COMBINATORIAL DESCRIPTION OF JUMPS IN SPECTRAL NETWORKS

ANASTASIA FROLOVA AND ALEXANDER VASIL'EV

Abstract. We describe a graph parametrization of rational quadratic differentials with presence of a simple pole, whose critical trajectories form a network depending on parameters focusing on the network topological jumps. Obtained bifurcation diagrams are associated with the Stasheff polytopes.

1. Introduction

The problem of BPS (Bogomol'nyi–Prasad–Sommerfield) wall crossing have received much attention the last decade, see e.g, [1, 2, 3, 4, 12, 13]. In physics terms, a supersymmetric particle may change from stable to unstable crossing loci (walls) in a parameter space. Considering four-dimensional $\mathcal{N} = 2$ theories coupled to surface defects, particularly the theories of class S, see [24], Gaiotto, Moore, and Neitzke [5] introduced *spectral networks* of trajectories on Riemann surfaces obeying certain local rules aiming at the characterization of the possible spectra of BPS states and their allowed changes under continuous deformations of the theory. Given a compact Riemann surface $\mathcal{R}$ with punctures and a Lie algebra $\mathfrak{g}$ of ADE type, e.g., SU(2) in our case, there exists a corresponding four-dimensional quantum field theory $S(\mathcal{R}, \mathfrak{g})$, see [12, 24]. The spectral network is defined by the critical trajectories of a quadratic differential q given by $q(z)dz^2$ in a local parameter z, which defines a singular measured foliation of $\mathcal{R}$ with singularities at the zeros and poles of q. The differential is holomorphic on $\mathcal{R}$ and has possible poles at the punctures. The trajectories emerging from the zeros form the spectral network. For certain values of the zeros, there occur critical trajectories starting and ending at them, and we say that the network undergoes jumps and splits $\mathcal{R}$ into cells. Generic small variation of zeros changes the network by isotopy whereas the jumps occur for certain values of them. Such critical trajectories we will call *short*. Counting the special trajectories is related to generalized Donaldson-Thomas invariants of the theory.

Short trajectories of q turn to play an important role also in potential theory, approximation theory and other branches of mathematics. For example, short trajectories of rational quadratic differentials describe limiting distributions of certain

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types of orthogonal polynomials, see e.g., [16, 17, 18]. Motivated by applications to minimal surfaces, Bruce and O'Shea published a preprint [9], where the short trajectories characterized umbilical points and the geometry of unfolding. Baryshnikov [6, 7] described the combinatorial structure of the Stokes sets for polynomials in one variable by bifurcation diagrams, and in particular, encoded the short trajectories of the differential q in the simplest case when $\mathcal{R} = \mathbb{C}$ and q is a versal deformation of $z^n dz^2$. It was proved in [14] that the versal deformation of $z^n\,dz^2$ is the family $(z^n + a_{n-2}z^{n-2} + ... + a_0)\,dz^2$, where a_k, $0 \le k \le n-2$, are complex parameters. It can be understood as a family which includes in a certain sense all quadratic differentials of the form $p_n(z)\,dz^2$, where $p_n(z)$ is a monic polynomial of degree n. The set of all parameters $(a_{n-2}, ..., a_0)$ in the parameter base space $\mathbb{C}^{n-2}$, for which the corresponding quadratic differential has a short trajectory, is the bifurcation diagram of the versal deformation, i.e., whenever a parameter $(a_{n-2}, ..., a_0)$ belongs to the bifurcation diagram, a small change of parameter causes a significant change in the trajectory structure. Using formal power series Bruce and O'Shea gave an explicit form of the bifurcation diagram for the case $n = 2$. They initiated the study of combinatorial structure of bifurcation diagrams for arbitrary n, which was completed by Baryshnikov [6, 7], who gave combinatorial and geometric descriptions of the set of polynomial quadratic differentials with short trajectories. He also established correspondence between polynomial quadratic differentials and weighted graphs, and used the connection between weighted graphs and the Stasheff polyhedra.

The combinatorial description of the Stokes sets above is equivalent to the description of admissible contours among which there is a solution to a max-min energy variational problem in logarithmic potential theory in the polynomial external field, and the extremal contour satisfies the S-property introduced and studied by Stahl, Gonchar, and Rakhmanov [10, 20]. The equilibrium measure is supported on a finite union of arcs of this extremal contour, see [15, Theorem 2.3].

The latter and physics motivation encouraged us to consider the case of quadratic differential with the presence of poles, in particular, the case of one simple pole. The domains in the trajectory structure of the differential in our approach contain ending domains [11, 23] (or half-plane domains in terminology of [22]) and strip domains. More poles destroy completely the proposed picture because even two simple poles guarantee new types of domains, i.e., ring domains and dense structures. We so far do not know what kind of graphs could parametrize them. So our result in some sense extends Baryshnikov's approach up to the end.

Let us remark that different graph encodings of quadratic differentials were also used as a tool for solving a number of other problems. For example, Bogatyrëv in [8] used certain graphs based on quadratic differentials in connection with the problem of description of extremal polynomials. Solynin [19] established the connection between weighted graphs and quadratic differentials with closed trajectories.

The outline of the paper is as follows. In Section 2, we introduce the correspondence between weighted chord diagrams and the Stasheff polyhedra through the balanced weights following [6]. In Section 3, we discuss briefly the trajectory structure of rational quadratic differentials with a simple pole. We establish one-to-one correspondence between weighted graphs and rational quadratic differentials with a simple pole in Section 4. Graphs and weighted chord diagrams identified with the quadratic differentials with short trajectories are described there. The latter allows us to use the correspondence between weighted chord diagrams and the Stasheff polyhedra to obtain an analogue of the bifurcation diagram for the case of rational quadratic differentials with a simple pole.

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2. Weighted chord diagrams and balanced weights

Following Baryshnikov [6, 7] we introduce weighted chord diagrams, Stasheff polyhedra, balanced weights, and describe the correspondence between them. Although this section does not contain essentially new results, we take the liberty to complete some details and proofs missing in [6, 7].

A polytope C in $\mathbb{R}^d$ is a convex hull of a certain number of points in $\mathbb{R}^d$. If C intersects a hyperplane H and lies entirely in one of the half-spaces defined by H, we call $H \cap C$ a face of C. The vertices and edges of a polytope C are $0-$ and $1-$dimentional faces of C respectively. Any given vector $v \in \mathbb{R}^d$ determines a face $F_v(C)$ of C:

$$F_v(C) = \{x \in C : x \cdot v \geq y \cdot v \ \ \forall y \in C\}.$$

$F_v(C)$ is an intersection of C with a hyperplane which goes through the point $\operatorname{argmax}_{x \in C} x \cdot v$ and has v as the normal vector. For $v = 0$ we obtain the entire polytope C. For any face F of C we define a normal cone $N_F(C)$ as

$$N_F(C) = \{v \in \mathbb{R}^d : F = F_v(C)\}.$$

Note that if the face F has dimension l and $l \leq d$, then the dimension of the normal cone $N_F(C)$ is $d - l$. The collection of all normal cones of C is called the normal fan of C.

Stasheff polyhedron (associahedron) K_n, see [21], is an $(n-2)-$dimensional polytope. Each vertex of K_n corresponds to a maximal bracketing of a string of n symbols, and each 1-D edge corresponds to a single application of associativity rule removing one bracket. Following applications correspond to $(n-k)$-D faces, $k = 0, 1, \ldots, n-1$. For example, K_3 consists of two vertices represented by $(ab)c$ and $a(bc)$ and one edge abs connecting them. Analogously, K_4 is a plane pentagon and K_5 is a 3-D polyhedron.

Alternatively, K_n can be realized as a polytope whose vertices represent triangulations of a regular $(n+1)-$gon and edges represent diagonal flips. Triangulation of a polygon is a collection of non-intersecting diagonals; it is said to be incomplete if the number of diagonals is not maximal. The vertices of the polytope dual to K_n correspond to incomplete triangulations of the $(n+1)-$gon.

Example 1. The triangulation realization of K_4 is shown on figure 1.

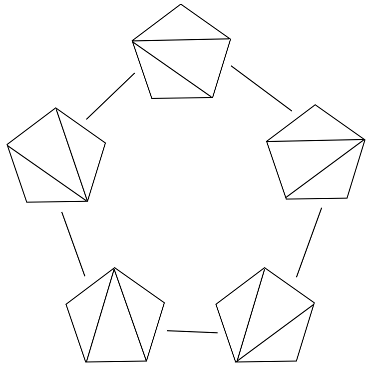

FIGURE 1. The triangulation realization of K_4.

Normal fan Σ_n to the Stasheff polytope K_n is called the Stasheff fan. The union of the cones of Σ_n constitutes $\mathbb{R}^{n-2}$. The number of full-dimensional cones is equal to the Catalan number $c_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$.

Example 2. The Stasheff fan Σ_4 is illustrated on Figure 2.

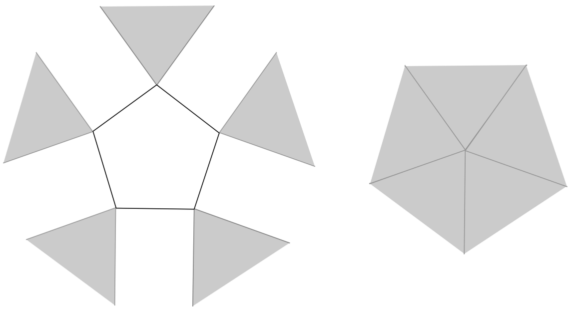

FIGURE 2. Normal fan Σ_4

Suppose we have a convex regular $(n+1)-$gon P. Together with some weighted non-intersecting diagonals it is called a weighted chord diagram. In this case we say that the weighted chord diagram is based on P.

A balanced weight is a function defined on the vertices of the $(n+1)-$gon P, such that the sum of its values at the vertices is zero and the geometric center of masses is at the origin. A balanced weight f is called degenerate if there exists a real linear function L and vertices $a_{1,2,3,4}$, such that L majorizes f and coincides with it at the vertices $a_{1,2,3,4}$.

Balanced weights form a linear space of real dimension $n-2$. According to Baryshnikov, the degenerate balanced weights form a fan Σ_n, which is a normal fan for the Stasheff polytope K_n.

In what follows, we describe the correspondence between weighted chord diagrams and balanced weights.

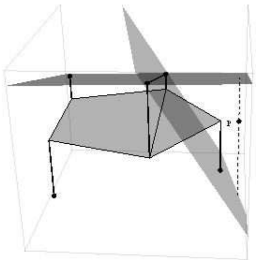

FIGURE 3. Construction of a chord.

Lemma 1. *There is one-to-one correspondence between balanced weights and weighted chord diagrams.*

Proof. Let us show that each balanced weight gives rise to a weighted chord diagram. We fix a point p lying on the plane of the polygon P and in general position with respect to P. Let a and c be two arbitrary non-adjacent vertices of P. We consider all the real linear functions L on the plane of P, such that $L(a) = f(a)$, $L(c) = f(c)$, and $L|_P(z) \geq f(z)$ for any vertex z of P. The values of such linear functions at p swipe out an interval of length v, $v \geq 0$. If $v > 0$, we construct a diagonal with weight v joining a and c. We go through this procedure for any pair of non-adjacent vertices of P and we construct all possible diagonals. Construction of a chord is illustrated in Figure 3.

Note that the diagonals in the resulting diagram do not intersect, i.e., we obtain a weighted chord diagram. Suppose we have constructed two intersecting diagonals a_1a_2 and b_1b_2. Then there exist linear functions L and Λ, satisfying the relations

$$\begin{aligned} &L|_P \geq f,\ L(a_{1,2}) = f(a_{1,2}), \\ &\Lambda|_P \geq f,\ \Lambda(b_{1,2}) = f(b_{1,2}). \end{aligned} \tag{1}$$

The latter gives us that

$$\begin{aligned} L(a_{1,2}) &= f(a_{1,2}) \leq \Lambda(a_{1,2}), \\ \Lambda(b_{1,2}) &= f(b_{1,2}) \leq L(b_{1,2}). \end{aligned}$$

Thus, the function $d(z) = \Lambda(z) - L(z)$ satisfies the inequalities $d(a_{1,2}) \geq 0$ and $d(b_{1,2}) \leq 0$. As a_1 and a_2 lie on different sides with respect to b_1b_2 and d is real linear, we obtain that d vanishes identically. Therefore, the diagonals a_1a_2 and b_1b_2 fail to exist and we arrive at a contradiction.

Analogously, for each weighted chord diagram there is a balanced weight corresponding to it. $\square$

Furthermore, there is a one-to-one correspondence between the degenerate balanced weights and weighted chord diagrams with incomplete triangulation. We separate the proof of this fact into three lemmas.

Lemma 2. *Suppose f is a degenerate balanced weight and P is the weighted chord diagram corresponding to f. Then P has an incomplete triangulation.*

Proof. Suppose we are given a degenerate weight f, i.e., there exists a linear function L and vertices $a_k \in P$, $1 \le k \le 4$, such that $L|_P \ge f$, $L(a_k) = f(a_k)$, $1 \le k \le 4$. The chord diagram corresponding to the weight f can not have a diagonal that intersects the interior of the quadrilateral Q formed by a_k, $1 \le k \le 4$, and thus, triangulation of P is not complete. To show this, we assume that P has non-adjacent vertices b_1 and b_2, such that the diagonal b_1b_2 intersects the interior of Q. As the diagonal b_1b_2 exists, there must be a linear function Λ, such that $\Lambda|_P \ge f$, $\Lambda(b_{1,2}) = f(b_{1,2})$. As L majorizes f, we have that $\Lambda(b_{1,2}) = f(b_{1,2}) \le L(b_{1,2})$. Thus, the real linear function d defined by $d(z) = \Lambda(z) - L(z)$ satisfies $d(b_{1,2}) \le 0$. Since b_1b_2 intersects the interior of Q, there are two vertices a_k and a_j, $1 \le k < j \le 4$, which are separated by the line containing the diagonal b_1b_2. Since Λ majorizes f, we obtain that $\Lambda(a_{k,j}) \ge f(a_{k,j}) = L(a_{k,j})$ and $d(a_{k,j}) \ge 0$. Such a behaviour of the sign of a linear function d is possible if and only if $d \equiv 0$. Therefore, there may be only one linear function majorizing f and coinciding with it at $b_{1,2}$, which contradicts the existence of the diagonal b_1b_2.

□

The converse to Lemma 2 is also true, and we need the following lemma to prove this.

Lemma 3. *Let a weighted chord diagram have a chord a_1a_2 with some positive weight. Let f be the balanced weight corresponding to the diagram. Then there are two distinct vertices a_3 and a_4, which lie on different sides with respect to a_1a_2, such that there exist distinct linear functions L and Λ majorizing f and satisfying the relations*

$$L|_P \ge f, \;\; L|_P\,(a_{1,2,3}) = f(a_{1,2,3}),$$
$$\Lambda|_P \ge f, \;\; \Lambda|_P\,(a_{1,2,4}) = f(a_{1,2,4}). \tag{2}$$

Proof. Indeed, there must exist distinct vertices a_3, a_4 and distinct linear functions L, Λ satisfying relations (2), because otherwise the chord does not have a positive weight. Let us assume now that a_3 and a_4 lie on one side with respect to the line segment a_1a_2. The relations (2) imply that

$$L(a_3) = f(a_3) \le \Lambda(a_3),$$

$$\Lambda(a_4) = f(a_4) \le L(a_4),$$

and thus, the linear function $d(z) = \Lambda(z) - L(z)$ satisfies the inequalities $d(a_3) \ge 0$ and $d(a_4) \le 0$. In addition we have that $d(a_{1,2}) = 0$. As the points a_3 and a_4 lie on one side with respect to the chord a_1a_2, the linear function d has to vanish identically, which contradicts the existence of the chord a_1a_2. □

Lemma 4. *A balanced weight corresponding to a weighted chord diagram with an incomplete triangulation is degenerate.*

Proof. Let f be a balanced weight which corresponds to the weighted chord diagram P. If P has no diagonals, there must be a linear function whose restriction to P is identically equal to f, and thus f is degenerate. Assume the contrary. We denote by a_1 and a_2 the vertices of P such that the values $f(a_1)$ and $f(a_2)$ are the biggest and second biggest values of f. Let a_3 and a_4 be the argument of the maximum of f among the vertices to the left and to the right from the line segment a_1a_2. Linear functions L_1 and L_2, which coincide with f at $a_{1,2,3}$ and $a_{1,2,4}$ respectively, majorize f. By assumption, L_1 and L_2 are not identically equal, and thus, the chord a_1a_2 must have a positive weight, and we arrive at a contradiction.

Assume now that the chord diagram has a chord a_1a_2 with a positive weight. Then there are a vertex a_3 and a linear function L, such that $L|_P \geq f$, $L|_P\,(a_{1,2,3}) = f(a_{1,2,3})$. The line segment a_1a_3 divides P into two parts. Denote by $b_1, \dots, b_m$ the vertices of the part of P which does not contain a_2. We define $b = \mathrm{argmax}_{1\leq k\leq m} f(b_k)$ and construct the linear function L_1 which coincides with f at vertices a_1, a_3 and b. The function L_1 majorizes f. If $L_1 \equiv L$, f is degenerate. If $L_1 \not\equiv L$, the chord a_1a_3 has a positive weight. Using Lemma 3 we continue this construction of the chords until we either discover degeneracy of f or obtain a complete triangulation of P. □

We summarize the results stated previously in the following theorem.

Theorem 1. *The balanced weights defined on an $(n+1)-$gon P constitute a vector space isomorphic to $\mathbb{R}^{n-2}$. The degenerate balanced weights form the Stasheff fan Σ_n. There is a one-to-one correspondence between the degenerate balanced weights and the weighted chord diagrams based on P with an incomplete triangulation.*

3. Quadratic differentials

A meromorphic quadratic differential q on a Riemann sphere S is a meromorphic section of the symmetric square of the complexified cotangent bundle over S. It is represented as $q(z)dz^2$ in the local parameter z by a meromorphic function $q(z)$ on S together with the following transition rule

$$q^*(\zeta) = q(z(\zeta)) \left(\frac{dz}{d\zeta}\right)^2,$$

in the common neigbourhood of the parameters z and ζ, where $q^*$is the same quadratic differential in terms of the local parameter ζ.

A *horizontal* (respectively, *vertical*) trajectory of quadratic differential is a maximal curve along which the inequality $q(z)\,dz^2 > 0$, (respectively, $q(z)\,dz^2 < 0$) holds. If the endpoint of a trajectory is a zero or a simple pole of q, such trajectory is called *critical*.

The zeros and poles of a quadratic differential are the *critical* points. All noncritical points are *regular*. In a neighborhood of a regular point horizontal and vertical trajectories are just straight horizontal and respectively vertical lines. The trajectory structure about the critical points is well-known, see e.g., [11, 22, 23]. Description of the global structure of a quadratic differentials is much more difficult.

We are interested in the following family of quadratic differentials:

$$\frac{z^k + a_1 z^{k-1} + \cdots + a_0}{z}\, dz^2 \tag{3}$$

where $k \geq 2$, $z \in \mathbb{C}$, $a_j \in \mathbb{C}$, $0 \leq j \leq k-2$.

Let $q(z)dz^2$ be a member of the family (3). It has k zeros (counting multiplicity), a simple pole at the origin and a pole p of order $m = k+3$ at infinity. Observe that unlike the versal deformation of a polynomial quadratic differential the coefficient a_1 in (3) is not necesserily vanishing, because the simple pole at the origin prevents to perform affine coordinate change.

Let us have a look at trajectory structure of $q(z)dz^2$ near infinity. If the infinite pole p is of order $m \geq 5$, then it is possible to find a neighbourhood U of p, such that any trajectory ray entering U stays in U. In this neighbourhood one can define $m-2$ so-called *principal directions*, such that the directions divide U into $m-2$ sectors of angles $\frac{2\pi}{m-2}$; and any trajectory ray that enters U tends to p in one of these directions.

Example 3. Quadratic differential of form $\frac{z^2-1}{z}\, dz^2$ has three principal directions at infinity. Figure 4 illustrates the trajectory structure about infinity in this case.

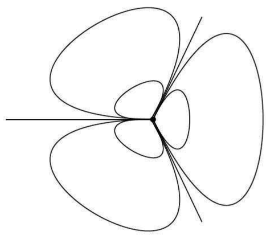

FIGURE 4. Trajectory structure near a pole of order 5

We denote by Φ_q the union of all critical trajectories of $q(z)\, dz^2$. Then $S \setminus \bar{\Phi}_q$ splits S into *strip* and *ending* domains. Strip and ending domains in the trajectory structure of $q(z)\, dz^2$ are simply connected domains, which can be mapped conformally by $w = \int \sqrt{q(z)}\, dz$ onto a strip $\{a < \Im w < b\}$ and a halfplane respectively. These domains are swept out by the trajectories starting and ending at infinity; the critical points of $q(z)\, dz^2$ belong to their boundaries.

4. GRAPH REPRESENTATION OF QUADRATIC DIFFERENTIALS

In this section we establish the one-to-one correspondence between weighted graphs of special type and quadratic differentials of the form (3).

4.1. **Assigning admissible graphs to quadratic differentials.** We describe an algorithm of assigning a pair of graphs to a quadratic differential.

Suppose we are given a quadratic differential $q(z)\,dz^2$ from the family (3). It has k zeros, a simple pole at the origin and a pole of order $k+3$ at infinity. Thus, the horizontal and vertical trajectory structures of $q(z)\,dz^2$ about the infinite pole have $k+1$ principal directions each.

We construct graphs G_h and G_v which represent the horizontal and vertical trajectory structure of $q(z)\,dz^2$ respectively.

The graph G_h (respectively, G_v) contains $k+1$ vertices and edges, which form a regular convex $(k+1)-$gon, denoted by Π_h (respectively, Π_v). In addition, each graph has a vertex O placed at the center of the interior of the $(k+1)-$gon. The vertices of Π_h (respectively, Π_v) represent the principal directions along which the critical trajectories tend to the infinite pole. The vertex O represents the finite pole.

The quadratic differential $q(z)\,dz^2$ is characterized uniquely by its structure in the large, i.e., the strip and half-plane domains in the trajectory structure. The half-plane domains are represented by the edges of Π_h (respectively, Π_v). In order to mark the strip domains of horizontal (respectively, vertical) trajectory structure of $q(z)\,dz^2$ on the G_h (respectively G_v) we construct additional edges so that all edges may intersect only at the vertices. Suppose that we have a strip domain S, which is to be marked on the graph G_h or G_v. As any strip domain it is swept out by trajectories whose ends approach infinity in certain principal directions. Suppose these two principal directions are represented by the vertices a and b of Π_h or Π_v. Observe that a and b may coincide. If the strip domain S does not have a finite pole on its boundary, we mark it with an edge joining the vertices a and b. If S has the finite pole on its boundary, we mark it with an edge joining the pole vertex O with a and an edge joining O with b. Finally, to each edge representing S we assign a weight w_S which is equal the width of S in the metric associated with the quadratic differential $q(z)\,dz^2$.

This way we mark all the strip domains of the horizontal (respectively vertical) trajectory structure of $q(z)\,dz^2$ on G_h (respectively G_v) so that the edges of G_h (respectively G_v) intersect only at the vertices.

Example 4. The quadratic differential $\frac{z^3-1}{z}\,dz^2$ has 3 simple zeros, a simple pole at the origin, and the infinite pole of order 6. The horizontal and vertical trajectory structures have 4 principal directions at infinity each. Figure 5 illustrates the graphs G_h and G_v together with the corresponding trajectory and orthogonal trajectory structures in the large (by the circles we just bound the picture in the plane).

4.2. **Admissible graphs.** Let us describe the graphs which may represent a quadratic differential. We call a graph *admissible* if we can associate a horizontal or vertical trajectory structure of a quadratic differential to it. An admissible graph Γ contains n, $n \geq 3$, vertices and edges which constitute a regular convex polygon Π_n. In addition, Γ has vertex O at the center of the interior of Π_n. There are two edges connecting the vertex O with some adjacent vertices a and b of Π_n. The vertices a and b may coincide and in this case, we treat them as two adjacent vertices with one and the same support. The edges of an admissible graph intersect only at vertices.

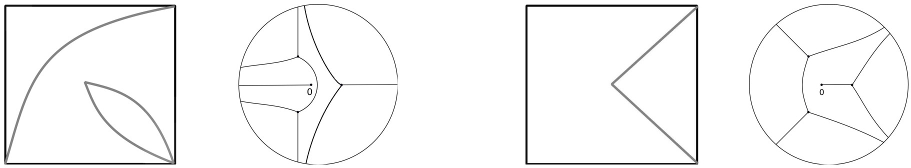

FIGURE 5. Graphs G_h and G_v and corresponding trajectory and orthogonal trajectory structures for $\frac{z^3-1}{z}\,dz^2$.

4.3. **Assigning a quadratic differential to a pair of admissible graphs.** Here we describe an algorithm of assigning the trajectory structure of a quadratic differential to a pair (G_h, G_v) of admissible graphs with $k+2$ vertices. Let us start with merging G_h and G_v into one and the same graph G, assigning to G_h and G_v different colours. Then, we place G_h over G_v in such a way, that the vertex O of G_h is right above the vertex O of G_v, and the vertices of the polygons Π_h and Π_v are interlacing. Furthermore, we erase the edges forming the polygons Π_h and Π_v, and join the $2(k+1)$ interlacing vertices with edges, so that a regular convex $2(k+1)-$gon Π is formed. Finally, we merge what is left of G_h and G_v with the $2(k+1)-$gon Π into a new graph G. Note, that the edges of G may intersect not only at vertices.

Example 5. The graphs G_h and G_v from Example 4 are admissible. The corresponding graph G is shown in Figure 6.

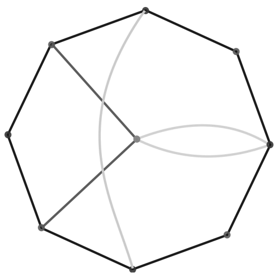

FIGURE 6. Graph G for Examples 4 and 5.

Further, let us describe an algorithm of the construction of an extended graph G_{ext}. The constructed edges represent further pieces of critical trajectories of a quadratic differential. Hence, we specify the correspondence between G_{ext} and the trajectory structure of a quadratic differential.

Remark 1. *For the construction we need the following rule: if the graph G has a double edge with ends at the vertex O and a vertex b of the polygon Π, then b counts as two vertices with one and the same support. This follows from the fact that in this case the strip domain in the trajectory structure starts and ends at one and the same image of infinity which is counted as 2 different points over the same support.*

4.4. **Algorithm of construction of G_{ext}.** By admissibility of G_h and G_v the interior of the polygon Π is divided by edges of G into at least four connected components. Pick a point in each connected component. We call these points *component centers*. The component centers represent points of intersection of critical trajectories of a quadratic differential. If the boundary of a component contains vertices of the polygon Π, then connect the component center with these vertices by line segments. Whenever the boundaries of two connected components share a piece of an edge of G, connect the component centres by a line segment.

After the completion of previous steps, the interior of the polygon Π is divided into triangles and quadrilaterals. Whenever the boundary of a quadrilateral contains the vertex O and two pieces of the edges of the same colour, we construct a line segment connecting O with the non-adjacent vertex of the quadrilateral. Such a line segment represents a piece of a critical trajectory. This completes the construction of G_{ext}.

The edges of G_{ext} divide the interior of the polygon Π into the following domains:

(a) Triangles including a side of Π as their side;
(b) Triangles having only one vertex of Π as a vertex. A piece of an edge of G_h or G_v constitutes one of the triangles sides;
(c) Quadrilateral having 2 pieces of edges of G of different color as adjacent sides;
(d) Triangle whose boundary contains the vertex O and a piece of an edge of G.

Each triangle of type (a) can be identified with a quadrant. The boundary of a triangle of type (b) contains a piece of an edge of G_h or G_v of weight v. Then the triangle is identified with a quarter of the strip $\{a < \Im w < b\}$, where $b - a = v$. The boundary of a quadrilateral of type (c) contains a piece of edge of G_h of weight v, and a piece of edge of G_v of weight u. Then the quadrilateral is identified with the rectangle with sides of length v and u.

The union of quadrilaterals and triangles with the vertex O at the boundary is identified with a rectangle, which is a part of a strip. The width of the rectangle is given by the weight of one of the coloured sides of the quadrilaterals and triangles.

Recall that each strip or ending domain of the trajectory structure of a quadratic differential is mapped conformally onto an infinite strip or a half-plane. The identification described above establishes the correspondence between the domains formed by G_{ext} and the domains of the trajectory structure of a quadratic differential.

Example 6. Figure 7 illustrates two copies of the graph G_{ext} corresponding to the graph G from Example 5. The dashed lines represent the critical trajectories. The right-hand side copy has the shadowed region representing a strip domain.

The position and weights of the coloured edges of G_{ext} define uniquely a quadratic differential representing the original pair of graphs (G_h, G_v). The position of coloured edges defines the relative position of the strip domains, while the weights fix their width in the natural metric. More precisely, the mapping $w = \int \sqrt{q(z)}dz$ maps the complex plane onto a Riemann surface branched at the images of the zeros of q and the regular trajectories are mapped onto the horizontal straight lines in the w-plane.

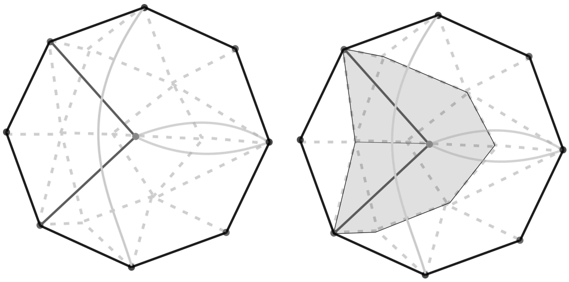

FIGURE 7. G_{ext} for the graph G from Example 5.

4.5. **Correspondence between triangulation and the short trajectories.** We established the one-to-one correspondence between quadratic differentials of the form (3) and pairs of admissible graphs in the previous sections. In what follows, let us specify the graphs which represent quadratic differentials with short trajectories.

We describe how an admissible graph G_h (respectively, G_v) gives rise to a weighted chord diagram. Suppose that the graph G_h (respectively, G_v) has $k+2$ vertices, and let the vertex O be connected with the vertices a and b by edges u and v. We erase the vertex O and replace u and v by a single edge joining a and b. If a and b have the same support, we disunite it, so that a and b become two separate adjacent vertices. The resulting graph Γ_h (respectively, Γ_v) has n vertices, where $n = k$ if a and b originally had different supports, and $n = k + 1$ if a and b originally had coinciding supports. The graph Γ_h (respectively, Γ_v) is isomorphic to a regular convex $n-$gon with weighted diagonals. This convex realisation of Γ_h (respectively, Γ_v) is exactly the desired weighted chord diagram. Analogously, a pair of weighted chord diagrams with an appropriate number of vertices gives rise to a pair of admissible graphs.

The diagonals of Γ_h (respectively, Γ_v) generate triangulation, possibly incomplete, of the $n-$gon. The following lemma provides characterization of quadratic differentials with short trajectories.

Lemma 5. *The trajectory structure represented by G_h (respectively, G_v) has a short trajectory joining two zeros if and only if the triangulation of the corresponding $n-$gon is incomplete.*

Example 7. Figure 7 shows the weighted diagrams Γ_h and Γ_v associated with the graphs G_h and G_v from Example 4. As we can see, the triangulations of Γ_h and Γ_v are incomplete. The corresponding trajectory structures have short trajectories.

4.6. **Parametric space.** Our goal is to characterize the set of parameters S in the parameter space $\Lambda \cong \mathbb{R}^{2k}$, for which the corresponding quadratic differential has a short trajectory joining two zeros. The set S naturally splits into the horizontal and the vertical components S_h and S_v.

Theorem 2. *The horizontal and vertical components of the set S have the following form:*

$$S_h = \left(\Sigma_k \times \mathbb{R}^{k+2}\right) \cup \left(\Sigma_{k+1} \times \mathbb{R}^{k+1}\right),$$

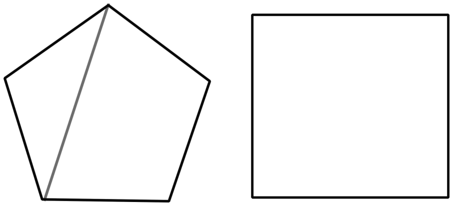

FIGURE 8. Γ_h and Γ_v.

$$S_v = \left(\mathbb{R}^{k+2} \times \Sigma_k\right) \cup \left(\mathbb{R}^{k+1} \times \Sigma_{k+1}\right).$$

Proof. By Theorem 1 and Lemma 5 a quadratic differential of the form (3) with a short trajectory can be identified with a point in the fan Σ_k or Σ_{k+1}. The set S has codimention 1, which leads us to the statement of the theorem. □

Remark 2. *Quadratic differentials of the form (3) contain a subfamily of quadratic differentials*

$$(z^{k-1} + a^*_{k-2} z^{k-2} + \cdots + a^*_0)\, dz^2, \tag{4}$$

where $a^*_{k-2}, \ldots, a^*_0$ are complex parameters. By Baryshnikov's result [7] *the bifurcation diagram S^* of this family consists of components $S^*_v = \mathbb{R}^{k-2} \times \Sigma_k$ and $S^*_h = \Sigma_k \times \mathbb{R}^{k-2}$ in the parameter space $\mathbb{R}^{2(k-2)}$. Therefore, S^* is exactly the subset of S corresponding to quadratic differentials of the form (4).*

REFERENCES

[1] S. Cecotti and C. Vafa, *On classification of $N = 2$ supersymmetric theories*, Commun. Math. Phys. **158** (1993), no. 3, 569–644.

[2] S. Cecotti and C. Vafa, *Classification of complete $N = 2$ supersymmetric theories in 4 dimensions*, Surveys in differential geometry. Geometry and topology, Surv. Differ. Geom., 18, Int. Press, Somerville, MA, 2013, 19–101.

[3] S. Cecotti, *Supersymmetric field theories. Geometric structures and dualities.* Cambridge University Press, Cambridge, 2015, 410 pp.

[4] D. Gaiotto, G. W. Moore, and A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation*, Adv. Math. **234** (2013), 239–403.

[5] D. Gaiotto, G. W. Moore, and A. Neitzke, *Spectral networks* Ann. Henri Poincaré **14** (2013), no. 7, 1643–1731.

[6] Yu. Baryshnikov, *Bifurcation diagrams of quadratic differentials*, C. R. Acad. Sci. Paris Sér. I Math., **325** (1997), no. 1, 71–76.

[7] Yu. Baryshnikov, *On Stokes sets*, NATO Sci. Ser. II Math. Phys. Chem., **21** (2001), 65–86.

[8] A. Bogatyrëv, *A combinatorial description of a moduli space of curves and of extremal polynomials*, Mat. Sb., **194** (2003), 27–48.

[9] J. W. Bruce and D. B. O'Shea, *On binary differential equations and minimal surfaces*, preprint, Liverpool, 1997.

[10] A A. Gonchar and E. A. Rakhmanov, *Equilibrium distributions and the rate of rational approximation of analytic functions*, Mat. Sb. (NS) **134(176)** (1987), no. 3, 306–352 (English translation in Math. USSR-Sb. **62** (1989), no. 2, 305–348).

[11] J. A. Jenkins, *Univalent functions and conformal mapping*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958, 169 pp.
[12] D. Joyce and Y. Song, *A theory of generalized Donaldson-Thomas invariants* Mem. Amer. Math. Soc. **217** (2012), no. 1020, 199 pp.
[13] M. Kontsevich and Y. Soibelman, *Motivic Donaldson-Thomas invariants: summary of results. Mirror symmetry and tropical geometry*, Contemp. Math., 527, Amer. Math. Soc., Providence, RI, 2010, 55–89.
[14] V. P. Kostov and S. K. Lando, *Versal deformations of powers of volume forms*, Comput. Algebraic Geom. **109** (1993), 143–162.
[15] A. B. J. Kuijlaars and G. Silva, *S-curves in polynomial external fields*, J. Approx. Theory **191** (2015), 1–37.
[16] A. Martínez-Finkelshtein and E. A. Rakhmanov, *Critical measures, quadratic differentials, and weak limits of zeros of Stieltjes polynomials*, Commun. Math. Phys. **302** (2011), no. 1, 53–111.
[17] M. J. Atia, A. Martínez-Finkelshtein, P. Martínez-González, and F. Thabet, *Quadratic differentials and asymptotics of Laguerre polynomials with varying complex parameters*, J. Math. Anal. Appl., **416** (2014), no. 1, 52–80.
[18] B. Shapiro, K. Takemura, and M. Tater, *On spectral polynomials of the Heun equation II*, Commun. Math. Phys. **311** (2012), no. 2, 277–300.
[19] A. Yu. Solynin, *Quadratic differentials and weighted graphs on compact surfaces*, Analysis and Mathematical Physics, Trends Math., Birkhäuser, Basel, 2009, 473–505.
[20] H. Stahl, *Extremal domains associated with an analytic function. I, II,* Complex Var. Theory Appl. **4** (1985), no. 4, 311–324; 325–338.
[21] J. D. Stasheff, *Homotopy associativity of H-spaces. I, II*, Trans. Amer. Math. Soc. **108** (1963), 275-292; 293–312.
[22] K. Strebel, *Quadratic differentials*, Springer-Verlag, Berlin, 1984, 184 pp.
[23] A. Vasil'ev, *Moduli of families of curves for conformal and quasiconformal mappings.* Lecture Notes in Mathematics, vol. 1788, Springer-Verlag, Berlin–New York, 2002, 211 pp.
[24] E. Witten, *Solutions of four-dimensional field theories via M-theory*, Nuclear Phys. B **500** (1997), no. 1-3, 3–42.

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