Department of APPLIED MATHEMATICS

A continous Dependence Result for nonlinear degenerate parabolic Equations with spatially dependent flux Function.

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A CONTINUOUS DEPENDENCE RESULT FOR NONLINEAR DEGENERATE PARABOLIC EQUATIONS WITH SPATIALLY DEPENDENT FLUX FUNCTION

STEINAR EVJE, KENNETH HVISTENDAHL KARLSEN, AND NILS HENRIK RISEBRO

ABSTRACT. We study entropy solutions of nonlinear degenerate parabolic equations of form $u_t + \operatorname{div}(k(x)f(u)) = \Delta A(u)$, where k(x) is a vector-valued function and f(u), A(u) are scalar functions. We prove a result concerning the continuous dependence on the initial data, the flux function k(x)f(u), and the diffusion function A(u). This paper complements previous work [7] by two of the authors, which contained a continuous dependence result concerning the initial data and the flux function k(x)f(u).

1. INTRODUCTION

In this paper we are concerned with entropy solutions of the initial value problem

(1.1)
$$u_t + \operatorname{div}(k(x)f(u)) = \Delta A(u), \quad u(x,0) = u_0(x)$$

for $(x,t) \in \Pi_T = \mathbb{R}^d \times (0,T)$ with T > 0 fixed. In (1.1), u(x,t) is the scalar unknown function that is sought, k(x)f(u) is the flux function, and A = A(u) is the diffusion function. We always assume that $k : \mathbb{R}^d \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, and $A : \mathbb{R} \to \mathbb{R}$ satisfy

(1.2)
$$\begin{cases} k \in W_{\text{loc}}^{1,1}(\mathbb{R}^d); \ k, \text{div}k \in L^{\infty}(\mathbb{R}^d); \ f \in \text{Lip}_{\text{loc}}(\mathbb{R}); \ f(0) = 0; \\ A \in \text{Lip}_{\text{loc}}(\mathbb{R}) \text{ and } A(\cdot) \text{ is nondecreasing with } A(0) = 0. \end{cases}$$

Since $A'(\cdot)$ is allowed to be zero on an interval $[\alpha, \beta]$ (the scalar conservation law is a special case of (1.1)), solutions may become discontinuous in finite time even with a smooth initial function. Consequently, one needs to interpret (1.1) in the weak sense. However, weak solutions are in general not uniquely determined by their initial data and an entropy condition must be imposed to single out the physically correct solution.

Definition 1.1. A measurable function u = u(x, t) is an entropy solution of (1.1) if

D.1 $u \in L^1(\Pi_T) \cap L^{\infty}(\Pi_T) \cap C(0,T; L^1(\mathbb{R}^d))$ and $A(u) \in L^2(0,T; H^1(\mathbb{R}^d))$.

Date: May 31, 2000.

Key words and phrases. nonlinear degenerate parabolic equation, spatially dependent flux function, entropy solution, continuous dependence.

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for $(x,t) \in \Pi_{T} = \mathbb{R}^{n} \times (0, T)$ with T = 0 then in (1,1) w((2,1)) is the scalar and share function that is rought, $\mathcal{H}_{(2)}(u)$ is the first function, and $\mathcal{A} = \mathcal{A}(u)$ is the difference function. We always resume that $t \in \mathbb{R}^{n} \to 0$, $f \in \mathbb{R} \to \mathbb{R}$ and $\mathcal{A} \in \mathbb{R}$ with u

 $\begin{bmatrix} \mathbf{k} \in W_{n,i} \\ (\mathbf{k}^{n}) \end{bmatrix} = \mathbf{k} \cdot \mathbf{d} \cdot \mathbf{k} \in \mathbb{R}^{n} \\ A \in Lap_{n,i}(\mathbf{R}) \end{bmatrix} = \mathbf{0} \quad (a) \in \mathbb{R}^{n} \\ A \in Lap_{n,i}(\mathbf{R}) \text{ and } A(-) \text{ is conditioned in positive with even (b) = 0.$

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Definition 1.1. A measurale privation are e(a, t) is an entropy of directly if () if D.1 we L¹(1)-101-2010 (1000 (1000) 7.50 (2010) and star a 1200 7.60 (2010)

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D.2 For all $c \in \mathbb{R}$ and all non-negative test functions in $C_0^{\infty}(\Pi_T)$,

(1.3)
$$\iint_{\Pi_T} \left(|u-c|\phi_t + \operatorname{sign} (u-c) k(x) (f(u) - f(c)) \cdot \nabla \phi + |A(u) - A(c)| \Delta \phi - \operatorname{sign} (u-c) \operatorname{div} k(x) f(c) \phi \right) dt \, dx > 0.$$

D.3 Essentially as $t \downarrow 0$, $||u(\cdot, t) - u_0(x)||_{L^1(\mathbb{R}^d)} \to 0$.

Following Kružkov [9] and the recent work of Carrillo [3], two of the authors proved in [7] a uniqueness result for entropy solutions of the more general equation

(1.4)
$$u_t + \operatorname{div} f(x, t, u) = \Delta A(u) + q(x, t, u),$$

where the flux function f = f(x, t, u) may have a non-smooth spatial dependence, see [7] for the precise assumptions on f and q in (1.4). Moreover, in the $L^{\infty}(0, T; BV(\mathbb{R}^d))$ class of entropy solutions, the authors of [7] proved continuous dependence on the initial function u_0 and flux function in the case f(x, t, u) = k(x)f(u). However, in [7] the question of continuous dependence with respect to the diffusion function A was left open. Recently, Cockburn and Gripenberg [4] have obtained such a result when k(x) = 1. Their result does *not*, however, imply uniqueness of the entropy solution (from reasons that will become apparent later). Let us also mention that results regarding continuous dependence on the flux function in scalar conservation laws ($A' \equiv 0$) have been obtained in [11, 1, 8].

The purpose of the present paper is to combine the ideas in [7] with those in [4] and prove a version of Theorem 1.3 in [7] which also includes continuous dependence on the diffusion function A. To state our continuous dependence result, let us introduce

(1.5)
$$v_t + \operatorname{div}(l(x)g(v)) = \Delta B(v), \quad v(x,0) = v_0(x).$$

We assume that $l : \mathbb{R}^d \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$, and $B : \mathbb{R} \to \mathbb{R}$ satisfy the same conditions as k, f, A, see (1.2). We now state our main result:

Theorem 1.1. Let $v, u \in L^{\infty}(0,T; BV(\mathbb{R}^d))$ be the unique entropy solutions of (1.5), (1.1) with initial data $v_0, u_0 \in L^1(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$, respectively. Suppose that v, u take values in in the closed interval $I \subset \mathbb{R}$ and define $V_v = \sup_{t \in (0,T)} |v(\cdot,t)|_{BV(\mathbb{R}^d)}$. Suppose $k \in \operatorname{Lip}(\mathbb{R}^d)$ and div $k \in BV(\mathbb{R}^d)$. Then for almost all $t \in (0,T)$,

(1.6)
$$\|v(\cdot,t) - u(\cdot,t)\|_{L^{1}(\mathbb{R}^{d})} \leq \|v_{0} - u_{0}\|_{L^{1}(\mathbb{R}^{d})} \\ + C_{\text{Conv}}t\Big(\|l-k\|_{L^{\infty}(\mathbb{R}^{d})} + \|l-k\|_{BV(\mathbb{R}^{d})} + \|g-f\|_{L^{\infty}(I)} + \|g-f\|_{\text{Lip}(I)}\Big) \\ + C_{\text{Diff}}\sqrt{t} \left\|\sqrt{B'} - \sqrt{A'}\right\|_{L^{\infty}(I)},$$

for some constants $C_{\text{Conv}}, C_{\text{Diff}}$. Here C_{Conv} depends on $V_v, ||k||_{L^{\infty}(\mathbb{R}^d)}, ||k||_{BV(\mathbb{R}^d)}, ||g||_{L^{\infty}(I)}, ||g||_{L^{\infty$

We remark that existence of $BV(\Pi_T)$ entropy solutions of (1.1) (or (1.4)) can be proved by the vanishing viscosity method provided f, A, q, u_0 are sufficiently smooth, see Vo'lpert

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We assume that $I : \mathbb{R}^{d} \to \mathbb{R}$, $g \in \mathbb{R}$ $\to \mathbb{R}$, and $B \in \mathbb{R}^{d} \to \mathbb{R}$ antidy the wine contributes as k, f, A, see (1.2). We now which out main result:

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We remark that existence of BV (Re) antrony solutions of (L1) for (L4)) con he proved by the version of the ver

and Hudjaev [13]. Existence of $L^{\infty}(0,T; BV(\mathbb{R}^d))$ entropy solutions of (1.1) is guaranteed if div $k \in BV(\mathbb{R}^d)$. This follows from the results obtained by Karlsen and Risebro [6], who proved convergence of finite difference schemes for degenerate parabolic equations with rough coefficients. For an overview of the literature dealing with numerical methods for approximating entropy solutions of degenerate parabolic equations, we refer to [5]. In this connection, we should mention that the arguments used to prove Theorem 1.1 can be used to prove error estimates for numerical methods. This will be discussed elsewhere.

For later use, we mention that the results in [6] can be used to prove the existence of $L^{\infty}(0,T;BV(\mathbb{R}^d))$ entropy solutions of (1.1) by the vanishing viscosity method. To this end, consider the uniformly parabolic problem

(1.7)
$$u_t^{\mu} + \operatorname{div}(k(x)f(u^{\mu})) = \Delta A(u^{\mu}) + \mu \Delta u^{\mu}, \qquad u^{\mu}(x,0) = u_0(x),$$

for $\mu > 0$. Provided k, f, A, u_0 are sufficiently smooth, it is well known that there exists a unique classical (and hence entropy) solution of (1.7) which possesses all the continuous derivatives occurring in the partial differential equation in (1.7). Using the space and time translation estimates derived in [6], it is not difficult to show that u^{μ} converges in $L^1_{\rm loc}(\Pi_T)$ as $\mu \downarrow 0$ to an entropy solution u of (1.1) (see also Vo'lpert and Hudjaev [13]). Convergence of the viscosity method and smoothness of the solution u^{μ} of (1.7) will be used in the proof of Theorem 1.1. Finally, to relax the smoothness assumptions on k, f, A, u_0 needed by the vanishing viscosity method to those actually required by Theorem 1.1, one can approximate k, f, A, u_0 by smoother functions and then use Theorem 1.1 to pass to the limit as the smoothing parameter tends to zero. We will not go into further details about this limiting operation but instead leave this as an exercise for the interested reader. Also, in this paper we have exclusively treated the initial value problem but it is possible to treat various initial-boundary value problems. For some work in this direction, we refer to Bürger, Evje, and Karlsen [2] and Rouvre and Gagneux [12].

Before ending this section, we present an immediate corollary of Theorem 1.1 concerning the convergence rate of the viscosity method.

Corollary 1.1. Let $u \in L^{\infty}(0,T; BV(\mathbb{R}^d))$ be the unique entropy solution of (1.1) with initial data $u_0 \in L^1(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ and let u^{μ} be the corresponding viscous approximation of u, i.e., u^{μ} is the unique classical solution of (1.7). Suppose $k \in \text{Lip}(\mathbb{R}^d)$ and $\text{div} k \in BV(\mathbb{R}^d)$. Then for almost all $t \in (0,T)$,

(1.8)
$$||u(\cdot,t) - u^{\mu}(\cdot,t)||_{L^1(\mathbb{R}^d)} \le C\sqrt{\mu}$$

for some non-negative constant C.

The remaining part of this paper is devoted to proving Theorem 1.1.

2. Some preliminaries

Let u be an entropy solution of (1.1). It easy to see from Definition 1.1 that the equality

(2.1)
$$\iint_{\Pi_T} \left(u\phi_t + \left[k(x)f(u) - \nabla A(u) \right] \cdot \nabla \phi \right) dt \, dx = 0$$

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and Hudger (13), Salatanes of L² (3,3, 5% (3,7), salatopy administration of (1.5) is glassianteed if dive C BV(B²). This follows from the results obtained by Marisea and Risebro [6], who proved convergence of halte difference attended in constants (strabolic equations with rough coefficience. For an overdew of the literature design, with managed random for the approximating subrous solutions of degree state paralials constructs, we refer at [5]. In this connection, we should be state the appliance is a factor to connect to the state of the bole of the factor connection. We should be state the appliance is used to the factor of the connection of the state of the factor connection.

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Corollary 1.1. Let $\mathbf{u} \in \mathcal{L}^{\infty}(\mathbb{R}^{d})$ Birds us a de de unque reduces contacts of (1.1) and mined data $\mathbf{u}_{0} \in \mathcal{L}'(\mathbb{R}^{d})$ Collective response dissocial for \mathbf{u}^{n} by this corresponding response approximation of \mathbf{u}_{1} i.e., \mathbf{u}^{n} by the response dissocial solution by (1.1): Beapule $\mathbf{k} \in \mathrm{Lig}(\mathbb{R}^{d})$ and divk $\in BV(\mathbb{R}^{d})$. Then for encasts div $\in [0, T]$

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holds for all $\phi \in L^2(0,T; H^1_0(\mathbb{R}^d)) \cap W^{1,1}(0,T; L^{\infty}(\mathbb{R}^d))$. Let $\langle \cdot, \cdot \rangle$ denote the usual pairing between $H^{-1}(\mathbb{R}^d)$ and $H^1_0(\mathbb{R}^d)$. From (2.1), we conclude that $\partial_t u \in L^2(0,T; H^{-1}(\mathbb{R}^d))$, so that the equality (2.1) can restated as

(2.2)
$$-\int_0^T \left\langle \partial_t u, \phi \right\rangle dt + \iint_{\Pi_T} \left(\left[k(x) f(u) - \nabla A(u) \right] \cdot \nabla \phi \right) dt \, dx = 0$$

for all $\phi \in L^2(0,T; H_0^1(\mathbb{R}^d)) \cap W^{1,1}(0,T; L^{\infty}(\mathbb{R}^d))$. The fact that an entropy solution u satisfies (2.2) is important for the uniqueness proof [3, 7]. We recall that after the recent work of Carrillo [3], the uniqueness proof for entropy solutions of degenerate parabolic equations has become very similar to the "doubling of variables" proof introduced by Kružkov [9] many years ago for first order hyperbolic equations. However, to apply the "doubling of variables" device to second order equations, one needs a version of an important lemma stated and proved first in [3]. This lemma indentifies a certain entropy dissipation term that must be taken into account if the "doubling device" is going to work.

Before stating this lemma, we need to introduce some notation. For $\varepsilon > 0$, set

$$\operatorname{sign}_{\varepsilon}(\tau) = \begin{cases} -1, & \tau < \varepsilon, \\ \tau/\varepsilon, & \varepsilon \le \tau \le \varepsilon, \\ 1 & \tau > \varepsilon. \end{cases}$$

Moreover, we let $A^{-1} : \mathbb{R} \to \mathbb{R}$ denote the unique left-continuous function satisfying $A^{-1}(A(u)) = u$ for all $u \in \mathbb{R}$. By E we denote the set $E = \{r : A^{-1}(\cdot) \text{ discontinuous at } r\}$. Note that E is associated with the set of points $\{u : A'(u) = 0\}$ at which the operator $u \mapsto \Delta A(u)$ is degenerate elliptic. We can now state following lemma:

Lemma 2.1 ([7]). Let u be an entropy solution of (1.1). Then, for any non-negative $\phi \in C_0^{\infty}(\Pi_T)$ and any $c \in \mathbb{R}$ such that $A(c) \notin E$, we have

$$\int_{\Pi_T} \iint \left(|u - c|\phi_t + \operatorname{sign}(u - c) \left[k(x) \left(f(u) - f(c) \right) - \nabla A(u) \right] \cdot \nabla \phi - \operatorname{sign}(u - c) \operatorname{div} k(x) f(c) \phi \right) dt \, dx = \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \left| \nabla A(u) \right|^2 \operatorname{sign}'_{\varepsilon} \left(A(u) - A(c) \right) \phi \, dt \, dx.$$

Note that if (1.1) is uniformly parabolic (A' > 0), then the set E is empty and a weak solution is automatically an entropy solution. The idea of the proof of Lemma 2.1 is to use $[\operatorname{sign}_{\varepsilon} (A(u) - A(c)) \phi] \in L^2(0, T; H^1_0(\mathbb{R}))$ as a test function in (2.2) together with a "weak chain rule" to deal with the time derivative and subsequently sending $\varepsilon \downarrow 0$. We refer to [7] for details on the proof of Lemma 2.1 (see Carrillo [3] when $k(x) \equiv 1$).

Although the identification of the entropy dissipation term (i.e., the right-hand side of (2.3)) is the cornerstone of the uniqueness proof as well as the proof of continuous dependence on the flux function k(x)f(u), it seems difficult to obtain continuous dependence on the diffusion function A(u) with this form of the dissipation term. However, it is possible

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holds for all $\phi \in L^{2}(0, T; M_{2}(\mathbb{R}^{2})) \cap W^{2}(0, T; L^{\infty}(\mathbb{R}^{2}))$. Let (...) denote the would belong between $R^{-1}(\mathbb{R}^{2})$ and $M_{2}(\mathbb{R}^{2})$. Show (2.1), we constude that $A \in \mathbb{R}^{2}(0, T; R^{-1}(\mathbb{R}^{2}))$, so that the equality (2.1) can restrict as

$$(23) = -\int (6\pi_{1}\phi) d\phi + \int \int (\pi_{1}(\mu_{1})) d\phi - \nabla \pi_{1}(\mu_{1}) \nabla \phi + \phi = 0$$

for all of 5 2°(0, 7. 16)(8°) (0, 17. 16) (0, 7. 17°(18°)). The task that at more solution class if the (2.2) is imposents for the uniquents proof [3. 7]. We much that at or the redent with of Cartillo [3], the uniquese plot (0, entropy solutions of intervente paralleli republic has become very similar to the theorem of the tendence of versions of intervente paralleli republic many years ago for first order by remove many second for the residues of intervente restables' device to second order examples are beed as version of an intervent stated and proved first is [3]. The intervente indentifies a second or an intervente that much proved first is [3]. The intervente device the second of the second of the stated and proved first is [3]. The intervente of the intervente of an intervente the second proved first is [3]. The intervente of the intervente of the second of the second of the stated and proved first is [3]. The intervente of the intervente of the second of the s

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Moreover, we let $A^{(1)} : B \to B$, denote the sample left-continuous function statistics $A^{(1)}(A(u)) = u$ for all $u \in B$. By E we denote the set $E = \{r \in A^{(1)}() | destination (v) \in B \}$ Note that E is associated with the set of points $\{u : A'(u) = 0\}$ at which the operator $u \to \Delta A(u)$ is described.

Lemma 2.1 ((7)). Let y be an energy solution of (1.1). Then, for any new solution $\phi \in C_{\mathbb{C}}^{\infty}(\Pi_{\mathbb{C}})$ and any $g \in \mathbb{R}$ much that $A(g) \in C_{\mathbb{C}}$ are have

$$(u - c(u) + s(u - u)(k(u))(f(u) + f(c)) - \nabla A(u))$$

$$= \sup_{i \in I} \left\{ u - c \right\} div k(z) / (c | z) di dz = \lim_{i \in I} \left\{ \left| \nabla A(u) \right|^2 dz dz \left(z | z - u | z | z - u | z | z - u | z | z - u | z | z - u | z | z - u | z | z - u | z | z - u | z | z - u | z | z - u | z - u | z | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u | z - u |$$

Note that if (1,1) is uniferrative ratialistic (A' > 0), thus the set B is stappy and a weak solution is antomatically an entropy solution. The idea of the proof of Dennia 1.1 is to tes [sign, (A(a) - A(c)) of $\in \mathbb{Z}^{2}(0, 2^{\circ}; B_{0}(b))$ as a tast function in (2,2) togother with a body chain rule^o to deal with the terms (2,1) for a rowards fill when the (2,3) togother with a body [7] for datalls on the proof of Lemma 2.1 (see Corrido fill when the (2,3) = 0.

Although the identification of the entropy dissipation term (i.e., the right hand alde of (2.3)) is the portectoria of the uniqueness proof as well as the proof of continuous densitied of denses on the flux function for a densitied to plate the portectorial of the denses of the dense of the densitied of the flux function of the trip of the dense of the d

to derive a version of (2.3) in which a different (form of the) entropy dissipation term appears. But this seems possible only if u is smooth or at least belongs to $L^2(0, T; H^1(\mathbb{R}^d))$.

Consequently, Theorem 1.1 does not yield uniqueness of the entropy solution.

Provided (1.1) is unformly parabolic and hence admits a unique classical solution, the following version of Lemma 2.1 holds:

Lemma 2.2. Suppose (1.1) is uniformly parabolic (i.e., A' > 0). Let u be a classical solution of (1.1). Then, for any non-negative $\phi \in C_0^{\infty}(\Pi_T)$ and any $c \in \mathbb{R}$, we have

(2.4)
$$\iint_{\Pi_{T}} \left(|u-c|\phi_{t} + \operatorname{sign}(u-c) \left[k(x) \left(f(u) - f(c) \right) - \nabla A(u) \right] \cdot \nabla \phi \right) \\ - \operatorname{sign}(u-c) \operatorname{div} k(x) f(c) \phi dt dx = \lim_{\varepsilon \downarrow 0} \iint_{\Pi_{T}} A'(u) |\nabla u|^{2} \operatorname{sign}_{\varepsilon}(u-c) \phi dt dx.$$

This lemma can proved by using $[\operatorname{sign}_{\varepsilon} (u-c) \phi]$ as a test function (this is indeed a test function since u is smooth!) in (2.2) and then sending $\varepsilon \downarrow 0$. The proof of Lemma 2.2 is similar to the proof of Lemma 2.1 and it is therefore omitted. Notice the difference between the entropy dissipation terms in (2.3) and (2.4).

3. Proof of Theorem 1.1

We are now interested in estimating the L^1 difference between the entropy solution v of (1.5) and the entropy solution u of (1.1). In view of the discussion in Section 1, we will prove Theorem 1.1 under the assumption that B', A' > 0 so that (1.5) and (1.1) become uniformly parabolic problems and therefore admit unique classical solutions. To treat the degenerate parabolic case $(B', A' \ge 0)$, we proceed via the vanishing viscosity method, i.e., we replace $\Delta A(u)$ and $\Delta B(v)$ in (1.1) and (1.5) by $\Delta A(u) + \eta \Delta u$ and $\Delta B(v) + \eta \Delta v$, respectively, and then send $\eta \downarrow 0$.

The argument given below is based on Lemma 2.2 and Kružkov's idea of doubling the number of dependent variables together with a penalization procedure. Moreover, it is inspired by Carrillo [3] and Cockburn and Gripenberg [4]. Strictly speaking, we could have carried out the argument below under the assumptions that the (entropy) solutions v, u belong to $L^2(0, T; H^1(\mathbb{R}^d))$, i.e., v, u need not be (entirely!) classical solutions.

Following [9, 10], we now specify a non-negative test function $\phi \in C_0^{\infty}(\Pi_T \times \Pi_T)$. To this end, introduce a nonnegative function $\delta \in C_0^{\infty}(\mathbb{R})$ which satisfies $\delta(\sigma) = \delta(-\sigma)$, $\delta(\sigma) \equiv 0$ for $|\sigma| \ge 1$, and $\int_{\mathbb{R}} \delta(\sigma) d\sigma = 1$. For $\rho_0 > 0$, let $\delta_{\rho_0}(\sigma) = \frac{1}{\rho_0} \delta(\frac{\sigma}{\rho_0})$. Pick two (arbitrary but fixed) Lebesgue points $\nu, \tau \in (0, T)$ of $||v(\cdot, t) - u(\cdot, t)||_{L^1(\mathbb{R}^d)}$.

For any $\alpha_0 \in (0, \min(\nu, T - \tau))$, let $W_{\alpha_0}(t) = H_{\alpha_0}(t - \nu) - H_{\alpha_0}(t - \tau)$, where $H_{\alpha_0}(t) = \int_{-\infty}^t \delta_{\alpha_0}(s) \, ds$. We then define $\phi = \phi(x, t, y, s)$ by

(3.1)
$$\phi(x, t, y, s) = W_{\alpha_0}(t)\delta_{\rho}(x-y)\delta_{\rho_0}(t-s), \qquad \rho, \rho_0 > 0.$$

Observe that $\phi_t + \phi_s = \left[\delta_{\alpha_0}(t-\nu) - \delta_{\alpha_0}(t-\tau)\right]\delta_{\rho}(x-y)\delta_{\rho_0}(t-s)$ and $\nabla_x \phi + \nabla_y \phi \equiv 0$.

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to derive a version of (2.3) in which a different (nerts of the) entropy designation term an pears. But this scenas provible entry if a is subset for a sole least belongs to (2.10, 7. 3 's le')). Conceptiontly, Theorem 1.1 does not yield desprises of the intropy substant.

following version of treatmin 2.1 holds:

Lemma 2.2: Suppose (1.1) is uniformly previous (i.e. A. > 0). Also a set assisted

This lemma can proved by using [sign, (n - a) of a sol that the proof of Lemma 2.2 function since was arrachilis in (2.2) and then evening 2.1.0. The proof of Lemma 2.3 is similar to the proof of Lemma 2.1 and it is therefore control. Notice the difference between the sutropy disargation forms in (2.3) and (7.4).

LA MARKARING ROOK LA

We are now interested in returning the L difference between the metric product of (1.5) and the contexposed in equation (1.5). In view of the discussion in Section L we will us prove Theorem 1.1 under the sector particles in the sector (1.5) because (1.5) and the contexposed in the sector (1.5) in (1.5) and (1.5) and the contexposed in the sector prove (1.5) in (1.5) are discussioned in the sector prove (1.5) in (1.5) and (1.5) because (1.5) because (1.5) and (1.5) because (1.5) becau

The argument grant being is based on Lemma 1.2 and Pressor's stea of doubling the number of dependent variables rotation with a penalization granting and the second is inspired by Carrillo [3] and Could are and Granesberg [4]. Strictly speaking we could inner carried out the argument below ander the seating of a first time (area argument), a belong to 1°(0, 7, 10°(10° 1), i.e. p. a med and be interpreted with the familiarity of a second on a

Following [3, 10], we now electify a non-negative destricted and $K \in C_{2}^{\infty}(W_{1}, \infty W_{1})$. In this end, introduce a momentative labeliant $\delta \in C_{2}^{\infty}(W)$ stars and stress $K_{2}^{0} = \delta(-\sigma)$. $\delta(\alpha) = 0$ for $|\sigma| \geq 1$, and $f_{0} \delta(\alpha)$ do -1. For $\mu > 0$, for $|\sigma| \geq 1$, $\delta(\alpha) = \int (\frac{1}{2})^{2} + \int (\frac{1}{2$

For any $a_0 \in \{0, and (n, T - T)\}$, let $W_{a_0}(t) = M_{a_0}(t - x) - M_{a_0}(t - T)$, where $M_{a_0}(t)$, a_0

Denote that $p_1 + p_2 = \{b_0, (1 - n), b_1, (1 - n)\}$ is (1 - n) and $V_1 + p_2$ and $V_2 + p_3$

8

Applying Lemma 2.2 with v = v(x, t) and c = u(y, s) and then integrating the resulting equation with respect to $(y, s) \in \Pi_T$, we get

$$-\iiint_{\Pi_T \times \Pi_T} \left(|v - u| \phi_t + \operatorname{sign} (v - u) \left[l(x) \left(g(v) - g(u) \right) - \nabla_x B(v) \right] \cdot \nabla_x \phi \right.$$
$$- \operatorname{sign} (v - u) \operatorname{div}_x l(x) g(u) \phi \right) dt \, dx \, ds \, dy$$
$$= -\lim_{\varepsilon \downarrow 0} \iiint_{\Pi_T \times \Pi_T} B'(v) |\nabla_x v|^2 \operatorname{sign}_{\varepsilon}' (v - u) \phi \, dt \, dx \, ds \, dy.$$

Similarly, applying Lemma 2.2 with u = u(y, s) and c = v(x, t) and then integrating the resulting equation with respect to $(x, t) \in \Pi_T$, we get

$$-\iiint_{\Pi_T \times \Pi_T} \left(|u - v|\phi_t + \operatorname{sign} (u - v) \left[k(y) \left(f(u) - f(v) \right) - \nabla_y A(u) \right] \cdot \nabla_y \phi \right. \\ \left. - \operatorname{sign} (u - v) \operatorname{div}_y k(y) f(v) \phi \right) dt \, dx \, ds \, dy \\ \left. = - \lim_{\varepsilon \downarrow 0} \iiint_{\Pi_T \times \Pi_T} A'(u) |\nabla_y u|^2 \operatorname{sign}'_{\varepsilon} (u - v) \phi \, dt \, dx \, ds \, dy. \right.$$

Following [7] when adding (3.2) and (3.3), we get

$$-\iiint_{\Pi_{T}\times\Pi_{T}}\left(|v-u|(\phi_{t}+\phi_{s})+I_{\text{Conv}}-I_{\text{Diff}}^{1}\right)dt\,dx\,ds\,dy$$

$$=-\lim_{\varepsilon\downarrow0}\iiint_{\Pi_{T}\times\Pi_{T}}\left(B'(v)|\nabla_{x}v|^{2}+A'(u)|\nabla_{y}u|^{2}\right)\operatorname{sign}_{\varepsilon}'(v-u)\,\phi\,dt\,dx\,ds\,dy,$$

$$=-\lim_{\varepsilon\downarrow0}\iiint_{\Pi_{T}\times\Pi_{T}}\left(\left(\sqrt{B'(v)}\nabla_{x}v-\sqrt{A'(u)}\nabla_{y}u\right)^{2}+2\sqrt{B'(v)}\sqrt{A'(u)}\nabla_{x}v\cdot\nabla_{y}u\right)\operatorname{sign}_{\varepsilon}'(v-u)\,\phi\,dt\,dx\,ds\,dy,$$

$$\leq-\lim_{\varepsilon\downarrow0}\iiint_{\Pi_{T}\times\Pi_{T}}I_{\text{Diff}}^{2}\,dt\,dx\,ds\,dy,$$

where

$$I_{\text{Conv}} = \operatorname{sign} \left(v - u \right) \left(\operatorname{div}_{x} \left[\left(k(y) f(u) - l(x) g(u) \right) \phi \right] - \operatorname{div}_{y} \left[\left(l(x) g(v) - k(y) f(v) \right) \phi \right] \right),$$

$$I_{\text{Diff}}^{1} = \operatorname{sign} \left(v - u \right) \nabla_{x} B(v) \cdot \nabla_{x} \phi + \operatorname{sign} \left(u - v \right) \nabla_{y} A(u) \cdot \nabla_{y} \phi,$$

$$I_{\text{Diff}}^{2} = \left(2 \sqrt{B'(v)} \sqrt{A'(u)} \nabla_{x} v \cdot \nabla_{y} u \right) \operatorname{sign}_{\varepsilon}' \left(v - u \right) \phi.$$

(3.2)

(3.3)

By the triangle inequality, we get

$$-\iiint_{\Pi_T \times \Pi_T} |v(x,t) - u(y,s)| (\phi_t + \phi_s) dt dx ds dy \le I + R^t + R^x.$$

where

$$I = -\iiint_{\Pi_T \times \Pi_T} |v(y,t) - u(y,t)| [\delta_{\alpha_0}(t-\nu) - \delta_{\alpha_0}(t-\tau)] \delta_{\rho}(x-y) \delta_{\rho_0}(t-s) dt dx ds dy,$$

$$R^t = -\iiint_{\Pi_T \times \Pi_T} |u(y,t) - u(y,s)| [\delta_{\alpha_0}(t-\nu) - \delta_{\alpha_0}(t-\tau)] \delta_{\rho}(x-y) \delta_{\rho_0}(t-s) dt dx ds dy,$$

$$R^x = -\iiint_{\Pi_T \times \Pi_T} |v(x,t) - v(y,t)| [\delta_{\alpha_0}(t-\nu) - \delta_{\alpha_0}(t-\tau)] \delta_{\rho}(x-y) \delta_{\rho_0}(t-s) dt dx ds dy.$$

It is fairly easy to see that $\lim_{\rho_0\downarrow 0} R^t = 0$ and

$$\lim_{\alpha_0 \downarrow 0} R^x = \int_{\mathbb{R}^d} \left(|v(x,\tau) - v(y,\tau)| - |v(x,\nu) - v(y,\nu)| \right) \delta_\rho(x-y) \, dx,$$

$$\leq 2\rho \sup_{t \in (\nu,\tau)} |v(\cdot,t)|_{BV(\mathbb{R}^d)},$$

$$\lim_{\alpha_0 \downarrow 0} I = \|v(\cdot,\tau) - u(\cdot,\tau)\|_{L^1(\mathbb{R}^d)} - \|v(\cdot,\nu) - u(\cdot,\nu)\|_{L^1(\mathbb{R}^d)}.$$

We therefore get the following approximation inequality

(3.5) $\|v(\cdot,\tau) - u(\cdot,\tau)\|_{L^{1}(\mathbb{R}^{d})} \\ \leq \|v(\cdot,\nu) - u(\cdot,\nu)\|_{L^{1}(\mathbb{R}^{d})} + 2\rho \sup_{t \in (\nu,\tau)} |v(\cdot,t)|_{BV(\mathbb{R}^{d})} + \lim_{\alpha_{0},\rho_{0}\downarrow 0} \Big(E_{\text{Conv}} + E_{\text{Diff}}\Big),$

where

$$E_{\text{Conv}} = \iiint_{\Pi_T \times \Pi_T} I_{\text{Conv}} \, dt \, dx \, ds \, dy,$$

$$E_{\text{Diff}} = - \iiint_{\Pi_T \times \Pi_T} I_{\text{Diff}}^1 \, dt \, dx \, ds \, dy - \lim_{\varepsilon \downarrow 0} \iiint_{\Pi_T \times \Pi_T} I_{\text{Diff}}^2 \, dt \, dx \, ds \, dy =: E_{\text{Diff}}^1 - E_{\text{Diff}}^2.$$

By the triangle inequality, we get

$$= \int \int \left[\int \left[h(z,z) - u(z,z) \right] \left((z_1, - z_2) \right] dz dz dz dz (z_1 + R + R) \right] dz dz dz dz dz dz (z_1 + R + R)$$

where

It is fairly easy to see that lim, is if = 0 and

We therefore get the following approximation inequality

$$\|v(\cdot, \tau) - v(\cdot, \tau)\|_{L^2(\Omega)}$$

$$\leq \|v(\cdot, t)\|_{L^2(\Omega)} + 2\nu \sup_{u \in [u]} \|v(\cdot, t)\|_{L^2(\Omega)} + \beta n$$

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Observe that

$$\begin{split} E_{\text{Diff}}^2 &= -\lim_{\varepsilon \downarrow 0} \iiint_{\Pi_T \times \Pi_T} 2\nabla_y \Big(\int_v^u \operatorname{sign}_{\varepsilon}' (v - \xi) \sqrt{A'(\xi)} \, d\xi \Big) \sqrt{B'(v)} \nabla_x v \phi \, dt \, dx \, ds \, dy \\ &= \lim_{\varepsilon \downarrow 0} \iiint_{\Pi_T \times \Pi_T} 2 \Big(\int_v^u \operatorname{sign}_{\varepsilon}' (v - \xi) \sqrt{A'(\xi)} \, d\xi \Big) \sqrt{B'(v)} \nabla_x v \nabla_y \phi \, dt \, dx \, ds \, dy \\ &= -\lim_{\varepsilon \downarrow 0} \iiint_{\Pi_T \times \Pi_T} 2 \Big(\int_v^u \operatorname{sign}_{\varepsilon}' (v - \xi) \sqrt{A'(\xi)} \, d\xi \Big) \sqrt{B'(v)} \nabla_x v \nabla_x \phi \, dt \, dx \, ds \, dy \\ &= \iiint_{\Pi_T \times \Pi_T} 2 \operatorname{sign} (v - u) \sqrt{B'(v)} \sqrt{A'(v)} \nabla_x v \nabla_x \phi \, dt \, dx \, ds \, dy. \end{split}$$

Writing sign $(u - v) \nabla_y A(u) = \nabla_y |A(u) - A(v)|$ and using integration parts twice as well as the relation $\Delta_x \phi = \Delta_y \phi$, one can easily show that

(3.7)
$$\iiint_{\Pi_T \times \Pi_T} \operatorname{sign} (u - v) \nabla_y A(u) \cdot \nabla_y \phi \, dt \, dx \, ds \, dy$$
$$= \iiint_{\Pi_T \times \Pi_T} \operatorname{sign} (v - u) \nabla_x A(v) \cdot \nabla_x \phi \, dt \, dx \, ds \, dy$$

From (3.6), (3.7), and $\|\nabla_x \delta_{\rho}(x-y)\|_{L^1(\mathbb{R}^d)} = \frac{2d}{\rho}$, we get

$$(3.8)$$

$$\lim_{\alpha_{0}\downarrow0} E_{\text{Diff}}$$

$$\leq -\lim_{\alpha_{0}\downarrow0} \iiint_{\Pi_{T}\times\Pi_{T}} \operatorname{sign}\left(v-u\right) \left(B'(v)+A'(v)-2\sqrt{B'(v)}\sqrt{A'(v)}\right) \nabla_{x}v \nabla_{x}\phi \, dt \, dx \, ds \, dy$$

$$\leq \lim_{\alpha_{0}\downarrow0} \iiint_{\Pi_{T}\times\Pi_{T}} \left(\sqrt{B'(v)}-\sqrt{A'(v)}\right)^{2} \left|\nabla_{x}v\right| \left|\nabla_{x}\delta_{\rho}(x-y)\right| W_{\alpha_{0}}(t)\delta_{\rho_{0}}(t-s) \, dt \, dx \, ds \, dy$$

$$\leq (\tau-\nu) \sup_{t\in(\nu,\tau)} |v(\cdot,t)|_{BV(\mathbb{R}^{d})} \frac{2d}{\rho} \left\|\sqrt{B'(v)}-\sqrt{A'(v)}\right\|_{L^{\infty}(\mathbb{R}^{d})}^{2}.$$

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(3.6)

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Arguing exactly as in [7], one can prove that

$$\begin{split} E_{\text{Conv}} &= \iiint_{\Pi_T \times \Pi_T} \left(\text{sign} \left(v - u \right) \left[\text{div}_y k(y) \left(f(v) - g(v) \right) - \left(\text{div}_y k(y) - \text{div}_x l(x) \right) g(v) \right] \right. \\ &+ \left(k(y) - l(x) \right) \cdot \nabla_x G(v, u) + k(y) \cdot \nabla_x \left(F(v, u) - G(v, u) \right) \right) \phi \, dt \, dx \, ds \, dy \\ &= \iiint_{\Pi_T \times \Pi_T} \left(\text{sign} \left(v - u \right) \left[\text{div}_y k(y) \left(f(v) - g(v) \right) - \left(\text{div}_x k(x) - \text{div}_x l(x) \right) g(v) \right] \right. \\ &+ \left(k(x) - l(x) \right) \cdot \nabla_x G(v, u) + k(y) \cdot \nabla_x \left(F(v, u) - G(v, u) \right) \right) \phi \, dt \, dx \, ds \, dy \\ &+ \iiint_{\Pi_T \times \Pi_T} \left. \text{sign} \left(v - u \right) \left(\text{div}_x k(x) - \text{div}_y k(y) \right) g(v) \phi \, dt \, dx \, ds \, dy \\ &+ \iiint_{\Pi_T \times \Pi_T} \left. \text{sign} \left(v - u \right) \left(k(y) - k(x) \right) \cdot \nabla_x G(v, u) \phi \, dt \, dx \, ds \, dy \\ &=: E_{\text{Conv}}^1 + E_{\text{Conv}}^2 + E_{\text{Conv}}^3. \end{split}$$

Following [7], we derive the estimate

 $\lim_{\alpha_0\downarrow 0} E^1_{\rm Conv}$

(3.9)

(3

$$(10) \leq (\tau - \nu) \Big(\|g\|_{\operatorname{Lip}} \sup_{t \in (\nu, \tau)} |v(\cdot, t)|_{BV(\mathbb{R}^d)} \|k - l\|_{L^{\infty}(\mathbb{R}^d)} + \|g\|_{L^{\infty}(I)} |k - l|_{BV(\mathbb{R}^d)} + \|k\|_{BV(\mathbb{R}^d)} \|f - g\|_{L^{\infty}(I)} + \|k\|_{L^{\infty}(\mathbb{R}^d)} \sup_{t \in (\nu, \tau)} |v(\cdot, t)|_{BV(\mathbb{R}^d)} \|f - g\|_{\operatorname{Lip}(I)} \Big).$$

Taking into account div $k \in BV(\mathbb{R}^d)$ and $k \in Lip(\mathbb{R}^d)$, it is easy to show that

(3.11)
$$\lim_{\alpha_0 \downarrow 0} E^2_{\operatorname{Conv}} \le |\operatorname{div} k|_{BV(\mathbb{R}^d)} ||g||_{L^{\infty}(I)} (\tau - \nu) \rho,$$

(3.12)
$$\lim_{\alpha_0 \downarrow 0} E^3_{\operatorname{Conv}} \le ||k||_{\operatorname{Lip}(\mathbb{R}^d)} ||g||_{\operatorname{Lip}(I)} \sup_{t \in (\nu, \tau)} |v(\cdot, t)|_{BV(\mathbb{R}^d)} (\tau - \nu) \rho.$$

Inserting (3.8), (3.10), (3.11), and (3.12) into (3.5), minimizing the result with respect to $\rho > 0$, and subsequently sending $\nu \downarrow 0$, we get (1.6). This concludes the proof of Theorem 1.1 when B', A' > 0. Note that (1.6) does not depend on the smoothness of v, u. Hence the proof in the general case $B', A' \ge 0$ can proceed via the L^1 convergence of the viscosity method (see Section 1).

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Arguing exactly as in [7], one can prove that

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inserting (3.6), (3.10), (5.11), and (3.12) and (3.5), which each with respect to $\rho \geq 0$, and subsequently sending $\rho \geq 0$, we get (1.6). This constructs the great all Theorem 1.1 when $B'_{1}A' \geq 0$, from the (1.5) does not denoted to the smoothenes of v, v. Hence the proof in the greated rate $B'_{1}A' \geq 0$, from $B'_{2}A' \geq 0$, from $B'_{2}A' \geq 0$, from $B'_{2}A' \geq 0$, the proof in the greated $A'_{2}A' \geq 0$ and prove the proof in the greated (3.5) and the proof in the greated (3.5).

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