Different aspects of the hodograph transform and Riemann invariants for the shallow water equations

Master thesis in Applied and Computational Mathematics

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Abstract

The aim of this work is to visualize irrotational long wave on a sloping beach by following the approach of Carrier and Greenspan [5]. We first derive the non-linear shallow-water equations for sloping beach and then find the Riemann invariants. The Riemann invariants are then used to implement a proper hodograph transformation in order to transform the equations into linear form. By using separation of variables the exact solutions of the linear equations are found and the results are plotted for different values of run-up and run-down time. Furthermore, in this study we obtain shallow-water equations for shear flow which are also called Benney equations [10]. These equations are written in a vector form [1] to find the characteristic form and the Riemann invariants of the shallow-water equations for shear flow over a flat bed.

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Chapter 1

Derivation of shallow water wave equations

The basic equations of fluid motion follow fundamental physical principles which can be formulated explicitly by applying Newton's second law of motion. These principles are defined by the conservation laws of mass, momentum and energy. If we apply these principles to a suitable model of the flow, then we will get mathematical equations which will represent such principles. As Carrier and Greenspan [5] used only conservation equations of mass and momentum to find non-breaking waves on a sloping beach, so in this chapter we will derive only conservation equations of mass and momentum including a flat bottom and an inclined bottom. We will not focus on energy conservation equation. We will choose a control volume as our model of the flow and apply the physical principles of mass and momentum to this model. This will give us the equations in integral form which can be later transformed into partial differential form. By deriving the conservation equations of mass and momentum for a flat bottom, we use our result to find the equations when bottom is inclined. We will also present some linearized conditions and the theory for shallow-water waves.

General assumptions

When we are dealing with the water wave problem we always make some assumptions, we assume that the fluid is incompressible and inviscid. Particularly, when the fluid is incompressible and the flow is irrotational then we get a velocity potential ϕ which satisfies the Laplace's equation. The shallow-water assumptions will be expanded to include rotational assumption in chapter three.

1.1 Conservation equations

There are two different approaches to describe fluid flow. They are Lagrangian method and Eulerian method. We will only apply Eulerian method to derive mass and momentum coservation equations. By following this method we will have fluid properties as a function of space and time. Suppose that the space coordinates are defined in Cartesian system and are denoted by $\mathbf{x} = (x, y, z)$ and the corresponding components of the velocity vector $\mathbf{u}$ by $\mathbf{u} = (u, v, w)$. Let time be denoted by t and the unit vectors along x , y and z axes be denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively. The velocity vector $\mathbf{u}$ can be defined in Cartesian coordinates as $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ where u , v and w are the x, y and z components of the velocity vector which are given as follows:

$$u = u(x, y, z, t),$$

$$v = v(x, y, z, t),$$

$$w = w(x, y, z, t).$$

Also, let $p(\mathbf{x}, t)$ be the pressure and $\rho(\mathbf{x}, t)$ be the density of the fluid. Let us suppose we have a volume V which is fixed and bounded by the surface A. According to the principle of conservation of mass we know that mass can neither be created nor destroyed. If we apply this principle for the fixed volume V, then we have:

Rate of change of mass in V = Mass flux across the boundary surface A. This can be written as

$$\frac{d}{dt}\int_V \rho dV = -\int_A \rho \mathbf{u}\cdot\mathbf{n}dA,$$

where $\mathbf{n}$ is the normal vector on the surface of V, $\int_V dV$ is the triple integral over V and $\int_A dA$ is the double integral over A. If we apply Leibnitz's rule to the left hand side of this equation, then we can write the time derivative $\frac{\partial}{\partial t}$ inside the integral and if we apply Gauss divergence theorem to the right hand side of this equation, then we can express the surface integral in the form of volume integral. This will give us the equation in the following form:

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{u}\right)dV = 0.$$

Since the control volume V is arbitrary the integrand must vanish at every point $(\mathbf{x}, t)$, so the equation becomes (see Appendix A).

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0. \tag{1.1}$$

By applying the chain rule we get $\nabla\cdot(\rho\mathbf{u}) = \rho(\nabla\cdot\mathbf{u}) + (\mathbf{u}\cdot\nabla)\rho$. If we apply this expression to (1.1), we get

$$\frac{\partial \rho}{\partial t} + \rho(\nabla\cdot\mathbf{u}) + (\mathbf{u}.\nabla)\rho = 0. \tag{1.2}$$

Let us define the material derivative as $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$. By using this relation we can write equation (1.2) in the following way:

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\mathbf{u} = 0.$$

For an incompressible flow $\frac{1}{\rho}\frac{D\rho}{Dt} = 0$ and therefore, the equation reduces to

$$\nabla\cdot\mathbf{u} = 0. \tag{1.3}$$

Now for the momentum conservation equation we apply Newton's second law $\sum \mathbf{F} = m\mathbf{a}$ to the control volume V which gives the momentum equation as

$$\sum \mathbf{F} = \frac{d}{dt}\int_V \rho\mathbf{u}dV + \int_A \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dA, \tag{1.4}$$

where $\int_V dV$ is the volume integral and $\int_A dA$ is the surface integral. Also $\mathbf{n}$ is the outward unit normal on A. The first term on the right hand side of the equation represents the rate of change of momentum inside the control volume V whereas the second term represents the flux of momentum across the surface A. The left hand side of the equation represents the sum of the external forces acting on the volume V which consist of body forces that act throughout the volume V and the surface forces that act on the surface A. Taking gravity as the body force the total body force acting on the control volume is defined as

$$\sum \mathbf{F}_b = \int_V \rho \mathbf{g} dV.$$

Surface forces can be split into normal components and tangential components to the contact area. So the total surface force acting on the surface A can be defined as

$$\sum \mathbf{F}_s = \int_A \tau_{ij} \cdot \mathbf{n} dA,$$

where τ_{ij} is a matrix which is also called a stress tensor. The first subscript i indicates the direction of the normal to the surface and the second subscript j indicates the direction of the stress. In a static fluid there are only normal components of stress on a surface which can be defined as $\tau_{ij} = -p\delta_{ij}$, where p is the pressure and δ_{ij} is the identity matrix. For a moving fluid there is additional components σ_{ij} of stress due to viscosity. Thus, when the fluid is in motion the stress tensor τ_{ij} can be split into normal components and tangential components which can be written as $\tau_{ij} = -p\delta_{ij} + \sigma_{ij}$. Now substituting the expressions of body force and surface force in equation (1.4) give the momentum equation as

$$\int_V \rho \mathbf{g} dV + \int_A (-p\delta_{ij} + \sigma_{ij}) \cdot \mathbf{n} dA = \frac{d}{dt} \int_V \rho \mathbf{u} dV + \int_A \rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) dA.$$

Applying Gauss divergence theorem to the surface integral on both sides of this equation and applying Leibnitz's rule to the first term on the right hand side of this equation, we can write the momentum equation as

$$\int_V \left(\rho \mathbf{g} + \nabla \cdot (-p\delta_{ij} + \sigma_{ij}) \right) dV = \int_V \left(\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) \right) dV.$$

Since the control volume is arbitrary, the integrand must vanish at every point. Hence, the equation reduces to

$$\rho\mathbf{g} + \nabla \cdot (-p\delta_{ij} + \sigma_{ij}) = \frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla \cdot (\rho\mathbf{u}\mathbf{u}).$$

By using the relation $\nabla \cdot (\rho\mathbf{u}\mathbf{u}) = \Big(\nabla \cdot (\rho\mathbf{u})\Big)\mathbf{u} + \rho\mathbf{u} \cdot \nabla\mathbf{u}$ and also by using the relation $\frac{\partial}{\partial t}(\rho\mathbf{u}) = \mathbf{u}\frac{\partial \rho}{\partial t} + \rho\frac{\partial \mathbf{u}}{\partial t}$, the above equation can be written as

$$\rho\mathbf{g} + \nabla \cdot (-p\delta_{ij} + \sigma_{ij}) = \rho\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u})\right] + \rho\mathbf{u} \cdot \nabla\mathbf{u}.$$

Substituting the value of equation (1.1) on the right side of this equation and using the material derivative operator previously defined, we can write this equation as

$$\rho\mathbf{g} + \nabla \cdot (-p\delta_{ij} + \sigma_{ij}) = \rho\frac{D\mathbf{u}}{Dt}. \tag{1.5}$$

Also for a Newtonian fluid, we can define the stress tensor τ_{ij} as [6]

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu(\nabla \cdot \mathbf{u})\delta_{ij}. \tag{1.6}$$

Since $\nabla \cdot \mathbf{u} = 0$ for an incompressible fluid, by substituting the value of stress tensor from equation (1.6) into equation (1.5), we have

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{u}, \tag{1.7}$$

where μ is the viscosity of the fluid and equation (1.7) is the Navier-Stokes equation for an incompressible fluid. If we apply this equation to an inviscid fluid, then we get

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g},$$

which is the Euler's equation. We now find the condition for irrotational flow by introducing the vorticity into this equation. Suppose that the density ρ is constant and the only body force that is defined in this equation is the gravity force. Let this force be conservative so that it can be expressed as the gradient of a potential function. Since the z-axis is directed vertically upward so $g_x = 0$, $g_y = 0$ and $g_z = -g$ and hence the potential function must be defined as $\Phi = -gz$ where g is the acceleration of gravity. Since $\mathbf{k}$ is the unit vector along z-axis so the body force becomes $\mathbf{g} = -g\mathbf{k}$. So, we have

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p - g\mathbf{k}, \tag{1.8}$$

where $(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})$ and $\nabla \times \mathbf{u}$ is the vorticity. For an irrotational flow we must have $\nabla \times \mathbf{u} = 0$ so that there exists a velocity potential ϕ such that $\mathbf{u} = \nabla\phi$. Hence, by the relations $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{u} = \nabla\phi$, the velocity potential ϕ satisfies the Laplace's equation $\nabla^2\phi = 0$. Now by substituting $\mathbf{u} = \nabla\phi$ and $(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u} \cdot \mathbf{u})$ into equation (1.8) and then by integrating with respect to $\mathbf{x}$, we have

$$\frac{p - p_0}{\rho} = B(t) - \phi_t - \frac{1}{2}(\nabla\phi)^2 - gz,$$

where p_0 is a constant and the term $B(t)$ is an arbitrary function which can be absorbed into ϕ by choosing a new potential. So we can now ignore the term $B(t)$ and therefore, the equation becomes

$$p - p_0 = -\rho\left\{\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz\right\}. \tag{1.9}$$

If we introduce the velocity potential in the continuity equation (1.3), we obtain

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0.$$

The solution of these equations can be found for the relevant boundary conditions.

Boundary conditions

We consider the air-water interface and describe it as $f(x, y, z, t) = 0$. The interface is defined in such a way that the fluid does not cross it. Therefore, the velocity of the fluid normal to the interface must be equal to the velocity of the interface normal to itself, which means that $(\mathbf{n} \cdot \mathbf{u})_{z=\eta} = \mathbf{n} \cdot \mathbf{U}_{interface}$, where $\mathbf{n}$ is the surface normal defined as $\mathbf{n} = \nabla f/|\nabla f|$. This equality also gives us the following condition [15]

$$\frac{Df}{Dt} = f_t + uf_x + vf_y + wf_z = 0. \tag{1.10}$$

However, it is convenient to describe the interface as $z = \eta(x, y, t)$ and choose $f(x, y, z, t) = \eta(x, y, t) - z$. By substituting this value in condition (1.10), we get

$$\eta_t + u\eta_x + v\eta_y = w \quad on\ z = \eta(x, y, t).$$

Writing u, v and w in terms of velocity potential ϕ, we have

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z \quad on\ z = \eta(x, y, t), \tag{1.11}$$

which is a kinematic condition on the boundary. If we neglect the motion of the air, then we can find another boundary condition by assuming $p = p_0$ at the surface, where p_0 is the atmospheric pressure. So we can get a boundary condition as

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta = 0 \quad on\ z = \eta(x, y, t), \tag{1.12}$$

which is a dynamic condition. Now we need to find the boundary condition on the bottom. We have the bottom which is defined on $z = -h(x, y)$. The fluid can not cross the solid fixed boundary, so the normal velocity of the fluid must vanish, which means that $\mathbf{n} \cdot \mathbf{u} = 0$. Hence, for $z = -h(x, y)$, equation (1.10) will give us

$$\phi_x + \phi_x h_x + \phi_y h_y = 0 \quad on\ z = \eta(x, y, t), \tag{1.13}$$

which is a bottom boundary condition.

Linear waves

We consider the small amplitude water waves such that the wave amplitude is much smaller than the wavelength. For small perturbations on the water surface, η and ϕ are small, so we can find the linearized free surface boundary conditions as follows:

$$\eta_t = \phi_z, \quad \phi_t + g\eta = 0 \quad on\ z = \eta(x, y, t). \tag{1.14}$$

Now we can linearize further by applying these conditions on $z = 0$. So differentiating the second condition with respect to t and then substituting the first condition, we have

$$\phi_{tt} + g\phi_z = 0 \quad on\ z = 0, \tag{1.15}$$

which is independent of η. The linear problem that consists of the Laplace's equation with the boundary conditions are as follows:

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad on \quad z = -h_0 < \ z \ < \ 0, \tag{1.16}$$

$$\phi_{tt} + g\phi_z = 0 \quad on \quad z = 0, \tag{1.17}$$

$$\phi_z = 0 \quad on \quad z = -h_0, \tag{1.18}$$

where h_0 is the constant undisturbed depth. The water waves are propagating horizontally in $(\mathbf{x}, t)$ direction so that the wave shape is expressed by the function $\eta(\mathbf{x}, t)$ and the solutions can be assumed to take the form

$$\eta = Ae^{i\boldsymbol{\kappa}\cdot\mathbf{x}-i\omega t}, \quad \phi = Z(z)e^{i\boldsymbol{\kappa}\cdot\mathbf{x}-i\omega t}, \tag{1.19}$$

where the vector $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)^T$ is the wave number, A is the wave amplitude and ω is the angular frequency. By using the form of solution (1.19), we can find the dispersion relation and the propagation speed of these water waves. So by substituting ϕ into equation (1.16) and then solving the equation by using the conditions (1.14) and (1.18), we obtain [15]

$$Z(z) = -\frac{ig}{\omega} A \frac{\cosh \kappa(h_0 + z)}{\cosh \kappa h_0}.$$

If we substitute this value for ϕ in equation (1.19), we get

$$\phi = -\frac{ig}{\omega} A \frac{\cosh \kappa(h_0 + z)}{\cosh \kappa h_0} e^{i\boldsymbol{\kappa}\cdot\mathbf{x}-i\omega t}. \tag{1.20}$$

Finally, by solving the equation (1.15) for ϕ and then using the boundary condition (1.18), we get the dispersion relation as

$$\omega^2 = g\kappa \tanh \kappa h_0, \tag{1.21}$$

where the wave number κ is defined by $\kappa = \dfrac{2\pi}{\lambda}$ and λ is the wavelength. Also by using the dispersion relation (1.21), we can find the propagation speed as

$$c = \frac{\omega}{\kappa} = \sqrt{\frac{g}{\kappa} \tanh \kappa h_0},$$

where the speed of propagation is dependent on the wavelength, so the water waves are dispersive. This implies that the water waves will propagate faster with longer wavelength.

Shallow-water assumptions

When the wavelength is much larger than the depth of the water then the waves are called shallow-water waves or long waves. For shallow-water waves we can write $\kappa h_0 \to 0$, where h_0 is the undisturbed depth of the water. To find the dispersion relation and the propagation speed of the shallow-water waves, we can define the hyperbolic functions as $\cosh(x) \approx 1$ and $\sinh(x) \approx \tanh(x) \approx x$. By substituting this approximation in equation (1.21), we get the dispersion relation of the shallow-water as $\omega^2 \approx g\kappa^2 h_0$. This approximation will give the phase speed of the shallow-water as

$$c = \sqrt{gh_0}, \tag{1.22}$$

where the propagation speed is independent of the wavelength. It depends, however, on the depth of the water which implies that the shallow-water waves are non-dispersive. Now we will consider one-dimensional shallow-water waves such that the waves are propagating along x-axis only. The water velocity of the one-dimensional waves that we are considering has two components,$\mathbf{u} = \mathbf{u}(u, w)$. Furthermore, we can see that the horizontal velocity component will depend only on x and t and hence it will be free from z.

For long wave approximation, we neglect the vertical acceleration terms in the Euler's equation (1.8) and derive the hydrostatic pressure by integrating the vertical component of equation (1.8). To obtain this, let us write the z-component of the Euler's equation (1.8) as

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

where we neglect the terms on the left hand side and find the vertical component as

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = 0.$$

Integrating both sides of this equation with respect to z as

$$\int_z^\eta \frac{\partial p}{\partial z} dz = \int_z^\eta \rho g dz.$$

By solving both sides of this equation we can find the equation of hydrostatic pressure as

$$p - p_0 = \rho g(\eta - z). \tag{1.23}$$

By substituting this value in equation (1.8), the horizontal components become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -g\frac{\partial \eta}{\partial x}. \tag{1.24}$$

Since the right hand side is independent of z, the rate of change of u is independent of z and this equation reduces to

$$u_t + uu_x + g\eta_x = 0, \tag{1.25}$$

which is the shallow-water equation for momentum. We can also find the shallow-water equation for mass by integrating equation (1.3) as

$$\int_{-h_0}^\eta \left\{ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right\} dz = 0,$$

where u is independent of z. By applying the Leibnitz's integral rule and the fundamental theorem of integration and then using the boundary conditions (1.10) and (1.13), this equation becomes

$$\eta_t + [u(\eta + h_0)]_x = 0. \tag{1.26}$$

The set of equations (1.25) and (1.26) that we have obtained are called the shallow water equations.

1.2 Flat bottom

In this section, we derive conservation equations of mass and momentum for flat bottom. So let us consider a one-dimensional flow where the water waves are propagating in x-direction. The fluid is assumed to be homogeneous, inviscid, incompressible and irrotational. The horizontal velocity component u depends on the space coordinate x and time t. The pressure is considered to be hydrostatic and the density ρ is constant. Let us define a control volume V as

$$V = \int_0^H \int_0^b \int_{x_1}^{x_2} \cdot \, dx \, dy \, dz \tag{1.27}$$

where the height of the control volume is from 0 to H, b is width which is assumed to be constant and x_1 and x_2 are limits of the interval along x-axis where the length of the control volume is defined.

Conservation equation of mass

According to the law of conservation of mass, we can write

$$\frac{dm}{dt} = \text{mass influx} - \text{mass outflux}. \tag{1.28}$$

Mass is given by density times volume, that is $m = \rho * V$, so substituting the value of V from equation (1.27) we have

$$m = \int_0^b \int_{x_1}^{x_2} \int_0^H \rho \; dzdxdy,$$

where the width b and density ρ are constants. By solving, we get

$$m = \int_{x_1}^{x_2} \rho b H dx. \tag{1.29}$$

By applying Leibnitz's rule, we have

$$\frac{dm}{dt} = \int_{x_1}^{x_2} \frac{\partial(\rho b H)}{\partial t} dx = \int_{x_1}^{x_2} (\rho b H)_t dx. \tag{1.30}$$

We need to find the mass influx and mass outflux. Mass flux is the rate at which mass crosses a control surface and flow rate is the rate at which volume of fluid crosses a control surface. The formula for flow rate is given by

$$\text{flow rate} = \text{velocity} * \text{area} = u * Hb,$$

where u is velocity component and Hb is area. Hence we have

$$\text{mass flux} = \text{density} * \text{flow rate} = \rho * u * Hb.$$

Therefore we have

$$\text{mass influx} = [\rho u H b]_{x_1}, \tag{1.31}$$

$$\text{mass outflux} = [\rho u H b]_{x_2}. \tag{1.32}$$

Now subtracting equation (1.32) from equation (1.31), we have

$$\text{mass influx} - \text{mass outflux} = -\left[\rho u H b\right]_{x_1}^{x_2} = -\int_{x_1}^{x_2} (\rho b H u)_x dx. \tag{1.33}$$

Finally, substituting equation (1.30) and equation (1.33) in equation (1.28), we have

$$\int_{x_1}^{x_2} (\rho b H)_t dx = -\int_{x_1}^{x_2} (\rho b H u)_x dx.$$

We can write this equation as

$$\int_{x_1}^{x_2} [(\rho b H)_t + (\rho b H u)_x] dx = 0.$$

Since ρ and b are constants, this equation becomes

$$\int_{x_1}^{x_2} [(H)_t + (H u)_x] dx = 0.$$

Since x_1 and x_2 are arbitrary, the final equality can only be possible if the integrand vanishes at every point in space. Thus, we must have

$$(H)_t + (Hu)_x = 0, \tag{1.34}$$

which is the conservation equation of mass for flat bottom.

Conservation equation of momentum

The momentum equation follows from the Newton's second law which states that the rate of change of momentum in a control volume is equal to the sum of external forces acting on it. We can write this as:

Time rate of change of momentum in control volume = (Rate of momentum inflow to control volume - Rate of momentum outflow from control volume) + Sum of forces acting on the control volume.

As Momentum = mass $*$ velocity, so by applying equation (1.29) in this expression, we get

$$\text{Momentum} = \left(\rho b \int_{x_1}^{x_2} H dx\right) * u = \rho b \int_{x_1}^{x_2} H u dx.$$

Using Leibnitz's rule, we have

$$\frac{d(\text{Momentum})}{dt} = \int_{x_1}^{x_2} (\rho b H u)_t dx. \tag{1.35}$$

Multiplying equation (1.31) and equation (1.32) by u, we get

$$\text{Momentum inflow} = \left[\rho u^2 H b\right]_{x_1} \tag{1.36}$$

$$\text{Momentum outflow} = \left[\rho u^2 H b\right]_{x_2}. \tag{1.37}$$

Subtracting equation (1.37) from equation (1.36), we get

$$\text{Momentum inflow} - \text{Momentum outflow} = -\int_{x_1}^{x_2} \left(\rho b H u^2\right)_x dx. \tag{1.38}$$

Now we need to find the sum of the forces acting on the control volume. The only force acting on the control volume is taken as the pressure force which is assumed to be hydrostatic and given by equation (1.23). Thus, we have

$$p = \int_0^H \rho g(H - z) b dz = \rho g b \frac{H^2}{2}.$$

Therefore, the pressure force acting on the volume between the interval x_1 and x_2 is given by

$$[p]_{x_1}^{x_2} = -\left[\rho g b \frac{H^2}{2}\right]_{x_1}^{x_2} = -\int_{x_1}^{x_2} \left(\rho g b \frac{H^2}{2}\right)_x dx. \tag{1.39}$$

Finally, substituting equation (1.35), equation (1.38) and equation (1.39) in the above definition of momentum equation, we have

$$\int_{x_1}^{x_2} (\rho b H u)_t \, dx = -\int_{x_1}^{x_2} (\rho b H u^2)_x \, dx - \int_{x_1}^{x_2} \left(\rho g b \frac{H^2}{2}\right)_x dx.$$

Since ρ and b are constants, we can write this equation as

$$\int_{x_1}^{x_2} \left[(Hu)_t + (Hu^2 + g\frac{H^2}{2})_x\right] dx = 0.$$

Again, since x_1 and x_2 are arbitrary, the final equality only holds if the integrand vanishes at every point in space, so this equation will reduce to

$$(Hu)_t + (Hu^2 + g\frac{H^2}{2})_x = 0, \tag{1.40}$$

which is the momentum conservation equation for flat bottom.

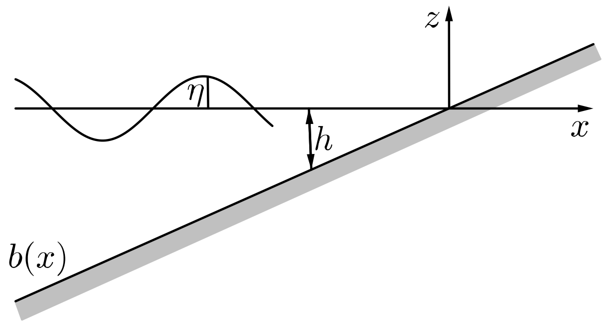

Figure 1.1: Shallow-water wave on a sloping beach.

1.3 Inclined bottom

In this section, we derive shallow-water equations for the case when the bottom is inclined. When bottom is inclined, there is an angle of inclination. Suppose that the angle of inclination is α and the beach profile be defined as $b(x) = \alpha x$ (see figure 1.1). Also, suppose that the surface elevation to be $\eta(x, t)$ and the undisturbed water depth is $h(x)$. Then, the total depth is given by

$$H(x, t) = \eta(x, t) + h(x).$$

The length of the control volume is defined between x_1 and x_2 and the height is from $-h(x)$ to $\eta(x, t)$ and the width b is constant.

Conservation of mass

By following the derivation of mass conservation equation for flat bottom we can write the mass conservation equation for an inclined bottom in the following way:

$$\frac{d}{dt}\int_{x_1}^{x_2}\int_{-h}^{\eta} \rho b \; dzdx + \left[\int_{-h}^{\eta} \rho u b \; dz\right]_{x_1}^{x_2} = 0.$$

Applying similar reasoning of the previous integral equation of mass conservation, we have

$$\eta_t + \Big[u(\eta + h)\Big]_x = 0, \tag{1.41}$$

which is the mass conservation equation for inclined bottom.

Conservation of momentum

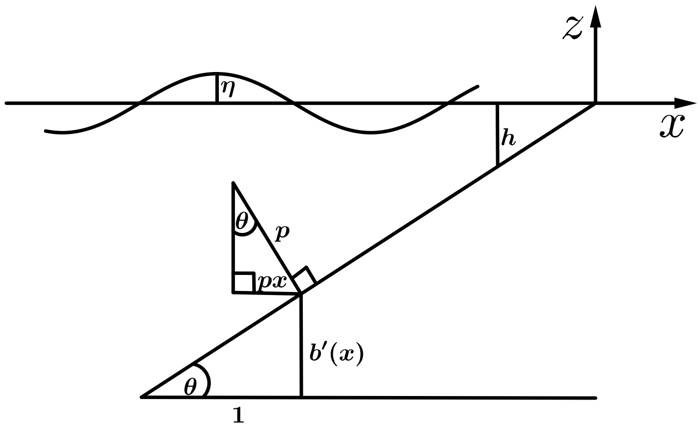

Figure 1.2: Construction of px.

As the fluid is flowing in one-dimension which is along x-axis so let us consider only the x-component of Newton's second law, then we have

$$F_x = ma_x. \tag{1.42}$$

The hydrostatic pressure force acting on the control volume can be defined as

$$\int_{-h}^{\eta} \rho g(\eta - z)b \, dz.$$

There is no gravitational force along x-axis but there is a force px which is in the negative direction of the flow. That force is along negative x-direction and is defined as $px = p \sin\theta$ (see figure 1.2). For shallow-water approximation, we write $b' << 1$, that is, the rate of change of seabed is very small. From figure, we can find expression for $\sin\theta$ as

$$\sin\theta = \frac{b'(x)}{\sqrt{1 + (b'(x))^2}},$$

which is approximately equal to α. We can now find the momentum equation as

$$\frac{d}{dt}\int_{x_1}^{x_2}\int_{-h}^{\eta}\rho ub\ dzdx+\left[\int_{-h}^{\eta}\rho u^2 b\ dz\right]_{x_1}^{x_2}+$$
$$\left[\int_{-h}^{\eta}\rho g(\eta-z)b\ dz\right]_{x_1}^{x_2}=-\int_{x_1}^{x_2}\alpha\rho g(\eta+h)b\ dx.$$

The term on the right hand side is the force which is obtained by taking pressure times the area, where the area can be found between x_1 and x_2 times b. Since ρ and b are constants and since $H=\eta+h$, we can write the third term as

$$\int_{-h}^{\eta} g(\eta-z)dz=\frac{1}{2}gH^2.$$

Since x_1 and x_2 are arbitrary, therefore the equation can be written as

$$(uH)_t+(u^2H)_x+(\frac{1}{2}gH^2)_x=-\alpha gH.$$

By using the mass conservation equation in (1.41), we can write this equation as

$$u_t+uu_x+gH_x=-\alpha g.$$

Recall that $h(x)=-\alpha x$, which gives $H(x,t)=\eta(x,t)+h=\eta(x,t)-\alpha x$, so the momentum equation becomes

$$u_t+uu_x+g\eta_x=0. \tag{1.43}$$

Chapter 2

Solutions of non-linear shallow-water equations

Carrier and Greenspan[5] investigated the behaviour of shallow-water wave as it climbs a sloping beach. They introduced the hodograph transformation to transform the non-linear shallow-water equations into a linear form with separable variables and finally found the exact solutions to show that there are waves that climb a sloping beach without breaking. In this chapter, we will derive non-linear shallow-water equations in linear form and the exact solutions of these equations will be found by following the idea of Carrier and Greenspan [5].

2.1 Linearization

The non-linear shallow water wave equations for an inclined bottom are given by the conservation equations of mass and momentum. To convert into linear form, at first, we will find the non-dimensional form of these equations by defining some non-dimensional quantities. Then, we will obtain the Riemann invariants and the characteristics form of these equations. By using the Riemann invariants, we will provide a proper hodograph transformation to interchange the role of dependent and independent variables. This will give a new coordinate system (σ, λ) where the potential function ϕ will be defined as $\phi(\sigma, \lambda)$. Finally, in terms of this potential function, we will be able to transform the non-linear shallow-water equations into linear form.

2.2 Non-dimensional quantities

In this section we will define non-dimensional quantities that can be used to find the non-dimensional form of the non-linear shallow-water equations. let us define the non-dimensional quantities as follows:

$$u^* = \frac{u}{u_0},$$
$$x^* = \frac{x}{l_0},$$
$$t^* = \frac{t}{T},$$
$$h^* = \frac{h}{\alpha l_0},$$
$$\eta^* = \frac{\eta}{\alpha l_0},$$

where l_0 is a characteristic length , $u_0 = \sqrt{g l_0 \alpha}$ and $T = \sqrt{\frac{l_0}{\alpha g}}$. Also $h(x) = -\alpha x$ is non-uniform depth. The characteristics speed is defined as $c = \sqrt{gH}$, where H(x,t) $= \eta(x,t)+h$ is the total depth of the fluid. The non-dimensional quantity for c is $c^* = \frac{c}{u_0}$.

2.3 Equations in non-dimensional form

The non-linear shallow-water equations for sloping beach are given by conservation equations of mass and momentum which are as follows:

$$\eta_t + [u(\eta + h)]_x = 0 \tag{2.1}$$

$$u_t + uu_x + g\eta_x = 0 \tag{2.2}$$

Substituting the non-dimensional quantities defined above in these equations, we get

$$\eta^*_{t^*} + [u^*(\eta^* - x^*)]_{x^*} = 0, \tag{2.3}$$

$$u^*_{t^*} + u^* u^*_{x^*} + \eta^*_{x^*} = 0. \tag{2.4}$$

These are the non-dimensional form of the shallow-water equations (2.1) and (2.2).

2.4 Characteristic form

In this section, we will find the characteristic form of the shallow-water equations (2.3) and (2.4) by finding the characteristics curves and their corresponding Riemann invariants.

Now to solve the non-dimensional equations (2.3) and (2.4) for Riemann invariants, we also need the values of η^*, $\eta^*_{x^*}$ and $\eta^*_{t^*}$. To find these values, let us calculate the non-dimensional characteristic speed c^* by using the following quantities:

$$c^* = \frac{c}{u_0}, \eta^* = \frac{\eta}{\alpha l_0}, x^* = \frac{x}{l_0}, h = -\alpha x, H = \eta + h$$

Now substituting all these quantities in characteristic speed $c = \sqrt{gH}$, we have

$$c = \sqrt{g(\eta + h)}.$$

Squaring both sides and putting value of h gives

$$c^2 = g(\eta - \alpha x).$$

Finally, inserting the values of c, x, α and η from the above quantities, we will get value of c^*. Thus,

$$c^{*2} u_0^2 = g(\eta^* \alpha l_0 - \alpha x^* l_0).$$

Factorizing αl_0 on the right hand side and substituting the value of u_0 on left hand side

$$c^{*2} \alpha l_0 g = \alpha l_0 g(\eta^* - x^*).$$

We can cancel $\alpha l_0 g$ on both sides. Therefore, we have

$$c^{*2} = (\eta^* - x^*).$$

Taking square root of c^*, we get

$$c^* = \sqrt{\eta^* - x^*},$$

which is the non-dimensional form of the wave speed $c = \sqrt{gH}$.

We will now ignore the asterisks (*) and write the non-dimensional equations of (2.3) and (2.4) in a simple way.

We have $c = \sqrt{\eta - x}$.

This gives $c^2 = \eta - x$ and $\eta = c^2 + x$. Differentiating η with respect to t and x, we have

$$\eta_t = 2cc_t$$
$$\eta_x = 2cc_x + 1.$$

Substituting the values of η , η_t and η_x in the non-dimensional equations (2.3) and (2.4) and solving equation(2.3), we get

$$2cc_t + c^2u_x + 2ucc_x = 0.$$

c is common in each term so this equtaion can be written as

$$2c_t + cu_x + 2uc_x = 0. \tag{2.5}$$

Solving equation (2.4) we have

$$u_t + uu_x + 2cc_x + 1 = 0. \tag{2.6}$$

Adding equation (2.5) and equation (2.6) yield

$$(u + 2c)_t + u(u + 2c)_x + c(u + 2c)_x + 1 = 0. \tag{2.7}$$

Subtracting equation (2.5) from equation (2.6), we get

$$(u - 2c)_t + u(u - 2c)_x - c(u - 2c)_x + 1 = 0. \tag{2.8}$$

We can write this pair of equations (2.7) and (2.8) into another form. If we define a function $f(x(t), t)$ along a curve $x = x(t)$ in the (x, t)-plane then in the form of total derivative we can write

$$\frac{df}{dt} = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial x}\frac{dx}{dt},$$

where dx/dt is a slope of the curve. Now, if we take a function as $f = u+2c+t$ along a curve C^+ in the (x, t)-plane and set $\frac{dx}{dt} = u + c$ and $\frac{dt}{dt} = 1$ we get

$$\frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial x}\frac{dx}{dt} = \left\{\frac{\partial}{\partial t} + (u + c)\frac{\partial}{\partial x}\right\}(u + 2c + t),$$

which is same as the left hand side of the equation (2.7). Hence we have

$$\frac{d}{dt}(u + 2c + t) = 0.$$

That is, along the curve C^+, the function $u + 2c + t$ is constant. Similarly, if we take a function as $f = u - 2c + t$ along a curve C^- in the (x, t)-plane and set $\frac{dx}{dt} = u - c$ and $\frac{dt}{dt} = 1$ we get from equation (2.8) that

$$\frac{d}{dt}(u - 2c + t) = 0.$$

This means that along the curve C^-, the function $u - 2c + t$ is constant. Thus, we have now different form of equations (2.7) and (2.8) as

$$\left\{\frac{\partial}{\partial t} + (u + c)\frac{\partial}{\partial x}\right\}(u + 2c + t) = 0, \tag{2.9}$$

$$\left\{\frac{\partial}{\partial t} + (u - c)\frac{\partial}{\partial x}\right\}(u - 2c + t) = 0, \tag{2.10}$$

which are known as the characteristic form of the equation. This set of equations can be solved by the method of characteristics [11] to give

$$\begin{cases} u + 2c + t = \text{constant on curves } C^+ : \frac{dx}{dt} = u + c, \\ u - 2c + t = \text{constant on curves } C^- : \frac{dx}{dt} = u - c. \end{cases} \tag{2.11}$$

The set of curves (C^+, C^-) are called characteristics curves and the functions $u \pm 2c + t$ which are constant on their respective curves are called Riemann invariants. These characteristic curves require that the characteristic speed c is relative to the velocity u. Equation (2.11) can be expressed as

$$\begin{cases} u + 2c + t = f(\alpha) \text{ , } \alpha \text{ constant on curves } C^+ : \frac{dx}{dt} = u + c, \\ u - 2c + t = g(\beta) \text{ , } \beta \text{ constant on curves } C^- : \frac{dx}{dt} = u - c, \end{cases} \tag{2.12}$$

where f and g are arbitrary functions and also α and β are characteristic variables. For the arbitrary functions f and g we can write

$$\alpha = u + 2c + t, \tag{2.13}$$

$$-\beta = u - 2c + t. \tag{2.14}$$

Adding equations (2.13) and (2.14), we have

$$u + t = \frac{\alpha - \beta}{2}.$$

Subtracting equation (2.14) from equation (2.13), we get

$$c = \frac{\alpha + \beta}{4}.$$

2.5 Interchanging variables

Carrier and Greenspan [5] found hodograph transformation useful in linearizing the shallow-water equations. It changes the role of dependent and independent variables to convert the equation into linear form. We have two nonlinear equations (2.9) and (2.10) where u and c are dependent variables and x and t are independent variables. While solving these equations by hodograph transformation we need to be careful of the Jacobian determinant. It must be dropped out from the equations and according to the Inverse Function Theorem, the Jacobian determinant need to be non-zero for the transformation. Therefore, in order to interchange the role of dependent and independent variables, we will cancel the Jacobian determinant carefully from the equations (2.9) and (2.10). Hence, we will take (x, t) as the independent variables and the characteristic variables (α, β) as the dependent variables and then apply the hodograph transformation. Thus, we transform $\alpha = \alpha(x, t)$ and $\beta = \beta(x, t)$ to

$$x = x(\alpha, \beta), \quad t = t(\alpha, \beta), \tag{2.15}$$

where now α and β are independent variables and x and t are dependent variables. To interchange the variables, we can differentiate equation (2.15) with respect to x and t by using the chain rule for partials. So differentiating each term of equation (2.15) with respect to x, we have

$$\frac{\partial x}{\partial x} = \frac{\partial x}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial x}{\partial \beta}\frac{\partial \beta}{\partial x}$$

Substituting the values of α and β from the equations (2.13) and (2.14), we obtain

$$\begin{aligned}
1 &= x_\alpha \frac{\partial(u+2c+t)}{\partial x} + x_\beta \frac{\partial(-u+2c-t)}{\partial x},\\
1 &= x_\alpha(u_x+2c_x) + x_\beta(-u_x+2c_x).
\end{aligned}$$

$$\text{Notice that} \quad \frac{\partial t}{\partial x} = \frac{\partial t}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial t}{\partial \beta}\frac{\partial \beta}{\partial x}, \quad \text{so that}$$

$$\begin{aligned}
0 &= t_\alpha \frac{\partial(u+2c+t)}{\partial x} + t_\beta \frac{\partial(-u+2c-t)}{\partial x},\\
0 &= t_\alpha(u_x+2c_x) + t_\beta(-u_x+2c_x)
\end{aligned}$$

Similarly, differentiating equation (2.15) with respect to t gives

$$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial x}{\partial \beta}\frac{\partial \beta}{\partial t},$$

$$\begin{aligned}
0 &= x_\alpha \frac{\partial(u+2c+t)}{\partial t} + x_\beta \frac{\partial(-u+2c-t)}{\partial t},\\
0 &= x_\alpha(u_t+2c_t+1) + x_\beta(-u_t+2c_t-1).
\end{aligned}$$

$$\text{If we use the fact that} \quad \frac{\partial t}{\partial t} = \frac{\partial t}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial t}{\partial \beta}\frac{\partial \beta}{\partial t}, \text{then we obatin}$$

$$\begin{aligned}
1 &= t_\alpha \frac{\partial(u+2c+t)}{\partial t} + t_\beta \frac{\partial(-u+2c-t}{\partial t},\\
1 &= t_\alpha(u_t+2c_t+1) + t_\beta(-u_t+2c_t-1).
\end{aligned}$$

To summarize, we find the following equations

$$\begin{aligned}
1 &= x_\alpha(u_x+2c_x) + x_\beta(-u_x+2c_x),\\
0 &= t_\alpha(u_x+2c_x) + t_\beta(-u_x+2c_x),\\
0 &= x_\alpha(u_t+2c_t+1) + x_\beta(-u_t+2c_t-1),\\
1 &= t_\alpha(u_t+2c_t+1) + t_\beta(-u_t+2c_t-1).
\end{aligned}$$

Solving these four equations, we get the following values

$$u_x = \frac{t_\alpha + t_\beta}{2J} \quad , \quad u_t = -1 - \frac{x_\alpha + x_\beta}{2J}$$

$$c_x = \frac{t_\beta - t_\alpha}{4J} \quad , \quad c_t = \frac{x_\alpha - x_\beta}{4J}$$

These are only true when the Jacobian, $J \neq 0$. The Jacobian of the transformation is defined as

$$J = \frac{\partial(x,t)}{\partial(\alpha,\beta)} = \begin{vmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial t}{\partial \alpha} & \frac{\partial t}{\partial \beta} \end{vmatrix} = x_\alpha t_\beta - x_\beta t_\alpha.$$

Substituting the values of u_x, u_t, c_x and c_t in equation (2.9), we have

$$-1 - \frac{x_\alpha + x_\beta}{2J} + 2\frac{x_\alpha - x_\beta}{4J} + 1 + (u+c)\Big(\frac{t_\alpha + t_\beta}{2J} + 2\frac{t_\beta - t_\alpha}{4J}\Big) = 0. \quad (2.16)$$

Again, substituting all those values in equation (2.10), we obtain

$$-1 - \frac{x_\alpha + x_\beta}{2J} - 2\frac{x_\alpha - x_\beta}{4J} + 1 + (u-c)\Big(\frac{t_\alpha + t_\beta}{2J} - 2\frac{t_\beta + t_\alpha}{4J}\Big) = 0. \quad (2.17)$$

In order to solve these equations we need to cancel the Jacobian J. We can see that the number 1 and -1 can be easily cancelled out from equations (2.16) and (2.17) and that leaves those equations free from J. Hence, we obtain

$$x_\beta - (u+c)t_\beta = 0, \quad (2.18)$$

$$x_\alpha - (u-c)t_\alpha = 0. \quad (2.19)$$

If we substitute the values of u and c by solving equations (2.13) and (2.14), then we can see that these equations (2.18) and (2.19) are still non-linear in t. Therefore, to make them linear we will change the independent variables by defining new independent variables which is done in next section.

2.6 New independent variables

In the non-linear equations (2.18) and (2.19), we have α and β as independent variables. To convert these equations into linear form, we define σ and λ as new independent variables and then change (α, β) to (σ, λ). For the values of σ and λ, we can solve the equations (2.13) and (2.14) and then find

$$u + t = \frac{\alpha - \beta}{2} = \frac{\lambda}{2}, \quad (2.20)$$

$$c = \frac{\alpha + \beta}{4} = \frac{\sigma}{4} \quad (2.21)$$

To change the independent variables of equations (2.18) and (2.19) we can apply the chain rule to obtain the following terms:

$$
\begin{aligned}
x_\beta &= \frac{\partial x}{\partial \beta} = \frac{\partial x}{\partial \lambda}\frac{\partial \lambda}{\partial \beta} + \frac{\partial x}{\partial \sigma}\frac{\partial \sigma}{\partial \beta} = x_\sigma - x_\lambda, \\
t_\beta &= \frac{\partial t}{\partial \beta} = \frac{\partial t}{\partial \lambda}\frac{\partial \lambda}{\partial \beta} + \frac{\partial t}{\partial \sigma}\frac{\partial \sigma}{\partial \beta} = t_\sigma - t_\lambda, \\
x_\alpha &= \frac{\partial x}{\partial \alpha} = \frac{\partial x}{\partial \lambda}\frac{\partial \lambda}{\partial \alpha} + \frac{\partial x}{\partial \sigma}\frac{\partial \sigma}{\partial \alpha} = x_\lambda + x_\sigma, \\
t_\alpha &= \frac{\partial t}{\partial \alpha} = \frac{\partial t}{\partial \lambda}\frac{\partial \lambda}{\partial \alpha} + \frac{\partial t}{\partial \sigma}\frac{\partial \sigma}{\partial \alpha} = t_\lambda + t_\sigma.
\end{aligned}
$$

Substituting all these terms in equations (2.18) and (2.19) and simplifying, we obtain

$$x_\sigma - ut_\sigma + ct_\lambda - x_\lambda + ut_\lambda - ct_\sigma = 0, \tag{2.22}$$

$$x_\lambda + ct_\sigma - ut_\lambda + x_\sigma - ut_\sigma + ct_\lambda = 0. \tag{2.23}$$

We now write these equations into another form. So suppose that $A = x_\sigma - ut_\sigma + ct_\lambda$ and $B = x_\lambda + ct_\sigma - ut_\lambda$. By using these values we can reduce the equations (2.22) and (2.23) into homogeneous system as

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the determinant of the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is non-zero, the system will have a trivial solution. This implies $A = 0$ and $B = 0$ and hence we obtain

$$x_\sigma - ut_\sigma + ct_\lambda = 0, \tag{2.24}$$

$$x_\lambda + ct_\sigma - ut_\lambda = 0. \tag{2.25}$$

These equations are still nonlinear in t. To make them linear we have to cancel the nonlinear terms ut_σ and ut_λ. If we cancel x from these equations then we are able to cancel these nonlinear terms [5] and can obtain a linear second order equation in t. To achieve this we differentiate equation (2.24) with respect to λ and equation (2.25) with respect to σ. By differentiating equation (2.24) with respect to λ we have

$$x_{\sigma\lambda} - u_\lambda t_\sigma - u t_{\sigma\lambda} c_\lambda t_\lambda + c t_{\lambda\lambda} = 0. \tag{2.26}$$

Differentiating equation (2.25) with respect to σ we get

$$x_{\lambda\sigma} + c_\sigma t_\sigma + c t_{\sigma\sigma} - u_\sigma t_\lambda - u t_{\lambda\sigma} = 0. \tag{2.27}$$

Assume that x and t are smooth functions so that $x_{\sigma\lambda} = x_{\lambda\sigma}$ and $t_{\sigma\lambda} = t_{\lambda\sigma}$. With these assumptions and by combining equations (2.26) and (2.27), we have

$$u_\lambda t_\sigma - u_\sigma t_\lambda - c_\lambda t_\lambda + c_\sigma t_\sigma = c(t_{\lambda\lambda} - t_{\sigma\sigma}) \tag{2.28}$$

From equations (2.20) and (2.21) we find that $u_\lambda = \frac{1}{2} - t_\lambda$, $u_\sigma = -t_\sigma$, $c_\lambda = 0$, $c_\sigma = \frac{1}{4}$ and $c = \frac{\sigma}{4}$. Substituting all these values in equation (2.28), we have

$$\sigma(t_{\lambda\lambda} - t_{\sigma\sigma}) - 3t_\sigma = 0, \tag{2.29}$$

which is a linear second-order partial differential equation in t and since $u + t = \frac{\lambda}{2}$, u must also satisfy equation (2.29). In order to find an expression for $u(\sigma, \lambda)$ and to solve equation (2.29), we introduce the potential function $\phi(\sigma, \lambda)$ in next section. We will also find expressions for $\eta(\sigma, \lambda)$, $x(\sigma, \lambda)$ and $t(\sigma, \lambda)$.

2.7 Another transformation

Let us define a potential function $\phi(\sigma, \lambda)$ as

$$u(\sigma, \lambda) = \frac{1}{\sigma}\phi_\sigma(\sigma, \lambda). \tag{2.30}$$

By using this expression we can transform equation (2.29) in terms of the potential function. By differentiating equation (2.20) with respect to λ and σ and obtain

$$\begin{aligned} t_\sigma &= -u_\sigma, \\ t_{\sigma\sigma} &= -u_{\sigma\sigma}, \\ t_{\lambda\lambda} &= -u_{\lambda\lambda}. \end{aligned}$$

Substituting these values in equation (2.29), we have

$$\sigma(u_{\sigma\sigma} - u_{\lambda\lambda}) + 3u_\sigma = 0, \tag{2.31}$$

where the values of u_σ, $u_{\sigma\sigma}$ and $u_{\lambda\lambda}$ can be obtained by differentiating equation (2.30) with respect to λ and σ. Finally, by putting all these values in equation (2.31), we have

$$\frac{-1}{\sigma^2}\phi_\sigma + \frac{1}{\sigma}\phi_{\sigma\sigma} + \phi_{\sigma\sigma\sigma} - \phi_{\sigma\lambda\lambda} = 0. \tag{2.32}$$

We can also write this equation in the following way:

$$\frac{\partial}{\partial\sigma}\left(\frac{1}{\sigma}\phi_\sigma\right) + \frac{\partial}{\partial\sigma}\phi_{\sigma\sigma} - \frac{\partial}{\partial\sigma}\phi_{\lambda\lambda} = 0.$$

Integrate it with respect to σ, we have

$$\Big(\sigma\phi_\sigma\Big)_\sigma - \sigma\phi_{\lambda\lambda} = 0, \tag{2.33}$$

which is a linear equation in terms of the potential function $\phi(\sigma, \lambda)$. Now we will find u, x, t and η in terms of (σ, λ).

For $u(\sigma, \lambda)$

The expression of u can be taken from the definition (2.30) as

$$u = \frac{1}{\sigma}\phi_\sigma. \tag{2.34}$$

For $t(\sigma, \lambda)$

From equation (2.20), we have $u + t = \frac{\lambda}{2}$ which gives us

$$t = \frac{\lambda}{2} - u. \tag{2.35}$$

For $x(\sigma, \lambda)$

By differentiating equation (2.20) with respect to σ and λ, we get $t_\sigma = -u_\sigma$ and $t_\lambda = \frac{1}{2} - u_\lambda$. Also by differentiating equation (2.30) with respect to σ and λ, we have $u_\sigma = \frac{-1}{\sigma^2}\phi_\sigma + \frac{1}{\sigma}\phi_{\sigma\sigma}$ and $u_\lambda = \frac{1}{\sigma}\phi_{\sigma\lambda}$. To find the expression for $x(\sigma, \lambda)$ substitute all these values and also $c = \frac{\sigma}{4}$ in equation (2.24) to obtain

$$x_\sigma - \left(\frac{1}{\sigma^3}(\phi_\sigma)^2 - \frac{1}{\sigma^2}\phi_\sigma\phi_{\sigma\sigma}\right) + \frac{\sigma}{8} - \frac{1}{4}\phi_{\sigma\lambda} = 0. \tag{2.36}$$

This equation can be solved by using the following expression:

$$\frac{1}{\sigma^3}(\phi_\sigma)^2 - \frac{1}{\sigma^2}\phi_\sigma\phi_{\sigma\sigma} = -\frac{1}{2}\frac{\partial}{\partial\sigma}\left(\frac{1}{\sigma^2}(\phi_\sigma)^2\right) = \frac{\partial}{\partial\sigma}\Big(-\frac{u^2}{2}\Big).$$

Finally, putting the last expression in equation (2.36) and simplifying, we have

$$\frac{\partial}{\partial\sigma}x = \frac{\partial}{\partial\sigma}\Big(-\frac{u^2}{2}\Big) - \frac{\partial}{\partial\sigma}\frac{\sigma^2}{16} + \frac{\partial}{\partial\sigma}\frac{\phi_\lambda}{4}.$$

Integrating it with respect to σ, we get

$$x = \frac{\phi_\lambda}{4} - \frac{\sigma^2}{16} - \frac{u^2}{2}. \tag{2.37}$$

For $\eta(\sigma,\lambda)$

We have $c = \sqrt{\eta - x}$ which implies that $\eta = c^2 + x$ and $c = \dfrac{\sigma}{4}$ which also implies that $c^2 = \dfrac{\sigma^2}{16}$. By substituting the value of c^2 and the expression for x from equation (2.37) we obtain

$$\eta = \frac{\sigma_\lambda}{4} - \frac{u^2}{2}. \tag{2.38}$$

2.8 Exact solutions

The non-linear set of equations (2.18) and (2.19) have been reduced to a linear equation (2.33) which can be solved by using separation of variables. So let us assume that the solution takes the form

$$\phi(\sigma,\lambda) = f(\sigma)g(\lambda). \tag{2.39}$$

Substituting this value into equation (2.33), we get

$$\frac{\sigma f''(\sigma) + f'(\sigma)}{\sigma f(\sigma)} = \frac{g''(\lambda)}{g(\lambda)} = -\omega^2,$$

where ω is an arbitrary constant. By combining the first and last expressions, we have

$$\sigma f''(\sigma) + f'(\sigma) + \omega^2\sigma f(\sigma) = 0. \tag{2.40}$$

Likewise, combining the second and last expressions, we obtain

$$g''(\lambda) + \omega^2 g(\lambda) = 0. \tag{2.41}$$

To proceed the simplification of equation (2.40), let us do a small change of variable to $x = \omega\sigma$. Then we obtain $f'(\sigma) = \omega f'(x)$ and $f''(\sigma) = \omega^2 f''(x)$. Putting these values in equation (2.40), we get

$$x^2 f''(x) + x f'(x) + x^2 f(x) = 0.$$

This is called the Bessel's equation of order zero [9]. Since the order is zero, the first kind of Bessel function is $J_0(x)$ and the second kind of Bessel function is $Y_0(x)$. The general solution of this equation is

$$f(x) = c_1 J_0(x) + c_2 Y_0(x),$$

where c_1 and c_2 are arbitrary constants. Substituting $x = \omega\sigma$ in this equation, we have

$$f(\sigma) = c_1 J_0(\omega\sigma) + c_2 Y_0(\omega\sigma).$$

In particular, the Bessel functions of the second kind are not bounded near zero. Therefore, the Bessel function of the second kind $Y_0(x)$ is not bounded when $\sigma \to 0$ which implies that $c_2 = 0$ and the solution of this equation becomes $f(\sigma) = c_1 J_0(\omega\sigma)$. Also, the general solution of equation (2.41) is

$$g(\lambda) = c_3 cos(\omega\lambda) + c_4 sin(\omega\lambda),$$

where c_3 and c_4 are arbitrary constants. Hence, we determine the bounded solution of equation (2.33) to be

$$\phi = A J_0(\omega\sigma) cos(\omega\lambda),$$

where A is an arbitrary constant. Without loss of generality, if we put $\omega = 1$, then we have

$$\phi = A J_0(\sigma) cos(\lambda). \tag{2.42}$$

The choice of a function $\phi(\sigma, \lambda)$ defines η, u, x and t in terms of the coordinates (σ, λ). If the Jacobian $\dfrac{\partial(x,t)}{\partial(\sigma,\lambda)}$ does not vanish in $\sigma > 0$, the solutions

$\eta(x,t)$ and $u(x,t)$ are single-valued and such solutions define waves which do not break. Thus, in order to see the non-breaking waves, we need to find the solutions in terms of (x,t) coordinates and then plot them. Let us fix a value for the matrix t say $t1$. We will also choose a tolerance value and then find all the indices for t such that $|t - t1| <$ tolerance value. Now, to plot the free surface $\eta(x,t)$, we will find the particular values of the matrices η and x in terms of the indices for t. We will make a pair of these corresponding values of η and x and then sort these pairs. Now we have obtained the values of η and x in order to plot $\eta(x,t)$. The similar process can be done for $u(x,t)$ as well. In particular, when $A \leq 1$ the Jacobian J vanishes nowhere in $\sigma > 0$, thus, we will find the plots for non-breaking waves for both cases $A < 1$ and $A = 1$.

At the shoreline the depth of the water is zero which is given by the line $\sigma = 0$. We can now solve equation (2.37) for x by putting $\sigma = 0$. We also need to find the value of u and ϕ_λ, so differentiating equation (2.42) with respect to σ and λ, we get

$$\phi_\sigma = -AJ_1(\sigma) cos\lambda,$$

$$\phi_\lambda = -AJ_0(\sigma)\sin(\lambda).$$

Putting the value of ϕ_σ in equation (2.34), we get

$$u = -\frac{1}{\sigma} AJ_1(\sigma) cos\lambda.$$

Finally, substituting the value of ϕ_λ and u in equation (2.37), we have

$$x(\sigma,\lambda) = -\frac{A}{4} J_0(\sigma)\sin\lambda - \frac{\sigma^2}{16} - \frac{1}{2}\left(\frac{A}{\sigma} J_1(\sigma) cos\lambda\right)^2.$$

When $\sigma = 0$ we have $J_0(0) = 1$ and $J_1(0) = 0$, so the above equation reduces to

$$x(0,\lambda) = -\frac{A}{4}\sin\lambda. \tag{2.43}$$

Since the value of sin lies between -1 and +1, we can write

$$-1 \leq \sin\lambda \leq +1$$

Multiplying the inequality by $\frac{-A}{4}$, we have

$$\frac{A}{4} \geq -\frac{A}{4} \sin \lambda \geq -\frac{A}{4}.$$

This inequality gives us the maximum value as $\frac{A}{4}$ and minimum value as $-\frac{A}{4}$. Hence the maximum run-up given by equation (2.43) is $\frac{A}{4}$ and the minimum run-down given by equation (2.43) is $-\frac{A}{4}$.

Now we need to find the time for maximum run-up and minimum run-down at the shoreline. So when we put $\sigma = 0$ for $t(\sigma, \lambda)$ in equation (2.35), we get

$$t(0, \lambda) = \frac{\lambda}{2}. \tag{2.44}$$

This equation shows that value of λ will determine the time for maximum run-up and minimum run-down. We can find the value of λ by using equation (2.43).

Time for maximum run-up

From equation (2.43) , we have maximum run-up as $\frac{A}{4}$. So in order to get maximum run-up as $\frac{A}{4}$ we must have $\lambda = \frac{3\pi}{2}$ in equation (2.43). Notice that $\sin \frac{3\pi}{2} = -1$ so if we multiply (2.43) by -1, the right hand side of equation (2.43) will be $\frac{A}{4}$ which is our maximum run-up. Hence, for maximum run-up we must have $\lambda = \frac{3\pi}{2}$. Therefore, if we put $\lambda = \frac{3\pi}{2}$ in equation (2.44) we get

$$t(0, \frac{3\pi}{2}) = \frac{3\pi}{4}.$$

Therefore the time for maximum run-up $\frac{A}{4}$ is $\frac{3\pi}{4}$.

Time for minimum run-down

Similarly, from equation (2.43) we have minimum run-down as $-\frac{A}{4}$ which we can get if we take $\lambda = \frac{\pi}{2}$ in equation (2.43). The choice of λ will make equation (2.43) negative which will give minimum run-down as $-\dfrac{A}{4}$. So, if we put $\lambda = \dfrac{\pi}{2}$ in equation (2.44), we have

$$t(0, \frac{\pi}{2}) = \frac{\pi}{4}.$$

Hence, the time for minimum run-down $-\dfrac{A}{4}$ is $\dfrac{\pi}{4}$.

Now we will take two different values of A as $A = 0.5$ and $A = 1$ and then we will show the wave shape $\eta(x,t)$ at different period of time on a sloping beach. At first for $A = 0.5$, we will get different shapes of wave which are as follow:

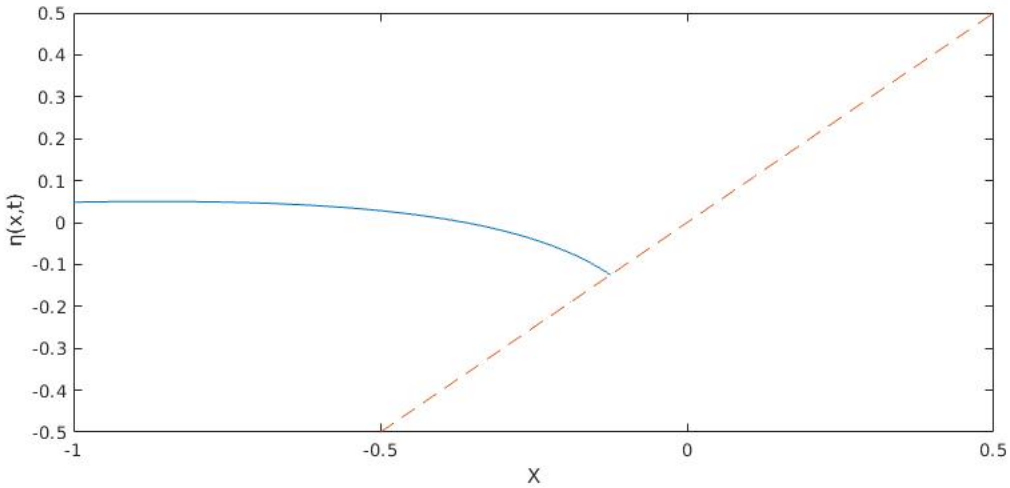

Figure 2.1: For $A = 0.5$ the wave is plotted at $t = \pi/4$ on a sloping beach.

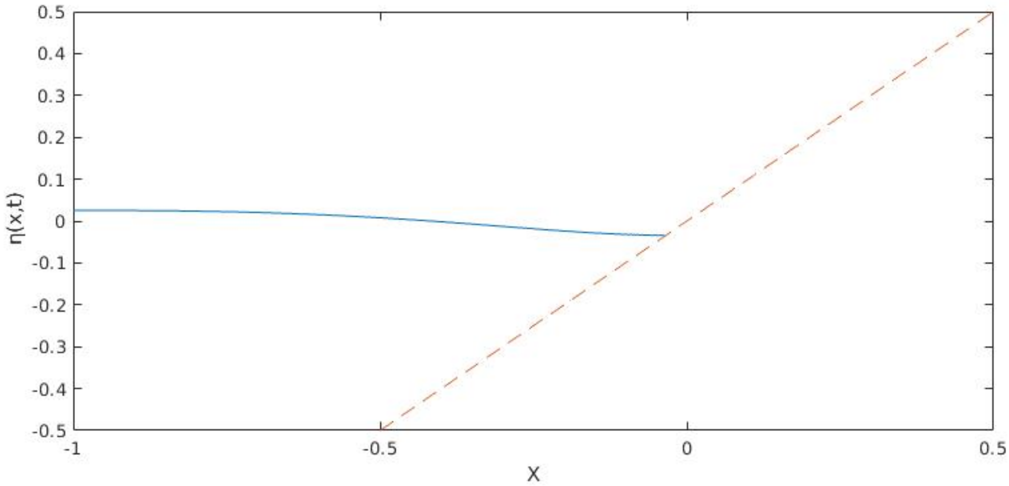

Figure 2.2: For $A = 0.5$ the wave is plotted at $t = 5\pi/12$ on a sloping beach.

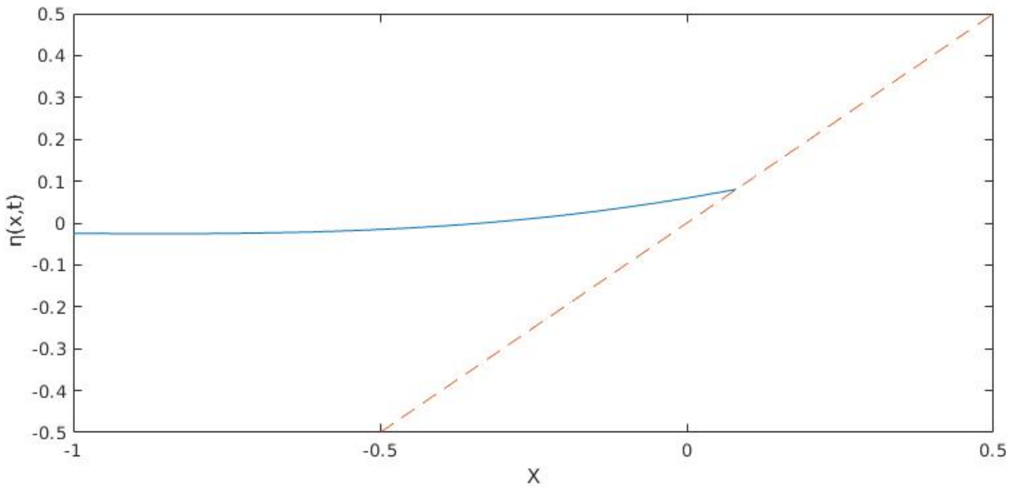

Figure 2.3: For $A = 0.5$ the wave is plotted at $t = 7\pi/12$ on a sloping beach.

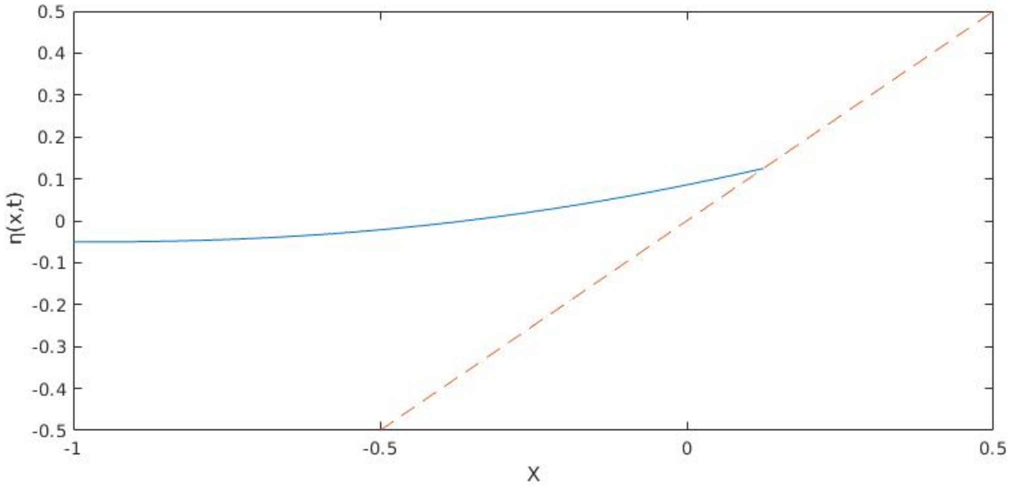

Figure 2.4: For $A = 0.5$ the wave is plotted at $t = 3\pi/4$ on a sloping beach.

Now for $A = 1$ we have different shapes of waves which are as follows:

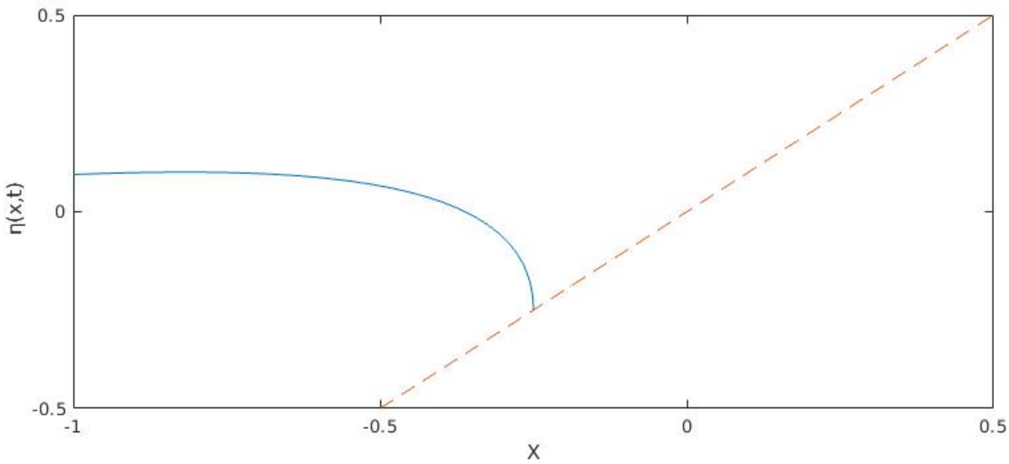

Figure 2.5: For $A = 1$ the wave is plotted at $t = \pi/4$ on a sloping beach.

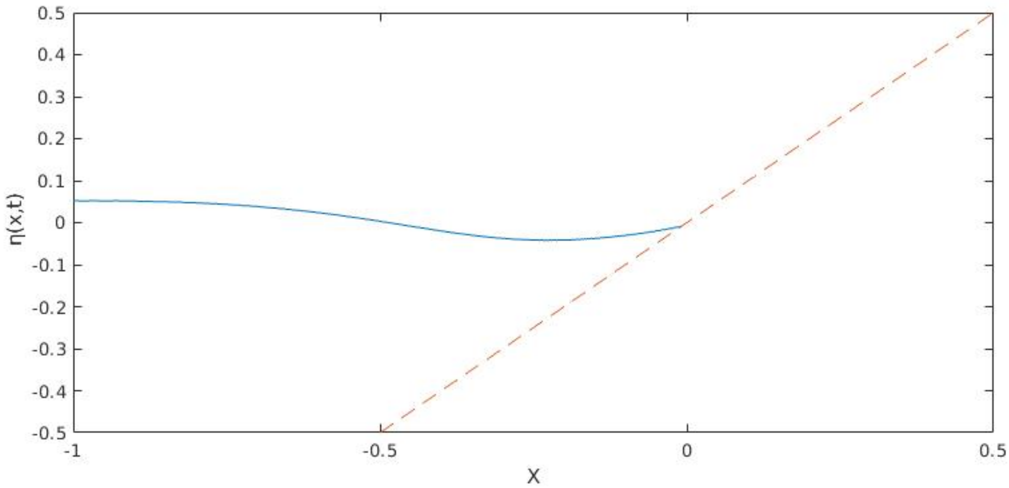

Figure 2.6: For $A = 1$ the wave is plotted at $t = 5\pi/12$ on a sloping beach.

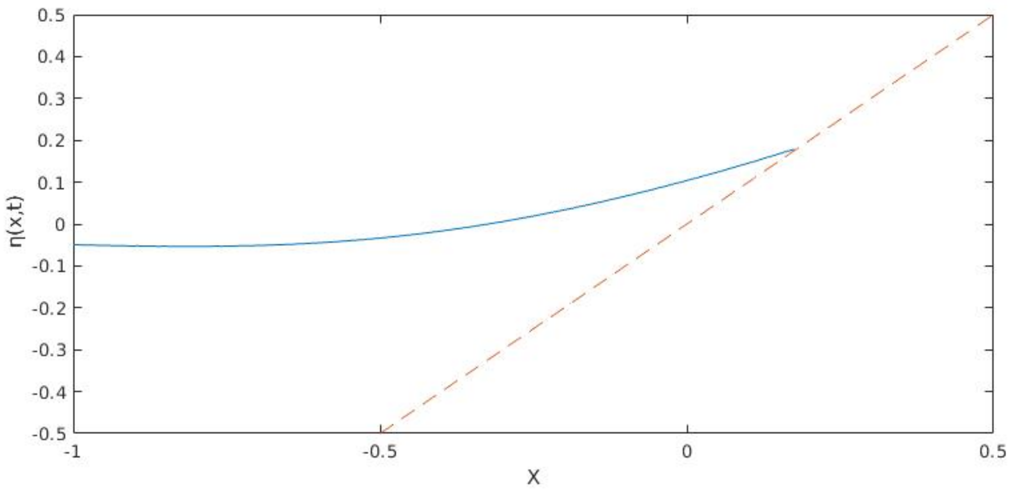

Figure 2.7: For $A = 1$ the wave is plotted at $t = 7\pi/12$ on a sloping beach.

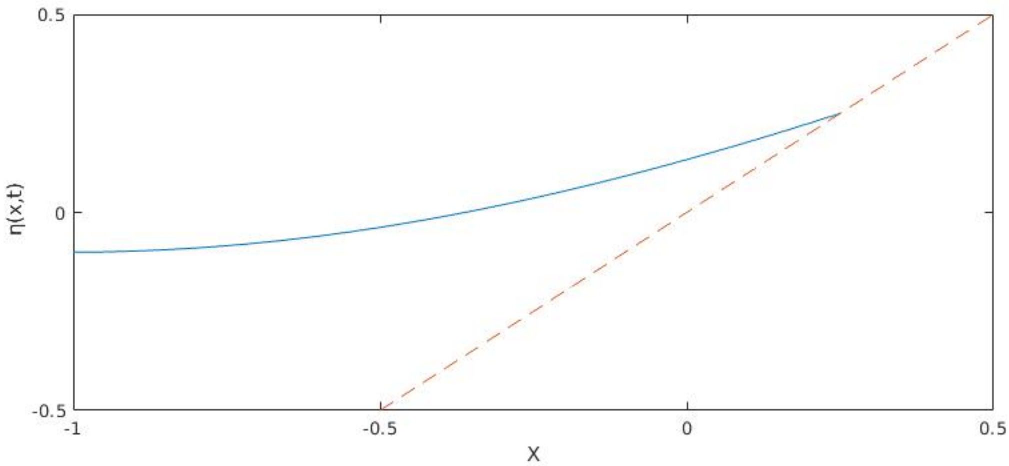

Figure 2.8: For $A = 1$ the wave is plotted at $t = 3\pi/4$ on a sloping beach.

Chapter 3

Long waves propagation on shear flows

This chapter represents a theory that is used in [2]. In the previous chapter, we derived the shallow-water equations for irrotational flow. In this chapter, we will include shear flow in our previous work and then find the shallow-water equations for the case of a flat bed. It should be noted that the perturbation flow is still assumed to be irrotational. We consider a homogeneous, inviscid and incompressible fluid where the fluid is propagating along x-axis. Because of the shear flow the velocity component in the x-direction will now depend on both horizontal and vertical variables x and z. Therefore, the velocity vector for the fluid can be written as $\mathbf{u} = u(x,z,t)\mathbf{i} + w(x,z,t)\mathbf{k}$. The derivation starts with the Euler's equation. The assumption of incompressibility (1.3) implies that the continuity equation is

$$u_x + w_z = 0. \tag{3.1}$$

To get the assumption of the hydrostatic pressure, the x-component of the Euler's equation (1.8) can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

Substituting the value of p from equation(1.23) and simplifying, we get

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -g\frac{\partial \eta}{\partial x}, \tag{3.2}$$

where $u(x, z, t)$ is the horizontal velocity component and $w(x, z, t)$ is the vertical velocity component. We can solve these equations by using proper boundary conditions.

Boundary conditions

Let us define the air-water interface by the property that the fluid can not cross it and describe it as $f(x, y, z, t) = 0$. In order to find the shallow water equations for shear flows it is convenient to describe the surface by $f(x, y, z, t) = h(x, t) - z$. At the free surface we can find a boundary condition that requires the velocity of the fluid normal to the interface must be the same as the velocity of the interface normal to itself. This can be expressed as $(\mathbf{n} \cdot \mathbf{u})_{z=\eta} = \mathbf{n} \cdot \mathbf{U}_{interface}$, where $\mathbf{n}$ is the surface normal defined as $\mathbf{n} = \nabla f / |\nabla f|$ and $U_{interface}$ is defined as $U_{interface} = h_t \mathbf{k}$. Therefore, the condition that these be equal is

$$\frac{Df}{Dt} = f_t + u f_x + v f_y + w f_z = 0. \tag{3.3}$$

By using this condition, we find the boundary condition at the free surface as

$$h_t + u h_x = w \quad on\ z = h(x, t), \tag{3.4}$$

which is a kinematic boundary condition. If we neglect the motion of the air then we can find another boundary condition by assuming $p = p_0$ at the surface, where p_0 is the atmospheric pressure. The boundary condition at the bottom $z = 0$ can be obtained by solving equation (3.3). Thus, we have

$$w = 0 \quad on\ z = 0, \tag{3.5}$$

which is a bottom boundary condition.

Now to find the shallow water equations for shear flows, integrate equation (3.1) with respect to z by taking the limit from 0 to h as

$$\int_0^h (u_x + w_z) dz = 0.$$

By applying the Leibnitz's integral rule to the first term and the fundamental theorem of integration to the second term of this integral, we get

$$\frac{\partial}{\partial x}\int_0^h udz - u(z=h)h_x + u(z=0)\cdot 0 + w(z=h) - w(z=0) = 0.$$

Using the boundary conditions (3.4) and (3.5) in this equation, we obtain

$$h_t + \frac{\partial}{\partial x}\int_0^h udz = 0.$$

To obtain another equation we can integrate equation (3.1) with respect to z by taking the limit from 0 to z as

$$\int_0^z w_z dz = -\int_0^z u_x dz.$$

Solving left hand side and then substituting the boundary condition (3.5), we get

$$w(x,z,t) = -\int_0^z u_x dz.$$

To summarize, the boundary conditions lead to the following system of equations:

$$h_t + \frac{\partial}{\partial x}\int_0^h udz = 0, \tag{3.6}$$

$$u_t + uu_x + wu_z + g\eta_x = 0, \tag{3.7}$$

$$w = -\int_0^z u_x dz. \tag{3.8}$$

These are called the Benney equations or shallow-water equations for shear

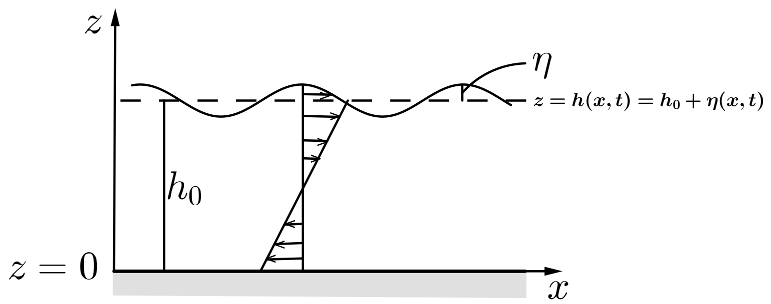

Figure 3.1: Background shear flow over a flat bottom. In this figure $\Omega > 0$.

flows [10], where the horizontal velocity component u depends on both x and z. If u is independent of the vertical variable z, then these equations will coincide with the shallow-water equations in chapter one. For solutions to these system of equations, we can introduce the velocity component as

$$u(x,z,t) = U(x,t) + \Omega\left(z - \frac{h(x,t)}{2}\right), \tag{3.9}$$

where $U(x,t)$ is the average over the depth which can be defined as $U(x,t) = \frac{1}{h}\int_0^h u dz$ and $\Omega\left(z - \frac{h(x,t)}{2}\right)$ defines the shear flow (see Figure 3.1). Solving equation (3.6), we can obtain

$$h_t + (hU)_x = 0, \tag{3.10}$$

which is the mass conservation equation. Now, we can find a conservation equation of momentum by solving equation (3.7) for the same velocity component (3.9). For this, we will determine the value of w by solving equation (3.8) as

$$w = -\int_0^z u_x dz = -U_x z + \frac{\Omega}{2}h_x z. \tag{3.11}$$

Substituting this value in equation (3.7) and simplifying we get

$$U_t - \frac{\Omega}{2}(h_t + hU_x + Uh_x) + UU_x + \frac{\Omega^2}{4}hh_x + gh_x = 0.$$

The terms inside the bracket becomes zero by the mass conservation equation (3.10), so the equation becomes

$$U_t + UU_x + \frac{\Omega^2}{4}hh_x + gh_x = 0. \tag{3.12}$$

We need to do one more step to find the momentum conservation equation by using this equation. For this, we will rearrange equation (3.10) by multiplying it by U and equation (3.12) by multiplying it by h. This will give us the following form of equations

$$Uh_t + hUU_x + h_x U^2 = 0, \tag{3.13}$$

$$hU_t + hUU_x + \frac{\Omega^2}{4}h^2 h_x + ghh_x = 0. \tag{3.14}$$

To solve equation (3.14) we need to find an expression for hU_t. We find this from the relation $(hU)_t = h_t U + U_t h$. This can also be written as $hU_t = (hU)_t - Uh_t$ and by substituting this expression in equation (3.14), we get

$$(hU)_t - Uh_t + hUU_x + ghh_x + \frac{\Omega^2}{4}h^2 h_x = 0.$$

From equation (3.13) we can find the value of Uh_t and then by including that value in this equation we have

$$(hU)_t + (hU^2)_x + ghh_x + \frac{\Omega^2}{4}h^2 h_x = 0.$$

Further, we define $P(h) = \frac{1}{2}gh^2 + \frac{\Omega^2 h^3}{12}$ and insert it in the above equation to give

$$(hU)_t + \left(hU^2 + P(h)\right)_x = 0, \tag{3.15}$$

which is a conservation of momentum equation. We will define equations (3.10) and (3.15) in a different form in order to find the characteristic form and the corresponding Riemann invariants. So let us write these equations in a vector form [1] as

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}, \tag{3.16}$$

where the vector function $\mathbf{f}(\mathbf{u})$ is defined as

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} hU \\ hU^2 + P(h) \end{pmatrix} = \begin{pmatrix} u_2 \\ \dfrac{u_2^2}{u_1} + P(u_1) \end{pmatrix}, \tag{3.17}$$

where $\mathbf{u} = [u_1, u_2]^T = [h, hU]^T$. We can write $\mathbf{f}(\mathbf{u})_x = \mathbf{f}'(\mathbf{u})\mathbf{u}_x$, where $\mathbf{f}'(\mathbf{u})$ is the Jacobian matrix

$$\mathbf{f}'(\mathbf{u}) = \begin{pmatrix} \dfrac{\partial f_1}{\partial u_1} & \dfrac{\partial f_1}{\partial u_2} \\ \dfrac{\partial f_2}{\partial u_1} & \dfrac{\partial f_2}{\partial u_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{u_2^2}{u_1^2} + \dfrac{dP}{du_1} & \dfrac{2u_2}{u_1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -U^2 + \dfrac{dP}{dh} & 2U \end{pmatrix}. \tag{3.18}$$

If we set $\dfrac{dP}{dh} = C^2$, then we have

$$\mathbf{f}'(\mathbf{u}) = \begin{pmatrix} 0 & 1 \\ C^2 - U^2 & 2U \end{pmatrix}. \tag{3.19}$$

The characteristic equation of $\mathbf{f}'(\mathbf{u})$ is $|\mathbf{f}'(\mathbf{u}) - \lambda I| = 0$ which will give us two eigenvalues as $\lambda_1 = U + C$ and $\lambda_2 = U - C$. By using these eigenvalues, we find the right eigenvectors as

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ U + C \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ U - C \end{pmatrix}. \tag{3.20}$$

If we define $R = [r_1, r_2]$, then the left eigenvectors are give by $L = R^{-1}$. Solving this we find left eigenvectors explicitly as

$$\mathbf{l}_1 = \begin{pmatrix} -U + C \\ 1 \end{pmatrix}, \quad \mathbf{l}_2 = \begin{pmatrix} -U - C \\ 1 \end{pmatrix}. \tag{3.21}$$

In order to derive the Riemann invariants w_1 and w_2 we can write equation (3.16) for eigenproblem $\mathbf{L}\mathbf{f}'(\mathbf{u}) = \mathbf{\Lambda}\mathbf{L}$ as

$$\mathbf{l}_i^T \mathbf{u}_t + \lambda_i \mathbf{l}_i^T \mathbf{u}_x = 0, \tag{3.22}$$

where $\lambda_i(\mathbf{u})$ is an eigenvalue of matrix $\mathbf{f}'(\mathbf{u})$ and $\mathbf{l}_i(\mathbf{u})$ is left eigenvector for $i = 1, 2$. If we define the auxiliary function $\mu(\mathbf{u})$ that satisfies

$$w_1'(\mathbf{u}) = \left[\frac{\partial w_1}{\partial u_1}, \frac{\partial w_1}{\partial u_2}\right] = \mu\left[-\frac{u_2}{u_1} + \sqrt{\frac{dP}{du_1}}, \quad 1\right] = \mu_1(\mathbf{u})\mathbf{l}_1^T, \tag{3.23}$$

$$w_2'(\mathbf{u}) = \left[\frac{\partial w_2}{\partial u_1}, \frac{\partial w_2}{\partial u_2}\right] = \mu\left[-\frac{u_2}{u_1} - \sqrt{\frac{dP}{du_1}}, \quad 1\right] = \mu_2(\mathbf{u})\mathbf{l}_2^T, \tag{3.24}$$

then for $i = 1, 2$,we can write equation (3.22) as

$$w_i'(\mathbf{u})\mathbf{u}_t + \lambda_i w_i'(\mathbf{u})\mathbf{u}_x = 0, \tag{3.25}$$

that is

$$\left\{\frac{\partial}{\partial t} + \lambda_i \frac{\partial}{\partial x}\right\} w_i(\mathbf{u}) = 0. \tag{3.26}$$

This is the characteristic form where the function $w_i(\mathbf{u})$ is constant along the characteristics $\frac{dx}{dt} = \lambda_i(\mathbf{u})$. Furthermore, we can assume that we have the relation $\frac{\partial^2 w_i}{\partial u_2 \partial u_1} = \frac{\partial^2 w_i}{\partial u_1 \partial u_2}$, which is satisfied. So if we suppose that $\mu = \frac{1}{u_1}$, then we can find the Riemann invariant w_1 from equation (3.23)

$$\frac{\partial w_1}{\partial u_1} = -\frac{u_2}{u_1^2} + \frac{1}{u_1}\sqrt{\frac{dP(u_1)}{du_1}}, \tag{3.27}$$

$$\frac{\partial w_1}{\partial u_2} = \frac{1}{u_1}. \tag{3.28}$$

Integrating equation (3.27) with respect to u_1 and then equation (3.28) with respect to u_2, we get

$$w_1 = \frac{u_2}{u_1} + \int \frac{1}{u_1}\sqrt{\frac{dP(u_1)}{du_1}} du_1 + K_1(u_2), \tag{3.29}$$

$$w_1 = \frac{u_2}{u_1} + K_2(u_1), \tag{3.30}$$

where $K_1(u_2)$ and $K_2(u_1)$ are constants of integration. By combining these two equations, we have

$$w_1 = U + \int \frac{1}{h}\sqrt{\frac{dP(h)}{dh}} dh = U + \int \frac{C(h)}{h} dh, \tag{3.31}$$

where $\mathbf{u} = [u_1, u_2]^T = [h, hu]^T$ and $C^2 = \frac{dP}{dh}$. In order to find another Riemann invariant w_2 we can solve equation (3.24) for $\mu = \frac{1}{u_1}$ as

$$\frac{\partial w_2}{\partial u_1} = -\frac{u_2}{u_1^2} - \frac{1}{u_1}\sqrt{\frac{dP(u_1)}{du_1}}, \tag{3.32}$$

$$\frac{\partial w_2}{\partial u_2} = \frac{1}{u_1}. \tag{3.33}$$

Integrating equation (3.32) with respect to u_1 and then equation (3.33) with respect to u_2, we get

$$w_2 = \frac{u_2}{u_1} - \int \frac{1}{u_1}\sqrt{\frac{dP(u_1)}{du_1}} du_1 + K_3(u_2), \tag{3.34}$$

$$w_2 = \frac{u_2}{u_1} + K_4(u_1), \tag{3.35}$$

where $K_3(u_2)$ and $K_4(u_1)$ are constants of integration. By combining these two equations, we have

$$w_2 = U - \int \frac{1}{h}\sqrt{\frac{dP(h)}{dh}}dh = U - \int \frac{C(h)}{h}dh. \tag{3.36}$$

We have obtained Riemann invariants w_1 and w_2 which are given by equations (3.31) and (3.36), but still we need to do one more step to find $\int \frac{C(h)}{h}dh$. We have $C^2 = \frac{dP}{dh}$ and also from equation (3.15) we have $P(h) = \frac{1}{2}gh^2 + \frac{\Omega^2 h^3}{12}$, so by solving these two we find that

$$C = \frac{1}{2}\sqrt{\Omega^2 h^2 + 4gh}. \tag{3.37}$$

Substituting this value in equation (3.31) and (3.36) and then simplifying, we have

$$w_1 = U + \frac{1}{2}\int \sqrt{\Omega^2 + \frac{4g}{h}}\, dh, \tag{3.38}$$

$$w_2 = U - \frac{1}{2}\int \sqrt{\Omega^2 + \frac{4g}{h}}\, dh \tag{3.39}$$

where we can now find $\frac{1}{2}\int \sqrt{\Omega^2 + \frac{4g}{h}}\, dh$. By using integration by parts and then by using the formula of integration for $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log(x + \sqrt{x^2 - a^2})$ together with integration by substitution, we get

$$\frac{1}{2}\int \sqrt{\Omega^2 + \frac{4g}{h}}dh = \frac{1}{2}\sqrt{\Omega^2 h^2 + 4gh} + \frac{g}{\Omega}\log\left(\sqrt{\Omega^2 h^2 + 4gh} + \Omega h + \frac{2g}{\Omega}\right). \tag{3.40}$$

Finally, substituting this expression in equation (3.31) and (3.36) we get the Riemann invariants as

$$\begin{aligned} w_1 &= U + \int \frac{c(h)}{h}\, dh \\ &= U + \left[\frac{1}{2}\sqrt{\Omega^2 h^2 + 4gh} + \frac{g}{\Omega}\log\left(\sqrt{\Omega^2 h^2 + 4gh} + \Omega h + \frac{2g}{\Omega}\right)\right], \end{aligned} \tag{3.41}$$

$$\begin{aligned} w_2 &= U - \int \frac{c(h)}{h}\, dh \\ &= U - \left[\frac{1}{2}\sqrt{\Omega^2 h^2 + 4gh} + \frac{g}{\Omega}\log\left(\sqrt{\Omega^2 h^2 + 4gh} + \Omega h + \frac{2g}{\Omega}\right)\right]. \end{aligned} \tag{3.42}$$

Also for two eigenvalues $\lambda_1 = U + C$ and $\lambda_2 = U - C$ if we substitute the value of C from (3.37) then we get

$$\lambda_1 = U + C = U + \frac{1}{2}\sqrt{\Omega^2 h^2 + 4gh}, \tag{3.43}$$

$$\lambda_2 = U - C = U - \frac{1}{2}\sqrt{\Omega^2 h^2 + 4gh}. \tag{3.44}$$

Using all these expressions (3.41), (3.42), (3.43) and (3.44) we can write equations (3.10) and (3.15) in characteristic form as

$$\left[\frac{\partial}{\partial t} + \lambda_{1,2}\frac{\partial}{\partial x}\right] w_{1,2} = 0. \tag{3.45}$$

To summarize, we have obtained the following equations:

$$\left[\frac{\partial}{\partial t} + \lambda_{1,2}\frac{\partial}{\partial x}\right] w_{1,2} = 0, \tag{3.46}$$

$$\lambda_{1,2} = U \pm C = U \pm \frac{1}{2}\sqrt{\Omega^2 h^2 + 4gh}, \tag{3.47}$$

$$\begin{aligned} w_{1,2} &= U \pm \int \frac{c(h)}{h} dh \\ &= U \pm \left[\frac{1}{2}\sqrt{\Omega^2 h^2 + 4gh} + \frac{g}{\Omega}\log(\sqrt{\Omega^2 h^2 + 4gh} + \Omega h + \frac{2g}{\Omega})\right]. \end{aligned} \tag{3.48}$$

Similar equations were found in [12].

Chapter 4

Summary and conclusion

We have begun the thesis by defining basic principles of physics for mass and momentum. An Eulerian method is used and the equations of mass and momentum are derived in a control volume V. The fluid is considered to be incompressible, inviscid and for the shallow-water theory, the pressure is assumed to be hydrostatic. We derived the conservation equations of mass and momentum for both flat and inclined bottom. In the second chapter, the non-linear shallow-water equations of mass and momentum for inclined bottom are transformed into linear form by following the idea of Carrier and Greenspan [5]. In order to transform the non-linear shallow-water equations into linear form we found the characteristic form and the Riemann invariants. The Riemann invariants are then used to apply a proper hodograph transformation which transforms the equations into linear form. By using the method of separation of variables, exact solutions of the linear equations are found and the results are plotted for different values. In addition, including background shear flow, the shallow-water equations for shear flow [10] are obtained in the third chapter. We wrote these equations in a vector form in order to find the characteristic form and the Riemann invariants of the shallow-water equations for shear flow over a flat bed.

It would be interesting to find the exact solutions of shallow-water equations for shear flow and analyse the results. It would be done by applying the same method used on the shallow-water equations for inclined bottom in the second chapter. Then, using the Riemann invariants it could be possible to implement a proper hodograph transformation to transform the equations into linear form. The method of separation of variables could then be used to find the exact solutions of the linear equations. It would be interesting to plot the results and see the long wave where a background shear flow is included. It is worth mentioning that long waves in shallow-water were also studied in the context of shear flows in [4, 12]. Exact solutions were found in [4]. The interaction of surface waves with compactly supported patches of vorticity was studied in [7, 14]. We found standing waves and established exact solutions by following the method used in Carrier and Greenspan [5]. It is worth mentioning that our approach could be applied to investigate the propagating long waves using ideas from [8].

Appendix A

In chapter one and two we used following lemma which can be found in [3].

Lemma 4.0.1. *If a continuous function $g(t)$, $t_0 \leq t \leq t_1$ satisfies $\int_{t_0}^{t_1} g(t)f(t)dt = 0$ for any continuous function $f(t)$ with $f(t_0) = f(t_1) = 0$, then $g(t) = 0$.*

Proof. Suppose $g(t^*) > 0$ for some t^*, $t_0 < t^* < t_1$. By the assumption in the lemma we have g is continuous, therefore $g(t) > \kappa$ in some neighbourhood Ω of the point t^* (see Figure 4.1) such that $t_0 < t^* - \alpha < t < t^* + \alpha < t_1$. Let $f(t)$ be defined such that the following conditions hold:

$$f(t) = 0 \quad \text{outside } \Omega, \tag{4.1}$$

$$f(t) = 1 \quad \text{in } \frac{\Omega}{2}, \tag{4.2}$$

$$f(t) > 0 \quad \text{in } \Omega. \tag{4.3}$$

Equation (4.1)-(4.3) imply that for t the inequality, $t^* - \frac{1}{2}\alpha < t < t^* + \frac{1}{2}\alpha$ is

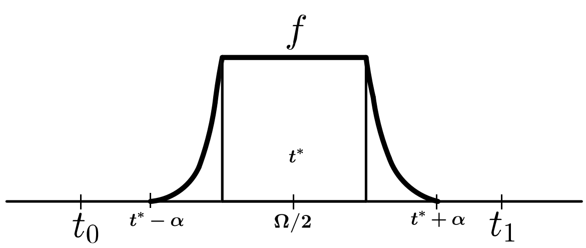

Figure 4.1: Construction of the function f

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satisfied. Then by integrating $g(t) > \kappa$, we obtain

$$\begin{aligned}\int_{t_0}^{t_1} g(t)f(t)dt &\geq \kappa \int_{t_0}^{t_1} f(t)dt \\ &\geq \kappa \int_{t^*-\alpha}^{t^*+\alpha} dt = \kappa(t^* + \alpha - t^* + \alpha) = 2\alpha\kappa \geq \alpha\kappa > 0, \quad (4.4)\end{aligned}$$

which contradicts that $g(t^*) = 0$ for all t^* , $t_0 < t^* < t_1$. $\square$

Appendix B

Listing 4.1: Matlab code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% Computing time development of a wave on a sloping beach
%
%
% by Dipti Acharya
%
% April 2018
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%



lamda= (0.002:0.01:20)';               % Independent variable 1
sigma= (0.001:0.01:20);                % Independent variable 2
A=0.5;                                 % Amplitude

A1=cos(lamda);
phi= A*A1*besselj(0,sigma);
B= besselj(0,sigma);
C= -besselj(1,sigma);
```

```
phi_sigma= A*cos(lamda)*C;
phi_lamda= -A*sin(lamda)*B;

u= phi_sigma./sigma;                        % Expressions for exact solutions
etta= phi_lamda/4-u.^2/2;
t= bsxfun(@minus,lamda/2,u);
x= bsxfun(@minus,etta,sigma.^2/16);

sx=(-4:0.01:4)                              % The slope
tol= 0.002;                                 % Tolerance


% Finding the indices to the t-values in the t-matrix
for i = pi/4:pi/6:3*pi/4
    D =find(abs(t-i)<tol);
    %numbers_tvalue=length(D);
% Using the indices to find the etta-values and x-values
    etta_1= etta(D);
    x_1= x(D);
    F_1= [x_1,etta_1];
    N= sortrows(F_1);                       % Sort the x-column
    N_1=N(:,1);
    N_2=N(:,2);
   figure;
    plot(N_1,N_2,sx,sx,'--');
axis([-1 0.5 -0.5 0.5]);
xlabel('X')                                 % x-axis label;
ylabel('\eta(x,t)')                         % y-axis label;
end
```

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