UNIVERSITY OF BERGEN
Faculty of Mathematics and Natural Sciences

Master's Thesis in Topology

Derived Logarithmic Hochschild Homology

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Contents

Introduction

In this thesis we study *logarithmic Hochschild homology* in the sense of Rognes [Rog09]. This is an extension of the usual Hochschild homology of algebras, and takes as input a *pre–logarithmic algebra* (an affine pre–logarithmic scheme in the sense of Fontaine-Illusie [Kat89]). From the perspective of topology, it is perhaps the idea of a pre–logarithmic structure on a ring being an "intermediate localization" which has had the greatest appeal. Historically speaking, this idea was first applied in homotopy theory by Hesselholt and Madsen [HM03]. In the context of discrete valuation rings A with residue field k and fraction field K, they found that the topological Hochschild homology $\mathrm{THH}(A)$ fits in a localization sequence

$$\mathrm{THH}(k) \to \mathrm{THH}(A) \to \mathrm{THH}(A|K) \to \Sigma\mathrm{THH}(k)$$

for a "relative construction" $\mathrm{THH}(A|K)$. Moreover, they found that this sequence is compatible with the localization sequence in algebraic K–theory arising from Quillen's localization theorem under the Dennis trace map. To some extent, this alleviates the failure of the localization theorem for (topological) Hochschild homology.

In the context of logarithmic topological Hochschild homology in the sense of Rognes [Rog09] and Rognes-Sagave-Schlichtkrull [RSS15], the authors construct several localization sequences involving logarithmic THH. There are two main goals of this thesis, one of which is to establish a linear version of one of these sequences:

Theorem (Theorem 3.2.12)**.** *Let R be a commutative ring and let A be a commutative R-algebra. For $\langle x \rangle$ the free commutative monoid on one generator x and $\alpha\colon \langle x \rangle \to (A, \cdot)$ a morphism of monoids such that $a := \alpha(x)$ is not a divisor of zero, there is a long exact sequence*

$$\cdots \to \pi_q \mathrm{HH}^R((A/(a))^{\mathrm{cof}}) \to \pi_q \mathrm{HH}^R(A^{\mathrm{cof}}) \to \pi_q \mathrm{HH}^R(A^{\mathrm{cof}}, \langle x \rangle) \to \pi_{q-1} \mathrm{HH}^R((A/(a))^{\mathrm{cof}}) \cdots .$$

The morphism α is by definition a pre-logarithmic structure on A, and the object $\mathrm{HH}^R(A^{\mathrm{cof}}, \langle x \rangle)$ is the logarithmic Hochschild homology of the pre-logarithmic algebra $(A, \langle x \rangle)$. The superscripts $^{\mathrm{cof}}$ denote choices of flat simplicial resolutions of the given algebra, so the theorem is really a statement about *derived* Hochschild homology. This is also called *Shukla homology* in the literature. In the context of discrete valuation rings, we show that this sequence coincides with that of Hesselholt and Madsen in low degrees (Section 3.2.3).

While it is not uncommon to simply define Hochschild homology for flat algebras, this would be far more restrictive for logarithmic Hochschild homology. The reason is that the flatness hypotheses should not be made directly over the ground ring, but rather the monoid ring $R[M]$, where $\alpha\colon M \to (A, \cdot)$ is a pre-logarithmic structure on the R-algebra A. This is why we choose to work in this derived context, without which many examples and results considered in the thesis would not be available.

One prominent feature of Hochschild homology is its relation to Kähler differentials through the Hochschild-Kostant-Rosenberg theorem. In logarithmic geometry, a common object of study are the *logarithmic Kähler differentials* $\Omega^1_{(A,M)/(R,N)}$. As the notation indicates, this construction allows for a pre-logarithmic structure on the ground ring R. In the thesis we give a construction of logarithmic Hochschild homology which allows for a pre-logarithmic structure on the ground ring (Section 4.1.3). As one would expect from such a construction, it coincides with the logarithmic Kähler differentials on π_1 (Theorem 4.1.23).

For any pre-logarithmic structure $\alpha\colon M \to (A, \cdot)$ one can form the *associated logarithmic structure*

$$\alpha^a\colon M^a \to (A, \cdot).$$

This ensures that the part of M^a which maps to the units $\mathrm{GL}_1(A)$ of A is isomorphic to $\mathrm{GL}_1(A)$. From the perspective of a logarithmic structure being an "intermediate localization", the interesting part of a logarithmic structure is then the part of M^a which maps to non-units of A. Many constructions in logarithmic geometry, for instance the logarithmic Kähler differentials, have the property that they are invariant under the logification construction. It is therefore natural to ask whether the same can be said for logarithmic Hochschild homology. For logarithmic topological Hochschild homology, this is proved by Rognes, Sagave and Schlichtkrull in [RSS15, Theorem 4.24].

The second main goal of the thesis is to extend the logification invariance result to our relative construction of logarithmic Hochschild homology. For this it seems to be necessary to consider a homotopy invariant version of the logification construction. In this context, we find that the below result is best stated in terms of general simplicial pre-logarithmic R-algebras:

Theorem (Theorem 4.2.13)**.** *Let $(R, N) \to (A, M)$ be a cofibration of simplicial pre-logarithmic R-algebras. The logification construction induces weak equivalences*

$$\mathrm{HH}^{(R,N)}(A, M) \xrightarrow{\simeq} \mathrm{HH}^{(R,N)}(A^a_N, M^a) \xrightarrow{\simeq} \mathrm{HH}^{(R,N^a)}(A^a_{N^a}, M^a).$$

Here A^a_N and $A^a_{N^a}$ are simplicial commutative R-algebras that are weakly equivalent to A. These replacements are necessary, as (A, M) being cofibrant over (R, N) does not necessarily imply that (A, M^a) is cofibrant over (R, N) and (R, N^a), which is required for our construction of relative logarithmic Hochschild homology.

Outline

The thesis is structured as follows:

In Section 1 we discuss preliminary definitions and results that will be used throughout the thesis.

In Section 2 we introduce logarithmic rings, or more generally, logarithmic schemes. We discuss basic properties and examples of pre-logarithmic rings, and also introduce logarithmic Kähler differentials.

In Section 3 we begin our study of logarithmic Hochschild homology. We first introduce Hochschild homology of algebras, with particular emphasis on its derived variant. After this we introduce logarithmic Hochschild homology in the sense of Rognes, and establish the aforementioned long exact sequence. We obtain explicit descriptions of the morphisms involved in this sequence in low degrees, which we later employ in computational examples.

In Section 4 we introduce our proposed definition of relative logarithmic Hochschild homology. This construction requires a bit of prerequisite knowledge, and we first discuss model structures on simplicial commutative monoids and simplicial commutative pre-logarithmic R-algebras. Once this is in place, we give our construction and discuss its relation with relative logarithmic Kähler differentials. We also discuss the notion of repletion of simplicial commutative monoids, a key ingredient in our construction of relative logarithmic Hochschild homology. Finally, we prove the aforementioned logification invariance result. We first translate the proof in the absolute case for log topological Hochschild homology in [RSS15] to our context, and then provide a generalization for our relative construction.

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1 Preliminaries

In this section we discuss a variety of topics which will be actively used throughout the thesis. This section should be considered more of a toolbox than an exposition, and we will only give references for the majority of the proofs.

1.1 Simplicial objects

Simplicial objects will often be used in this thesis. In particular we will often use a model structure on the category of simplicial commutative R-algebras (for a commutative ring R), and we will state the basic facts about this model category. We will assume familiarity with simplicial sets and Quillen model categories; introductions to these topics are provided in e.g. [GJ09], [DS95] and [Hov99].

For a category $\mathcal{C}$ and objects X and Y of $\mathcal{C}$, we denote by $\mathcal{C}(X,Y)$ the set of morphisms between X and Y in $\mathcal{C}$. We write $\mathcal{C}^{\mathrm{op}}$ for the opposite category with the same objects as $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}(X,Y) := \mathcal{C}(Y,X)$. We denote by $\mathrm{s}\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$, that is, the category of functors $\Delta^{\mathrm{op}} \to \mathcal{C}$. Here Δ denotes the simplex category with

$$[n] = \{0, 1, \dots, n\}$$

for each natural number n as objects and order-preserving maps as morphisms.

1.1.1 Simplicial modules and the Dold-Kan correspondence

Let R be a commutative ring, and write Mod_R for the category of modules over R. We consider the category sMod_R of simplicial R-modules and discuss the classical result that there is an equivalence of categories

$$\mathrm{sMod}_R \simeq \mathrm{Ch}_{\geq 0}(R)$$

between simplicial R-modules and positively graded chain complexes of R-modules.

Definition 1.1.1. Let M be a simplicial R-module. The *Moore complex* of M, denoted M_*, is the chain complex with q-chains M_q, the q-simplices of R, and differential

$$\partial_q^{M_*} \colon M_q \to M_{q-1}$$

given by the alternating sum of the face maps $d_i^M \colon M_q \to M_{q-1}$:

$$\partial_q^{M_*} := \sum_{i=0}^{q} (-1)^i d_i^M.$$

Remark 1.1.2. The terminology varies quite a bit in the literature, and the term "Moore complex" is sometimes used for other chain complexes arising from a simplicial module. The terminology used here coincides with that of [GJ09]. For aesthetic purposes we will denote by $\pi_q(M)$ the q–th homology group of the Moore complex of the simplicial R-module M. While there is always an isomorphism $H_q(M_*) \cong \pi_q(|M|)$ between the homology groups of the Moore complex of M and the homotopy groups of the geometric realization of M (cf. [GJ09, Chapter 3, Corollary 2.5]), we stress that we always mean the homology groups of the Moore complex throughout this thesis.

Definition 1.1.3. Let M and N be simplicial R-modules. We denote by $M \boxtimes_R N$ the simplicial R-module with q-simplices $M_q \otimes_R N_q$ with face and degeneracy maps inherited degreewise from M and N, i.e.

$$d_i^{M \boxtimes_R N} \colon (M \boxtimes_R N)_q \to (M \boxtimes_R N)_{q-1}$$

is given on elementary tensors by

$$d_i^{M\boxtimes_R N}(m_q \otimes n_q) := d_i^M(m_q) \otimes d_i^N(n_q)$$

for $d_i^M\colon M_q \to M_{q-1}$ and $d_i^N\colon N_q \to N_{q-1}$, and likewise for the degeneracies.

If M_* and N_* are chain complexes of R-modules, the tensor product $M_* \otimes_R N_*$ has q-chains

$$\bigoplus_{i+j=q} M_i \otimes_R N_j$$

with differential

$$\partial_q^{M_*\otimes_R N_*}(m_i \otimes n_j) := \partial_i^{M_*}(m_i) \otimes n_j + (-1)^i n_i \otimes \partial_j^{N_*}(n_j).$$

The following result, the *Eilenberg-Zilber theorem*, relates the Moore complex $(M \boxtimes_R N)_*$ with $M_* \otimes_R N_*$ for simplicial R-modules M and N:

Theorem 1.1.4. [Wei94, Chapter 8.5] *Let M and N be simplicial R-modules. There are mutually inverse homotopy equivalences*

$$M_* \otimes_R N_* \underset{\beta}{\overset{\alpha}{\rightleftarrows}} (M \boxtimes_R N)_*.$$

□

We record the explicit definitions of the morphisms involved in the above result. The map α, called the *shuffle map*, is given in degree q by

$$\alpha_q(m_i \otimes n_j) = \sum_{(\mu,\nu)\in \mathrm{Sh}(i,j)} \mathrm{sgn}(\mu,\nu)\, s_\nu^M(m_i) \otimes s_\mu^N(n_j),$$

where $\mathrm{Sh}(i,j)$ denotes the (i,j)-shuffles of the set $\{0,1,\dots,q-1\}$, i.e. the permutations

$$(\mu_1,\dots,\mu_i,\nu_1,\dots,\nu_j)$$

such that $\mu_1 < \mu_2 \cdots < \mu_i$ and $\nu_1 < \nu_2 \cdots < \nu_j$, while s_ν^M denotes the composite

$$s_\nu^M := s_{\nu_j}^M \cdots s_{\nu_2}^M s_{\nu_1}^M.$$

The map β is given by the *Alexander-Whitney map*, given in degree q by

$$\beta_q(m_q \otimes n_q) = \sum_{i=0}^{q} \tilde{d}^{q-i}(m_q) \otimes d_0^i(n_q).$$

Here d_0^i denotes the i-fold composite of the zeroth face map d_0^N, while $\tilde{d}^{q-i}$ denotes the composition $d_q^M d_{q-1}^M \cdots d_{q-i}^M$ of the last face maps.

We will often be in a situation where an object of interest has two simplicial directions:

Definition 1.1.5. Let $M\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Mod}_R$ be a bisimplicial R-module. The *Moore bicomplex* $M_{*,*}$ has as (p,q)-chains the (p,q)-simplices $M_{p,q}$. The horizontal differential is given by

$${}_h\partial_p^{M_{*,*}} := \sum_{i=0}^{p} (-1)^i\, {}_h d_i^M\colon M_{p,q} \to M_{p-1,q}$$

for ${}_h d_i^M : M_{p,q} \to M_{p-1,q}$ the horizontal face maps, and the vertical differential is given by

$$ {}_v\partial_q^{M_{*,*}} := \sum_{i=0}^{q} (-1)^{p+i}\, {}_v d_i^M : M_{p,q} \to M_{p,q-1}, $$

where ${}_v d_i^M : M_{p,q} \to M_{p,q-1}$ denotes the vertical face maps.

As always we have spectral sequences

$$ E^2_{p,q} = H_p(H_q(M_{*,*})) \implies H_{p+q}(\mathrm{Tot}(M_{*,*})) $$

converging to the homology of the total complex $\mathrm{Tot}(M_{*,*})$, see e.g. [Wei94, Chapter 5.6]. This will be useful to us, particularly in light of the following result:

Theorem 1.1.6. [GJ09, Theorem IV.2.4] *Let M be a bisimplicial R-module. There is a chain homotopy equivalence between the total complex* $\mathrm{Tot}(\mathrm{M}_{*,*})$ *and* $\mathrm{d}(M)_*$*, the Moore complex of the diagonal of M. This equivalence is natural in bisimplicial R-modules.* □

Notice that Theorem 1.1.6 is a generalization of the Eilenberg-Zilber Theorem 1.1.4: if M and N are simplicial R-modules, let X be the bisimplicial R-module with $X_{p,q} = M_p \otimes_R N_q$. Then the Eilenberg-Zilber theorem states precisely that the Moore complex of the diagonal $\mathrm{d}(X)_*$ is homotopy equivalent to the total complex of the Moore bicomplex $\mathrm{Tot}(X_{*,*})$.

We now wish to establish a model structure on the category sMod_R of simplicial R-modules. We first establish an (honest) equivalence of categories between simplicial R-modules and the category of positively graded chain complexes of R-modules using the *normalization functor*:

Definition 1.1.7. For a simplicial R-module M, let NM denote the *normalized chain complex*, with

$$ (NM)_q := M_q / s_0 M_{q-1} + \cdots + s_{q-1} M_{q-1} $$

and differential

$$ \partial_q^{NM} : (NM)_q \to (NM)_{q-1} $$

defined as the alternating sum of the face maps.

Remark 1.1.8. The construction of NM is well-defined by the simplicial identities. Given a degenerate simplex $s_j x_{q-1}$ in $(NM)_q$, we have

$$ \begin{aligned} \sum_{i=0}^{q} (-1)^i d_i s_j x_{q-1} &= \left(\sum_{i=0}^{j-1} (-1)^i s_{j-1} d_i x_{q-1} \right) + (-1)^j x_{q-1} + (-1)^{j+1} x_{q-1} + \left(\sum_{i=j+2}^{q} (-1)^i s_j d_{i-1} x_{q-1} \right) \\ &= \left(\sum_{i=0}^{j-1} (-1)^i s_{j-1} d_i x_{q-1} \right) + \left(\sum_{i=j+2}^{q} (-1)^i s_j d_{i-1} x_{q-1} \right) \\ &= s_{j-1} \left(\sum_{i=0}^{j-1} (-1)^i d_i x_{q-1} \right) + s_j \left(\sum_{i=j+2}^{q} (-1)^i d_{i-1} x_{q-1} \right). \end{aligned} $$

The statement of the Dold-Kan correspondence is the following:

Theorem 1.1.9. [GJ09, Corollary III.2.3] *The normalization functor is an equivalence of categories*

$$ N : \mathrm{sMod}_R \to \mathrm{Ch}_{\geq 0}(R). $$

We we will consider the following model structure on the category $\mathrm{Ch}_{\geq 0}(R)$: a map $f\colon C \to D$ of chain complexes is

- a weak equivalence if it is a quasi-isomorphism, i.e. $H_q(f)\colon H_q(C) \to H_q(D)$ is an isomorphism for each q;
- a fibration if it is a surjection in all strictly positive degrees;
- a cofibration if it is a degreewise injection with degreewise projective cokernel.

A proof that this is in fact a model structure on $\mathrm{Ch}_{\geq 0}(R)$ is given in e.g. [DS95, Section 7].

Example 1.1.10. It follows directly from the definition that the cofibrant chain complexes, i.e., those for which the map $0 \to C$ is a cofibration, are the degreewise projective complexes. Let P be a cofibrant chain complex, and suppose $f\colon C \to D$ is a cofibration. Then there is a short exact sequence

$$0 \to C \to D \to Q \to 0$$

of chain complexes with Q degreewise projective and, since P is degreewise projective, a short exact sequence

$$0 \to P \otimes_R C \to P \otimes_R D \to P \otimes_R Q \to 0.$$

As $P \otimes_R Q$ is degreewise projective, we conclude that the functor $P \otimes_R -$ preserves cofibrations. If $f\colon C \to D$ is a quasi-isomorphism, then there is a short exact sequence

$$0 \to C \to D \to K \to 0$$

with K an acyclic complex (i.e. $H_q(K) \cong 0$ for all q), and again there is a short exact sequence

$$0 \to P \otimes_R C \to P \otimes_R D \to P \otimes_R K \to 0.$$

The complex $P \otimes_R K$ is again acyclic, as it is for instance seen by applying the Künneth spectral sequence

$$E^2_{p,q} = \bigoplus_{i+j=q} \mathrm{Tor}^R_p(H_i(P), H_j(K)) \implies H_{p+q}(P \otimes_R K),$$

see e.g. [Wei94, Exercise 5.7.5], which is applicable since P is degreewise flat. We conclude that $P \otimes_R -$ preserves (acyclic) cofibrations, i.e. that it is a left Quillen functor.

We will need an explicit description of the inverse functor in the Dold-Kan correspondence:

Construction 1.1.11. Any chain complex C can be considered a *semi-simplicial object*, that is, a diagram defined on $\Delta' \subset \Delta$ consisting of only injective morphisms: if $i\colon [p] \to [q]$ is an injection, we obtain a morphism $i^*\colon C_q \to C_p$ defined by

$$i^* = \begin{cases} 0, & \text{if } p < q-1 \text{ or } p = q-1, i \neq d^q \\ \partial^C_q, & \text{if } p = q-1 \text{ and } i = d^q \end{cases}$$

where $d^q\colon [q-1] \to [q]$ denotes the last coface map. For a chain complex of R-modules C, consider the simplicial R-module $K(C)$ with q-simplices

$$K(C)_q = \bigoplus_{\phi\colon [q] \twoheadrightarrow [k]} C_k,$$

whose simplicial structure we now describe. The direct sum runs over all surjections in Δ. Let $\psi\colon [p] \to [q]$ be a morphism in Δ. For any epimorphism $\phi\colon [q] \twoheadrightarrow [k]$, factor the composite

$$[p] \xrightarrow{\psi} [q] \xrightarrow{\phi} [k]$$

as an epimorphism followed by a monomorphism:

$$[p] \xrightarrow{\sigma} [s] \xrightarrow{\tau} [k]$$

We then define $\psi^*\colon K(C)_q \to K(C)_p$ as follows: on the summand corresponding to $\phi\colon [q] \twoheadrightarrow [k]$, it is the composite

$$C_k \xrightarrow{\tau^*} C_s \longrightarrow K(C)_p,$$

where the latter map is the inclusion into the summand corresponding to σ. That this construction is functorial and is indeed inverse the normalization functor is verified in e.g. [GJ09, Chapter III.2].

We now consider the model structure on sMod_R obtained by directly transporting the structure from the Dold-Kan correspondence: that is, a morphism $f\colon A \to B$ of simplicial R-modules is a (co)fibration or weak equivalence if and only if that is the case for the morphism $Nf\colon NA \to NB$ of normalized chain complexes. With this definition, for a cofibration of simplicial R-modules there is a short exact sequence

$$0 \to NA \to NB \to P \to 0$$

with P degreewise projective. In each degree, this splits as a sequence of R-modules, and applying the inverse functor K to this sequence we obtain that the map f in each degree is isomorphic to an inclusion

$$A_q \to A_q \oplus \left(\bigoplus_{\phi\colon [q]\twoheadrightarrow[k]} P_k \right)$$

for projective R-modules P_k. We now ask how this splitting behaves with the simplicial structure: degreewise split sequences of chain complexes are not necessarily split as chain complexes, and as the above construction of $K(C)$ involves the differential of C, we do not obtain a splitting of simplicial R-modules. However, the differentials are not involved if we restrict ourselves to the surjective morphisms in Δ: given an epimorphism $\psi\colon [p] \twoheadrightarrow [q]$, the induced map

$$\psi^*\colon K(C)_q \to K(C)_p$$

is merely the inclusion sending the summand corresponding $\phi\colon [q] \twoheadrightarrow [k]$ to the one corresponding to $\phi \circ \psi$. We conclude that we obtain a splitting on the *underlying degeneracy diagrams*, where we restrict all simplicial objects to the subcategory $\Delta_+ \subset \Delta$ consisting of only surjective morphisms. We have the following:

Theorem 1.1.12. [GS06, Proposition 4.2] *There is a model structure on the category of simplicial R-modules, sMod_R, where the weak equivalences and fibrations coincide with those of the underlying simplicial sets, and $f\colon A \to B$ is a cofibration if and only if the morphism of underlying degeneracy diagrams is isomorphic to an inclusion*

$$A \to A \oplus K(P),$$

where P is a degreewise projective chain complex. □

1.1.2 Commutative simplicial algebras

We now aim to describe a model structure on the category sAlg_R of simplicial commutative R-algebras by "lifting" the model structure on sMod_R along a free-forgetful adjunction. This is a common way of constructing model structures applicable under certain technical hypotheses, see e.g. [GS06, Theorem 3.6]. We recall the definition of the symmetric algebra:

Definition 1.1.13. Let R be a commutative ring and let M be an R-module. The *symmetric R-algebra* $S_R(M)$ on M is defined by the following universal property: there is a morphism $M \to S_R(M)$ of R-modules, such that, for any morphism $M \to A$ with A an R-algebra, there is a unique map $S_R(M) \to A$ of R-algebras such that the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & A \\ S_R(M) \arrow[ur, dashed] & \end{tikzcd}$$

commutes. A model for $S_R(M)$ is given by

$$R \oplus M \oplus (M^{\otimes 2}/\Sigma_2) \oplus (M^{\otimes 3}/\Sigma_3) \oplus \cdots$$

where the Σ_n-action permutes the tensor factors and the multiplication is given by concatenation.

The model structure on sAlg_R is obtained from the model structure on sMod_R through the adjunction

$$\mathrm{sMod}_R \underset{U}{\overset{S_R(-)}{\rightleftarrows}} \mathrm{sAlg}_R,$$

where U denotes the forgetful functor. To describe the cofibrations of the model structure we obtain, we need to introduce some terminology:

Definition 1.1.14. Let $\mathcal{C}$ be a category with coproducts and let I be a small category. Denote by I^δ the subcategory of I with the same objects and only the identity morphisms. A diagram $X\colon I \to \mathcal{C}$ is *free* if there exists a diagram $Z\colon I^\delta \to \mathcal{C}$ such that X is naturally isomorphic to $K(Z)\colon I \to \mathcal{C}$, where

$$K(C)_i := \bigsqcup_{j \to i} Z_j,$$

where the coproduct runs over all morphisms $j \to i$ in I. For a morphism $i \to k$ in I, we obtain a morphism $K(C)_i \to K(C)_k$ by sending the factor corresponding to $j \to i$ to the one corresponding to $j \to i \to k$. We say that X is *free* on the objects $\{Z_j\}$. A morphism $f\colon X \to Y$ of diagrams $I \to \mathcal{C}$ is free if it is naturally isomorphic to an inclusion

$$X \to X \bigsqcup F,$$

where F is free.

Example 1.1.15. A cofibration $f\colon A \to B$ of simplicial R-modules is Δ_+^{op}-free: indeed, it is naturally isomorphic to an inclusion

$$A \to A \bigsqcup K(P),$$

where P is considered as a diagram $P\colon (\Delta_+^{\mathrm{op}})^\delta \to \mathrm{Mod}_R$, and each P_n is projective. In this language, a cofibration of simplicial R-modules is a Δ_+^{op}-free morphism on a set of projective modules.

We now specialize to the case of simplicial commutative R-algebras:

Definition 1.1.16. A simplicial object $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ with $\mathcal{C}$ a category with coproducts is *s-free* if the underlying degeneracy diagram

$$X_+\colon \Delta_+^{\mathrm{op}} \to \mathcal{C}$$

is free, and a morphism $f\colon X \to Y$ of simplicial objects is s-free if it is isomorphic to an inclusion $X \to X \sqcup F$ where F is s-free. A *free morphism* of simplicial commutative R-algebras $f\colon A \to B$ is by definition s-free on a set of objects $\{S_R(P_k)\}$, where each P_k is a projective R-module. In each degree, this means that f is of the form

$$A_q \to A_q \otimes_R \left(\bigotimes_{\phi\colon [q] \twoheadrightarrow [k]} S_R(P_k) \right).$$

We will use the following model structure on the category of simplicial commutative R-algebras:

Theorem 1.1.17. [GS06, Theorem 4.17, Proposition 4.20] *There is a model structure on* sAlg_R *where the weak equivalences and fibrations coincide with those of the underlying simplicial sets, and a morphism is a cofibration precisely when it is a retract of a free morphism. Moreover, any morphism of simplicial commutative R-algebras can be factored as a free morphism and an acyclic fibration.* □

Remark 1.1.18. We notice that symmetric algebras on projective R-modules are themselves projective R-modules: if $P \oplus Q$ is a free R-module, then $S_R(P \oplus Q)$ is a polynomial algebra on a generating set for $P \oplus Q$. Since $S_R(-)$ is a left adjoint, and hence commutes with coproducts, we have that $S_R(P\oplus Q) \cong S_R(P) \otimes_R S_R(Q)$. As $S_R(Q) = R \oplus Q \oplus Q^{\otimes 2}/\Sigma_2 \oplus \cdots$, we find that $S_R(P) \otimes_R S_R(Q) \cong S_R(P) \oplus M$ for some R-module M, which displays $S_R(P)$ as a direct summand of a free R-module. In particular, Theorem 1.1.17 tells us that, for any commutative R-algebra A, there is always a factorization

$$R \rightarrowtail A^{\mathrm{cof}} \xrightarrow{\simeq} A$$

of the unit map $R \to A$ in the category sAlg_R as a free morphism followed by an acyclic fibration. By definition, A^{cof} is degreewise of the form

$$A_q^{\mathrm{cof}} \cong \bigotimes_{\phi\colon [q] \twoheadrightarrow [k]} S_R(P_k)$$

and so it is degreewise projective. This ensures the existence of simplicial flat resolutions (that is, weak equivalences $A^{\mathrm{cof}} \to A$ with A^{cof} degreewise flat) of any commutative R-algebra, a fact which will often be used throughout this thesis.

1.2 Discrete valuation rings

Discrete valuation rings provide an interesting source of examples of logarithmic structures, and we will frequently use them for computational examples throughout this thesis. In this section we provide the basic definitions and results, with particular emphasis on the structure of discrete valuation rings in mixed characteristic. Our main reference for this section is [Ser79, Chapters 1 and 2].

1.2.1 Definition and examples

The following is a list of the basic definitions which we will be working with:

Definition 1.2.1. Let A be a commutative ring.

- The ring A is a *discrete valuation ring* if it is a local principal ideal domain which is not a field.
- Given a discrete valuation ring A, denote by $\mathfrak{m}_A$ its maximal ideal. The field $A/\mathfrak{m}_A$ is the *residue field* of A.
- A choice of generator for the ideal $\mathfrak{m}_A$, which we typically denote by π, is a *uniformizer* for A. Notice that π is uniquely determined up to multiplication with a unit of A.

Example 1.2.2. Let p be a prime number and let $\mathbb{Z}_{(p)}$ denote the localization of $\mathbb{Z}$ at p. Then $\mathbb{Z}_{(p)}$ is a discrete valuation ring with residue field $\mathbb{F}_p$. A choice of uniformizer is any element x where the exponent of p in the prime decomposition of x is 1, that is, x is of the form $p \cdot a$ where both the numerator and denominator of a is coprime to p. Indeed, in this situation we have that

$$\mathbb{Z}_{(p)}/(p \cdot a) \cong \mathbb{Z}_{(p)}/(p) \cong \mathbb{F}_p,$$

since a is a unit in $\mathbb{Z}_{(p)}$.

Remark 1.2.3. We justify the use of the word "valuation" in this context. Let A be a discrete valuation ring, and let us choose a uniformizer π. Let K denote the field of fractions of A. Define a function

$$\nu\colon K \to \mathbb{Z} \cup \{\infty\}$$

as follows: for any non-zero element x of K, write $x = \pi^m \cdot a$ for an integer m, and define $\nu(x)$ to be m. By convention, set $\nu(0) = \infty$. We notice that ν

1. is a surjective group homomorphism as a map $\nu\colon \mathrm{GL}_1(K) \to \mathbb{Z}$;
2. satisfies the inequality $\nu(x + y) \geq \min(\nu(x), \nu(y))$.

If K is a field, a function $\nu\colon \mathrm{GL}_1(K) \to \mathbb{Z}$ satisfying the properties above is a *valuation* on K. Then there is a correspondence: given a valuation ν on a field K, let $A := \{x \in K \mid \nu(x) \geq 0\}$. This is again a discrete valuation ring, and one can choose any element x with $\nu(x) = 1$ as uniformizer, see [Ser79, Chapter 1, Proposition 1].

Example 1.2.4. Let $K = \mathbb{Q}$, the field of rational numbers. For any rational number x, write $\nu_p(x)$ for the exponent of p in the prime decomposition of x. This defines a valuation on $\mathbb{Q}$ whose associated discrete valuation ring, i.e., the ring A defined by

$$A = \{x \in \mathbb{Q} \mid \nu_p(x) \geq 0\},$$

is precisely $\mathbb{Z}_{(p)}$.

Definition 1.2.5. Let A be a discrete valuation ring and let π be a choice of uniformizer of A. The *completion* of A is the discrete valuation ring

$$\hat{A} := \lim_{\leftarrow}(A/\pi^n A).$$

This is again a discrete valuation ring by [AM69, Proposition 10.16]. By the cited proposition, we have that

$$\hat{A}/\pi\hat{A} = A/\pi A$$

for a choice of uniformizer π of A, and so we have that (the image of) π in $\hat{A}$ is a uniformizer of $\hat{A}$. We say that A is *complete* if the morphism $A \to \hat{A}$ is an isomorphism.

Example 1.2.6. Consider the discrete valuation ring $\mathbb{Z}_{(p)}$ from Example 1.2.2. Its completion is the limit

$$\hat{\mathbb{Z}}_{(p)} = \lim_{\leftarrow}(\mathbb{Z}/p^n\mathbb{Z}).$$

These are, by definition, the *p-adic integers* $\mathbb{Z}_p$. We then see that a p-adic integer consists of a sequence

$$(a_1, a_2, \dots)$$

where each a_i is an element of $\mathbb{Z}/p^i\mathbb{Z}$, with the requirement that $a_i \equiv a_j \pmod{p^i}$ for $i \leq j$, that is, a_j maps to a_i under the canonical map

$$\mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}.$$

As noted in Definition 1.2.5, a choice of uniformizer for $\mathbb{Z}_p$ is the sequence $(0, p, p, \dots)$, the image of p under the completion map $\mathbb{Z} \to \mathbb{Z}_p$.

1.2.2 Structure in mixed characteristic

In this section we specialize to the case where A is a discrete valuation ring of characteristic 0 with residue field k of finite characteristic p. In such a situation, the integers are a subring of A, and so we can make the following definition:

Definition 1.2.7. Let A be a discrete valuation ring of characteristic 0 with residue field k of characteristic $p > 0$. The *absolute ramification index* of A is the valuation $\nu(p)$.

We notice that the valuation $\nu(p)$ is necessarily strictly positive, as the elements of valuation 0 are precisely the units of A. Since p is sent to 0 by the ring map

$$A \to A/\mathfrak{m}_A = k,$$

the element $p \in A$ is not a unit.

Definition 1.2.8. Let A be a discrete valuation ring of characteristic 0 with residue field k of characteristic $p > 0$. We say that A is *absolutely unramified* if the absolute ramification index of A is 1.

For absolutely unramified discrete valuation rings in mixed characteristic, we have the following structure theorem:

Theorem 1.2.9. [Ser79, Chapter 2, Theorem 3] *Let k be a perfect field of positive characteristic $p > 0$. Then there exists a complete discrete valuation ring $W(k)$ of characteristic 0 which is absolutely unramified with residue field k. This ring is unique up to isomorphism.* □

The discrete valuation ring $W(k)$ appearing in the above theorem are the *p-typical Witt vectors* over k. An introduction to Witt vectors is provided in e.g. [Ser79, Chapter 2.6]. We will not need the explicit construction; we do however record the fact that the Witt vectors $W(k)$ have quotients $W_n(k)$, the *Witt vectors of length n*, and that we can realize $W(k)$ as an inverse limit

$$W(k) = \lim_{\leftarrow} W_n(k).$$

We allow ourselves to be very brief on this point, as we shall only use it for the following:

Example 1.2.10. [Ser79, Corollary of Theorem 8, Chapter 2] If $k = \mathbb{F}_p$ is the field with p elements, then the p-typical Witt vectors $W_n(\mathbb{F}_p)$ of length n are isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. The morphisms $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$ in the inverse system are the canonical ones, and the p-typical Witt vectors

$$W(\mathbb{F}_p) = \lim_{\leftarrow} W_n(\mathbb{F}_p)$$

are the p-adic integers $\mathbb{Z}_p$ of Example 1.2.6.

The following is the structure theorem which we will use on numerous occasions throughout this thesis:

Theorem 1.2.11. [Ser79, Chapter 2, Theorem 4] *Let A be a complete discrete valuation ring of characteristic 0 with perfect residue field k of positive characteristic p. Then, up to isomorphism, A is a truncated polynomial ring*

$$W(k)[x]/(\phi(x)),$$

where $\phi(x)$ is a monic Eisenstein polynomial of degree given by the absolute ramification index of A. Conversely, any such truncated polynomial ring is a discrete valuation ring. A choice of uniformizer is the class of x in A. □

The condition that $\phi(x)$ is a *monic Eisenstein polynomial* means that it is of the form

$$\phi(x) = x^e + a_1 x^{e-1} + \cdots + a_e,$$

where p divides all the coefficients a_i, but p^2 does not divide a_e.

Example 1.2.12. Let $k = \mathbb{F}_5$ and let $\phi(x) = x^2 + 5$. Then

$$A = W(\mathbb{F}_5)[x]/(x^2 + 5) = \mathbb{Z}_5[x]/(x^2 + 5)$$

is a discrete valuation ring with residue field $\mathbb{F}_5$ and absolute ramification index 2.

2 Logarithmic structures

In this section we introduce logarithmic structures on schemes, which in the affine case will serve as the input of the main topic of this thesis, logarithmic Hochschild homology. In Section 2.1 we set up some preliminaries on the category of commutative monoids before providing a general introduction to log schemes in Section 2.2. In addition to the general theory, we discuss the notion of *repletion* as introduced by Rognes in [Rog09], which is a key ingredient in the definition of logarithmic Hochschild homology. Finally, we introduce logarithmic Kähler differentials in Section 2.3, emphasizing certain descriptions of them in the affine case.

2.1 The category of commutative monoids

We introduce the category of commutative monoids, with particular emphasis on the (surprisingly esoteric) description of pushouts in this category. We have used the lecture notes of Arthur Ogus [Ogu06] as a reference for this section.

Definition 2.1.1. A *commutative monoid* $(M, \cdot)$ is a set M with a binary operation $\cdot$ on M which is unital and associative, and a morphism of commutative monoids preserves this structure. We will denote by Mon the resulting category of commutative monoids.

Example 2.1.2. We consider some natural first examples of commutative monoids:

- The natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ is a commutative monoid under addition. It is the free commutative monoid on one generator.
- Given commutative monoids M and N, the commutative monoid $M \oplus N$ has $M \times N$ as its underlying set, and the binary operation is defined coordinatewise. More generally, for a collection of monoids $\{M_i\}_{i \in I}$ indexed over a potentially infinite set I, we demand that elements of the commutative monoid
$$\bigoplus_{i \in I} M_i$$
has all but finitely coordinates equal to the unit 1_{M_i}. The operation $\oplus$ is the coproduct in the category of commutative monoids.
- The k-fold coproduct $\mathbb{N}^{\oplus k}$ is the free monoid on k generators.
- Any commutative ring R has an underlying commutative monoid $(R, \cdot)$ by forgetting the additive structure. This gives an adjunction
$$\mathrm{Mon} \underset{(-,\cdot)}{\overset{\mathbb{Z}[-]}{\rightleftarrows}} \mathrm{Ring},$$
where Ring denotes the category of commutative rings and $\mathbb{Z}[M]$ denotes the *monoid ring* of M. More generally, for any ground ring R one may consider the category Alg_R of commutative R-algebras. The adjunctions
$$\mathrm{Mon} \underset{(-,\cdot)}{\overset{\mathbb{Z}[-]}{\rightleftarrows}} \mathrm{Ring} = \mathrm{Alg}_{\mathbb{Z}} \underset{U}{\overset{R\otimes_{\mathbb{Z}}-}{\rightleftarrows}} \mathrm{Alg}_R$$
with U the forgetful functor shows that forming the monoid algebra $R[-]$ is left adjoint to the forgetful functor $\mathrm{Alg}_R \to \mathrm{Mon}$.

We now explain how to construct coequalizers in the category of commutative monoids. Once this is in place, the existence of coproducts (Example 2.1.1) and coequalizers allows for the construction of all small colimits, see e.g. [ML78, Theorem V.2.1].

Construction 2.1.3. Define a *congruence relation* on a commutative monoid M to be a submonoid K of $M \times M$ which is also an equivalence relation. Then there is a unique commutative monoid structure on the quotient set M/K such that the projection $p\colon M \to M/K$ of sets is a morphism of commutative monoids. For any congruence relation K on M, we see that the quotient M/K fits in a coequalizer diagram

$$K \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} M \xrightarrow{p} M/K,$$

where $\mathrm{pr}_i\colon K \subset M \times M \to M$ denotes the projections. Consider now two morphisms of commutative monoids

$$\varphi_1,\ \varphi_2\colon M \to N.$$

We may describe the coequalizer of these two morphisms as the quotient N/S, where S is the congruence relation generated by

$$R = \{(\varphi_1(m), \varphi_2(m)) \mid m \in M\} \subset N \times N.$$

A more explicit description of S may be given as follows: Consider the set

$$R_N = \{(\varphi_1(m) \cdot n, \varphi_2(m) \cdot n) \mid m \in M, n \in N\}.$$

Then we have that $R_N \subset S$, and moreover R_N is closed under the multiplication by the diagonal $\Delta(N) \subset N \times N$. Then we may consider the *equivalence* relation generated by R_N, and we claim that such an equivalence relation is necessarily a congruence relation: indeed, it contains the identity $(1, 1)$, and for two elements (x, y) and (z, w) of R_N, we have that $(x \cdot z, y \cdot z)$ and $(z \cdot y, w \cdot y)$ also are elements of R_N. Commutativity of N and transitivity of the equivalence relation generated by R_N then imply that $(x \cdot z, y \cdot w)$ is in this equivalence relation, and hence it is a congruence relation. We conclude that S, the congruence relation generated by R, is the *equivalence* relation generated by R_N.

Example 2.1.4 (Pushouts in Mon)**.** Let $\phi\colon M \to N_1$ and $\psi\colon M \to N_2$ be morphisms of commutative monoids. The pushout of the diagram

$$N_1 \xleftarrow{\phi} M \xrightarrow{\psi} N_2,$$

denoted by $N_1 \oplus_M N_2$, can be described by the coequalizer diagram

$$M \underset{(1,\psi)}{\overset{(\phi,1)}{\rightrightarrows}} N_1 \oplus N_2 \longrightarrow N_1 \oplus_M N_2.$$

The following lemma gives an explicit description of pushouts in the category of commutative monoids, provided that one of the monoids involved is actually a group. While the lemma is proved in [Ogu06, Chapter 1], we choose to write out the details here as the argument in *loc. cit.* is not streamlined towards this result by itself.

Lemma 2.1.5. *Consider a diagram of commutative monoids*

$$N_1 \xleftarrow{\varphi_1} M \xrightarrow{\varphi_2} N_2$$

where at least one of M, N_1 and N_2 is a group. The pushout of the diagram can be described as a quotient

$$(N_1 \oplus N_2)/\sim$$

where $(n_1, n_2) \sim (n_1', n_2')$ *if and only if there is a pair* $(m, m') \in M \times M$ *such that* $n_1 \cdot \varphi_1(m) = n_1' \cdot \varphi_1(m')$ *and* $n_2 \cdot \varphi_2(m') = n_2' \cdot \varphi_2(m)$*.*

In the setting of Lemma 2.1.5, we will adopt the terminology that "the pairs (n_1, n_2) and (n_1', n_2') are equivalent through (m, m')".

Proof. Suppose $(n_1, n_2) \sim (n_1', n_2')$ through (m, m'). The same pair (m, m') then shows that $(n_1 \cdot n_1'', n_2 \cdot n_2'') \sim (n_1' \cdot n_1'', n_2' \cdot n_2'')$ for any (n_1'', n_2''), so the equivalence relation $\sim$ is in particular closed under multiplication by the diagonal. We have already noted in Construction 2.1.3 that such an equivalence relation is a congruence relation. Moreover, $(\varphi_1(m), 1) \sim (1, \varphi_2(m))$ through $(1, m)$, so that the relevant congruence relation for the pushout is contained in $\sim$. Consider the two natural morphisms $i_j \colon N_j \to N_1 \oplus_M N_2$ for $j = 1, 2$. As at least one of the monoids involved is a group, the composite $i := i_j \varphi_j$ factors through the units $(N_1 \oplus_M N_2)^*$. Applying i_1 and i_2 to the defining equations of the equivalence $(n_1, n_2) \sim (n_1', n_2')$, we obtain

$$i_1(n_1) \cdot i_1\varphi_1(m) = i_1(n_1') \cdot i_1\varphi_1(m') \quad \text{and} \quad i_2(n_2) \cdot i_2\varphi_2(m') = i_2(n_2') \cdot i_2\varphi_2(m)$$

from which it readily follows, using that $i = i_j \varphi_j$ lands in $(N_1 \oplus_M N_2)^*$, that

$$i_1(n_1) \cdot i_2(n_2) = i_1(n_1') \cdot i_2(n_2').$$

Hence the pairs (n_1, n_2) and (n_1', n_2') become equal in $N_1 \oplus_M N_2$, which concludes the proof. $\square$

Example 2.1.6. Consider the inclusion of additive monoids $\mathbb{N} \to \mathbb{Z}$. The pushout of the diagram

$$0 \longleftarrow \mathbb{N} \longrightarrow \mathbb{Z}$$

is trivial: since $\mathbb{Z}$ is a group, Lemma 2.1.5 applies. Consider a pair $(0, n)$. If n is positive, then $(0, n) \sim (0, 0)$ through $(n, 0)$, while if n is negative, $(0, n) \sim (0, 0)$ through $(0, -n)$. This behaviour is somewhat eccentric compared to that of e.g. abelian groups, where the pushout of the diagram

$$0 \longleftarrow A \longrightarrow B$$

with $A \to B$ a monomorphism is trivial precisely when $A \to B$ is an isomorphism.

The following example describes the group completion of a monoid M as a pushout in the category of commutative monoids:

Example 2.1.7. Let M be a commutative monoid. Recall (from e.g. [Ros94, Theorem 1.1.3]) that the *group completion* of M, denoted by M^{gp}, comes with a morphism $\gamma_M \colon M \to M^{\mathrm{gp}}$ and satisfies the universal property that for any morphism $\phi \colon M \to G$ of commutative monoids, where G is an abelian group, there is a unique morphism $M^{\mathrm{gp}} \to G$ making the diagram

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\gamma_M"'] & G \\ M^{\mathrm{gp}} \arrow[ur, dashed] & \end{tikzcd}$$

commute. An explicit description of M^{gp} is given by

$$(M \times M)/\sim_{\mathrm{gp}},$$

where $(m_1, m_2) \sim_{\mathrm{gp}} (m_1', m_2')$ if and only if there exists an element k of M such that

$$m_1 \cdot m_2' \cdot k = m_2 \cdot m_1' \cdot k.$$

We claim that the group completion can also be described by the pushout diagram

$$\{1\} \longleftarrow M \xrightarrow{\Delta} M \times M$$

in the category of commutative monoids, where $\Delta\colon M \to M \times M$ denotes the diagonal map $\Delta(m) = (m, m)$. Indeed, suppose that $(m_1, m_2) \sim_{\mathrm{gp}} (m_1', m_2')$, i.e. $m_1 \cdot m_2' \cdot k = m_1' \cdot m_2 \cdot k$ for some k in M. Then, with the terminology of Lemma 2.1.5 (which is applicable as $\{1\}$ is a group), we have that $(1, (m_1, m_2)) \sim (1, (m_1', m_2'))$ through $(m_2 \cdot k, m_2' \cdot k)$, as

$$(m_1, m_2) \cdot \Delta(m_2' \cdot k) = (m_1 \cdot m_2' \cdot k, m_2 \cdot m_2' \cdot k) = (m_1' \cdot m_2 \cdot k, m_2' \cdot m_2 \cdot k) = (m_1', m_2') \cdot \Delta(m_2 \cdot k).$$

Conversely, if $(1, (m_1, m_2)) \sim (1, (m_1', m_2'))$ through (m, m'), then we obtain the equality

$$(m_1 \cdot m', m_2 \cdot m') = (m_1' \cdot m, m_2' \cdot m)$$

which implies that

$$m_1 \cdot m_2' \cdot (m \cdot m') = m_1' \cdot m_2 \cdot (m \cdot m')$$

so that $(m_1, m_2) \sim_{\mathrm{gp}} (m_1', m_2')$.

2.2 Log schemes

We review the central definitions and constructions for logarithmic schemes. While it is the affine case of logarithmic rings which will be of interest to us in subsequent sections, we choose to give this exposition at the level of schemes, as the core idea of a logarithmic ring being an "intermediate localization" is perhaps expressed more clearly in this language. Familiarity with schemes will be assumed throughout this section; an introduction is provided in [Hart77, Chapter 2]. In addition to [Ogu06], we have consulted [Kat89] and [Rog09] for this section.

We have been careful to state all definitions both in the language of schemes and in the language of rings, and the reader is free to only read the parts concerning rings.

Definition 2.2.1. Let $X = (X, \mathcal{O}_X)$ be a scheme. A *pre-log structure* on X is a morphism α from a sheaf of monoids $\mathcal{M}$ to the underlying multiplicative sheaf of monoids $(\mathcal{O}_X, \cdot)$:

$$\alpha\colon \mathcal{M} \to (\mathcal{O}_X, \cdot).$$

A pre-log structure on X is a *log structure* if, in addition, the map $\tilde{\alpha}$ in the pullback square

$$\begin{array}{ccc} \alpha^{-1}\mathcal{O}_X^* & \xrightarrow{\tilde{\alpha}} & \mathcal{O}_X^* \\ {\scriptstyle \tilde{i}}\downarrow & & \downarrow{\scriptstyle i} \\ \mathcal{M} & \xrightarrow[\alpha]{} & \mathcal{O}_X \end{array}$$

is an isomorphism. A morphism of pre-log structures $(\mathcal{M}, \alpha) \to (\mathcal{N}, \beta)$ is a morphism of sheaves of monoids $\varphi\colon \mathcal{M} \to \mathcal{N}$ compatible with the structure maps:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\varphi} & \mathcal{N} \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ \mathcal{O}_X & = & \mathcal{O}_X. \end{array}$$

We denote the categories of pre-log and log structures on a scheme X by PreLog_X and Log_X respectively.

Example 2.2.2. Consider the constant sheaf of monoids $\{1\}$ on X. The inclusion $\{1\} \subset (\mathcal{O}_X, \cdot)$ gives the *trivial pre-log structure* on X. The inclusion $\mathcal{O}_X^* \subset \mathcal{O}_X$ gives the *trivial log structure* on X.

Example 2.2.3. Let k be a field and let $X = \operatorname{Spec} k$. Suppose M is a *sharp* monoid, i.e. the group of units M^* is trivial, and consider it as a sheaf of monoids on X. A log structure on X is given by

$$\alpha\colon k^* \oplus M \to (k, \cdot)$$
$$(x, m) \mapsto \begin{cases} x, & \text{if } m = 1, \\ 0, & \text{else.} \end{cases}$$

This is a log structure on X, which is often (particularly when $M = (\mathbb{N}, +)$) called a *log point*. Note that the sharpness condition is necessary, both for the proposed map to be a morphism of monoids, and for there to be any hope for the map to define a log (as opposed to a pre-log) structure.

Definition 2.2.4. A *pre-log scheme* is a scheme X equipped with a pre-log structure $\alpha\colon \mathcal{M} \to (\mathcal{O}_X, \cdot)$. A morphism of pre-log schemes

$$(f, f^\#, f^\flat)\colon (X, \mathcal{M}, \alpha) \to (Y, \mathcal{N}, \beta)$$

consists of a morphism of schemes $(f, f^\#)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and a morphism of sheaves of monoids $f^\flat\colon \mathcal{N} \to f_*\mathcal{M}$ such that the diagram

$$\begin{tikzcd} \mathcal{N} \arrow[r, "f^\flat"] \arrow[d, "\beta"'] & f_*\mathcal{M} \arrow[d, "f_*\alpha"] \\ (\mathcal{O}_Y, \cdot) \arrow[r, "f^\#"'] & (f_*\mathcal{O}_X, \cdot). \end{tikzcd}$$

commutes. We denote the categories of pre-log and log schemes by PreLog and Log respectively.

The situation in which we will primarily work is the following: let $X = \operatorname{Spec} A$ be an affine scheme and let M be a constant sheaf of monoids on X. Then a morphism of sheaf of monoids $\alpha\colon M \to (\mathcal{O}_X, \cdot)$ is completely determined by the morphism $\alpha\colon M \to (A, \cdot)$ on global sections. We call (A, M, α) a *pre-log ring*, and a morphism $(f, f^\flat)\colon (A, M, \alpha) \to (B, N, \beta)$ of pre-log rings consists of a commutative diagram

$$\begin{tikzcd} M \arrow[r, "f^\flat"] \arrow[d, "\alpha"'] & N \arrow[d, "\beta"] \\ (A, \cdot) \arrow[r, "(f, \cdot)"'] & (B, \cdot). \end{tikzcd}$$

This gives the categories of pre-log and log rings, opposite to affine pre-log and log schemes.

Remark 2.2.5. For morphisms $(f, f^\#)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in the category of schemes, one demands that the induced morphisms on stalks

$$f_x^\#\colon \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X, x}$$

is a local morphism of local rings, that is, the preimage of the unique maximal ideal of $\mathcal{O}_{X,x}$ is that of $\mathcal{O}_{Y,f(x)}$. An *ideal* of a monoid M is a subset $I \subset M$ such that $m \cdot n$ is in I if either m or n are. One then sees that any monoid is a "local ring" in the sense that there is a unique maximal ideal: $M^+ := M - M^*$. Given a morphism of *log* schemes, demanding that $f_x^\flat$ should be a local morphism is redundant on account of the commutativity of the diagram

$$\begin{tikzcd} \mathcal{N}_{f(x)} \arrow[r, "f_x^\flat"] \arrow[d, "\beta_x"'] & \mathcal{M}_x \arrow[d, "\alpha_x"] \\ \mathcal{O}_{Y,f(x)} \arrow[r, "f_x^\#"'] & \mathcal{O}_{X,x}. \end{tikzcd}$$

Indeed, the defining property of a log structure is that the structure map induces an isomorphism on the groups of units. Then we have

$$(f_x^\flat)^{-1}(\mathcal{M}_x^+) = (f_x^\flat)^{-1}(\alpha_x^{-1}(\mathfrak{m}_x)) = \beta_x^{-1}((f_x^\#)^{-1}(\mathfrak{m}_x)) = \beta_x^{-1}(\mathfrak{m}_{f(x)}) = \mathcal{N}_{f(x)}^+$$

where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ and $\mathfrak{m}_{f(x)} \subset \mathcal{O}_{Y,f(x)}$ denote the respective maximal ideals.

2.2.1 The logification functor

We introduce a functor

$$\mathrm{PreLog} \to \mathrm{Log}$$

associating to any pre-log scheme $(X, \mathcal{M}, \alpha)$ a log scheme $(X, \mathcal{M}^a, \alpha^a)$. An often omitted point in expositions on log geometry is the tedious but straightforward diagram gymnastics proving that this construction is functorial. We write out the details of this below.

Definition 2.2.6. Let $\alpha\colon \mathcal{M} \to (\mathcal{O}_X, \cdot)$ be a pre-log structure. Define a (a priori pre-)log structure

$$\alpha^a\colon \mathcal{M}^a \to (\mathcal{O}_X, \cdot)$$

on X, the *logification* of α, as the map induced by the pushout square

$$\begin{tikzcd} \alpha^{-1}\mathcal{O}_X^* \arrow[r, "\tilde{\alpha}"] \arrow[d, "\tilde{i}"'] & \mathcal{O}_X^* \arrow[d] \arrow[ddr, bend left, "i"] & \\ \mathcal{M} \arrow[r] \arrow[drr, bend right, "\alpha"'] & \mathcal{M}^a \arrow[dr, dashed, "\alpha^a"] & \\ & & \mathcal{O}_X. \end{tikzcd}$$

In the affine case, if $\alpha\colon M \to (A, \cdot)$ is a pre-log structure, the logification $\alpha^a\colon M^a \to (A, \cdot)$ is then induced by the pushout square

$$\begin{tikzcd} \alpha^{-1}\,\mathrm{GL}_1(A) \arrow[r, "\tilde{\alpha}"] \arrow[d, "\tilde{i}"'] & \mathrm{GL}_1(A) \arrow[d] \\ M \arrow[r] & M^a, \end{tikzcd}$$

where $\alpha^{-1}\,\mathrm{GL}_1(A)$ is by definition the pullback of

$$M \xrightarrow{\alpha} (A, \cdot) \xleftarrow{i} \mathrm{GL}_1(A).$$

Example 2.2.7. Consider the trivial pre-log structure $\{1\} \subset (\mathcal{O}_X, \cdot)$ from Example 2.2.2. Its logification is the trivial log structure $\mathcal{O}_X^* \subset (\mathcal{O}_X, \cdot)$.

Example 2.2.8. Let A be a discrete valuation ring with uniformizer π, and denote by $\langle\pi\rangle$ the free monoid generated by π. Consider the pre-log structure on A given by the inclusion $\langle\pi\rangle \to (A,\cdot)$. Its logification is given by the pushout square

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d] & \mathrm{GL}_1(A) \arrow[d] \\
\langle\pi\rangle \arrow[r] & A - \{0\},
\end{tikzcd}$$

where we are using that any non-zero element of A can be written on the form $u\pi^n$ for a unit u. This example will be used in subsequent sections, and our preferred notation for $A - \{0\}$ is $A \cap \mathrm{GL}_1(K)$, where K denotes the fraction field of A.

Proposition 2.2.9. *Let $\alpha\colon \mathcal{M} \to (\mathcal{O}_X,\cdot)$ be a pre-log structure. Then the logification α^a is a log structure.*

Proof. It is enough to check the isomorphism on each stalk, and as $\mathcal{O}_X^*$ is a sheaf of groups, Lemma 2.1.5 applies. Recall the diagram from Definition 2.2.6. Denote the equivalence class of (m,x) by $[m,x] \in \mathcal{M}^a$. Then $\alpha^a([m,x]) = \alpha(m)\cdot i(x)$, and as $i(x)$ is a unit, $\alpha^a([m,x])$ is a unit precisely when $\alpha(m)$ is. If $\alpha(m)$ is a unit, then $[m,x] = [1,\alpha(m)\cdot x]$ through the pair $((1,1),(m,\alpha(m)))$ of elements in $\alpha^{-1}\mathcal{O}_X^*$ in the terminology of Lemma 2.1.5. Hence

$$\tilde{\alpha}^a\colon (\alpha^a)^{-1}\mathcal{O}_X^* \to \mathcal{O}_X^*$$

is an isomorphism with inverse sending $x \in \mathcal{O}_X^*$ to $([1,x],x) \in (\alpha^a)^{-1}\mathcal{O}_X^*$. ☐

Lemma 2.2.10. *The logification construction from Definition 2.2.6 gives functors*

1. $\mathrm{PreLog}_X \to \mathrm{Log}_X$
2. $\mathrm{PreLog} \to \mathrm{Log}$.

Proof. The argument is an elementary (but tedious) diagram chase:

1. Let $\varphi\colon (\mathcal{M},\alpha) \to (\mathcal{N},\beta)$ be a morphism of pre-log structures. Provided that the outer diagram commutes, a map $\varphi^a\colon \mathcal{M}^a \to \mathcal{N}^a$ is given by the defining pushout square of $\mathcal{M}^a$:

$$\begin{tikzcd}
\alpha^{-1}\mathcal{O}_X^* \arrow[r, "\tilde{\alpha}"] \arrow[d] & \mathcal{O}_X^* \arrow[d] \arrow[ddr, bend left] & \\
\mathcal{M} \arrow[r] \arrow[dr, bend right, "\varphi"'] & \mathcal{M}^a \arrow[dr, dashed, "\varphi^a"] & \\
& \mathcal{N} \arrow[r] & \mathcal{N}^a.
\end{tikzcd}$$

To see that the outer diagram commutes, note that φ being a morphism of pre-log structures gives a map $\tilde{\varphi}$ satisfying $\tilde{\beta}\tilde{\varphi} = \tilde{\alpha}$:

$$\begin{tikzcd}
\alpha^{-1}\mathcal{O}_X^* \arrow[d] \arrow[dr, dashed, "\tilde{\varphi}"] \arrow[drr, bend left, "\tilde{\alpha}"] & & \\
\mathcal{M} \arrow[dr, bend right, "\varphi"'] & \beta^{-1}\mathcal{O}_X^* \arrow[r, "\tilde{\beta}"] \arrow[d] & \mathcal{O}_X^* \arrow[d, "i"] \\
& \mathcal{N} \arrow[r, "\beta"] & \mathcal{O}_X.
\end{tikzcd}$$

Indeed, the outer diagram above is the defining pullback diagram of $\alpha^{-1}\mathcal{O}_X^*$. Then the diagram

$$\begin{tikzcd}
\alpha^{-1}\mathcal{O}_X^* \arrow[r, "\tilde{\varphi}"] \arrow[d] & \beta^{-1}\mathcal{O}_X^* \arrow[r, "\tilde{\beta}"] \arrow[d] & \mathcal{O}_X^* \arrow[d] \\
\mathcal{M} \arrow[r, "\varphi"] & \mathcal{N} \arrow[r] & \mathcal{N}^a
\end{tikzcd}$$

commutes, the right-hand square being the defining pushout square of $\mathcal{N}^a$. That this determines a morphism of log structures, i.e. that $\beta^a\varphi^a = \alpha^a$, follows from uniqueness of α^a.

2. Let $(f, f^\flat)\colon (X, \mathcal{M}, \alpha) \to (Y, \mathcal{N}, \beta)$ be a morphism of pre-log schemes. Provided that the outer diagram below commutes, we obtain a morphism $f^{\flat a}\colon \mathcal{N}^a \to f_*\mathcal{M}^a$:

$$\begin{tikzcd}
\beta^{-1}\mathcal{O}_Y^* \arrow[r, "\tilde{\beta}"] \arrow[d] & \mathcal{O}_Y^* \arrow[d] \arrow[dr, bend left, "f^{\#}_{|\mathcal{O}_Y^*}"] & \\
\mathcal{N} \arrow[r] \arrow[dr, bend right, "f^\flat"'] & \mathcal{N}^a \arrow[dr, dashed, "f^{\flat a}"] & f_*\mathcal{O}_X^* \arrow[d] \\
 & f_*\mathcal{M} \arrow[r] & f_*\mathcal{M}^a.
\end{tikzcd}$$

As f_* is a right adjoint, $f_*\alpha^{-1}\mathcal{O}_X^* \cong (f_*\alpha)^{-1}f_*\mathcal{O}_X^*$, so we obtain a map $\beta^{-1}\mathcal{O}_Y^* \to f_*\alpha^{-1}\mathcal{O}_X^*$ in the diagram

$$\begin{tikzcd}
\beta^{-1}\mathcal{O}_Y^* \arrow[r, bend left, "\tilde{\beta}"] \arrow[d, bend right] \arrow[dr, dashed] & \mathcal{O}_Y^* \arrow[dr, bend left, "f^{\#}_{|\mathcal{O}_Y^*}"] & \\
\mathcal{N} \arrow[dr, bend right, "f^\flat"'] & f_*\alpha^{-1}\mathcal{O}_X^* \arrow[r, "f_*\tilde{\alpha}"] \arrow[d] & f_*\mathcal{O}_X^* \arrow[d] \\
 & f_*\mathcal{M} \arrow[r, "f_*\alpha"] & f_*\mathcal{O}_X.
\end{tikzcd}$$

Commutativity of the above outer diagram is a consequence of $(f, f^\flat)$ being a morphism of pre-log schemes: By definition we have that $(f_*\alpha)f^\flat = f^{\#}\beta$, so that commutativity of the outer diagram is equivalent to commutativity of

$$\begin{tikzcd}
\beta^{-1}\mathcal{O}_Y^* \arrow[r, "\tilde{\beta}"] \arrow[d] & \mathcal{O}_Y^* \arrow[r, "f^{\#}_{|\mathcal{O}_Y^*}"] \arrow[d, "i"] & f_*\mathcal{O}_X^* \arrow[d, "f_*i"] \\
\mathcal{N} \arrow[r, "\beta"] & \mathcal{O}_Y \arrow[r, "f^{\#}"] & f_*\mathcal{O}_X.
\end{tikzcd}$$

By construction of the morphism $\beta^{-1}\mathcal{O}_Y^* \to f_*\alpha^{-1}\mathcal{O}_X^*$ it follows that we have a commutative diagram

$$\begin{tikzcd}
\beta^{-1}\mathcal{O}_Y^* \arrow[r, equal] \arrow[d] & \beta^{-1}\mathcal{O}_Y^* \arrow[r, "\tilde{\beta}"] \arrow[d] & \mathcal{O}_Y^* \arrow[d, "f^{\#}_{|\mathcal{O}_Y^*}"] \\
\mathcal{N} \arrow[d, "f^\flat"] & f_*\alpha^{-1}\mathcal{O}_X^* \arrow[r, "f_*\tilde{\alpha}"] \arrow[d] & f_*\mathcal{O}_X^* \arrow[d] \\
f_*\mathcal{M} \arrow[r, equal] & f_*\mathcal{M} \arrow[r] & f_*\mathcal{M}^a.
\end{tikzcd}$$

Here the lower right-hand square commutes by functoriality, while the upper right-hand square and the left-hand rectangle commutes by construction. The entire square is precisely the diagram we wanted to commute. That the map $f^{\flat a}$ determines a map of log schemes follows from uniqueness of maps out of $\mathcal{N}^a$ and that $(f, f^\flat)$ is a morphism of pre-log schemes. Indeed, the following two diagrams coincide:

$$\begin{tikzcd}
\beta^{-1}\mathcal{O}_Y^* \arrow[r, "\tilde{\beta}"] \arrow[d] & \mathcal{O}_Y^* \arrow[d] \arrow[dr, bend left, "f^{\#}_{|\mathcal{O}_Y^*}"] & & \\
\mathcal{N} \arrow[r] \arrow[dr, bend right, "f^\flat"'] & \mathcal{N}^a \arrow[dr, "f^{\flat a}"] & f_*\mathcal{O}_X^* \arrow[ddr, bend left, "f_*i"] & \\
& f_*\mathcal{M} \arrow[drr, bend right, "f_*\alpha"'] & f_*\mathcal{M}^a \arrow[dr, "f_*\alpha^a"] & \\
& & & f_*\mathcal{O}_X,
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta^{-1}\mathcal{O}_Y^* \arrow[r, "\tilde{\beta}"] \arrow[d] & \mathcal{O}_Y^* \arrow[d] \arrow[dr, bend left, "f^{\#}_{|\mathcal{O}_Y^*}"] & & \\
\mathcal{N} \arrow[r] \arrow[dr, bend right, "\beta"'] & \mathcal{N}^a \arrow[dr, "\beta^a"] & f_*\mathcal{O}_X^* \arrow[ddr, bend left, "f_*i"] & \\
& \mathcal{O}_Y \arrow[drr, bend right, "f^{\#}"'] & \mathcal{O}_Y \arrow[dr, "f^{\#}"] & \\
& & & f_*\mathcal{O}_X.
\end{tikzcd}$$

□

2.2.2 Constructions and examples

We introduce the direct and inverse image log structures and highlight the idea of a log structure as an "intermediate localization."

Definition 2.2.11. Let $f\colon X \to Y$ be a morphism of schemes.

1. Let $(X, \mathcal{M}, \alpha)$ be a log scheme. Define the *direct image* log structure on Y to be the pullback square

$$\begin{tikzcd}
f_*^{\log}\mathcal{M} \arrow[r, "f_*^{\log}\alpha"] \arrow[d] & \mathcal{O}_Y \arrow[d, "f^{\#}"] \\
f_*\mathcal{M} \arrow[r, "f_*\alpha"'] & f_*\mathcal{O}_X.
\end{tikzcd}$$

2. Let $(Y, \mathcal{N}, \beta)$ be a log scheme. Define the *inverse image* log structure on X as the logification of the pre-log structure
$$f^{-1}\mathcal{N} \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X,$$
where the latter map is adjoint to $f^{\#}$. Denote this log structure by $f^*\beta$.

Example 2.2.12. The direct image log structure gives rise to a class of interesting log structures as follows: jet $j\colon U \to X$ be the inclusion of a Zariski open U, and endow U with the trivial log structure (Example 2.2.2). Give X the direct image log structure through j:

$$\begin{tikzcd} \mathcal{M}_{U/X} \arrow[r, "\alpha_{U/X}"] \arrow[d] & \mathcal{O}_X \arrow[d, "j^\#"] \\ j_*\mathcal{O}_U^* \arrow[r] & j_*\mathcal{O}_U. \end{tikzcd}$$

Explicitly $\mathcal{M}_{U/X}$ is the subsheaf of $(\mathcal{O}_X, \cdot)$ consisting of those sections that restrict to units on U. For example, $\alpha_{X/X}$ is the trivial log structure.

Example 2.2.13. Building on Example 2.2.12, we recreate the log structure discussed in Example 2.2.8. Let A be a discrete valuation ring with fraction field K. Then $\operatorname{Spec} A$ consists of two points: one open and generic point (the zero ideal) and one closed point (the maximal ideal). The direct image log structure associated to the inclusion of the open point consists of those sections which restrict to units on the open point: that is, it consists of those elements of A which map to a unit through the localization map

$$A \to A_{(0)} = K.$$

Hence the direct image log structure is the inclusion $A - \{0\} = A \cap \mathrm{GL}_1(K) \to A$, which is precisely the log structure discussed in Example 2.2.8.

Remark 2.2.14. Let X be a scheme and let $U \subset X$ be Zariski open, and consider both U and X as trivial log schemes. The open immersion $j\colon U \to X$ has a non-trivial factorization in the category of log schemes, namely

$$(U, \mathcal{O}_U^*) \longrightarrow (X, \mathcal{M}_{U/X}) \longrightarrow (X, \mathcal{O}_X^*).$$

With respect to Example 2.2.13, we obtain a factorization, after taking global sections:

$$A \longrightarrow (A, A \cap \mathrm{GL}_1(K)) \longrightarrow K.$$

We think of the log ring $(A, A \cap \mathrm{GL}_1(K))$ as an "intermediate localization" of A. From the perspective of log geometry, this is thought of as a "compactification" of the Zariski open U, which is the main idea from log geometry which has been used by topologists. In Section 3.2.1 we construct a long exact sequence relating the Hochschild homology of a ring to the "Hochschild homology" of such an intermediate localization.

Example 2.2.15. Let (A, M) be a pre-log ring, and consider the ring $A[M^{-1}] = A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M^{\mathrm{gp}}]$. Following [Rog09], we define the *trivial locus* of (A, M) to be the pre-log ring $(A[M^{-1}], M^{\mathrm{gp}})$. If we endow A with the trivial pre-log structure, we obtain a factorization

$$(A, \{1\}) \to (A, M) \to (A[M^{-1}], M^{\mathrm{gp}}).$$

Applying the logification functor we obtain the factorization

$$(A, \mathrm{GL}_1(A)) \to (A, M^a) \to (A[M^{-1}], \mathrm{GL}_1(A[M^{-1}])),$$

as the logification of $(A[M^{-1}], M^{\mathrm{gp}})$ is trivial since M^{gp} is a group. Specializing to the case of a discrete valuation ring A with the pre-log structure induced by a uniformizer π, this recovers the factorization considered in Remark 2.2.14.

This is a good occasion to make the following observation:

Remark 2.2.16. Let M be a commutative monoid and let M^{gp} be its group completion. Then the pushout $M^{\mathrm{gp}} \sqcup_M M^{\mathrm{gp}}$ of commutative monoids is isomorphic to M^{gp}: any two morphisms $\phi_1, \phi_2 \colon M^{\mathrm{gp}} \to N$ must factor through the group of units N^* of N, and the morphisms $M^{\mathrm{gp}} \to N^*$ are uniqely determined by the composites $M \to M^{\mathrm{gp}} \to N^*$ by adjunction. So if ϕ_1 and ϕ_2 satisfy $\phi_1 \circ \gamma_M = \phi_2 \circ \gamma_M$ for $\gamma_M \colon M \to M^{\mathrm{gp}}$ the group completion map, then $\phi_1 = \phi_2$. In particular, if (A, M, α) is a pre-log structure such that the structure map α factors through the group completion map M^{gp}, then

$$A[M^{-1}] = A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M^{\mathrm{gp}}] \cong A \otimes_{\mathbb{Z}[M^{\mathrm{gp}}]} (\mathbb{Z}[M^{\mathrm{gp}}] \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M^{\mathrm{gp}}]) \cong A,$$

as one would expect.

2.2.3 Exact and replete morphisms

We introduce exact and replete morphisms of commutative monoids. The notion of repletion will be an essential ingredient in the definition of logarithmic Hochschild homology. For this section we have consulted [Rog09] in addition to [Ogu06].

Definition 2.2.17. A morphism $\varphi \colon M \to N$ of commutative monoids is *exact* if the square

$$\begin{array}{ccc} M & \xrightarrow{\gamma_M} & M^{\mathrm{gp}} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi^{\mathrm{gp}}} \\ N & \xrightarrow[\gamma_N]{} & N^{\mathrm{gp}} \end{array}$$

is a pullback square.

Example 2.2.18. Let M be a commutative monoid. We characterize the conditions that make the diagonal map $\Delta \colon M \to M \times M$ exact. The pullback P in the diagram

$$\begin{array}{ccc} P & \longrightarrow & M^{\mathrm{gp}} \\ \downarrow & & \downarrow \\ M \times M & \xrightarrow[\gamma_{M \times M}]{} & (M \times M)^{\mathrm{gp}} \cong M^{\mathrm{gp}} \times M^{\mathrm{gp}} \end{array}$$

consists of pairs $((m_1, m_2), \gamma_M(m))$ such that

$$(\gamma_M(m_1), \gamma_M(m_2)) = (\gamma_M(m), \gamma_M(m)).$$

We then see that we may identify P with M precisely when $\gamma_M \colon M \to M^{\mathrm{gp}}$ is injective. In turn, this occurs if and only if M is integral (cancellative): indeed, M is integral if and only if $m \cdot k = m' \cdot k$ implies $m = m'$, or equivalently $\gamma_M(m) = \gamma_M(m')$ implies $m = m'$, cf. Example 2.1.7. Exact morphisms are in general better behaved in the category of integral monoids; many examples can be found in [Ogu06].

Definition 2.2.19. Let $\varphi \colon M \to N$ be a morphism of commutative monoids.

1. The morphism φ is *virtually surjective* if the morphism φ^{gp} of abelian groups is surjective.

2. The morphism φ is *replete* if it is both exact and virtually surjective.

3. If φ is virtually surjective, define the *repletion* M^{rep} of M by the right-hand pullback square

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\varphi"'] & M^{\mathrm{rep}} \arrow[r] \arrow[d, "\varphi^{\mathrm{rep}}"] & M^{\mathrm{gp}} \arrow[d, "\varphi^{\mathrm{gp}}"] \\ N \arrow[r, equal] & N \arrow[r, "\gamma_N"'] & N^{\mathrm{gp}}. \end{tikzcd}$$

The map $M \to M^{\mathrm{rep}}$ is given by the universal property of the right-hand pullback square along φ and γ_M.

The notion of virtual surjectivity will play an essential role in the simplicial setting of Section 4, see e.g. Lemma 4.1.13.

Lemma 2.2.20. [Rog09, Lemma 3.8] *Let $\varphi\colon M \to N$ be a virtually surjective morphism of commutative monoids. Then φ^{rep} is replete.* □

We will prove a simplicial analogue of the above lemma in Proposition 4.1.16.

2.3 Log Kähler differentials

In this section we discuss *logarithmic Kähler differentials*, an extension of Kähler differentials to logarithmic rings, or more generally, schemes. The main references for this section are [Rog09] and [Ogu06], in which the definitions and results stated can be found. In Example 2.3.11 we give a description of the logarithmic Kähler differentials $\Omega^1_{(A,M)/(R,N)}$ for a morphism of pre-log rings $(f, f^\flat)\colon (R,N) \to (A,M)$, which will motivate the definition that we give for "relative" logarithmic Hochschild homology in Section 4.1.3. In Lemma 2.3.10 we prove that, in the case of pre-log rings, the module of log Kähler differentials is invariant under logification.

Definition 2.3.1. Let A be a commutative R-algebra and let M be an A-module. An *A-derivation* is an R-linear map $\mathrm{d}\colon A \to M$ which satisfies the Leibniz rule

$$\mathrm{d}(ab) = a\mathrm{d}(b) + \mathrm{d}(a)b.$$

Denote by $\mathrm{Der}(A, M)$ the A-derivations $A \to M$.

Remark 2.3.2. We recall the basic properties of Kähler differentials, allowing ourselves to be very brief as they will be presented in a more general setting in Proposition 2.3.9. More details can be found in e.g. [Hart77, Section 2.8]. The functor $\mathrm{Der}(A, -)$ is representable, the representing object being the module of relative differentials $\Omega^1_{A/R}$, defined by the universal property that there is an A-derivation $\mathrm{d}\colon A \to \Omega^1_{A/R}$, through which any A-derivation factors uniquely. This gives the mentioned natural isomorphism

$$\mathrm{Mod}_A(\Omega^1_{A/R}, -) \xrightarrow{\simeq} \mathrm{Der}(A, -)$$

by precomposition with d, while its inverse is provided by the universal property of $\Omega^1_{A/R}$.

Example 2.3.3. Suppose A is a polynomial algebra $R[x_1, \ldots, x_n]$. Then $\Omega^1_{A/R}$ is the free A-module $A\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ on n generators $\mathrm{d}x_i$. The universal A-derivation $A \to A\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ sends x_i to the generator $\mathrm{d}x_i$, cf. [Hart77, Example 8.2.1].

Example 2.3.4. Let $A = R[x, y]/(x^2)$. Then $\Omega^1_{A/R} = A\{\mathrm{d}x, \mathrm{d}y\}/(2x\,\mathrm{d}x)$. This can be seen using the exact sequence

$$I/I^2 \to B/I \otimes_B \Omega^1_{B/R} \to \Omega^1_{(B/I)/R} \to 0$$

for any R-algebra B and ideal $I \subset B$, where the first morphism maps b to $1 \otimes \mathrm{d}b$, see [Hart77, Proposition 8.4A]. Setting $B = R[x,y]$ and $I = (x^2)$, we find that the middle term is

$$A \otimes_{R[x,y]} R[x,y]\{\mathrm{d}x, \mathrm{d}y\} \cong A\{\mathrm{d}x, \mathrm{d}y\},$$

and the cokernel of the left-hand map is precisely

$$A\{\mathrm{d}x, \mathrm{d}y\}/(2x\,\mathrm{d}x),$$

since this morphism maps x^2 to

$$1 \otimes \mathrm{d}x^2 = 1 \otimes 2x\,\mathrm{d}x = 2x \otimes \mathrm{d}x.$$

Example 2.3.5. Let I be the kernel of the multiplication map $A \otimes_R A \to A$ for a commutative R-algebra A. There is an isomorphism of A-modules $\Omega^1_{A/R} \cong I/I^2$ under which the universal A-derivation is sent to the map

$$a \mapsto 1 \otimes a - a \otimes 1.$$

Indeed, consider a lifting problem

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\mathrm{d}"'] & M \\ I/I^2. \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

with δ an A-derivation. Clearly, a lift is given by $\phi(a \otimes b) = a\delta(b)$, which is well-defined since δ is an A-derivation. There is no other choice, for if ϕ is a lift, then

$$\phi(a \otimes b) = \phi((a \otimes 1)(1 \otimes b - b \otimes 1) + ab \otimes 1) = \phi(a\mathrm{d}b) = a\delta(b).$$

The above example motivates the definition often given for the module $\Omega^1_{X/Y}$ for a morphism of schemes $f \colon X \to Y$, namely, as the pullback $\Delta^*(\mathcal{J}/\mathcal{J}^2)$ where $\Delta \colon X \to X \times_Y X$ is the diagonal map and $\mathcal{J}$ is the sheaf of ideals of the image $\Delta(X)$. See [Hart77, Section 2.8] for details.

Definition 2.3.6. Let $(f, f^\flat) \colon (X, \mathcal{M}, \alpha) \to (Y, \mathcal{N}, \beta)$ be a morphism of pre-log schemes. A *log derivation* is a pair of maps

$$(D \colon \mathcal{O}_X \to \mathcal{F},\ \ \delta \colon \mathcal{M} \to \mathcal{F})$$

with $\mathcal{F}$ an $\mathcal{O}_X$-module such that

1. $D(st) = sD(t) + tD(s)$ for sections s and t,
2. $D(f^{-1}(c)) = 0$ for $c \in f^{-1}\mathcal{O}_Y$,
3. $D(\alpha(m)) = \alpha(m)\delta(m)$,
4. $\delta(\tilde{f}^\flat(n)) = 0$ for $n \in f^{-1}\mathcal{N}$, where $\tilde{f}^\flat$ is adjoint to $f^\flat$.

Denote by $\mathrm{Der}_{X/Y} = \mathrm{Der}_{(X,\mathcal{M})/(Y,\mathcal{N})}(\mathcal{F})$ the log derivations $(\mathcal{O}_X \to \mathcal{F}, \mathcal{M} \to \mathcal{F})$.

Definition 2.3.7. Let $(f, f^\flat)$ be a morphism of pre-log schemes as above. The *sheaf of log Kähler differentials* $\Omega^1_{(X,\mathcal{M})/(Y,\mathcal{N})}$ is given by

$$\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{\mathrm{gp}})/R,$$

where R is the $\mathcal{O}_X$-submodule generated by $(\mathrm{d}\alpha(m), -\alpha(m) \otimes m)$ and $(0, 1 \otimes \tilde{f}^\flat(n))$. Here $\tilde{f}^\flat$ is adjoint to $f^\flat$. The *universal log derivation* $(D, \mathrm{d}\log)$ is given by $D(x) = (\mathrm{d}x, 0)$ and $\mathrm{d}\log(m) = (0, 1 \otimes m)$. We let $\mathcal{O}_X$ act diagonally on $\Omega^1_{X/Y}$.

We translate the above definition the pre-log rings. If $(f, f^\flat)\colon (R, N, \beta) \to (A, M, \alpha)$ is a morphism of pre-log rings, then the log Kähler differentials are given by

$$\Omega^1_{(A,M)/(R,N)} = (\Omega^1_{A/R} \oplus (A \otimes_{\mathbb{Z}} M^{\mathrm{gp}}))/\sim,$$

where the equivalence relation is A-linearly generated by $(\mathrm{d}\alpha(m), 0) \sim (0, \alpha(m) \otimes \gamma_M(m))$ and

$$0 \sim (0, 1 \otimes \gamma_M(f^\flat(n))).$$

Also in this setting, we will write $\mathrm{d}\log m$ for the element $(0, 1 \otimes \gamma_M(m))$ of $\Omega^1_{(A,M)/(R,N)}$.

Remark 2.3.8. The universal log derivation is indeed a log derivation, as

$$D(\alpha(m)) = (\mathrm{d}\alpha(m), 0) = (0, \alpha(m) \otimes m) = \alpha(m)(0, 1 \otimes m) = \alpha(m)\mathrm{d}\log(m).$$

If $(X, \mathcal{M}, \alpha)$ is a log scheme, the universal log derivation provides some justification for the term "logarithmic". We have the identity $\mathrm{d}\log(m_1 \cdot m_2) = \mathrm{d}\log(m_1) + \mathrm{d}\log(m_2)$. Moreover, since the map α induces an isomorphism on the groups of units, if we call the inverse map $\lambda\colon \mathcal{O}_X^* \to \mathcal{M}$, we have that

$$\begin{aligned}\mathrm{d}\log(\lambda(x)) = (0, 1 \otimes \lambda(x)) &= x^{-1}(0, x \otimes \lambda(x)) \\ &= x^{-1}(0, \alpha(\lambda(x)) \otimes \lambda(x)) = x^{-1}(\mathrm{d}\alpha(\lambda(x)), 0) \\ &= x^{-1}(\mathrm{d}x, 0) = x^{-1}D(x).\end{aligned}$$

Naming the map λ was only a matter of notational convenience during the computation; we conclude that the identity

$$\mathrm{d}\log(x) = x^{-1}D(x)$$

holds for units $x \in \mathcal{O}_X^* \cong \mathcal{M}^*$.

Proposition 2.3.9. *If $f\colon (X, \mathcal{M}, \alpha) \to (Y, \mathcal{N}, \beta)$ is a morphism of log schemes, there is a natural isomorphism*

$$\mathrm{Mod}_{\mathcal{O}_X}(\Omega^1_{(X,\mathcal{M})/(Y,\mathcal{N})}, -) \xrightarrow{\simeq} \mathrm{Der}_{(X,\mathcal{M})/(Y,\mathcal{N})}(-).$$

Proof. A morphism of $\mathcal{O}_X$-modules $\Omega^1_{(X,\mathcal{M})/(Y,\mathcal{N})} \to \mathcal{F}$ determines a log derivation by precomposition with the universal log derivation. Conversely, given a log derivation $(D'\colon \mathcal{O}_X \to \mathcal{F}, \delta'\colon \mathcal{M} \to \mathcal{F})$, define a morphism

$$\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{\mathrm{gp}}) \to \mathcal{F}$$

as follows: the universal properties of $\Omega^1_{X/Y}$ and $\mathcal{M}^{\mathrm{gp}}$ provide factorizations

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "D'"] \arrow[d, "\mathrm{d}"] & \mathcal{F} \\ \Omega^1_{X/Y} \arrow[ur, dashed, "\rho"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M} \arrow[r, "\delta'"] \arrow[d, "\gamma"] & \mathcal{F} \\ \mathcal{M}^{\mathrm{gp}}. \arrow[ur, dashed, "(\delta')^{\mathrm{gp}}"'] & \end{tikzcd}$$

We then define the map by

$$(\mathrm{d}x, x' \otimes g) \mapsto \rho(\mathrm{d}x) + x' \cdot (\delta')^{\mathrm{gp}}(g).$$

This defines a map $\chi\colon \Omega^1_{(X,\mathcal{M})/(Y,\mathcal{N})} \to \mathcal{F}$, as

$$(\mathrm{d}\alpha(m), -\alpha(m)\otimes\gamma(m)) \mapsto \rho(\mathrm{d}\alpha(m)) - \alpha(m)\cdot(\delta')^{\mathrm{gp}}(\gamma(m)) = D'(\alpha(m)) - \alpha(m)\cdot\delta'(m) = 0$$

since (D',δ') is a log derivation.

We check that the two constructions are mutually inverse. Given a morphism of $\mathcal{O}_X$-modules $\varphi\colon \Omega^1_{(X,\mathcal{M})/(Y,\mathcal{N})} \to \mathcal{F}$, we obtain the log derivation

$$(\varphi D\colon \mathcal{O}_X \to \mathcal{F}, \varphi d\log\colon \mathcal{M} \to \mathcal{F}).$$

The morphisms φD and $\varphi\mathrm{d}\log$ fit in the diagrams

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "D"] \arrow[d, "\mathrm{d}"] & \Omega^1_{(X,\mathcal{M})/(Y,\mathcal{N})} \arrow[r, "\varphi"] & \mathcal{F} \\
\Omega^1_{X/Y} \arrow[ur] \arrow[urr, dashed, "\rho"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M} \arrow[r, "\mathrm{d}\log"] \arrow[d, "\gamma"] & \Omega^1_{(X,\mathcal{M})/(Y,\mathcal{N})} \arrow[r, "\varphi"] & \mathcal{F}. \\
\mathcal{M}^{\mathrm{gp}} \arrow[ur, dashed] \arrow[urr, dashed] & &
\end{tikzcd}$$

The morphism χ we obtain from the construction in the previous paragraph then coincides with φ, as direct computation gives that

$$\chi(\mathrm{d}x, x'\otimes\gamma(m)) = \rho(\mathrm{d}x) + x'\cdot(\varphi\mathrm{d}\log)^{\mathrm{gp}}(\gamma(m)) = \varphi(\mathrm{d}x,0) + x'\cdot\varphi(0,1\otimes\gamma(m)) = \varphi(\mathrm{d}x, x'\otimes\gamma(m)).$$

Conversely, if $(D'\colon \mathcal{O}_X \to \mathcal{F}, \delta'\colon \mathcal{M}\to\mathcal{F})$ is a log derivation which induces a map $\chi\colon \Omega^1_{(X,\mathcal{M})/(Y,\mathcal{N})} \to \mathcal{F}$ as in the previous paragraph, the associated log derivation $(\chi D, \chi\mathrm{d}\log)$ satisfies

$$\chi(D(x)) = \chi(\mathrm{d}x, 0) = \rho(\mathrm{d}x) = D'(x), \quad \chi(\mathrm{d}\log(m)) = \chi(0, 1\otimes\gamma(m)) = (\delta')^{\mathrm{gp}}(\gamma(m)) = \delta'(m)$$

where $\rho\colon \Omega^1_{X/Y} \to \mathcal{F}$ now denotes the unique map satisfying $\rho\mathrm{d} = D'$. $\square$

We have allowed for pre-log schemes in the definition of log Kähler differentials. It is however true that the sheaf of log Kähler differentials is invariant under logification. While this is also true for schemes ([Kat89]), we only prove it at the level of pre-log rings.

Lemma 2.3.10. *Let $(f, f^\flat)\colon (R,N,\beta) \to (A,M,\alpha)$ be a morphism of pre-log rings. There are isomorphisms*

$$\Omega^1_{(A,M)/(R,N)} \cong \Omega^1_{(A,M^a)/(R,N)} \cong \Omega^1_{(A,M^a)/(R,N^a)}.$$

Proof. Consider the morphism

$$\Omega^1_{(A,M)/(R,N)} \to \Omega^1_{(A,M^a)/(R,N)}$$

sending $\mathrm{d}a \mapsto \mathrm{d}a$ and $\mathrm{d}\log m \mapsto \mathrm{d}\log([m,1])$, where we use the brackets to denote equivalence classes in the logification M^a. This is well-defined, as

$$\mathrm{d}\alpha(m) \mapsto \mathrm{d}\alpha(m) = \mathrm{d}\alpha^a([m,1]) = \alpha^a([m,1])\mathrm{d}\log([m,1]) = \alpha(m)\mathrm{d}\log([m,1]),$$

which is precisely the image of $\alpha(m)\mathrm{d}\log m$. Moreover,

$$\mathrm{d}\log(f^\flat(n)) \mapsto \mathrm{d}\log([f^\flat(n),1]) = 0,$$

as the morphism $N \to M^a$ factors through M by definition.

We define the inverse map as follows: for a logarithmic differential $\mathrm{d}\log([m,x])$ in $\Omega^1_{(A,M^a)/(R,N)}$, write $\mathrm{d}\log([m,x]) = \mathrm{d}\log([m,1]) + \mathrm{d}\log([1,x])$. As $[1,x]$ corresponds to the unit x in $\mathrm{GL}_1(A) \cong (M^a)^*$, this can be written as $\mathrm{d}\log([m,1]) + x^{-1}\mathrm{d}x$, cf. Remark 2.3.8. We now define the inverse map on logarithmic differentials by

$$\mathrm{d}\log([m,x]) \mapsto \mathrm{d}\log(m) + x^{-1}\mathrm{d}x.$$

This is well-defined, as

$$\mathrm{d}\log([f^\flat(n),1]) \mapsto \mathrm{d}\log(f^\flat(n)) = 0,$$

and

$$\mathrm{d}\alpha^a([m,x]) \mapsto \mathrm{d}\alpha^a([m,x]) = \mathrm{d}(\alpha(m)\cdot x) = \alpha(m)\mathrm{d}x + x\mathrm{d}\alpha(m) = \alpha(m)\mathrm{d}x + x\alpha(m)\mathrm{d}\log(m)$$

which is precisely the image of

$$\alpha^a([m,x])\mathrm{d}\log([m,x]) = \alpha^a([m,x])(\mathrm{d}\log([m,1]) + x^{-1}\mathrm{d}x) = \alpha(m)x\mathrm{d}\log([m,1]) + \alpha(m)\mathrm{d}x.$$

We now define a morphism

$$\Omega^1_{(A,M^a)/(R,N)} \to \Omega^1_{(A,M^a)/(R,N^a)}$$

by sending $\mathrm{d}a \mapsto \mathrm{d}a$ and $\mathrm{d}\log([m,x]) \mapsto \mathrm{d}\log([m,x])$. This is well-defined, as

$$\mathrm{d}\log([f^\flat(n),1]) \mapsto \mathrm{d}\log([f^\flat(n),1]) = \mathrm{d}\log((f^\flat)^a([n,1])) = 0.$$

The obvious inverse map is also well-defined, as

$$\mathrm{d}\log((f^\flat)^a([n,u])) = \mathrm{d}\log([f^\flat(n),\beta(u)]) = \mathrm{d}\log([f^\flat(n),1]) + \beta(u)^{-1}\mathrm{d}\beta(u) = \mathrm{d}\log([f^\flat(n),1])$$

where u is in $\mathrm{GL}_1(R)$. This is sent to $\mathrm{d}\log([f^\flat(n),1]) = 0$ in $\Omega^1_{(A,M^a)/(R,N)}$. ☐

Example 2.3.11. Let $(f,f^\flat)\colon (R,N,\beta) \to (A,M,\alpha)$ be a morphism of pre-log rings. We find that the absolute differentials $\Omega^1_{(A,M)} = \Omega^1_{(A,M)/(\mathbb{Z},\{1\})}$ are given by

$$(\Omega^1_A \oplus (A \otimes_{\mathbb{Z}} M^{\mathrm{gp}}))/\sim,$$

where the equivalence relation is A-linearly generated by $(\mathrm{d}\alpha(m),0) \sim (0,\alpha(m)\otimes m)$. This recovers [Rog09, Definition 4.25]. In *loc. cit.* one uses the description of $\Omega^1_{(A,M)}$ as the pushout of the diagram

$$\Omega^1_A \xleftarrow{\phi} A \otimes_{\mathbb{Z}[M]} \Omega^1_{\mathbb{Z}[M]} \xrightarrow{\psi_M} A \otimes_{\mathbb{Z}} M^{\mathrm{gp}}.$$

Here $\phi(a\otimes \mathrm{d}m) = a\mathrm{d}\alpha(m)$ and $\psi_M(a\otimes \mathrm{d}m) = a\alpha(m)\otimes\gamma(m)$. We shall explain how to generalize this perspective to relative differentials of pre-log rings. Note first that the relative differentials over a trivial pre-log ring, $\Omega^1_{(A,M)/(R,\{1\})}$ may be described by the pushout square

$$\begin{tikzcd}
A \otimes_{R[M]} \Omega^1_{R[M]} \arrow[r, "\psi_M"] \arrow[d, "\phi"] & A \otimes_{\mathbb{Z}} M^{\mathrm{gp}} \arrow[d] \\
\Omega^1_{A/R} \arrow[r] & \Omega^1_{(A,M)/(R,\{1\})},
\end{tikzcd}$$

where the maps ψ_M and ϕ are as above. Note that ψ_M can be described as the image of the derivation $R[M] \to A\otimes_{\mathbb{Z}} M^{\mathrm{gp}}$ sending $m \mapsto \alpha(m)\otimes\gamma_M(m)$ under the identifications

$$\mathrm{Der}(R[M], A\otimes_{\mathbb{Z}} M^{\mathrm{gp}}) \cong \mathrm{Mod}_{R[M]}(\Omega^1_{R[M]/R}, A\otimes_{\mathbb{Z}} M^{\mathrm{gp}}) \cong \mathrm{Mod}_A(A\otimes_{R[M]}\Omega^1_{R[M]/R}, A\otimes_{\mathbb{Z}} M^{\mathrm{gp}}).$$

Likewise, ϕ is obtained from the derivation $R[M] \to \Omega^1_{A/R}$ sending m to $\mathrm{d}\alpha(m)$.

Suppose now that $(f, f^\flat)\colon (R, N, \beta) \to (A, M, \alpha)$ is a morphism of pre-log rings. To describe the relative differentials $\Omega^1_{(A,M)/(R,N)}$, we replace $A \otimes_{R[M]} \Omega^1_{R[M]/R}$ with $A \otimes_{R[M]} \Omega^1_{(R[M],N)/(R,\{1\})}$ in the above definition. Indeed, by the previous paragraph we have that $\Omega^1_{(R[M],N)/(R,\{1\})}$ fits into a pushout square

$$\begin{tikzcd}
R[M] \otimes_{R[N]} \Omega^1_{R[N]/R} \arrow[r] \arrow[d] & R[M] \otimes_{\mathbb{Z}} N^{\mathrm{gp}} \arrow[d] \\
\Omega^1_{R[M]/R} \arrow[r] & \Omega^1_{(R[M],N)/(R,\{1\})}.
\end{tikzcd}$$

Applying the (left adjoint) functor $A \otimes_{R[M]} -$ we get a pushout square

$$\begin{tikzcd}
A \otimes_{R[N]} \Omega^1_{R[N]/R} \arrow[r] \arrow[d] & A \otimes_{\mathbb{Z}} N^{\mathrm{gp}} \arrow[d] \\
A \otimes_{R[M]} \Omega^1_{R[M]/R} \arrow[r] & A \otimes_{R[M]} \Omega^1_{(R[M],N)/(R,\{1\})}.
\end{tikzcd}$$

We obtain maps from the pushout to $A \otimes_{\mathbb{Z}} M^{\mathrm{gp}}$ and $\Omega^1_{A/R}$ as follows:

$$\begin{tikzcd}
A \otimes_{R[N]} \Omega^1_{R[N]/R} \arrow[r, "\psi_N"] \arrow[d] & A \otimes_{\mathbb{Z}} N^{\mathrm{gp}} \arrow[d] \arrow[rdd, bend left, "\mathrm{id}_A \otimes (f^\flat)^{\mathrm{gp}}"] & \\
A \otimes_{R[M]} \Omega^1_{R[M]/R} \arrow[r] \arrow[rrd, bend right, "\psi_M"'] & A \otimes_{R[M]} \Omega^1_{(R[M],N)/(R,\{1\})} \arrow[rd, dashed] & \\
& & A \otimes_{\mathbb{Z}} M^{\mathrm{gp}},
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes_{R[N]} \Omega^1_{R[N]/R} \arrow[r, "\psi_N"] \arrow[d] & A \otimes_{\mathbb{Z}} N^{\mathrm{gp}} \arrow[d] \arrow[rdd, bend left, "0"] & \\
A \otimes_{R[M]} \Omega^1_{R[M]/R} \arrow[r] \arrow[rrd, bend right, "\phi"'] & A \otimes_{R[M]} \Omega^1_{(R[M],N)/(R,\{1\})} \arrow[rd, dashed] & \\
& & \Omega^1_{A/R}.
\end{tikzcd}$$

Both of the outer diagrams commute, as, for the first square, we have

$$\begin{aligned}
(\mathrm{id}_A \otimes (f^\flat)^{\mathrm{gp}})(\psi_N(a \otimes \mathrm{d}n)) &= (\mathrm{id}_A \otimes (f^\flat)^{\mathrm{gp}})(af(\beta(n)) \otimes \gamma_N(n)) = af(\beta(n)) \otimes \gamma_M(f^\flat(n)) \\
&= a\alpha(f^\flat(n)) \otimes \gamma_M(f^\flat(n)) = \psi_M(a \otimes \mathrm{d}f^\flat(n)),
\end{aligned}$$

while for the second one we have

$$\phi(a \otimes \mathrm{d}f^\flat(n)) = a\mathrm{d}\alpha(f^\flat(n)) = a\mathrm{d}(f\beta(n)) = 0,$$

since $f(\beta(n))$ is in R. Now the relative differentials $\Omega^1_{(A,M)/(R,N)}$ may be realized as the lower pushout square in the diagram

$$\begin{tikzcd}
A \otimes_{R[N]} \Omega^1_{R[N]/R} \arrow[r] \arrow[d] & A \otimes_{\mathbb{Z}} N^{\mathrm{gp}} \arrow[d] & \\
A \otimes_{R[M]} \Omega^1_{R[M]/R} \arrow[r] & A \otimes_{R[M]} \Omega^1_{(R[M],N)/(R,\{1\})} \arrow[r] \arrow[d] & A \otimes_{\mathbb{Z}} M^{\mathrm{gp}} \arrow[d] \\
& \Omega^1_{A/R} \arrow[r] & \Omega^1_{(A,M)/(R,N)}.
\end{tikzcd}$$

Indeed, taking this as the definition, we have that

$$\Omega^1_{(A,M)/(R,N)} = (\Omega^1_{A/R} \oplus (A \otimes_{\mathbb{Z}} M^{\mathrm{gp}}))/\sim$$

where the equivalence relation is A-linearly generated by $a\mathrm{d}\alpha(m) \sim a\alpha(m) \otimes \gamma_M(m)$ and $0 \sim a \otimes \gamma_M(f^\flat(n))$, as desired.

3 Logarithmic Hochschild homology

In this section we begin discussing logarithmic Hochschild homology, an extension of classical Hochschild homology for algebras for pre-log rings. We review Hochschild homology for classical rings in Section 3.1, as well as its derived variant, which is sometimes called *Shukla homology*. In Section 3.2 we discuss the definition of logarithmic Hochschild homology as introduced in [Rog09]. We then give a proof for a long exact sequence in log Hochschild homology in analogy with the cofibration sequence for log topological Hochschild homology constructed in [RSS15, Theorem 5.5, Example 5.7]. In Section 3.2.2 we establish an isomorphism between the first log Hochschild homology group of a pre-log ring (A, M) and the module of log Kähler differentials $\Omega^1_{(A,M)/R}$. We apply both of the aforementioned results in Section 3.2.3, where we study the long exact sequence constructed in Section 3.2.1 in low degrees, comparing the output with the one of Hesselholt-Madsen [HM03] in the case of discrete valuation rings.

3.1 Hochschild homology of commutative algebras

We introduce the Hochschild homology of a commutative algebra A over a commutative ground ring R, with particular emphasis on its relation to Kähler differentials and the case of A a discrete valuation ring. Both will be actively employed when we consider logarithmic Hochschild homology. An introduction to Hochschild homology is provided in e.g. [Lod98].

Definition 3.1.1. Let Fin denote the category of finite sets and let Alg_R denote the category of commutative R-algebras. Define a functor

$$\Lambda_{(-)}-\colon \mathrm{Fin} \times \mathrm{Alg}_R \to \mathrm{Alg}_R$$

as follows: for every finite set X and commutative R-algebra A, let

$$\Lambda_X A := \bigotimes_{x \in X} A,$$

where the tensor product denotes the coproduct in commutative R-algebras. From a morphism

$$(f, \varphi)\colon (X, A) \to (Y, B)$$

we obtain a morphism

$$\Lambda_f \varphi\colon \Lambda_X A \to \Lambda_Y B$$

by sending $\otimes_{x\in X} a_x$ to $\otimes_{y \in Y} b_y$, where b_y is defined to be φ applied to the product of the elements whose index map to y through f, or we insert the unit in R if no such element exists. Symbolically,

$$b_y := \begin{cases} \prod_{x \in f^{-1}(\{y\})} \varphi(a_x), & \text{if } f^{-1}(\{y\}) \neq \emptyset, \\ 1_R, & \text{if } f^{-1}(\{y\}) = \emptyset. \end{cases}$$

The resulting functor is called the *algebraic Loday functor*.

Remark 3.1.2. We note that the above construction is only functorial when working in the category of commutative R-algebras: the definition of $\Lambda_f \varphi$ is only well-defined if A is commutative. Note also that, since the covariant Hom-functor sends colimits to limits, there is a natural isomorphism

$$\mathrm{Alg}_R(\Lambda_X A, B) \cong \mathrm{Fin}(X, \mathrm{Alg}_R(A, B)).$$

In the language of enriched category theory, the above isomorphism is the definition of Alg_R being *tensored* over Fin, and $\Lambda_X A$ reads in this language as merely "apply the tensor."

Definition 3.1.3. For any category $\mathcal{C}$, let $\mathrm{s}\mathcal{C}$ denote the category of simplicial objects in $\mathcal{C}$. Consider the functor

$$\Lambda_{(-)}-\colon \mathrm{sSet} \times \mathrm{Alg}_R \to \mathrm{sAlg}_R$$

obtained by extending the Loday functor degreewise, i.e. $(\Lambda_X A)_n = \Lambda_{X_n} A$. While we only defined the algebraic Loday functor for finite sets, one can extend the definition by writing a set as a colimit of its finite subsets. This is unnecessary for our purposes; indeed, X will always be a finite simplicial set. If A is a flat commutative R-algebra, define the *Hochschild homology* of A to be the simplicial commutative R-algebra

$$\mathrm{HH}^R(A) := \Lambda_{S^1} A,$$

where $S^1 = \Delta[1]/\partial\Delta[1]$ denotes the simplicial circle. If A is not R-flat, choose a factorization

$$R \rightarrowtail A^{\mathrm{cof}} \xrightarrow{\simeq} A$$

of the unit map $R \to A$ as a free morphism followed by an acyclic fibration in sAlg_R (see Remark 1.1.18). Then A^{cof} is degreewise flat. We define the Hochschild homology of A to be the diagonal of the resulting bisimplicial commutative R-algebra:

$$\mathrm{HH}^R(A^{\mathrm{cof}}) := \mathrm{d}(\Lambda_{S^1} A).$$

We will always use the somewhat tedious notation $\mathrm{HH}^R(A^{\mathrm{cof}})$ for the latter case, as we do not want to take a cofibrant replacement if A is already R-flat.

Remark 3.1.4. We spell out the above definition, recovering the usual Hochschild complex in the case where A is R-flat. First recall the structure of the simplicial circle: The q-simplices are given by

$$S^1_q = \{x_0, \ldots, x_q\}$$

with face maps $d_i^{S^1}\colon S^1_q \to S^1_{q-1}$ given by

$$d_i^{S^1}(x_t) = \begin{cases} x_t, & \text{if } t < i \text{ or } t = i \neq q, \\ x_{t-1}, & \text{if } t > i. \end{cases}$$

The degeneracy maps $s_j^{S^1}\colon S^1_q \to S^1_{q+1}$ are given by

$$s_j^{S^1}(x_t) = \begin{cases} x_t, & \text{if } t \leq j, \\ x_{t+1}, & \text{if } t > j. \end{cases}$$

The unspecified identity $d_q^{S^1}(x_q) = x_0$ follows from the simplicial identity $d_i s_j = s_j d_{i-1}$ for $i > j+1$, as we have

$$d_q^{S^1}(x_q) = d_q^{S^1} s_0^{S^1} \cdots s_0^{S^1}(x_1) = s_0^{S^1} \cdots s_0^{S^1} d_1^{S^1}(x_1) = s_0^{S^1} \cdots s_0^{S^1}(x_0) = x_0,$$

where we have used $q-1$ iterations of $s_0^{S^1}$ and that $d_1^{S^1}(x_1) = x_0$ for $d_1^{S^1}\colon S^1_1 \to S^1_0 = \{x_0\}$. If A is R-flat, then the q-simplices of $\mathrm{HH}^R(A)$ are given by

$$\mathrm{HH}^R(A)_q = A^{\otimes(q+1)},$$

where the tensor product is over the ground ring R. The structure of the simplicial circle and the definition of the algebraic Loday functor (Definition 3.1.1) gives that the face maps

$$d_i^{\mathrm{HH}^R(A)}\colon \mathrm{HH}^R(A)_q \to \mathrm{HH}^R(A)_{q-1}$$

are given by

$$d_i^{\mathrm{HH}^R(A)}(a_0 \otimes \cdots \otimes a_q) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_q, & \text{if } 0 \leq i < q, \\ a_q a_0 \otimes \cdots \otimes a_{q-1}, & \text{if } i = q. \end{cases}$$

Taking the Moore complex, we now obtain the usual Hochschild complex

$$\cdots \xrightarrow{\partial_{q+1}^{\mathrm{HH}^R(A)}} A^{\otimes(q+1)} \xrightarrow{\partial_q^{\mathrm{HH}^R(A)}} A^{\otimes q} \longrightarrow \cdots \xrightarrow{\partial_2^{\mathrm{HH}^R(A)}} A^{\otimes 2} \xrightarrow{\partial_1^{\mathrm{HH}^R(A)}} A \longrightarrow 0.$$

In the case where we take a cofibrant replacement, the q-simplices are given by

$$\mathrm{HH}^R(A^{\mathrm{cof}})_q = (A_q^{\mathrm{cof}})^{\otimes(q+1)}.$$

The homology groups of the Moore complex associated to $\mathrm{HH}^R(A^{\mathrm{cof}})$ coincides with the homology groups of the total complex associated to the Moore bicomplex of $\Lambda_{S^1} A^{\mathrm{cof}}$ by Theorem 1.1.6, and so we can for instance see that $\pi_0 \mathrm{HH}^R(A^{\mathrm{cof}}) \cong A$ by considering the relevant part of the Moore bicomplex:

$$\begin{tikzcd} & A_1^{\mathrm{cof}} \arrow[d, "\partial_1^{A_*^{\mathrm{cof}}}"] \\ A_0^{\mathrm{cof}} \otimes_R A_0^{\mathrm{cof}} \arrow[r, "0"'] & A_0^{\mathrm{cof}}. \end{tikzcd}$$

In Section 3.1.1 we will see that we can also identify $\pi_1 \mathrm{HH}^R(A)$ and $\pi_1 \mathrm{HH}^R(A^{\mathrm{cof}})$. Finally we note that the relation between the Moore bicomplex of $\Lambda_{S^1} A^{\mathrm{cof}}$ and $\mathrm{HH}^R(A^{\mathrm{cof}})_*$ gives spectral sequences

$$E_{p,q}^2 = H_p(H_q((A_*^{\mathrm{cof}})^{\otimes(*+1)})) \implies \pi_{p+q} \mathrm{HH}^R(A^{\mathrm{cof}}),$$

see e.g. [Wei94, Chapter 5.6]. While the Moore bicomplex is of the form

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots \arrow[r] & (A_2^{\mathrm{cof}})^{\otimes 3} \arrow[r] \arrow[d] & (A_2^{\mathrm{cof}})^{\otimes 2} \arrow[r] \arrow[d] & A_2^{\mathrm{cof}} \arrow[r] \arrow[d] & 0 \\ \cdots \arrow[r] & (A_1^{\mathrm{cof}})^{\otimes 3} \arrow[r] \arrow[d] & (A_1^{\mathrm{cof}})^{\otimes 2} \arrow[r] \arrow[d] & A_1^{\mathrm{cof}} \arrow[r] \arrow[d] & 0 \\ \cdots \arrow[r] & (A_0^{\mathrm{cof}})^{\otimes 3} \arrow[r] \arrow[d] & (A_0^{\mathrm{cof}})^{\otimes 2} \arrow[r] \arrow[d] & A_0^{\mathrm{cof}} \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0, & \end{tikzcd}$$

so that the vertical chain complexes are the degreewise tensor products $(A_*^{\mathrm{cof}})^{\boxtimes(q+1)}$ (see Definition 1.1.3), the Eilenberg-Zilber Theorem 1.1.4 gives equivalences

$$(A_*^{\mathrm{cof}})^{\boxtimes(q+1)} \simeq (A_*^{\mathrm{cof}})^{\otimes(q+1)}.$$

The term *Shukla homology* is sometimes used in the literature to mean this derived version $\mathrm{HH}^R(A^{\mathrm{cof}})$ of Hochschild homology.

We first observe that the derived version coincides with the usual definition of Hochschild homology if A is already R-flat:

Lemma 3.1.5. *Let A be a flat commutative R-algebra. Then the map $\mathrm{HH}^R(A^{\mathrm{cof}}) \to \mathrm{HH}^R(A)$ induced by the cofibrant replacement map $A^{\mathrm{cof}} \to A$ is a weak equivalence.*

Proof. Since A^{cof} is degreewise flat, the Künneth spectral sequence

$$E^2_{p,q} = \bigoplus_{i+j=q} \mathrm{Tor}^R_p(H_i(A^{\mathrm{cof}}_*), H_j(A^{\mathrm{cof}}_*)) \implies H_{p+q}(A^{\mathrm{cof}}_* \otimes A^{\mathrm{cof}}_*)$$

of [Wei94, Exercise 5.7.5] is applicable. Since the homology groups of A^{cof}_* are concentrated in degree zero, $E^2_{p,q} = 0$ for $q > 0$. But since A is R-flat, $E^2_{p,0} = 0$ for $p > 0$. Hence the only surviving term is $E^2_{0,0} = A \otimes_R A$, since the homology groups of A^{cof}_* is concentrated as A in degree 0 and A is R-flat. Similarly we obtain $H_0((A^{\mathrm{cof}}_*)^{\otimes(p+1)}) = A^{\otimes(p+1)}$, while it vanishes in all higher degrees.

We now consider the spectral sequence

$$E^2_{p,q} = H_p(H_q((A^{\mathrm{cof}}_*)^{\otimes(*+1)})) \implies \pi_{p+q}\mathrm{HH}^R(A^{\mathrm{cof}}).$$

We have that $E^2_{p,0} = \pi_p\mathrm{HH}^R(A)$ while $E^2_{p,q} = 0$ for $q > 0$ by the previous paragraph, so that $\pi_p\mathrm{HH}^R(A^{\mathrm{cof}}) \cong \pi_p\mathrm{HH}^R(A)$. To see that this isomorphism is induced by the cofibrant replacement map, note that this map induces isomorphisms

$$H_p(H_q((A^{\mathrm{cof}}_*)^{\otimes(*+1)})) \xrightarrow{\cong} H_p(H_q(A^{\otimes(*+1)}_*))$$

for all p and q. Then the result follows from [Wei94, Theorem 5.2.12], which in our case states that a morphism $\pi_*\mathrm{HH}^R(A^{\mathrm{cof}}) \to \pi_*\mathrm{HH}^R(A)$ compatible with a map of spectral sequences which induces an isomorphism on $E^r_{p,q}$ for some r and all p, q is an isomorphism. $\square$

We now discuss that Hochschild homology is independent of the choice of cofibrant replacement, or more generally, of simplicial resolution:

Lemma 3.1.6. *Let $A \to B$ be a weak equivalence of commutative and degreewise flat simplicial R-algebras. Then the induced map $\mathrm{HH}^R(A) \to \mathrm{HH}^R(B)$ is a weak equivalence.*

Recall that for a simplicial commutative R-algebra A, we define $\mathrm{HH}^R(A)$ as the diagonal $\mathrm{d}(\Lambda_{S^1} A)$.

Proof. Since A and B are both degreewise flat, the Künneth spectral sequences are applicable. Since $A \to B$ is a weak equivalence, we have isomorphisms

$$\mathrm{Tor}^R_p(H_i(A_*), H_j(A_*)) \xrightarrow{\cong} \mathrm{Tor}^R_p(H_i(B_*), H_j(B_*))$$

for all p, i and j, and consequently isomorphisms $H_q(A^{\otimes 2}_*) \to H_q(B^{\otimes 2}_*)$ for all q, again by [Wei94, Theorem 5.2.12] as in the proof of Lemma 3.1.5. More generally we have isomorphisms

$$H_q(A^{\otimes(p+1)}_*) \xrightarrow{\cong} H_q(B^{\otimes(p+1)}_*)$$

for all p and q. Hence, there are isomorphisms

$$H_p(H_q(A^{\otimes(*+1)}_*)) \xrightarrow{\cong} H_p(H_q(B^{\otimes(*+1)}_*))$$

for all p and q, which implies that

$$\pi_{p+q}\mathrm{HH}^R(A) \to \pi_{p+q}\mathrm{HH}^R(B)$$

is an isomorphism, again by [Wei94, Theorem 5.2.12]. $\square$

We record two important consequences of Lemma 3.1.6: firstly, the definition of $\mathrm{HH}^R(A^{\mathrm{cof}})$ is independent on the choice of cofibrant replacement; the desired equivalence is obtained as a lift in the diagram

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d, tail] & \tilde{A}^{\mathrm{cof}} \arrow[d, two heads, "\simeq"] \\ A^{\mathrm{cof}} \arrow[r, two heads, "\simeq"'] \arrow[ur, dashed, "\simeq"] & A. \end{tikzcd}$$

Secondly, we could equivalently define the derived version of Hochschild homology using *any* choice of simplicial flat resolution. Indeed, given a diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & C \arrow[d, "\simeq"] \\ B \arrow[r, "\simeq"'] & A \end{tikzcd}$$

of simplicial commutative R-algebras, we may choose factorizations in sAlg_R to obtain a lift

$$\begin{tikzcd} R \arrow[rr] \arrow[dd, tail] & & C \arrow[d, tail, "\simeq"] \\ & & C' \arrow[d, two heads, "\simeq"] \\ B' \arrow[r, two heads, "\simeq"'] \arrow[urr, dashed, "\simeq"] & B \arrow[r, "\simeq"'] & A, \end{tikzcd}$$

so Lemma 3.1.6 implies that $\mathrm{HH}^R(B)$ and $\mathrm{HH}^R(C)$ are weakly equivalent provided that both B and C are degreewise R-flat. This in particular applies in the case where $B = A^{\mathrm{cof}}$.

Example 3.1.7 (Tor-interpretation of Hochschild homology)**.** Suppose that A is flat over R. Resolve A by the *bar resolution*

$$\cdots \xrightarrow{\partial'} A^{\otimes n+2} \xrightarrow{\partial'} A^{\otimes n+1} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} A^{\otimes 2}.$$

Here $\partial' \colon A^{\otimes n+1} \to A^{\otimes n}$ is given by $\sum_{i=0}^{n-1}(-1)^i d_i^{\mathrm{HH}^R(A)}$ and the augmentation map is given by the multiplication map

$$\mu \colon A \otimes_R A \to A.$$

Denote by A^{op} the *opposite algebra* of A with multiplication $a \cdot b := \mu(b, a)$, and denote by A^{e} the *enveloping algebra* $A \otimes_R A^{\mathrm{op}}$. We show that the bar resolution is an A^{e}-flat resolution A. As A is R-flat, so is $A^{\otimes n}$. Consider $A^{\otimes(n+2)}$ as an A^e-module by $(a \otimes a') \cdot (a_0 \otimes \cdots \otimes a_{n+1}) = aa_0 \otimes \cdots \otimes a_{n+1}a'$. There is a natural isomorphism

$$- \otimes_{A^e} A^{\otimes(n+2)} \to - \otimes_R A^{\otimes n}$$

where for any A-bimodule M the map is given by

$$m \otimes a_0 \otimes \cdots \otimes a_{n+1} \mapsto a_{n+1} m a_0 \otimes a_1 \otimes \cdots \otimes a_n,$$

where the inverse simply inserts 1 in the first and last coordinates. This is an inverse due to the A^e-module structure chosen on $A^{\otimes(n+2)}$. Naturality follows from the definition. We now show that the proposed resolution is in fact a resolution of A. The map $s\colon A^{\otimes n} \to A^{\otimes(n+1)}$ inserting 1 in the first coordinate is a null homotopy, as the composite $\partial' s + s\partial'$ evaluated on $a_0 \otimes \cdots \otimes a_{n-1}$ reads

$$\sum_{i=0}^{n-1}(-1)^i d_i(1\otimes a_0\otimes\cdots\otimes a_{n-1}) + 1\otimes\sum_{i=0}^{n-2}(-1)^i d_i(a_0\otimes\cdots\otimes a_{n-1}) = d_0(1\otimes a_0\otimes\cdots\otimes a_{n-1}) = a_0\otimes\cdots\otimes a_{n-1}.$$

Finally, our chosen isomorphism $A\otimes_{A^e} A^{\otimes(n+2)} \cong A\otimes_R A^{\otimes n} = A^{\otimes(n+1)}$ is a chain map, i.e., the diagram

$$\begin{tikzcd} A^{\otimes(n+1)} \arrow[r, "\partial_n^{\mathrm{HH}^R(A)}"] \arrow[d, "\cong"'] & A^{\otimes n} \arrow[d, "\cong"] \\ A\otimes_{A^e} A^{\otimes(n+2)} \arrow[r, "\mathrm{id}\otimes\partial'"'] & A\otimes_{A^e} A^{\otimes(n+1)} \end{tikzcd}$$

commutes. We conclude that

$$\pi_i\mathrm{HH}^R(A) \cong \mathrm{Tor}_i^{A^e}(A, A).$$

One can define Hochschild homology with coefficients in any A-bimodule M, by letting the bimodule play the role of the first coordinate in the Hochschild complex; and the argument for the Tor-interpretation works without modification.

3.1.1 Hochschild homology and Kähler differentials

We discuss the relation between Hochschild homology and Kähler differentials, and also provide a brief introduction to André-Quillen homology.

Lemma 3.1.8. *Let A be a flat R-algebra. The map*

$$\pi_1\mathrm{HH}^R(A) \to \Omega^1_{A/R}$$

sending $a_0\otimes a_1$ to $a_0\mathrm{d}a_1$ is an isomorphism of A-modules.

We specify that we consider we consider $A^{\otimes(q+1)}$ as an A-module by multiplication in the zeroth coordinate.

Proof. From the Moore complex $\mathrm{HH}^R(A)_*$ we obtain

$$\pi_1\mathrm{HH}^R(A) = (A\otimes_R A)/(a_0a_1\otimes a_2 - a_0\otimes a_1a_2 + a_2a_0\otimes a_1),$$

and so the proposed isomorphism is well-defined by the Leibniz rule for differentials. An inverse map is given by sending $a_0\mathrm{d}a_1$ to $a_0\otimes a_1$, which is well-defined as

$$\mathrm{d}(a_0a_1) \mapsto 1\otimes a_0a_1 = a_0\otimes a_1 + a_1\otimes a_0$$

in $\pi_1\mathrm{HH}^R(A)$, and this is precisely the image of $a_0\mathrm{d}a_1 + a_1\mathrm{d}a_0$. □

Remark 3.1.9. It is clear from the proof above that the flatness hypothesis on A is never used. One could make the same identification by considering the usual Hochschild complex described in Remark 3.1.4 without any flatness hypotheses. We choose to be consistent on always exchanging A with a simplicial flat resolution A^{cof} if A is not already R-flat. This also avoids confusion with the following proposition:

Proposition 3.1.10. *Let A be a commutative R-algebra. There is an isomorphism of A-modules*

$$\pi_1\mathrm{HH}^R(A^{\mathrm{cof}}) \cong \Omega^1_{A/R}.$$

Before presenting the proof of the above proposition we will develop some machinery, namely *André-Quillen homology*, our reference for which is [GS06, Section 4.4]. A more direct argument will be given in Remark 3.2.22.

Lemma 3.1.11. *Let A be a commutative R-algebra. The functor*

$$A \otimes_{(-)} \Omega^1_{-/R} \colon \mathrm{sAlg}_R/A \to \mathrm{sMod}_A$$

is a left Quillen functor.

Proof. We first construct a right adjoint. For any A-module M, let $A \ltimes M$ denote the R-algebra $A \oplus M$ with multiplication $(a, m) \cdot (a', m') = (aa', am' + a'm)$. Note then that

$$\mathrm{Alg}_R/A(B, A \ltimes M) \simeq \mathrm{Der}_R(B, M).$$

Indeed, an element of $\varphi \in \mathrm{Alg}_R/A(B, A \ltimes M)$ is a commutative triangle

$$\begin{tikzcd} B \arrow[rr, "\varphi"] \arrow[dr, "\iota_B"'] & & A \ltimes M \arrow[dl, "\mathrm{pr}_A"] \\ & A. & \end{tikzcd}$$

Hence φ is of the form (ι_B, ∂), and $\partial \colon B \to M$ is a derivation, as

$$\partial(bb') = (\iota_B(b)\partial(b') + \iota_B(b')\partial(b)).$$

Conversely, any derivation ∂ gives such a triangle by setting $\varphi = (\iota_B, \partial)$. Hence we have established that

$$\mathrm{Alg}_R/A(B, A \ltimes M) \simeq \mathrm{Mod}_B(\Omega^1_{B/R}, M) \simeq \mathrm{Mod}_A(A \otimes_B \Omega^1_{B/R}, M).$$

We now extend levelwise to obtain the functor $A \otimes_{(-)} \Omega_{-/R}$ with the indicated right adjoint. As the fibrations and weak equivalences on both categories are those of the underlying simplicial sets, the right adjoint preserves (acyclic) fibrations. $\square$

Definition 3.1.12. Define the *cotangent complex* of A to be the simplicial A-module

$$\mathbb{L}_{A/R} := A \boxtimes_{A^{\mathrm{cof}}} \Omega^1_{A^{\mathrm{cof}}/R}$$

for a cofibrant replacement $A^{\mathrm{cof}} \to A$ of simplicial commutative R-algebras. We denote by

$$\bigwedge_A^q \mathbb{L}_{A/R} = A \boxtimes_{A^{\mathrm{cof}}} \Omega^q_{A^{\mathrm{cof}}/R}$$

the wedge powers of the cotangent complex. The groups $\pi_* \mathbb{L}_{A/R}$ are the *André-Quillen homology* groups of A.

Proof of Proposition 3.1.10. We will use that there is a convergent spectral sequence

$$E^2_{p,q} = \pi_p(\bigwedge_A^q \mathbb{L}_{A/R}) \implies \pi_{p+q} \mathrm{HH}^R(A^{\mathrm{cof}})$$

relating André-Quillen homology and Hochschild homology, cf. [Qui70, Section 8.1]. By definition,

$$E^2_{0,1} = \pi_0(A \boxtimes_{A^{\mathrm{cof}}} \Omega^1_{A^{\mathrm{cof}}/R}) = (A \otimes_{A_0^{\mathrm{cof}}} \Omega^1_{A^{\mathrm{cof}}/R})/(a \otimes \mathrm{d}(\partial_1^{A_*^{\mathrm{cof}}}(a^1))$$

is the zeroth homology group of the Moore complex of $\mathbb{L}_{A/R}$. Here we use the superscript a^q to denote elements of A_q^{cof}. Let $p\colon A^{\mathrm{cof}} \to A$ denote the choice of cofibrant replacement of A. We claim that there is an isomorphism

$$E^2_{0,1} \to \Omega^1_{A/R}$$

sending $[a \otimes \mathrm{d}(a^0)]$ to $a\mathrm{d}(p_0(a_0))$. This is well-defined, as $a\mathrm{d}(p_0(\partial_1^{A_*^{\mathrm{cof}}}(a^1))) = 0$ by exactness of the sequence

$$A_1^{\mathrm{cof}} \xrightarrow{\partial_1^{A_*^{\mathrm{cof}}}} A_0^{\mathrm{cof}} \xrightarrow{p_0} A \longrightarrow 0.$$

It is an isomorphism as an inverse map is given by sending $a\mathrm{d}b$ to $[a \otimes \mathrm{d}\tilde{b}]$, where $\tilde{b} \in A_0^{\mathrm{cof}}$ is any element satisfying $p_0(\tilde{b}) = b$. This is well-defined, for if $\tilde{b}$ and $\tilde{b}'$ are two choices of such lifts, we have

$$[a \otimes \mathrm{d}(\tilde{b} - \tilde{b}')] = [a \otimes \mathrm{d}(\partial_1^{A_*^{\mathrm{cof}}}(a^1))] = 0$$

for some a^1 in A_1^{cof}, again by exactness of the above sequence. Finally, we notice that

$$E_{p,0} = \pi_p(A \boxtimes_{A^{\mathrm{cof}}} A^{\mathrm{cof}}) = 0$$

for all $p > 0$. Then $E^2_{1,0} = 0$, so there is nothing more in bidegree 1, and $E^2_{2,0} = 0$, so $E^2_{0,1}$ is not hit by $E^2_{2,0}$. This concludes the proof. □

We end this section with a brief discussion on formal smoothness and the statement of the Hochschild-Kostant-Rosenberg theorem:

Definition 3.1.13. A commutative R-algebra A is *smooth* over R if for every square-zero extension $S \to T$ (that is, $S \to T$ is a surjection of R-algebras with square-zero kernel), the lifting problem

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & T \end{tikzcd}$$

can always be solved in the category of commutative R-algebras.

Example 3.1.14. Let k be a field and consider the coordinate axes $k[x,y]/(xy)$. This is *not* a smooth k-algebra: Consider the lifting problem

$$\begin{tikzcd} k \arrow[r] \arrow[d] & k[\epsilon]/(\epsilon^3) \arrow[d] \\ k[x,y]/(xy) \arrow[r, "g"'] \arrow[ur, dashed, "h"] & k[\epsilon]/(\epsilon^2). \end{tikzcd}$$

Here g is defined by $g(x) = g(y) = \epsilon$, and the right-hand map is the canonical projection. If such a lift existed, we would have that

$$0 = h(xy) = \epsilon^2,$$

a contradiction.

Remark 3.1.15. The above example has a geometric interpretation: considering elements of $k[\epsilon]/(\epsilon^2)$ as "a point with a tangent direction", the problem is that we cannot lift those tangent vectors which do not lie on the coordinate axes themselves. Indeed, had we defined g above by $g(x) = \epsilon$ and $g(y) = 0$, a lift would be given by $h(x) = \epsilon$ and $h(y) = 0$.

Theorem 3.1.16. [Wei94, Theorem 9.4.7, Exercise 9.4.2] *Let A be a smooth R-algebra of finite type. There is an isomorphism of graded R-algebras*

$$\Omega^*_{A/R} \to \pi_* \mathrm{HH}^R(A).$$

Remark 3.1.17. We have not explained how to give $\pi_* \mathrm{HH}^R(A)$ the structure of a graded algebra. This is done in e.g. [Wei94, Chapter 9.5] through the *shuffle product*. Here $\Omega^*_{A/R}$ denotes the exterior algebra with $\Omega^q_{A/R} = \bigwedge^q_A \Omega^1_{A/R}$. We will not make use of the Hochschild-Kostant-Rosenberg in the generality it is given above, in particular we will not make use of the graded ring structure on Hochschild homology.

Example 3.1.18. Let $A = R[x_1, \ldots, x_n]$ be the free commutative R-algebra on n generators. Then $\Omega^1_{A/R} = A\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$, and so Theorem 3.1.16 gives

$$\pi_q \mathrm{HH}^R(A) = \bigwedge_A^q \Omega^1_{A/R} \cong A^{\oplus \binom{n}{q}},$$

the free A-module of rank $\binom{n}{q}$.

3.1.2 Hochschild homology of discrete valuation rings

Recall from Theorem 1.2.10 that a complete discrete valuation ring A of characteristic 0 with perfect residue field k of positive characteristic is of the form

$$A = W(k)[x]/(\phi(x)),$$

where $W(k)$ denotes the p-typical Witt vectors over k and ϕ is an Eisenstein polynomial of degree the absolute ramification index e of A. We have the following description of the Hochschild homology groups of such truncated polynomial rings:

Lemma 3.1.19. *Let R be a commutative ring and let $A = R[x]/(\phi(x))$ for a monic polynomial ϕ. Then the Hochschild homology groups of A is given by*

$$\pi_q \mathrm{HH}^R(A) = \begin{cases} A, & \text{if } q = 0, \\ A/(\phi'(x)), & \text{if } q = 2k+1, \\ \ker(\cdot \phi'(x) \colon A \to A), & \text{otherwise.} \end{cases}$$

Here $\phi'(x)$ denotes the formal derivative of $\phi(x)$.

We note that A is a free R-module of rank the degree of ϕ, and so there is no need to pick a simplicial resolution by Lemma 3.1.5. We will often be in a situation when A is an integral domain (as is the case for the discrete valuation rings), so that $\ker(\cdot \phi'(x)) = 0$.

Proof. By Example 3.1.7 we can make the identification

$$\pi_q \mathrm{HH}^R(A) \cong \mathrm{Tor}_q^{A^e}(A, A),$$

where $A^e = A \otimes_R A \cong R[y,z]/(\phi(y), \phi(z))$ is the enveloping algebra of A. We will apply the free two-periodic A^e-resolution of A

$$\cdots \xrightarrow{\cdot(y-z)} A^e \xrightarrow{\cdot\frac{\phi(y)-\phi(z)}{y-z}} A^e \xrightarrow{\cdot(y-z)} A^e \xrightarrow{\mu} A \longrightarrow 0,$$

where $\mu\colon A^e = A \otimes_R A \to A$ denotes the multiplication map, established in e.g. [BAG91, Proposition 1.3]. Notice that $y^q - z^q$ is divisible by $y - z$ for all $q > 0$, so that $\phi(y) - \phi(z)$ is divisible by $y - z$. Under the isomorphisms $A \otimes_{A \otimes_R A} (A \otimes_R A) \cong A$, the map $\cdot(y - z)$ becomes $x - x = 0$, while the quotient becomes the formal derivative of ϕ: indeed, we have the identity

$$\frac{y^q - z^q}{y - z} = \sum_{i=0}^{q-1} y^{(q-1)-i} z^i,$$

which corresponds to qx^{q-1} under the isomorphism $A \otimes_{A \otimes_R A} (A \otimes_R A) \cong A$. Hence we are left to compute the homology of the complex

$$\cdots \xrightarrow{0} A \xrightarrow{\cdot\phi'(x)} A \xrightarrow{0} A \longrightarrow 0,$$

which gives the desired result. □

In the case of discrete valuation rings, we will sometimes need the following descriptions of the ideal $(\phi'(x))$:

Proposition 3.1.20. [Ser79, Chapter 3, Proposition 13] *Let A be a discrete valuation ring as described in the beginning of this section, and let e denote the absolute ramification index of A. We have the following descriptions of the ideal $(\phi'(x))$:*

$$(\phi'(x)) = \begin{cases} A, & \text{if } e = 1, \\ (\pi^m), & \text{if } p \mid e, \\ (\pi^{e-1}), & \text{if } p \nmid e. \end{cases}$$

Here π denotes a uniformizer of A and m is a natural number greater than or equal to e. □

3.2 Hochschild homology of logarithmic algebras

Following [Rog09] we introduce the logarithmic Hochschild homology of a pre-log R-algebra (A, M), by which we mean an R-algebra A and a commutative monoid M with a morphism of commutative monoids

$$\alpha\colon M \to (A, \cdot).$$

We will give a proof for a linear version of the cofibration sequence constructed in [RSS15, Theorem 5.5, Example 5.7] in Section 3.2.1. In Section 3.2.3 we study the behaviour of the sequence constructed in Section 3.2.1 in detail in low degrees, with particular emphasis on the case of discrete valuation rings. For this we apply the isomorphism between the first log Hochschild homology group and the module of log Kähler differentials established in Section 3.2.2.

We first discuss three different “bar constructions”:

Definition 3.2.1. Let M be a commutative monoid, and let X and Y be sets with a right and left unital and associative action from M, respectively. The *bar construction* $B(X,M,Y)$ is the simplicial set with q-simplices

$$B_q(X,M,Y) = X \times M^{\times q} \times Y$$

with face maps $d_i^{B(X,M,Y)}\colon B_q(X,M,Y) \to B_{q-1}(X,M,Y)$ given by

$$d_i^{B(X,M,Y)}(x,m_1,\ldots,m_q,y) = \begin{cases} (xm_1,\ldots,m_{q-1},m_q,y), & \text{if } i=0, \\ (x,m_1,\ldots,m_im_{i+1},\ldots,m_{q-1},m_q,y), & \text{if } 0<i<q, \\ (x,m_1,\ldots,m_qy), & \text{if } i=q. \end{cases}$$

The degeneracies $s_j\colon B_q(X,M,Y) \to B_{q+1}(X,M,Y)$ are given by

$$s_j(x,m_1,\ldots,m_q,y) = (x,m_1,\ldots,m_j,1,m_{j+1},\ldots,m_q,y).$$

In the case $X=Y=*$ we simply denote $B(*,M,*)$ by BM, and this is a simplicial commutative monoid by coordinatewise multiplication.

Remark 3.2.2. The above definition is better expressed in the language of monoidal categories, which we have not discussed in this thesis. The above is a special case for the monoidal category $(\mathrm{Set},\times,*)$.

Example 3.2.3. We have already used a version of the bar construction in Example 3.1.7. There we considered a flat R-algebra A, and showed that the Moore complex

$$B(A,A,A)_* \to A$$

served as an $A^e = A\otimes_R A$-flat resolution of A.

Definition 3.2.4. Let M be a commutative monoid. The *cyclic bar construction* of M is defined as the simplicial commutative monoid

$$B^{\mathrm{cy}}M := \Lambda_{S^1}M,$$

defined in analogy with $\Lambda_{S^1}A$ from Definition 3.1.3. The *replete bar construction* $B^{\mathrm{rep}}M$ of M is defined by the right-hand pullback square in the diagram

$$\begin{tikzcd} B^{\mathrm{cy}}M \arrow[r, dashed, "\rho_M"] \arrow[d] & B^{\mathrm{rep}}M \arrow[r] \arrow[d] & B^{\mathrm{cy}}M^{\mathrm{gp}} \arrow[d] \\ M \arrow[r, equal] & M \arrow[r, "\gamma_M"'] & M^{\mathrm{gp}} \end{tikzcd}$$

of simplicial commutative monoids. Here γ_M is the group completion map and the outer vertical maps are the multiplication maps $B^{\mathrm{cy}}M \to M$. The *repletion map* ρ_M is induced by the universal property of the pullback along $B^{\mathrm{cy}}\gamma_M\colon B^{\mathrm{cy}}M \to B^{\mathrm{cy}}M^{\mathrm{gp}}$ and the map $B^{\mathrm{cy}}M \to M$.

Remark 3.2.5. We unravel the definition of $B^{\mathrm{rep}}M$. By definition, the q-simplices are given by

$$B_q^{\mathrm{rep}}M = \{(m,g_0,\ldots,g_q) \mid \gamma_M(m) = g_0\cdots g_q\}$$

for $m\in M$ and $g_i \in M^{\mathrm{gp}}$. This implies that one of the coordinates can be written in terms of the others, e.g. any q-simplex can be written as $(m,\gamma_M(m)(g_1\cdots g_q)^{-1},g_1,\ldots g_q)$. Then we obtain an isomorphism of simplicial commutative monoids

$$B^{\mathrm{rep}}M \cong M\times BM^{\mathrm{gp}}$$

by forgetting the superfluous coordinate, where BM is the bar construction introduced in Definition 3.2.1. Under this isomorphism, the repletion map $\rho_M\colon B^{\mathrm{cy}}M \to M\times BM^{\mathrm{gp}}$ is given by

$$\rho_M(m_0,\ldots,m_q) = (m_0\cdots m_q,\gamma_M(m_1),\ldots,\gamma_M(m_q)).$$

Example 3.2.6. We compute $\pi_1 R[B^{\mathrm{rep}}M]$. The isomorphism $B^{\mathrm{rep}}M \cong M \times BM^{\mathrm{gp}}$ describes $R[B^{\mathrm{rep}}_q M]$ as $R[M] \otimes_R R[M^{\mathrm{gp}}]^{\otimes q}$. Then the Moore complex $R[B^{\mathrm{rep}}M]_*$ and the above description of the simplicial structure of BM^{gp} give that

$$\pi_1 R[B^{\mathrm{rep}}M] = (R[M] \otimes_R R[M^{\mathrm{gp}}])/\sim,$$

where the equivalence relation is R-linearly generated by the relation $m \otimes g_1 g_2 = m \otimes g_1 + m \otimes g_2$. As we have isomorphisms

$$R[M] \otimes_R R[M^{\mathrm{gp}}] \cong R[M] \otimes_R R \otimes_{\mathbb{Z}} \mathbb{Z}[M^{\mathrm{gp}}] \cong R[M] \otimes_{\mathbb{Z}} \mathbb{Z}[M^{\mathrm{gp}}],$$

we obtain an isomorphism

$$\pi_1 R[B^{\mathrm{rep}}M] \to R[M] \otimes_{\mathbb{Z}} M^{\mathrm{gp}}$$

by sending $[m \otimes g]$ to $m \otimes g$. In light of Example 2.3.11 and Definition 3.2.7 below, this suggests that $R[B^{\mathrm{rep}}M]$ should play the role of the logarithmic differentials on π_1.

Definition 3.2.7 ([Rog09])**.** Let A be a commutative R-algebra and let $\alpha\colon M \to (A, \cdot)$ be a pre-log structure on A. This induces a map on cyclic bar constructions

$$B^{\mathrm{cy}}\alpha\colon B^{\mathrm{cy}}M \to B^{\mathrm{cy}}(A, \cdot) \cong (\mathrm{HH}^R(A), \cdot).$$

This map is adjoint to a map $\phi\colon R[B^{\mathrm{cy}}M] \to \mathrm{HH}^R(A)$. If A is flat over the monoid ring $R[M]$, we define the *log Hochschild homology* of (A, M) to be the simplicial commutative R-algebra given by the pushout square

$$\begin{tikzcd} R[B^{\mathrm{cy}}M] \arrow[r, "\psi"] \arrow[d, "\phi"'] & R[B^{\mathrm{rep}}M] \arrow[d, "\bar{\phi}"] \\ \mathrm{HH}^R(A) \arrow[r, "\bar{\psi}"'] & \mathrm{HH}^R(A, M) \end{tikzcd}$$

in the category of commutative simplicial R-algebras, where $\psi := R[\rho_M]$ is induced by the repletion map ρ_M introduced in Definition 3.2.4. If A is not $R[M]$-flat, we replace A with a simplicial resolution $A^{\mathrm{cof}} \to A$ of flat $R[M]$-algebras, e.g. a factorization

$$R[M] \rightarrowtail A^{\mathrm{cof}} \xrightarrow{\simeq} A$$

in the model structure on $\mathrm{sAlg}_{R[M]}$, where the first map is a free morphism, see Remark 1.1.18. We then define $\mathrm{HH}^R(A^{\mathrm{cof}}, M)$ to be the diagonal of the resulting commutative bisimplical algebra.

Remark 3.2.8. As the q-simplices of $\mathrm{HH}^R(A, M)$ are given by

$$A^{\otimes(q+1)} \otimes_{R[M]^{\otimes(q+1)}} R[M] \otimes_R R[M^{\mathrm{gp}}]^{\otimes q},$$

we make the flatness hypotheses over $R[M]$. As $R[M]$ is already free over R, $A^{\mathrm{cof}} \to A$ is also an R-flat simplicial resolution, and so the definition of $\mathrm{HH}^R(A^{\mathrm{cof}})$ is not affected by Lemma 3.1.6. Note that $R[B^{\mathrm{cy}}M] \cong \mathrm{HH}^R(R[M])$ and $R[B^{\mathrm{rep}}M] \cong \mathrm{HH}^R(R[M], M)$, so that the above pushout square from Definition 3.2.7 may be written as

$$\begin{tikzcd} \mathrm{HH}^R(R[M]) \arrow[r] \arrow[d] & \mathrm{HH}^R(R[M], M) \arrow[d] \\ \mathrm{HH}^R(A) \arrow[r] & \mathrm{HH}^R(A, M). \end{tikzcd}$$

Example 3.2.9. If A is $R[M]$-flat, we have $\pi_0 \mathrm{HH}^R(A,M) \cong A$ for any monoid M. The relevant part of the Moore complex is

$$A^{\otimes 2} \otimes_{R[M]^{\otimes 2}} R[M] \otimes_R R[M^{\mathrm{gp}}] \xrightarrow{\partial_1^{\mathrm{HH}^R(A,M)}} A \otimes_{R[M]} R[M] \cong A \longrightarrow 0,$$

and it is readily seen that $\partial_1^{\mathrm{HH}^R(A,M)}$ is trivial in light of commutativity of A and the simplicial structure of $B^{\mathrm{rep}} M \cong M \times BM^{\mathrm{gp}}$. In analogy with Remark 3.1.4 we find that $\pi_0 \mathrm{HH}^R(A^{\mathrm{cof}}, M) \cong A$: This time the relevant part of the Moore bicomplex is

$$\begin{tikzcd} & A_1^{\mathrm{cof}} \arrow[d, "\partial_1^{A^{\mathrm{cof}}}"] \\ (A_0^{\mathrm{cof}})^{\otimes 2} \otimes_{R[M]^{\otimes 2}} R[M] \otimes_R R[M^{\mathrm{gp}}] \arrow[r, "0"'] & A_0^{\mathrm{cof}}. \end{tikzcd}$$

As log Hochschild homology is defined as a coproduct of commutative simplicial algebras, the following spectral sequence will prove useful to us on numerous occasions:

Theorem 3.2.10. [Qui67, Page 6.8, Theorem 6 (b)] *Let R be a simplicial ring and let M and N be simplicial R-modules. If either M or N are degreewise flat over R, there is a first quadrant spectral sequence*

$$E^2_{p,q} = \mathrm{Tor}_p^{\pi_* R}(\pi_* M, \pi_* N)_q \implies \pi_{p+q}(M \boxtimes_R N),$$

where $\mathrm{Tor}(-,-)_q$ denotes the homogeneous submodule of degree q.

In Section 1.1 we only considered the product $\boxtimes_R$ where R was constant. Unsurprisingly, $M \boxtimes_R N$ for R a simplicial ring is defined to be the simplicial R-module with q-simplices $(M \boxtimes_R N)_q = M_q \otimes_{R_q} N_q$, and simplicial structure maps inherited degreewise from R, M and N.

A simple example of how one can apply Theorem 3.2.10 is given by the following analogue of Lemma 3.1.6:

Lemma 3.2.11. *The definition of $\mathrm{HH}^R(A^{\mathrm{cof}}, M)$ is independent of the choice of $R[M]$-flat resolution up to weak equivalence.*

Proof. Let B and C be two choices of $R[M]$-flat resolutions of A. As remarked after the proof of Lemma 3.1.6, we can reduce to the case where there is a weak equivalence $B \to C$. As Hochschild homology is independent of the choice of flat resolution, we have isomorphisms

$$\mathrm{Tor}_p^{\pi_* R[B^{\mathrm{cy}} M]}(\pi_* \mathrm{HH}^R(B), \pi_*(R[B^{\mathrm{rep}} M]))_q \xrightarrow{\cong} \mathrm{Tor}_p^{\pi_* R[B^{\mathrm{cy}} M]}(\pi_* \mathrm{HH}^R(C), \pi_*(R[B^{\mathrm{rep}} M]))_q$$

for all p and q, and so the result follows from Theorem 3.2.10 and [Wei94, Theorem 5.2.12] as in the proof of Lemma 3.1.5. □

3.2.1 A long exact sequence

In this section we establish a long exact sequence in log Hochschild homology in the case of a pre-log structure $\alpha \colon \langle x \rangle \to (A, \cdot)$, where $\langle x \rangle$ denotes the free commutative monoid on the one generator x which is assumed to map to a non-zero divisor $a \in A$. This is a linear version of the cofibration sequence constructed in [RSS15, Theorem 5.5, Example 5.7]. The exposition given here makes all quasi-isomorphisms used in the construction completely explicit, and so one could in principle give concrete descriptions of all maps in the sequence in the derived category of A-modules. The result is the following:

Theorem 3.2.12. *Let A be a commutative R-algebra, and let $\alpha\colon \langle x\rangle \to (A, \cdot)$ be a pre-log structure on A under which $\alpha(x)$ is not a divisor of zero. Then there is a long exact sequence*

$$\cdots \to \pi_q \mathrm{HH}^R((A/(a))^{\mathrm{cof}}) \to \pi_q \mathrm{HH}^R(A^{\mathrm{cof}}) \to \pi_q(\mathrm{HH}^R(A^{\mathrm{cof}}, \langle x\rangle)) \to \pi_{q-1}((A/(a))^{\mathrm{cof}}) \to \cdots.$$

We recall that A^{cof} denotes a choice of a simplicial resolution $A^{\mathrm{cof}} \to A$ of flat $R[x]$-algebras. As $R[x]$ is R-flat, this resolution can be used to compute both $\mathrm{HH}^R(A^{\mathrm{cof}}, \langle x\rangle)$ and $\mathrm{HH}^R(A^{\mathrm{cof}})$. Here $(A/(a))^{\mathrm{cof}}$ denotes a choice of a simplicial resolution $(A/(a))^{\mathrm{cof}} \to A/(a)$ of flat R-algebras.

Theorem 3.2.12 is proved by constructing a short exact sequence of chain complexes of the form

$$0 \to \mathrm{HH}^R(A^{\mathrm{cof}})_* \to \mathrm{HH}^R(A^{\mathrm{cof}}, \langle x\rangle)_* \to \mathrm{HH}^R((A/(a))^{\mathrm{cof}})[1]_* \to 0$$

up to quasi-isomorphism. For a chain complex X_*, we denote by $X[1]_*$ the shifted complex with $X[1]_q := X_{q-1}$ and differential $\partial_q^{X[1]_*} := -\partial_{q-1}^X$.

Definition 3.2.13. Define a subobject $\hat{B}^{\mathrm{rep}}\langle x\rangle$ of the replete bar construction $B^{\mathrm{rep}}\langle x\rangle \cong \langle x\rangle \times B\langle x\rangle^{\mathrm{gp}}$ (cf. Remark 3.2.5) by

$$\hat{B}_q^{\mathrm{rep}}\langle x\rangle = \{(x^i, g_1, \ldots, g_q) \mid g_1 = \cdots = g_q = 1 \text{ if } i = 0\}.$$

This gives a factorization of the repletion map $\rho_{\langle x\rangle}$:

$$\begin{tikzcd} B^{\mathrm{cy}}\langle x\rangle \arrow[r, "\hat{\rho}_{\langle x\rangle}"] \arrow[rr, bend right, "\rho_{\langle x\rangle}"'] & \hat{B}^{\mathrm{rep}}\langle x\rangle \arrow[r] & B^{\mathrm{rep}}\langle x\rangle. \end{tikzcd}$$

Define an action of $\hat{B}^{\mathrm{rep}}\langle x\rangle$ on $B\langle x\rangle^{\mathrm{gp}}$ by

$$(x^i, g_1, \ldots, g_q) \cdot (g_1', \ldots, g_q') = \begin{cases} (g_1', \ldots, g_q'), & \text{if } i = 0, \\ (1, \ldots, 1), & \text{if } i > 0. \end{cases}$$

We have a map from the simplicial quotient $B^{\mathrm{rep}}\langle x\rangle / \hat{B}^{\mathrm{rep}}\langle x\rangle$ to $B\langle x\rangle^{\mathrm{gp}}$, induced by the map

$$\pi\colon B^{\mathrm{rep}}\langle x\rangle \to B\langle x\rangle^{\mathrm{gp}}$$

defined by

$$\pi(x^i, g_1, \ldots, g_q) = \begin{cases} (g_1, \ldots, g_q), & \text{if } i = 0, \\ (1, \ldots, 1), & \text{if } i > 0. \end{cases}$$

In particular, we see that π is equivariant with respect to the $\hat{B}^{\mathrm{rep}}$-action defined above, and that the induced map $B^{\mathrm{rep}}\langle x\rangle / \hat{B}^{\mathrm{rep}}\langle x\rangle \to B\langle x\rangle^{\mathrm{gp}}$ is an isomorphism with inverse inserting 1 in the first coordinate.

Remark 3.2.14. The above construction gives a short exact sequence of simplicial $R[B^{\mathrm{cy}}\langle x\rangle]$-modules

$$0 \to R[\hat{B}^{\mathrm{rep}}\langle x\rangle] \to R[B^{\mathrm{rep}}\langle x\rangle] \to \tilde{R}[B\langle x\rangle^{\mathrm{gp}}] \to 0.$$

If $(X, *)$ is a pointed simplicial set, $\tilde{R}[X]$ denotes the quotient $R[X]/R[*]$. Tensoring with $\mathrm{HH}^R(A^{\mathrm{cof}})$ we obtain a short exact sequence

$$0 \to \mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} R[\hat{B}^{\mathrm{rep}}\langle x\rangle] \to \mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} R[B^{\mathrm{rep}}\langle x\rangle] \to \mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} \tilde{R}[B\langle x\rangle^{\mathrm{gp}}] \to 0$$

of simplicial A-modules. By definition, the middle term is the log Hochschild homology $\mathrm{HH}^R(A^{\mathrm{cof}}, \langle x \rangle)$. The remainder of this section is devoted to constructing quasi-isomorphisms

$$\mathrm{HH}^R(A^{\mathrm{cof}})_* \simeq (\mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x \rangle]} R[\hat{B}^{\mathrm{rep}}\langle x \rangle])_*$$

and

$$\mathrm{HH}^R((A/(a))^{\mathrm{cof}})[1]_* \simeq (\mathrm{HH}^R(A^{\mathrm{cof}}) \otimes_{R[B^{\mathrm{cy}}\langle x \rangle]} \tilde{R}[B\langle x \rangle^{\mathrm{gp}}])_*.$$

We first prove that tensoring with $\tilde{R}[B\langle x \rangle^{\mathrm{gp}}]$ corresponds to a degree shift:

Lemma 3.2.15. *The map*

$$R[1] \to \tilde{R}[B\langle x \rangle^{\mathrm{gp}}]_*$$

sending 1 *to* x *is a quasi-isomorphism.*

Proof. The map induces an isomorphism $\pi_1 R[1] \to \pi_1 \tilde{R}[B\langle x \rangle^{\mathrm{gp}}]$, as this map fits in a diagram

$$\begin{tikzcd}
\pi_1 R[1] = R \arrow[r] \arrow[ddr, equal] & \pi_1 \tilde{R}[B\langle x \rangle^{\mathrm{gp}}] = R[\langle x \rangle^{\mathrm{gp}}]/(x^{i+j} \sim x^i + x^j) \arrow[d, "\cong"] \\
 & R \otimes_{\mathbb{Z}} \langle x \rangle^{\mathrm{gp}} \arrow[d, "\cong"] \\
 & R.
\end{tikzcd}$$

We now argue that $\pi_q \tilde{R}[B\langle x \rangle^{\mathrm{gp}}]$ vanishes for all other q. Consider the bar construction $B(\{1\}, M, M)$ defined in Definition 3.2.1. The Moore complex $R[B(\{1\}, M, M)]_*$ is a free resolution of R with augmentation map $\eta \colon R[M] \to R$ defined by $m \mapsto 1$, which is also our chosen $R[M]$-module structure on R. That this is in fact a resolution is checked in e.g. [Wei94, 8.6.14]. We then note that

$$R[B(\{1\}, M, M)] \otimes_{R[M]} R \cong R[BM]$$

and so $\pi_q R[BM] \cong \mathrm{Tor}_q^{R[M]}(R, R)$. Specializing to the case where $M = \langle x \rangle^{\mathrm{gp}}$, we have

$$\pi_q R[B\langle x \rangle^{\mathrm{gp}}] \cong \mathrm{Tor}_q^{R[x, x^{-1}]}(R, R).$$

This can be computed with the free resolution

$$0 \longrightarrow R[x, x^{-1}] \xrightarrow{\cdot(x-1)} R[x, x^{-1}] \xrightarrow{\eta} R \longrightarrow 0,$$

and applying $- \otimes_{R[x, x^{-1}]} R$ we are left to compute the homology of the complex

$$0 \longrightarrow R \xrightarrow{0} R \longrightarrow 0$$

which gives the result, as this implies that $\pi_q R[B\langle x \rangle^{\mathrm{gp}}]$ is concentrated as R in degrees 0 and 1, and so $\pi_q \tilde{R}[B\langle x \rangle^{\mathrm{gp}}]$ is concentrated as R in degree 1. $\square$

We now check that $\mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x \rangle]} R[\hat{B}^{\mathrm{rep}}\langle x \rangle]$ can be used as a model for $\mathrm{HH}^R(A^{\mathrm{cof}})$:

Lemma 3.2.16. *Let* $\hat{\rho}_{\langle x \rangle} \colon B^{\mathrm{cy}}\langle x \rangle \to \hat{B}^{\mathrm{rep}}\langle x \rangle$ *denote the map from Definition 3.2.13. The induced map*

$$R[\hat{\rho}_{\langle x \rangle}] \colon R[B^{\mathrm{cy}}\langle x \rangle] \to R[\hat{B}^{\mathrm{rep}}\langle x \rangle]$$

is a weak equivalence.

Proof. By Example 3.1.18, we know that $\pi_q R[B^{\mathrm{cy}}\langle x\rangle] = \pi_q \mathrm{HH}^R(R[x])$ is concentrated as $R[x]$ in degrees 0 and 1. The map is an isomorphism in degree 0, as the repletion map

$$R[B_0^{\mathrm{cy}}\langle x\rangle] = R[x] \to R[\hat{B}_0^{\mathrm{rep}}\langle x\rangle] = R[x]$$

is the identity, and both differentials from degree 1 to 0 are trivial. We now argue that the map is an isomorphism in degree 1. An isomorphism

$$\pi_1 R[B^{\mathrm{cy}}\langle x\rangle] = \pi_1 \mathrm{HH}^R(R[x]) \to \Omega^1_{R[x]/R} \cong R[x]\{\mathrm{d}x\}$$

is provided by Lemma 3.1.8, sending the Hochschild class $(1, x)$ to $\mathrm{d}x$. The map $\pi_1 R[\hat{\rho}_{\langle x\rangle}]$ then factors through the as

$$\pi_1 R[B^{\mathrm{cy}}\langle x\rangle] \to R[x]\{\mathrm{d}x\} \to \pi_1 R[\hat{B}^{\mathrm{rep}}\langle x\rangle],$$

the latter map sending $\mathrm{d}x$ to (x, x). Using Example 3.2.6, we have that $\pi_1 R[B^{\mathrm{rep}}\langle x\rangle] \cong R[x]\{\mathrm{d}\log x\}$, the free $R[x]$-module on a logarithmic differential $\mathrm{d}\log x$ (cf. Section 2.3). Consider now the long exact sequence induced by the short exact sequence of R-modules from Remark 3.2.14:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_2 R[\hat{B}^{\mathrm{rep}}\langle x\rangle] \arrow[r] & \pi_2 R[B^{\mathrm{rep}}\langle x\rangle] \arrow[r] & \pi_2 \tilde{R}[B\langle x\rangle^{\mathrm{gp}}] \arrow[dll, out=-30, in=150] & \\
& \pi_1 R[\hat{B}^{\mathrm{rep}}\langle x\rangle] \arrow[r] & \pi_1 R[B^{\mathrm{rep}}\langle x\rangle] \arrow[r] & \pi_1 \tilde{R}[B\langle x\rangle^{\mathrm{gp}}] \arrow[dll, out=-30, in=150] & \\
& \pi_0 R[\hat{B}^{\mathrm{rep}}\langle x\rangle] \arrow[r] & \pi_0 R[B^{\mathrm{rep}}\langle x\rangle] \arrow[r] & \pi_0 \tilde{R}[\langle x\rangle^{\mathrm{gp}}] \arrow[r] & 0.
\end{tikzcd}$$

As $R[\hat{B}_0^{\mathrm{rep}}\langle x\rangle] = R[B_0^{\mathrm{rep}}\langle x\rangle]$ and both differentials from degree 1 to 0 are trivial, the map

$$\pi_0 R[\hat{\rho}_{\langle x\rangle}] \colon \pi_0 R[\hat{B}^{\mathrm{rep}}\langle x\rangle] \to \pi_0 R[B^{\mathrm{rep}}\langle x\rangle]$$

is an isomorphism. As $\pi_2 \tilde{R}[B\langle x\rangle^{\mathrm{gp}}] \cong 0$ by Lemma 3.2.15, we obtain a short exact sequence

$$0 \to \pi_1 R[\hat{B}^{\mathrm{rep}}\langle x\rangle] \to \pi_1 R[B^{\mathrm{rep}}\langle x\rangle] \to \pi_1 \tilde{R}[B\langle x\rangle^{\mathrm{gp}}] \to 0.$$

We argue that this fits in a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & R[x]\{\mathrm{d}x\} \arrow[r, "\cdot x"] \arrow[d] & R[x]\{\mathrm{d}\log x\} \arrow[r] \arrow[d, "\cong"] & R \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \pi_1 R[\hat{B}^{\mathrm{rep}}\langle x\rangle] \arrow[r] & \pi_1 R[B^{\mathrm{rep}}\langle x\rangle] \arrow[r] & \pi_1 \tilde{R}[B\langle x\rangle^{\mathrm{gp}}] \arrow[r] & 0
\end{tikzcd}$$

which will conclude the argument. The left square commutes, as the two composites read

$$\mathrm{d}x \mapsto x\mathrm{d}\log x \mapsto (x, x) \text{ and } \mathrm{d}x \mapsto (x, x) \mapsto (x, x).$$

The right square commutes, as the two composites read

$$\mathrm{d}\log x \mapsto 1 \mapsto x \text{ and } \mathrm{d}\log x \mapsto (1, x) \mapsto x,$$

cf. the definition of the map $\pi \colon B^{\mathrm{rep}}\langle x\rangle \to B\langle x\rangle^{\mathrm{gp}}$ from Definition 3.2.13. Hence $\pi_1 \hat{\rho}_{\langle x\rangle}$ is an isomorphism. Finally, we argue that $\pi_q R[\hat{B}^{\mathrm{rep}}\langle x\rangle]$ vanishes for $q \geq 2$. By the long exact sequence above and the fact that $\pi_q \tilde{R}[B\langle x\rangle^{\mathrm{gp}}] = 0$ for $q \geq 2$, we have that $\pi_q R[\hat{B}^{\mathrm{rep}}\langle x\rangle] \cong \pi_q R[B^{\mathrm{rep}}\langle x\rangle]$ for $q \geq 2$, and the identification $R[B^{\mathrm{rep}}\langle x\rangle] \cong R[x] \otimes_R R[B\langle x\rangle^{\mathrm{gp}}]$ combined with the proof of Lemma 3.2.15 concludes the proof, as $\pi_q R[B\langle x\rangle^{\mathrm{gp}}] \cong 0$ for $q \geq 2$. $\square$

Corollary 3.2.17. *The induced map*

$$\mathrm{HH}^R(A^{\mathrm{cof}}) \cong \mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} R[B^{\mathrm{cy}}\langle x\rangle] \to \mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} R[\hat{B}^{\mathrm{rep}}\langle x\rangle]$$

is a weak equivalence.

Proof. By Lemma 3.2.16, we have an isomorphism

$$\mathrm{Tor}_p^{\pi_* R[B^{\mathrm{cy}}\langle x\rangle]}(\pi_* \mathrm{HH}^R(A^{\mathrm{cof}}), \pi_* R[B^{\mathrm{cy}}\langle x\rangle])_q \to \mathrm{Tor}_p^{\pi_* R[B^{\mathrm{cy}}\langle x\rangle]}(\pi_* \mathrm{HH}^R(A^{\mathrm{cof}}), \pi_* R[\hat{B}^{\mathrm{rep}}\langle x\rangle])_q$$

for all p and q. The result then follows from the spectral sequence in Theorem 3.2.10, which is applicable since $\mathrm{HH}^R(A^{\mathrm{cof}})$ is degreewise flat over $R[B^{\mathrm{cy}}\langle x\rangle]$, and [Wei94, Theorem 5.2.12] as in the proof of Lemma 3.1.5. □

Our next goal is to prove that $(\mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} \tilde{R}[B\langle x\rangle^{\mathrm{gp}}])_*$ is a model for $\mathrm{HH}^R((A/(a))^{\mathrm{cof}})[1]_*$. To this end we will have to show that a flat simplicial resolution $A^{\mathrm{cof}} \to A$ naturally gives a flat simplicial resolution $(A/(a))^{\mathrm{cof}} \to A/(a)$. We use the following remark to set some notation:

Remark 3.2.18. Let A be a commutative R-algebra equipped with a pre-log structure $\alpha\colon \langle x\rangle \to (A, \cdot)$. Then α is adjoint to a map $\eta_A\colon R[x] \to A$, which gives A the structure of an $R[x]$-algebra with unit η_A. Let $A^{\mathrm{cof}} \to A$ be a choice of simplicial resolution of A of flat $R[x]$-algebras. Then each A_q^{cof} is a pre-log algebra with pre-log structure $\alpha_q\colon \langle x\rangle \to (A_q^{\mathrm{cof}}, \cdot)$ adjoint to the unit map $\eta_{A_q^{\mathrm{cof}}}\colon R[x] \to A_q^{\mathrm{cof}}$. As every face and degeneracy map of A_q^{cof} is a morphism of $R[x]$-algebras, by adjunction they are also morphisms of pre-log algebras. Denote by $A^{\mathrm{cof}}/(\alpha(x))$ the simplicial commutative R-algebra with q-simplices $A_q^{\mathrm{cof}}/(\alpha_q(x))$ and face and degeneracy maps induced by A^{cof}. Upon taking Moore complexes, we obtain a short exact sequence of chain complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(\alpha_*(x)) := \cdots & \longrightarrow & (\alpha_2(x)) & \longrightarrow & (\alpha_1(x)) & \longrightarrow & (\alpha_0(x)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
A_*^{\mathrm{cof}} = \cdots & \longrightarrow & A_2^{\mathrm{cof}} & \longrightarrow & A_1^{\mathrm{cof}} & \longrightarrow & A_0^{\mathrm{cof}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(A^{\mathrm{cof}}/\alpha(x))_* = \cdots & \longrightarrow & A_2^{\mathrm{cof}}/(\alpha_2(x)) & \longrightarrow & A_1^{\mathrm{cof}}/(\alpha_1(x)) & \longrightarrow & A_0^{\mathrm{cof}}/(\alpha_0(x)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Lemma 3.2.19. *With notation as in Remark 3.2.18, assume that $\alpha(x) := a \in A$ is not a divisor of zero. The morphism $A^{\mathrm{cof}}/(\alpha(x)) \to A/(a)$ is simplicial resolution of flat commutative R-algebras.*

Proof. Each $A_q^{\mathrm{cof}}/(\alpha_q(x))$ is R-flat, as there is a natural isomorphism

$$- \otimes_R A_q^{\mathrm{cof}}/(\alpha_q(x)) \simeq - \otimes_R R \otimes_{R[x]} A_q^{\mathrm{cof}} \simeq - \otimes_{R[x]} A_q^{\mathrm{cof}}$$

of functors from Mod_R. We now check that the map $A^{\mathrm{cof}}/(\alpha(x)) \to A/(a)$ is a weak equivalence. Notice that since each A_q^{cof} is $R[x]$-flat, we have that each $\alpha_q(x)$ is not a divisor of zero, otherwise the exactness of

$$0 \longrightarrow R[x] \xrightarrow{\cdot x} R[x] \longrightarrow R \longrightarrow 0$$

would not be preserved under $A_q^{\mathrm{cof}} \otimes_{R[x]} -$. In particular, as the differentials of $(\alpha_*(x))$ are induced by those of A_*^{cof}, we have that the homology of $(\alpha_*(x))$ is concentrated as (a) in degree 0. More explicitly, the differential of a generic element $a_q\alpha_q(x)$ in $(\alpha_q(x))$ is given by

$$\sum_{i=0}^{q}(-1)^i d_i^{A^{\mathrm{cof}}}(a_q\alpha_q(x)) = \alpha_{q-1}(x)\sum_{i=0}^{q}(-1)^i d_i^{A^{\mathrm{cof}}}(a_q),$$

where we have used that the face maps are morphisms of $R[x]$-algebras and that α_q is adjoint to the unit map $\eta_{A_q^{\mathrm{cof}}} \colon R[x] \to A_q^{\mathrm{cof}}$. This vanishes precisely when the differential of a_q does, which gives the claim on the homology of $(\alpha_*(x))$. The long exact sequence of the diagram in Remark 3.2.18 now reduces to

$$0 \to H_1(A^{\mathrm{cof}}/(\alpha(x))_*) \to (a) \to A \to H_0(A^{\mathrm{cof}}/(\alpha(x))_*) \to 0.$$

We are done if $H_1(A^{\mathrm{cof}}/(\alpha(x))_*) = 0$, and we prove this by showing that the connecting homomorphism $H_1(A^{\mathrm{cof}}/(\alpha(x))_*) \to H_0(\alpha_*(x))$ is trivial. A homology class in $H_1(A^{\mathrm{cof}}/(\alpha(x))_*)$ is represented by an element a_1 in A_1^{cof} which is mapped to an element of the form $a_0 \cdot \alpha_0(x)$ in A_0^{cof}. This is sent to $d(a_0)\cdot a = 0$ under the augmentation map $d \colon A_0^{\mathrm{cof}} \to A$, so $d(a_0) = 0$ as a is not a divisor of zero. Hence there is an element a_1' in A_1^{cof} which hits a_0, and consequently $a_1' \cdot \alpha_1(x)$ hits $a_0 \cdot \alpha_0(x)$, so the latter represents the trivial class in homology. This proves that $A^{\mathrm{cof}}/(\alpha(x)) \to A/(a)$ is a flat simplicial resolution of $A/(a)$. $\square$

We checked in Lemma 3.2.15 that $\tilde{R}[B\langle x\rangle^{\mathrm{gp}}]$ is quasi-isomorphic to a shift $R[1]$. We check that this behaves as expected before we conclude this section.

Lemma 3.2.20. *There are isomorphisms*

$$\pi_q(\mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} \tilde{R}[B\langle x\rangle^{\mathrm{gp}}]) \cong \pi_{q-1}\mathrm{HH}^R((A/(a))^{\mathrm{cof}}).$$

Proof. We first establish a simplicial isomorphism

$$\mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} \tilde{R}[B\langle x\rangle^{\mathrm{gp}}] \simeq \mathrm{HH}^R((A/(a))^{\mathrm{cof}}) \boxtimes_R \tilde{R}[B\langle x\rangle^{\mathrm{gp}}]$$

of simplicial algebras. The $R[B^{\mathrm{cy}}\langle x\rangle]$-module structure on $\tilde{R}[B\langle x\rangle^{\mathrm{gp}}]$ is given by the composite

$$R[B^{\mathrm{cy}}\langle x\rangle] \xrightarrow{R[\rho_{\langle x\rangle}]} R[B^{\mathrm{rep}}\langle x\rangle] \xrightarrow{R[\pi]} \tilde{R}[B\langle x\rangle^{\mathrm{gp}}]$$

of the repletion map $\rho_{\langle x\rangle}$ and projection map π, cf. Definitions 3.2.4 and 3.2.13. As this composite sends any q-simplex $(x^{i_0}, \dots, x^{i_q})$ of $R[B^{\mathrm{cy}}\langle x\rangle]$ with at least one i_j positive to 0, it factors through the map

$$R[B^{\mathrm{cy}}\langle x\rangle] \to R$$

sending a q-simplex to 1 if all i_j are 0, and to 0 otherwise. Hence there is an isomorphism

$$\mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} \tilde{R}[B\langle x\rangle^{\mathrm{gp}}] \cong \mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} R \boxtimes_R \tilde{R}[B\langle x\rangle^{\mathrm{gp}}].$$

As the $R[B^{\mathrm{cy}}\langle x\rangle]$-module structure on $\mathrm{HH}^R(A^{\mathrm{cof}})$ is given by the map sending $(x^{i_0}, \dots, x^{i_q})$ to

$$(\alpha_q(x^{i_0}), \dots, \alpha_q(x^{i_q}))$$

in $(A_q^{\mathrm{cof}})^{\otimes(q+1)}$ with α_q adjoint to the unit map $\eta_{A_q^{\mathrm{cof}}}\colon R[x] \to A_q^{\mathrm{cof}}$, we have that

$$\mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} R \cong \mathrm{HH}^R(A^{\mathrm{cof}}/\alpha(x))$$

in the notation of Remark 3.2.18. We verified in Lemma 3.2.19 that this is $\mathrm{HH}^R((A/(a))^{\mathrm{cof}})$ up to weak equivalence. Hence we are left to compute the homology of

$$(\mathrm{HH}^R((A/(a))^{\mathrm{cof}}) \boxtimes_R \tilde{R}[B\langle x\rangle^{\mathrm{gp}}])_* \cong (\mathrm{HH}^R((A/(a))^{\mathrm{cof}}))_* \otimes_R \tilde{R}[B\langle x\rangle^{\mathrm{gp}}]_*$$

where the equivalence is from the Eilenberg-Zilber Theorem 1.1.4. Since $R[1]$, $\tilde{R}[B\langle x\rangle^{\mathrm{gp}}]$ and $\mathrm{HH}^R((A/(a))^{\mathrm{cof}})$ are all degreewise projective, hence cofibrant in $\mathrm{Ch}_{\geq 0}(R)$, the quasi-isomorphism $R[1] \to \tilde{R}[B\langle x\rangle^{\mathrm{gp}}]$ of Lemma 3.2.15 extends to a quasi-isomorphism

$$\mathrm{HH}^R((A/(a))^{\mathrm{cof}})[1] \cong \mathrm{HH}^R((A/(a))^{\mathrm{cof}}) \otimes_R R[1] \to \mathrm{HH}^R((A/(a))^{\mathrm{cof}}) \otimes_R \tilde{R}[B\langle x\rangle^{\mathrm{gp}}],$$

cf. Example 1.1.10. This concludes the proof. □

This finishes the construction of the long exact sequence. We summarize the proof below:

Proof of Theorem 3.2.12. By Remark 3.2.14, there is a short exact sequence

$$0 \to \mathrm{HH}^R(A^{\mathrm{cof}})\boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} R[\hat{B}^{\mathrm{rep}}\langle x\rangle] \to \mathrm{HH}^R(A^{\mathrm{cof}})\boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} R[B^{\mathrm{rep}}\langle x\rangle] \to \mathrm{HH}^R(A^{\mathrm{cof}})\boxtimes_{R[B^{\mathrm{cy}}\langle x\rangle]} \tilde{R}[B\langle x\rangle^{\mathrm{gp}}] \to 0$$

of simplicial A-modules. By definition, the middle term is $\mathrm{HH}^R(A^{\mathrm{cof}}, \langle x\rangle)$. By Corollary 3.2.17, the first term is weakly equivalent to $\mathrm{HH}^R(A^{\mathrm{cof}})$, and by Lemma 3.2.20, the third term acts as a shift of $\mathrm{HH}^R((A/(a))^{\mathrm{cof}})$. □

3.2.2 Logarithmic Hochschild homology and log Kähler differentials

In this section we prove the following:

Proposition 3.2.21. *There is an isomorphism of A-modules*

$$\Omega^1_{(A,M)/R} \to \pi_1 \mathrm{HH}^R(A^{\mathrm{cof}}, M)$$

sending $\mathrm{d}a$ *to* $\pi_1\bar{\psi}(1 \otimes a)$ *and* $\mathrm{d}\log m$ *to* $\pi_1\bar{\phi}(1 \otimes \gamma_M(m))$*, with the notation from Definition 3.2.7.*

We extend this result to relative log Kähler differentials $\Omega^1_{(A,M)/(R,N)}$ in Section 4.1.23. Notice that we are now treating elements of $\pi_1\mathrm{HH}^R(A^{\mathrm{cof}})$ as being represented by elements of $A^{\otimes 2}$. We explain that we can do this in the remark below, which will also be actively used in Section 3.2.3.

Remark 3.2.22. Consider the following part of the total complex of the Moore bicomplex of $\Lambda_{S^1} A^{\mathrm{cof}}$:

$$A_2^{\mathrm{cof}} \oplus (A_1^{\mathrm{cof}})^{\otimes 2} \oplus (A_0^{\mathrm{cof}})^{\otimes 3} \longrightarrow A_1^{\mathrm{cof}} \oplus (A_0^{\mathrm{cof}})^{\otimes 2} \longrightarrow A_0^{\mathrm{cof}}.$$

The kernel of the last map is

$$\ker(\partial_1^{A_*^{\mathrm{cof}}}) \oplus (A_0^{\mathrm{cof}})^{\otimes 2},$$

where $\partial_1^{A_*^{\mathrm{cof}}}\colon A_1^{\mathrm{cof}} \to A_0^{\mathrm{cof}}$ denotes the differential of A_*^{cof}. We now consider the image of the last map. Only A_2^{cof} and $(A_1^{\mathrm{cof}})^{\otimes 2}$ maps to A_1^{cof}. As the Hochschild differential

$$\partial_1^{\mathrm{HH}^R(A_1^{\mathrm{cof}})_*}\colon (A_1^{\mathrm{cof}})^{\otimes 2} \to A_1^{\mathrm{cof}}$$

is trivial, and the image of $\partial_2^{A_*^{\mathrm{cof}}}: A_2^{\mathrm{cof}} \to A_1^{\mathrm{cof}}$ equals the kernel of $\partial_1^{A_*^{\mathrm{cof}}}$, we conclude that the first homology group of the bicomplex equals $(A_0^{\mathrm{cof}})^{\otimes 2}$ modulo the image of the morphism

$$(A_1^{\mathrm{cof}})^{\otimes 2} \oplus (A_0^{\mathrm{cof}})^{\otimes 3} \to (A_0^{\mathrm{cof}})^{\otimes 2}. \tag{1}$$

By definition, we have that

$$\pi_0(A^{\mathrm{cof}} \boxtimes_R A^{\mathrm{cof}}) = (A_0^{\mathrm{cof}})^{\otimes 2}/(\partial_1^{A^{\mathrm{cof}} \boxtimes A^{\mathrm{cof}}}(b_0 \otimes b_1)),$$

where b_0 and b_1 denote elements of A_1^{cof}. If we let $p\colon A^{\mathrm{cof}} \to A$ denote the simplicial flat resolution, we see that p induces an isomorphism $\pi_0(A^{\mathrm{cof}} \boxtimes_R A^{\mathrm{cof}}) \cong A \otimes_R A$ by for instance appealing to the spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^{\pi_* R}(\pi_*(A^{\mathrm{cof}}), \pi_*(A^{\mathrm{cof}}))_q \implies \pi_{p+q}(A^{\mathrm{cof}} \boxtimes_R A^{\mathrm{cof}})$$

of Theorem 3.2.10. Under this isomorphism, the image of the Hochschild differential $\partial_2^{\mathrm{HH}^R(A_0^{\mathrm{cof}})}(a_0, a_1, a_2)$ corresponds to $\partial_2^{\mathrm{HH}^R(A)}(p_0(a_0), p_0(a_1), p_0(a_2))$, where $p_0\colon A_0^{\mathrm{cof}} \to A$ is p in degree 0. We conclude that the cokernel of the morphism (1) is isomorphic to

$$\pi_1 \mathrm{HH}^R(A^{\mathrm{cof}}) \cong A^{\otimes 2}/(\partial_1^{\mathrm{HH}^R(A)}(p_0(a_0), p_0(a_1), p_0(a_2))).$$

This gives the desired description of $\pi_1 \mathrm{HH}^R(A^{\mathrm{cof}})$, and also a direct proof of Proposition 3.1.10: the morphism $\pi_1 \mathrm{HH}^R(A^{\mathrm{cof}}) \to \Omega^1_{A/R}$ sending $[1 \otimes a]$ to $\mathrm{d}a$ is an isomorphism, where the inverse is well-defined by the same argument as in Lemma 3.1.8 and surjectivity of p_0.

Proof of Proposition 3.2.21. We apply the spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^{\pi_* \mathrm{HH}^R(R[M])}(\pi_* \mathrm{HH}^R(A^{\mathrm{cof}}), \pi_* R[B^{\mathrm{rep}} M])_q \implies \pi_{p+q} \mathrm{HH}^R(A^{\mathrm{cof}}, M)$$

of Theorem 3.2.10. Notice first that

$$E^2_{p,0} = \mathrm{Tor}^{\pi_0 \mathrm{HH}^R(R[M])}(\pi_0 \mathrm{HH}^R(A^{\mathrm{cof}}), \pi_0 R[B^{\mathrm{rep}}(M)]) \cong \mathrm{Tor}_p^{R[M]}(A, R[M]) \cong 0$$

for all $p > 0$. In particular $E^2_{0,1}$ is not hit by $E^2_{2,0}$. The term $E^2_{0,1}$ is by definition given by the colimit of the diagram

$$\begin{array}{c} \bigoplus_{i+j+k=1} \pi_i \mathrm{HH}^R(A^{\mathrm{cof}}) \otimes_R \pi_j \mathrm{HH}^R(R[M]) \otimes_R \pi_k R[B^{\mathrm{rep}} M] \\ \downarrow\downarrow \\ \bigoplus_{p+q=1} \pi_p \mathrm{HH}^R(A^{\mathrm{cof}}) \otimes_R \pi_q R[B^{\mathrm{rep}} M]. \end{array} \tag{2}$$

We first consider the part where the $(1,0,0)$-summand maps to the $(1,0)$-summand. By Proposition 3.1.10, $\pi_1 \mathrm{HH}^R(A^{\mathrm{cof}}) \cong \Omega^1_{A/R}$, and by Remark 3.1.4, $\pi_0 \mathrm{HH}^R(R[M]) \cong R[M]$. Finally, we know that $\pi_0 R[B^{\mathrm{rep}}(M)] \cong R[M]$. The resulting diagram

$$\Omega^1_{A/R} \otimes_R R[M] \otimes_R R[M] \rightrightarrows \Omega^1_{A/R} \otimes_R R[M]$$

is the defining coequalizer diagram for $\Omega^1_{A/R} \otimes_{R[M]} R[M] \cong \Omega^1_{A/R}$. We now consider the part where the $(0,0,1)$-summand maps to the $(0,1)$-summand. By Example 3.2.6 we have that $\pi_1 R[B^{\mathrm{rep}}(M)] \cong R[M] \otimes_{\mathbb{Z}} M^{\mathrm{gp}}$, and the resulting diagram

$$A \otimes_R R[M] \otimes_R (R[M] \otimes_{\mathbb{Z}} M^{\mathrm{gp}}) \rightrightarrows A \otimes_R (R[M] \otimes_{\mathbb{Z}} M^{\mathrm{gp}})$$

is the defining coequalizer diagram for $A \otimes_{R[M]} (R[M] \otimes_{\mathbb{Z}} M^{\mathrm{gp}}) \cong A \otimes_{\mathbb{Z}} M^{\mathrm{gp}}$. We now consider the part where the $(0,1,0)$-summand maps to both summands. By Lemma 3.1.8, we have that $\pi_1 \mathrm{HH}^R(R[M]) \cong \Omega^1_{R[M]/R}$. The morphism

$$A \otimes_R \Omega^1_{R[M]/R} \otimes_R R[M] \to \Omega^1_{A/R} \otimes_R R[M]$$

sends $(a, \mathrm{d}m, m')$ to $(a\mathrm{d}\alpha(m), m')$, while the morphism

$$A \otimes_R \Omega^1_{R[M]/R} \otimes_R R[M] \to A \otimes_R (R[M] \otimes_{\mathbb{Z}} M^{\mathrm{gp}})$$

sends $(a, \mathrm{d}m, m')$ to $(a, \alpha(m)m', \gamma_M(m))$, cf. Example 2.3.11. We conclude that

$$E^2_{0,1} = (\Omega^1_{A/R} \oplus (A \otimes_{\mathbb{Z}} M^{\mathrm{gp}}))/\sim,$$

where $\sim$ is A-linearly generated by the relation $(\mathrm{d}\alpha(m), 0) \sim (0, \alpha(m) \otimes \gamma_M(m))$. This is by definition the log Kähler differentials $\Omega^1_{(A,M)/R}$. Hence we have established that $\Omega^1_{(A,M)/R} \cong \pi_1 \mathrm{HH}^R(A^{\mathrm{cof}}, M)$. The proposed isomorphism is an isomorphism, as the universal property of the coequalizer (2) above gives a dashed isomorphism

$$\begin{tikzcd}
\bigoplus_{p+q=1} \pi_p \mathrm{HH}^R(A^{\mathrm{cof}}) \otimes_R \pi_q R[B^{\mathrm{rep}}(M)] \arrow[r] \arrow[dr] & \Omega^1_{(A,M)/R} \arrow[d, dashed, "\cong"] \\
& \pi_1 \mathrm{HH}^R(A^{\mathrm{cof}}, M).
\end{tikzcd}$$

By construction, the elements $[1 \otimes a]$ in $\pi_1 \mathrm{HH}^R(A^{\mathrm{cof}})$ and $1 \otimes \gamma_M(m)$ in $\pi_1 R[B^{\mathrm{rep}}(M)] \cong R[M] \otimes_{\mathbb{Z}} M^{\mathrm{gp}}$ are sent to $\mathrm{d}a$ and $\mathrm{d}\log(m) = (0, 1 \otimes \gamma_M(m))$ in $\Omega^1_{(A,M)/R}$, respectively. By commutativity of the diagram, the dashed isomorphism sends $\mathrm{d}a$ to $\pi_1 \bar{\psi}([1 \otimes a])$ and $\mathrm{d}\log(m)$ to $\pi_1 \bar{\phi}(1 \otimes \gamma_M(m))$, which concludes the proof. $\square$

3.2.3 Low-degree computations and discrete valuation rings

In this section we give an explicit description of the map

$$\pi_1 \mathrm{HH}^R(A^{\mathrm{cof}}, \langle x \rangle) \to \pi_0 \mathrm{HH}^R((A/(a))^{\mathrm{cof}})$$

arising from the long exact sequence from Section 3.2.1. Identifying (log) Hochschild homology with Kähler differentials (cf. Lemma 3.1.8 and Proposition 3.2.21), we obtain a short exact sequence

$$0 \to \Omega^1_{A/R} \to \Omega^1_{(A,\langle x \rangle)/R} \to A/(a) \to 0.$$

In particular, injectivity of the map $\Omega^1_{A/R} \to \Omega^1_{(A,\langle x \rangle)/R}$ implies that the "transfer map"

$$\pi_1 \mathrm{HH}^R((A/(a))^{\mathrm{cof}}) \to \pi_1 \mathrm{HH}^R(A^{\mathrm{cof}})$$

arising from the long exact sequence is trivial. In the setting of discrete valuation rings, the above sequence coincides with the extension discussed in [HM03, Proposition 2.2.2].

Throughout this section we will freely use Remark 3.2.22, which allows us to think of elements of $\pi_i \mathrm{HH}^R(A^{\mathrm{cof}})$ as being represented by elements of A and $A^{\otimes 2}$ for $i = 0, 1$ respectively.

Lemma 3.2.23. *The map*

$$\pi_1 \mathrm{HH}^R(A^{\mathrm{cof}}, \langle x \rangle) \to \pi_0 \mathrm{HH}^R((A/(a))^{\mathrm{cof}})$$

from Theorem 3.2.12 sends the log Hochschild class $(a_0, a_1) \otimes (x^i, x^j)$ *to* $[ja_1a_0]$*, where* $[a]$ *denotes the class of* a *in* $A/(a)$*.*

Proof. From the construction of the long exact sequence in Section 3.2.1, we find that the map can be described by the commutative diagram

$$\begin{tikzcd}
\pi_1 \mathrm{HH}^R(A^{\mathrm{cof}}, \langle x \rangle) \arrow[rr] \arrow[d] & & \pi_0 \mathrm{HH}^R((A/(a))^{\mathrm{cof}}) \\
\pi_1(\mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}\langle x \rangle]} \tilde{R}[B\langle x \rangle^{\mathrm{gp}}]) \arrow[d, "\cong"'] & & \bigoplus_{p+q=1} H_p(\mathrm{HH}^R_*((A/(a))^{\mathrm{cof}})) \otimes_R H_q(\tilde{R}[B_*\langle x \rangle^{\mathrm{gp}}]) \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
\pi_1(\mathrm{HH}^R((A/(a))^{\mathrm{cof}}) \boxtimes_R \tilde{R}[B\langle x \rangle^{\mathrm{gp}}]) \arrow[rr, "\cong"'] & & H_1(\mathrm{HH}^R_*((A/(a))^{\mathrm{cof}}) \otimes_R \tilde{R}[B_*\langle x \rangle^{\mathrm{gp}}]).
\end{tikzcd}$$

Here the lower map is the Alexander-Whitney map (cf. Theorem 1.1.4), and the isomorphism

$$\bigoplus_{p+q=1} H_p(\mathrm{HH}^R((A/(a))^{\mathrm{cof}})_*) \otimes_R H_q(\tilde{R}[B\langle x \rangle^{\mathrm{gp}}]_*) \to H_1(\mathrm{HH}^R((A/(a))^{\mathrm{cof}}))_*$$

is given by the Künneth theorem, cf. [Wei94, Theorem 3.6.3]. We need an explicit description of the isomorphism

$$H_1 \tilde{R}[B_*\langle x \rangle^{\mathrm{gp}}] \to R.$$

Recall that we are implicitly identifying $B\langle x \rangle^{\mathrm{gp}}$ with the simplicial quotient $B^{\mathrm{rep}}\langle x \rangle / \hat{B}^{\mathrm{rep}}\langle x \rangle$ by the discussion in Definition 3.2.13. Looking at the Moore complex (before making the aforementioned identification) gives

$$H_1 \tilde{R}[B_*\langle x \rangle^{\mathrm{gp}}] = R[B\langle x \rangle^{\mathrm{gp}}]/(g_1g_2 \sim g_1 + g_2) \cong R \otimes_{\mathbb{Z}} \langle x \rangle^{\mathrm{gp}} \cong R,$$

the isomorphism sending $rg = rx^i$ to ir for an integer i. Then the composite

$$H_1 R[B^{\mathrm{rep}}\langle x \rangle / \hat{B}^{\mathrm{rep}}\langle x \rangle] \to H_1 R[B\langle x \rangle^{\mathrm{gp}}] \to R$$

maps $r[x^i, x^j]$ to jr.

We now chase the diagram above. A log Hochschild class $(a_0, a_1) \otimes (x^i, x^j)$ is sent to $([a_0], [a_1]) \otimes [x^i, x^j]$ in $\pi_1(\mathrm{HH}^R((A/(a))^{\mathrm{cof}}) \boxtimes_R \tilde{R}[B\langle x \rangle^{\mathrm{gp}}])$. Under the Alexander-Whitney map from Theorem 1.1.4, only $[a_1a_0] \otimes [x^i, x^j]$ (one application of the last face map in the first coordinate and zero applications of the zeroth face map in the second) survives to the direct sum under the Künneth isomorphism, as the homology groups of $\tilde{R}[B\langle x \rangle^{\mathrm{gp}}]$ are concentrated in degree 1 by Lemma 3.2.15. This element is in turn is sent to $[ja_1a_0]$ by the previous paragraph. □

Lemma 3.2.24. *Let* $(A, \langle x \rangle)$ *be a pre-log ring with* $\alpha(x) = a$ *not a divisor of zero in* A*. There is a short exact sequence*

$$0 \to \Omega^1_{A/R} \to \Omega^1_{(A, \langle x \rangle)/R} \to A/(a) \to 0,$$

where the first map is the canonical inclusion while the last map sends $b\,d\log x^i$ *to* $[ib]$*.*

Proof. If $b\mathrm{d}\log x^i = ib\mathrm{d}\log x$ is sent to 0, then $ib = b'a$ for some b' in A. Hence

$$b\mathrm{d}\log x^i = ib\mathrm{d}\log x = b'a\mathrm{d}\log x = b'\alpha(x)\mathrm{d}\log x = b'\mathrm{d}\alpha(x) = b'\mathrm{d}a,$$

as $\mathrm{d}\alpha(x) = \alpha(x)\mathrm{d}\log x$ in $\Omega^1_{(A,\langle x\rangle)/R}$. It is clear that the first map injective and that the second map is surjective. □

Proposition 3.2.25. *The diagram*

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_{A/R} \arrow[r] \arrow[d, "\cong"] & \Omega^1_{(A,\langle x\rangle)} \arrow[r] \arrow[d, "\cong"] & A/(a) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \pi_1\mathrm{HH}^R(A^{\mathrm{cof}}) \arrow[r] & \pi_1\mathrm{HH}^R(A^{\mathrm{cof}},\langle x\rangle) \arrow[r] & \pi_0\mathrm{HH}^R((A/(a))^{\mathrm{cof}}) \arrow[r] & 0
\end{tikzcd}$$

commutes.

Here the upper sequence is described in Lemma 3.2.24, the lower sequence comes from the long exact sequence of Theorem 3.2.12, and the identifications with Kähler differentials are described in Example 3.1.8 and Proposition 3.2.21.

Proof. The left square commutes, as $a_0\mathrm{d}a_1$ is sent to the Hochschild class (a_0, a_1), which in turn is sent to $(a_0, a_1) \otimes (1, 1)$. The right square commutes, as a differential a_0da_1 is sent to the log Hochschild class $(a_0, a_1) \otimes (1, 1 = x^0)$, which is sent to 0 in $\pi_0\mathrm{HH}^R((A/(a))^{\mathrm{cof}})$ by Lemma 3.2.23. Consider now a logarithmic differential $a_0\mathrm{d}\log x^i$. It is sent to the log Hochschild class $a_0((1, 1) \otimes (1, x^i))$, which, by Lemma 3.2.23, is sent to $[ia_0]$. □

Remark 3.2.26. Recall from Theorem 1.2.10 that a complete discrete valuation ring of characteristic 0 with perfect residue field k of characteristic p is a truncated polynomial ring

$$A = W(k)[x]/(\phi(x)).$$

In the setting of such discrete valuation rings, Hesselholt and Madsen consider the short exact sequence [HM03, Proposition 2.2.2]

$$0 \to \Omega^1_{A/W(k)} \to \Omega^1_{(A,A\cap\mathrm{GL}_1(K))/W(k)} \to k \to 0,$$

the latter map sending $a\mathrm{d}\log b$ to $[a\nu(b)]$, where ν denotes the valuation on K. They find that, for their relative construction $\mathrm{THH}(A|K)$, one recovers the above sequence from the cofiber sequence [HM03, Theorem 1.5.6]

$$\mathrm{THH}(k) \to \mathrm{THH}(A) \to \mathrm{THH}(A|K) \to \Sigma\mathrm{THH}(k)$$

in degrees 0 and 1 [HM03, Proof of Proposition 2.3.4]. This is compatible with the above results: The diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_{A/W(k)} \arrow[r] \arrow[d, "\cong"] & \Omega^1_{(A,A\cap\mathrm{GL}_1(K))/W(k)} \arrow[r] \arrow[d, "\cong"] & k \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \pi_1\mathrm{HH}^{W(k)}(A^{\mathrm{cof}}) \arrow[r] & \pi_1\mathrm{HH}^{W(k)}(A^{\mathrm{cof}},\langle\pi\rangle) \arrow[r] & \pi_0\mathrm{HH}^{W(k)}(k^{\mathrm{cof}}) \arrow[r] & 0
\end{tikzcd}$$

commutes, where π is a uniformizer for A. Here the middle map factors as

$$\Omega^1_{(A,A\cap\mathrm{GL}_1(K))/W(k)} \to \Omega^1_{(A,\langle\pi\rangle)/W(k)} \to \pi_1\mathrm{HH}^{W(k)}(A^{\mathrm{cof}},\langle\pi\rangle),$$

where the first map is the isomorphism realizing invariance of log Kähler differentials under logification (cf. Lemma 2.3.10 and Example 2.2.8). Explicitly this map is given as follows: for a logarithmic differential $a\mathrm{d}\log b$, write $b = u\pi^{\nu(b)}$ for a unit u and choice of uniformizer π. Then

$$a\mathrm{d}\log b = a\mathrm{d}\log u\pi^{\nu(b)} = a\mathrm{d}\log u + \nu(b)a\mathrm{d}\log \pi = au^{-1}\mathrm{d}u + \nu(b)a\mathrm{d}\log \pi$$

which maps to $[\nu(b)a]$ in $\pi_0\mathrm{HH}^{W(k)}(k^{\mathrm{cof}})$.

Example 3.2.27. Consider a complete discrete valuation ring A of characteristic 0 with perfect residue field k of characteristic p. Assume for instance that p does not divide the absolute ramification index e. By Proposition 3.1.20 we have that

$$\pi_1\mathrm{HH}^{W(k)}(A) \cong A/(\pi^{e-1})$$

for a uniformizer π, or in terms of Kähler differentials,

$$\Omega^1_{A/W(k)} \cong A\{\mathrm{d}\pi\}/(\pi^{e-1}\mathrm{d}\pi).$$

We compute $\pi_1\mathrm{HH}^{W(k)}(A, \langle\pi\rangle)$. By definition we have that

$$\Omega^1_{(A,\langle\pi\rangle)/W(k)} = \Omega^1_{A/W(k)} \oplus (A \otimes_{\mathbb{Z}} \langle\pi\rangle^{\mathrm{gp}}),$$

modulo the relation $\mathrm{d}\pi \sim \pi\mathrm{d}\log\pi = \pi \otimes \pi$, so that $0 = \pi^{e-1}\mathrm{d}\pi \sim \pi^e\mathrm{d}\log\pi$. This describes the relative log differentials as

$$\Omega^1_{(A,\langle\pi\rangle)/W(k)} = A\{\mathrm{d}\log\pi\}/(\pi^e\mathrm{d}\log\pi),$$

which is isomorphic to $\pi_1\mathrm{HH}^{W(k)}(A, \langle\pi\rangle)$ by Proposition 3.2.21. Note that this fits into the short exact sequence

$$0 \to A\{\mathrm{d}\pi\}/(\pi^{e-1}\mathrm{d}\pi) \to A\{\mathrm{d}\log\pi\}/(\pi^e\mathrm{d}\log\pi) \to k \to 0,$$

of Lemma 3.2.24, the first map sending $\mathrm{d}\pi$ to $\pi\mathrm{d}\log\pi$.

3.3 Computational examples

As far as concrete computations go, we have mostly been focusing on the fact that the derived and classical (logarithmic) Hochschild homology coincide in degrees 0 and 1. In this section we study two examples: the log Hochschild homology $\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}^{\mathrm{cof}}, \langle p\rangle)$, where p is a prime number, and the log Hochschild homology $\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]^{\mathrm{cof}}, \langle 1+i\rangle)$. For both of these examples we will use the long exact sequence of Theorem 3.2.12, and we will be dependent on the following result:

Proposition 3.3.1. [NS17, Proposition IV.4.3] *The (derived) Hochschild homology groups of $\mathbb{F}_p$ are given by*

$$\pi_i\mathrm{HH}^{\mathbb{Z}}(\mathbb{F}_p^{\mathrm{cof}}) = \begin{cases} \mathbb{F}_p, & \text{if } i \geq 0 \text{ even,} \\ 0, & \text{if } i \text{ odd.} \end{cases}$$

□

Recall that $\mathbb{F}_p^{\mathrm{cof}} \to \mathbb{F}_p$ is a simplicial resolution of $\mathbb{F}_p$ of flat $\mathbb{Z}$-algebras. This computation requires a bit more knowledge of the cotangent complex (see Definition 3.1.12) than we have developed in this thesis. The idea is to use that for a surjection of rings $A \to A/(a)$ with a a non-zero divisor, the cotangent complex $\mathbb{L}_{(A/(a))/A}$ is quasi-isomorphic to $(A/(a))[1]$, a copy of $A/(a)$ concentrated in degree 1. In the case of $\mathbb{Z} \to \mathbb{F}_p$, one has that the wedge powers satisfy

$$\bigwedge_{\mathbb{F}_p}^{q} \mathbb{L}_{\mathbb{F}_p/\mathbb{Z}} \simeq \mathbb{F}_p[q],$$

and consequently the spectral sequence

$$E^2_{s,q} = \pi_s(\bigwedge_{\mathbb{F}_p}^{q} \mathbb{L}_{\mathbb{F}_p/\mathbb{Z}}) \implies \pi_{s+q}\mathrm{HH}^{\mathbb{Z}}(\mathbb{F}_p^{\mathrm{cof}})$$

of [Qui70, Section 8.1] is concentrated as $\mathbb{F}_p$ in bidegrees (q,q), which gives the result.

We now proceed with the aforementioned examples:

Example 3.3.2. Consider the integers $\mathbb{Z}$ with the pre-log structure given by the inclusion $\langle p\rangle \to (\mathbb{Z},\cdot)$, where $\langle p\rangle$ is the free monoid generated by a prime number p. We consider the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_2\mathrm{HH}^{\mathbb{Z}}(\mathbb{F}_p^{\mathrm{cof}}) \arrow[r] & \pi_2\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}^{\mathrm{cof}}) \arrow[r] & \pi_2\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}^{\mathrm{cof}}, \langle p\rangle) \arrow[dll, in=north, out=south] & \\
& \pi_1\mathrm{HH}^{\mathbb{Z}}(\mathbb{F}_p^{\mathrm{cof}}) \arrow[r] & \pi_1\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}^{\mathrm{cof}}) \arrow[r] & \pi_1\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}^{\mathrm{cof}}, \langle p\rangle) \arrow[dll, in=north, out=south] & \\
& \pi_0\mathrm{HH}^{\mathbb{Z}}(\mathbb{F}_p^{\mathrm{cof}}) \arrow[r, "0"] & \pi_0\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}^{\mathrm{cof}}) \arrow[r, "\cong"] & \pi_0\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}^{\mathrm{cof}}, \langle p\rangle) \arrow[r] & 0
\end{tikzcd}$$

established in Theorem 3.2.12. By Example 3.2.9, $\pi_0\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}^{\mathrm{cof}}, \langle p\rangle) \cong \mathbb{Z}$. Since $\mathbb{Z}$ is already $\mathbb{Z}$-flat, we have that

$$\pi_q\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}^{\mathrm{cof}}) \cong \pi_q\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } q = 0. \\ 0, & \text{if } q > 0 \end{cases}$$

by Lemma 3.1.5. We conclude that for all $q > 0$, there are isomorphisms

$$\pi_q\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}^{\mathrm{cof}}, \langle p\rangle) \cong \pi_{q-1}\mathrm{HH}^{\mathbb{Z}}(\mathbb{F}_p^{\mathrm{cof}}).$$

By Proposition 3.3.1, we conclude that

$$\pi_q\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}^{\mathrm{cof}}, \langle p\rangle) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ \mathbb{F}_p, & \text{if } q \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Example 3.3.3. We now consider the Gaussian integers $\mathbb{Z}[i]$ with the pre-log structure generated by $\langle 1+i\rangle$. We first recall that

$$\mathbb{Z}[i]/(1+i) \cong \mathbb{F}_2.$$

As the Gaussian integers is a truncated polynomial ring

$$\mathbb{Z}[i] = \mathbb{Z}[x]/(x^2+1),$$

its Hochschild homology is known from Lemma 3.1.19:

$$\pi_q\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]) = \begin{cases} \mathbb{Z}[i], & \text{if } q = 0, \\ \mathbb{Z}[i]/(2i), & \text{if } q = 2k+1, \\ 0, & \text{if } q = 2k > 0. \end{cases}$$

In terms of Kähler differentials, we in particular have that

$$\pi_1 \mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]) \cong \Omega^1_{\mathbb{Z}[i]/\mathbb{Z}} \cong \mathbb{Z}[i]\{\mathrm{d}i\}/(2i\,\mathrm{d}i).$$

Considering now the log Kähler differentials, we have (by definition) that

$$\Omega^1_{(\mathbb{Z}[i],\langle 1+i\rangle)/\mathbb{Z}} = \Omega^1_{\mathbb{Z}[i]/\mathbb{Z}} \oplus (\mathbb{Z}[i] \otimes_{\mathbb{Z}} \langle 1+i\rangle^{\mathrm{gp}})/\sim,$$

where $\mathrm{d}i = \mathrm{d}(1+i) \sim (1+i)\mathrm{d}\log(1+i)$. Notice then that

$$0 = 2i\,\mathrm{d}i \sim 2i(1+i)\mathrm{d}\log(1+i) = (1+i)^3\mathrm{d}\log(1+i),$$

as $2i = (1+i)^2$. This describes the log Kähler differentials as

$$\mathbb{Z}[i]\{\mathrm{d}\log(1+i)\}/((1+i)^3\mathrm{d}\log(1+i)).$$

By Proposition 3.2.21, this is isomorphic to $\pi_1\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i],\langle 1+i\rangle)$. We note that this fits in the short exact sequence

$$0 \to \mathbb{Z}[i]\{\mathrm{d}i\}/((1+i)^2\mathrm{d}i) \to \mathbb{Z}[i]\{\mathrm{d}\log(1+i)\}/((1+i)^3\mathrm{d}\log(1+i)) \to \mathbb{F}_2 \to 0 \tag{3}$$

of Lemma 3.2.24, the first map sending $\mathrm{d}i$ to $(1+i)\mathrm{d}\log(1+i)$. We now consider the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_2\mathrm{HH}^{\mathbb{Z}}(\mathbb{F}_2^{\mathrm{cof}}) \arrow[r] & \pi_2\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]^{\mathrm{cof}}) \arrow[r] & \pi_2\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]^{\mathrm{cof}},\langle 1+i\rangle) \arrow[dll] & \\
& \pi_1\mathrm{HH}^{\mathbb{Z}}(\mathbb{F}_2^{\mathrm{cof}}) \arrow[r] & \pi_1\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]^{\mathrm{cof}}) \arrow[r] & \pi_1\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]^{\mathrm{cof}},\langle 1+i\rangle) \arrow[dll] & \\
& \pi_0\mathrm{HH}^{\mathbb{Z}}(\mathbb{F}_2^{\mathrm{cof}}) \arrow[r, "0"] & \pi_0\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]^{\mathrm{cof}}) \arrow[r, "\cong"] & \pi_0\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]^{\mathrm{cof}},\langle 1+i\rangle) \arrow[r] & 0
\end{tikzcd}$$

established in Theorem 3.2.12. As $\pi_*\mathrm{HH}^{\mathbb{Z}}(\mathbb{F}_2^{\mathrm{cof}})$ is known from Proposition 3.3.1, we see that we have short exact sequences

$$0 \to \pi_{2k+1}\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]^{\mathrm{cof}}) \to \pi_{2k+1}\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]^{\mathrm{cof}},\langle 1+i\rangle) \to \pi_{2k}\mathrm{HH}^{\mathbb{Z}}(\mathbb{F}_2^{\mathrm{cof}}) \to 0$$

in odd degrees $2k+1$, while $\pi_{2k}\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]^{\mathrm{cof}},\langle 1+i\rangle) = 0$ for all $2k > 0$. From this we expect that

$$\pi_q\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]^{\mathrm{cof}},\langle 1+i\rangle) = \begin{cases} \mathbb{Z}[i], & \text{if } q = 0, \\ \mathbb{Z}[i]/(1+i)^3, & \text{if } q = 2k+1, \\ 0, & \text{if } q = 2k > 0, \end{cases}$$

provided that the above extension

$$0 \to \mathbb{Z}[i]/(2i) \to \pi_{2k+1}\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[i]^{\mathrm{cof}},\langle 1+i\rangle) \to \mathbb{F}_2 \to 0$$

is isomorphic to (3).

4 Hochschild homology over pre-logarithmic ground rings

In this section we extend the definition of logarithmic Hochschild homology $\mathrm{HH}^R(A, M)$ from Definition 3.2.7 to allow for pre-logarithmic ground rings (R, N). This definition depends heavily on a simplicial analogue of the notion of *repletion* from Section 2.2.3, which we discuss in Section 4.1.2. We then give the construction of the relative log Hochschild homology in Section 4.1.3. In Section 4.1.4 we provide some evidence as to why the definition is "correct" in that we prove that there is an isomorphism between the first "relative" log Hochschild homology group and the relative log Kähler differentials $\Omega^1_{(A,M)/(R,N)}$ from Example 2.3.11. While the mentioned material is alluded to in [Rog09, Definition 5.27], there does not seem to be a definition of $\mathrm{HH}^{(R,N)}(A, M)$ given in the literature.

In Section 4.2.2 we compare the relative construction $\mathrm{HH}^{(R,N)}(A, M)$ to its logified versions $\mathrm{HH}^{(R,N)}(A, M^a)$ and $\mathrm{HH}^{(R,N^a)}(A, M^a)$, see Section 2.2.1. In order to do so, it seems to be necessary to consider a homotopy invariant version of the logification functor, where the defining pushout square of the logification is replaced by a homotopy pushout. In this context, we are no longer ensured that the logification of a discrete monoid remains discrete, and so it is important that our definition of relative log Hochschild homology allows for not necessarily discrete pre-log ground rings. This point will be discussed on numerous occasions in this section.

4.1 Relative logarithmic Hochschild homology

In this section we introduce the relative log Hochschild homology $\mathrm{HH}^{(R,N)}(A^{\mathrm{cof}}, M^{\mathrm{cof}})$, working in the context of a cofibrant replacement $(A^{\mathrm{cof}}, M^{\mathrm{cof}}) \to (A, M)$ of the pre-logarithmic R-algebra (A, M) in a model structure to be discussed in Section 4.1.1. In this setting, we discuss the relation between $\mathrm{HH}^{(R,N)}(A^{\mathrm{cof}}, M^{\mathrm{cof}})$ and the relative log Kähler differentials $\Omega^1_{(A,M)/(R,N)}$ in Section 4.1.4.

4.1.1 Model structures on simplicial monoids and pre-log algebras

The construction and study of the object $\mathrm{HH}^{(R,N)}(A^{\mathrm{cof}}, M^{\mathrm{cof}})$ to be introduced in Section 4.1.3 requires a significant change in context: as indicated in the notation, we now need to take a replacement of the monoid M, and so it is necessary to use a model structure on simplicial commutative monoids.

Proposition 4.1.1. [SSV16, Proposition 2.1] *There is a model structure on the category of simplicial commutative monoids,* sMon*, where a morphism is a weak equivalence or fibration precisely when the underlying morphism of simplicial sets is.*

Notice then that the adjunction

$$\mathrm{sMon} \underset{U}{\overset{R[-]}{\rightleftarrows}} \mathrm{sAlg}_R$$

with U the forgetful functor is a Quillen adjunction, as the right adjoint preserves (acyclic) fibrations. In particular, a cofibration $M \to N$ of simplicial commutative monoids gives a cofibration $R[M] \to R[N]$ of simplicial commutative R-algebras. We also notice that the functor $R[-]$ preserves all weak equivalences, as the weak equivalences in both sMon and sAlg_R are those of the underlying simplicial sets. Given a weak equivalence $M \to N$ in the above model structure, the induced morphism

$$H_*(M, R) = \pi_* R[M] \to \pi_* R[N] = H_*(N, R)$$

of simplicial homology groups is then necessarily an isomorphism, and so $R[M] \to R[N]$ is a weak equivalence.

We refer to the above model structure on simplicial commutative monoids as the *standard* model structure, as to avoid confusion with the *group completion* model structure on sMon considered below:

Proposition 4.1.2. [SSV16, Proposition 2.6] *There is a model structure on* sMon *in which the cofibrations coincide with those of the standard model structure. The weak equivalences $M \to N$ are the morphisms for which the induced map of bar constructions $B(M) \to B(N)$ is a weak equivalence of simplicial sets. The fibrant objects are precisely those which are fibrant as simplicial sets and grouplike.*

Recall that a simplicial commutative monoid M is *grouplike* if $\pi_0(M)$ is a group. As fibrant objects in the group completion model structure are grouplike, the following definition is natural:

Definition 4.1.3. Let M be a simplicial commutative monoid. The *group completion* of M is a fibrant replacement

$$M \overset{\gamma_M}{\rightarrowtail} M^{\mathrm{gp}} \twoheadrightarrow *$$

in the group completion model structure of Proposition 4.1.2.

If M is a discrete commutative monoid, then the usual group completion M^{gp} of Example 2.1.7 is fibrant in the group completion model structure (being fibrant as a simplicial set and grouplike), and the induced morphism $BM \to BM^{\mathrm{gp}}$ is a weak equivalence of simplicial sets by [SSV16, Lemma 2.10]. In fact, the cited lemma states that one could naively form the group completion of a simplicial commutative monoid degreewise, obtaining a fibrant, grouplike simplicial commutative monoid for which the induced map of bar constructions is a weak equivalence of simplicial sets. This uses results of Quillen [FM94, Appendix Q].

Remark 4.1.4. We will often use the fact that a weak equivalence $M \to N$ in the group completion model structure of grouplike simplicial commutative monoids is also a weak equivalence in the standard model structure. The idea is that there is a group completion of M intrinsic to the standard model structure: there is a Quillen adjunction

$$\mathrm{sMon} \underset{\Omega(-)}{\overset{B(-)}{\rightleftarrows}} \mathrm{sMon}$$

with $B(-)$ the (diagonal of) the bar construction considered in Definition 3.2.1. In the language of enriched category theory, this uses that sMon has a zero object (the discrete trivial monoid), so that the simplicial model category sMon is not only enriched over simplicial sets, but also pointed simplicial sets. From this perspective, the bar construction $B(M)$ is the pointed tensor $S^1 \otimes M$ (while the tensor over unpointed simplicial sets would give the usual cyclic bar construction of M). A definition of the group completion of M intrinsic to the standard model structure on sMon is then given by the morphism

$$M \to \Omega(B(M)^{\mathrm{fib}}),$$

adjoint to a fibrant replacement $B(M) \to B(M)^{\mathrm{fib}}$. One shows that, for M grouplike, this group completion map is a weak equivalence, and considers the square

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \Omega((B(M))^{\mathrm{fib}}) \arrow[d] \\ N \arrow[r] & \Omega((B(N))^{\mathrm{fib}}). \end{tikzcd}$$

If M and N are grouplike, the horizontal arrows are weak equivalences, and the right-hand vertical arrow is a weak equivalence by the characterization of weak equivalences in the group completion model structure. For more details, see e.g. [SSV16, Section 2].

The following remark should be considered a "cheat sheet" on Bousfield localizations:

Remark 4.1.5. The group completion model structure is a left *Bousfield localization* of the standard model structure, see e.g. [SSV16, Proof of Proposition 2.6]. While we will not introduce this concept, we list useful consequences of this fact below, giving the appropriate references in [Hir03]:

1. [Hir03, Proposition 3.3.5] Consider a diagram

$$\begin{tikzcd} M \arrow[rr, "\simeq"] \arrow[dr, two heads] & & N \arrow[dl, two heads] \\ & K, & \end{tikzcd}$$

   where the arrows are decorated with respect to the group completion model structure of Proposition 4.1.2. In this setting, the horizontal morphism is also a weak equivalence in the standard model structure of Proposition 4.1.1.

2. [Hir03, Proposition 3.3.16] Let $f\colon M \to N$ be a morphism between fibrant objects in the group completion model structure. Then f is a fibration in the group completion model structure if and only if it is a fibration in the standard model structure.

We will also need to use the fact that some of the model categories that we consider are *proper*, a property which is particularly useful when working with homotopy pushouts and pullbacks. We recall the definition and list some useful consequences of this fact in the following remark. Proofs of the claims can be found in [Hir03, Section 13].

Remark 4.1.6. The standard model structure on simplicial commutative monoids from Proposition 4.1.1 and the one simplicial commutative R-algebras from Theorem 1.1.17 are *proper*, in that they are both left and right proper (see e.g. [SSV16, Proposition 2.1]). By definition, a model category is *left proper* if weak equivalences are preserved under pushouts along cofibrations. Dually, a model category is *right proper* is weak equivalences are preserved under pullbacks along fibrations. For a right proper model category, we define the *homotopy pullback* of the diagram

$$X \longrightarrow Z \longleftarrow Y$$

by picking functorial factorizations in the given model category and forming the usual pullback:

$$\begin{tikzcd} & & Y \arrow[d, tail, "\simeq"] \\ & X' \times_Z Y' \arrow[r] \arrow[d] & Y' \arrow[d, two heads] \\ X \arrow[r, tail, "\simeq"] & X' \arrow[r, two heads] & Z. \end{tikzcd}$$

Dually, for a left proper model category, one forms the *homotopy pushout* of

$$Y \longleftarrow X \longrightarrow Z$$

by replacing the morphisms out of X with cofibrations. Up to natural weak equivalence, both constructions are independent of the chosen factorization, and only one of the morphisms need to be replaced by a (co)fibration, in the sense that the usual pullback of

$$X \twoheadrightarrow Z \longleftarrow Y$$

represents the homotopy pullback if $X \to Z$ is a fibration. The dual statement holds for homotopy pushouts. As the name suggests, these constructions are homotopy invariant, in the sense that a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\simeq"'] & Z \arrow[d, "\simeq"] & Y \arrow[l] \arrow[d, "\simeq"] \\ X' \arrow[r] & Z' & Y' \arrow[l] \end{tikzcd}$$

in which all vertical morphisms are weak equivalences induces a weak equivalence on homotopy pullbacks. The dual statement holds for homotopy pushouts. We call a square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

a *homotopy pullback square* if the morphism from X to the homotopy pullback of $Y \to Z \leftarrow W$ is a weak equivalence, and dually for homotopy pushouts.

In analogy with [RSS15, Construction 3.11], we introduce the replete bar construction of a simplicial commutative monoid:

Definition 4.1.7. Let M be a simplicial commutative monoid, and pick a fibrant replacement

$$M \overset{\gamma_M}{\rightarrowtail} M^{\mathrm{gp}} \twoheadrightarrow *$$

of M in the group completion model structure of Proposition 4.1.2. Define the *replete bar construction* $B^{\mathrm{rep}}(M)$ of M by the right-hand pullback square

$$\begin{tikzcd} B^{\mathrm{cy}}(M) \arrow[r, dashed, "\rho_M"] \arrow[d] & B^{\mathrm{rep}}(M) \arrow[r, two heads] \arrow[d] & B^{\mathrm{cy}}(M^{\mathrm{gp}}) \arrow[d] \\ M \arrow[r, tail, "\simeq"'] & M' \arrow[r, two heads] & M^{\mathrm{gp}}. \end{tikzcd}$$

For a simplicial commutative monoid M, the cyclic bar construction $B^{\mathrm{cy}}(M)$ is the diagonal of the bisimplicial commutative monoid obtained by applying the cyclic bar construction to M degreewise. Here the lower map is a functorial factorization of γ_M in the standard model structure of Proposition 4.1.1, the outer vertical arrows are the degreewise multiplication maps, and the *repletion map* ρ_M is induced by the universal property of the pullback along γ_M and $B^{\mathrm{cy}}(\gamma_M)$. In light of Remark 4.1.6, the definition of $B^{\mathrm{rep}}(M)$ is a functorial model for the homotopy pullback of

$$M \xrightarrow{\gamma_M} M^{\mathrm{gp}} \longleftarrow B^{\mathrm{cy}}(M^{\mathrm{gp}})$$

in the standard model structure.

Remark 4.1.8. If M is a discrete simplicial commutative monoid, we recover (up to weak equivalence) the replete bar construction of Definition 3.2.4, which was given by the right-hand pullback square in the diagram

$$\begin{tikzcd} B^{\mathrm{cy}}M \arrow[r, dashed, "\rho_M"] \arrow[d] & B^{\mathrm{rep}}M \arrow[r, two heads] \arrow[d] & B^{\mathrm{cy}}M^{\mathrm{gp}} \arrow[d] \\ M \arrow[r, equal] & M \arrow[r, two heads, "\gamma_M"'] & M^{\mathrm{gp}}. \end{tikzcd}$$

By the discussion after Proposition 4.1.2, we can use the usual group completion of Example 2.1.7 to model M^{gp}, and the morphism $\gamma_M\colon M \to M^{\mathrm{gp}}$ is a Kan fibration of simplicial sets, since both M and M^{gp} are discrete.

For the usual replete bar construction of Definition 3.2.4, it is clear that $B^{\mathrm{cy}}(M) \cong B^{\mathrm{rep}}(M)$ if M is a group. We record the following simplicial analogue of that fact:

Lemma 4.1.9. *Let M be a grouplike simplicial commutative monoid. Then the repletion map*

$$\rho_M\colon B^{\mathrm{cy}}(M) \to B^{\mathrm{rep}}(M)$$

is a weak equivalence in the standard model structure of Proposition 4.1.1.

Proof. Since M is grouplike, the group completion map $\gamma_M\colon M \to M^{\mathrm{gp}}$ is a weak equivalence in the standard model structure of Proposition 4.1.1 by Remark 4.1.4. As γ_M is by definition an acyclic cofibration in the group completion model structure of Proposition 4.1.2 whose cofibrations coincide with those of the standard model structure, we conclude that γ_M is an acyclic cofibration in the standard model structure. Hence every horizontal morphism in the defining square

$$\begin{tikzcd}
B^{\mathrm{cy}}(M) \arrow[r] \arrow[d] & B^{\mathrm{rep}}(M) \arrow[r] \arrow[d] & B^{\mathrm{cy}}(M^{\mathrm{gp}}) \arrow[d] \\
M \arrow[r, tail, "\simeq"] & M' \arrow[r, two heads] & M^{\mathrm{gp}}
\end{tikzcd}$$

of $B^{\mathrm{rep}}(M)$ is a weak equivalence: by the two–out–of–three property, $M' \to M^{\mathrm{gp}}$ is a weak equivalence. As acyclic fibrations are stable under pullbacks, the morphism $B^{\mathrm{rep}}(M) \to B^{\mathrm{cy}}(M^{\mathrm{gp}})$ is a weak equivalence. Since $\gamma_M\colon M \to M^{\mathrm{gp}}$ is an acyclic cofibration, so is $B^{\mathrm{cy}}(\gamma_M)$: this follows from the fact that the category of simplicial commutative monoids with its standard model structure is *simplicial* (see e.g. [SSV16, Proposition 2.1]), and in particular the tensor $B^{\mathrm{cy}}(-) = S^1 \otimes -$ is a left Quillen functor. By the two–out–of–three property again, this implies that $B^{\mathrm{cy}}(M) \to B^{\mathrm{rep}}(M)$ is a weak equivalence. ☐

Define a *simplicial pre-log R-algebra* (A, M, α) to be a simplicial object in the category of pre-log R-algebras; that is, a simplicial commutative R-algebra A with a morphism $\alpha\colon M \to (A, \cdot)$ for a simplicial commutative monoid M. We now introduce the model structure which we will use on the category of simplicial pre-log R-algebras, $\mathrm{s\,PreLog}_R$:

Proposition 4.1.10. [SSV16, Proposition 3.3] *There is a model structure on the category of simplicial pre-log R-algebras, $\mathrm{s\,PreLog}_R$, where a morphism $(f, f^\flat)\colon (A, M) \to (B, N)$ of pre-log rings (cf. Definition 2.2.4) is a weak equivalence or fibration precisely when both f and $f^\flat$ is a weak equivalence or fibration of simplicial commutative R-algebras or simplicial commutative monoids (in the standard model structure). The morphism $(f, f^\flat)$ is a cofibration if $f^\flat$ is a cofibration of simplicial commutative monoids and the morphism*

$$R[N] \boxtimes_{R[M]} A \to B$$

is a cofibration of simplicial commutative R-algebras.

Recall from Definition 1.1.3 that $\boxtimes$ denotes the degreewise tensor product of simplicial commutative R-algebras. Since $R[-]$ is a left Quillen functor, we have that a cofibration $(f, f^\flat)\colon (A, M, \alpha) \to (B, N, \beta)$ of simplicial pre-log R-algebras gives a cofibration $R[f^\flat]\colon R[M] \to R[N]$ of simplicial commutative R-algebras. Moreover, in the diagram

$$\begin{tikzcd}
R[M] \arrow[r, "\bar{\alpha}"] \arrow[d, tail, "{R[f^\flat]}"'] & A \arrow[d, tail] \arrow[ddr, bend left, "f"] & \\
R[N] \arrow[r] \arrow[drr, bend right, "\bar{\beta}"'] & R[N] \boxtimes_{R[M]} A \arrow[dr, tail, dashed] & \\
& & B
\end{tikzcd}$$

we see that $f\colon A \to B$ is a cofibration of simplicial commutative R-algebras. Here we denote by e.g. $\bar{\alpha}$ the morphism adjoint to $\alpha\colon M \to (A, \cdot)$.

As a final preliminary fact, we will recall the Bousfield-Friedlander theorem. The reader may recall that this depends on the somewhat technical π_*-Kan condition; however, we will only apply the result in special cases where it is well-known that this condition holds.

Theorem 4.1.11. [BF74, Theorem B.4] *Consider a square*

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & X \arrow[d] \\
W \arrow[r] & Y
\end{tikzcd}$$

of bisimplicial sets satisfying the following conditions:

- *the square*

$$\begin{tikzcd}
V_{q,\bullet} \arrow[r] \arrow[d] & X_{q,\bullet} \arrow[d] \\
W_{q,\bullet} \arrow[r] & Y_{q,\bullet}
\end{tikzcd}$$

  is a homotopy pullback for each $q \geq 0$;

- *the bisimplicial sets X and Y satisfy the π_*-Kan condition;*

- *the map $X \to Y$ is a Kan fibration on vertical path components; that is, the morphism*

$$([p] \mapsto \pi_0(X_{\bullet,p})) \to ([p] \mapsto \pi_0(Y_{\bullet,p}))$$

  is a Kan fibration of simplicial sets.

Then the square of diagonals

$$\begin{tikzcd}
\mathrm{d}V \arrow[r] \arrow[d] & \mathrm{d}X \arrow[d] \\
\mathrm{d}W \arrow[r] & \mathrm{d}Y
\end{tikzcd}$$

is a homotopy pullback square.

Remark 4.1.12. If M is a grouplike simplicial commutative monoid, then it is well-known that the π_*-Kan condition holds for M. This is checked in for instance [AJM02, Corollary 6.15]. While we first discussed virtual surjectivity for discrete commutative monoids in Section 2.2.3, it is in this simplicial setting that we see why this is a good definition: define a morphism $\epsilon\colon N \to M$ of commutative simplicial monoids to be *virtually surjective* if the morphism

$$\pi_0(N^{\mathrm{gp}}) \to \pi_0(M^{\mathrm{gp}})$$

is surjective. The following lemma allows us to apply the Bousfield-Friedlander theorem in the cases that we are interested in:

Lemma 4.1.13. *Let $\epsilon\colon N \to M$ be a virtually surjective morphism of simplicial commutative monoids. Then the morphism*

$$B(\epsilon^{\mathrm{gp}})\colon B(N^{\mathrm{gp}}) \to B(M^{\mathrm{gp}})$$

of bar constructions is a Kan fibration on vertical path components.

Proof. The induced morphism on vertical path components is in each degree given by

$$\pi_0(N^{\mathrm{gp}})^{\times p} \to \pi_0(M^{\mathrm{gp}})^{\times p},$$

which is a surjection as ϵ was assumed to be virtually surjective. We conclude that $B(\epsilon^{\mathrm{gp}})$ is a degreewise surjection between simplicial abelian groups, which is always a Kan fibration by e.g. [Sta, Lemma 14.31.7, Tag 08P0]. □

4.1.2 Repletion for simplicial commutative monoids

In this section we consider a generalization of the notion of repletion for commutative monoids discussed in Section 2.2.3 to simplicial commutative monoids. We then prove a technical result relating this notion of repletion to the replete bar construction from Definition 4.1.7, which will prove to be essential in the next section.

Definition 4.1.14. Let $\epsilon\colon N \to M$ be a morphism of simplicial commutative monoids. The *repletion* N^{rep} of N over M is defined by a functorial factorization

$$N \underset{\rho_N}{\overset{\simeq}{\rightarrowtail}} N^{\mathrm{rep}} \twoheadrightarrow M$$

in the group completion model structure of Proposition 4.1.2.

As we have defined the group completion of a simplicial commutative monoid to be a fibrant replacement in the group completion model structure, the definition above should be thought of as a "relative group completion" of N over M.

The following is an analogue of [RSS15, Proposition 3.15].

Theorem 4.1.15. *Let M be a simplicial commutative monoid. There is a chain of weak equivalences in the standard model structure of Proposition 4.1.1*

$$B^{\mathrm{rep}}(M) \simeq B^{\mathrm{cy}}(M)^{\mathrm{rep}}$$

relating the replete bar construction of Definition 4.1.7 to the repletion of $B^{\mathrm{cy}}(M)$ over M. This chain is under $B^{\mathrm{cy}}(M)$ and over M', where M' is the simplicial commutative monoid appearing in a functorial factorization

$$M \overset{\simeq}{\rightarrowtail} M' \longrightarrow M^{\mathrm{gp}}$$

of the group completion map in the standard model structure, see Definition 4.1.7.

We provide a proof of this result at the end of this section. The theorem is useful as it gives a model for the replete bar construction under which the repletion map is an acyclic cofibration in the group completion model structure of Proposition 4.1.2. We will mostly apply the theorem as follows: if (A, M) is a cofibrant simplicial pre-log R-algebra, we have that $R[B^{\mathrm{cy}}(M)] \to \mathrm{HH}^R(A)$ is a cofibration. As the chain of equivalences are over $R[B^{\mathrm{cy}}(M)]$, we then obtain a chain of equivalences

$$\mathrm{HH}^R(A) \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[B^{\mathrm{rep}}(M)] \simeq \mathrm{HH}^R(A) \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}].$$

Under the cofibrancy hypothesis, the right-hand side will be the definition that we give for $\mathrm{HH}^{(R,\{1\})}(A, M)$ in Definition 4.1.19. This may seem unnatural with respect to the definition of log Hochschild homology given in Definition 3.2.7. We will see, however, that it is more convenient to work with the repletion $B^{\mathrm{cy}}(M)^{\mathrm{rep}}$ in the context of relative log Hochschild homology. The theorem will also play an important role when the morphism $R[B^{\mathrm{cy}}(M)] \to \mathrm{HH}^R(A)$ is not necessarily a cofibration, as we point out in Remark 4.2.6.

The following proposition relates the notion of repletion for simplicial commutative monoids from Definition 4.1.14 to the one considered in Section 2.2.3. This is an analogue of [RSS15, Lemma 3.17]. Recall that a morphism $\epsilon\colon N \to M$ of simplicial commutative monoids is *virtually surjective* if the morphism $\pi_0(N^{\mathrm{gp}}) \to \pi_0(M^{\mathrm{gp}})$ is a surjection.

Proposition 4.1.16. *Let $\epsilon\colon N \to M$ be a virtually surjective morphism of simplicial commutative monoids. Consider the solid diagram*

$$\begin{tikzcd} N \arrow[r, "\rho_N"] \arrow[d, "\gamma_N"'] & N^{\mathrm{rep}} \arrow[r] \arrow[d, dashed] & M \arrow[d, "\gamma_M"] \\ N^{\mathrm{gp}} \arrow[r, tail, "\simeq"'] & (N^{\mathrm{gp}})' \arrow[r, two heads] & M^{\mathrm{gp}}, \end{tikzcd}$$

where the upper horizontal composite is a factorization in the group completion model structure as in Definition 4.1.14, and the lower horizontal composite is a factorization of $N^{\mathrm{gp}} \to M^{\mathrm{gp}}$ in the standard model structure of Proposition 4.1.1. Then there exists a dashed map $N^{\mathrm{rep}} \to (N^{\mathrm{gp}})'$ such that the right-hand square is a homotopy pullback with respect to the standard model structure.

Proof. Since the morphism $(N^{\mathrm{gp}})' \to M^{\mathrm{gp}}$ is a fibration in the standard model structure of fibrant objects in the group completion model structure, it follows from [Hir03, Proposition 3.3.16] that $(N^{\mathrm{gp}})' \to M^{\mathrm{gp}}$ is a fibration in the group completion model structure (see Remark 4.1.5). Hence we obtain a morphism $N^{\mathrm{rep}} \to (N^{\mathrm{gp}})'$ by the lifting properties of the group completion model structure:

$$\begin{tikzcd} N \arrow[r] \arrow[d, tail, "\simeq"] & N^{\mathrm{gp}} \arrow[r] & (N^{\mathrm{gp}})' \arrow[d] \\ N^{\mathrm{rep}} \arrow[r] \arrow[urr, dashed] & M \arrow[r] & M^{\mathrm{gp}} \end{tikzcd}$$

Notice that $N^{\mathrm{rep}} \to (N^{\mathrm{gp}})'$ is a weak equivalence in the group completion model structure, as this is the case for both of the upper horizontal arrows. Consider now the diagram

$$\begin{tikzcd} N^{\mathrm{rep}} \arrow[dr, dashed] \arrow[drr, bend left, "\simeq"] \arrow[ddr, bend right, two heads] & & \\ & N' \arrow[r] \arrow[d, two heads] & (N^{\mathrm{gp}})' \arrow[d, two heads] \\ & M \arrow[r, "\gamma_M"'] & M^{\mathrm{gp}} \end{tikzcd} \tag{4}$$

where the square is a pullback and the arrows are decorated with respect to the group completion model structure. Since $(N^{\mathrm{gp}})' \to M^{\mathrm{gp}}$ is a fibration, this represents the homotopy pullback. If we are able to show that $N^{\mathrm{rep}} \to N'$ is a weak equivalence in the group completion model structure, then it is also one in the standard model structure by [Hir03, Proposition 3.3.5] applied to the lower triangle in the diagram (see Remark 4.1.5). In order to prove this, we show that $N' \to (N^{\mathrm{gp}})'$ is a weak equivalence in the group completion model structure. For this we show that the square

$$\begin{tikzcd} B(N') \arrow[r] \arrow[d] & B((N^{\mathrm{gp}})') \arrow[d] \\ B(M) \arrow[r, "\simeq"] & B(M^{\mathrm{gp}}) \end{tikzcd}$$

is a homotopy pullback. As the morphism $B(M) \to B(M^{\mathrm{gp}})$ is a weak equivalence by the definition of weak equivalences in the group completion model structure (Proposition 4.1.2), it will follow that $B(N') \to B((N^{\mathrm{gp}})')$ is also, hence $N' \to (N^{\mathrm{gp}})'$ will be a weak equivalence in the group completion model structure and the result will follow.

In order to show that the square is a homotopy pullback, we apply the Bousfield-Friedlander Theorem 4.1.11. As noted in Remark 4.1.12, we have that $B((N^{\mathrm{gp}})')$ and $B(M^{\mathrm{gp}})$ satisfy the π_*-Kan condition since both simplicial commutative monoids involved are grouplike. Since the square in the diagram (4) is a homotopy pullback, we have that the square of the bar constructions is pointwise a homotopy pullback. Finally, the induced morphism on vertical path components is a Kan fibration by Lemma 4.1.13, since $N \to M$ is virtually surjective: this implies that the composite

$$\pi_0(N^{\mathrm{gp}}) \to \pi_0((N^{\mathrm{gp}})') \to \pi_0(M^{\mathrm{gp}})$$

is surjective, and consequenly the latter map is surjective, so that $B((N^{\mathrm{gp}})') \to B(M^{\mathrm{gp}})$ is a Kan fibration on vertical path components. $\square$

As a corollary we obtain that the definition of repletion given in Definition 4.1.14 coincides with the one in Section 2.2.3 for constant simplicial commutative monoids up to weak equivalence: that is, if $\epsilon\colon N \to M$ is a virtually surjective morphism of simplicial commutative monoids, then the square

$$\begin{tikzcd} N^{\mathrm{rep}} \arrow[r] \arrow[d] & N^{\mathrm{gp}} \arrow[d] \\ M \arrow[r, "\gamma_M"'] & M^{\mathrm{gp}} \end{tikzcd}$$

is a homotopy pullback with respect to the standard model structure. Before we prove Theorem 4.1.15, we establish the following analogue of [RSS15, Lemma 3.19]:

Lemma 4.1.17. *Let M be a simplicial commutative monoid. There is a chain of weak equivalences*

$$B^{\mathrm{cy}}(M^{\mathrm{gp}}) \simeq B^{\mathrm{cy}}(M)^{\mathrm{gp}}$$

as objects under $B^{\mathrm{cy}}(M)$ and over M^{gp}.

Proof. Consider the diagram

$$\begin{tikzcd}
B^{\mathrm{cy}}M \arrow[r, tail, "\gamma_{B^{\mathrm{cy}}M}", "\simeq"'] \arrow[dd, "B^{\mathrm{cy}}(\gamma_M)"'] & B^{\mathrm{cy}}(M)^{\mathrm{gp}} \arrow[d, tail, "\simeq"] \\
 & (B^{\mathrm{cy}}M)' \arrow[d, two heads] \\
B^{\mathrm{cy}}(M^{\mathrm{gp}}) \arrow[r] & M^{\mathrm{gp}},
\end{tikzcd}$$

where the arrows are decorated with respect to the group completion model structure, and the right-hand vertical composite is a factorization of the map $B^{\mathrm{cy}}(M)^{\mathrm{gp}} \to M^{\mathrm{gp}}$ in this model structure. By [SSV16, Proposition 2.6], the category of simplicial commutative monoids with the group completion model structure is *simplicial*, and the cyclic bar construction is merely the tensor with the simplicial circle, as alluded to in Remark 3.1.2. As a consequence, the acyclic cofibration γ_M gives an acyclic cofibration $B^{\mathrm{cy}}(\gamma_M) = S^1 \otimes \gamma_M$ in the group completion model structure, as $S^1 \otimes -$ is a left Quillen functor. Consequently there is a lift $B^{\mathrm{cy}}(M^{\mathrm{gp}}) \to (B^{\mathrm{cy}}M)'$, which is a weak equivalence in the group completion model structure by the two–out–of–three property. We will show that $B^{\mathrm{cy}}(M^{\mathrm{gp}})$ is grouplike, which will imply that $B^{\mathrm{cy}}(M^{\mathrm{gp}}) \to (B^{\mathrm{cy}}M)'$ is a weak equivalence in the standard model structure, it being a weak equivalence between fibrant objects in the group completion model structure (see Remark 4.1.4). We can describe $\pi_0(M^{\mathrm{gp}})$ as M_0^{gp} modulo the equivalence relation

$$(d_1, d_0)\colon M_1^{\mathrm{gp}} \to M_0^{\mathrm{gp}} \times M_0^{\mathrm{gp}}.$$

We then find that the 0-simplices of M^{gp} which are equivalent in $\pi_0(M^{\mathrm{gp}})$ are also equivalent in $\pi_0(B^{\mathrm{cy}}(M^{\mathrm{gp}}))$, as this can be described as $B^{\mathrm{cy}}(M^{\mathrm{gp}})_0 = M_0^{\mathrm{gp}}$ modulo the equivalence relation

$$B^{\mathrm{cy}}(M^{\mathrm{gp}})_1 = M_1^{\mathrm{gp}} \times M_1^{\mathrm{gp}} \to M_0^{\mathrm{gp}} \times M_0^{\mathrm{gp}}$$

induced by the face maps of $B^{\mathrm{cy}}(M^{\mathrm{gp}})$. Hence $\pi_0(B^{\mathrm{cy}}(M^{\mathrm{gp}}))$ is a group if $\pi_0(M^{\mathrm{gp}})$ is. This concludes the proof. □

Finally, we prove Theorem 4.1.15:

Proof of Theorem 4.1.15. By Lemma 4.1.17, the replete bar construction of Definition 4.1.7 fits in the homotopy pullback of

$$M \longrightarrow M^{\mathrm{gp}} \longleftarrow B^{\mathrm{cy}}(M)^{\mathrm{gp}}.$$

By the remark following the proof of Proposition 4.1.16, we are done if the morphism $B^{\mathrm{cy}}(M) \to M$ is virtually surjective: the morphism $B^{\mathrm{cy}}(M)^{\mathrm{gp}} \to M^{\mathrm{gp}}$ has a section, namely the inclusion of the 0-simplices $M^{\mathrm{gp}} = B_0^{\mathrm{cy}}(M)^{\mathrm{gp}} \to B^{\mathrm{cy}}(M)^{\mathrm{gp}}$, which concludes the proof. □

4.1.3 The construction of relative log Hochschild homology

Recall from Definition 3.2.7 that we defined $\mathrm{HH}^R(A^{\mathrm{cof}}, M)$ by choosing a cofibrant replacement of A in the category of simplicial commutative $R[M]$-algebras. In order to define $\mathrm{HH}^{(R,N)}(A^{\mathrm{cof}}, M^{\mathrm{cof}})$, we need to choose a cofibrant replacement of the pre-log ring (A, M) over (R, N) in the model structure in simplicial pre-log R-algebras from Proposition 4.1.10. The following construction is motivated by the construction of the relative log Kähler differentials $\Omega^1_{(A,M)/(R,N)}$ in Example 2.3.11:

Construction 4.1.18. Let $(f, f^\flat)\colon (R, N, \beta) \to (A, M, \alpha)$ be a morphism of discrete pre-log rings, and factor $(f, f^\flat)$ as

$$(R, N, \beta) \xrightarrow{(f^{\mathrm{cof}}, (f^\flat)^{\mathrm{cof}})} (A^{\mathrm{cof}}, M^{\mathrm{cof}}, \alpha^{\mathrm{cof}}) \xrightarrow{\simeq} (A, M, \alpha)$$

in the model structure on $\mathrm{s}\,\mathrm{PreLog}_R$ from Proposition 4.1.10. Consider the coproduct

$$\begin{tikzcd} B^{\mathrm{cy}}(N) \arrow[r, tail] \arrow[d, tail] & B^{\mathrm{cy}}(N)^{\mathrm{rep}} \arrow[d, tail] \\ B^{\mathrm{cy}}(M^{\mathrm{cof}}) \arrow[r, tail] & B^{\mathrm{cy}}(M^{\mathrm{cof}}) \sqcup_{B^{\mathrm{cy}}(N)} B^{\mathrm{cy}}(N)^{\mathrm{rep}} \end{tikzcd}$$

of simplicial commutative monoids. Recall that $(R, N) \to (A^{\mathrm{cof}}, M^{\mathrm{cof}})$ being a cofibration of simplicial pre-log algebras implies (by definition) that $N \to M^{\mathrm{cof}}$ is a cofibration of simplicial commutative monoids, and so the morphism $B^{\mathrm{cy}}(N) \to B^{\mathrm{cy}}(M^{\mathrm{cof}})$ is also a cofibration. The morphism $N \to M^{\mathrm{cof}}$ induces a morphism

$$B^{\mathrm{cy}}(M^{\mathrm{cof}}) \sqcup_{B^{\mathrm{cy}}(N)} B^{\mathrm{cy}}(N)^{\mathrm{rep}} \to B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}},$$

and by applying $R[-]$ we obtain a morphism

$$\psi\colon R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \to R[B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}]$$

of simplicial commutative R-algebras. We now construct a morphism

$$\phi\colon R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \to \mathrm{HH}^R(A^{\mathrm{cof}}).$$

Recall that the repletion $B^{\mathrm{cy}}(N)^{\mathrm{rep}}$ is by definition given by a functorial factorization

$$B^{\mathrm{cy}}(N) \xrightarrow{\simeq} B^{\mathrm{cy}}(N)^{\mathrm{rep}} \twoheadrightarrow N$$

of the multiplication map $B^{\mathrm{cy}}(N) \to N$ in the group completion model structure (Proposition 4.1.2). We consider the composite

$$R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \longrightarrow R[N] \xrightarrow{R[(f^\flat)^{\mathrm{cof}}]} R[M^{\mathrm{cof}}] \xrightarrow{\bar{\alpha}^{\mathrm{cof}}} A^{\mathrm{cof}} \longrightarrow \mathrm{HH}^R(A^{\mathrm{cof}}),$$

where $\bar{\alpha}^{\mathrm{cof}}$ is the morphism induced by $\alpha^{\mathrm{cof}}\colon M^{\mathrm{cof}} \to (A^{\mathrm{cof}}, \cdot)$ and the last map is the inclusion of the zero-simplices. As the composite

$$R[B^{\mathrm{cy}}(N)] \to R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \to R[N]$$

is merely the multiplication map, and the composite

$$R[N] \to R[M^{\mathrm{cof}}] \to A^{\mathrm{cof}}$$

factors through R (since $(f^{\mathrm{cof}}, (f^\flat)^{\mathrm{cof}})$ is a morphism of simplicial pre-log R-algebras), we find that this composite coincides with

$$R[B^{\mathrm{cy}}(N)] \longrightarrow R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \longrightarrow \mathrm{HH}^R(A^{\mathrm{cof}}).$$

Hence the universal property of the coproduct gives the desired morphism

$$\phi\colon R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \to \mathrm{HH}^R(A^{\mathrm{cof}})$$

We now give our proposed definition of $\mathrm{HH}^{(R,N)}(A^{\mathrm{cof}}, M^{\mathrm{cof}})$:

Definition 4.1.19. With the notation from Construction 4.1.18, factor the morphism

$$B^{\mathrm{cy}}(M^{\mathrm{cof}}) \sqcup_{B^{\mathrm{cy}}(N)} B^{\mathrm{rep}}(N) \longrightarrow B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}$$

as a cofibration followed by an acyclic fibration

$$B^{\mathrm{cy}}(M^{\mathrm{cof}}) \sqcup_{B^{\mathrm{cy}}(N)} B^{\mathrm{rep}}(N) \rightarrowtail \tilde{B}^{\mathrm{rep}}(M^{\mathrm{cof}}) \xrightarrow{\simeq} B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}$$

in the standard model structure of Proposition 4.1.1. Applying $R[-]$ gives a factorization of the morphism

$$\psi\colon R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \to R[B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}]$$

of the form

$$R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \rightarrowtail R[\tilde{B}^{\mathrm{rep}}(M^{\mathrm{cof}})] \xrightarrow{\simeq} R[B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}]$$

in the category of simplicial commutative R-algebras, since $R[-]$ is a left Quillen functor and preserves all weak equivalences. Define $\mathrm{HH}^{(R,N)}(A^{\mathrm{cof}}, M^{\mathrm{cof}})$ by the pushout square

$$\begin{tikzcd}
R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \arrow[r, tail] \arrow[d, "\phi"'] & R[\tilde{B}^{\mathrm{rep}}(M^{\mathrm{cof}})] \arrow[d] \\
\mathrm{HH}^{R}(A^{\mathrm{cof}}) \arrow[r] & \mathrm{HH}^{(R,N)}(A^{\mathrm{cof}}, M^{\mathrm{cof}}).
\end{tikzcd}$$

Remark 4.1.20. We notice that if $N = \{1\}$, there is no need to factor the morphism

$$\psi\colon R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \to R[B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}],$$

as we then obtain a weak equivalence of (homotopy) pushout data

$$\begin{tikzcd}
\mathrm{HH}^{R}(A^{\mathrm{cof}}) \arrow[d, equal] & R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \arrow[l, tail] \arrow[r, tail] \arrow[d, equal] & R[\tilde{B}^{\mathrm{rep}}(M^{\mathrm{cof}})] \arrow[d, "\simeq"] \\
\mathrm{HH}^{R}(A^{\mathrm{cof}}) & R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \arrow[l, tail] \arrow[r, tail] & R[B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}],
\end{tikzcd}$$

which induces a weak equivalence on (homotopy) pushouts (see Remark 4.1.6). We also notice that in this situation, since $(R, N) \to (A^{\mathrm{cof}}, M^{\mathrm{cof}})$ is a cofibration, we have (by definition) that $R[M^{\mathrm{cof}}] \to A^{\mathrm{cof}}$ is a cofibration. Then $R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \to \mathrm{HH}^{R}(A^{\mathrm{cof}})$ is a cofibration, and Theorem 4.1.15 is applicable: the chain of weak equivalences $R[B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}] \simeq R[B^{\mathrm{rep}}(M^{\mathrm{cof}})]$ are under $R[B^{\mathrm{cy}}(M^{\mathrm{cof}})]$, and so we have a chain of weak equivalences

$$\mathrm{HH}^{R}(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}(M^{\mathrm{cof}})]} R[B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}] \simeq \mathrm{HH}^{R}(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}(M^{\mathrm{cof}})]} R[B^{\mathrm{rep}}(M^{\mathrm{cof}})],$$

cf. Remark 4.1.6. We see that, in the case of $N = \{1\}$, we can use the replete bar construction of Definition 4.1.7 to model $\mathrm{HH}^{(R,\{1\})}(A^{\mathrm{cof}}, M^{\mathrm{cof}})$, and this is the perspective used for log topological Hochschild homology in [RSS15]. The reason why we rather choose to work with the repletion $B^{\mathrm{cy}}(M)^{\mathrm{rep}}$ is that it admits a functorial morphism to M, which we used to define relative log Hochschild homology. While, for discrete monoids N, it is true that the replete bar construction can be modeled by the (homotopy) pullback of

$$N \xrightarrow{\gamma_N} N^{\mathrm{gp}} \longleftarrow B^{\mathrm{cy}}(N),$$

where γ_N denotes the usual group completion map of discrete monoids, we no longer have a direct map $B^{\mathrm{rep}}(N) \to N$ when N is not discrete. This will be important when we study invariance under logification, as we will have to use a homotopy invariant version of the logification functor.

One potential drawback of our construction is that, as opposed to $B^{\mathrm{cy}}(M)^{\mathrm{rep}}$, the replete bar construction $B^{\mathrm{rep}}(M)$ has a cyclic action [RSS15, After Proposition 3.15].

It is not immediately clear that $\mathrm{HH}^{(R,\{1\})}(A^{\mathrm{cof}}, M^{\mathrm{cof}})$ coincides with the usual log Hochschild homology $\mathrm{HH}^R(A^{\mathrm{cof}}, M)$ from Definition 3.2.7. Our definition of $\mathrm{HH}^{(R,\{1\})}(A^{\mathrm{cof}}, M^{\mathrm{cof}})$ is now closer to the definition of logarithmic topological Hochschild homology in [RSS15] for cofibrant pre-log ring spectra ([RSS15, Definition 4.5]). We check that these two constructions coincide in a manner analogous to [RSS15, Section 5.1]:

Proposition 4.1.21. *There is a chain of weak equivalences*

$$\mathrm{HH}^{(R,\{1\})}(A^{\mathrm{cof}}, M^{\mathrm{cof}}) \simeq \mathrm{HH}^R(\tilde{A}^{\mathrm{cof}}, M),$$

where $(A^{\mathrm{cof}}, M^{\mathrm{cof}})$ is a cofibrant replacement of the discrete pre-log R-algebra (A, M) in the model structure described in Proposition 4.1.10, while $\tilde{A}^{\mathrm{cof}}$ is a cofibrant replacement of A in the category of simplicial commutative $R[M]$-algebras.

In order to prove the result above we need to know that cofibrant replacements of commutative monoids behave well with respect to the cyclic and replete bar constructions:

Lemma 4.1.22. *Let M be a commutative monoid and let $M^{\mathrm{cof}} \to M$ be a cofibrant replacement of M in the standard model structure of Proposition 4.1.1. Then $R[B^{\mathrm{cy}}(M^{\mathrm{cof}})]$ is weakly equivalent to $R[B^{\mathrm{cy}}(M)]$, and $R[B^{\mathrm{rep}}(M^{\mathrm{cof}})]$ is weakly equivalent to $R[B^{\mathrm{rep}}(M)]$, where $B^{\mathrm{rep}}(M)$ denotes the usual replete bar construction of a commutative monoid from Definition 3.2.4.*

Proof. As we noted after Proposition 4.1.1, the cofibrant replacement map $M^{\mathrm{cof}} \to M$ induces a weak equivalence $R[M^{\mathrm{cof}}] \to R[M]$ of simplicial commutative R-algebras. As both simplicial algebras involved are degreewise R-flat, Lemma 3.1.5 gives a weak equivalence

$$R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] = \mathrm{HH}^R(R[M^{\mathrm{cof}}]) \to R[B^{\mathrm{cy}}(M)] = \mathrm{HH}^R(R[M]),$$

which proves the first statement.

Taking group completions and (co)fibrant replacements in the standard model structure on simplicial commutative monoids from Proposition 4.1.1, we obtain a solid diagram

$$\begin{tikzcd}
M \arrow[d, "\gamma_M"'] & M^{\mathrm{cof}} \arrow[l, two heads, "\simeq"'] \arrow[r, equal] \arrow[d, "(\gamma_M)^{\mathrm{cof}}"] & M^{\mathrm{cof}} \arrow[r, equal] \arrow[d] & M^{\mathrm{cof}} \arrow[d, "\gamma_{M^{\mathrm{cof}}}"] \\
M^{\mathrm{gp}} & (M^{\mathrm{gp}})^{\mathrm{cof}} \arrow[l, two heads, "\simeq"] \arrow[r, tail, "\simeq"'] & ((M^{\mathrm{gp}})^{\mathrm{cof}})^{\mathrm{fib}} & (M^{\mathrm{cof}})^{\mathrm{gp}}. \arrow[l, dashed]
\end{tikzcd} \tag{5}$$

The existence of the dashed arrow making the diagram commute follows from the lifting properties of the group completion model structure, since the group completion map is by definition an acyclic cofibration in this structure, and the object $((M^{\mathrm{gp}})^{\mathrm{cof}})^{\mathrm{fib}}$ is grouplike and fibrant as a simplicial set, hence fibrant in the group completion model structure:

$$\begin{tikzcd} M^{\mathrm{cof}} \arrow[r] \arrow[d, tail, "\gamma_{M^{\mathrm{cof}}}"', "\simeq"] & ((M^{\mathrm{gp}})^{\mathrm{cof}})^{\mathrm{fib}} \arrow[d, two heads] \\ (M^{\mathrm{cof}})^{\mathrm{gp}} \arrow[r] \arrow[ur, dashed] & *, \end{tikzcd}$$

where the arrows are decorated with respect to the group completion model structure. By construction, every vertical morphism in the diagram (5) are weak equivalences in the group completion model structure, and the terms on the lower row are all grouplike. As weak equivalences between fibrant objects in the group completion model structure are equivalences in the standard model structure by Remark 4.1.4, all of the horizontal morphisms in the diagram are weak equivalences in the standard model structure. We then obtain a diagram

$$\begin{tikzcd} B^{\mathrm{cy}}(M^{\mathrm{gp}}) \arrow[d] & B^{\mathrm{cy}}((M^{\mathrm{gp}})^{\mathrm{cof}}) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & B^{\mathrm{cy}}(((M^{\mathrm{gp}})^{\mathrm{cof}})^{\mathrm{fib}}) \arrow[d] & B^{\mathrm{cy}}((M^{\mathrm{cof}})^{\mathrm{gp}}) \arrow[l, "\simeq"'] \arrow[d] \\ M^{\mathrm{gp}} & (M^{\mathrm{gp}})^{\mathrm{cof}} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & ((M^{\mathrm{gp}})^{\mathrm{cof}})^{\mathrm{fib}} & (M^{\mathrm{cof}})^{\mathrm{gp}} \arrow[l, "\simeq"'] \\ M \arrow[u, two heads, "\gamma_M"] & M^{\mathrm{cof}} \arrow[l, two heads, "\simeq"] \arrow[u] \arrow[r, equal] & M^{\mathrm{cof}} \arrow[u] \arrow[r, equal] & M^{\mathrm{cof}}, \arrow[u] \end{tikzcd}$$

where the arrows are decorated with respect to the standard model structure. The morphisms in the upper row are weak equivalences since the ones in the middle row are: the morphisms in the middle row are weak equivalences after applying $B^{\mathrm{cy}}_q(-)$. Hence the morphisms in the upper row come from pointwise weak equivalences of bisimplicial sets, and so the induced morphisms on diagonals are a weak equivalences by e.g. [GJ09, Chapter IV, Proposition 1.7]. Since the usual group completion map γ_M is a Kan fibration, the left-hand pullback data represents the homotopy pullback $B^{\mathrm{rep}}(M)$. The replete bar construction $B^{\mathrm{rep}}(M^{\mathrm{cof}})$ of Definition 4.1.7 is by definition the homotopy pullback of the right-hand pullback data, and so we obtain the desired chain of equivalences. □

We are now in position to prove Proposition 4.1.21:

Proof of Proposition 4.1.21. We obtain a weak equivalence $A^{\mathrm{cof}} \to \tilde{A}^{\mathrm{cof}}$ as a lift in the diagram

$$\begin{tikzcd} R[M^{\mathrm{cof}}] \arrow[r] \arrow[d, tail] & R[M] \arrow[r] & \tilde{A}^{\mathrm{cof}} \arrow[d, two heads, "\simeq"] \\ A^{\mathrm{cof}} \arrow[rr, two heads, "\simeq"'] \arrow[urr, dashed, "\simeq"'] & & A. \end{tikzcd}$$

By Lemma 3.1.6 we have that $\mathrm{HH}^R(A^{\mathrm{cof}}) \to \mathrm{HH}^R(\tilde{A}^{\mathrm{cof}})$ is a weak equivalence, and consequently there is a weak equivalence

$$\mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}M^{\mathrm{cof}}]} R[B^{\mathrm{rep}}M^{\mathrm{cof}}] \simeq \mathrm{HH}^R(\tilde{A}^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}M]} R[B^{\mathrm{rep}}M]$$

by Lemma 4.1.22. By definition, $\mathrm{HH}^{(R,\{1\})}(A^{\mathrm{cof}}, M^{\mathrm{cof}})$ is the coproduct

$$\mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}(M^{\mathrm{cof}})]} R[B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}],$$

and since $R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \to \mathrm{HH}^R(A^{\mathrm{cof}})$ is a cofibration, we have a chain of equivalences

$$\mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}(M^{\mathrm{cof}})]} R[B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}] \simeq \mathrm{HH}^R(A^{\mathrm{cof}}) \boxtimes_{R[B^{\mathrm{cy}}(M^{\mathrm{cof}})]} R[B^{\mathrm{rep}}(M^{\mathrm{cof}})]$$

by Lemma 4.1.15. This concludes the proof. □

4.1.4 Relation to relative log Kähler differentials

The aim of this section is to prove the following analogue of Proposition 3.2.21:

Theorem 4.1.23. *Let $(f, f^\flat)\colon (R, N, \beta) \to (A, M, \alpha)$ be a morphism of discrete pre-log rings, and assume that N is cofibrant as a discrete simplicial commutative monoid in the model structure of Proposition 4.1.1. Then there is an isomorphism*

$$\pi_1 \mathrm{HH}^{(R,N)}(A^{\mathrm{cof}}, M^{\mathrm{cof}}) \cong \Omega^1_{(A,M)/(R,N)}$$

relating the relative log Hochschild homology from Definition 4.1.19 to the relative log Kähler differentials of Example 2.3.11.

We add the cofibrancy hypothesis on N so that M^{cof} is a cofibrant simplicial commutative monoid, making Proposition 4.1.21 applicable. It seems likely that this hypothesis can be dropped by an argument similar to that of Remark 3.2.22, as we are only interested in the homotopy groups of $R[B^{\mathrm{rep}}(M^{\mathrm{cof}})]$ in degrees 0 and 1. We choose, however, to make this simplification as N is free (and hence cofibrant, as noted in [Rog09, Proof of Lemma 4.8]) in the examples that we have in mind.

Proof of Theorem 4.1.23. We apply the spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^{\pi_*(R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}])}(\pi_* \mathrm{HH}^R(A^{\mathrm{cof}}), \pi_* R[\tilde{B}^{\mathrm{rep}}(M^{\mathrm{cof}})])_q \implies \pi_{p+q} \mathrm{HH}^{(R,N)}(A^{\mathrm{cof}}, M^{\mathrm{cof}})$$

of Theorem 3.2.10. We first notice that, since $N \to M^{\mathrm{cof}}$ is a cofibration, there is a chain of weak equivalences

$$R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \simeq R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{rep}}(N)] = \mathrm{HH}^R(R[M^{\mathrm{cof}}], N)$$

by Theorem 4.1.15. Recall from Definition 4.1.19 that we have chosen a factorization

$$R[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}] \rightarrowtail R[\tilde{B}^{\mathrm{rep}}(M^{\mathrm{cof}})] \xrightarrow{\simeq} R[B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}].$$

Since the discrete simplicial commutative monoid N is cofibrant, $M^{\mathrm{cof}} \to M$ is a cofibrant replacement of M. Hence Proposition 4.1.21 is applicable, and

$$\pi_* R[\tilde{B}^{\mathrm{rep}}(M^{\mathrm{cof}})] \cong \pi_* R[B^{\mathrm{cy}}(M^{\mathrm{cof}})^{\mathrm{rep}}] \cong \pi_* R[B^{\mathrm{rep}}(M^{\mathrm{cof}})] \cong \pi_* R[B^{\mathrm{rep}}(M)],$$

where the second isomorphism follows from Theorem 4.1.15. We now notice that

$$E^2_{1,0} \cong \mathrm{Tor}_1^{\pi_0 \mathrm{HH}^R(R[M^{\mathrm{cof}}],N)}(\pi_0 \mathrm{HH}^R(A^{\mathrm{cof}}), \pi_0 R[B^{\mathrm{rep}}(M^{\mathrm{cof}})]) \cong \mathrm{Tor}_1^{R[M]}(A, R[M]) \cong 0$$

as $\pi_0 \mathrm{HH}^R(R[M^{\mathrm{cof}}], N) \cong R[M]$ by Example 3.2.9, $\pi_0 \mathrm{HH}^R(A^{\mathrm{cof}}) \cong A$ by Remark 3.1.4, and we know that $\pi_0 R[B^{\mathrm{rep}}(M)] \cong R[M]$. Similarly, $E^2_{p,0} = 0$ for all $p > 0$, so that $E_{2,0}$ does not hit $E^2_{0,1}$. We now compute

$$E^2_{0,1} \cong \mathrm{Tor}_0^{\pi_* \mathrm{HH}^R(R[M^{\mathrm{cof}}],N)}(\pi_* \mathrm{HH}^R(A^{\mathrm{cof}}), \pi_* R[B^{\mathrm{rep}}(M^{\mathrm{cof}})])_1.$$

By definition, this is the colimit of the diagram

$$\begin{array}{c} \bigoplus_{i+j+k=1} \pi_i \mathrm{HH}^R(A^{\mathrm{cof}}) \otimes_R \pi_j \mathrm{HH}^R(R[M^{\mathrm{cof}}], N) \otimes_R \pi_k R[B^{\mathrm{rep}}(M^{\mathrm{cof}})] \\ \downdownarrows \\ \bigoplus_{p+q=1} \pi_p \mathrm{HH}^R(A^{\mathrm{cof}}) \otimes_R \pi_q R[B^{\mathrm{rep}}(M^{\mathrm{cof}})]. \end{array}$$

Consider first the $(1,0,0)$-summand mapping to the $(1,0)$-summand. By Proposition 3.1.8,

$$\pi_1 \mathrm{HH}^R(A^{\mathrm{cof}}) \cong \Omega^1_{A/R},$$

and the resulting diagram is merely the defining coequalizer diagram for $\Omega^1_{A/R} \otimes_{R[M]} R[M] \cong \Omega^1_{A/R}$. We now consider the $(0,0,1)$-summand mapping to the $(0,1)$-summand. As

$$\pi_1 R[B^{\mathrm{rep}}(M)] \cong R[M] \otimes_{\mathbb{Z}} M^{\mathrm{gp}}$$

by the discussion in the beginning of this proof and Example 3.2.6, we find that the resulting diagram is the defining coequalizer diagram for

$$A \otimes_{R[M]} (R[M] \otimes_{\mathbb{Z}} M^{\mathrm{gp}}) \cong A \otimes_{\mathbb{Z}} M^{\mathrm{gp}}.$$

Finally, we consider the $(0,1,0)$-summand mapping to both summands. The reader may want to look at Example 2.3.11 for comparison. By Proposition 3.2.21, we have that $\pi_1 \mathrm{HH}^R(R[M^{\mathrm{cof}}], N) \cong \Omega^1_{(R[M],N)/R}$. One of the morphisms is the map

$$A \otimes_R \Omega^1_{(R[M],N)/R} \otimes_R R[M] \to \Omega^1_{A/R} \otimes_R R[M]$$

sending $a \otimes \mathrm{d}m \otimes m'$ to $a\mathrm{d}\alpha(m) \otimes m'$ and $a \otimes \mathrm{d}\log n \otimes m'$ to $a\mathrm{d}\alpha(f^\flat(n)) \otimes m' = a\mathrm{d}f(\beta(n)) \otimes m' = 0$. The other is the map

$$A \otimes_R \Omega^1_{(R[M],N)/R} \otimes_R R[M] \to A \otimes_R (R[M] \otimes_{\mathbb{Z}} M^{\mathrm{gp}})$$

sending $a \otimes \mathrm{d}m \otimes m'$ to $a\alpha(m) \otimes (m' \otimes \gamma_M(m))$, while it sends $a \otimes \mathrm{d}\log n \otimes m'$ to $a \otimes (m' \otimes \gamma_M(f^\flat(n)))$. In conclusion, we find that

$$E^2_{0,1} = (\Omega^1_{A/R} \oplus (A \otimes_{\mathbb{Z}} M^{\mathrm{gp}}))/\sim,$$

where the relations are A-linearly generated by $(\alpha(m), 0) \sim (0, \alpha(m) \otimes \gamma_M(m))$ and $(0, 1 \otimes \gamma_M(f^\flat(n)) \sim (\mathrm{d}\alpha(f^\flat(n)), 0) = (\mathrm{d}f(\beta(n)), 0) = 0$. These are the relative log Kähler differentials $\Omega^1_{(A,M)/(R,N)}$ of Example 2.3.11. □

Example 4.1.24. Let $f\colon A \to B$ be a morphism of discrete valuation rings with uniformizers π_A and π_B. Write $f(\pi_A) = u\pi_B^n$ for some unit u of B and natural number n. Suppose for simplicity that $u = 1$, and consider the associated morphism of pre-log rings $(A, \langle \pi_A \rangle) \to (B, \langle \pi_B \rangle)$. In this situation, Theorem 4.1.23 is applicable, and so we have an isomorphism

$$\pi_1 \mathrm{HH}^{(A,\langle \pi_A \rangle)}(B, \langle \pi_B \rangle) \cong \Omega^1_{(B,\langle \pi_B \rangle)/(A,\langle \pi_A \rangle)}.$$

A discussion of log Kähler differentials in this context is provided in [Rog09, Example 4.32], emphasizing how log structures extend the range of formally smooth morphisms. From this perspective, Theorem 4.1.23 tells us that logarithmic Hochschild homology can detect this behavior.

4.2 Invariance under logification

The aim of this section is to prove that the construction of relative logarithmic Hochschild homology from the previous section is invariant under the logification functor of Definition 2.2.6 (more precisely, a homotopy invariant version of this functor). This is done for logarithmic topological Hochschild homology in [RSS15, Theorem 4.24], and in Section 4.2.1 we translate the relevant parts of *loc. cit.* to simplicial commutative monoids using the model structures of Propositions 4.1.1 and 4.1.2. In this setting we extend the result to the relative construction of logarithmic Hochschild homology of Definition 4.1.19 in Section

4.2.2. We prove these results for general simplicial pre-logarithmic algebras which are suitably cofibrant, as we do not find a significant simplification of the proofs in the case our input is a cofibrant replacement of a constant object in the case of relative log Hochschild homology $\mathrm{HH}^{(R,N)}(A, M)$. In Remark 4.2.4 we hint at how the argument may be simplified for usual log Hochschild homology $\mathrm{HH}^{R}(A, M)$ under the assumption that the monoid M is *integral* (cancellative).

4.2.1 Invariance for absolute log Hochschild homology

The aim of this section is to establish the following analogue of [RSS15, Theorem 4.24]:

Theorem 4.2.1. *Let (A, M, α) be a simplicial pre-log R-algebra which is cofibrant in the model structure of Proposition 4.1.10. Then there is a weak equivalence*

$$\mathrm{HH}^{(R,\{1\})}(A, M) \xrightarrow{\simeq} \mathrm{HH}^{(R,\{1\})}(A^a, M^a)$$

induced by the logification construction, where (A^a, M^a) is defined in Construction 4.2.2.

Here we have defined $\mathrm{HH}^{(R,\{1\})}(A, M)$ for cofibrant (A, M) in analogy with the definition for discrete pre-log algebras in Definition 4.1.19, i.e. as a pushout square

$$\begin{tikzcd}
R[B^{\mathrm{cy}}(M)] \arrow[r] \arrow[d] & R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] \arrow[d] \\
\mathrm{HH}^{R}(A) \arrow[r] & \mathrm{HH}^{(R,\{1\})}(A, M).
\end{tikzcd}$$

As noted in Remark 4.1.20, there is no need to replace the morphism $R[B^{\mathrm{cy}}(M)] \to R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}]$ with a cofibration in this absolute case: indeed, both of the morphisms out of $R[B^{\mathrm{cy}}(M)]$ are cofibrations in this setting. We do not find a significant simplification of the argument in the case our simplicial pre-log R-algebra is a cofibrant replacement of a discrete one, and therefore we state the result in the above generality. For the usual definition of log Hochschild homology of Definition 3.2.7, we obtain an analogue of the above theorem under a cofibrancy condition on M in Corollary 4.2.7. We have not explained what is meant by the simplicial pre-log algebra (A^a, M^a), nor the meaning of M^a in the simplicial setting. We do this in the following construction:

Construction 4.2.2. Let A be a simplicial commutative R-algebra. Define the units $\mathrm{GL}_1(A)$ of A to be the grouplike simplicial commutative monoid given by the pullback square

$$\begin{tikzcd}
\mathrm{GL}_1(A) \arrow[r] \arrow[d] & (A, \cdot) \arrow[d] \\
\mathrm{GL}_1(\pi_0(A)) \arrow[r] & (\pi_0(A), \cdot).
\end{tikzcd}$$

If $\alpha\colon M \to (A, \cdot)$ is a simplicial pre-log structure, define $\alpha^{-1}\,\mathrm{GL}_1(A)$ by the pullback square

$$\begin{tikzcd}
\alpha^{-1}\,\mathrm{GL}_1(A) \arrow[r] \arrow[d] & \mathrm{GL}_1(A) \arrow[d] \\
M \arrow[r, "\alpha"'] & (A, \cdot).
\end{tikzcd}$$

Choose a factorization

$$\begin{tikzcd} \alpha^{-1}\operatorname{GL}_1(A) \arrow[r, tail] & G \arrow[r, two heads, "\simeq"] & \operatorname{GL}_1(A) \end{tikzcd}$$

in the standard model structure on simplicial commutative monoids of Proposition 4.1.1, and define the logification M^a of M by the pushout square

$$\begin{tikzcd} \alpha^{-1}\operatorname{GL}_1(A) \arrow[r, tail] \arrow[d] & G \arrow[d] \\ M \arrow[r, tail] & M^a. \end{tikzcd}$$

Since the upper map is a cofibration and the standard model structure on simplicial commutative monoids is proper, this represents the homotopy pushout. Suppose now that the simplicial pre-log structure (A, M, α) is a cofibrant pre-log R-algebra. Consider the commutative diagram

$$\begin{tikzcd} R[M] \arrow[r, tail] \arrow[d, tail] & R[M^a] \arrow[d, tail] & \\ A \arrow[r, tail] & A \boxtimes_{R[M]} R[M^a] \arrow[r, tail] \arrow[d] & A^a \arrow[d, two heads, "\simeq"] \\ & A \boxtimes_{R[M]} A \arrow[r] & A. \end{tikzcd}$$

Here the upper square is a pushout, and in the lower square one composite is a factorization of the other in the category of simplicial commutative R-algebras. Letting $\alpha^a \colon M^a \to (A^a, \cdot)$ be adjoint to $R[M^a] \to A^a$, this provides a simplicial pre-log R-algebra for which both maps in the composite $R[M] \to R[M^a] \to A^a$ are cofibrations. In particular, $\operatorname{HH}^R(A) \simeq \operatorname{HH}^R(A^a)$ by Lemma 3.1.6.

We give a very rough outline of the proof of Theorem 4.2.1, omitting all homotopical details. The strategy is to construct a pushout square of the form

$$\begin{tikzcd} \operatorname{HH}^R(A, \alpha^{-1}\operatorname{GL}_1(A)) \arrow[r] \arrow[d] & \operatorname{HH}^R(A, \operatorname{GL}_1(A)) \arrow[d] \\ \operatorname{HH}^R(A, M) \arrow[r] & \operatorname{HH}^R(A, M^a) \end{tikzcd}$$

from the defining pushout square for M^a. As the morphism $\alpha^{-1}\operatorname{GL}_1(A) \to A$ factors through $\operatorname{GL}_1(A)$, which is grouplike, we can show that the upper morphism is an equivalence (from the perspective of a pre-log structure as an "intermediate localization", we are inverting elements of A which are already units), and under cofibrancy hypotheses we obtain that the lower map is an equivalence. The following lemma is a first step in making this formal:

Lemma 4.2.3. *Let (A, M, α) be a simplicial pre-log R-algebra for which the structure map α factors through the group completion $\gamma_M \colon M \to M^{\mathrm{gp}}$ of M. Then there is a weak equivalence*

$$\operatorname{HH}^R(A) \xrightarrow{\simeq} \operatorname{HH}^R(A) \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}].$$

In particular, if (A, M) is cofibrant, the codomain of the equivalence is $\operatorname{HH}^{(R,\{1\})}(A, M)$ by definition.

Proof. By the assumption on α we have a pushout square

$$\begin{tikzcd}
R[B^{\mathrm{cy}}(M)] \arrow[r] \arrow[d] & R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] \arrow[d] \\
R[B^{\mathrm{cy}}(M^{\mathrm{gp}})] \arrow[r] \arrow[d] & R[B^{\mathrm{cy}}(M^{\mathrm{gp}})] \boxtimes_{R[B^{\mathrm{cy}}M]} R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] \arrow[d] \\
\mathrm{HH}^R(A) \arrow[r] & \mathrm{HH}^R(A) \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}].
\end{tikzcd}$$

This leads us to consider the pushout square

$$\begin{tikzcd}
B^{\mathrm{cy}}(M) \arrow[r, "\rho_{B^{\mathrm{cy}}(M)}"] \arrow[d, "B^{\mathrm{cy}}(\gamma_M)"'] & B^{\mathrm{cy}}(M)^{\mathrm{rep}} \arrow[d] \\
B^{\mathrm{cy}}(M^{\mathrm{gp}}) \arrow[r] & B^{\mathrm{cy}}(M^{\mathrm{gp}}) \sqcup_{B^{\mathrm{cy}}(M)} B^{\mathrm{cy}}(M)^{\mathrm{rep}}
\end{tikzcd} \tag{6}$$

of simplicial commutative monoids. By definition, the repletion map $\rho_{B^{\mathrm{cy}}(M)}$ is an acyclic cofibration in the group completion model structure of Proposition 4.1.2, and so it follows that the lower map in the diagram is an acyclic cofibration in the group completion model structure. We now argue that the lower map is a weak equivalence in the standard model structure. Since M^{gp} is grouplike, there is a homotopy fiber sequence

$$B(M^{\mathrm{gp}}) \to B^{\mathrm{cy}}(M^{\mathrm{gp}}) \to M^{\mathrm{gp}},$$

as noted in for instance [BHM93, Section 2]. Recall from Lemma 4.1.16 that the repletion $B^{\mathrm{cy}}(M)^{\mathrm{rep}}$ fits in a commutative diagram

$$\begin{tikzcd}
B^{\mathrm{cy}}(M) \arrow[r, "\rho_{B^{\mathrm{cy}}(M)}"] \arrow[d] & B^{\mathrm{cy}}(M)^{\mathrm{rep}} \arrow[r] \arrow[d] & B^{\mathrm{cy}}(M^{\mathrm{gp}}) \arrow[d] \\
M \arrow[r, tail, "\simeq"] & M' \arrow[r, two heads] & M^{\mathrm{gp}},
\end{tikzcd}$$

where the lower composite is a factorization of the group completion map in the standard model structure and the right-hand square is a homotopy pullback. This identifies the homotopy fiber fib of $B^{\mathrm{cy}}(M)^{\mathrm{rep}} \to M'$ with $B(M^{\mathrm{gp}})$ up to weak equivalence, as in the diagram

$$\begin{tikzcd}
\mathrm{fib} \arrow[r] \arrow[d] & B^{\mathrm{cy}}(M)^{\mathrm{rep}} \arrow[r] \arrow[d] & B^{\mathrm{cy}}(M^{\mathrm{gp}}) \arrow[d] \\
* \arrow[r] & M' \arrow[r] & M^{\mathrm{gp}}
\end{tikzcd}$$

both squares are homotopy pullbacks, so the entire rectangle is a homotopy pullback. From the map of homotopy fiber sequences

$$\begin{tikzcd}
\mathrm{fib} \arrow[r] \arrow[d, "\simeq"] & B^{\mathrm{cy}}(M)^{\mathrm{rep}} \arrow[r] \arrow[d] & M' \arrow[d, equal] \\
B(M^{\mathrm{gp}}) \arrow[r] & M' \times B(M^{\mathrm{gp}}) \arrow[r] & M',
\end{tikzcd}$$

we find that $B^{\mathrm{cy}}(M)^{\mathrm{rep}} \to M' \times B(M^{\mathrm{gp}})$ is a weak equivalence. As $B(M^{\mathrm{gp}})$ is connected, we have that $\pi_0(B^{\mathrm{cy}}(M)^{\mathrm{rep}}) \cong \pi_0(M')$, and from the square

$$\begin{tikzcd}
\pi_0(B^{\mathrm{cy}}(M)) \arrow[r, "\pi_0(\rho_{B^{\mathrm{cy}}(M)})"] \arrow[d] & \pi_0(B^{\mathrm{rep}}(M)) \arrow[d, "\cong"] \\
\pi_0(M) \arrow[r, "\cong"] & \pi_0(M')
\end{tikzcd}$$

we find that the repletion map $\rho_{B^{\mathrm{cy}}(M)}$ is a surjection on π_0. Consequently the lower morphism in the diagram (6) is a surjection on π_0, so that its codomain is grouplike, since $B^{\mathrm{cy}}(M^{\mathrm{gp}})$ is. It being a weak equivalence in the group completion model structure between grouplike objects, we conclude that the lower map is an equivalence in the standard model structure of Proposition 4.1.1 by Remark 4.1.4. Since $R[-]$ is a left Quillen functor from the standard model structure on simplicial commutative monoids, the morphism

$$R[B^{\mathrm{cy}}(M^{\mathrm{gp}})] \to R[B^{\mathrm{cy}}(M^{\mathrm{gp}})] \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}]$$

is an acyclic cofibration. Then the result follows, as acyclic cofibrations are stable under cobase change. □

Remark 4.2.4. Working with the usual definition of log Hochschild homology $\mathrm{HH}^R(A^{\mathrm{cof}}, M)$ from Definition 3.2.7, the above argument has a simplification in the case M is *integral*, i.e. cancellative. As we noted in Example 2.2.18, being integral is equivalent to the usual group completion map $\gamma_M \colon M \to M^{\mathrm{gp}}$ being injective, which allows us to think of the replete bar construction $B^{\mathrm{rep}}(M)$ as a subobject of the cyclic bar construction $B^{\mathrm{cy}}(M^{\mathrm{gp}})$ of the group completion of M. By Remark 2.2.16, an inverse of the morphism

$$R[B^{\mathrm{cy}}(M^{\mathrm{gp}})] \to R[B^{\mathrm{cy}}(M^{\mathrm{gp}})] \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[B^{\mathrm{rep}}(M)]$$

is then given by the multiplication map, and we in fact obtain such a simplicial isomorphism. Of course, the conclusion $\mathrm{HH}^R(A^{\mathrm{cof}}, M) \simeq \mathrm{HH}^R((A^{\mathrm{cof}})^a, M^a)$ would still only hold up to weak equivalence, as we are changing the choice of cofibrant replacement of A.

As a final step before proving Theorem 4.2.1, we will use the following analogue of [RSS15, Lemma 4.26], where the proof in this context is similar to that in *loc. cit.*:

Lemma 4.2.5. *Consider the commutative cube*

$$\begin{tikzcd}
& N_2 \arrow[rr] \arrow[dd] & & M_2 \arrow[dd] \\
N_1 \arrow[ur] \arrow[rr] \arrow[dd] & & M_1 \arrow[ur] \arrow[dd] & \\
& N_4 \arrow[rr] & & M_4 \\
N_3 \arrow[ur] \arrow[rr] & & M_3 \arrow[ur] &
\end{tikzcd}$$

of cofibrant simplicial commutative monoids. Name the horizontal morphisms $\epsilon_i \colon N_i \to M_i$. If the left and right-hand face are homotopy cocartesian and each ϵ_i is virtually surjective, the diagram

$$\begin{tikzcd}
N_1^{\mathrm{rep}} \arrow[r] \arrow[d] & N_2^{\mathrm{rep}} \arrow[d] \\
N_3^{\mathrm{rep}} \arrow[r] & N_4^{\mathrm{rep}}
\end{tikzcd}$$

of repletions of each ϵ_i (see Definition 4.1.14) is a homotopy pushout. □

Proof of Theorem 4.2.1. We must prove that the morphism

$$\mathrm{HH}^R(A) \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] \to \mathrm{HH}^R(A^a) \boxtimes_{R[B^{\mathrm{cy}}(M^a)]} R[B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}]$$

induced by the logification construction is a weak equivalence. Pick a cofibrant replacement $P \to \alpha^{-1}\,\mathrm{GL}_1(A)$ in the standard model structure of Proposition 4.1.1 (we do this so that Lemma 4.2.5 is applicable). Since the morphism $P \to A$ factors through the grouplike object $\mathrm{GL}_1(A)$, Lemma 4.2.3 gives that the morphism

$$\mathrm{HH}^R(A) \to \mathrm{HH}^R(A) \boxtimes_{R[B^{\mathrm{cy}}(P)]} R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}]$$

is a weak equivalence. Choose a factorization

$$P \rightarrowtail G \xrightarrow{\simeq} \mathrel{\!\!\rightarrow}\! \mathrm{GL}_1(A)$$

in the standard model structure. We can still model the logification M^a by the pushout

$$M \longleftarrow P \rightarrowtail G$$

using the following argument: recall from Construction 4.2.2 that the logification M^a of M is given by a pushout square

$$\begin{tikzcd}
\alpha^{-1}\,\mathrm{GL}_1(A) \arrow[r, tail] \arrow[d] & G' \arrow[r, two heads, "\simeq"] \arrow[d] & \mathrm{GL}_1(A) \\
M \arrow[r, tail] & M^a,
\end{tikzcd}$$

where the upper composite is a factorization in the standard model structure. The lifting axioms gives a weak equivalence $G \to G'$:

$$\begin{tikzcd}
P \arrow[r, two heads] \arrow[d, tail] & \alpha^{-1}\,\mathrm{GL}_1(A) \arrow[r, tail] & G' \arrow[d, two heads, "\simeq"] \\
G \arrow[urr, dashed, "\simeq"'] \arrow[rr, two heads, "\simeq"'] & & \mathrm{GL}_1(A)
\end{tikzcd}$$

Hence we obtain a weak equivalence of homotopy pushout data

$$\begin{tikzcd}
M \arrow[d, equal] & P \arrow[l] \arrow[r, tail] \arrow[d, "\simeq"] & G \arrow[d, "\simeq"] \\
M & \alpha^{-1}\,\mathrm{GL}_1(A) \arrow[l] \arrow[r, tail] & G'
\end{tikzcd}$$

which gives a weak equivalence on pushouts. We now consider the commutative cube

$$\begin{tikzcd}[row sep=small, column sep=small]
& R[B^{\mathrm{cy}}(G)] \arrow[rr] \arrow[dd] & & R[B^{\mathrm{cy}}(M^a)] \arrow[dd] \\
R[B^{\mathrm{cy}}(P)] \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & R[B^{\mathrm{cy}}(M)] \arrow[ur] & \\
& R[B^{\mathrm{cy}}(G)^{\mathrm{rep}}] \arrow[rr] & & R[B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}] \\
R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] \arrow[ur] \arrow[rr] & & R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}]. \arrow[ur] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

The upper face is a homotopy pushout since both $B^{\mathrm{cy}}(-)$ and $R[-]$ are left Quillen functors, while the lower face is a homotopy pushout by Lemma 4.2.5. We proceed to study the left-hand face.

Since G is grouplike, $R[B^{\mathrm{cy}}(G)] \to R[B^{\mathrm{cy}}(G)^{\mathrm{rep}}]$ is a weak equivalence by Lemma 4.1.9. We then notice that this implies that the lower map in the pushout square

$$\begin{tikzcd}
R[B^{\mathrm{cy}}(G)] \boxtimes_{R[B^{\mathrm{cy}}(P)]} R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] \arrow[r] \arrow[d] & R[B^{\mathrm{cy}}(G)^{\mathrm{rep}}] \arrow[d] \\
\mathrm{HH}^R(A) \boxtimes_{R[B^{\mathrm{cy}}(P)]} R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] \arrow[r] & X
\end{tikzcd}$$

is a weak equivalence, where the left-hand map is induced by $R[B^{\mathrm{cy}}(G)] \to \mathrm{HH}^R(A)$. Indeed, the above square fits in the diagram

$$\begin{tikzcd}
R[B^{\mathrm{cy}}(P)] \arrow[r, tail] \arrow[d, tail] & R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] \arrow[d, tail] & \\
R[B^{\mathrm{cy}}(G)] \arrow[r, tail] \arrow[d] & R[B^{\mathrm{cy}}(G)] \boxtimes_{R[B^{\mathrm{cy}}(P)]} R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] \arrow[r] \arrow[d] & R[B^{\mathrm{cy}}(G)^{\mathrm{rep}}] \arrow[d] \\
\mathrm{HH}^R(A) \arrow[r, tail, "\simeq"'] & \mathrm{HH}^R(A) \boxtimes_{R[B^{\mathrm{cy}}(P)]} R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] \arrow[r] & X
\end{tikzcd} \tag{7}$$

where all squares are pushouts. Here the upper map is a cofibration since $B^{\mathrm{cy}}(P) \to B^{\mathrm{cy}}(P)^{\mathrm{rep}}$ is (by Definition 4.1.14) and $R[-]$ is a left Quillen functor from the standard model structure of Proposition 4.1.1, whose cofibrations coincide with those of the group completion model structure of Proposition 4.1.2. This implies that the morphism

$$R[B^{\mathrm{cy}}(G)] \to R[B^{\mathrm{cy}}(G)] \boxtimes_{R[B^{\mathrm{cy}}(P)]} R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}]$$

is a cofibration. By Lemma 4.1.9, the middle horizontal composite is an acyclic cofibration since $R[-]$ is a left Quillen functor. This implies that the lower horizontal composite is an acyclic cofibration. As we already have that

$$\mathrm{HH}^R(A) \to \mathrm{HH}^R(A) \boxtimes_{R[B^{\mathrm{cy}}(P)]} R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}]$$

is a weak equivalence by Lemma 4.2.3, this implies that

$$\mathrm{HH}^R(A) \boxtimes_{R[B^{\mathrm{cy}}(P)]} R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] \to X$$

is a weak equivalence by the two–out–of–three property. As diagram (7) displays X as merely $\mathrm{HH}^R(A) \boxtimes_{R[B^{\mathrm{cy}}(G)]} R[B^{\mathrm{cy}}(G)^{\mathrm{rep}}]$, we conclude that the morphism

$$\mathrm{HH}^R(A) \boxtimes_{R[B^{\mathrm{cy}}(P)]} R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] \to \mathrm{HH}^R(A) \boxtimes_{R[B^{\mathrm{cy}}(G)]} R[B^{\mathrm{cy}}(G)^{\mathrm{rep}}]$$

is a weak equivalence. We now consider the homotopy pushout of

$$\begin{tikzcd}
\mathrm{HH}^R(A^a) \boxtimes_{R[B^{\mathrm{cy}}(P)]} R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] \arrow[r, "\simeq"] \arrow[d] & \mathrm{HH}^R(A^a) \boxtimes_{R[B^{\mathrm{cy}}(G)]} R[B^{\mathrm{cy}}(G)^{\mathrm{rep}}] \\
\mathrm{HH}^R(A^a) \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}].
\end{tikzcd}$$

As noted in Construction 4.2.2, changing A for A^a makes no difference on Hochschild homology up to weak equivalence; we only make the change so that $\mathrm{HH}^R(A^a) \boxtimes_{R[B^{\mathrm{cy}}(M^a)]} R[B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}]$ is a model for $\mathrm{HH}^{(R,\{1\})}(A^a, M^a)$. By commutativity of the cube, the homotopy pushout above can be computed as the homotopy pushout

$$\begin{array}{ccc} h(R[B^{\mathrm{cy}}(M)] \leftarrow R[B^{\mathrm{cy}}(P)] \to R[B^{\mathrm{cy}}(G)]) & \longrightarrow & h(R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] \leftarrow R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] \to R[B^{\mathrm{cy}}(G)^{\mathrm{rep}}]) \\ \downarrow & & \\ h(\mathrm{HH}^R(A^a) = \mathrm{HH}^R(A^a) = \mathrm{HH}^R(A^a)), & & \end{array}$$

where we have used $h(-)$ to denote the homotopy pushout of the given data. As remarked after the introduction of the cube, this is the homotopy pushout of

$$\begin{array}{ccc} R[B^{\mathrm{cy}}(M^a)] & \longrightarrow & R[B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}] \\ \rightarrowtail\!\!\downarrow & & \\ \mathrm{HH}^R(A^a), & & \end{array} \tag{8}$$

which, since the left-hand map is a cofibration, is $\mathrm{HH}^R(A^a) \boxtimes_{R[B^{\mathrm{cy}}(M^a)]} R[B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}]$. In conclusion, there is a homotopy pushout square

$$\begin{array}{ccc} \mathrm{HH}^R(A^a) \boxtimes_{R[B^{\mathrm{cy}}(P)]} R[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] & \xrightarrow{\simeq} & \mathrm{HH}^R(A^a) \boxtimes_{R[B^{\mathrm{cy}}(G)]} R[B^{\mathrm{cy}}(G)^{\mathrm{rep}}] \\ \downarrow & & \downarrow \\ \mathrm{HH}^R(A^a) \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] & \longrightarrow & \mathrm{HH}^R(A^a) \boxtimes_{R[B^{\mathrm{cy}}(M^a)]} R[B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}], \end{array}$$

and the lower horizontal morphism is a weak equivalence since the upper one is. This concludes the proof. □

The following remark is essential for the upcoming proof of Theorem 4.2.9:

Remark 4.2.6. In diagram (8), we chose to highlight that the morphism $R[B^{\mathrm{cy}}(M^a)] \to \mathrm{HH}^R(A^a)$ is a cofibration, so that the pushout not only represents the homotopy pushout, but also has the correct cofibrancy conditions for $\mathrm{HH}^{(R,\{1\})}(A^a, M^a)$. We could have equally well have noticed that the morphism $R[B^{\mathrm{cy}}(M)] \to R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}]$ is a cofibration: by definition (see Definition 4.1.14), the morphism

$$\rho_{B^{\mathrm{cy}}(M)} \colon B^{\mathrm{cy}}(M) \to B^{\mathrm{cy}}(M)^{\mathrm{rep}}$$

is an acyclic cofibration in the group completion model structure of Proposition 4.1.2, and hence a cofibration in the standard model structure of Proposition 4.1.1, as the cofibrations of these two model structures coincide. Since $R[-]$ is a left Quillen functor from the standard model structure on simplicial commutative monoids, we conclude that $R[B^{\mathrm{cy}}(M)] \to R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}]$ is a cofibration. In particular, it is not necessary for A^a to be cofibrant over $R[M]$ and $R[M^a]$ for the morphism

$$\mathrm{HH}^R(A^a) \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] \to \mathrm{HH}^R(A^a) \boxtimes_{R[B^{\mathrm{cy}}(M^a)]} R[B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}]$$

to be a weak equivalence, although the two objects no longer model e.g. $\mathrm{HH}^{(R,\{1\})}(A^a, M^a)$. This will be important to us later, as a morphism of simplicial pre-log rings $(R, N) \to (A, M)$ being a cofibration does not imply that $R[M] \to A$ is a cofibration.

The statement of Theorem 4.2.1 was made for a general cofibrant simplicial pre-log R-algebra (A, M) in the model structure of Proposition 4.1.10, and in particular we used a homotopy invariant version of the logification construction as described in Construction 4.2.2. Given a discrete pre-log R-algebra (A, M), it is not immediately clear how this leads to a comparison of $\mathrm{HH}^R(A^{\mathrm{cof}}, M)$ and $\mathrm{HH}^R(A^{\mathrm{cof}}, M^a)$, where M^a denotes the usual logification of M. The problem is that the pushout

$$M \longleftarrow \alpha^{-1}\,\mathrm{GL}_1(A) \longrightarrow \mathrm{GL}_1(A)$$

of discrete commutative monoids is not necessarily a homotopy pushout. The reason why e.g. the replete bar construction is well-behaved when passing to cofibrant resolutions of constant objects (Lemma 4.1.22) is that the usual group completion map $\gamma_M \colon M \to M^{\mathrm{gp}}$ is a Kan fibration, so that the defining pullback square of the usual replete bar construction (Definition 3.2.4) is already a homotopy pullback. Not all is lost, however, as the conditions of the following corollary hold in many situations of interest:

Corollary 4.2.7. *Let (A, M, α) be a discrete pre-log R-algebra. Assume that in the defining pushout square*

$$\begin{tikzcd}
\alpha^{-1}\,\mathrm{GL}_1(A) \arrow[r] \arrow[d] & \mathrm{GL}_1(A) \arrow[d] \\
M \arrow[r] & M^a
\end{tikzcd}$$

of M^a, the morphism $\alpha^{-1}\,\mathrm{GL}_1(A) \to M$ is a cofibration of discrete simplicial commutative monoids in the standard model structure of Proposition 4.1.1. Then there is a chain of weak equivalences

$$\mathrm{HH}^R(A^{\mathrm{cof}}, M) \simeq \mathrm{HH}^R(\tilde{A}^{\mathrm{cof}}, M^a),$$

where $\mathrm{HH}^R(-,-)$ denotes the usual definition of log Hochschild homology of Definition 3.2.7 and M^a denotes the usual definition of logification of Definition 2.2.6.

Here $\tilde{A}^{\mathrm{cof}} \to A$ denotes a cofibrant replacement $\tilde{A}^{\mathrm{cof}} \to A$ of commutative simplicial $R[M^a]$-algebras. Common examples of pre-log structures come from free commutative monoids M mapping to non-units of A. For example, this is the case for a discrete valuation ring A and $M = \langle \pi \rangle$, the free commutative monoid on a uniformizer π (see Example 2.2.8). In these examples, the hypotheses of the corollary hold: the pullback $\alpha^{-1}\,\mathrm{GL}_1(A)$ is the trivial monoid, and the demand is then that M is cofibrant as a (discrete) simplicial commutative monoid. For example, it follows from the corollary that

$$\mathrm{HH}^R(A^{\mathrm{cof}}, \langle \pi \rangle) \simeq \mathrm{HH}^R(A^{\mathrm{cof}}, A - \{0\})$$

for discrete valuation rings A, cf. Example 2.2.8.

Proof of Corollary 4.2.7. Let $(A^{\mathrm{c}}, M^{\mathrm{c}}) \to (A, M)$ be a cofibrant replacement of (A, M) in the model structure of Proposition 4.1.10. By Theorem 4.2.1, there is a weak equivalence

$$\mathrm{HH}^{(R,\{1\})}(A^{\mathrm{c}}, M^{\mathrm{c}}) \xrightarrow{\simeq} \mathrm{HH}^{(R,\{1\})}((A^{\mathrm{c}})^a, (M^{\mathrm{c}})^a),$$

where $(M^{\mathrm{c}})^a$ denotes the logification of the simplicial commutative monoid M^c as in Construction 4.2.2. By Proposition 4.1.21, there is a chain of weak equivalences

$$\mathrm{HH}^{(R,\{1\})}(A^c, M^c) \simeq \mathrm{HH}^R(A^{\mathrm{cof}}, M).$$

Recall from Construction 4.2.2 that the logification $(M^c)^a$ of M^c is defined by a pushout square

$$\begin{tikzcd}
(\alpha^c)^{-1}\,\mathrm{GL}_1(A^c) \arrow[r, tail] \arrow[d] & G \arrow[r, two heads, "\simeq"] \arrow[d] & \mathrm{GL}_1(A^c) \\
M^c \arrow[r, tail] & (M^c)^a,
\end{tikzcd}$$

where the upper composite is a factorization in the standard model structure of Proposition 4.1.1. Notice that there is a weak equivalence $\mathrm{GL}_1(A^c) \to \mathrm{GL}_1(A)$: in the defining pullback square

$$\begin{tikzcd}
\mathrm{GL}_1(A^c) \arrow[r, two heads] \arrow[d, two heads, "\simeq"'] & (A^c, \cdot) \arrow[d, two heads, "\simeq"] \\
\mathrm{GL}_1(A) \arrow[r, two heads] & (A, \cdot),
\end{tikzcd}$$

we have that the lower map is a Kan fibration as it is a morphism of discrete simplicial sets. The right-hand map is an acyclic fibration since $A^c \to A$ is one of simplicial commutative R-algebras, from which the forgetful functor to simplicial commutative monoids is a right Quillen functor. As the property of being an (acyclic) fibration is stable under pullbacks, we obtain morphisms as indicated in the diagram. It follows that we have a weak equivalence

$$\begin{tikzcd}
M^c \arrow[r] \arrow[d, "\simeq"'] & (A^c, \cdot) \arrow[d, two heads, "\simeq"] & \mathrm{GL}_1(A^c) \arrow[l, two heads] \arrow[d, two heads, "\simeq"] \\
M \arrow[r] & (A, \cdot) & \mathrm{GL}_1(A) \arrow[l, two heads]
\end{tikzcd}$$

of (homotopy) pullback data, which induces a weak equivalence

$$(\alpha^c)^{-1}\,\mathrm{GL}_1(A^c) \to \alpha^{-1}\,\mathrm{GL}_1(A)$$

of (homotopy) pullbacks, see Remark 4.1.6. Consequently there is a weak equivalence

$$\begin{tikzcd}
M^c \arrow[dd, "\simeq"'] & (\alpha^c)^{-1}\,\mathrm{GL}_1(A^c) \arrow[l] \arrow[r, tail] \arrow[dd, "\simeq"] & G \arrow[d, "\simeq"] \\
 & & \mathrm{GL}_1(A^c) \arrow[d, "\simeq"] \\
M & \alpha^{-1}\,\mathrm{GL}_1(A) \arrow[l, tail] \arrow[r] & \mathrm{GL}_1(A)
\end{tikzcd}$$

of pushout data, and by assumption the morphism $\alpha^{-1}\,\mathrm{GL}_1(A) \to M$ is a cofibration, so that both represent the respective homotopy pushouts. We conclude that there is a weak equivalence $(M^c)^a \to M^a$, and consequently a chain of weak equivalences

$$\mathrm{HH}^R(\tilde{A}^{\mathrm{cof}}, M^a) \simeq \mathrm{HH}^{(R,\{1\})}((A^c)^a, (M^c)^a)$$

by Proposition 4.1.21. By Theorem 4.2.1, we know that there is a weak equivalence

$$\mathrm{HH}^{(R,\{1\})}(A^c, M^c) \xrightarrow{\simeq} \mathrm{HH}^{(R,\{1\})}((A^c)^a, (M^c)^a),$$

which concludes the proof. $\square$

Example 4.2.8. Let $\mathbb{N}_{>0}$ be the multiplicative monoid of strictly positive integers, and consider the inclusion $\mathbb{N}_{>0} \to (\mathbb{Z}, \cdot)$. The defining pushout of the logification of this pre-log structure is

$$\mathbb{N}_{>0} \longleftarrow \{1\} \longrightarrow \mathrm{GL}_1(\mathbb{Z}).$$

In this situation neither of the monoids involved are free, and so the hypotheses of Corollary 4.2.7 do not hold. It therefore seems necessary to consider the homotopy invariant version of the logification functor from Construction 4.2.2 to obtain invariance under logification for log Hochschild homology.

4.2.2 Invariance for relative log Hochschild homology

The aim of this section is to extend Theorem 4.2.1 to the relative definition of log Hochschild homology of Definition 4.1.19. The first result is that, despite having a non-trivial log structure on the ground ring R, $\mathrm{HH}^{(R,N)}(A, M)$ is still invariant under logification of the simplicial commutative monoid M:

Theorem 4.2.9. *Let (A, M) be a simplicial pre-log ring which is cofibrant over the pre-log ring (R, N) in the model structure of Proposition 4.1.10. Then there is a weak equivalence*

$$\mathrm{HH}^{(R,N)}(A, M) \xrightarrow{\simeq} \mathrm{HH}^{(R,N)}(A^a_N, M^a),$$

where A^a_N is defined in Construction 4.2.10.

As was the case before, we have defined $\mathrm{HH}^{(R,N)}(A, M)$ for simplicial pre-log R-algebras (A, M) which are cofibrant over (R, N) by the pushout square

$$\begin{tikzcd}
R[B^{\mathrm{cy}}(M)] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \arrow[r, tail] \arrow[d] & R[\tilde{B}^{\mathrm{rep}}(M)] \arrow[r, "\simeq"] \arrow[d] & R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] \\
\mathrm{HH}^R(A) \arrow[r] & \mathrm{HH}^{(R,N)}(A, M) &
\end{tikzcd}$$

as in Construction 4.1.18. As we now have a non-trivial pre-log structure on the ground ring R, it is necessary to factor the morphism

$$B^{\mathrm{cy}}(M) \sqcup_{B^{\mathrm{cy}}(N)} B^{\mathrm{cy}}(N)^{\mathrm{rep}} \to B^{\mathrm{cy}}(M)^{\mathrm{rep}}$$

as in Definition 4.1.19. While the meaning of the logification M^a of M remains the same as in Construction 4.2.2, we now need to construct an object A^a_N which is compatible with the pre-log structure on R:

Construction 4.2.10. Let $(R, N) \to (A, M)$ be a morphism of simplicial pre-log rings, where (R, N) is discrete. Assume that this morphism is a cofibration in the model structure on simplicial pre-log R-algebras of Proposition 4.1.10. Recall from Construction 4.2.2 that the logification M^a of M is constructed so that the morphism $M \to M^a$ is a cofibration in the standard model structure of Proposition 4.1.1, and consequently $R[M] \to R[M^a]$ is a cofibration. Therefore the lower map in the pushout diagram

$$\begin{tikzcd}
R[M] \arrow[r, tail] \arrow[d] & R[M^a] \arrow[d] \\
R[M] \boxtimes_{R[N]} R \arrow[r, tail] & R[M^a] \boxtimes_{R[N]} R
\end{tikzcd}$$

is also a cofibration. Recall that cofibrancy over (R, N) in the model structure of Proposition 4.1.10 implies that the morphism

$$R[M] \boxtimes_{R[N]} R \to A$$

is a cofibration. We then imitate the definition of A^a from Construction 4.2.2, and define A^a_N by

$$\begin{tikzcd}
R[M] \boxtimes_{R[N]} R \arrow[r, tail] \arrow[d, tail] & R[M^a] \boxtimes_{R[N]} R \arrow[d, tail] & \\
A \arrow[r, tail] & A \boxtimes_{R[M] \boxtimes_{R[N]} R} (R[M^a] \boxtimes_{R[N]} R) \arrow[r, tail] \arrow[d] & A^a_N \arrow[d, two heads, "\simeq"] \\
& A \boxtimes_{R[M] \boxtimes_{R[N]} R} A \arrow[r] & A,
\end{tikzcd}$$

where the upper square is a pushout and in the lower square one composite is a factorization of the other. This gives a simplicial pre-log algebra A^a_N such that both morphisms in the composite

$$R[M] \boxtimes_{R[N]} R \to R[M^a] \boxtimes_{R[N]} R \to A^a_N$$

are cofibrations. In particular, (A^a_N, M^a) is cofibrant over (R, N) in the model structure of Proposition 4.1.10.

Before we proceed with the proof of Theorem 4.2.9, we give a brief outline of the argument, again omitting all homotopical details. The idea for the proof of Theorem 4.2.1 was to construct a pushout square of the form

$$\begin{tikzcd}
\mathrm{HH}^R(A^a, \alpha^{-1}\,\mathrm{GL}_1(A)) \arrow[r, "\simeq"] \arrow[d] & \mathrm{HH}^R(A^a, \mathrm{GL}_1(A)) \arrow[d] \\
\mathrm{HH}^R(A^a, M) \arrow[r, "\simeq"'] & \mathrm{HH}^R(A^a, M^a)
\end{tikzcd}$$

from the defining pushout square of M^a. The idea is then to prove that the relative construction fits in a pushout square

$$\begin{tikzcd}
\mathrm{HH}^R(A^a_N, \alpha^{-1}\,\mathrm{GL}_1(A)) \arrow[r, "\simeq"] \arrow[d] & \mathrm{HH}^R(A^a_N, \mathrm{GL}_1(A)) \arrow[d] \\
\mathrm{HH}^R(A^a_N, M) \arrow[r, "\simeq"] \arrow[d] & \mathrm{HH}^R(A^a_N, M^a) \arrow[d] \\
\mathrm{HH}^{(R,N)}(A^a_N, M) \arrow[r] & \mathrm{HH}^{(R,N)}(A^a_N, M^a),
\end{tikzcd}$$

so that the desired equivalence follows from Theorem 4.2.1.

Remark 4.2.11. One technicality when passing from $\mathrm{HH}^{(R,\{1\})}(A, M)$ to $\mathrm{HH}^{(R,N)}(A, M)$ is that it is now necessary to change the morphism

$$R[B^{\mathrm{cy}}(M)] \sqcup_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \to R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}]$$

with a cofibration as in Definition 4.1.19. As we will see in the proof of Theorem 4.2.9, it will be essential that the resulting morphism $R[B^{\mathrm{cy}}(M)] \to R[\tilde{B}^{\mathrm{rep}}(M)]$ is a cofibration. This would not have been the case if we defined $\mathrm{HH}^{(R,N)}(A, M)$ in terms of the replete bar construction of Definition 4.1.7, and so this is another reason why we prefer to work with the repletion $B^{\mathrm{cy}}(M)^{\mathrm{rep}}$ in this relative context.

Proof of Theorem 4.2.9. With the notation from Definition 4.1.19, we first argue that the morphism

$$\mathrm{HH}^R(A^a_N) \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[\tilde{B}^{\mathrm{rep}}(M)] \to \mathrm{HH}^R(A^a_N) \boxtimes_{R[B^{\mathrm{cy}}(M^a)]} R[\tilde{B}^{\mathrm{rep}}(M^a)]$$

induced by the logification construction is a weak equivalence. By Theorem 4.2.1 and Remark 4.2.6, we know that the morphism

$$\mathrm{HH}^R(A^a_N) \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] \to \mathrm{HH}^R(A^a_N) \boxtimes_{R[B^{\mathrm{cy}}(M^a)]} R[B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}]$$

induced by the logification construction is a weak equivalence. By construction, we obtain a weak equivalence of homotopy pushout data

$$\begin{tikzcd}
\mathrm{HH}^R(A^a_N) \arrow[d, equal] & R[B^{\mathrm{cy}}(M)] \arrow[l] \arrow[r, tail] \arrow[d, equal] & R[\tilde{B}^{\mathrm{rep}}(M)] \arrow[d, "\simeq"] \\
\mathrm{HH}^R(A^a_N) & R[B^{\mathrm{cy}}(M)] \arrow[l] \arrow[r, tail] & R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}],
\end{tikzcd}$$

which induces a weak equivalence on (homotopy) pushouts, and similarly for M replaced with M^a. Here $R[B^{\mathrm{cy}}(M)] \to R[\tilde{B}^{\mathrm{rep}}(M)]$ is a cofibration by construction (Construction 4.1.18 and Definition 4.1.19), as it is the composite of the cofibrations

$$R[B^{\mathrm{cy}}(M)] \to R[B^{\mathrm{cy}}(M)] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \to R[\tilde{B}^{\mathrm{rep}}(M)].$$

By the two–out–of–three property applied to the diagram

$$\begin{tikzcd}
\mathrm{HH}^R(A^a_N) \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[\tilde{B}^{\mathrm{rep}}(M)] \arrow[r] \arrow[d, "\simeq"'] & \mathrm{HH}^R(A^a_N) \boxtimes_{R[B^{\mathrm{cy}}(M^a)]} R[\tilde{B}^{\mathrm{rep}}(M^a)] \arrow[d, "\simeq"] \\
\mathrm{HH}^R(A^a_N) \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] \arrow[r, "\simeq"'] & \mathrm{HH}^R(A^a_N) \boxtimes_{R[B^{\mathrm{cy}}(M^a)]} R[B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}]
\end{tikzcd} \tag{9}$$

we obtain the desired equivalence. In light of Remark 4.2.11, it is worthwhile to notice that we needed $R[B^{\mathrm{cy}}(M)] \to R[\tilde{B}^{\mathrm{rep}}(M)]$ to be a cofibration. Consider now the commutative cube

$$\begin{tikzcd}
& R[B^{\mathrm{cy}}(M)] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \arrow[rr] \arrow[dd] & & R[B^{\mathrm{cy}}(M^a)] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \arrow[dd] \\
R[B^{\mathrm{cy}}(M)] \arrow[ur] \arrow[rr] \arrow[dd] & & R[B^{\mathrm{cy}}(M^a)] \arrow[ur] \arrow[dd] & \\
& R[\tilde{B}^{\mathrm{rep}}(M)] \arrow[rr] \arrow[dl, equal] & & R[\tilde{B}^{\mathrm{rep}}(M^a)] \arrow[dl, equal] \\
R[\tilde{B}^{\mathrm{rep}}(M)] \arrow[rr] & & R[\tilde{B}^{\mathrm{rep}}(M^a)]. &
\end{tikzcd}$$

Both the lower and upper face are pushouts, and in fact homotopy pushouts since $R[B^{\mathrm{cy}}(M)] \to R[B^{\mathrm{cy}}(M^a)]$ is a cofibration. We now compute the homotopy pushout of

$$\begin{tikzcd}
\mathrm{HH}^R(A_N^a) \boxtimes_{R[B^{\mathrm{cy}}(M)]} R[\tilde{B}^{\mathrm{rep}}(M)] \arrow[r, "\simeq"] \arrow[d] & \mathrm{HH}^R(A_N^a) \boxtimes_{R[B^{\mathrm{cy}}(M^a)]} R[\tilde{B}^{\mathrm{rep}}(M^a)] \\
\mathrm{HH}^R(A_N^a) \boxtimes_{R[B^{\mathrm{cy}}(M)] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}]} R[\tilde{B}^{\mathrm{rep}}(M)], &
\end{tikzcd} \tag{10}$$

where the upper equivalence was established in diagram (9) above. By commutativity of the cube, this is the homotopy pushout of

$$\begin{tikzcd}
h(R[\tilde{B}^{\mathrm{rep}}(M)] = R[\tilde{B}^{\mathrm{rep}}(M)] \to R[\tilde{B}^{\mathrm{rep}}(M^a)]) \\
h(R[B^{\mathrm{cy}}(M)] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \leftarrow R[B^{\mathrm{cy}}(M)] \to R[B^{\mathrm{cy}}(M^a)]) \arrow[u] \arrow[d] \\
h(\mathrm{HH}^R(A_N^a) = \mathrm{HH}^R(A_N^a) = \mathrm{HH}^R(A_N^a)),
\end{tikzcd}$$

where we have used $h(-)$ to denote the homotopy pushout of the given data. As remarked after the introduction of the cube, this is the homotopy pushout of

$$\begin{tikzcd}
R[B^{\mathrm{cy}}(M^a)] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \arrow[r, tail] \arrow[d] & R[\tilde{B}^{\mathrm{rep}}(M^a)] \\
\mathrm{HH}^R(A_N^a), &
\end{tikzcd}$$

which, since the upper map is a cofibration, is

$$\mathrm{HH}^R(A_N^a) \boxtimes_{R[B^{\mathrm{cy}}(M^a)] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}]} R[\tilde{B}^{\mathrm{rep}}(M^a)],$$

which in turn, by definition (see Definition 4.1.19), is $\mathrm{HH}^{(R,N)}(A_N^a, M^a)$. By the diagram (10), we conclude that the logification construction induces a weak equivalence

$$\mathrm{HH}^{(R,N)}(A_N^a, M) \to \mathrm{HH}^{(R,N)}(A_N^a, M^a).$$

This concludes the proof, as the morphism $\mathrm{HH}^R(A, M) \to \mathrm{HH}^R(A_N^a, M)$ is a weak equivalence: there is a weak equivalence of homotopy pushout data

$$\begin{tikzcd}
\mathrm{HH}^R(A) \arrow[d, "\simeq"] & R[B^{\mathrm{cy}}(M)] \arrow[l, tail] \arrow[r, tail] \arrow[d, equal] & R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] \arrow[d, equal] \\
\mathrm{HH}^R(A_N^a) & R[B^{\mathrm{cy}}(M)] \arrow[l] \arrow[r, tail] & R[B^{\mathrm{cy}}(M)^{\mathrm{rep}}],
\end{tikzcd}$$

where the left-hand morphism is a weak equivalence since $A \to A_N^a$ is, so that Lemma 3.1.5 applies. $\square$

We now start comparing $\mathrm{HH}^{(R,N)}(A, M^a)$ and $\mathrm{HH}^{(R,N^a)}(A, M^a)$. We first construct an object $A_{N^a}^a$ in a manner analogous to that of the constructions in 4.2.2 and 4.2.10:

Construction 4.2.12. Let $(R, N, \beta) \to (A, M, \alpha)$ be a cofibration of simplicial pre-log R-algebras, where (R, N) is discrete. Denote by M^a and N^a the logifications of M and N as in Construction 4.2.2: in particular both of the morphisms $N \to N^a$ and $M \to M^a$ are cofibrations in the standard model structure on simplicial commutative monoids of Proposition 4.1.1. First define a simplicial commutative monoid $\tilde{M}^a$ by the diagram

$$\begin{tikzcd}
N \arrow[r, tail] \arrow[d, tail] & N^a \arrow[d, tail] & \\
M^a \arrow[r, tail] & M^a \sqcup_N N^a \arrow[r, tail] \arrow[d] & \tilde{M}^a \arrow[d, two heads, "\simeq"] \\
& M^a \sqcup_N M^a \arrow[r] & M^a,
\end{tikzcd}$$

where the upper square is a pushout and in the lower square one composite is a factorization of the other. Notice that, by applying $R[-]$ to the above diagram, the simplicial commutative R-algebra $R[\tilde{M}^a]$ is a model for $R[M^a]^a$, as defined in Construction 4.2.2. We then define $A^a_{N^a}$, starting from A^a_N from Construction 4.2.10, to be given by the diagram

$$\begin{tikzcd}
R[M^a] \boxtimes_{R[N]} R \arrow[r] \arrow[d, tail] & R[\tilde{M}^a] \boxtimes_{R[N^a]} R \arrow[d, tail] & \\
A^a_N \arrow[r] & A^a_N \boxtimes_{R[M^a] \boxtimes_{R[N]} R} (R[\tilde{M}^a] \boxtimes_{R[N^a]} R) \arrow[r, tail] \arrow[d] & A^a_{N^a} \arrow[d, two heads, "\simeq"] \\
& A^a_N \boxtimes_{R[M^a] \boxtimes_{R[N^a]} R} A^a_N \arrow[r] & A^a_N,
\end{tikzcd}$$

where the upper square is a pushout and in the lower square one composite is a factorization of the other.

Theorem 4.2.13. *Let $(R, N) \to (A, M)$ be a cofibration of simplicial pre-log R-algebras in the model structure of Proposition 4.1.10, where (R, N) is discrete. The logification construction gives weak equivalences*

$$\mathrm{HH}^{(R,N)}(A, M) \xrightarrow{\simeq} \mathrm{HH}^{(R,N)}(A^a_N, M^a) \xrightarrow{\simeq} \mathrm{HH}^{(R,N^a)}(A^a_{N^a}, M^a).$$

Notice that we do not have any assumptions on N; in particular N^a is not necessarily a discrete commutative monoid. The reader may have noticed that the statement of the theorem does not involve the simplicial commutative monoid $\tilde{M}^a$ from Construction 4.2.12. The theorem could have equally well have been stated with M^a replaced with $\tilde{M}^a$: the proof goes through with only notational modifications. It is worthwhile to notice that $\mathrm{HH}^{(R,N^a)}(A^a_{N^a}, M^a)$ is defined as a coproduct over

$$R[B^{\mathrm{cy}}(M^a)] \boxtimes_{R[B^{\mathrm{cy}}(N^a)]} R[B^{\mathrm{cy}}(N^a)^{\mathrm{rep}}],$$

and we do not necessarily have that the morphism $R[B^{\mathrm{cy}}(N^a)] \to R[B^{\mathrm{cy}}(M^a)]$ is a cofibration. In particular, we may not appeal to the chain of equivalences $R[B^{\mathrm{cy}}(N^a)^{\mathrm{rep}}] \simeq R[B^{\mathrm{rep}}(N^a)]$ of Theorem 4.1.15 to obtain that the coproduct is weakly equivalent to $R[B^{\mathrm{cy}}(M^a)] \boxtimes_{R[B^{\mathrm{cy}}(N^a)]} R[B^{\mathrm{rep}}(N^a)]$. For this reason, it may in practice be convenient to replace M^a by the weakly equivalent simplicial commutative monoid $\tilde{M}^a$, for which the morphism $N^a \to \tilde{M}^a$ is a cofibration by construction.

Proof. The equivalence

$$\mathrm{HH}^{(R,N)}(A, M) \xrightarrow{\simeq} \mathrm{HH}^{(R,N)}(A^a_N, M^a)$$

was established in the proof of Theorem 4.2.9. We now establish the second equivalence, and we will use the notation $\tilde{B}^{\mathrm{rep}}(M^a)^a$ for the object appearing in a functorial factorization

$$B^{\mathrm{cy}}(M^a) \sqcup_{B^{\mathrm{cy}}(N^a)} B^{\mathrm{cy}}(N^a)^{\mathrm{rep}} \rightarrowtail \tilde{B}^{\mathrm{rep}}(M^a)^a \xrightarrow{\simeq} B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}.$$

By Theorem 4.2.1 and Remark 4.2.6, we have a weak equivalence

$$R[B^{\mathrm{cy}}(M^a)] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \xrightarrow{\simeq} R[B^{\mathrm{cy}}(M^a)] \boxtimes_{R[B^{\mathrm{cy}}(N^a)]} R[B^{\mathrm{cy}}(N^a)^{\mathrm{rep}}]$$

induced by the logification construction. The lifting axioms provide the dashed weak equivalence $\tilde{B}^{\mathrm{rep}}(M^a) \to \tilde{B}^{\mathrm{rep}}(M^a)^a$ in the following diagram:

$$\begin{tikzcd}
B^{\mathrm{cy}}(M^a) \sqcup_{B^{\mathrm{cy}}(N)} B^{\mathrm{cy}}(N)^{\mathrm{rep}} \arrow[r] \arrow[d, tail] & B^{\mathrm{cy}}(M^a) \sqcup_{B^{\mathrm{cy}}(N^a)} B^{\mathrm{cy}}(N^a)^{\mathrm{rep}} \arrow[r, tail] & \tilde{B}^{\mathrm{rep}}(M^a)^a \arrow[d, two heads, "\simeq"] \\
\tilde{B}^{\mathrm{rep}}(M^a) \arrow[urr, dashed, "\simeq"] \arrow[rr, two heads, "\simeq"'] & & B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}
\end{tikzcd}$$

Hence, we have a weak equivalence of (homotopy) pushout data

$$\begin{tikzcd}
\mathrm{HH}^R(A^a_N) \arrow[d, "\simeq"'] & R[B^{\mathrm{cy}}(M^a)] \boxtimes_{R[B^{\mathrm{cy}}(N)]} R[B^{\mathrm{cy}}(N)^{\mathrm{rep}}] \arrow[l] \arrow[r, tail] \arrow[d, "\simeq"] & R[\tilde{B}^{\mathrm{rep}}(M^a)] \arrow[d, "\simeq"] \\
\mathrm{HH}^R(A^a_{N^a}) & R[B^{\mathrm{cy}}(M^a)] \boxtimes_{R[B^{\mathrm{cy}}(N^a)]} R[B^{\mathrm{cy}}(N^a)^{\mathrm{rep}}] \arrow[l] \arrow[r, tail] & R[\tilde{B}^{\mathrm{rep}}(M^a)^a]
\end{tikzcd}$$

which induces a weak equivalence on (homotopy) pushouts. By definition, this is a weak equivalence

$$\mathrm{HH}^{(R,N)}(A^a_N, M^a) \xrightarrow{\simeq} \mathrm{HH}^{(R,N^a)}(A^a_{N^a}, M^a),$$

which concludes the proof. □

In conclusion, using the homotopy invariant definition of logification of Construction 4.2.2, we find that our definition $\mathrm{HH}^{(R,N)}(A, M)$ of relative log Hochschild homology from Definition 4.1.19 is invariant under logification for both N and M. As a final remark, we notice that in the setting of Construction 4.2.12, the morphism

$$R[M^a] \boxtimes_{R[N^a]} R \to A^a_{N^a}$$

is not necessarily a cofibration, while this is the case is we replace M^a with $\tilde{M}^a$. This is another reason as to why it may be convenient to work with $\tilde{M}^a$ in this context, as we then obtain that $(A, \tilde{M}^a)$ is cofibrant over (R, N^a) in the model structure of Proposition 4.1.10.

Example 4.2.14. Let $f\colon A \to B$ be a morphism of discrete valuation rings with uniformizers π_A and π_B respectively. Write $f(\pi_A) = u\pi_B^n$ for some unit u and natural number n, and suppose for simplicity that $u = 1$. Consider the associated morphism of pre-log rings $(A, \langle\pi_A\rangle) \to (B, \langle\pi_B\rangle)$. In this case, the we have

$$\mathrm{HH}^{(A,\langle\pi_A\rangle)}(B, \langle\pi_B\rangle) \simeq \mathrm{HH}^{(A,\langle\pi_A\rangle)}(B, B - \{0\}) \simeq \mathrm{HH}^{(A,A-\{0\})}(B, B - \{0\}),$$

where we have suppressed the cofibrant replacements from the notation (see Example 2.2.8). We remark that as that the monoids $\langle\pi_A\rangle$ and $\langle\pi_B\rangle$ are free, and hence cofibrant, we have that e.g. the defining pushout square

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d, tail] & \mathrm{GL}_1(A) \arrow[d] \\ \langle \pi_A \rangle \arrow[r] & A - \{0\} \end{tikzcd}$$

of the logification of the discrete commutative monoid $\langle \pi_A \rangle$ is already a homotopy pushout.

There are examples of pre-log ring spectra for which the analogue of Theorem 4.2.1 for logarithmic THH has proven useful for computations [RSS15, Example 4.25]. The hope is that the above results generalize and interesting examples pop up in this context.

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