABS64]) that the Clifford algebras of the form $\mathrm{Cl}(r, s)$ are periodic, in the following sense:

$$\begin{aligned} \mathrm{Cl}(r, s+8) &\simeq \mathrm{Cl}(r, s) \otimes \mathbb{R}(16) \\ \mathrm{Cl}(r+8, s) &\simeq \mathrm{Cl}(r, s) \otimes \mathbb{R}(16) \\ \mathrm{Cl}(r+4, s+4) &\simeq \mathrm{Cl}(r, s) \otimes \mathbb{R}(16) \end{aligned}$$

where $\mathbb{R}(16)$ represents the matrices 16×16 with real entries. This property made it possible (see for example [LM89]) to classify all the Clifford algebras of the form $\mathrm{Cl}(r, s)$.

Definition 2.23. Given a Clifford algebra $\mathrm{Cl}(r, s)$, we define a **$\mathrm{Cl}(r, s)$-module** as a vector space γ which is the carrier space for a representation $J : \gamma \to J_\gamma$ of $\mathrm{Cl}(r, s)$.

Definition 2.24. Given a $\mathrm{Cl}(r,s)$-module V and a scalar product $\langle -,-\rangle_V$ on V, we call the pair $(V, \langle -,-\rangle_V)$ an **admissible** (r,s)**-module** if the operators J_Z are skew-adjoint for all $Z \in \mathfrak{z}$, i.e., if

$$\langle J_Z X, Y\rangle_V = -\langle X, J_Z Y\rangle_V. \tag{2.12}$$

In [Cia00] there is a proof of the following lemma.

Lemma 2.25. *Let $\mathfrak{z}$ be a real vector space of dimension k endowed with a scalar product $\langle -,-\rangle_{\mathfrak{z}}$ of signature (r,s) with $r+s=n$ and let V be a $\mathrm{Cl}(r,s)$-module. Then the algebra $\mathfrak{n} = \mathfrak{z} \oplus V$ is a pseudo H-type Lie algebra if and only if there exists a scalar product $\langle -,-\rangle : V \times V \to \mathbb{R}$ such that $(V, \langle -,-\rangle_V)$ is admissible.*

We now state the theorem by P. Ciatti which proves that, for any two integers r, s and for any $\mathrm{Cl}(r,s)$-module V, at least one between V and $V \oplus V$ can be endowed with a scalar product which satisfies (2.12). We remark that Ciatti in [Cia00] uses a slightly different notation from ours.

Theorem 2.26. *For all (r,s) there exists at least one admissible $Cl(r,s)$-module.*

In particular, P. Ciatti finds an admissible module $(V, \langle -,-\rangle)$ for the cases $r = 3 \pmod 4$ and for the cases $s = 0 \pmod 4$; for all the other cases he constructs an admissible module over $V \oplus V$. This result does not exclude a priori the existence of admissible modules V where the existence is verified by $V \oplus V$.

Remark 2.27. From now on, we will only consider the minimal admissible modules. In general the classification will depend on the dimension on the module.

So far we have followed [Cia00]; a result by I. Markina and K. Furutani gives us more information about admissible modules; this will depend on the value of $r-s$.

Definition 2.28. Let us consider an orthonormal basis $\{Z_1, \ldots, Z_{r+s}\}$ of $\mathbb{R}^{r,s}$, and let $J_{Z_1}, \ldots, J_{Z_{r+s}}$ be the corresponding representation maps. We define the **volume form**

$$\Omega^{r,s} := \prod_{i=1}^{r+s} Z_i.$$

In the case $r-s \equiv 3 \pmod 4$, the volume form is such that $J_{(\Omega^{r,s})^2} = \mathrm{Id}$. This implies the existence of two non-equivalent irreducible modules V^+ and V^-, on which the volume form acts respectively as Id and $-\mathrm{Id}$. If neither of them is admissible, then the direct sum of two of them is. We can hence summarize the possible structures of minimal admissible modules for every case [FM17]:

$r-s \neq 3 \pmod 4$	$r-s = 3 \pmod 4$		
any s	s is even	s is even	s is odd
V or $V \oplus V$	V^+ or V^-	$V^+ \oplus V^+$ or $V^- \oplus V^-$	$V^+ \oplus V^-$

2.3.1 Block structure of an admissible module

As r and s increase, the dimension of the minimal admissible modules gets larger and larger, and so the computations necessary for the classification of $\mathrm{Aut}^0(\mathfrak{n})$ will become more challenging. In particular, the cases we need to study would have a dimension ranging between 2 and 64. We will prove that it is enough to consider only one part (a "block") of the admissible module in order to classify $\mathrm{Aut}^0(\mathfrak{n})$. We will explain how a module can be divided into blocks and what these blocks represent. The following results will not apply to the case $\mathfrak{n}^{3,3}$; in fact, as we have seen in Remark 2.12, in this case the matrix C does not satisfy $C^T C = \mathrm{Id}$, and only consider the cases where this condition holds. In the case $\mathfrak{n}^{3,3}$ the condition satisfied by C is $C^T C = -\mathrm{Id}$, and thus we need to treat it separately.

What follows is an adaptation of [FM17]. As in Section 2.2, the original results refer to isomorphism groups, while here they are adapted to the automorphism groups.

Definition 2.29. We define **involution** a linear map $P : V \to V$ such that $P^2 = \mathrm{Id}$. In particular we will consider involutions that are a product of a number of linear operators J_i. We will denote with E^k with $k \in \{-1, 1\}$ the eigenspace of P, according to the eigenvalue.

Remark 2.30. By Lemma 2.16, we can observe that involutions that are a product of four elements will commute with the matrix $A \in Aut^0(\mathfrak{n})$.

The following is a corollary of Lemma 2.16.

Corollary 2.31. *Let $r \neq s$ or $r = s \neq 3 \pmod 4$, and let $\Phi = A \oplus \mathrm{Id}_m : \mathfrak{n}^{r,s} \to \mathfrak{n}^{r,s}$. Let P_j for $j = 1, \ldots, N$ be mutually commuting isometric involutions on $V^{r,s}$ obtained by the product of some J_i's. Then the map A can be written as $A = \bigoplus A_I$ where $A_I : E^I \to E^I$ for any choice of $I = (k_1, \ldots, k_N)$, $k_l \in \{\pm 1\}$ for all l. Moreover,*

$$\begin{aligned} A_I \prod_{i=1}^{2m} J_i &= \prod_{i=1}^{2m} J_i A_I \\ A_I \prod_{i=1}^{2m+1} J_i &= \prod_{i=1}^{2m+1} J_i (A_I^T)^{-1} \end{aligned} \tag{2.13}$$

Theorem 2.32. *Let P_j, $j = 1, \ldots, N$ be mutually commuting isometric involutions on $V^{r,s}$, obtained by the product of some J_i's. Let $E^1 := \bigcap_{j=1}^N E^1_{P_j}$. We assume there exists $G_I : E^1 \to E^I$ for all multi-indices I written as $G_I = \prod J_i$. We also assume that there exists $A_1 : E^1 \to E^1$ satisfying (2.13). Then there exists a map $A : V^{r,s} \to V^{r,s}$ such that $A \oplus \mathrm{Id}_m : \mathfrak{n}^{r,s} \to \mathfrak{n}^{r,s}$ is a Lie algebra automorphism.*

Proof. We define the maps $A_I : E^I \to E^I$ by:

$$A_I = \begin{cases} G_I (A_1^{-1})^T G_I^{-1} & \text{if } \ G_I = \prod_{i=1}^{p=2m+1} J_i \\ G_I A_1 G_I^{-1} & \text{if } \ G_I = \prod_{i=1}^{p=2m} J_i. \end{cases}$$

Then we can write the adjoint maps as

$$A_I^T = \begin{cases} G_I (A_1^{-1}) G_I^{-1} & \text{if } \ G_I = \prod_{i=1}^{p=2m+1} J_i \\ G_I A_1^T G_I^{-1} & \text{if } \ G_I = \prod_{i=1}^{p=2m} J_i. \end{cases}$$

We set $A := \bigoplus A_I$. We need to check the condition $A J_i A^T = J_i$ for any Z_i in the orthonormal basis for $\mathbb{R}^{r,s}$.

One can observe that the spaces E^I are mutually orthogonal; in fact, if $P_j(X) = X$ and $P_j(Y) = -Y$ for some isometry P_j, then

$$\langle X, -Y \rangle_{V^{r,s}} = \langle P_j(X), P_j(Y) \rangle_{V^{r,s}} = \langle X, Y \rangle_{V^{r,s}}.$$

This implies that $\langle X,Y\rangle_{V^{r,s}}=0$. Thus $V^{r,s}=\bigoplus E^I$, where the direct sums are orthogonal. The maps G_I are invertible and

$$G_I^{-1}=(\prod_{i=1}^{p}J_i)^{-1}=(-1)^p\prod_{i=1}^{p}\langle z_i,z_i\rangle^{-1}\prod_{k=0}^{p-1}J_{p-k}.$$

From Lemma 2.16 we know that the following relationships hold:

$$(A_1^{-1})^T\prod_{i=1}^{2m+1}J_iA_1^{-1}=\prod_{i=1}^{2m+1}J_i,\qquad A_1\prod_{i=1}^{2m+1}J_iA_1^T=\prod_{i=1}^{2m+1}J_i \tag{2.14}$$

$$(A_1^{-1})^T\prod_{i=1}^{2m}J_iA_1^T=\prod_{i=1}^{2m}J_i,\qquad A_1\prod_{i=1}^{2m}J_iA_1^{-1}=\prod_{i=1}^{2m}J_i \tag{2.15}$$

For arbitrary J_{i_0} and $Y\in V^{r,s}$, we can write $Y=\bigoplus Y_I$ with $Y_I\in E^I$. For the multi-index I we find a multi-index K such that $G_K^{-1}J_{i_0}G_I$ leaves invariant the space E^1. Since G_I and G_K can be product of an even or an odd number of J_k's, we consider the different cases:

$$AJ_{i_0}A^Ty_I=A_KJ_{j_0}A_I^TY_I=$$
$$=\begin{cases}G_K(A_1^{-1})^TG_K^{-1}J_{i_0}G_IA_1^{-1}G_I^{-1}Y_I & \text{if } G_I=\prod_{i=1}^{2m+1}J_i,\quad G_K=\prod_{l=1}^{2k+1}J_l\\ G_KA_1G_K^{-1}J_{i_0}G_IA_1^{-1}G_I^{-1}Y_I & \text{if } G_I=\prod_{i=1}^{2m+1}J_i,\quad G_K=\prod_{l=1}^{2k}J_l\\ G_K(A_1^{-1})^TG_K^{-1}J_{i_0}G_IA_1^TG_I^{-1}Y_I & \text{if } G_I=\prod_{i=1}^{2m}J_i,\quad G_K=\prod_{l=1}^{2k+1}J_l\\ G_KA_1G_K^{-1}J_{i_0}G_IA_1^TG_I^{-1}Y_I & \text{if } G_I=\prod_{i=1}^{2m}J_i,\quad G_K=\prod_{l=1}^{2k}J_l\end{cases}$$

which we get by the definition of A_I and A_I^T. Counting the elements J_i in every product $G_K^{-1}J_{i_0}G_I$, we can apply the formulas in (2.14) and (2.15) to obtain:

$$AJ_{i_0}A^TY_I=$$
$$=\begin{cases}G_KG_K^{-1}J_{i_0}G_IG_I^{-1}Y_I & \text{if } G_I=\prod_{i=1}^{2m+1}J_i,\quad G_K=\prod_{l=1}^{2k+1}J_l\\ G_KG_K^{-1}J_{i_0}G_IG_I^{-1}Y_I & \text{if } G_I=\prod_{i=1}^{2m+1}J_i,\quad G_K=\prod_{l=1}^{2k}J_l\\ G_KG_K^{-1}J_{i_0}G_IG_I^{-1}Y_I & \text{if } G_I=\prod_{i=1}^{2m}J_i,\quad G_K=\prod_{l=1}^{2k+1}J_l\\ G_KG_K^{-1}J_{i_0}G_IG_I^{-1}Y_I & \text{if } G_I=\prod_{i=1}^{2m}J_i,\quad G_K=\prod_{l=1}^{2k}J_l\end{cases}$$

Thus $AJ_{i_0}A^TY_I=J_{i_0}Y_I$. □

2.3.2 The group $\mathrm{Aut}^0(\mathfrak{n}^{r,0})$

The pseudo H-type Lie algebras with positive definite scalar product have already been studied, in particular by L. Saal [Saa96]. These Lie algebras have already been classified, and here we provide the proposition which summarizes this classification. The first proposition is a general result, while the corollary is its restriction to minimal admissible modules, which is the case we will focus on.

Proposition 2.33. *Let* $\mathfrak{n}^{r,0}=\mathbb{R}^{r,0}\oplus V$ *be an algebra of H-type, let* $\dim_{\mathbb{R}}(V)=n$*. Then*

$\mathrm{Aut}^0(\mathfrak{n}^{r,0})$ *is isomorphic to*

$$\begin{cases} \mathrm{Sp}(n \cdot 2^{-(\frac{r+1}{2})}, \mathbb{R}) & \textit{if } r = 1 \ (\textit{mod } 8) \\ \mathrm{Sp}(n \cdot 2^{-(\frac{r+2}{2})}, \mathbb{C}) & \textit{if } r = 2 \ (\textit{mod } 8) \\ \mathrm{U}(n_1 \cdot 2^{-(\frac{r+1}{2})}, n_{-1} \cdot 2^{-(\frac{r+1}{2})}, \mathbb{H}) & \textit{if } r = 3 \ (\textit{mod } 8) \\ \mathrm{GL}(n \cdot 2^{-(\frac{r+2}{2})}, \mathbb{H}) & \textit{if } r = 4 \ (\textit{mod } 8) \\ \mathrm{SO}^*(2n \cdot 2^{-(\frac{r+1}{2})}) & \textit{if } r = 5 \ (\textit{mod } 8) \\ \mathrm{O}(n \cdot 2^{-(\frac{r}{2})}, \mathbb{C}) & \textit{if } r = 6 \ (\textit{mod } 8) \\ \mathrm{O}(n_1 \cdot 2^{-(\frac{r-1}{2})}, n_{-1} \cdot 2^{-(\frac{r-1}{2})}, \mathbb{R}) & \textit{if } r = 7 \ (\textit{mod } 8) \\ \mathrm{GL}(n \cdot 2^{-(\frac{r}{2})}, \mathbb{R}) & \textit{if } r = 0 \ (\textit{mod } 8) \end{cases}$$

where n_1 *and* $n_{-1} = n - n_1$ *are the dimensions of the eigenspaces of* V *with respect to* $\Omega^{r,0}$*, and* $\mathrm{SO}^*(2l) := \mathrm{GL}(l, \mathbb{H}) \cap \mathrm{O}(2l, \mathbb{C})$.

Corollary 2.34. *If we assume* V *to be of minimal dimension, we have:*

$$\begin{cases} \mathrm{Sp}(1, \mathbb{R}) & \textit{if } r = 1 \ (\textit{mod } 8) \\ \mathrm{Sp}(1, \mathbb{C}) & \textit{if } r = 2 \ (\textit{mod } 8) \\ \mathrm{U}(1, 0, \mathbb{H}) & \textit{if } r = 3 \ (\textit{mod } 8) \\ \mathrm{GL}(1, \mathbb{H}) & \textit{if } r = 4 \ (\textit{mod } 8) \\ \mathrm{SO}^*(2) & \textit{if } r = 5 \ (\textit{mod } 8) \\ \mathrm{O}(1, \mathbb{C}) & \textit{if } r = 6 \ (\textit{mod } 8) \\ \mathrm{O}(1, 0, \mathbb{R}) & \textit{if } r = 7 \ (\textit{mod } 8) \\ \mathrm{GL}(1, \mathbb{R}) & \textit{if } r = 0 \ (\textit{mod } 8) \end{cases}$$

Chapter 3

Classification of $\mathrm{Aut}^0(\mathfrak{n}^{\mathrm{r,s}})$

In this chapter we will classify the automorphism groups of pseudo H-type Lie algebras. Because of the periodicity of Clifford algebras described in Remark 2.22, we will only need to study the cases $\mathfrak{n}^{r,s}$ for r, s corresponding to the non-empty cells in this table:

s										
8	1									
7	2	4	8	8						
6	4	4	4	4						
5	8	8	4	2						
4	4	4	2	1	1					
3	8	8	4		2	4	8	8		
2	4	4	4	4	4	4	4	4		
1	2	4	8	8	8	8	4	2		
0		2	4	4	4	4	2	1	1	
	0	1	2	3	4	5	6	7	8	r

Table 3.1

The number written in every position is the dimension of the minimal admissible module of $\mathfrak{n}^{r,s}$, or, if the pseudo H-type Lie algebra admits mutually commuting isometric involutions, the dimension of its minimal eigenspace. The cases coloured in blue are the ones which admit an involution which is a product of three linear operators.

For the cases that admit one or more involutions, we will consider only the first common eigenspace as in Theorem 2.32, since by that theorem we know that this is enough to provide a structure of the entirety of $\mathrm{Aut}^0(\mathfrak{n})$. The first eigenspace will be one-, two-, four- or eight-dimensional; we will study these four cases separately. Some of them will yield the same group $\mathrm{Aut}^0\,\mathfrak{n}$, even though the starting pseudo H-type Lie algebras are not isomorphic; these cases will be treated together.

We will make extensive use of the following lemmas.

Lemma 3.1. *Consider three operators J_i, J_k and J_l and assume they satisfy the conditions $AJ_iJ_k = J_iJ_kA$ and $A^TJ_lA = J_l$. Then $A^TJ_iJ_kJ_lA = J_iJ_kJ_l$.*

Proof. We have the chain of implications:

$$A^TJ_lA = J_l \Rightarrow A^TJ_lAJ_iJ_k = J_lJ_iJ_k \Rightarrow A^TJ_lJ_iJ_kA = J_lJ_iJ_k \Rightarrow A^TJ_iJ_kJ_lA = J_iJ_kJ_l.$$

□

Lemma 3.2. *Let Ω_n be the matrix as in the definition of the symplectic group of degree $2n$ and let σ_n be the matrix as in the definition of the group* T *of degree n (as in Definition 1.7). Consider the matrix*

$$\eta := \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & -\mathrm{Id}_n \end{pmatrix}.$$

Then $\eta \cdot \Omega_n = -\sigma_n$ and $\eta \cdot \sigma_n = -\Omega_n$ (by computation). □

The main tool employed in our analysis is the knowledge of a basis for each admissible module of $\mathfrak{n}^{r,s}$ and the subdivision of such bases in common eigenspaces. This data, the collection of which represents an important part of this thesis, is presented in the tables in Appendix A. In the same tables, all the involutions of each admissible module of a pseudo H-type Lie algebra are also listed. We notice that the metric of every first common eigenspace can either be neutral or sign definite.

The techniques we adopted in order to gather the data included in those tables are explained in Appendix A.

As shown in Table 3.1, we reduce our analysis to the study of $n \times n$ real matrices, with $n = 1, 2, 4, 8$. The linear operators J_i that we will encounter will be represented as real matrices of the appropriate dimension. The isomorphisms presented in Section 1.2 will further reduce these cases to the study of matrices of lower dimensions in other fields (such as $\mathbb{C}$ and $\mathbb{H}$).

Remark 3.3. As we can see from Table 3.1, many admissible modules (marked in blue) admit involutions which are a product of three linear operators. These involutions do not satisfy the conditions of Theorem 2.32, so we will not consider them when subdividing the modules into common eigenspaces. Nevertheless, it follows from Lemma 2.16 that any involution P which is a product of three J_i's satisfies $A^T PA = P$.

Proposition 3.4. *Let E be a common eigenspace of mutually commuting isometric involutions and let $A : E \to E$ be as in Theorem 2.32 (hence we know that A is an invertible real matrix). Let P be an involution which is a product of three linear operators. Then, the condition $A^T PA = P$ implies $A^T A = \mathrm{Id}$.*

Proof. Let us first observe that the condition $A^T PA(x) = P(x)$ for every $x \in E$ is equivalent to the condition $A^T PA = P$ as matrices, by the definition of a linear operator. We want to prove that $A^T A = \mathrm{Id}$. Observe that P, being an involution, divides the space into two eigenspaces, E^+ and E^-, on which it acts respectively as Id and $-\mathrm{Id}$.
The claim is proved by contradiction. Assume first that there exists $x_0 \in E$ such that

$$A^T A x_0 = y \neq x_0 \text{ with } y \neq 0. \tag{3.1}$$

First, assume $x_0 \in E^+$. Let $y \in E$; for every $x \in E^+$, the relation $A^T PAx = Px = x$ holds; Hence $A^T PA = \mathrm{Id}$ holds on E^+. Multiplying on both sides by A^T, we obtain $A^T PAA^T = A^T$. Hence $A^T PAA^T x_0 = A^T x_0$, which implies, by (3.1):

$$A^T Py = A^T x_0. \tag{3.2}$$

If $y \in E^+$, it follows that $A^T y = A^T x_0$. Since A is invertible, then A^T is as well, hence we come to the contradiction $y = x_0$. If instead $y \in E^-$, then the equation (3.2) becomes

$-A^T y = A^T x_0$. Since A^T is invertible, we obtain $y = -x_0$. But we are assuming $x_0 \in E^+$, so $-x_0 \in E^+$. Since $y \in E^-$, also in this case we come to a contradiction.
Assume now that $x_0 \in E^-$. We can observe that $A^T P A x = P x = -x$ for every $x \in E^-$. Hence, $A^T P A = -\mathrm{Id}$ on E^-. We can multiply on the right both sides by A^T we obtain $A^T P A A^T = -A^T$. Then $A^T P A A^T x_0 = -A^T x_0$, which implies, by (3.1):

$$A^T P y = -A^T x_0. \tag{3.3}$$

If $y \in E^-$, then $-A^T y = -A^T x_0$. Since A is invertible, then A^T is as well, hence we come to the contradiction $-y = -x_0$, i.e. $y = x_0$. If instead $y \in E^+$, then the equation (3.3) becomes $A^T y = -A^T x_0$. Since A^T is invertible, we obtain $y = -x_0$. But we are assuming $x_0 \in E^-$, so $-x_0 \in E^-$. Since $y \in E^+$, we come to a contradiction.

□

Remark 3.5. Let P be an involution which is a product of three operators, and let E be the first common eigenspace with respect to the involutions which are products of four linear operators; if P has two non-trivial eigenspaces E^+ and E^- on which it acts respectively as Id and $-$Id, then, since P is a simmetric operator, we know that E^+ and E^- are orthogonal. Moreover, Proposition 3.4 proves that A maps each eigenspace in itself, i.e. $A : E^+ \oplus E^- \to E^+ \oplus E^-$. Hence, if E^+ and E^- are isomorphic to $\mathrm{GL}(n, \mathbb{F})$ with n $\mathbb{F}$ fields, then A represents the orthogonal matrices over $\mathbb{F}$. In particular, if the metric is sign definite we conclude that $A \in \mathrm{O}(2n, \mathbb{F})$; if instead the metric is neutral, we conclude that $A \in \mathrm{O}(n, n, \mathbb{F})$.
One cannot conclude the same if one between E^+ and E^- is trivial.

What follows is a table which summarizes the results obtained in this chapter.

8	$\mathrm{GL}(1,\mathbb{R})$								
7	$\mathrm{T}(2,\mathbb{R})$	$\overline{\mathrm{Sp}}(2,\mathbb{C})$	$\overline{\mathrm{Sp}}(2,\mathbb{H})$	$\overline{\mathrm{Sp}}^\dagger(2,\mathbb{H})$					
6	$\mathrm{GL}^\sharp(2,\mathbb{C})$	$\overline{\mathrm{Sp}}^*(2,\mathbb{C})$	$\mathrm{GL}(1,\mathbb{H})$	$\mathrm{GL}^*_{\mathbb{C}}(1,\mathbb{H})$					
5	$\overline{\mathrm{Sp}}(2,\mathbb{H})$	$\overline{\mathrm{Sp}}^\dagger(2,\mathbb{H})$	$\overline{\mathrm{Sp}}^*(2,\mathbb{C})$	$\mathrm{Sp}^*(2,\mathbb{R})$					
4	$\mathrm{GL}(1,\mathbb{H})$	$\mathrm{GL}^*_{\mathbb{C}}(1,\mathbb{H})$	$\mathrm{O}(1,\mathbb{C})$	$\mathrm{O}(1,\mathbb{R})$	$\mathrm{GL}(1,\mathbb{R})$				
3	$\mathrm{T}(2,\mathbb{H})$	$\widetilde{\mathrm{Sp}}^*(2,\mathbb{SH})$	$\mathrm{GL}^*_{\mathbb{C}}(1,\mathbb{SH})$		$\mathrm{T}(2,\mathbb{R})$	$\overline{\mathrm{Sp}}(2,\mathbb{C})$	$\overline{\mathrm{Sp}}(2,\mathbb{H})$	$\overline{\mathrm{Sp}}^\dagger(2,\mathbb{H})$	
2	$\mathrm{Sp}(2,\mathbb{C})$	$\mathrm{GL}^*_{\mathbb{C}}(1,\mathbb{SH})$	$\mathrm{GL}(1,\mathbb{H})$	$\mathrm{GL}^*_{\mathbb{C}}(1,\mathbb{SH})$	$\mathrm{GL}^\sharp(2,\mathbb{C})$	$\overline{\mathrm{Sp}}^*(2,\mathbb{C})$	$\mathrm{GL}(1,\mathbb{H})$	$\mathrm{GL}^*_{\mathbb{C}}(1,\mathbb{H})$	
1	$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{Sp}(2,\mathbb{SC})$	$\widetilde{\mathrm{Sp}}(2,\mathbb{SH})$	$\overline{\mathrm{Sp}}^*(2,\mathbb{H})$	$\overline{\mathrm{Sp}}(2,\mathbb{H})$	$\overline{\mathrm{Sp}}^\dagger(2,\mathbb{H})$	$\overline{\mathrm{Sp}}^*(2,\mathbb{C})$	$\mathrm{Sp}^*(2,\mathbb{R})$	
0		$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{Sp}(2,\mathbb{C})$	$\mathrm{U}(1,\mathbb{H})$	$\mathrm{GL}(1,\mathbb{H})$	$\mathrm{GL}^*_{\mathbb{C}}(1,\mathbb{H})$	$\mathrm{O}(1,\mathbb{C})$	$\mathrm{O}(1,\mathbb{R})$	$\mathrm{GL}(1,\mathbb{R})$
	0	1	2	3	4	5	6	7	8

In the table we have used the following notation. Given G a group, we define:
- $G^*(2n, \mathbb{F}) := G(2n, \mathbb{F}) \cap \{A \in \mathrm{GL}(2n, \mathbb{F}) \mid A^t \eta A = \eta\}$.
- $G^*_{\mathbb{F}'}(n, \mathbb{F}) := G(n, \mathbb{F}') \cap \{A \in \mathrm{GL}(2n, \mathbb{F}) \mid A^t \eta A = \eta\}$, for $\mathbb{F}$, $\mathbb{F}'$ different fields.
- $G^\dagger(2n, \mathbb{F}) := G(2n, \mathbb{F}) \cap \{A \in \mathrm{GL}(2n, \mathbb{F}) \mid \overline{A^t} \eta A = \eta\}$.
- $G^\dagger_{\mathbb{F}'}(n, \mathbb{F}) := G(n, \mathbb{F}) \cap \{A \in \mathrm{GL}(2n, \mathbb{F}') \mid \overline{A^t} \eta A = \eta\}$, for $\mathbb{F}$, $\mathbb{F}'$ different fields.
- $\mathrm{GL}^\sharp(2, \mathbb{C}) := GL(2, \mathbb{C}) \cap \mathrm{T}(4, \mathbb{R})$.
- We denote denote with the color red all the cases where the first common eigenspace (or the module itself if there is no division in eigenspaces) has a sign definite metric.

3.1 One-dimensional common eigenspaces

As we can see from Table 3.1, there are only five one-dimensional cases. In particular, $\mathfrak{n}^{8,0}$ and $\mathfrak{n}^{0,8}$ are isomorphic pseudo H-type Lie algebras, so we expect their respective

$\mathrm{Aut}^0(\mathfrak{n})$'s to be isomorphic. It turns out that also $\mathfrak{n}^{4,4}$ has the same $\mathrm{Aut}^0(\mathfrak{n})$.
The other two cells in the table correspond to $\mathfrak{n}^{7,0}$ and $\mathfrak{n}^{3,4}$, which satisfy $r-s=3 \pmod 4$. Hence, both cases admit two non-equivalent irreducible admissible modules V^+ and V^-; they will turn out to have the same $\mathrm{Aut}^0(\mathfrak{n})$.

s										
8	(1)									
7	2	4	8	8						
6	4	4	4	4						
5	8	8	4	2						
4	4	4	2	(1)	(1)					
3	8	8	4		2	4	8	8		
2	4	4	4	4	4	4	4	4		
1	2	4	8	8	8	8	4	2		
0		2	4	4	4	4	2	(1)	(1)	
	0	1	2	3	4	5	6	7	8	r

Table 3.1, one-dimensional cases circled.

3.1.1 Cases $\mathfrak{n}^{7,0}$ and $\mathfrak{n}^{3,4}$

The minimal admissible modules of $\mathfrak{n}^{7,0}$ and $\mathfrak{n}^{3,4}$ have a very similar structure. Indeed, they both admit four different involutions, three of which are products of four linear operators, while the last one is the product of three linear operators (see Subsections A.1.1, A.1.2). The first three involutions, named P_1, P_2 and P_3, divide the admissible module into one-dimensional blocks, while the fourth one, named P_4, has a different role. We know, in fact, that in the case $r-s \equiv 3 \pmod 4$ the pseudo H-type Lie algebra has two non-equivalent minimal admissible modules, on which the volume form acts as Id or $-\mathrm{Id}$. The involution P_4 acts differently on the two admissible modules, which we call V^+ and V^-.
We will analyse the case $\mathfrak{n}^{7,0}$, as the case $\mathfrak{n}^{3,4}$ is analogous, and we will prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{O}(1,\mathbb{R})$.

First we will show that $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{O}(1,\mathbb{R})$. Consider an element v such that $\langle v, v\rangle = 1$ and belonging to the first block in V^+. Then we have the following chain of implications:

$$\begin{aligned}\Omega v = v &\Rightarrow J_1J_2J_3J_4J_5J_6J_7v = v \Rightarrow J_5J_6J_7J_1J_2J_3J_4v = v \Rightarrow J_5J_6J_7v = v\\ &\Rightarrow J_1^2J_2^2J_5J_6J_7v = v \Rightarrow -J_1J_2J_7J_1J_2J_5J_6v = v \Rightarrow -J_1J_2J_7v = v\\ &\Rightarrow P_4v = -v.\end{aligned}$$

Similarly, if we choose an element w in the first block of V^-, we can obtain that $P_4w = w$.
We will proceed to study V^+. Following the tables in Subsections A.1.1, A.1.2, we can construct the basis of V^+:

$$\{v, J_7v, J_6v, J_5v, J_4v, J_3v, J_2v, J_1v\}$$

Since we managed to divide the admissible module into one-dimensional blocks, we know that $A \in \mathrm{GL}(1,\mathbb{R})$. We can use the involution P_4 to obtain more information. In fact, by Lemma 2.16, we have that $A^TP_4A = P_4$. We are currently considering the module V^+, on which P_4 acts as $-\mathrm{Id}$. Hence we obtain $-A^TA = -\mathrm{Id}$, which implies that $A^TA = \mathrm{Id}$. This

implies that $A \in \mathrm{O}(1, \mathbb{R})$, so, in particular, $A \in \{\pm 1\}$. This result is not in contradiction with the one provided by Saal (Corollary 2.34): indeed, we have proven in Remark 1.9 that $\mathrm{O}(1, 0, \mathbb{R}) \simeq \mathrm{O}(1, \mathbb{R})$. Hence, $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{O}(1, \mathbb{R})$.
Note that, even though the admissible module of $\mathrm{Cl}^{3,4}$ has a neutral metric, the first block has a definite positive metric, so one can apply exactly the same computations as the ones for the case $\mathfrak{n}^{7,0}$.

We now want to prove that $\mathrm{O}(1, \mathbb{R}) \subset \mathrm{Aut}^0(\mathfrak{n})$. Let $M \in \mathrm{O}(1, \mathbb{R})$, so $M \in \{\pm 1\}$. We need to check that it satisfies the condition $M^T J_1 J_2 J_7 M = J_1 J_2 J_7$, where $J_1 J_2 J_7$ acts as $-\mathrm{Id}$. In particular, we obtain the condition $M^T \,\mathrm{Id}\, M = \mathrm{Id}$, but since M is a real number, its transpose is simply M itself and it also commutes with any matrix. Hence, we obtain $MM = \mathrm{Id}$. Now since $M = \{\pm 1\}$, $MM = 1$, so the condition holds and $\mathrm{O}(1, \mathbb{R}) \subset \mathrm{Aut}^0(\mathfrak{n})$.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{O}(1, \mathbb{R})$ for the considered cases.

3.1.2 Cases $\mathfrak{n}^{8,0}$, $\mathfrak{n}^{0,8}$ and $\mathfrak{n}^{4,4}$

We already know that $\mathfrak{n}^{8,0}$ and $\mathfrak{n}^{0,8}$ are isomorphic, so we can expect a similar behaviour once we consider their automorphism groups; we will see that $\mathfrak{n}^{4,4}$ will also have the same automorphism group, which we will prove to be isomorphic to $\mathrm{GL}(1, \mathbb{R})$.
From Subsections A.1.3, A.1.4 and A.1.5, we see that these three cases all have four involutions which are products of four linear operators. These involutions divide the admissible module into one-dimensional blocks. We will study the case $\mathfrak{n}^{8,0}$, as the two other ones will behave in the same way.

First, we will prove the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{GL}(1, \mathbb{R})$. As shown in the table in Subsection A.1.3, a basis for the first eigenspace is given by $\{v\}$, where v is an element in the first eigespace such that $\langle v, v \rangle = 1$. As there is no further condition on the basis, we conclude that $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{GL}(1, \mathbb{R})$.

Proving the inclusion $\mathrm{GL}(1, \mathbb{R}) \subset \mathrm{Aut}^0(\mathfrak{n})$ is trivial: as an element in $\mathrm{GL}(1, \mathbb{R})$ is simply a number in $\mathbb{R}^\times$, it trivially acts as an automorphism on the first block.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{GL}(1, \mathbb{R})$ for the considered cases.

3.2 Two-dimensional common eigenspaces

We identify in Table 3.1 eight two-dimensional cases. We will see that they can actually be gathered in four different classes. Some of the results we obtained were expected – it is known, for example, that $\mathfrak{n}^{1,0}$ and $\mathfrak{n}^{0,1}$ are isomorphic – while other ones are surprising. As the dimension of the minimal admissible module of the common eigenspaces is 2, the automorphism groups will be subgroups of $\mathrm{GL}(2, \mathbb{R})$ or $\mathrm{GL}(1, \mathbb{C})$.

3.2.1 Cases $\mathfrak{n}^{1,0}$ and $\mathfrak{n}^{0,1}$

The pseudo H-type Lie algebras $\mathfrak{n}^{1,0}$ and $\mathfrak{n}^{0,1}$ are isomorphic, but we must study them separately. In both cases we have a single linear operator J_1. The admissible module is two-dimensional, and in the tables in Subsections A.2.1 and A.2.2 we construct a basis of it, given by $\{v, J_1 v\}$. We will show that in both cases, the automorphism groups are isomorphic to $\mathrm{Sp}(2, \mathbb{R})$.

First, we will prove the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{Sp}(2, \mathbb{R})$. Since there is obviously no even product of operators in the basis, we can say that $A \in \mathrm{GL}(2, \mathbb{R})$. From the condition $A^T J_1 A = J_1$, we obtain that $\eta A^t \eta J_1 A = J_1$, and hence $A^t (\eta J_1) A = (\eta J_1)$.
In the case $\mathfrak{n}^{1,0}$, we have that $J_1^2 = -\mathrm{Id}$, so its matrix representation has the same form as Ω_1. The condition $A^T J_1 A = J_1$ resolves into $A^t \Omega_1 A = \Omega_1$, since the metric is definite

s									
8	1								
7	2	4	8	8					
6	4	4	4	4					
5	8	8	4	②					
4	4	4	②	1	1				
3	8	8	4		②	4	8	8	
2	4	4	4	4	4	4	4	4	
1	②	4	8	8	8	8	4	②	
0		②	4	4	4	4	②	1	1
	0	1	2	3	4	5	6	7	8 r

Table 3.1, two-dimensional cases circled.

positive. Hence, $A \in \mathrm{Sp}(2,\mathbb{R})$.
In the case $\mathfrak{n}^{0,1}$, we have that $J_1^2 = \mathrm{Id}$, so its matrix representation is of the form as σ_1. In this case, though, the metric is neutral, so its matrix representation is

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By Lemma 3.2, $\eta \cdot \sigma_1 = -\Omega_1$. So the condition $A^T J_1 A = J_1$ gives again $A^t(-\Omega_1)A = -\Omega_1$, so $A^t \Omega_1 A = \Omega_1$. Hence $A \in \mathrm{Sp}(2,\mathbb{R})$. So, in both the cases $\mathfrak{n}^{1,0}$ and $\mathfrak{n}^{0,1}$, we have that $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{Sp}(2,\mathbb{R})$.

We will now prove the inclusion $\mathrm{Sp}(2,\mathbb{R}) \subset \mathrm{Aut}^0(\mathfrak{n})$. Let $M \in \mathrm{Sp}(2,\mathbb{R})$; we need to check if it satisfies the condition $M^T J_1 M = J_1$, which in both cases becomes $M^t \Omega_1 M = \Omega_1$ because $J_1^2 = \pm\,\mathrm{Id}$. This is also the necessary condition for M to be in $\mathrm{Sp}(2,\mathbb{R})$; hence, the inclusion is trivially verified.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{Sp}(2,\mathbb{R})$ for the considered cases.

3.2.2 Cases $\mathfrak{n}^{6,0}$ and $\mathfrak{n}^{2,4}$

The admissible modules of these two pseudo H-type Lie algebras admit three involutions each; two of them are products of four operators, while the last one is a product of three linear operators. The module has dimension 8, and can be divided in two-dimensional blocks by the first two involutions. We will study the case $\mathfrak{n}^{6,0}$, as the case $\mathfrak{n}^{2,4}$ is analogous. We will in particular prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{O}(1,\mathbb{C})$.

We start by proving $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{O}(1,\mathbb{C})$. In the tables in Subsection A.2.3 we have constructed a basis for the minimal admissible module; in particular, a basis for the first block is given by $\{v, J_1J_2v\}$. Note that $(J_1J_2)^2 = -\,\mathrm{Id}$, and can be described as a matrix as Ω_1. Moreover, J_1J_2 is the product of two operators; hence, it commutes with the matrix A. It is known that Ω_1 is a matrix representation for i; hence by Remark 1.11 it follows that $A \in \mathrm{GL}(1,\mathbb{C})$. We now need to consider the ulterior condition $A^T P_3 A = P_3$, given by the involution P_3, as described in Subsection A.2.3. By Proposition 3.4, this condition implies $A^T A = \mathrm{Id}$. Note that P_3 divides the first common eigenspaces in two non-trivial eigenspaces which are both isomorphic to $\mathrm{GL}(1,\mathbb{R})$; hence, because of Remark 3.5, we obtain that $A \in \mathrm{O}(2,\mathbb{R})$. We know that $\mathrm{GL}(1,\mathbb{C}) \cap \mathrm{O}(2,\mathbb{R}) \cong \mathrm{O}(1,\mathbb{C})$; hence,we have proven $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{O}(1,\mathbb{C})$.

Note that also in this case, as in the case $\mathfrak{n}^{3,4}$ studied above, the first eigenspace has a definite positive metric even though the metric of the admissible module of $\mathfrak{n}^{2,4}$ is neutral; hence, the condition $A^T A = \mathrm{Id}$ becomes $A^t A = \mathrm{Id}$ in both $\mathfrak{n}^{6,0}$ and $\mathfrak{n}^{2,4}$.

We now want to prove the inclusion $\mathrm{O}(1, \mathbb{C}) \subset \mathrm{Aut}^0(\mathfrak{n})$. We need to check that any matrix A in $\mathrm{O}(1, \mathbb{C})$ commutes with $J_1 J_2$ and that $A^T A = \mathrm{Id}$. The first condition follows from the fact that A is a complex matrix, and the matrix form of $J_1 J_2$ is one of the equivalent matrix form for i. The condition $A^T A = \mathrm{Id}$ resolves into $A^t A = \mathrm{Id}$, which is the defining condition of the group $\mathrm{O}(1, \mathbb{C})$; hence it is trivially satisfied.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{O}(1, \mathbb{C})$ for the considered cases.

3.2.3 Cases $\mathfrak{n}^{0,7}$ and $\mathfrak{n}^{4,3}$

The admissible modules of both $\mathfrak{n}^{0,7}$ and $\mathfrak{n}^{4,3}$ admit three involutions; all of them are products of four linear operators. These three involutions subdivide the admissible module in two-dimensional common eigenspaces. We can study $\mathfrak{n}^{0,7}$ and $\mathfrak{n}^{4,3}$ together because, as one can see from the tables of Subsection A.2.5 and A.2.6, their bases coincide. We want to prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{T}(2, \mathbb{R})$.

We first prove the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{T}(2, \mathbb{R})$. The basis of the first common eigenspace constructed in the tables is given by $\{v, J_1 J_2 J_7 v\}$. Note that $(J_1 J_2 J_7)^2 = -\mathrm{Id}$, but since this is the product of three operators, it does not commute with A. Hence, $A \in \mathrm{GL}(2, \mathbb{R})$ because of the dimension of the eigenspace. We need, moreover, to consider the relation $A^T J_1 J_2 J_7 A = J_1 J_2 J_7$, which implies $\eta A^t \eta J_1 J_2 J_7 A = J_1 J_2 J_7$, where η is the matrix representation of the metric, and lastly becomes $A^t (\eta J_1 J_2 J_7) A = \eta J_1 J_2 J_7$. In particular, since $(J_1 J_2 J_7)^2 = -\mathrm{Id}$, its matrix representation is Ω_1. Note that the metric is neutral; hence, by Lemma 3.1, we obtain $\eta \cdot \Omega_1 = -\sigma_1$. We can hence conclude that $A \in \mathrm{T}(2, \mathbb{R})$, so $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{T}(2, \mathbb{R})$.

We now want to prove the inclusion $\mathrm{T}(2, \mathbb{R}) \subset \mathrm{Aut}^0(\mathfrak{n})$. Given a matrix $M \in \mathrm{T}(2, \mathbb{R})$, we want to prove that it satisfies the condition $M^T J_1 J_2 J_7 M = J_1 J_2 J_7$; by the construction of $J_1 J_2 J_7$, we know that this condition is equivalent to ask $M^t \sigma_1 M = \sigma_1$, which is satisfied by hypothesis.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{T}(2, \mathbb{R})$ for the considered cases.

3.2.4 Cases $\mathfrak{n}^{7,1}$ and $\mathfrak{n}^{3,5}$

The admissible module of $\mathfrak{n}^{7,1}$ and $\mathfrak{n}^{3,5}$ both admit four involutions, three of which are a product of four J_i's, while the last one is a product of three operators. The first three involutions divide the module into two-dimensional eigenspaces. We will only consider $\mathfrak{n}^{7,1}$, as the two cases are analogous. We will prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{Sp}(2, \mathbb{R}) \cap \mathrm{O}(1, 1, \mathbb{R})$.

First, we prove the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{Sp}(2, \mathbb{R}) \cap \mathrm{O}(1, 1, \mathbb{R})$. As one can see from the tables in Subsection A.2.7, a basis of the first common eigenspace of the minimal admissible module of $\mathfrak{n}^{7,1}$ is $\{v, J_8 v\}$, with $J_8^2 = \mathrm{Id}$. Since J_8 is not a product of an even number of operators, A does not commute with it; hence $A \in \mathrm{GL}(2, \mathbb{R})$. We have two more relations that we need to consider: first of all, the existence of an involution which is a product of three operators implies the condition $A^T A = \mathrm{Id}$ by Proposition 3.4. Note that since the metric of the first eigenspace is neutral, the condition becomes $\eta A^t \eta A = \mathrm{Id}$, which implies $A^t \eta A = \eta$; by definition, this means that $A \in \mathrm{O}(1, 1, \mathbb{R})$. Secondly, we have the chain of implications:

$$A^T J_8 A = J_8 \Rightarrow \eta A^t \eta J_8 A = J_i \Rightarrow A^t (\eta J_8) A = \eta J_8,$$

where η is the matrix representation of the metric. In this case the metric is neutral; moreover, since $J_8^2 = \mathrm{Id}$, it is easy to see that its matrix representation is σ_1. Hence by Lemma 3.1, $\eta \cdot J_1 = -\Omega_1$. What we obtain is hence $A^t \Omega_1 A = \Omega_1$, which implies $A \in \mathrm{Sp}(2, \mathbb{R})$. We have hence proven $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{Sp}(2, \mathbb{R}) \cap \mathrm{O}(1, 1, \mathbb{R}) =: \mathrm{Sp}^*(2, \mathbb{R})$.

In order to conclude the proof, we need to show that $\mathrm{Sp}^*(2, \mathbb{R}) \subset \mathrm{Aut}^0(\mathfrak{n})$. In particular we want to show that any matrix $M \in \mathrm{Sp}^*(2, \mathbb{R})$ satisfies the conditions $M^T M = \mathrm{Id}$ and $M^t \Omega_1 M = \Omega_1$. As these are the defining conditions of $\mathrm{Sp}^*(2, \mathbb{R})$, the inclusion is trivially verified.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{Sp}^*(2, \mathbb{R})$ for the considered cases.

3.3 Four-dimensional common eigenspaces

As we can see from Table 3.1, most of the admissible modules are four-dimensional or can be divided into eigenspaces of dimension 4. In this setting, we start encountering groups constructed over quaternion numbers, but also split-quaternion and split-complex numbers. Here we also start to use extensively the isomorphisms described in Section 1.2. As usual, some of the results of our study are expected: for example, $\mathfrak{n}^{2,0}$ and $\mathfrak{n}^{0,2}$ are isomorphic and share the same automorphism group. However, we also find isomorphic automorphism groups for non-isomorphic pseudo H-type Lie algebras, as for example those of $\mathfrak{n}^{3,2}$ and $\mathfrak{n}^{2,3}$. Moreover, we find pseudo H-type Lie algebras of different dimensions which have the same automorphism groups, as for example $\mathfrak{n}^{4,0}$ and $\mathfrak{n}^{6,2}$; this happens because their mutual commuting isometric involutions subdivide the admissible module in eigenspaces of the same dimension.

s \ r	0	1	2	3	4	5	6	7	8
8	1								
7	2	(4)	8	8					
6	(4)	(4)	(4)	(4)					
5	8	8	(4)	2					
4	(4)	(4)	2	1	1				
3	8	8	(4)		2	(4)	8	8	
2	(4)	(4)	(4)	(4)	(4)	(4)	(4)	(4)	
1	2	(4)	8	8	8	8	(4)	2	
0		2	(4)	(4)	(4)	(4)	2	1	1

Table 3.1, four-dimensional cases circled.

3.3.1 Cases $\mathfrak{n}^{2,0}$ and $\mathfrak{n}^{0,2}$

We want to prove that if $\mathfrak{n} = \mathfrak{n}^{2,0}$ or $\mathfrak{n} = \mathfrak{n}^{0,2}$, then $\mathrm{Aut}^0(\mathfrak{n}) \cong \overline{\mathrm{Sp}}(2, \mathbb{C})$. It is enough to prove the statement for $\mathfrak{n} = \mathfrak{n}^{2,0}$; once we have done it, the proof for $\mathfrak{n} = \mathfrak{n}^{0,2}$ will follow, similarly to the cases treated in Section 3.2.1 for $\mathfrak{n}^{0,1}$ and $\mathfrak{n}^{1,0}$.

We start by proving $\mathrm{Aut}^0(\mathfrak{n}) \subset \overline{\mathrm{Sp}}(2, \mathbb{C})$. In the tables in Subsection A.3.1 we have constructed a basis for the admissibe module, given by $\{v, J_2 J_1 v, J_2 v, J_1 v\}$. Note that the matrix form of J_2J_1 is

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In particula,r J_1J_2 is of the form I described in Proposition 1.12. Hence, since we need to impose that A commutes with J_1J_2, we can apply the proposition; if we denote with $\mathcal{A}$ the matrix A through the isomorphism (1.5), then $\mathcal{A} \in \mathrm{GL}(2,\mathbb{C})$. We can observe that the matrix form of J_2 is $-\Omega_2 \in \mathrm{GL}(4,\mathbb{R})$. The metric is positive definite, and Ω_2 is mapped via the isomorphism (1.5) into $\Omega_1 \in \mathrm{GL}(2,\mathbb{C})$. Hence, the condition $A^T J_2 A = J_2$ becomes $\overline{\mathcal{A}^t}\Omega_1\mathcal{A} = \Omega_1$ and we can conclude that $\mathcal{A} \in \overline{\mathrm{Sp}}(2,\mathbb{C})$. We have then proven that $\mathrm{Aut}^0(\mathfrak{n}) \subset \overline{\mathrm{Sp}}(2,\mathbb{C})$.

In order to prove the inclusion $\overline{\mathrm{Sp}}(2,\mathbb{C}) \subset \mathrm{Aut}^0(\mathfrak{n})$ it is enough to prove that a matrix $M \in \mathrm{Sp}(2,\mathbb{C})$ commutes with J_1J_2 and satisfies the condition $M^T J_1 M = J_1$. The first condition follows from the fact that M has complex entries and the matrix form of J_1J_2 is one of the equivalent matrix descriptions for i. The second one follows from the fact that J_2 can be written as a matrix as Ω_1 via the isomorphism (1.5), so the condition is trivially satisfied.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \overline{\mathrm{Sp}}(2,\mathbb{C})$ for the considered cases.

3.3.2 Case $\mathfrak{n}^{1,1}$

The admissible module of $\mathfrak{n}^{1,1}$ is four-dimensional and does not admit any involution. We want to prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{Sp}(2,\mathbb{SC})$.

Let us start by proving the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{Sp}(2,\mathbb{SC})$. In the tables of Subsection A.3.3 we have found a basis for the admissible module, given by $\{v, J_1v, J_2v, J_1J_2v\}$, where $i^* := J_1J_2$ satisfies $i^{*2} = \mathrm{Id}$. Since i^* is the product of two linear operators, A must commute with it. In particular, one can see that the matrix form of i^* is as in I from Proposition 1.15; hence, because of the same theorem, the image of A via the isomorphism (1.10), which we call $\mathcal{A}$, belongs to $\mathrm{GL}(2,\mathbb{SC})$. Observe that the matrix form of J_1 is $\sigma_2 \in \mathrm{GL}(4,\mathbb{R})$. Since the metric of the module is neutral, $\eta \cdot J_1 = \Omega_2 \in \mathrm{GL}(4,\mathbb{R})$ by Lemma 3.2; moreover,Ω_2 is mapped to $\Omega_1 \in \mathrm{GL}(2,\mathbb{C})$ via the isomorphism (1.10). The condition $A^T J_1 A = J_1$ becomes hence $\mathcal{A}^t\Omega_1\mathcal{A} = \Omega_1$; this implies that $\mathcal{A} \in \mathrm{Sp}(2,\mathbb{SC})$. We can moreover observe that $J_1J_2 \cdot J_1 = -J_2$, hence, by Lemma 3.1, J_2 does not provide any ulterior conditions. We have hence proven that $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{Sp}(2,\mathbb{SC})$.

We now want to prove that $\mathrm{Sp}(2,\mathbb{SC}) \subset \mathrm{Aut}^0(\mathfrak{n})$. Consider $M \in \mathrm{Sp}(2,\mathbb{SC})$; since the matrix form of J_1J_2 is one of the equivalent matrix which describes i^*, it follows trivially that M commutes with it. Note that, once we map the product $\eta \cdot J_1$ into $\mathrm{Sp}(2,\mathbb{SC})$ via the isomorphism (1.10), the condition $M^t\eta \cdot J_1 M = \eta \cdot J_1$ is satisfied because M is symplectic.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{Sp}(2,\mathbb{SC})$ for the considered case.

3.3.3 Case $\mathfrak{n}^{3,0}$

The admissible module of $\mathfrak{n}^{3,0}$ is four-dimensional and admits an involution which is a product of three linear operators. We want to prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{U}(1,\mathbb{H})$.

Let us first prove the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{U}(1,\mathbb{H})$. As one can see in the tables in Subsection A.3.4, we have constructed the basis $\{v, J_1J_2v, J_2J_3v, J_3J_1v\}$, where $i := J_1J_2$, $j := J_2J_3$ and $k := J_3J_1$ all satisfy the conditions to be quaternion units; since they all are product of an even number of linear operators, they all commute with A. In particular, by Remark 1.11 we know that $A \in \mathrm{GL}(1,\mathbb{H})$. The module admits an involution which is a product of three operators and acts as Id on the entire module. Hence, we obtain the condition $A^T A = \mathrm{Id}$, which becomes $\overline{A^t} A = \mathrm{Id}$ since the metric is positive. This implies that $A \in \mathrm{U}(1, \mathbb{H})$. We have hence proven the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{U}(1, \mathbb{H})$.

We now want to prove $\mathrm{U}(1, \mathbb{H}) \subset \mathrm{Aut}^0(\mathfrak{n})$. This inclusion follows trivially: since any element $M \in \mathrm{U}(1, \mathbb{H})$ is simply a quaternion number, it trivially commutes with any matrix; moreover, being in $\mathrm{U}(1, \mathbb{H})$ grants the fullfillment of the condition $\overline{A^t} A = \mathrm{Id}$.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{U}(1, \mathbb{H})$ for the considered cases.

3.3.4 Cases $n^{1,2}$, $n^{3,2}$ and $n^{2,3}$

The admissible module of $\mathfrak{n}^{1,2}$ is four-dimensional and admits a single involution, which is a product of three linear operators. The admissible modules of $\mathfrak{n}^{3,2}$ and $\mathfrak{n}^{2,3}$ both admit two involutions; one of them is a product of three operators, while the other one is a product of four operators and divides the module into two four-dimensional common eigenspaces. We will study $\mathfrak{n}^{1,2}$, as the two other cases are analogous; our aim is to prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{GL}(1, \mathbb{SH}) \cap \mathrm{O}(1,1,\mathbb{C})$.

In the tables in Subsection A.3.5, we have constructed the basis $\{v, J_2J_3v, J_1J_2v, J_3J_1v\}$, where $i^* = J_2J_3$, $j^* = J_1J_2$ and $k^* = J_3J_1$ all satisfy the conditions to be split-quaternion units. Since i^*, j^* and k^* are all product of an even number of operators, they commute with A; this is equivalent to say that $A \in \mathrm{GL}(1, \mathbb{SH})$, by Remark 1.11. The existence of an involution which is a product of three linear operators imposes the condition $A^T A = \mathrm{Id}$ by Propostion 3.4; this resolves into $A^t \eta A = \eta$. Note that the said involution divides the module in two non-trivial eigenspaces, each of which isomorphic to $\mathrm{GL}(1, \mathbb{C})$. We can hence apply Remark 3.5 and conclude that $A \in \mathrm{O}(1,1,\mathbb{C})$, since the metric is neutral.We have hence proven the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{GL}(1, \mathbb{SH}) \cap \mathrm{O}(1,1,\mathbb{C}) =: \mathrm{GL}^*_{\mathbb{C}}(1, \mathbb{SH})$.

We want to prove the inclusion $\mathrm{GL}^*_{\mathbb{C}}(1, \mathbb{SH}) \subset \mathrm{Aut}^0(\mathfrak{n})$. Let $M \in \mathrm{GL}(1, \mathbb{SH}) \cap \mathrm{O}(1,1,C)$. It trivially commutes with any triple of split-quaternion units, so in particular it commutes with J_2J_3, J_1J_2 and J_3J_1. Moreover, the condition $M^T \eta M = \eta$ is trivially satisfied by any matrix in $\mathrm{O}(1,1,\mathbb{C})$.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{GL}(1, \mathbb{SH}) \cap \mathrm{O}(1,1,\mathbb{C})$ for the considered cases.

3.3.5 Cases $\mathfrak{n}^{4,0}$, $\mathfrak{n}^{2,2}$, $\mathfrak{n}^{0,4}$, $\mathfrak{n}^{6,2}$ and $\mathfrak{n}^{2,6}$

The admissible module associated to the pseudo H-type Lie algebras $\mathfrak{n}^{4,0}$, $\mathfrak{n}^{2,2}$ and $\mathfrak{n}^{0,4}$ are eight-dimensional and admit one involution each, which is a product of four linear operators and divides the admissible module in two four-dimensional eigenspaces. The admissible modules of $\mathfrak{n}^{6,2}$ and $\mathfrak{n}^{2,6}$ are both 32-dimensional and admit three involutions which are a product of four linear operators and divide the module in eight four-dimensional eigenspaces. We will study the case $\mathfrak{n}^{4,0}$, as the procedure is the same. We will prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{GL}(1, \mathbb{H})$.

We start by proving the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{GL}(1, \mathbb{H})$. In the tables in Subsection A.1.5 we constructed a basis $\{v, J_1J_2v, J_2J_3v, J_3J_1v\}$, where $i := J_1J_2$, $j := J_2J_3$ and $k := J_3J_1$ all satisfy the conditions to be quaternion units. Since i, j and k are products of two linear operators, A must commute with them; hence, by Remark 1.11, $A \in \mathrm{GL}(1, \mathbb{H})$. Since there is no other condition which A needs to satisfy, we conclude that $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{GL}(1, \mathbb{H})$.
The admissible module of $\mathfrak{n}^{6,2}$ and $\mathfrak{n}^{2,6}$ have as a basis $\{v, J_1J_2v, J_1J_3J_5J_7v, J_2J_3J_5J_7v\}$ (see Subsections A.3.21 and A.3.22). In particular, products of four linear operators appear in the basis. Note that the condition $A \prod_{i=1}^{p} J_i = \prod_{i=1}^{p} J_i A$ described in Lemma 2.16 holds for every even p; hence, the previous computations still make sense.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{GL}(1, \mathbb{H})$ for the considered cases.

3.3.6 Cases $\mathfrak{n}^{5,0}$ and $\mathfrak{n}^{1,4}$

The minimal admissible modules of $\mathfrak{n}^{5,0}$ and $\mathfrak{n}^{1,4}$ are both eight-dimensional and admit two involutions; one of them is a product of three linear operators, while the other one is the product of four operators and divides the module into two four-dimensional blocks. We can study the case $\mathfrak{n}^{5,0}$, as considering $\mathfrak{n} = \mathfrak{n}^{1,4}$ leads to analogous computations. In particular, we want to prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{GL}(1, \mathbb{H}) \cap \mathrm{O}(2, \mathbb{C})$.

Let us first prove the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{GL}(1, \mathbb{H}) \cap \mathrm{O}(2, \mathbb{C})$. As one can see in the tables in Subsection A.3.9, we have constructed the basis $\{v, J_1J_2v, J_2J_3v, J_3J_1v\}$, where $i := J_1J_2$, $j := J_2J_3$ and $k := J_3J_1$ all satisfy the conditions to be quaternion units; since they all are product of an even number of linear operators, they all commute with A. In particular, by Remark 1.11 we know that $A \in \mathrm{GL}(1, \mathbb{H})$. The module admits an involution which is a product of three operators; this said involution divides the first common eigenspace in two eigenspaces, each isomorphic to $\mathrm{GL}(1, \mathbb{C})$; hence, we can apply Remark 3.5, and obtain the condition $A \in \mathrm{O}(2, \mathbb{C})$, since the metric is sign definite. We have hence proven the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{GL}(1, \mathbb{H}) \cap \mathrm{O}(2, \mathbb{C}) =: \mathrm{GL}^*_{\mathbb{C}}(1, \mathbb{H})$.
We can observe that, although $\mathfrak{n}^{1,4}$ has an admissible module of neutral metric, the first block has a sign definite metric, we can still conclude $A \in \mathrm{O}(2, \mathbb{C})$.

We now want to prove $\mathrm{GL}^*_{\mathbb{C}}(1, \mathbb{H}) \subset \mathrm{Aut}^0(\mathfrak{n})$. This inclusion follows trivially: since any element $M \in \mathrm{GL}^*_{\mathbb{C}}(1, \mathbb{H})$ is simply a quaternion number, it trivially commutes with any matrix representation of i, j and k; moreover, being in $\mathrm{O}(2, \mathbb{C})$ grants the fullfillment of the condition $A^tA = \mathrm{Id}$.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{GL}^*_{\mathbb{C}}(1, \mathbb{H})$ for the considered cases.

3.3.7 Cases $\mathfrak{n}^{0,6}$ and $\mathfrak{n}^{4,2}$

The minimal admissible module of both $\mathfrak{n}^{0,6}$ and $\mathfrak{n}^{4,2}$ admit two involutions which are a product of four linear operators and divide the admissible module in four dimensional common eigenspaces. We will consider the numerical example of $\mathfrak{n}^{0,6}$ as $\mathfrak{n}^{4,2}$ is analogous. We will prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{GL}(2, \mathbb{C}) \cap \mathrm{T}(4, \mathbb{R})$.

We start by proving the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{GL}(2, \mathbb{C}) \cap \mathrm{T}(4, \mathbb{R})$. As one can see in the tables in Section A.3.13 , a basis for the first common eigenspace is given by $\{v, J_2J_1v, J_2J_3J_5v, J_1J_3J_5v\}$, where $i := J_1J_2$ satisfies $i^2 = -\mathrm{Id}$ and the other two terms $S := J_2J_3J_5$ and $Q := J_1J_3J_5$ squared are again $-\mathrm{Id}$. We separate i from S and Q because the first one is the only one which is a product of an even number of terms; in particular we know that i commutes with A, and the matrix form of i is the same as in Proposition 1.13. We can hence apply the preposition, and the image of the matrix A via the isomorphism 1.5 - namely $\mathcal{A}$ - belongs to $\mathrm{GL}(2, \mathbb{C})$. Consider now S; its matrix form is given by Ω_2. Since S is product of three linear operators, we need to apply the condition $A^TSA = S$, which resolves into $A^t(\eta S)A = (\eta S)$. We know from Lemma 3.2 that $\eta \cdot \Omega_2 = -\sigma_2$; note that we can't map σ_2 through the isomorphism 1.7; we hence conclude that $A \in \mathrm{T}(4, \mathbb{R})$. We have hence proven $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{GL}(2, \mathbb{C}) \cap \mathrm{T}(4, \mathbb{R}) =: \mathrm{GL}^\sharp(2, \mathbb{C})$.
Note that Q doesn't give any more informations: in fact

$$i \cdot S = J_1J_2J_1J_3J_5 = -J_1J_3J_5 = -Q$$

hence by Lemma 3.1 the condition $A^TQA = Q$ follows immediately.

We will now prove the inclusion $\mathrm{GL}(2, \mathbb{C}) \cap \mathrm{T}(4, \mathbb{R}) \subset Aut^0(\mathfrak{n})$. Consider $M \in \mathrm{GL}(2, \mathbb{C}) \cap \mathrm{T}(4, \mathbb{R})$; since it has complex entries, it commutes with J_1J_2 as it has a matrix form which is equivalent to a matrix form of i. The condition $M^TSM = S$ is trivially satisfied by the fact that $\eta \cdot S = \sigma_2$ and $M \in \mathrm{T}(4, \mathbb{R})$.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{GL}(2,\mathbb{C}) \cap \mathrm{T}(4,\mathbb{R})$ for the considered cases.

3.3.8 Cases $\mathfrak{n}^{6,1}$, $\mathfrak{n}^{1,6}$, $\mathfrak{n}^{5,2}$ and $\mathfrak{n}^{2,5}$

The minimal admissible modules of $\mathfrak{n}^{6,1}$ and $n^{2,5}$ both admit four involutions; three of them are a product of four linear operators, while the last one is a product of three operators. We can study the two cases together, as the basis of the first common eigenspace is the same. The cases $\mathfrak{n}^{1,6}$ and $\mathfrak{n}^{5,2}$ follows, since we know the isomorphisms $\mathfrak{n}^{1,6} \cong \mathfrak{n}^{6,1}$ and $\mathfrak{n}^{5,2} \cong \mathfrak{n}^{2,5}$ by [FM17]. We will prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \overline{\mathrm{Sp}}(2,\mathbb{C}) \cap \mathrm{O}(1,1,\mathbb{C})$.

We begin by proving the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \overline{\mathrm{Sp}}(2,\mathbb{C}) \cap \mathrm{O}(1,1,\mathbb{C})$. As shown in the tables in Subsections A.3.17 and A.3.18, a possible basis is given by $\{v, J_1J_2v, J_7v, J_1J_2J_7v\}$, where $(J_1J_2)^2 = -\mathrm{Id}$, $J_7^2 = \mathrm{Id}$ and $(J_1J_2J_7)^2 = -\mathrm{Id}$. The matrix form of J_1J_2 is as I of Proposition 1.12; since A needs to commute with J_1J_2, we can apply the isomoprhism (1.5); in particular, the image of A through the isomorphism, which we can call $\mathcal{A}$, belongs to $\mathrm{GL}(2,\mathbb{C})$. The matrix representation of J_7 is σ_2, and since we have a neutral metric, we conclude that the condition $A^TJ_7A = J_7$ becomes $\overline{\mathcal{A}^t}\Omega_1\mathcal{A} = \Omega_1$ by means of Lemma 3.2 and the isomorphism (1.5), which maps $\Omega_2 \in \mathrm{GL}(4,\mathbb{R})$ to $\Omega_1 \in \mathrm{GL}(2,\mathbb{C})$. Hence, $\mathcal{A} \in \overline{\mathrm{Sp}}(2,\mathbb{C})$. Observe that $J_1J_2J_7$ does not provide any other condition, as $(J_1J_2)J_7 = J_1J_2J_7$ and one can apply Lemma 3.1. To complete the case, we need to consider the remaining involution product of three operators. Observe that said involution has two non-trvial eigenspaces, which are both isomorphic to $\mathrm{GL}(1,\mathbb{C})$; we can hence apply Remark 3.5, and obtain $A \in \mathrm{O}(1,1,\mathbb{C})$, since the metric is neutral. We have hence proven the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \overline{\mathrm{Sp}}(2,\mathbb{C}) \cap \mathrm{O}(1,1,\mathbb{C}) =: \overline{\mathrm{Sp}}^*(2,\mathbb{C})$.

We now want to prove the inclusion $\overline{\mathrm{Sp}}^*(2,\mathbb{C}) \subset \mathrm{Aut}^0(\mathfrak{n})$. Consider $M \in \overline{\mathrm{Sp}}^*(2,\mathbb{C})$; since it has complex entries, it commutes with J_1J_2, whose matrix representation is isomorphic to one of the equivalent matrix representation of i. Asking that A satisfies the condition $M^TJ_7M = J_7$ is equivalent, by means of the isomorphism (1.5), to require $\overline{M^t}\Omega_1M = \Omega_1$, which is trivially satisfied by construction. Moreover, the orthogonality condition is satisfied by the fact that M belongs to $\mathrm{O}(1,1,\mathbb{C})$.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \overline{\mathrm{Sp}}^*(2,\mathbb{C})$ for the considered cases.

3.3.9 Cases $\mathfrak{n}^{1,7}$ and $\mathfrak{n}^{5,3}$

THe admissible module of $\mathfrak{n}^{1,7}$ and $\mathfrak{n}^{5,3}$ both admit three involutions which are a product of four linear operators and subdivide the 32-dimensional minimal admissible module into four-dimensional eigenspaces. We will study the case $\mathfrak{n}^{1,7}$, since the other one behaves very similarly. We will prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \overline{\mathrm{Sp}}(2,\mathbb{C})$.

We start by proving the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \overline{\mathrm{Sp}}(2,\mathbb{C})$. A basis for the first common eigenspace is given by $\{v, J_1v, J_1J_6J_7J_8v, J_6J_7J_8v\}$, where $(J_1J_6J_7J_8)^2 = -\mathrm{Id}$ (see Subsection A.3.19). Since $J_1J_6J_7J_8$ is the product of four linear operators, it commutes with A; in particular its matrix form is as J of Proposition 1.13. By the same proposition, the image of A via the isomorphism (1.7), which we call $\mathcal{A}$, belongs to $\mathrm{GL}(2,\mathbb{C})$. We impose the condition $A^TJ_1A = J_1$, which becomes $A^t(\eta \cdot J_1)A = \eta \cdot J_1$. One can write J_1 and η as four-dimensional matrices; their product is given by

$$\eta \cdot J_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which via the isomorphism (1.7) is isomorphic to $\Omega_1 \in \mathrm{GL}(2,\mathbb{C})$. Hence, the condition becomes $\overline{\mathcal{A}^t}\Omega_1\mathcal{A} = \Omega_1$, which implies $\mathcal{A} \in \overline{\mathrm{Sp}}(2,\mathbb{C})$. We have then proven the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \overline{\mathrm{Sp}}(2,\mathbb{C})$.

In order to prove the inclusion $\overline{\mathrm{Sp}}(2,\mathbb{C}) \subset \mathrm{Aut}^0(\mathfrak{n})$, we consider $M \in \overline{\mathrm{Sp}}(2,\mathbb{C})$. Since it is a complex matrix, it commutes with any linear complex unit, so in particular it commutes with $J_1J_6J_7J_8$, which has a matrix representation isomoprhic to i. By construction, moreover, the condition $M^TJ_1M = J_1$ is satisfied, since $M \in \overline{\mathrm{Sp}}(2,\mathbb{C})$, once we recall that $\overline{M^t} \in \mathrm{GL}(2,\mathbb{C})$ is mapped to $M^t \in \mathrm{GL}(4,\mathbb{R})$.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \overline{\mathrm{Sp}}(2,\mathbb{C})$ for the considered cases.

3.3.10 Cases $\mathfrak{n}^{3,6}$ and $\mathfrak{n}^{7,2}$

The admissible modules of $\mathfrak{n}^{3,6}$ and $\mathfrak{n}^{7,2}$ both admit four involutions; three of them are a product of four linear operators, while the last one is a product of three linear operators. The first three involutions divide the module into four-dimensional common eigenspaces. We will consider only the case $\mathfrak{n} = \mathfrak{n}^{3,6}$, since $\mathfrak{n} = \mathfrak{n}^{7,2}$ behaves analogously. We will prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{GL}(1,\mathbb{H}) \cap \mathrm{O}(1,1,\mathbb{C})$.

We start by proving $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{GL}(1,\mathbb{H}) \cap \mathrm{O}(1,1,\mathbb{C})$. One can see from the tables in Subsection A.3.23 that a basis for the first eigenspace is $\{v, J_1J_2v, J_1J_4J_7J_8v, J_2J_4J_7J_8v\}$. We define $i := J_1J_2$, $j := J_1J_4J_7J_8$ and $k := J_2J_4J_7J_8$ and we can observe that they satisfy all the conditions in order to be quaternion units. In particular, since i, j and k are all product of an even number of operators, it follows that A commutes with the three of them; because of Remark 1.11, this implies that A it is isomorphic to some $\mathcal{A} \in \mathrm{GL}(1,\mathbb{H})$. The involution which is a product of three operators divides the block in two non-trivial eigenspaces, both of which are isomorphic to $\mathrm{GL}(1,\mathbb{C})$. Hence, by Remark 3.5, we can conclude that $A \in \mathrm{O}(1,1,\mathbb{C})$, since the metric is neutral.We have proven the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{GL}(1,\mathbb{H}) \cap \mathrm{O}(1,1,\mathbb{C}) =: \mathrm{GL}^*_{\mathbb{C}}(1,\mathbb{H})$.

We want to prove the inclusion $\mathrm{GL}^*_{\mathbb{C}}(1,\mathbb{H}) \subset \mathrm{Aut}^0(\mathfrak{n})$. Let $M \in \mathrm{GL}(1,\mathbb{H}) \cap \mathrm{O}(1,1,C)$. It trivially commutes with any triple of quaternion units, so in particular it commutes with J_1J_2, $J_1J_4J_7J_8$ and $J_2J_4J_7J_8$. Moreover, the orthogonality condition is trivially satisfied by any matrix in $\mathrm{O}(1,1,\mathbb{C})$.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{GL}^*_{\mathbb{C}}(1,\mathbb{H})$ for the considered cases.

3.4 Eight-dimensional common eigenspaces

From Table 3.1, we can distinguish twelve eight-dimensional cases, which correspond to pseudo H-type Lie algebras featuring very different automorphism groups. Our analysis shows that these are all subgroups of $\mathrm{GL}(2,\mathbb{H})$ or $\mathrm{GL}(2,\mathbb{SH})$.

3.4.1 Case $\mathfrak{n}^{0,3}$

The minimal admissible module of $\mathfrak{n}^{0,3}$ does not admit any involution, hence we will not consider its common eigenspaces. We will prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{T}(2,\mathbb{H})$.

We begin by proving the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{U}(1,\mathbb{H})$. A basis for the admissible module is $\{v, J_2J_1v, J_3J_2v, J_1J_3v, J_1J_2J_3v, J_3v, J_1v, J_2v\}$ (see tables in Subsection A.4.1). The products $i := J_2J_1$, $j := J_3J_2$ and $k := J_1J_3$ all satisfy the conditions to be quaternion units; moreover, the matrix forms of i and j are as I and J of Proposition 1.17. Since we require A to commute with i, j and k, we can apply Proposition 1.17 and obtain that the image $\mathcal{A}$ of A via the isomorphism (1.13) belongs to $\mathrm{GL}(2,\mathbb{H})$. Observe then that $(J_1J_2J_3)^2 = -\mathrm{Id}$, which has $\Omega_4 \in \mathrm{GL}(8,\mathbb{R})$ as matrix representation. The metric of

s									
8	1								
7	2	4	(8)	(8)					
6	4	4	4	4					
5	(8)	(8)	4	2					
4	4	4	2	1	1				
3	(8)	(8)	4		2	4	(8)	(8)	
2	4	4	4	4	4	4	4	4	
1	2	4	(8)	(8)	(8)	(8)	4	2	
0		2	4	4	4	4	2	1	1
	0	1	2	3	4	5	6	7	8 r

Table 3.1, eight-dimensional cases circled.

the admissible module is neutral; the product $\eta \cdot J_1J_2J_3 = \sigma_4$ by Lemma 3.2; via the isomorphism (1.13), σ_4 is mapped into $\sigma_1 \in \mathrm{GL}(2,\mathbb{H})$; hence, the condition $A^T J_1J_2J_3 A = J_1J_2J_3$ becomes $\overline{\mathcal{A}^t}\sigma_1\mathcal{A} = \sigma_1$. This implies that $\mathcal{A} \in \mathrm{T}(2,\mathbb{H})$. We have hence proven the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \mathrm{U}(1,\mathbb{H})$.

We now want to prove the inclusion $\mathrm{T}(2,\mathbb{H}) \subset \mathrm{Aut}^0(\mathfrak{n})$. Let $M \in \mathrm{T}(2,\mathbb{H})$; then it commutes with i, j and k; we know that J_1J_2, $J_1J_4J_7J_8$ and $J_2J_4J_7J_8$ all satisfy the properties to be quaternion units, hence A commutes with them. The condition $M^T J_1J_2J_3M = J_1J_2J_3$ becomes $M^t\sigma_4 M = \sigma_4$ by construction of $J_1J_2J_3$; applying the isomorphism in Proposition 1.17, the new condition is trivially satisfied by any $M \in \mathrm{T}(2,\mathbb{H})$, once we recall that $\overline{M^t} \in \mathrm{GL}(2,\mathbb{H})$ is mapped to $M^t \in \mathrm{GL}(8,\mathbb{R})$.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \mathrm{T}(2,\mathbb{H})$ for the considered case.

3.4.2 Case $\mathfrak{n}^{2,1}$

The admissible module of $\mathfrak{n}^{2,1}$ is eight-dimensional and does not admit any involution. We want to prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \widetilde{\mathrm{Sp}}(2,\mathbb{SH})$.

We start by proving the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \widetilde{\mathrm{Sp}}(2,\mathbb{SH})$. In the tables in Subsection A.4.2 we have found a basis $\{v, J_3v, J_2J_3v, J_3J_1v, J_1J_2J_3v, J_1J_2v, J_1v, J_2J_3v\}$, where $i^* = J_1J_2$, $j^* = J_1J_3$ and $k^* = J_2J_3$ satisfy the conditions to be split-quaternion units. In particular, i^* and k^* have the same matrix form as I^* and J^* of Proposition 1.20. If we call $\mathcal{A}$ the image of A via the isomorphism (1.17), then $\mathcal{A} \in \mathrm{GL}(2,\mathbb{SH})$ by Proposition 1.20. We can moreover observe that the matrix form of $\eta \cdot J_1J_2J_3$ is

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

which, via the isomorphism (1.17, is mapped to $k^*\cdot\Omega_1$. The condition $A^T J_1J_2J_3A = J_1J_2J_3$ becomes hence $\widetilde{\mathcal{A}^t}\Omega_1\mathcal{A} = \Omega_1$; hence, we can conclude that $\mathcal{A} \in \widetilde{\mathrm{Sp}}(2,\mathbb{SH})$. We have hence proven the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \widetilde{\mathrm{Sp}}(2, \mathbb{SH})$.

We will now prove the inclusion $\widetilde{\mathrm{Sp}}(2, \mathbb{SH}) \subset \mathrm{Aut}^0(\mathfrak{n})$.Let $M \in \overline{\mathrm{Sp}}(2, \mathbb{SH})$; it commutes with any triple of split-quaternion units, such as J_1J_2, J_2J_3 and J_3J_1. By construction, moreover, the condition $M^T J_1J_2J_3M = J_1J_2J_3$ is satisfied, since $M \in \widetilde{\mathrm{Sp}}(2, \mathbb{SH})$, once we recall that $\widetilde{M^t} \in \mathrm{GL}(2, \mathbb{SH})$ is mapped to $M^t \in \mathrm{GL}(8, \mathbb{R})$.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \widetilde{\mathrm{Sp}}(2, \mathbb{SH})$ for the considered cases.

3.4.3 Case $\mathfrak{n}^{1,3}$

We want to prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \widetilde{\mathrm{Sp}}(2, \mathbb{SH}) \cap \mathrm{O}(1, 1, \mathbb{SH})$ for $\mathfrak{n} = \mathfrak{n}^{1,3}$.

We start by proving the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \widetilde{\mathrm{Sp}}(2, \mathbb{SH}) \cap \mathrm{O}(1, 1, \mathbb{SH})$. We can follow the proof of Subsection 3.4.2, and prove that $\mathcal{A} \in \overline{\mathrm{Sp}}(2, \mathbb{SH})$, with $\mathcal{A}$ the image if A via the isomorphism (1.17). The existence of an involution which is a product of three operators and which divides the module in two non-trivial eigenspaces, each isomorphic to $\mathrm{GL}(1, \mathbb{SH})$, provides the extra condition $A \in \mathrm{O}(1, 1, \mathbb{SH})$, since the metric is neutral. We can conclude that $\mathrm{Aut}^0(\mathfrak{n}) \subset \widetilde{\mathrm{Sp}}(2, \mathbb{SH}) \cap \mathrm{O}(1, 1, \mathbb{SH})$.

In order to prove $\widetilde{\mathrm{Sp}}(2, \mathbb{SH}) \cap \mathrm{O}(4, 4, \mathbb{R}) \subset \mathrm{Aut}^0(\mathfrak{n})$ one can follow the proof of the previous case. The extra condition $A^T P_1 A = P_1$ resolves into $A^T A = \mathrm{Id}$, which is satisfied by construction by any matrix in $\mathrm{O}(1, 1, \mathbb{SH})$.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \widetilde{\mathrm{Sp}}(2, \mathbb{SH}) \cap \mathrm{O}(1, 1, \mathbb{H})$ for the considered cases.

3.4.4 Cases $\mathfrak{n}^{0,5}$, $\mathfrak{n}^{4,1}$, $\mathfrak{n}^{2,7}$ and $\mathfrak{n}^{6,3}$

The admissible modules of $\mathfrak{n}^{0,5}$ and $\mathfrak{n}^{4,1}$ admit one single involution which is a product of four linear operators. The admissible modules of $\mathfrak{n}^{2,7}$ and $\mathfrak{n}^{6,3}$ admit three involutions, all of which are a product of four operators. We will consider the case $\mathfrak{n}^{0,5}$, as the other ones are analogous. We will prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \overline{\mathrm{Sp}}(2, \mathbb{H})$.

We will start by proving the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \overline{\mathrm{Sp}}(2, \mathbb{H})$. In the tables in Subsection A.4.5 we find a basis for the first common eigenspace as

$$\{v, J_1J_2v, J_2J_4v, J_1J_4v, J_5v, J_1J_2J_5v, J_2J_4J_5v, J_1J_4J_5v\},$$

where $i := J_1J_2$, $j := J_2J_4$ and $k := J_1J_4$ satisfy the conditions to be quaternion units. Since all three of them are products of two elements, the matrix A commutes with them; in particular, J_1J_2 and J_2J_4 are in the forms I and J of Proposition 1.17, hence its image $\mathcal{A}$ via the isomorphism (1.13) belongs to $\mathrm{GL}(2, \mathbb{H})$. We need to consider the condition $A^T J_5 A = J_5$; one can observe that the matrix representation of J_5 is σ_4. The metric of the eigenspace is neutral, hence, by Lemma 3.2, we obtain the condition $A^T \Omega_4 A = \Omega_4$ which via the same isomorphism becomes $\overline{A^t} \Omega_1 A = \Omega_1$. We can conclude that $A \in \overline{\mathrm{Sp}}(2, \mathbb{H})$. We have hence proven that $\mathrm{Aut}^0(\mathfrak{n}) \subset \overline{\mathrm{Sp}}(2, \mathbb{H})$.
Observe that $J_1J_2J_5$, $J_2J_4J_5$ and $J_1J_4J_5$ can all be obtained by the product of i, j and k with J_5; hence, by Lemma 3.1, they do not provide any new information. Moreover, for the cases $\mathfrak{n}^{2,7}$ and $\mathfrak{n}^{6,3}$ we need to use the isomorphism (1.15), since A commutes with J_2J_1 and $J_1J_3J_6J_8$, which are of the form I and J in Proposition 1.18. Through that isomorphism the product $\eta \cdot J_2J_1J_9$ is mapped to Ω_1, so the outcome does not change.

We now want to prove the inclusion $\overline{\mathrm{Sp}}(2, \mathbb{H}) \subset \mathrm{Aut}^0(\mathfrak{n})$. In order to do so, one can simply follow the proof of Subsection 3.4.2, replacing split-quaternion numbers with quaternion numbers.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \overline{\mathrm{Sp}}(2, \mathbb{H})$ for the considered cases.

3.4.5 Case $n^{3,1}$

The admissible module of $\mathfrak{n}^{3,1}$ admits a single involution, which is a product of three operators. We will prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \overline{\mathrm{Sp}}(2,\mathbb{H}) \cap \mathrm{O}(1,1,\mathbb{H})$.

First, we prove the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \overline{\mathrm{Sp}}(2,\mathbb{H}) \cap \mathrm{O}(1,1,\mathbb{H})$. We have constructed a basis for the admissible module of $\mathfrak{n}^{3,1}$ in the tables in Subsection A.4.4; this is given by $\{v, J_1J_2v, J_2J_3v, J_3J_1v, J_4v, J_1J_4v, J_2J_4v, J_3J_4v\}$, where $i := J_1J_2$, $j := J_2J_3$ and $k := J_3J_1$ are quaternion units. In particular, the matrix forms of J_1J_2 and J_2J_3 are as I and J from Proposition 1.17. Hence, the image $\mathcal{A}$ of A via the isomorphism (1.13) belongs to $\mathrm{GL}(2,\mathbb{H})$. We now need to impose the condition $A^TJ_4A = J_4$. The element J_4 has σ_4 as matrix form and the admissible module has a neutral metric. By Lemma 3.2, the condition $A^TJ_4A = J_4$ becomes $A^t\Omega_4A = \Omega_4$, which is mapped via the isomorphism (1.13) to the condition $\overline{A^t}\Omega_1A = \Omega_1$; hence, $A \in \overline{\mathrm{Sp}}(2,\mathbb{H})$. This admissible module admits an involution which is a product of three linear operators; in particular, it divides the module in two non-trivial eigenspaces, both of which are isomorphic to $\mathrm{GL}(2,\mathbb{H})$. Hence, by Remark 3.5, we can conclude that $A \in \mathrm{O}(1,1,\mathbb{H})$ since the metric is neutral. We have hence proven the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \overline{\mathrm{Sp}}(2,\mathbb{H}) \cap \mathrm{O}(1,1,\mathbb{H}) =: \overline{\mathrm{Sp}}^*(2,\mathbb{H})$.

In order to prove $\overline{\mathrm{Sp}}^*(2,\mathbb{H}) \subset \mathrm{Aut}^0(\mathfrak{n})$, it is enough to follow the proof in Subsection 3.4.3, replacing split-quaternion numbers with quaternion numbers, and considering the standard quaternion conjugation $\overline{}$ instead of the split-quaternion conjugation $\widetilde{}$.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \overline{\mathrm{Sp}}^*(2,\mathbb{H})$ for the considered cases.

3.4.6 Cases $\mathfrak{n}^{5,1}$, $\mathfrak{n}^{1,5}$, $\mathfrak{n}^{7,3}$ and $\mathfrak{n}^{3,7}$

The minimal admissible modules of these four pseudo H-type Lie algebras look very different from each other; nevertheless, similarities emerge once we consider their first common eigenspaces. In particular, all the considered admissible modules admit one involution which is a product of three linear operators. The admissible modules of $\mathfrak{n}^{5,1}$ and $\mathfrak{n}^{1,5}$ both admit one more involution which is a product of four linear operators, while the admissible modules of $\mathfrak{n}^{3,7}$ and $\mathfrak{n}^{7,3}$ admit three involutions which are products of four linear operators. We will study the case $\mathfrak{n}^{5,1}$, as the other four are completely analogous. In particular, we will prove that $\mathrm{Aut}^0(\mathfrak{n}) \cong \overline{\mathrm{Sp}}(2,\mathbb{H}) \cap \mathrm{U}(1,1,\mathbb{H})$.

First, we prove the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \overline{\mathrm{Sp}}(2,\mathbb{H}) \cap \mathrm{U}(1,1,\mathbb{H})$. We have constructed a basis for the admissible module of $\mathfrak{n}^{5,1}$ in the tables in Subsection A.4.4; this is given by $\{v, J_1J_2v, J_1J_3v, J_2J_3v, J_6v, J_1J_2J_6v, J_1J_3J_6v, J_2J_3J_6v\}$, where $i := J_1J_2$, $j := J_2J_3$ and $k := J_1J_3$ are quaternion units. In particular, the matrix forms of J_1J_2 and J_2J_3 are as I and J from Proposition 1.17. Hence, the image $\mathcal{A}$ of A via the isomorphism (1.13) belongs to $\mathrm{GL}(2,\mathbb{H})$. We now need to impose the condition $A^TJ_6A = J_6$. The element J_6 has σ_4 as matrix form and the admissible module has a neutral metric. By Lemma 3.2, the condition $A^TJ_6A = J_6$ becomes $A^t\Omega_4A = \Omega_4$, which is mapped via the isomorphism (1.13) to the condition $\overline{A^t}\Omega_1A = \Omega_1$; hence, $A \in \overline{\mathrm{Sp}}(2,\mathbb{H})$. This admissible module admits an involution which is a product of three linear operators; in particular, it divides the module in two non-trivial eigenspaces; these two eigenspaces do not have neither a complex nor a quaternion structure; hence, we can't apply Remark 3.5. Instead, the condition $A^TA = \mathrm{Id}$ becomes via the isomorphism (1.13) $\overline{\mathcal{A}^t}\mathcal{A} = \mathrm{Id}$; hence we conclude $A \in \mathrm{U}(1,1,\mathbb{H})$ since the metric is neutral. We have hence proven the inclusion $\mathrm{Aut}^0(\mathfrak{n}) \subset \overline{\mathrm{Sp}}(2,\mathbb{H}) \cap \mathrm{U}(1,1,\mathbb{H}) =: \overline{\mathrm{Sp}}^\dagger(2,\mathbb{H})$. Observe that, when working with the admissible modules of $\mathfrak{n}^{7,3}$ and $\mathfrak{n}^{3,7}$, we need to use the isomorphism (1.15); in fact, the matrix representation of the operators J_2J_1 and $J_1J_4J_6J_8$ are the same as the matrices I and J of Proposition 1.18. In any case, the element $\eta \cdot J_2J_1J_{10}$ is mapped to Ω_1 via the latter isomorphism; hence, the condition is still satisfied.

we now want to prove $\overline{\mathrm{Sp}}^{\dagger}(2,\mathbb{H}) \subset \mathrm{Aut}^0(\mathfrak{n})$. Let $M \in \overline{\mathrm{Sp}}^{\dagger}(2,\mathbb{H})$; then it commutes with J_1J_2, J_1J_3 and J_2J_3, since they are quaternion units. The condition $M^TJ_6M = J_6$ is the defining condition of the group $\overline{\mathrm{Sp}}(2,\mathbb{H})$, once we recall that $\overline{M^t} \in \mathrm{GL}(2,\mathbb{H})$ is mapped into $M^t \in \mathrm{GL}(8,\mathbb{R})$. By the same reasoning the condition $M^TM = \mathrm{Id} \in \mathrm{GL}(8,\mathbb{R})$ is trivially satisfied by any matrix $M \in \mathrm{U}(1,1,\mathbb{H})$.

We conclude that $\mathrm{Aut}^0(\mathfrak{n}) \cong \overline{\mathrm{Sp}}^{*}(2,\mathbb{H})$ for the considered cases.

Appendix A

Tables for the constructions of the bases

We include here a collection of tables for every considered pseudo H-type Lie algebra $\mathfrak{n}^{r,s}$. The data provided with these tables accounts to the following:

- the dimension of the minimal admissible module V of $\mathfrak{n}^{r,s}$;
- a list of its involutions;
- a table of commutations for every linear operator J_i that belong to $\mathfrak{n}^{r,s}$ with the involutions;
- a possible basis for the admissible module V, subdivided into common eigenspaces if involutions are admitted.

The bases will always have either a positive definite or neutral metric; in particular, we will mark in black the elements $w \in V$ such that $\langle w, w \rangle = 1$ and in red the elements $w' \in V$ such that $\langle w', w' \rangle = -1$.

We will now describe how such data can be determined.

Firstly, the dimension of the minimal admissible module V is known (see, for example, [FM17]). From this, we can obtain the number p of involutions that V admits that can be written as a product of three or four linear operators. Indeed, we know that the dimension of V is 2^{r+s-p}, where p is the number of involutions. Hence, $p = r + s - \log_2(n)$ From this, a complete list of involutions can be obtained combinatorially:

- the involutions P that can be written as a product $P = J_1 \cdots J_m$, where $m = 3, 4$, need to be such that an even number of operators J_i satisfy $J_i^2 = \mathrm{Id}$;
- any two such involutions need to have exactly two operators J_i, J_j in common.

The tables of commutation are obtained by simple computations, recalling the property of skew-adjointness that the operators need to satisfy.

Lastly, we will show how to construct a basis for V using the data collected above. We will only need to know the list of involutions of V that are products of four operators: indeed, only those satisfy the necessary conditions of Corollary 2.31. These involutions divide the admissible module into common eigenspaces; a basis for V will be constructed upon the bases for each of those eigenspaces. To construct a basis for every common eigenspace, we rely on the commutation tables obtained as above: an element w in E^I commutes with the involutions that act as Id on W and anticommutes with the involutions that act as $-\mathrm{Id}$ on w. In particular, we will always choose a v in E^1, the eigenspace on which every involution acts as Id, as a starting point to construct our basis.

We will also provide the products of operators acting as G_I as explained in Theorem 2.32. These appear as a last, separate line in the tables relative to the construction of the bases.

A.1 One-dimensional cases

A.1.1 Case $\mathfrak{n}^{7,0}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_5J_6$ $P_3 = J_1J_3J_5J_7$	$P_4 = J_1J_2J_7$	8

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7
P_1	a	a	a	a	c	c	c
P_2	a	a	c	c	a	a	c
P_3	a	c	a	c	a	c	a
P_4	c	c	a	a	a	a	c

P_1V	+				-				dim $= 4$
$P_2(P_1V)$	+		-		+		-		dim $= 2$
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim $= 1$
basis	v	J_7v	J_6v	J_5v	J_4v	J_3v	J_2v	J_1v	

Observe that $E^1 = E^+_{P_4}$.

A.1.2 Case $\mathfrak{n}^{3,4}$

Involutions product of four	Involutions product of three	$dim(V)$
$P_1 = J_4J_5J_6J_7$ $P_2 = J_2J_3J_4J_5$ $P_3 = J_1J_3J_5J_7$	$P_4 = J_1J_4J_5$	8

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7
P_1	c	c	c	a	a	a	a
P_2	c	a	a	a	a	c	c
P_3	a	c	a	c	a	c	a
P_4	c	a	a	c	c	a	a

P_1V	+				-				dim $= 4$
$P_2(P_1V)$	+		-		+		-		dim $= 2$
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim $= 1$
basis	v	J_1v	J_2v	J_3v	J_6v	J_7v	J_4v	J_5v	

Observe that $E^1 = E^+_{P_4}$.

A.1.3 Case $\mathfrak{n}^{8,0}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$		16
$P_2 = J_1J_2J_5J_6$		
$P_3 = J_1J_2J_7J_8$		
$P_4 = J_1J_3J_5J_7$		

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
P_1	a	a	a	a	c	c	c	c
P_2	a	a	c	c	a	a	c	c
P_3	a	a	c	c	c	c	a	a
P_4	a	c	a	c	a	c	a	c

Construction of the basis:

P_1V								+								-	dim $=8$
$P_2(P_1V)$				+				-				+				-	dim $=4$
$P_3(P_2(P_1V))$		+		-		+		-		+		-		+		-	dim $=2$
$P_4(P_3(P_2(P_1V)))$	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	dim $=1$
basis	v	J_1J_2v	J_8v	J_7v	J_6v	J_5v	J_1J_3v	J_1J_4v	J_4v	J_3v	J_1J_5v	J_1J_6v	J_1J_7v	J_1J_8v	J_2v	J_1v	

A.1.4 Case $\mathfrak{n}^{0,8}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$		16
$P_2 = J_1J_2J_5J_6$		
$P_3 = J_1J_2J_7J_8$		
$P_4 = J_1J_3J_5J_7$		

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
P_1	a	a	a	a	c	c	c	c
P_2	a	a	c	c	a	a	c	c
P_3	a	a	c	c	c	c	a	a
P_4	a	c	a	c	a	c	a	c

Construction of the basis:

P_1V								+								-	dim = 8
$P_2(P_1V)$				+				-				+				-	dim = 4
$P_3(P_2(P_1V))$		+		-		+		-		+		-		+		-	dim = 2
$P_4(P_3(P_2(P_1V)))$	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	dim = 1
basis	v	J_1J_2v	J_8v	J_7v	J_6v	J_5v	J_1J_3v	J_1J_4v	J_4v	J_3v	J_1J_5v	J_1J_6v	J_1J_7v	J_1J_8v	J_2v	J_1v	

A.1.5 Case $\mathfrak{n}^{4,4}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$		16
$P_2 = J_1J_2J_5J_6$		
$P_3 = J_1J_2J_7J_8$		
$P_4 = J_1J_3J_5J_7$		

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
P_1	a	a	a	a	c	c	c	c
P_2	a	a	c	c	a	a	c	c
P_3	a	a	c	c	c	c	a	a
P_4	a	c	a	c	a	c	a	c

Construction of the basis:

P_1V								+								-	dim = 8
$P_2(P_1V)$				+				-				+				-	dim = 4
$P_3(P_2(P_1V))$		+		-		+		-		+		-		+		-	dim = 2
$P_4(P_3(P_2(P_1V)))$	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	dim = 1
basis	v	J_1J_2v	J_8v	J_7v	J_6v	J_5v	J_1J_3v	J_1J_4v	J_4v	J_3v	J_1J_5v	J_1J_6v	J_1J_7v	J_1J_8v	J_2v	J_1v	

A.2 Two-dimensional cases

A.2.1 Case $\mathfrak{n}^{1,0}$

Involutions product of four	Involutions product of three	dim(V)
		2

Costruction of the basis:

V		dim=2
basis	v $J_1 v$	

A.2.2 Case $\mathfrak{n}^{0,1}$

Involutions product of four	Involutions product of three	dim(V)
		2

Costruction of the basis:

V		$dim = 2$
basis	v $J_1 v$	

A.2.3 Case $\mathfrak{n}^{6,0}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1 J_2 J_3 J_4$ $P_2 = J_1 J_2 J_5 J_6$	$P_3 = J_1 J_3 J_5$	8

table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6
P_1	a	a	a	a	c	c
P_2	a	a	c	c	a	a
P_3	c	a	c	a	c	a

Construction of the basis:

$P_1 V$		+		-	dim $= 4$
$P_2(P_1 V)$	+	-	+	-	dim $= 2$
basis	v $J_1 J_2 v$	$J_5 v$ $J_6 v$	$J_4 v$ $J_3 v$	$J_1 v$ $J_2 v$	
G_I		J_5	J_4	J_1	

Observe that $E^1 = E^+_{P_3} \oplus E^-_{P_3}$, with

$$E^+_{P_3} = \text{span}\{v\}$$
$$E^-_{P_3} = \text{span}\{J_1 J_2 v\}.$$

A.2.4 Case $\mathfrak{n}^{2,4}$

Involutions product of four	Involutions product of three	$\dim(V)$
$P_1 = J_1J_2J_3J_4$	$P_3 = J_1J_3J_5$	8
$P_2 = J_1J_2J_5J_6$		

table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6
P_1	a	a	a	a	c	c
P_2	a	a	c	c	a	a
P_3	c	a	c	a	c	a

Construction of the basis:

P_1V		+		-	$\dim = 4$
$P_2(P_1V)$	+	-	+	-	$\dim = 2$
basis	v	J_5v	J_1v	J_4v	
	J_1J_2v	J_6v	J_2v	J_3v	
G_I		J_5	J_1	J_4	

Observe that $E^1 = E^+_{P_3} \oplus E^-_{P_3}$, with

$$E^+_{P_3} = \operatorname{span}\{v\}$$
$$E^-_{P_3} = \operatorname{span}\{J_1J_2v\}.$$

A.2.5 Case $\mathfrak{n}^{0,7}$

Involutions product of four	Involutions product of three	$\dim(V)$
$P_1 = J_1J_2J_3J_4$		16
$P_2 = J_1J_2J_5J_6$		
$P_3 = J_1J_3J_5J_7$		

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7
P_1	a	a	a	a	c	c	c
P_2	a	a	c	c	a	a	c
P_3	a	c	a	c	a	c	a

Construction of the basis:

P_1V				+				-	$\dim = 8$
$P_2(P_1V)$		+		-		+		-	$\dim = 4$
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	$\dim = 2$
basis	v	J_7v	J_6v	J_5v	J_4v	J_3v	J_2v	J_1v	
	$J_1J_2J_7v$	J_1J_2v	J_5J_7v	J_6J_7v	J_3J_7v	J_4J_7v	J_1J_7v	J_2J_7v	
G_I		J_7	J_6	J_5	J_4	J_3	J_2	J_1	

A.2.6 Case $\mathfrak{n}^{4,3}$

Involutions product of four	Involutions product of three	$\dim(V)$
$P_1 = J_1J_2J_3J_4$		16
$P_2 = J_1J_2J_5J_6$		
$P_3 = J_1J_3J_5J_7$		

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7
P_1	a	a	a	a	c	c	c
P_2	a	a	c	c	a	a	c
P_3	a	c	a	c	a	c	a

Construction of the basis:

P_1V				+				-	dim = 8
$P_2(P_1V)$		+		-		+		-	dim = 4
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim = 2
basis	v	J_7v	J_6v	J_5v	J_4v	J_3v	J_2v	J_1v	
	$J_1J_2J_7v$	J_1J_2v	J_5J_7v	J_6J_7v	J_3J_7v	J_4J_7v	J_1J_7v	J_2J_7v	
G_I		J_7	J_6	J_5	J_4	J_3	J_2	J_1	

A.2.7 Case $\mathfrak{n}^{7,1}$

Involutions product of four	Involutions product of three	$\dim(V)$
$P_1 = J_1J_2J_3J_4$	$P_4 = J_1J_2J_7$	16
$P_2 = J_1J_2J_5J_6$		
$P_3 = J_1J_3J_5J_7$		

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
P_1	a	a	a	a	c	c	c	c
P_2	a	a	c	c	a	a	c	c
P_3	a	c	a	c	a	c	a	c
P_4	c	c	a	a	a	a	c	a

Construction of the basis:

P_1V				+				-	dim = 8
$P_2(P_1V)$		+		-		+		-	dim = 4
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim = 2
	v	J_7v	J_6v	J_5v	J_4v	J_3v	J_2v	J_1v	
basis	J_8v	J_7J_8v	J_6J_8v	J_5J_8v	J_4J_8v	J_3J_8v	J_2J_8v	J_1J_8v	
G_I		J_7	J_6	J_5	J_4	J_3	J_2	J_1	

Observe that $E^1 = E^+_{P_4} \oplus E^-_{P_4}$, with

$$E^+_{P_4} = \operatorname{span}\{v\}$$
$$E^-_{P_4} = \operatorname{span}\{J_8v\}.$$

A.2.8 Case $\mathfrak{n}^{3,5}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_4J_5$ $P_2 = J_1J_2J_6J_7$ $P_3 = J_1J_3J_5J_7$	$P_4 = J_1J_2J_3$	16

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
P_1	a	a	c	a	a	c	c	c
P_2	a	a	c	c	c	a	a	c
P_3	a	c	a	c	a	c	a	c
P_4	c	c	c	a	a	a	a	a

Construction of the basis:

P_1V	+							-	dim = 8
$P_2(P_1V)$	+			-	+			-	dim = 4
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim = 2
Basis	v J_8v	J_3v J_3J_8v	J_6v J_6J_8v	J_7v J_7J_8v	J_4v J_4J_8v	J_5v J_5J_8v	J_2v J_2J_8v	J_1v J_1J_8v	
G_I		J_3	J_6	J_7	J_4	J_5	J_2	J_1	

Observe that $E^1 = E^+_{P_4} \oplus E^-_{P_4}$, with

$$E^+_{P_4} = \text{span}\{v\}$$
$$E^-_{P_4} = \text{span}\{J_8v\}.$$

A.3 Four-dimensional cases

A.3.1 Case $\mathfrak{n}^{2,0}$

Involutions product of four	Involutions product of three	dim(V)
		4

Construction of the basis:

V		dim=4
basis	v J_2J_1v J_2v J_1v	

A.3.2 Case $\mathfrak{n}^{0,2}$

Involutions product of four	Involutions product of three	dim(V)
		4

Construction of the basis:

V		$\dim = 4$
basis	v J_2J_1v J_1v J_2v	

A.3.3 Case $\mathfrak{n}^{1,1}$

Involutions product of four	Involutions product of three	$\dim(V)$
		4

Construction of the basis:

V		$dim = 4$
basis	v J_1v J_1J_2v J_2v	

A.3.4 Case $\mathfrak{n}^{3,0}$

Involutions product of four	Involutions product of three	$\dim(V)$
	$P_1 = J_1J_2J_3$	4

Table of commutativity:

	J_1	J_2	J_3
P_1	c	c	c

Construction of the basis:

V		dim=4
basis	v J_1J_2v J_2J_3v J_3J_1v	

Observe that $E^1 = E^+_{P_3}$.

A.3.5 Case $\mathfrak{n}^{1,2}$

Involutions product of four	Involutions product of three	$\dim(V)$
	$P_1 = J_1J_2J_3$	4

Table of commutativity:

	J_1	J_2	J_3
P_1	c	c	c

Construction of the basis:

V		$\dim = 4$
basis	v J_2J_3v J_1J_2v J_3J_1v	

Observe that $E^1 = E^+_{P_3}$.

A.3.6 Case $\mathfrak{n}^{4,0}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$		8

Table of commutativity:

	J_1	J_2	J_3	J_4
P_1	a	a	a	a

P_1V	+	-	dim = 4
basis	v	J_3v	
	J_1J_2v	J_4v	
	J_2J_3v	J_2v	
	J_3J_1v	J_1v	
G_I		J_3	

A.3.7 Case $\mathfrak{n}^{0,4}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$		8

Table of commutativity:

	J_1	J_2	J_3	J_4
P_1	a	a	a	a

P_1V	+	-	dim = 4
basis	v	J_3v	
	J_1J_2v	J_4v	
	J_2J_3v	J_2v	
	J_3J_1v	J_1v	
G_I		J_3	

A.3.8 Case $\mathfrak{n}^{2,2}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$		8

Table of commutativity:

	J_1	J_2	J_3	J_4
P_1	a	a	a	a

P_1V	+	-	dim = 4
basis	v	J_3v	
	J_1J_2v	J_4v	
	J_2J_3v	J_2v	
	J_3J_1v	J_1v	
G_I		J_3	

A.3.9 Case $\mathfrak{n}^{5,0}$

Involutions product of four	Involutions product of three	$\dim(V)$
$P_1 = J_1J_2J_3J_4$	$P_2 = J_1J_2J_5$	8

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5
P_1	a	a	a	a	c
P_2	c	c	a	a	c

P_1V	+	-	dim = 4
basis	v	J_3v	
	J_1J_2v	J_4v	
	J_2J_3v	J_2v	
	J_3J_1v	J_1v	
G_I		J_3	

Observe that $E^1 = E^+_{P_2} \oplus E^-_{P_2}$, with

$$E^+_{P_2} = \operatorname{span}\{v, J_1J_2v\}$$
$$E^-_{P_2} = \operatorname{span}\{J_2J_3v, J_3J_1v\}.$$

A.3.10 Case $\mathfrak{n}^{1,4}$

Involutions product of four	Involutions product of three	$\dim(V)$
$P_1 = J_2J_3J_4J_5$	$P_2 = J_1J_2J_3$	8

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5
P_1	c	a	a	a	a
P_2	c	c	c	a	a

Construction of the basis:

P_1V	+	-	dim = 4
basis	v	J_4v	
	J_2J_3v	J_5v	
	J_3J_4v	J_3v	
	J_2J_4v	J_2v	
G_I		J_4	

Observe that $E^1 = E^+_{P_2} \oplus E^-_{P_2}$, with

$$E^+_{P_2} = \operatorname{span}\{v, J_2J_3v\}$$
$$E^-_{P_2} = \operatorname{span}\{J_3J_4v, J_2J_4v\}.$$

A.3.11 Case $\mathfrak{n}^{3,2}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_4J_5$	$P_2 = J_3J_4J_5$	8

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5
P_1	a	a	c	a	a
P_2	a	a	c	c	c

Construction of the basis:

P_1V	+	-	dim = 4
basis	v	J_1v	
	J_4J_5v	J_2v	
	J_1J_4v	J_4v	
	J_1J_5v	J_5v	
G_I		J_1	

Observe that $E^1 = E^+_{P_2} \oplus E^-_{P_2}$, with

$$E^+_{P_2} = \text{span}\{v, J_4J_5v\}$$
$$E^-_{P_2} = \text{span}\{J_1J_4v, J_1J_5v\}.$$

A.3.12 Case $\mathfrak{n}^{2,3}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$	$P_2 = J_1J_4J_5$	8

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5
P_1	a	a	a	a	c
P_2	c	a	a	c	c

Costruction of the basis:

P_1V	+	-	dim = 4
	v	J_2v	
	J_3J_4v	J_1v	
basis	J_2J_4v	J_4v	
	J_2J_3v	J_3v	
G_I		J_2	

Observe that $E^1 = E^+_{P_2} \oplus E^-_{P_2}$, with

$$E^+_{P_2} = \text{span}\{v, J_2J_3v\}$$
$$E^-_{P_2} = \text{span}\{J_2J_4v, J_3J_4v\}.$$

A.3.13 Case $\mathfrak{n}^{0,6}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_5J_6$		16

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6
P_1	a	a	a	a	c	c
P_2	a	a	c	c	a	a

Construction of the basis:

P_1V		+		-	dim = 8
$P_2(P_1V)$	+	-	+	-	dim = 4
basis	v J_2J_1v $J_2J_3J_5v$ $J_1J_3J_5v$	J_5v J_6v J_2J_3v J_1J_3v	J_3v J_4v J_2J_5v J_1J_5v	J_1v J_2v J_4J_5v J_3J_5v	
G_I		J_5	J_3	J_1	

A.3.14 Case $\mathfrak{n}^{4,2}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_5J_6$		16

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6
P_1	a	a	a	a	c	c
P_2	a	a	c	c	a	a

Construction of the basis:

P_1V		+		-	dim = 8
$P_2(P_1V)$	+	-	+	-	dim = 4
basis	v J_1J_2v $J_2J_3J_5v$ $J_1J_3J_5v$	J_5v J_6v J_2J_3v J_1J_3v	J_3v J_4v J_2J_5v J_1J_5v	J_1v J_2v J_4J_5v J_3J_5v	
G_I		J_5	J_3	J_1	

A.3.15 Case $\mathfrak{n}^{1,6}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_2J_3J_4J_5$ $P_2 = J_2J_3J_6J_7$	$P_3 = J_1J_2J_3$	16

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7
P_1	c	a	a	a	a	c	c
P_2	c	a	a	c	c	a	a
P_3	c	c	c	a	a	a	a

Construction of the basis:

P_1V		+		-	dim = 8
$P_2(P_1V)$	+	-	+	-	dim = 4
basis	v J_2J_3v $J_2J_4J_6v$ $J_3J_4J_6v$	J_6v J_7v J_2J_4v J_3J_4v	J_4v J_5v J_2J_6v J_3J_6v	J_2v J_3v J_4J_6v J_5J_6v	
G_I		J_6	J_4	J_2	

Observe that $E^1 = E^+_{P_3}$.

A.3.16 Case $\mathfrak{n}^{5,2}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_6J_7$	$P_3 = J_1J_2J_5$	16

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7
P_1	a	a	a	a	c	c	c
P_2	a	a	c	c	c	a	a
P_3	c	c	a	a	c	a	a

Construction of the basis:

P_1V		+		-	dim = 8
$P_2(P_1V)$	+	-	+	-	dim = 4
basis	v J_1J_2v $J_2J_3J_6v$ $J_1J_3J_6v$	J_6v J_5J_6v J_2J_3v J_1J_3v	J_3v J_3J_5v J_2J_6v J_1J_6v	J_1v J_1J_5v J_4J_6v J_3J_6v	
G_I		J_6	J_3	J_1	

Observe that $E^1 = E^+_{P_3}$.

A.3.17 Case $\mathfrak{n}^{6,1}$

Involutions product of four	Involutions product of three	$dim(V)$
$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_5J_6$	$P_3 = J_1J_3J_5$	16

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7
P_1	a	a	a	a	c	c	c
P_2	a	a	c	c	a	a	c
P_3	c	a	c	a	c	a	a

Construction of the basis:

P_1V		+		-	$dim = 8$
$P_2(P_1V)$	+	-	+	-	$dim = 4$
basis	v J_1J_2v J_7v $J_1J_2J_7v$	J_5v J_6v J_5J_7v J_6J_7v	J_3v J_4v J_3J_7v J_4J_7v	J_1v J_2v J_1J_7v J_2J_7v	
G_I		J_5	J_3	J_1	

Observe that $E^1 = E^+_{P_3} \oplus E^-_{P_3}$, with

$$E^+_{P_3} = \text{span}\{v, J_1J_2J_7v\}$$
$$E^-_{P_3} = \text{span}\{J_1J_2v, J_7v\}.$$

A.3.18 Case $\mathfrak{n}^{2,5}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_5J_6$	$P_3 = J_1J_3J_5$	16

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7
P_1	a	a	a	a	c	c	c
P_2	a	a	c	c	a	a	c
P_3	c	a	c	a	c	a	a

Construction of the basis:

P_1V		+		-	dim = 8
$P_2(P_1V)$	+	-	+	-	dim = 4
basis	v J_1J_2v J_7v $J_1J_2J_7v$	J_6v J_5v J_6J_7v J_5J_7v	J_4v J_3v J_4J_7v J_3J_7v	J_2v J_1v J_2J_7v J_1J_7v	
G_I		J_6	J_4	J_2	

Observe that $E^1 = E^+_{P_3} \oplus E^-_{P_3}$, with

$$E^+_{P_3} = \text{span}\{v, J_1J_2J_7v\}$$
$$E^-_{P_3} = \text{span}\{J_1J_2v, J_7v\}.$$

A.3.19 Case $\mathfrak{n}^{1,7}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_2J_3J_4J_5$ $P_2 = J_2J_3J_6J_7$ $P_3 = J_2J_4J_6J_8$		32

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
P_1	c	a	a	a	a	c	c	c
P_2	c	a	a	c	c	a	a	c
P_3	c	a	c	a	c	a	c	a

Construction of the basis:

P_1V				+				-	dim = 16
$P_2(P_1V)$		+		-		+		-	dim = 8
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim = 4
basis	v J_1v $J_1J_6J_7J_8v$ $J_6J_7J_8v$	J_8v J_1J_8v $J_1J_6J_7v$ J_6J_7v	J_7v J_1J_7v $J_1J_6J_8v$ J_6J_8v	J_6v J_1J_6v $J_1J_7J_8v$ J_7J_8v	J_5v J_1J_5v $J_1J_4J_8v$ J_4J_8v	J_4v J_1J_4v $J_1J_2J_7v$ J_2J_7v	J_3v J_1J_3v $J_1J_2J_8v$ J_2J_8v	J_2v J_1J_2v $J_1J_3J_8v$ J_3J_8v	
G_I		J_8	J_7	J_6	J_5	J_4	J_3	J_2	

A.3.20 Case $\mathfrak{n}^{5,3}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_7J_8$ $P_3 = J_2J_4J_6J_8$		32

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
P_1	a	a	a	a	c	c	c	c
P_2	a	a	c	c	c	c	a	a
P_3	c	a	c	a	c	a	c	a

Construction of the basis:

P_1V	+							-	dim = 16
$P_2(P_1V)$	+			-		+		-	dim = 8
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim = 4
Basis	v J_5v $J_1J_4J_5J_8v$ $J_1J_4J_8v$	J_6v J_5J_6v $J_1J_2J_5v$ J_1J_2v	J_7v J_5J_7v $J_2J_4J_5v$ J_2J_4v	J_8v J_5J_8v $J_1J_4J_5v$ J_1J_4v	J_3v J_5J_3v $J_2J_5J_8v$ J_2J_8v	J_4v J_5J_4v $J_1J_5J_8v$ J_1J_8v	J_1v J_5J_1v $J_4J_5J_8v$ J_4J_8v	J_2v J_5J_2v $J_3J_5J_8v$ J_3J_8v	
G_I		J_6	J_7	J_8	J_3	J_4	J_1	J_2	

A.3.21 Case $\mathfrak{n}^{6,2}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$		32
$P_2 = J_1J_2J_5J_6$		
$P_3 = J_1J_2J_7J_8$		

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
P_1	a	a	a	a	c	c	c	c
P_2	a	a	c	c	a	a	c	c
P_3	a	a	c	c	c	c	a	a

Construction of the basis:

P_1V				+				-	dim = 16
$P_2(P_1V)$		+		-		+		-	dim = 8
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim = 4
basis	v J_1J_2v $J_1J_3J_5J_7v$ $J_2J_3J_5J_7v$	J_7v J_8v $J_1J_3J_5v$ $J_2J_3J_5v$	J_5v J_6v $J_1J_3J_7v$ $J_2J_3J_7v$	J_1J_3v J_1J_4v J_5J_7v J_6J_7v	J_3v J_4v $J_1J_5J_7v$ $J_2J_5J_7v$	J_1J_5v J_1J_6v J_3J_7v J_4J_7v	J_1J_7v J_1J_8v J_3J_5v J_4J_5v	J_1v J_2v $J_3J_5J_7v$ $J_4J_5J_7v$	
G_I		J_7	J_5	J_1J_3	J_3	J_1J_5	J_1J_7	J_1	

A.3.22 Case $\mathfrak{n}^{2,6}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_5J_6$ $P_3 = J_1J_2J_7J_8$		32

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
P_1	a	a	a	a	c	c	c	c
P_2	a	a	c	c	a	a	c	c
P_3	a	a	c	c	c	c	a	a

Construction of the basis:

P_1V				+				-	dim = 16
$P_2(P_1V)$		+		-		+		-	dim = 8
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim = 4
basis	v J_1J_2v $J_1J_3J_5J_7v$ $J_2J_3J_5J_7v$	J_7v J_8v $J_1J_3J_5v$ $J_2J_3J_5v$	J_5v J_6v $J_1J_3J_7v$ $J_2J_3J_7v$	J_1J_3v J_1J_4v J_5J_7v J_6J_7v	J_3v J_4v $J_1J_5J_7v$ $J_2J_5J_7v$	J_1J_5v J_1J_6v J_3J_7v J_4J_7v	J_1J_7v J_1J_8v J_3J_5v J_4J_5v	J_1v J_2v $J_3J_5J_7v$ $J_4J_5J_7v$	
G_I		J_7	J_5	J_1J_3	J_3	J_1J_5	J_1J_7	J_1	

A.3.23 Case $\mathfrak{n}^{3,6}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_4J_5$ $P_2 = J_1J_2J_6J_7$ $P_3 = J_1J_2J_8J_9$	$P_4 = J_1J_2J_3$	32

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9
P_1	a	a	c	a	a	c	c	c	c
P_2	a	a	c	c	c	a	a	c	c
P_3	a	a	c	c	c	c	c	a	a
P_4	c	c	c	a	a	a	a	a	a

Construction of the basis:

P_1V	+							-	dim = 16
$P_2(P_1V)$	+		-		+			-	dim = 8
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim = 4
basis	v J_1J_2v $J_1J_4J_7J_8v$ $J_2J_4J_7J_8v$	J_8v J_3J_8v $J_1J_4J_7v$ $J_2J_4J_7v$	J_7v J_3J_7v $J_1J_4J_8v$ $J_2J_4J_8v$	J_1J_5v J_2J_5v J_6J_8v J_7J_8v	J_5v J_3J_5v $J_2J_7J_8v$ $J_1J_7J_8v$	J_1J_7v J_2J_7v J_4J_8v J_5J_8v	J_1J_8v J_2J_8v J_4J_7v J_5J_7v	J_1v J_2v $J_4J_7J_8v$ $J_5J_7J_8v$	
G_I		J_8	J_7	J_1J_5	J_5	J_1J_7	J_1J_8	J_1	

Observe that $E^1 = E^+_{P_4} \oplus E^-_{P_4}$, with

$$E^+_{P_4} = \text{span}\{v, J_1J_2v\}$$
$$E^-_{P_4} = \text{span}\{J_1J_4J_7J_8v, J2J_4J_7J_8v\}.$$

A.3.24 Case $\mathfrak{n}^{7,2}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_8J_9$ $P_2 = J_3J_4J_8J_9$ $P_3 = J_5J_6J_8J_9$	$P_4 = J_7J_8J_9$	32

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9
P_1	a	a	c	c	c	c	c	a	a
P_2	c	c	a	a	c	c	c	a	a
P_3	c	c	c	c	a	a	c	a	a
P_4	a	a	a	a	a	a	c	c	c

Construction of the basis:

P_1V	+							-	dim = 16
$P_2(P_1V)$	+		-		+			-	dim = 8
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim = 4
basis	v J_8J_9v $J_1J_3J_5J_9v$ $J_1J_3J_5J_8v$	J_5v J_6v $J_1J_3J_9v$ $J_1J_3J_8v$	J_3v J_4v $J_1J_5J_9v$ $J_1J_5J_8v$	J_1J_8v J_2J_8v J_3J_6v J_4J_6v	J_1v J_2v $J_3J_5J_9v$ $J_3J_5J_8v$	J_3J_8v J_4J_8v J_1J_6v J_1J_5v	J_5J_8v J_6J_8v J_2J_3v J_1J_3v	J_8v J_9v $J_1J_3J_6v$ $J_1J_3J_5v$	
		J_5	J_3	J_1J_8	J_1	J_3J_8	J_5J_8	J_8	

Observe that $E^1 = E^+_{P_4} \oplus E^-_{P_4}$, with

$$E^+_{P_4} = \text{span}\{v, J_8J_9v\}$$
$$E^-_{P_4} = \text{span}\{J_1J_3J_5J_9v, J_1J_3J_5J_8v\}.$$

A.4 Eight-dimensional case

A.4.1 Case $\mathfrak{n}^{0,3}$

Involutions product of four	Involutions product of three	dim(V)
		8

Construction of the basis:

V		dim = 8
basis	v J_2J_1v J_3J_2v J_1J_3v $J_1J_2J_3v$ J_3v J_1v J_2v	

A.4.2 Case $\mathfrak{n}^{2,1}$

Involutions product of four	Involutions product of three	dim(V)
		8

Construction of the basis:

V		dim = 8
basis	v J_2v J_1v J_1J_2v J_3v J_2J_3v J_1J_3v $J_1J_2J_3v$	

A.4.3 Case $\mathfrak{n}^{1,3}$

Involutions product of four	Involutions product of three	dim(V)
	$P_1 = J_1J_2J_3$	8

Table of commutativity:

	J_1	J_2	J_3	J_4
P_1	c	c	c	a

Construction of the basis:

V		dim = 8
basis	v $J_1J_2J_4v$ $J_1J_3J_4v$ J_2J_3v J_4v J_1J_2v J_1J_3v $J_2J_3J_4v$	

Observe that $E^1 = E^+_{P_1} \oplus E^-_{P_1}$, with

$$E^+_{P_1} = \mathrm{span}\{v, J_2J_3v, J_1J_2v, J_1J_3v\}$$
$$E^-_{P_1} = \mathrm{span}\{J_4v, J_2J_3J_4v, J_1J_2J_4v, J_1J_3J_4v\}.$$

A.4.4 Case $\mathfrak{n}^{3,1}$

Involutions product of four	Involutions product of three	dim(V)
	$P_1 = J_1J_2J_3$	8

Table of commutativity:

	J_1	J_2	J_3	J_4
P_1	c	c	c	a

Construction of the basis:

V		dim = 8
basis	v	
	J_1J_2v	
	J_2J_3v	
	J_3J_1v	
	J_4v	
	$J_1J_2J_4v$	
	$J_2J_3J_4v$	
	$J_3J_1J_4v$	

Observe that $E^1 = E^+_{P_1} \oplus E^-_{P_1}$, with

$$E^+_{P_1} = \mathrm{span}\{v, J_2J_3v, J_1J_2v, J_1J_3v\}$$
$$E^-_{P_1} = \mathrm{span}\{J_4v, J_2J_3J_4v, J_1J_2J_4v, J_1J_3J_4v\}.$$

A.4.5 Case $\mathfrak{n}^{0,5}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$		16

	J_1	J_2	J_3	J_4	J_5
P_1	a	a	a	a	c

Construction of the basis:

P_1V	+	-	dim = 8
	v	J_1v	
	J_1J_2v	J_2v	
	J_4J_2v	J_3v	
	J_1J_4v	J_4v	
	J_5v	J_1J_5v	
	$J_1J_2J_5v$	J_2J_5v	
basis	$J_4J_2J_5v$	J_3J_5v	
	$J_1J_4J_5v$	J_4J_5v	
G_I		J_1	

A.4.6 Case $\mathfrak{n}^{4,1}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$		16

	J_1	J_2	J_3	J_4	J_5
P_1	a	a	a	a	c

Construction of the basis:

P_1V	+	-	dim = 8
basis	v	J_1v	
	J_1J_2v	J_2v	
	J_2J_4v	J_3v	
	J_1J_4v	J_4v	
	J_5v	J_1J_5v	
	$J_1J_2J_5v$	J_2J_5v	
	$J_2J_4J_5v$	J_3J_5v	
	$J_1J_4J_5v$	J_4J_5v	
G_I		J_1	

A.4.7 Case $\mathfrak{n}^{5,1}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$	$P_2 = J_1J_2J_5$	16

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6
P_1	a	a	a	a	c	c
P_2	c	c	a	a	c	a

Construction of the basis:

P_1V	+	-	dim = 8
basis	v	J_3v	
	J_1J_2v	J_4v	
	J_1J_3v	J_1v	
	J_2J_3v	J_2v	
	J_6v	J_3J_6v	
	$J_1J_2J_6v$	J_4J_6v	
	$J_1J_3J_6v$	J_1J_6v	
	$J_2J_3J_6v$	J_2J_6v	
G_I		J_3	

Observe that $E^1 = E^+_{P_2} \oplus E^-_{P_2}$, with

$$E^+_{P_2} = \mathrm{span}\{v, J_1J_2v, J_1J_3J_6v, J_2J_3J_6v\}$$
$$E^-_{P_1} = \mathrm{span}\{J_6v, J_2J_3v, J_1J_3v, J_1J_2J_6v\}.$$

A.4.8 Case $\mathfrak{n}^{1,5}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_2J_3J_4J_5$	$P_2 = J_1J_2J_3$	16

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6
P_1	c	a	a	a	a	c
P_2	c	c	c	a	a	a

Construction of the basis:

P_1V	+	-	dim = 8
basis	v	J_5v	
	J_2J_3v	J_4v	
	J_2J_4v	J_3v	
	J_3J_4v	J_2v	
	J_6v	J_5J_6v	
	$J_2J_3J_6v$	J_4J_6v	
	$J_2J_4J_6v$	J_3J_6v	
	$J_3J_4J_6v$	J_2J_6v	
G_I		J_5	

Observe that $E^1 = E^+_{P_2} \oplus E^-_{P_2}$, with

$$E^+_{P_2} = \mathrm{span}\{v, J_2J_3v, J_2J_4J_6v, J_3J_4J_6v\}$$
$$E^-_{P_1} = \mathrm{span}\{J_6v, J_2J_4v, J_3J_4v, J_2J_3J_6v\}.$$

A.4.9 Case $\mathfrak{n}^{6,3}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_5J_6$ $P_3 = J_1J_2J_7J_8$		64

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9
P_1	a	a	a	a	c	c	c	c	c
P_2	a	a	c	c	a	a	c	c	c
P_3	a	a	c	c	c	c	a	a	c

Construction of the basis:

P_1V	+							-	dim = 32
$P_2(P_1V)$	+		-		+		-		dim = 16
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim = 8
basis	v $J_1J_3J_6J_8J_9v$ $J_2J_3J_6J_8J_9v$ J_2J_1v J_9v $J_1J_3J_6J_8v$ $J_2J_3J_6J_8v$ $J_2J_1J_9v$	J_7v $J_2J_3J_6J_9v$ $J_1J_3J_6J_9v$ J_8v J_7J_9v $J_2J_3J_6v$ $J_1J_3J_6v$ J_8J_9v	J_5v $J_2J_3J_8J_9v$ $J_1J_3J_8J_9v$ J_6v J_5J_9v $J_2J_3J_8v$ $J_1J_3J_8v$ J_6J_9v	J_1J_3v $J_6J_8J_9v$ $J_5J_8J_9v$ J_1J_4v $J_1J_3J_9v$ J_6J_8v J_5J_8v $J_1J_4J_9v$	J_3v $J_1J_6J_8J_9v$ $J_2J_6J_8J_9v$ J_4v J_3J_9v $J_1J_6J_8v$ $J_2J_6J_8v$ J_4J_9v	J_1J_5v $J_4J_8J_9v$ $J_3J_8J_9v$ J_1J_6v $J_1J_5J_9v$ J_4J_8v J_3J_8v $J_1J_6J_9v$	J_1J_7v $J_3J_5J_9v$ $J_3J_6J_9v$ J_1J_8v $J_1J_7J_9v$ J_3J_5v J_3J_6v $J_1J_8J_9v$	J_1v $J_3J_6J_8J_9v$ $J_4J_6J_8J_9v$ J_2v J_1J_9v $J_3J_6J_8v$ $J_4J_6J_8v$ J_2J_9v	
G_I		$\cdot J_7$	$\cdot J_5$	$\cdot J_1J_3$	$\cdot J_3$	$\cdot J_1J_5$	$\cdot J_1J_7$	$\cdot J_1$	

A.4.10 Case $\mathfrak{n}^{2,7}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_5J_6$ $P_3 = J_1J_2J_7J_8$		64

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9
P_1	a	a	a	a	c	c	c	c	c
P_2	a	a	c	c	a	a	c	c	c
P_3	a	a	c	c	c	c	a	a	c

Construction of the basis:

P_1V				+				-	dim = 32
$P_2(P_1V)$		+		-		+		-	dim = 16
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim = 8
basis	v $J_1J_3J_6J_8J_9v$ $J_2J_3J_6J_8J_9v$ J_2J_1v J_9v $J_1J_3J_6J_8v$ $J_2J_3J_6J_8v$ $J_2J_1J_9v$	J_7v $J_2J_3J_6J_9v$ $J_1J_3J_6J_9v$ J_8v J_7J_9v $J_2J_3J_6v$ $J_1J_3J_6v$ J_8J_9v	J_5v $J_2J_3J_8J_9v$ $J_1J_3J_8J_9v$ J_6v J_5J_9v $J_2J_3J_8v$ $J_1J_3J_8v$ J_6J_9v	J_1J_3v $J_6J_8J_9v$ $J_5J_8J_9v$ J_1J_4v $J_1J_3J_9v$ J_6J_8v J_5J_8v $J_1J_4J_9v$	J_3v $J_1J_6J_8J_9v$ $J_2J_6J_8J_9v$ J_4v J_3J_9v $J_1J_6J_8v$ $J_2J_6J_8v$ J_4J_9v	J_1J_5v $J_4J_8J_9v$ $J_3J_8J_9v$ J_1J_6v $J_1J_5J_9v$ J_4J_8v J_3J_8v $J_1J_6J_9v$	J_1J_7v $J_3J_5J_9v$ $J_3J_6J_9v$ J_1J_8v $J_1J_7J_9v$ J_3J_5v J_3J_6v $J_1J_8J_9v$	J_1v $J_3J_6J_8J_9v$ $J_4J_6J_8J_9v$ J_2v J_1J_9v $J_3J_6J_8v$ $J_4J_6J_8v$ J_2J_9v	
G_I		$\cdot J_7$	$\cdot J_5$	$\cdot J_1J_3$	$\cdot J_3$	$\cdot J_1J_5$	$\cdot J_1J_7$	$\cdot J_1$	

A.4.11 Case $\mathfrak{n}^{7,3}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_4J_5$ $P_2 = J_1J_2J_6J_7$ $P_3 = J_1J_2J_8J_9$	$P_4 = J_1J_2J_3$	64

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9	J_{10}
P_1	a	a	a	a	c	c	c	c	c	c
P_2	a	a	c	c	a	a	c	c	c	c
P_3	a	a	c	c	c	c	a	a	c	c
P_4	c	c	c	a	a	a	a	a	a	a

Construction of the basis:

P_1V				+				-	dim = 32
$P_2(P_1V)$		+		-		+		-	dim = 16
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim = 8
basis	v $J_1J_4J_6J_8J_{10}v$ $J_2J_4J_6J_8J_{10}v$ J_2J_1v $J_{10}v$ $J_1J_4J_6J_8v$ $J_2J_4J_6J_8v$ $J_2J_1J_{10}v$	J_8v $J_1J_4J_6J_{10}v$ $J_2J_4J_6J_{10}v$ J_9v $J_8J_{10}v$ $J_1J_4J_6v$ $J_2J_4J_6v$ $J_9J_{10}v$	J_6v $J_1J_4J_8J_{10}v$ $J_2J_4J_8J_{10}v$ J_7v $J_6J_{10}v$ $J_1J_4J_8v$ $J_2J_4J_8v$ $J_7J_{10}v$	J_1J_4v $J_6J_8J_{10}v$ $J_7J_8J_{10}v$ J_1J_5v $J_1J_4J_{10}v$ J_6J_8v J_7J_8v $J_1J_5J_{10}v$	J_4v $J_1J_6J_8J_{10}v$ $J_2J_6J_8J_{10}v$ J_5v $J_4J_{10}v$ $J_1J_6J_8v$ $J_2J_6J_8v$ $J_5J_{10}v$	J_1J_6v $J_4J_8J_{10}v$ $J_5J_8J_{10}v$ J_1J_7v $J_1J_6J_{10}v$ J_4J_8v J_5J_8v $J_1J_7J_{10}v$	J_1J_8v $J_4J_6J_{10}v$ $J_5J_6J_{10}v$ J_1J_9v $J_1J_8J_{10}v$ J_4J_6v J_5J_6v $J_1J_9J_{10}v$	J_1v $J_4J_6J_8J_{10}v$ $J_5J_6J_8J_{10}v$ J_2v $J_1J_{10}v$ $J_4J_6J_8v$ $J_5J_6J_8v$ $J_2J_{10}v$	
G_I		J_8	J_6	J_1J_4	J_4	J_1J_6	J_1J_8	$\cdot J_1$	

Observe that $E^1 = E^+_{P_4} \oplus E^-_{P_4}$, with

$$E^+_{P_4} = \text{span}\{v, J_2J_1v, J_1J_4J_6J_8J_{10}v, J_2J_4J_6J_8J_{10}v\}$$
$$E^-_{P_4} = \text{span}\{J_{10}v, J_1J_4J_6J_8v, J_2J_4J_6J_8v, J_2J_1J_{10}v\}.$$

A.4.12 Case $\mathfrak{n}^{3,7}$

Involutions product of four	Involutions product of three	dim(V)
$P_1 = J_1J_2J_4J_5$ $P_2 = J_1J_2J_6J_7$ $P_3 = J_1J_2J_8J_9$	$P_4 = J_1J_2J_3$	64

Table of commutativity:

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9	J_{10}
P_1	a	a	a	a	c	c	c	c	c	c
P_2	a	a	c	c	a	a	c	c	c	c
P_3	a	a	c	c	c	c	a	a	c	c
P_4	c	c	c	a	a	a	a	a	a	a

Construction of the basis:

P_1V	+							-	dim = 32
$P_2(P_1V)$	+		-		+			-	dim = 16
$P_3(P_2(P_1V))$	+	-	+	-	+	-	+	-	dim = 8
basis	v $J_1J_4J_6J_8J_{10}v$ $J_2J_4J_6J_8J_{10}v$ J_2J_1v $J_{10}v$ $J_1J_4J_6J_8v$ $J_2J_4J_6J_8v$ $J_2J_1J_{10}v$	J_8v $J_1J_4J_6J_{10}v$ $J_2J_4J_6J_{10}v$ J_9v $J_8J_{10}v$ $J_1J_4J_6v$ $J_2J_4J_6v$ $J_9J_{10}v$	J_6v $J_1J_4J_8J_{10}v$ $J_2J_4J_8J_{10}v$ J_7v $J_6J_{10}v$ $J_1J_4J_8v$ $J_2J_4J_8v$ $J_7J_{10}v$	J_1J_4v $J_6J_8J_{10}v$ $J_7J_8J_{10}v$ J_1J_5v $J_1J_4J_{10}v$ J_6J_8v J_7J_8v $J_1J_5J_{10}v$	J_4v $J_1J_6J_8J_{10}v$ $J_2J_6J_8J_{10}v$ J_5v $J_4J_{10}v$ $J_1J_6J_8v$ $J_2J_6J_8v$ $J_5J_{10}v$	J_1J_6v $J_4J_8J_{10}v$ $J_5J_8J_{10}v$ J_1J_7v $J_1J_6J_{10}v$ J_4J_8v J_5J_8v $J_1J_7J_{10}v$	J_1J_8v $J_4J_6J_{10}v$ $J_5J_6J_{10}v$ J_1J_9v $J_1J_8J_{10}v$ J_4J_6v J_5J_6v $J_1J_9J_{10}v$	J_1v $J_4J_6J_8J_{10}v$ $J_5J_6J_8J_{10}v$ J_2v $J_1J_{10}v$ $J_4J_6J_8v$ $J_5J_6J_8v$ $J_2J_{10}v$	
		$\cdot J_8$	$\cdot J_6$	$\cdot J_1J_4$	$\cdot J_4$	$\cdot J_1J_6$	$\cdot J_1J_8$	$\cdot J_1$	

Observe that $E^1 = E^+_{P_4} \oplus E^-_{P_4}$, with

$$E^+_{P_4} = \text{span}\{v, J_2J_1v, J_1J_4J_6J_8J_{10}v, J_2J_4J_6J_8J_{10}v\}$$
$$E^-_{P_4} = \text{span}\{J_{10}v, J_1J_4J_6J_8v, J_2J_4J_6J_8v, J_2J_1J_{10}v\}.$$

Bibliography

[ABS64] M. F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology*, 3(suppl. 1):3–38, 1964.

[Cia00] Paolo Ciatti. Scalar products on Clifford modules and pseudo-H-type Lie algebras. *Ann. Mat. Pura Appl. (4)*, 178:1–31, 2000.

[Ebe94] Patrick Eberlein. Geometry of 2-step nilpotent groups with a left invariant metric. In *Annales scientifiques de l'Ecole normale supérieure*, volume 27, pages 611–660, 1994.

[FM17] Kenro Furutani and Irina Markina. Complete classification of pseudo h-type lie algebras: I. *Geometriae Dedicata*, pages 1–29, 2017.

[Kap80] Aroldo Kaplan. Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Trans. Amer. Math. Soc.*, 258(1):147–153, 1980.

[KT13] Aroldo Kaplan and Alejandro Tiraboschi. Automorphisms of non-singular nilpotent Lie algebras. *J. Lie Theory*, 23(4):1085–1100, 2013.

[LM89] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[M^+] Guy Métivier et al. Hypoellipticité analytique sur des groupes nilpotents de rang 2.

[Saa96] L. Saal. The automorphism group of a Lie algebra of Heisenberg type. *Rend. Sem. Mat. Univ. Politec. Torino*, 54(2):101–113, 1996.