# *Department of APPLIED MATHEMATICS*

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nonlinear degenerate parabolic Equations
with spatially dependent flux Function.

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Steinar Evje, Kenneth Hvistendahl Karlsen
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Report no. 146 June 2000

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ISSN 0084-778x

# A continous Dependence Result for nonlinear degenerate parabolic Equations with spatially dependent flux Function.

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# A CONTINUOUS DEPENDENCE RESULT FOR NONLINEAR DEGENERATE PARABOLIC EQUATIONS WITH SPATIALLY DEPENDENT FLUX FUNCTION

STEINAR EVJE, KENNETH HVISTENDAHL KARLSEN, AND NILS HENRIK RISEBRO

ABSTRACT. We study entropy solutions of nonlinear degenerate parabolic equations of form $u_t + \operatorname{div}\big(k(x)f(u)\big) = \Delta A(u)$, where $k(x)$ is a vector-valued function and $f(u), A(u)$ are scalar functions. We prove a result concerning the continuous dependence on the initial data, the flux function $k(x)f(u)$, and the diffusion function $A(u)$. This paper complements previous work [7] by two of the authors, which contained a continuous dependence result concerning the initial data and the flux function $k(x)f(u)$.

## 1. INTRODUCTION

In this paper we are concerned with entropy solutions of the initial value problem

$$u_t + \operatorname{div}\big(k(x)f(u)\big) = \Delta A(u), \quad u(x,0) = u_0(x) \tag{1.1}$$

for $(x,t) \in \Pi_T = \mathbb{R}^d \times (0,T)$ with $T > 0$ fixed. In (1.1), $u(x,t)$ is the scalar unknown function that is sought, $k(x)f(u)$ is the flux function, and $A = A(u)$ is the diffusion function. We *always* assume that $k : \mathbb{R}^d \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, and $A : \mathbb{R} \to \mathbb{R}$ satisfy

$$\begin{cases} k \in W^{1,1}_{\text{loc}}(\mathbb{R}^d);\ k, \operatorname{div} k \in L^\infty(\mathbb{R}^d);\ f \in \operatorname{Lip}_{\text{loc}}(\mathbb{R});\ f(0) = 0; \\ A \in \operatorname{Lip}_{\text{loc}}(\mathbb{R}) \text{ and } A(\cdot) \text{ is nondecreasing with } A(0) = 0. \end{cases} \tag{1.2}$$

Since $A'(\cdot)$ is allowed to be zero on an interval $[\alpha, \beta]$ (the scalar conservation law is a special case of (1.1)), solutions may become discontinuous in finite time even with a smooth initial function. Consequently, one needs to interpret (1.1) in the weak sense. However, weak solutions are in general not uniquely determined by their initial data and an entropy condition must be imposed to single out the physically correct solution.

**Definition 1.1.** *A measurable function $u = u(x,t)$ is an entropy solution of* (1.1) *if*

**D.1** $u \in L^1(\Pi_T) \cap L^\infty(\Pi_T) \cap C(0,T; L^1(\mathbb{R}^d))$ *and* $A(u) \in L^2(0,T; H^1(\mathbb{R}^d))$.

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*Date*: May 31, 2000.
*Key words and phrases.* nonlinear degenerate parabolic equation, spatially dependent flux function, entropy solution, continuous dependence.

**D**.2 *For all $c \in \mathbb{R}$ and all non-negative test functions in $C_0^\infty(\Pi_T)$,*

$$\iint_{\Pi_T} \Big( |u-c|\phi_t + \operatorname{sign}(u-c)\, k(x)\big(f(u)-f(c)\big)\cdot\nabla\phi + |A(u)-A(c)|\Delta\phi \tag{1.3}$$
$$- \operatorname{sign}(u-c)\operatorname{div}k(x) f(c)\phi \Big)\, dt\, dx \geq 0.$$

**D**.3 *Essentially as $t\downarrow 0$, $\|u(\cdot,t)-u_0(x)\|_{L^1(\mathbb{R}^d)} \to 0$.*

Following Kružkov [9] and the recent work of Carrillo [3], two of the authors proved in [7] a uniqueness result for entropy solutions of the more general equation

$$u_t + \operatorname{div} f(x,t,u) = \Delta A(u) + q(x,t,u), \tag{1.4}$$

where the flux function $f = f(x,t,u)$ may have a non-smooth spatial dependence, see [7] for the precise assumptions on $f$ and $q$ in (1.4). Moreover, in the $L^\infty(0,T;BV(\mathbb{R}^d))$ class of entropy solutions, the authors of [7] proved continuous dependence on the initial function $u_0$ and flux function in the case $f(x,t,u) = k(x)f(u)$. However, in [7] the question of continuous dependence with respect to the diffusion function $A$ was left open. Recently, Cockburn and Gripenberg [4] have obtained such a result when $k(x) = 1$. Their result does *not*, however, imply uniqueness of the entropy solution (from reasons that will become apparent later). Let us also mention that results regarding continuous dependence on the flux function in scalar conservation laws ($A' \equiv 0$) have been obtained in [11, 1, 8].

The purpose of the present paper is to combine the ideas in [7] with those in [4] and prove a version of Theorem 1.3 in [7] which also includes continuous dependence on the diffusion function $A$. To state our continuous dependence result, let us introduce

$$v_t + \operatorname{div}\big(l(x)g(v)\big) = \Delta B(v), \quad v(x,0) = v_0(x). \tag{1.5}$$

We assume that $l:\mathbb{R}^d\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$, and $B:\mathbb{R}\to\mathbb{R}$ satisfy the same conditions as $k, f, A$, see (1.2). We now state our main result:

**Theorem 1.1.** *Let $v,u \in L^\infty(0,T;BV(\mathbb{R}^d))$ be the unique entropy solutions of* (1.5), (1.1) *with initial data $v_0,u_0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)\cap BV(\mathbb{R}^d)$, respectively. Suppose that $v,u$ take values in in the closed interval $I\subset\mathbb{R}$ and define $V_v = \sup_{t\in(0,T)} |v(\cdot,t)|_{BV(\mathbb{R}^d)}$. Suppose $k\in \operatorname{Lip}(\mathbb{R}^d)$ and $\operatorname{div}k \in BV(\mathbb{R}^d)$. Then for almost all $t\in(0,T)$,*

$$\begin{aligned} \|v(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbb{R}^d)} &\leq \|v_0-u_0\|_{L^1(\mathbb{R}^d)} \\ &+ C_{\text{Conv}}\, t\Big(\|l-k\|_{L^\infty(\mathbb{R}^d)} + |l-k|_{BV(\mathbb{R}^d)} + \|g-f\|_{L^\infty(I)} + \|g-f\|_{\operatorname{Lip}(I)}\Big) \\ &+ C_{\text{Diff}}\sqrt{t}\,\Big\|\sqrt{B'}-\sqrt{A'}\Big\|_{L^\infty(I)}, \end{aligned} \tag{1.6}$$

*for some constants $C_{\text{Conv}}, C_{\text{Diff}}$. Here $C_{\text{Conv}}$ depends on $V_v$, $\|k\|_{L^\infty(\mathbb{R}^d)}$, $|k|_{BV(\mathbb{R}^d)}$, $\|g\|_{L^\infty(I)}$, $\|g\|_{\operatorname{Lip}(I)}$ and $C_{\text{Diff}}$ depends on $V_v$, $\|k\|_{\operatorname{Lip}(\mathbb{R}^d)}$, $|\operatorname{div}k|_{BV(\mathbb{R}^d)}$, $\|g\|_{L^\infty(I)}$, $\|g\|_{\operatorname{Lip}(I)}$. The explicit form of the constants $C_{\text{Conv}}, C_{\text{Diff}}$ can be traced from the proof of Theorem 1.1.*

We remark that existence of $BV(\Pi_T)$ entropy solutions of (1.1) (or (1.4)) can be proved by the vanishing viscosity method provided $f, A, q, u_0$ are sufficiently smooth, see Vo'lpert

D.2 For all $c \in \mathbb{R}$ and all nonnegative test functions in $C_0^\infty(\Pi_T)$,

$$
(1.3)\qquad \iint_{\Pi_T} \Big( |u - c|\phi_t + \operatorname{sgn}(u-c)\big[k(x)(f(u) - f(c)) \cdot \nabla\phi + |A(u) - A(c)|\Delta\phi\big] - \operatorname{sgn}(u-c)\operatorname{div}(k(x)) f(c)\phi \Big)\, dt\, dx \geq 0.
$$

D.3 Essentially as $t \downarrow 0$, $\|u(\cdot,t) - u_0(x)\|_{L^1_{loc}(\mathbb{R}^d)} \to 0$.

Following Kružkov [9] and the recent work of Carrillo [3], two of the authors proved in [7] a uniqueness result for entropy solutions of the more general equation

$$
(1.4)\qquad u_t + \operatorname{div} f(x,t,u) = \Delta A(u) + q(x,t,u),
$$

where the flux function $f = f(x,t,u)$ may have a non-smooth spatial dependence, see [7] for the precise assumptions on $f$ and $q$ in (1.4). Moreover, in the $L^\infty(0,T;BV(\mathbb{R}^d))$ class of entropy solutions, the authors of [7] proved continuous dependence on the initial function $u_0$ and flux function in the case $f(x,t,u) = k(x)f(u)$. However, in [7] the question of continuous dependence with respect to the diffusion function $A$ was left open. Recently, Cockburn and Gripenberg [4] have obtained such a result when $k(x) \equiv 1$. Their result does not, however, imply uniqueness of the entropy solution (for reasons that will become apparent later). Let us also mention that results regarding continuous dependence on the flux function in scalar conservation laws ($A \equiv 0$) have been obtained in [11, 3, 6].

The purpose of the present paper is to combine the ideas in [7] with those in [4] and prove a version of Theorem 1.3 in [7] which also includes continuous dependence on the diffusion function $A$. To state our continuous dependence result, let us introduce

$$
(1.5)\qquad v_t + \operatorname{div}(l(x)g(v)) = \Delta B(v), \qquad v(x,0) = v_0(x).
$$

We assume that $l : \mathbb{R}^d \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $B : \mathbb{R} \to \mathbb{R}$ satisfy the same conditions as $k, f, A$, see (1.2). We now state our main result.

**Theorem 1.1.** *Let $u, v \in L^\infty(0,T;BV(\mathbb{R}^d))$ be the unique entropy solutions of (1.1), (1.5) with initial data $u_0, v_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$, respectively. Suppose that $u, v$ take values in the closed interval $I \subset \mathbb{R}$ and define $V_f = \sup_{u \in I} |f'(u)|$. Suppose $k \in \mathrm{Lip}(\mathbb{R}^d)$ and $\operatorname{div} k \in BV(\mathbb{R}^d)$. Then for almost all $t \in (0,T)$,*

$$
\begin{aligned}
\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R}^d)} &\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} \\
&\quad + C_{k,l,f,g}\, t \Big( \|f - g\|_{\mathrm{Lip}(I)} + \|k - l\|_{L^\infty(\mathbb{R}^d)} + |k - l|_{BV(\mathbb{R}^d)} + \|\operatorname{div}(k - l)\|_{L^1(\mathbb{R}^d)} \Big) \qquad (1.6)\\
&\quad + C_{BV}\sqrt{t}\, \Big\| \sqrt{A'} - \sqrt{B'} \Big\|_{L^\infty(I)},
\end{aligned}
$$

*for some constants $C_{k,l,f,g}$, $C_{BV}$. Here $C_{k,l,f,g}$ depends on $V_f$, $\|k\|_{L^\infty(\mathbb{R}^d)}$, $|u_0|_{BV(\mathbb{R}^d)}$, $\|\operatorname{div} k\|_{L^1(\mathbb{R}^d)}$, and $C_{BV}$ depends on $V_f$, $|u_0|_{BV(\mathbb{R}^d)}$, $|v_0|_{BV(\mathbb{R}^d)}$. The explicit form of the constants $C_{k,l,f,g}$, $C_{BV}$ can be derived from the proof of Theorem 1.1.*

We remark that existence of BV ($L^1$) entropy solutions of (1.1) (or (1.4)) can be proved by the vanishing viscosity method provided $f, k, q, u_0$ are sufficiently smooth, see Volpert and Hudjaev [13]. Existence of $L^\infty(0,T;BV(\mathbb{R}^d))$ entropy solutions of (1.1) is guaranteed if $\mathrm{div}k \in BV(\mathbb{R}^d)$. This follows from the results obtained by Karlsen and Risebro [6], who proved convergence of finite difference schemes for degenerate parabolic equations with rough coefficients. For an overview of the literature dealing with numerical methods for approximating entropy solutions of degenerate parabolic equations, we refer to [5]. In this connection, we should mention that the arguments used to prove Theorem 1.1 can be used to prove error estimates for numerical methods. This will be discussed elsewhere.

For later use, we mention that the results in [6] can be used to prove the existence of $L^\infty(0,T;BV(\mathbb{R}^d))$ entropy solutions of (1.1) by the vanishing viscosity method. To this end, consider the uniformly parabolic problem

$$u_t^\mu + \mathrm{div}\big(k(x)f(u^\mu)\big) = \Delta A(u^\mu) + \mu\Delta u^\mu, \qquad u^\mu(x,0) = u_0(x), \tag{1.7}$$

for $\mu > 0$. Provided $k, f, A, u_0$ are sufficiently smooth, it is well known that there exists a unique classical (and hence entropy) solution of (1.7) which possesses all the continuous derivatives occurring in the partial differential equation in (1.7). Using the space and time translation estimates derived in [6], it is not difficult to show that $u^\mu$ converges in $L^1_{\mathrm{loc}}(\Pi_T)$ as $\mu \downarrow 0$ to an entropy solution $u$ of (1.1) (see also Vo'lpert and Hudjaev [13]). Convergence of the viscosity method and smoothness of the solution $u^\mu$ of (1.7) will be used in the proof of Theorem 1.1. Finally, to relax the smoothness assumptions on $k, f, A, u_0$ needed by the vanishing viscosity method to those actually required by Theorem 1.1, one can approximate $k, f, A, u_0$ by smoother functions and then use Theorem 1.1 to pass to the limit as the smoothing parameter tends to zero. We will not go into further details about this limiting operation but instead leave this as an exercise for the interested reader. Also, in this paper we have exclusively treated the initial value problem but it is possible to treat various initial-boundary value problems. For some work in this direction, we refer to Bürger, Evje, and Karlsen [2] and Rouvre and Gagneux [12].

Before ending this section, we present an immediate corollary of Theorem 1.1 concerning the convergence rate of the viscosity method.

**Corollary 1.1.** *Let $u \in L^\infty(0,T;BV(\mathbb{R}^d))$ be the unique entropy solution of* (1.1) *with initial data $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ and let $u^\mu$ be the corresponding viscous approximation of $u$, i.e., $u^\mu$ is the unique classical solution of* (1.7). *Suppose $k \in \mathrm{Lip}(\mathbb{R}^d)$ and $\mathrm{div}k \in BV(\mathbb{R}^d)$. Then for almost all $t \in (0,T)$,*

$$\|u(\cdot,t) - u^\mu(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le C\sqrt{\mu}, \tag{1.8}$$

*for some non-negative constant $C$.*

The remaining part of this paper is devoted to proving Theorem 1.1.

## 2. Some preliminaries

Let $u$ be an entropy solution of (1.1). It easy to see from Definition 1.1 that the equality

$$\iint_{\Pi_T} \Big(u\phi_t + \big[k(x)f(u) - \nabla A(u)\big]\cdot\nabla\phi\Big)\,dt\,dx = 0 \tag{2.1}$$

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**Corollary 1.1.** *Let $u \in L^\infty(0,T;BV(\mathbb{R}^d))$ be the unique entropy solution of (1.1) with initial data $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$, and let $u^\mu$ be the corresponding viscous approximation of $u$, i.e., $u^\mu$ is the unique classical solution of (1.7). Suppose $k \in \operatorname{Lip}(\mathbb{R}^d)$ and $\operatorname{div} k \in BV(\mathbb{R}^d)$. Then for almost all $t \in (0,T)$*

$$\|u(\cdot,t) - u^\mu(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le C\sqrt{\mu}, \tag{1.8}$$

*for some non-negative constant $C$.*

The remaining part of this paper is devoted to proving Theorem 1.1.

## 2. Some preliminaries

Let $u$ be an entropy solution of (1.1). It easy to see from Definition 1.1 that the equality

$$\iint_{\Pi_T} \Big( u\phi_t + [k(x)f(u) - \nabla A(u)] \cdot \nabla\phi \Big)\, dt\, dx = 0 \tag{2.1}$$

holds for all $\phi \in L^2(0,T;H_0^1(\mathbb{R}^d)) \cap W^{1,1}(0,T;L^\infty(\mathbb{R}^d))$. Let $\langle \cdot, \cdot \rangle$ denote the usual pairing between $H^{-1}(\mathbb{R}^d)$ and $H_0^1(\mathbb{R}^d)$. From (2.1), we conclude that $\partial_t u \in L^2(0,T;H^{-1}(\mathbb{R}^d))$, so that the equality (2.1) can restated as

$$
-\int_0^T \langle \partial_t u, \phi \rangle \, dt + \iint_{\Pi_T} \Big( \big[k(x)f(u) - \nabla A(u)\big] \cdot \nabla \phi \Big) \, dt\, dx = 0 \tag{2.2}
$$

for all $\phi \in L^2(0,T;H_0^1(\mathbb{R}^d)) \cap W^{1,1}(0,T;L^\infty(\mathbb{R}^d))$. The fact that an entropy solution $u$ satisfies (2.2) is important for the uniqueness proof [3, 7]. We recall that after the recent work of Carrillo [3], the uniquness proof for entropy solutions of degenerate parabolic equations has become very similar to the "doubling of variables" proof introduced by Kružkov [9] many years ago for first order hyperbolic equations. However, to apply the "doubling of variables" device to second order equations, one needs a version of an important lemma stated and proved first in [3]. This lemma indentifies a certain entropy dissipation term that must be taken into account if the "doubling device" is going to work.

Before stating this lemma, we need to introduce some notation. For $\varepsilon > 0$, set

$$
\operatorname{sign}_\varepsilon(\tau) = \begin{cases} -1, & \tau < \varepsilon, \\ \tau/\varepsilon, & \varepsilon \le \tau \le \varepsilon, \\ 1 & \tau > \varepsilon. \end{cases}
$$

Moreover, we let $A^{-1} : \mathbb{R} \to \mathbb{R}$ denote the unique left-continuous function satisfying $A^{-1}(A(u)) = u$ for all $u \in \mathbb{R}$. By $E$ we denote the set $E = \{r : A^{-1}(\cdot) \text{ discontinuous at } r\}$. Note that $E$ is associated with the set of points $\{u : A'(u) = 0\}$ at which the operator $u \mapsto \Delta A(u)$ is degenerate elliptic. We can now state following lemma:

**Lemma 2.1** ([7]). *Let $u$ be an entropy solution of* (1.1). *Then, for any non-negative $\phi \in C_0^\infty(\Pi_T)$ and any $c \in \mathbb{R}$ such that $A(c) \notin E$, we have*

$$
\begin{aligned}
&\iint_{\Pi_T} \Big( |u-c|\phi_t + \operatorname{sign}(u-c)\big[k(x)\big(f(u)-f(c)\big) - \nabla A(u)\big] \cdot \nabla\phi \\
&\qquad - \operatorname{sign}(u-c)\operatorname{div} k(x) f(c)\phi \Big)\, dt\, dx = \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \big|\nabla A(u)\big|^2 \operatorname{sign}_\varepsilon'(A(u)-A(c))\,\phi\, dt\, dx.
\end{aligned} \tag{2.3}
$$

Note that if (1.1) is uniformly parabolic ($A' > 0$), then the set $E$ is empty and a weak solution is automatically an entropy solution. The idea of the proof of Lemma 2.1 is to use $[\operatorname{sign}_\varepsilon(A(u) - A(c))\,\phi] \in L^2(0,T;H_0^1(\mathbb{R}))$ as a test function in (2.2) together with a "weak chain rule" to deal with the time derivative and subsequently sending $\varepsilon \downarrow 0$. We refer to [7] for details on the proof of Lemma 2.1 (see Carrillo [3] when $k(x) \equiv 1$).

Although the identification of the entropy dissipation term (i.e., the right-hand side of (2.3)) is the cornerstone of the uniqueness proof as well as the proof of continuous dependence on the flux function $k(x)f(u)$, it seems difficult to obtain continuous dependence on the diffusion function $A(u)$ with this form of the dissipation term. However, it is possible holds for all $\phi \in L^2(0,T;H_0^1(\mathbb{R}^d)) \cap W^{1,1}(0,T;L^\infty(\mathbb{R}^d))$. Let $\langle \cdot, \cdot \rangle$ denote the usual pairing between $H^{-1}(\mathbb{R}^d)$ and $H_0^1(\mathbb{R}^d)$. From (2.1), we conclude that $\partial_t u \in L^2(0,T;H^{-1}(\mathbb{R}^d))$, so that the equality (2.1) can restated as

$$-\int_0^T \langle \partial_t u, \phi \rangle \, dt + \iint_{\Pi_T} \Big( [k(x)f(u) - \nabla A(u)] \cdot \nabla \phi \Big) \, dx \, dt = 0 \tag{2.2}$$

for all $\phi \in L^2(0,T;H_0^1(\mathbb{R}^d)) \cap W^{1,1}(0,T;L^\infty(\mathbb{R}^d))$. The fact that an entropy solution satisfies (2.2) is important for the uniqueness proof [3, 7]. We recall that after the recent work of Carrillo [3], the uniqueness proof for entropy solutions of degenerate parabolic equations has become very similar to the "doubling of variables" proof introduced by Kružkov [9] many years ago for first order hyperbolic equations. However, to apply the "doubling of variables" device to second order equations, one needs a version of an important lemma stated and proved first in [3]. This lemma identifies a certain entropy dissipation term that must be taken into account if the "doubling device" is going to work.

Before stating this lemma, we need to introduce some notation. For $\varepsilon > 0$, set

$$\operatorname{sign}_\varepsilon(r) = \begin{cases} -1, & r < -\varepsilon, \\ r/\varepsilon, & -\varepsilon \le r \le \varepsilon, \\ 1, & r > \varepsilon. \end{cases}$$

Moreover, we let $A^{-1} : \mathbb{R} \to \mathbb{R}$ denote the unique left-continuous function satisfying $A^{-1}(A(u)) = u$ for all $u \in \mathbb{R}$. By $E$ we denote the set $E = \{r : A^{-1}(\cdot) \text{ discontinuous at } r\}$. Note that $E$ is associated with the set of points $\{u : A'(u) = 0\}$ at which the operator $u \mapsto \Delta A(u)$ is degenerate elliptic. We can now state following lemma:

**Lemma 2.1** ([7]). *Let $u$ be an entropy solution of (1.1). Then, for any non-negative $\phi \in C_0^\infty(\Pi_T)$ and any $c \in \mathbb{R}$ such that $A(c) \notin E$, we have*

(2.3)

$$\iint_{\Pi_T} \Big( |u - c|\phi_t + \operatorname{sign}(u - c)\,[k(x)(f(u) - f(c)) - \nabla A(u)] \cdot \nabla \phi$$

$$- \operatorname{sign}(u - c)\operatorname{div}(k(x)f(c))\phi \Big) \, dx \, dt = \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} |\nabla A(u)|^2 \operatorname{sign}_\varepsilon'(A(u) - A(c))\,\phi \, dt \, dx.$$

Note that if (1.1) is uniformly parabolic ($A' > 0$), then the set $E$ is empty and a weak solution is automatically an entropy solution. The idea of the proof of Lemma 2.1 is to use $\operatorname{sign}_\varepsilon(A(u) - A(c))\phi \in L^2(0,T;H_0^1(\mathbb{R}^d))$ as a test function in (2.2) together with a "weak chain rule" to deal with the time derivative and subsequently sending $\varepsilon \downarrow 0$. We refer to [7] for details on the proof of Lemma 2.1 (see Carrillo [3] when $k(x) \equiv 1$).

Although the identification of the entropy dissipation term (i.e., the right hand side of (2.3)) is the cornerstone of the uniqueness proof as well as the proof of continuous dependence on the flux function $k(x)f(u)$, it seems difficult to obtain continuous dependence on the diffusion function $A(u)$ with this form of the dissipation term. However, it is possible to derive a version of (2.3) in which a different (form of the) entropy dissipation term appears. But this seems possible only if $u$ is smooth or at least belongs to $L^2(0,T;H^1(\mathbb{R}^d))$. Consequently, Theorem 1.1 does *not* yield uniqueness of the entropy solution.

Provided (1.1) is unformly parabolic and hence admits a unique classical solution, the following version of Lemma 2.1 holds:

**Lemma 2.2.** *Suppose* (1.1) *is uniformly parabolic (i.e., $A' > 0$). Let $u$ be a classical solution of* (1.1). *Then, for any non-negative $\phi \in C_0^\infty(\Pi_T)$ and any $c \in \mathbb{R}$, we have*

$$\iint_{\Pi_T} \Big( |u-c|\phi_t + \operatorname{sign}(u-c)\,\big[k(x)\big(f(u)-f(c)\big) - \nabla A(u)\big]\cdot\nabla\phi \\ - \operatorname{sign}(u-c)\operatorname{div} k(x) f(c)\phi\Big)\,dt\,dx = \lim_{\varepsilon\downarrow 0}\iint_{\Pi_T} A'(u)\big|\nabla u\big|^2 \operatorname{sign}_\varepsilon'(u-c)\,\phi\,dt\,dx. \tag{2.4}$$

This lemma can proved by using $[\operatorname{sign}_\varepsilon(u-c)\,\phi]$ as a test function (this is indeed a test function since $u$ is smooth!) in (2.2) and then sending $\varepsilon \downarrow 0$. The proof of Lemma 2.2 is similar to the proof of Lemma 2.1 and it is therefore omitted. Notice the difference between the entropy dissipation terms in (2.3) and (2.4).

## 3. Proof of Theorem 1.1

We are now interested in estimating the $L^1$ difference between the entropy solution $v$ of (1.5) and the entropy solution $u$ of (1.1). In view of the discussion in Section 1, we will prove Theorem 1.1 under the assumption that $B', A' > 0$ so that (1.5) and (1.1) become uniformly parabolic problems and therefore admit unique classical solutions. To treat the degenerate parabolic case ($B', A' \geq 0$), we proceed via the vanishing viscosity method, i.e., we replace $\Delta A(u)$ and $\Delta B(v)$ in (1.1) and (1.5) by $\Delta A(u) + \eta\Delta u$ and $\Delta B(v) + \eta\Delta v$, respectively, and then send $\eta \downarrow 0$.

The argument given below is based on Lemma 2.2 and Kružkov's idea of doubling the number of dependent variables together with a penalization procedure. Moreover, it is inspired by Carrillo [3] and Cockburn and Gripenberg [4]. Strictly speaking, we could have carried out the argument below under the assumptions that the (entropy) solutions $v, u$ belong to $L^2(0,T;H^1(\mathbb{R}^d))$, i.e., $v, u$ need not be (entirely!) classical solutions.

Following [9, 10], we now specify a non-negative test function $\phi \in C_0^\infty(\Pi_T \times \Pi_T)$. To this end, introduce a nonnegative function $\delta \in C_0^\infty(\mathbb{R})$ which satisfies $\delta(\sigma) = \delta(-\sigma)$, $\delta(\sigma) \equiv 0$ for $|\sigma| \geq 1$, and $\int_{\mathbb{R}} \delta(\sigma)\,d\sigma = 1$. For $\rho_0 > 0$, let $\delta_{\rho_0}(\sigma) = \frac{1}{\rho_0}\delta\big(\frac{\sigma}{\rho_0}\big)$. Pick two (arbitrary but fixed) Lebesgue points $\nu, \tau \in (0,T)$ of $\|v(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R}^d)}$.

For any $\alpha_0 \in \big(0, \min(\nu, T-\tau)\big)$, let $W_{\alpha_0}(t) = H_{\alpha_0}(t-\nu) - H_{\alpha_0}(t-\tau)$, where $H_{\alpha_0}(t) = \int_{-\infty}^t \delta_{\alpha_0}(s)\,ds$. We then define $\phi = \phi(x,t,y,s)$ by

$$\phi(x,t,y,s) = W_{\alpha_0}(t)\delta_\rho(x-y)\delta_{\rho_0}(t-s), \qquad \rho, \rho_0 > 0. \tag{3.1}$$

Observe that $\phi_t + \phi_s = \big[\delta_{\alpha_0}(t-\nu) - \delta_{\alpha_0}(t-\tau)\big]\delta_\rho(x-y)\delta_{\rho_0}(t-s)$ and $\nabla_x\phi + \nabla_y\phi \equiv 0$.

to derive a version of (2.3) in which a different (form of the) entropy dissipation term appears. But this seems possible only if $u$ is smooth or at least belongs to $L^2(0,T;H^2(\mathbb{R}^d))$. Consequently, Theorem 1.1 does not yield uniqueness of the entropy solution.

Provided (1.1) is uniformly parabolic and hence admits a unique classical solution, the following version of Lemma 2.2 holds:

**Lemma 2.2.** *Suppose (1.1) is uniformly parabolic (i.e., $A' > 0$). Let $u$ be a classical solution of (1.1). Then, for any non-negative $\phi \in C_0^\infty(\Pi_T)$ and any $c \in \mathbb{R}$, we have*

$$\iint_{\Pi_T}\Big(|u-c|\phi_t + \operatorname{sign}(u-c)\big[k(x)(f(u)-f(c)) - \nabla A(u)\big]\cdot\nabla\phi$$

(2.4)

$$- \operatorname{sign}(u-c)\operatorname{div}\big(k(x)f(c)\big)\phi\Big)\,dx\,dt = \lim_{\varepsilon\downarrow 0}\iint_{\Pi_T} A'(u)|\nabla u|^2\operatorname{sign}_\varepsilon'(u-c)\,\phi\,dx\,dt.$$

This lemma can proved by using $\operatorname{sign}_\varepsilon(u-c)\phi$ as a test function (this is indeed a test function since $u$ is smooth) in (2.2) and then sending $\varepsilon \downarrow 0$. The proof of Lemma 2.2 is similar to the proof of Lemma 2.1 and it is therefore omitted. Notice the difference between the entropy dissipation terms in (2.3) and (2.4).

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The argument given below is based on Lemma 2.2 and Kružkov's idea of doubling the number of dependent variables together with a regularization procedure. Moreover, it is inspired by Carrillo [3] and Cockburn and Gripenberg [4]. Strictly speaking, we could have carried out the argument below under the assumption that the (entropy) solutions $v$, $u$ belong to $L^2(0,T;H^2(\mathbb{R}^d))$, i.e., $v$, $u$ need not be (entirely) classical solutions.

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For any $\alpha_0 \in (0, \min(\nu, T-\tau))$, let $W_{\alpha}(t) = H_{\alpha}(t-\nu) - H_{\alpha}(t-\tau)$, where $H_{\alpha}(t) = \int_{-\infty}^{t}\delta_{\alpha}(s)\,ds$. We then define $\phi = \phi(x,t,y,s)$ by

$$\phi(x,t,y,s) = W_\alpha(t)\delta_r(x-y)\delta_{r_0}(t-s), \qquad r, r_0 > 0. \tag{3.1}$$

Observe that $\phi_t + \phi_s = [\delta_\alpha(t-\nu) - \delta_\alpha(t-\tau)]\delta_r(x-y)\delta_{r_0}(t-s)$ and $\nabla_x\phi + \nabla_y\phi = 0$.

Applying Lemma 2.2 with $v = v(x,t)$ and $c = u(y,s)$ and then integrating the resulting equation with respect to $(y,s) \in \Pi_T$, we get

$$
\begin{aligned}
-\iiiint\limits_{\Pi_T\times\Pi_T} \Big( |v-u|\phi_t + \operatorname{sign}(v-u)\big[l(x)\big(g(v)-g(u)\big) - \nabla_x B(v)\big]\cdot\nabla_x\phi \\
- \operatorname{sign}(v-u)\operatorname{div}_x l(x) g(u)\phi\Big)\, dt\,dx\,ds\,dy \qquad (3.2) \\
= -\lim_{\varepsilon\downarrow 0}\iiiint\limits_{\Pi_T\times\Pi_T} B'(v)\big|\nabla_x v\big|^2 \operatorname{sign}'_\varepsilon(v-u)\,\phi\,dt\,dx\,ds\,dy.
\end{aligned}
$$

Similarly, applying Lemma 2.2 with $u = u(y,s)$ and $c = v(x,t)$ and then integrating the resulting equation with respect to $(x,t) \in \Pi_T$, we get

$$
\begin{aligned}
-\iiiint\limits_{\Pi_T\times\Pi_T} \Big( |u-v|\phi_t + \operatorname{sign}(u-v)\big[k(y)\big(f(u)-f(v)\big) - \nabla_y A(u)\big]\cdot\nabla_y\phi \\
- \operatorname{sign}(u-v)\operatorname{div}_y k(y) f(v)\phi\Big)\, dt\,dx\,ds\,dy \qquad (3.3) \\
= -\lim_{\varepsilon\downarrow 0}\iiiint\limits_{\Pi_T\times\Pi_T} A'(u)\big|\nabla_y u\big|^2 \operatorname{sign}'_\varepsilon(u-v)\,\phi\,dt\,dx\,ds\,dy.
\end{aligned}
$$

Following [7] when adding (3.2) and (3.3), we get

$$
\begin{aligned}
-\iiiint\limits_{\Pi_T\times\Pi_T} \Big( |v-u|\big(\phi_t+\phi_s\big) + I_{\text{Conv}} - I^1_{\text{Diff}}\Big)\, dt\,dx\,ds\,dy \\
= -\lim_{\varepsilon\downarrow 0}\iiiint\limits_{\Pi_T\times\Pi_T} \Big(B'(v)\big|\nabla_x v\big|^2 + A'(u)\big|\nabla_y u\big|^2\Big)\operatorname{sign}'_\varepsilon(v-u)\,\phi\,dt\,dx\,ds\,dy, \\
= -\lim_{\varepsilon\downarrow 0}\iiiint\limits_{\Pi_T\times\Pi_T} \Big(\Big(\sqrt{B'(v)}\nabla_x v - \sqrt{A'(u)}\nabla_y u\Big)^2 \qquad (3.4) \\
+ 2\sqrt{B'(v)}\sqrt{A'(u)}\nabla_x v\cdot\nabla_y u\Big)\operatorname{sign}'_\varepsilon(v-u)\,\phi\,dt\,dx\,ds\,dy, \\
\le -\lim_{\varepsilon\downarrow 0}\iiiint\limits_{\Pi_T\times\Pi_T} I^2_{\text{Diff}}\,dt\,dx\,ds\,dy,
\end{aligned}
$$

where

$$
\begin{aligned}
I_{\text{Conv}} &= \operatorname{sign}(v-u)\Big(\operatorname{div}_x\big[\big(k(y)f(u) - l(x)g(u)\big)\phi\big] - \operatorname{div}_y\big[\big(l(x)g(v) - k(y)f(v)\big)\phi\big]\Big), \\
I^1_{\text{Diff}} &= \operatorname{sign}(v-u)\,\nabla_x B(v)\cdot\nabla_x\phi + \operatorname{sign}(u-v)\,\nabla_y A(u)\cdot\nabla_y\phi, \\
I^2_{\text{Diff}} &= \Big(2\sqrt{B'(v)}\sqrt{A'(u)}\nabla_x v\cdot\nabla_y u\Big)\operatorname{sign}'_\varepsilon(v-u)\,\phi.
\end{aligned}
$$

By the triangle inequality, we get

$$-\iiiint_{\Pi_T\times\Pi_T} |v(x,t)-u(y,s)|\big(\phi_t+\phi_s\big)\,dt\,dx\,ds\,dy \le I + R^t + R^x,$$

where

$$I = -\iiiint_{\Pi_T\times\Pi_T} |v(y,t)-u(y,t)|\big[\delta_{\alpha_0}(t-\nu)-\delta_{\alpha_0}(t-\tau)\big]\delta_\rho(x-y)\delta_{\rho_0}(t-s)\,dt\,dx\,ds\,dy,$$

$$R^t = -\iiiint_{\Pi_T\times\Pi_T} |u(y,t)-u(y,s)|\big[\delta_{\alpha_0}(t-\nu)-\delta_{\alpha_0}(t-\tau)\big]\delta_\rho(x-y)\delta_{\rho_0}(t-s)\,dt\,dx\,ds\,dy,$$

$$R^x = -\iiiint_{\Pi_T\times\Pi_T} |v(x,t)-v(y,t)|\big[\delta_{\alpha_0}(t-\nu)-\delta_{\alpha_0}(t-\tau)\big]\delta_\rho(x-y)\delta_{\rho_0}(t-s)\,dt\,dx\,ds\,dy.$$

It is fairly easy to see that $\lim_{\rho_0\downarrow 0} R^t = 0$ and

$$\begin{aligned}\lim_{\alpha_0\downarrow 0} R^x &= \int_{\mathbb{R}^d}\Big(|v(x,\tau)-v(y,\tau)| - |v(x,\nu)-v(y,\nu)|\Big)\delta_\rho(x-y)\,dx,\\ &\le 2\rho \sup_{t\in(\nu,\tau)} |v(\cdot,t)|_{BV(\mathbb{R}^d)},\\ \lim_{\alpha_0\downarrow 0} I &= \|v(\cdot,\tau)-u(\cdot,\tau)\|_{L^1(\mathbb{R}^d)} - \|v(\cdot,\nu)-u(\cdot,\nu)\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

We therefore get the following approximation inequality

$$\begin{aligned}&\|v(\cdot,\tau)-u(\cdot,\tau)\|_{L^1(\mathbb{R}^d)}\\ &\quad\le \|v(\cdot,\nu)-u(\cdot,\nu)\|_{L^1(\mathbb{R}^d)} + 2\rho \sup_{t\in(\nu,\tau)} |v(\cdot,t)|_{BV(\mathbb{R}^d)} + \lim_{\alpha_0,\rho_0\downarrow 0}\Big(E_{\text{Conv}} + E_{\text{Diff}}\Big),\end{aligned} \tag{3.5}$$

where

$$E_{\text{Conv}} = \iiiint_{\Pi_T\times\Pi_T} I_{\text{Conv}}\,dt\,dx\,ds\,dy,$$

$$E_{\text{Diff}} = -\iiiint_{\Pi_T\times\Pi_T} I^1_{\text{Diff}}\,dt\,dx\,ds\,dy - \lim_{\varepsilon\downarrow 0}\iiiint_{\Pi_T\times\Pi_T} I^2_{\text{Diff}}\,dt\,dx\,ds\,dy =: E^1_{\text{Diff}} - E^2_{\text{Diff}}.$$

Observe that

$$
\begin{aligned}
E^2_{\mathrm{Diff}} &= -\lim_{\varepsilon\downarrow 0} \iiiint\limits_{\Pi_T\times\Pi_T} 2\nabla_y\Big(\int_v^u \mathrm{sign}'_\varepsilon\,(v-\xi)\,\sqrt{A'(\xi)}\,d\xi\Big)\sqrt{B'(v)}\nabla_x v\phi\,dt\,dx\,ds\,dy\\
&= \lim_{\varepsilon\downarrow 0} \iiiint\limits_{\Pi_T\times\Pi_T} 2\Big(\int_v^u \mathrm{sign}'_\varepsilon\,(v-\xi)\,\sqrt{A'(\xi)}\,d\xi\Big)\sqrt{B'(v)}\nabla_x v\nabla_y\phi\,dt\,dx\,ds\,dy\\
&= -\lim_{\varepsilon\downarrow 0} \iiiint\limits_{\Pi_T\times\Pi_T} 2\Big(\int_v^u \mathrm{sign}'_\varepsilon\,(v-\xi)\,\sqrt{A'(\xi)}\,d\xi\Big)\sqrt{B'(v)}\nabla_x v\nabla_x\phi\,dt\,dx\,ds\,dy\\
&= \iiiint\limits_{\Pi_T\times\Pi_T} 2\mathrm{sign}\,(v-u)\,\sqrt{B'(v)}\sqrt{A'(v)}\nabla_x v\nabla_x\phi\,dt\,dx\,ds\,dy.
\end{aligned}
\tag{3.6}
$$

Writing $\mathrm{sign}\,(u-v)\,\nabla_y A(u) = \nabla_y\big|A(u)-A(v)\big|$ and using integration parts twice as well as the relation $\Delta_x\phi = \Delta_y\phi$, one can easily show that

$$
\begin{aligned}
&\iiiint\limits_{\Pi_T\times\Pi_T} \mathrm{sign}\,(u-v)\,\nabla_y A(u)\cdot\nabla_y\phi\,dt\,dx\,ds\,dy\\
&\qquad= \iiiint\limits_{\Pi_T\times\Pi_T} \mathrm{sign}\,(v-u)\,\nabla_x A(v)\cdot\nabla_x\phi\,dt\,dx\,ds\,dy.
\end{aligned}
\tag{3.7}
$$

From (3.6), (3.7), and $\big\|\nabla_x\delta_\rho(x-y)\big\|_{L^1(\mathbb{R}^d)} = \frac{2d}{\rho}$, we get

$$
\begin{aligned}
&\lim_{\alpha_0\downarrow 0} E_{\mathrm{Diff}}\\
&\le -\lim_{\alpha_0\downarrow 0} \iiiint\limits_{\Pi_T\times\Pi_T} \mathrm{sign}\,(v-u)\,\Big(B'(v)+A'(v)-2\sqrt{B'(v)}\sqrt{A'(v)}\Big)\nabla_x v\nabla_x\phi\,dt\,dx\,ds\,dy\\
&\le \lim_{\alpha_0\downarrow 0} \iiiint\limits_{\Pi_T\times\Pi_T} \Big(\sqrt{B'(v)}-\sqrt{A'(v)}\Big)^2\,\big|\nabla_x v\big|\,\big|\nabla_x\delta_\rho(x-y)\big|W_{\alpha_0}(t)\delta_{\rho_0}(t-s)\,dt\,dx\,ds\,dy\\
&\le (\tau-\nu)\sup_{t\in(\nu,\tau)} |v(\cdot,t)|_{BV(\mathbb{R}^d)}\frac{2d}{\rho}\Big\|\sqrt{B'(v)}-\sqrt{A'(v)}\Big\|^2_{L^\infty(\mathbb{R}^d)}.
\end{aligned}
\tag{3.8}
$$

Arguing exactly as in [7], one can prove that

$$
\begin{aligned}
E_{\mathrm{Conv}} &= \iiiint\limits_{\Pi_T\times\Pi_T} \Big(\operatorname{sign}(v-u)\left[\operatorname{div}_y k(y)\big(f(v)-g(v)\big) - \big(\operatorname{div}_y k(y) - \operatorname{div}_x l(x)\big)g(v)\right] \\
&\qquad + \big(k(y)-l(x)\big)\cdot\nabla_x G(v,u) + k(y)\cdot\nabla_x\big(F(v,u)-G(v,u)\big)\Big)\phi\,dt\,dx\,ds\,dy \\
&= \iiiint\limits_{\Pi_T\times\Pi_T} \Big(\operatorname{sign}(v-u)\left[\operatorname{div}_y k(y)\big(f(v)-g(v)\big) - \big(\operatorname{div}_x k(x) - \operatorname{div}_x l(x)\big)g(v)\right] \\
&\qquad + \big(k(x)-l(x)\big)\cdot\nabla_x G(v,u) + k(y)\cdot\nabla_x\big(F(v,u)-G(v,u)\big)\Big)\phi\,dt\,dx\,ds\,dy \\
&\quad + \iiiint\limits_{\Pi_T\times\Pi_T} \operatorname{sign}(v-u)\big(\operatorname{div}_x k(x) - \operatorname{div}_y k(y)\big)g(v)\phi\,dt\,dx\,ds\,dy \\
&\quad + \iiiint\limits_{\Pi_T\times\Pi_T} \operatorname{sign}(v-u)\big(k(y)-k(x)\big)\cdot\nabla_x G(v,u)\phi\,dt\,dx\,ds\,dy \\
&=: E^1_{\mathrm{Conv}} + E^2_{\mathrm{Conv}} + E^3_{\mathrm{Conv}}.
\end{aligned}
\tag{3.9}
$$

Following [7], we derive the estimate

$$
\begin{aligned}
&\lim_{\alpha_0\downarrow 0} E^1_{\mathrm{Conv}} \\
&\le (\tau-\nu)\Big(\|g\|_{\mathrm{Lip}} \sup_{t\in(\nu,\tau)} |v(\cdot,t)|_{BV(\mathbb{R}^d)}\,\|k-l\|_{L^\infty(\mathbb{R}^d)} + \|g\|_{L^\infty(I)}\,|k-l|_{BV(\mathbb{R}^d)} \\
&\qquad + |k|_{BV(\mathbb{R}^d)}\,\|f-g\|_{L^\infty(I)} + \|k\|_{L^\infty(\mathbb{R}^d)} \sup_{t\in(\nu,\tau)} |v(\cdot,t)|_{BV(\mathbb{R}^d)}\|f-g\|_{\mathrm{Lip}(I)}\Big).
\end{aligned}
\tag{3.10}
$$

Taking into account $\operatorname{div}k \in BV(\mathbb{R}^d)$ and $k\in \mathrm{Lip}(\mathbb{R}^d)$, it is easy to show that

$$
\lim_{\alpha_0\downarrow 0} E^2_{\mathrm{Conv}} \le |\operatorname{div}k|_{BV(\mathbb{R}^d)}\|g\|_{L^\infty(I)}(\tau-\nu)\rho, \tag{3.11}
$$

$$
\lim_{\alpha_0\downarrow 0} E^3_{\mathrm{Conv}} \le \|k\|_{\mathrm{Lip}(\mathbb{R}^d)}\|g\|_{\mathrm{Lip}(I)} \sup_{t\in(\nu,\tau)} |v(\cdot,t)|_{BV(\mathbb{R}^d)}(\tau-\nu)\rho. \tag{3.12}
$$

Inserting (3.8), (3.10), (3.11), and (3.12) into (3.5), minimizing the result with respect to $\rho>0$, and subsequently sending $\nu\downarrow 0$, we get (1.6). This concludes the proof of Theorem 1.1 when $B', A' > 0$. Note that (1.6) does not depend on the smoothness of $v,u$. Hence the proof in the general case $B', A' \ge 0$ can proceed via the $L^1$ convergence of the viscosity method (see Section 1).

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