ON THE SHOALING OF SOLITARY WAVES IN THE KDV EQUATION

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The waveheight change in surface waves with a sufficiently slow variation in depth is examined. Using a new formulation of the energy flux associated to waves modeled by the Korteweg-de Vries equation, a system of three coupled equations is derived for the determination of the local wave properties as waves propagate over gently changing depth. The system of equations is solved numerically, and the resulting shoaling curves are compared to previous results on long wave shoaling.

Keywords: Solitary wave propagation; wave shoaling; long wave theory; KdV equation

INTRODUCTION

The study of surface water waves is of fundamental scientific importance, and has practical implications for a diverse number of fields ranging from ship building to wave forecasting and coastal engineering. One particular problem of interest is the development of surface waves as they propagate shorewards and experience a decrease in the water depth. The study of such shoaling waves is of importance in the maintenance of beaches, and in the design of coastal structures, and the literature on shoaling waves contains a considerable number of experimental and numerical investigations.

Most theoretical results attempting to predict the waveheight change of shoaling waves can be roughly classified as treating either periodic waves oscillating around a mean undisturbed level, or as treating solitary waves. Indeed, two classical results in the theory of long wave shoaling are Green's law which concerns the shoaling of long periodic waves in the linear limit, and Boussinesq's law which applies to the shoaling of solitary waves. Green's law states that the waveheight of a periodic wavetrain experiencing a decrease in depth is proportional to $h^{\frac{-1}{4}}$, where h is the local depth. Boussinesq's law states that the waveheight of a shoaling solitary wave is proportional to h^{-1}. On the other hand, the linear theory of shoaling applies not only to long waves, but more generally to periodic wavetrains of arbitrary wavelength but very small wave steepness. As laid out for example in (Dean and Dalrymple, 1991) and (Sorensen, 1993), the linear theory is based on conservation of the wave frequency and the energy flux. An extension of the linear shoaling theory to weakly nonlinear waves was developed by (Ostrovskiy and Pelinovskiy, 1970) who computed shoaling curves for nonlinear waves described by steady cnoidal solutions of the Korteweg-de Vries (KdV) equation. However, the definition for the energy flux of the weakly nonlinear waves used in this work was the same as in the linear theory. (Svendsen and Brink-Kjær, 1972) also built a cnoidal theory of wave shoaling with the additional feature that linear shoaling theory was employed for waves originating in deep water with initial wavelength L_0. A matching point of $\frac{h}{L_0} = 0.1$ was used to transition from the linear to the cnoidal theory, but the expression for the energy flux used in the cnoidal theory was still identical to the expression in the linear case. Therefore, while this strategy allowed the computations of waveheights for shoaling waves originating in deep water, it gave a discontinuity in waveheight at the matching point. In order to fix this problem, (Svendsen and Buhr Hansen, 1977) imposed continuity in waveheight, but now at the expense of introducing a discontinuity in the energy flux.

While the studies mentioned above used definitions of wave energy and energy flux which are identical to the expressions valid in the linear approximation, the present study outlines the beginnings of a theory which makes use of a genuinely nonlinear definition of the energy flux. As will be shown in the body of this paper, the use of such a nonlinear expression for the energy flux eliminates the problem of discontinuity in waveheight at the matching point. When applied to very long waves, the theory yields good agreement with numerical and experimental results of shoaling solitary waves.

Previous theoretical results on the evolution of solitary waves over a gently sloping bottom include the results of (Grimshaw, 1971) who found that for small values of initial waveheight, the shoaling rates exhibit a certain deviation from Boussinesq's law. However, as also confirmed by (Kalisch and Senthilkumar, 2013), the shoaling rates approach Boussinesq's law in the limit of zero waveheight. Further analytical results for the shoaling and run-up of solitary waves can be found in (Synolakis, 1987), and a shoaling analysis using numerical solutions of the shallow-water equations was reported by (Hibberd and Peregrine, 1979).

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Many experimental campaigns, including the early studies of (Ippen and Kulin, 1954) and (Camfield and Street, 1969) focussed partly on comparison with classical shoaling laws such as the laws of Green and Boussinesq. However, most experimental work on wave shoaling has shown that actual shoaling rates differ significantly from the predictions of both Green's and Boussinesq's law. Recent studies have concentrated both on experiments (Grilli et al., 1994), and on direct numerical simulations based on potential flow theory for the Euler equations (Grilli et al., 1997) or numerical studies of certain higher-order Boussinesq systems (Wei et al., 1995). These studies paint a much finer picture of different shoaling regimes, and predict a variety of scaling relations for the local waveheight ahead and beyond the breaking point. For instance, (Grilli et al., 1997) have found that the waveheight of a shoaling solitary wave initially increases at a lower rate than that predicted by Green's law, but then increases. In (Synolakis and Skjelbreia, 1993), the solitary wave shoaling on a plane slope was found to feature four separate regions each with a different dependence of the waveheight on the sloping bottom.

Concerning the shoaling analysis of periodic waves, among the most accurate theoretical results are the studies of (Svendsen and Buhr Hansen, 1978), (Sakai and Battjes, 1980) and (Swift and Dixon, 1987). (Svendsen and Buhr Hansen, 1978) employed a time-periodic solution of the KdV equation of the second-order approximation to represent the deformation of periodic long waves due to the sloping bottom. (Sakai and Battjes, 1980) applied Cokelet's nonlinear numerical theory to investigate wave shoaling, and compared their results to existing shoaling curves calculated from different finite amplitude waves theories. Cokelet's theory is based on a high-order approximation of surface waves, and the results of (Sakai and Battjes, 1980) were compared both to experiments and shoaling theory based on streamfunction theory (Swift and Dixon, 1987), and good agreement was found.

In the present study the evolution of a cnoidal wave solution of the KdV equation is under review. The new feature of our study is that we require nonlinear energy conservation. This approach is made possible by a recent advance in the interpretation of Boussinesq type model equations in the context of approximate mass, momentum and energy conservation (Ali and Kalisch, 2012). This theory has proved successful in the study of the energy budget in an undular bore (Ali and Kalisch, 2010), and is now brought to bear on the problem of wave shoaling in the context of the KdV equation. The KdV equation is given in dimensional variables by

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + \frac{c_0 h_0^2}{6}\eta_{xxx} = 0, \tag{1}$$

where $\eta(x,t)$ stands for the excursion of the free surface, h_0 denotes the undisturbed water depth, g is the gravitational acceleration and $c_0 = \sqrt{gh_0}$ is the limiting long-wave speed. In the classical long-wave scaling, it is assumed that $\beta = \frac{h_0^2}{l^2}$ and $\alpha = \frac{a}{h_0}$ are small parameters and $\frac{\alpha}{\beta} = O(1)$, where l represents a typical wave length and a a typical wave amplitude. Equation (1) has stationary solution in the form of the so-called cnoidal waves

$$\eta = f_2 + (f_1 - f_2)\mathrm{cn}^2\left(\sqrt{\frac{3(f_1 - f_3)}{4h_0^3}}(x - Ct); m\right), \tag{2}$$

which are given in terms of the Jacobian elliptic function cn with modulus $m = \frac{f_1 - f_2}{f_1 - f_3}$. These solutions depend on three parameters f_1, f_2 and f_3 which may be taken in the ordering $f_1 > f_2 > f_3$. The wave speed C and the wavelength L are given by

$$C = c_0\left(1 + \frac{f_1 + f_2 + f_3}{2h_0}\right) \quad \text{and} \quad L = K(m)\sqrt{\frac{16h_0^3}{3(f_1 - f_3)}}, \tag{3}$$

where $K(m)$ is the complete elliptic integral of the first kind. In this setup, f_1 and f_2 represent the wave crest and the wave trough, respectively. The parameter f_3 affects only the wave length and wave speed. The aim of the present investigation is to use the energy flux associated to the evolution equation (1), in order to develop a method to predict the wave amplitude of a wave as it propagates from a region with undisturbed depth h_0 to a region with a different (generally smaller) undisturbed depth h. It is assumed that the transition is very gentle, so that wave reflection can be neglected, and the waves readjust adiabatically to the new depth. These requirements are approximately met if the wavelength L of the wave running up the slope is much smaller than the characteristic length of the depth variation. With these provisos, the exact solutions (2) can be used together with the requirements that frequency, mass and energy flux be conserved to arrive at a set of equations which can be solved numerically to describe the variation of the waveheight of a shoaling wave.

The disposition of the paper is as follows. In the next section, a brief resume of the considerations that lead to the formulation of an approximate energy balance for the KdV equation is given. In the third section, a shoaling theory based on the energy flux corresponding to the KdV equation is put forward. In the forth section, shoaling curves for periodic waves are computed, and the results are compared to the linear theory, the results of (Svendsen and Brink-Kjær, 1972), (Svendsen and Buhr Hansen, 1977), and finally to the higher-order computations of (Sakai and Battjes, 1980). Next, the shoaling of solitary waves is investigated. Results based on the new theory are compared to numerical results obtained by direct numerical integration of the full Euler equations (Grilli et al., 1997). Finally, a short conclusion is given.

ENERGY BALANCE

For the convenience of the reader, a brief review of the process of approximating the energy balance in the context of the KdV equation is given. For more details, the reader is referred to (Ali and Kalisch, 2014). Consider an inviscid, incompressible fluid of unit density. Cartesian coordinates (x, z) are chosen so that the $x-$axis is in the direction of wave propagation and the $z-$axis pointing vertically upwards, and the free surface is described by $z = \eta(x, t)$. The fluid domain at time $t \geq 0$ is given by $\{(x, z) \in \mathbb{R}^2 \mid x \in \mathbb{R}, \ -h_0 < z < \eta(x, t)\}$. Let $\mathbf{u} = (u(x, z, t), w(x, z, t))$ be the velocity field, let $P(x, z, t)$ be the pressure, and let $\mathbf{g} = (0, -g)$ be the gravitational forcing. The surface water-wave problem is generally given by the Euler equations

$$\mathbf{u}_t + (\mathbf{u}\cdot \nabla)\mathbf{u} + \nabla P = \mathbf{g}, \tag{4}$$

$$\nabla\cdot \mathbf{u} = 0, \tag{5}$$

with no-flow conditions at the bottom and appropriate kinematic and dynamic boundary conditions at the free surface. Using the incompressibility of the fluid and assuming irrotational flow, the problem can be written in terms of the Laplace equation for a velocity potential ϕ. The complete problem is then given by

$$\triangle\phi = 0, \text{ in } -h_0 < z < \eta(x, t), \tag{6}$$

$$\eta_t + \phi_x\eta_x - \phi_z = 0, \text{ on } z = \eta(x, t), \tag{7}$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0, \text{ on } z = \eta(x, t). \tag{8}$$

$$\phi_z = 0, \text{ on } z = -h_0. \tag{9}$$

We perform the change of variables

$$\tilde{x} = \frac{x}{l}, \qquad \tilde{z} = \frac{z + h_0}{h_0}, \qquad \tilde{\eta} = \frac{\eta}{a}, \qquad \tilde{t} = \frac{c_0 t}{l}, \qquad \tilde{\phi} = \frac{c_0}{gal}\phi,$$

to transform the problem into non-dimensional form. Following the method explained in (Bona, Chen and Saut, 2002) and (Whitham, 1974), the KdV equation is found to be

$$\tilde{\eta}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \frac{3}{2}\alpha\tilde{\eta}\tilde{\eta}_{\tilde{x}} + \frac{1}{6}\beta\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}} = O(\alpha^2, \alpha\beta, \beta^2),$$

and the non-dimensional velocity field $(\tilde{\phi}_{\tilde{x}}, \tilde{\phi}_{\tilde{z}})$ is given by

$$\tilde{\phi}_{\tilde{x}}(\tilde{x}, \tilde{z}, \tilde{t}) = \tilde{\eta} + \frac{1}{4}\alpha\tilde{\eta}^2 + \beta\Big(\frac{1}{3} - \frac{\tilde{z}^2}{2}\Big)\tilde{\eta}_{\tilde{x}\tilde{x}} + O(\alpha^2, \alpha\beta, \beta^2), \tag{10}$$

$$\tilde{\phi}_{\tilde{z}}(\tilde{x}, \tilde{z}, \tilde{t}) = \beta\tilde{z}\tilde{\eta}_{\tilde{x}} + O(\alpha\beta, \beta^2). \tag{11}$$

The dynamic pressure P', which measures the deviation from hydrostatic pressure, is given by

$$P' = P - P_{atm} + \rho g z = -\rho\phi_t - \frac{\rho}{2}|\nabla\phi|^2.$$

Using the scaling $\rho g a\tilde{P}' = P'$, the dynamic pressure becomes

$$\tilde{P}' = \tilde{\eta} + \frac{1}{2}\beta(\tilde{z}^2 - 1)\tilde{w}_{\tilde{x}\tilde{t}} + O(\alpha\beta, \beta^2).$$

Now, we examine energy balance in the KdV approximation. If it is assumed that potential energy is zero when there is no wave motion, the energy balance in the fluid is given by the equation

$$\frac{\partial}{\partial t}\left(\int_{-h_0}^{\eta} \frac{1}{2}|\nabla\phi|^2\, dz + \int_0^{\eta} gz\, dz\right) + \frac{\partial}{\partial x}\int_{-h_0}^{\eta}\left\{\frac{1}{2}|\nabla\phi|^2 + gz + P\right\}\phi_x\, dz = 0.$$

Using non-dimensional variables and performing an integration with respect to $\tilde{z}$, the equation becomes

$$\frac{\partial}{\partial \tilde{t}}\left(\alpha^2\tilde{\eta}^2 + \frac{\alpha^3}{4}\tilde{\eta}^3 + \frac{\alpha^2\beta}{6}\tilde{\eta}\tilde{\eta}_{\tilde{x}\tilde{x}} + \frac{\alpha^2\beta}{6}\tilde{\eta}_{\tilde{x}}^2\right) + \frac{\partial}{\partial \tilde{x}}\left(\alpha^2\tilde{\eta}^2 + \frac{5}{4}\alpha^3\tilde{\eta}^3 + \frac{\alpha^2\beta}{2}\tilde{\eta}\tilde{\eta}_{\tilde{x}\tilde{x}}\right) = O(\alpha^4, \alpha^3\beta, \alpha^2\beta^2).$$

>From this relation, the terms needed in the expression for the energy density and the energy flux (including the work done by pressure forces) can be read off. Indeed, dividing by α^2, it becomes plain that if energy conservation is to be valid to the same order in α and β as the evolution equation (1), then the energy density must be given by

$$\tilde{E}^* = \alpha^2\tilde{\eta}^2 + \frac{\alpha^3}{4}\tilde{\eta}^3 + \frac{\alpha^2\beta}{6}\tilde{\eta}\tilde{\eta}_{\tilde{x}\tilde{x}} + \frac{\alpha^2\beta}{6}\tilde{\eta}_{\tilde{x}}^2,$$

and the energy flux should be

$$\tilde{q}_E^* = \alpha^2\tilde{\eta}^2 + \frac{5}{4}\alpha^3\tilde{\eta}^3 + \frac{\alpha^2\beta}{2}\tilde{\eta}\tilde{\eta}_{\tilde{x}\tilde{x}}.$$

The dimensional forms of these quantities are then given by

$$E^* = c_0^2\left(\frac{1}{h_0}\eta^2 + \frac{1}{4h_0^2}\eta^3 + \frac{h_0}{6}\eta\eta_{xx} + \frac{h_0}{6}\eta_x^2\right),$$

and

$$q_E^* = c_0^3\left(\frac{1}{h_0}\eta^2 + \frac{5}{4h_0^2}\eta^3 + \frac{h_0}{2}\eta\eta_{xx}\right). \tag{12}$$

Comparing (12) to the well known expression for the energy flux in the linear theory, it appears that in the limit as amplitude approaches zero, and the wavelength becomes large, the expression reduces to the linear energy flux if it is averaged in the appropriate way.

DETERMINATION OF THE WAVEHEIGHT

It is customary to base the analysis of a shoaling wavetrain on the assumption that the frequency of the waves is preserved. In addition, as indicated in Figure 1, the pressure force acting on the fluid at the sloping bottom is perpendicular to the fluid velocity, so that the energy flux of the waves is also constant. Finally, conservation of mass is used. Thus if the wave motion at a certain water depth h_A is given, the waveheight and other wave parameters at a new water depth h are determined by the following equations:

$$\frac{C_A}{L_A} = \frac{C}{L}, \tag{13}$$

$$\int_0^T q_{E_A}^*\, dt = \int_0^T q_E^*\, dt, \tag{14}$$

$$\int_0^L \eta_A dx = \int_0^L \eta\, dx. \tag{15}$$

Using the expression (12) for q_E^* and the exact solution η given by (2) with celerity C and wavelength L

given by (3), we arrive at three nonlinear equations for the three parameters f_1, f_2 and f_3. In order to write the equations in compact form, note first that

$$\int \mathrm{cn}^2(u)du = u - \frac{u}{m^2} + \frac{E(\mathrm{am}(u), m^2)}{m^2},$$
$$\int \mathrm{cn}^4(u)du = \frac{1}{3m^4}\Big[(2-3m^2)(1-m^2)u + 2(2m^2-1)E(\mathrm{am}(u), m^2) + m^2\mathrm{sn}(u)\mathrm{cn}(u)\mathrm{dn}(u)\Big],$$
$$\int \mathrm{cn}^6(u)du = \frac{1}{5m^2}\Big[4(2m^2-1)\int \mathrm{cn}^4(u)du + 3(1-m^2)\int \mathrm{cn}^2(u)du + \mathrm{cn}^3(u)\mathrm{sn}(u)\mathrm{dn}(u)\Big].$$

Here $E(\cdot, m)$ is the elliptic integral of the second kind, and $\mathrm{am}(\cdot)$ is the amplitude function (Lawden, 1989). As mentioned above, the wave parameters at water depth h_A are considered given, and therefore the expressions on the left-hand side of equations (13), (14) and (15) are given by the constant values

$$A_0 = \frac{C_A}{L_A}, \qquad A_1 = \int_0^T q^*_{E_A} dt, \qquad A_2 = \int_0^L \eta_A dx.$$

With these definitions, the equation (13) becomes

$$A_0 = \frac{c_0\left(1 + \frac{f_1+f_2+f_3}{2h_0}\right)}{K(m)\sqrt{\frac{16h_0^3}{3(f_1-f_3)}}},$$

the equation (14) becomes

$$A_1 = c_0^3\Bigg[T\Bigg(B_0 + \frac{(m^2-1)}{m^2}B_2 + \frac{B_4}{3m^4}(2-3m^2)(1-m^2) + B_6\Bigg\{\frac{-3}{5m^4}(m^2-1)^2 + \frac{4}{15m^6}(2m^2-1)(2-3m^2)(1-m^2)\Bigg\}\Bigg)$$

$$-\left(B_2 + \frac{3B_6}{5m^2}(1-m^2) + \frac{2B_4}{3m^2}(2m^2-1) + \frac{8B_6}{15m^4}(2m^2-1)^2\right)\frac{E(m)\sqrt{4h^3}}{c_0 m^2\left(1 + \frac{f_1+f_2+f_3}{2h_0}\right)\sqrt{3(f_1-f_3)}}\Bigg],$$

and the equation (15) reads

$$A_2 = K(m)\sqrt{\frac{16h_0^3}{3(f_1-f_3)}}\left(f_2 + \frac{(m^2-1)(f_1-f_2)}{m^2}\right) + \frac{(f_1-f_2)E(m)\sqrt{4h^3}}{m^2\sqrt{3(f_1-f_3)}}.$$

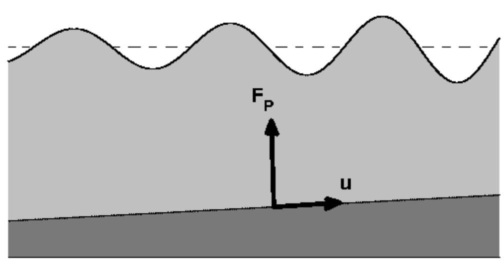

Figure 1: In the inviscid theory, the fluid velocity near the solid bottom is parallel to the boundary. However, the pressure force is perpendicular to the boundary. In this configuration, the energy flux is invariant.

In these equations, $E(m)$ denotes the complete elliptic integral of the second kind, and we have used the additional constants

$$B_0 = \frac{f_2^2}{h} + \frac{5f_2^3}{4h^2} + \frac{3f_2}{4h^2}(1-m^2)(f_1-f_2)(f_1-f_3),$$
$$B_2 = \frac{2f_2}{h}(f_1-f_2) + \frac{15f_2^2}{4h^2}(f_1-f_2) + \frac{3}{4h^2}(f_1-f_2)(f_1-f_3)\Big[2f_2(2m^2-1) + (1-m^2)(f_1-f_2)\Big],$$
$$B_4 = \frac{(f_1-f_2)^2}{h} + \frac{15f_2}{4h^2}(f_1-f_2)^2 + \frac{3}{4h^2}(f_1-f_2)(f_1-f_3)\Big[2(2m^2-1)(f_1-f_2) - 3m^2 f_2\Big],$$
$$B_6 = \frac{(f_1-f_2)^2}{4h^2}\Big[5(f_1-f_2) - 9m^2(f_1-f_3)\Big].$$

The system of equations (13), (14) and (15) can then be solved numerically to obtain the parameters f_1, f_2 and f_3 defining the cnoidal wave at an arbitrary water depth h.

RESULTS

We first apply the method explained above to the shoaling of a periodic wavetrain originating in deep water, and then look at the shoaling of solitary waves. In the case of periodic wave shoaling, it was noted for example in (Svendsen and Buhr Hansen, 1977) that there is a deep-water limit beyond which cnoidal solutions of the the KdV equation cannot be used to describe periodic wave trains. Because of this limitation, it is necessary in the shoaling problem to compute the initial transition from deep water to intermediate depths by a different theory. In (Svendsen and Brink-Kjær, 1972) and (Svendsen and Buhr Hansen, 1977), linear wave theory was used for this initial transition, and then continued with cnoidal theory as soon as the depth became small enough for the shallow-water assumption to be valid. However, one problem which the authors of (Svendsen and Brink-Kjær, 1972) faced was that at the point where linear and cnoidal theory were to be matched, a discontinuity in waveheight appeared in the shoaling curve (see Figure 2). This problem was overcome later in (Svendsen and Buhr Hansen, 1977) by imposing continuity in waveheight directly, but at the cost of incurring a discontinuity in the energy flux. In addition, the use of tables was required for the construction of the shoaling curves.

The main difference between the present contribution and the work of (Svendsen and Brink-Kjær, 1972) and (Svendsen and Buhr Hansen, 1977) is that a genuinely nonlinear definition of the energy flux is used in the cnoidal region. Indeed, while (Svendsen and Brink-Kjær, 1972) and (Svendsen and Buhr Hansen, 1977) used the same definition of the energy flux for both the linear and the cnoidal solutions, we use the linear flux only for the linear part of the curve, while we use (12) for the nonlinear part of the shoaling curve. In particular, as shown in Figure 2, the use of q_E^* eliminates the problem of discontinuities at the matching point, and both waveheight and energy flux can be described continuously as the shoaling curve changes from linear to cnoidal theory. We assume that the waves have a waveheight H_0 and wavelength L_0 in deep water. In the linear theory, the waveheight change is described by the formula

$$\frac{H}{H_0} = \sqrt{\frac{C_{g,0}}{C_g}}$$

where $C_{g,0} = \frac{1}{2}\sqrt{\frac{gL_0}{2\pi}}$ is the deep-water group velocity, and $C_g = \frac{\omega}{2k}(1 + \frac{2kh}{\sinh(2kh)})$ is the local group velocity. The latter can be computed numerically since the local depth is known, and it is assumed that the period of the shoaling waves is constant, so that the wavenumber k can be computed from the dispersion relation $\omega^2 = gk\tanh(kh)$.

Concerning the computations shown in Figure 2, they were performed for different values of the deep-water steepness $\frac{H_0}{L_0}$ in such a way as to allow comparison to the work of (Svendsen and Brink-Kjær, 1972), (Svendsen and Buhr Hansen, 1977) and (Sakai and Battjes, 1980). Figure 2 shows the shoaling curves for three periodic wave profiles with $\frac{H_0}{L_0}$ equal to 0.004, 0.002 and 0.001. In all three cases, the wavelength is much larger than 10 times the water depth at the matching point, so that that the long-wave assumption is justified. Note also that the curves due to (Svendsen and Brink-Kjær, 1972), which were obtained by matching the calculated energy flux values of the cnoidal theory and linear theory at $\frac{h}{L_0} = 0.1$, have a discontinuity in waveheight. On the other hand, in the shoaling curves due to (Svendsen and Buhr Hansen,

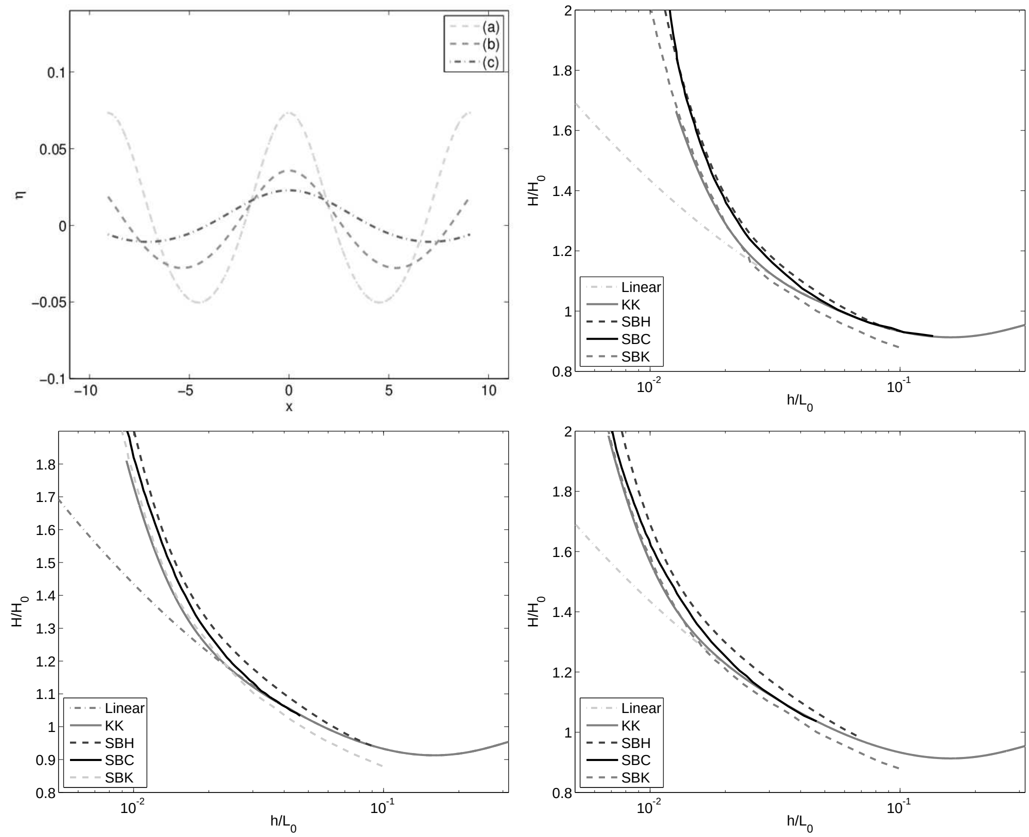

Figure 2: Wave profile and the corresponding shoaling curves.
Upper left panel: (*a*) $\frac{H_0}{L_0} = 0.004$; (*b*) $\frac{H_0}{L_0} = 0.002$; (*c*) $\frac{H_0}{L_0} = 0.001$. **Upper right panel:** $\frac{H_0}{L_0} = 0.004$**; lower left panel:** $\frac{H_0}{L_0} = 0.002$**; lower right panel** $\frac{H_0}{L_0} = 0.001$ **. The gray solid curve, KK, is the shoaling curve based on the present paper. The black solid curve, SBC, is the shoaling curve after Sakai and Battjes calculated from Cokelet's theory. The gray dashed curve, SBK, is the shoaling curve after Svendsen and Brink-Kjær. The black dashed curve, SBH, is the shoaling curve after Svendsen and Buhr Hansen.**

1977) the waveheights are manually matched at the point $\frac{h}{L_0} = 0.1$ through the use of conversion tables. As a result of this mechanical adjustment, the energy flux is discontinuous at the matching point, and it appears that the calculated waveheight values further up on the curve are affected. In fact, it appears that especially for larger amplitude waves, the shoaling curves due to (Svendsen and Buhr Hansen, 1977) show significantly higher values than all other curves shown in this figure. If it is assumed that the shoaling curves due to (Sakai and Battjes, 1980) using Cokelet's theory are the most accurate since they are computed fro the highest-order theory, then the comparison shows that shoaling of smaller amplitude waves is computed most accurately from (Svendsen and Buhr Hansen, 1977), while shoaling of larger amplitude waves is computed most accurately from (Svendsen and Brink-Kjær, 1972) and the present theory. However, as already noted, the present theory does not suffer from a discontinuity of either waveheight or energy flux at the matching point.

Next, the shoaling of solitary waves is discussed. In Figure 3, a wave profile close to a solitary wave with initial waveheight H_0 is 0.2 is shown. The wavelength is chosen in a similar way as explained in (Grilli et al., 1997), and the initial depth is h_0. A comparison between the present shoaling result and the numerical results of (Grilli et al., 1997) and also Boussinesq's law, $H \propto h^{-1}$, and Green's law, $H \propto h^{-1/4}$ is shown in the right panel of Figure 3. It can be seen that the waveheight increases initially more slowly than predicted by Green's law, but the shoaling curve then turns up, and reaches a slope similar to Boussinesq's law just before the breaking point. In Figure 4, a wave with initial waveheight $H_0 = 0.4$, is chosen. The wave profile is shown in the left panel, and the shoaling curve is shown in the right panel. In this case,

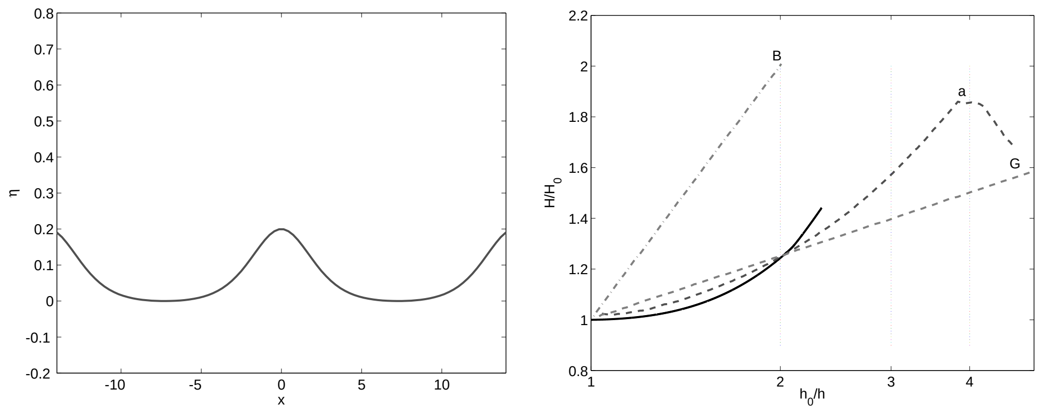

Figure 3: Wave profile and the corresponding shoaling curve with $\frac{L_0}{h_0} = 14.5$, $\frac{H_0}{h_0} = 0.2$, $\frac{H_0}{L_0} = 0.013$. a: shoaling curve after Grilli et al., black solid curve: shoaling curve of the present paper, G: Green's law and B: Boussinesq's law.

the comparison with the shoaling curve of (Grilli et al., 1997) is even better than in the the case of smaller initial waveheight. Finally in Figure 5, we consider a wave with the initial waveheight $H_0 = 0.6$, and the corresponding shoaling curve of the present work which is illustrated in the right panel, is also in good agreement with the numerical results of (Grilli et al., 1997). Note also that the wave steepness $\frac{H_0}{L_0}$ of the incident solitary waves is small in all three cases. One aspect in which the comparison is not favorable is the location of the breaking point. While the shoaling curve for smaller waveheights terminates too soon because of a lack of numerical accuracy, the shoaling curve for smaller waveheights overshoots the numerical breaking point of (Grilli et al., 1997). It is expected that a close study using a breaking criterion such as the one put forward in (Bjørkavåg and Kalisch, 2011) would improve the comparison, but this point has not been investigated further so far.

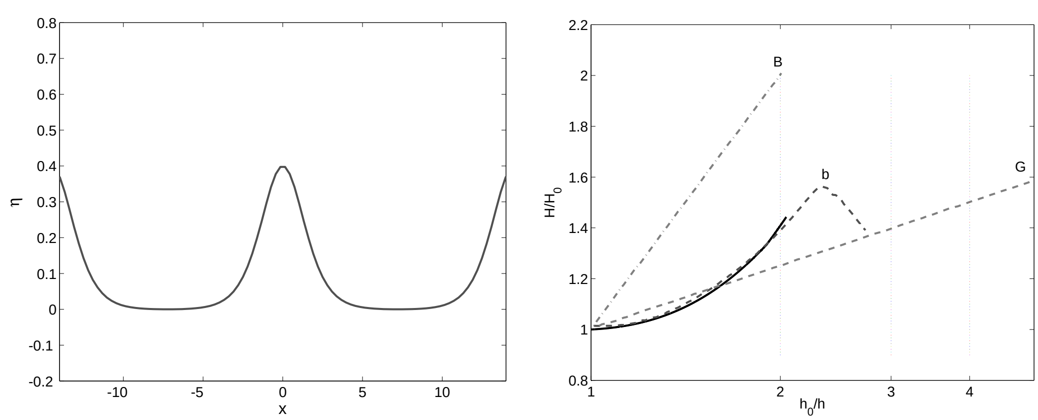

Figure 4: Wave profile and the corresponding shoaling curve with $\frac{L_0}{h_0} = 14.5$, $\frac{H_0}{h_0} = 0.4$, $\frac{H_0}{L_0} = 0.027$. b: shoaling curve after Grilli et al., black solid curve: shoaling curve of the present paper, G: Green's law and B: Boussinesq's law.

CONCLUSION

Changes in waveheight of surface waves propagating over decreasing fluid depth have been the focus of this article. A nonlinear definition of the energy flux in the KdV equation, and the conservation of wave period, mass and energy flux lead to three coupled equations which can be solved numerically by

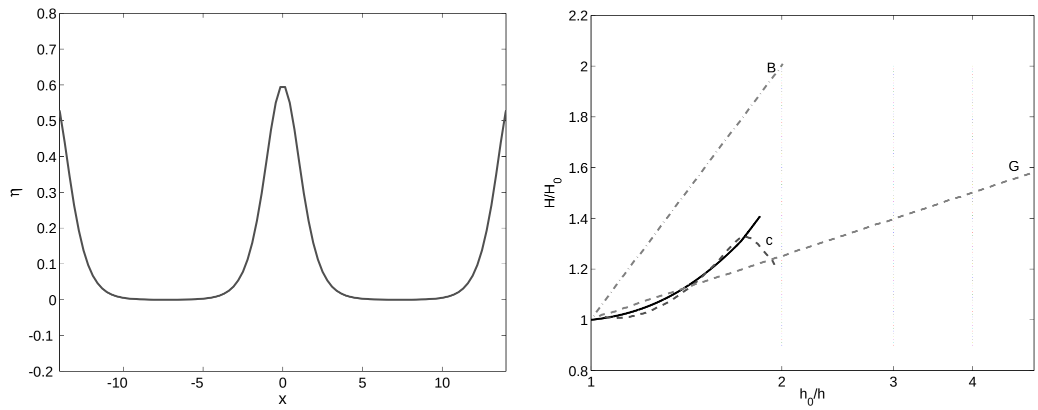

Figure 5: Wave profile and the corresponding shoaling curve with $\frac{L_0}{h_0} = 14.5$, $\frac{H_0}{h_0} = 0.6$, $\frac{H_0}{L_0} = 0.041$. c: shoaling curve after Grilli et al., black solid curve: shoaling curve of the present paper, G: Green's law and B: Boussinesq's law.

Newton's method. Numerical solutions of these equations have been used to construct shoaling curves for both periodic wavetrains and solitary waves. In the periodic case, the calculated values of $\frac{H}{L_0}$ have been plotted against $\frac{h}{L_0}$, and the results have been compared with three previous methods of construction shoaling curves: the methods of (Svendsen and Brink-Kjær, 1972), of (Svendsen and Buhr Hansen, 1977), and of (Sakai and Battjes, 1980). The nonlinear definition of the energy flux has been instrumental in removing the discontinuity in the shoaling curve which appeared when the linear definition of the energy flux is used in both the linear and the KdV regime.

Concerning the shoaling of solitary waves, the computations have been performed for different values for the waveheight of the incident wave. The calculated values of $\frac{H}{H_0}$ have been plotted versus $\frac{h}{h_0}$, and it has been found that the variation in waveheight is below Green's law at first, but then increases as the water depth keeps decreasing. These findings are in line with the results of (Grilli et al., 1997).

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References

Ali, A. and Kalisch, H., 2010, Energy balance for undular bores, *C R Mecanique*, 338, 67–70

Ali, A. and Kalisch, H., 2012, Mechanical balance laws for Boussinesq models of surface water waves, *J. Nonlinear Sci.*, 22, 371–398

Ali, A. and Kalisch, H., 2014, On the formulation of mass, momentum and energy conservation in the KdV equation, *Acta Appl. Math.*, 133, 113–131

Bjørkavåg, M. and Kalisch, H., 2011, Wave breaking in Boussinesq models for undular bores, *Phys. Lett. A*, 375, 1570–1578

Bona, J.L., Chen, M. and Saut, J.-C., 2002, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory, *J. Nonlinear Sci.*, 12, 283–318

Camfield, F.E. and Street, R.L., 1969, Shoaling of solitary waves on small slopes, *J. Wtrwy., Port, Coast., and Oc. Engrg, ASCE*, 95, 1–22

Dean, R.G. and Dalrymple, R.A., *Water wave mechanics for engineers and scientists,* (World Scientific, Singapore, 1991)

Grilli, S.T., Subramanya, R., Svendsen, I.A. and Veeramony, J., 1994, Shoaling of solitary waves on plane beaches, *J. Wtrwy., Port, Coast., and Oc. Engrg.*, 120, 609–628

Grilli, S.T., Svendsen, I.A. and Subramanya, R., 1997, Breaking criterion and characteristics for solitary waves on slopes, *J. Wtrwy., Port, Coast., and Oc. Engrg.*, 123, 102–112

Grimshaw, R., 1971, The solitary wave in water of variable depth, part 2, *J. Fluid Mech.*, 46, 611–622

Hibberd, S. and Peregrine, D.H., 1979, Surf and runup on a beach: a uniform bore, *J. Fluid Mech.*, 95, 323–345

Ippen, A.T. and Kulin, G., 1954, The shoaling and breaking of the solitary wave, *Proc. 5th Conf. On Coast. Eng., Grenoble*, 27–47

Kalisch, H. and Senthilkumar A., 2013, Derivation of Boussinesq's shoaling law using a coupled BBM system, *Nonlin. Processes Geophys.*, 20, 213–219

Lawden, D.F., *Elliptic functions and applications,* (Springer, New York, 1989)

Ostrovskiy, L.A. and Pelinovskiy, E.N., 1970, Wave transformation on the surface of a fluid on variable depth, *Atmos. Ocean. Phys.*, 6, 552–555

Papanicolaou, P. and Raichlen, F., 1987, Wave characteristics in the surf zone, *Proc. Coast. Hydrodynamics*, (R. A. Dalrymple, ed.), ASCE, New York, 765–780

Sakai, T. and Battjes, J., 1980, Wave shoaling calculated from Cokelet's theory, *Coastal Engineering*, 4, 65–84

Sorensen, R.M., *Basic wave mechanics: for coastal and ocean engineers,* (Wiley, New York, 1993)

Svendsen, I.A. and Brink-Kjær, O., 1972, Shoaling of cnoidal waves, *Proc. 13th Conf. Coastal Engng, Vancouver*, 365–383

Svendsen, I.A. and Buhr Hansen, J., 1977, The wave height variation for regular waves in shoaling water, *Coastal Engineering*, 1, 261–284

Svendsen, I.A. and Buhr Hansen, J., 1978, On the deformation of periodic long waves over a gently sloping bottom, *J. Fluid Mech.*, 87, 433–448

Swift, R.H. and Dixon, J.C., 1987, Transformation of regular waves, *Proc. Instn Civ. Engrs, Part2*, 83, 359–380

Synolakis, C.E., 1987, The runup of solitary wave, *J. Fluid Mech.*, 185, 523–545

Synolakis, C.E. and Skjelbreia, J.E., 1993, Evolution of maximum amplitude of solitary waves on plane beaches, *J. Wtrwy., Port, Coast., and Oc. Engrg., ASCE,* 119, 323–342

Wei, G., Kirby, J.T., Grilli, S.T. and Subramanya, R., 1995, A fully nonlinear Boussinesq model for surface waves. Part1. Highly nonlinear unsteady waves, *J. Fluid Mech.*, 294, 71–92

Whitham, G.B., *Linear and nonlinear waves,* (Wiley, New York, 1974)