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Cubical and cosimplicial descent

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Abstract

We prove that algebraic K-theory, topological Hochschild homology and topological cyclic homology satisfy cubical and cosimplicial descent at connective structured ring spectra along 1-connected maps of such ring spectra.

1. *Introduction*

In this paper we extend the techniques used in [Dun97] to prove that algebraic K-theory, topological Hochschild homology and topological cyclic homology of connective structured ring spectra all satisfy descent along 1-connected maps of such ring spectra.

Theorem 1.1 (Cubical descent). *Let R be a connective commutative $\mathbb{S}$-algebra and let A and B be connective R-algebras. Suppose that the unit map $\iota\colon R \to B$ is 1-connected. Then the functors $F = K$, THH and TC satisfy cubical descent at A along $R \to B$, in the sense that in each case the natural map*

$$\eta\colon F(A) \xrightarrow{\simeq} \operatorname*{holim}_{T \in P} F(X(T))$$

is an equivalence of spectra. Here P denotes the partially ordered set of nonempty finite subsets $T = \{t_0 < \cdots < t_q\}$ of $\mathbb{N}$, and $X(T) \cong A \wedge_R B \wedge_R \cdots \wedge_R B$, with $(q+1)$ copies of B.

When the cubical diagram $T \mapsto X(T)$ arises from a cosimplicial spectrum $[q] \mapsto Y^q$, the homotopy limit over P can be replaced by a homotopy limit over the category Δ of nonempty finite totally ordered sets $[q] = \{0 < \cdots < q\}$. This happens, for instance, when the R-algebra B is commutative.

Theorem 1.2 (Cosimplicial descent). *Let R be a connective commutative $\mathbb{S}$-algebra, let A be a connective R-algebra, and let B be a connective commutative R-algebra. Suppose that the unit map $\iota\colon R \to B$ is 1-connected. Then the functors $F = K$, THH and TC satisfy cosimplicial descent at A along $R \to B$, meaning that in each case the natural map*

$$\eta\colon F(A) \xrightarrow{\simeq} \operatorname*{holim}_{[q] \in \Delta} F(Y^q)$$

is an equivalence of spectra. Here $Y^q = A \wedge_R B \wedge_R \cdots \wedge_R B$, with $(q+1)$ copies of B.

These results will be proved in Theorems 2.4, 2.5 and 3.7. They apply, in particular, at any connective $\mathbb{S}$-algebra A along the unit map $\iota\colon \mathbb{S} \to MU$ for complex bordism. In the case of a group $\mathbb{S}$-algebra $A = \mathbb{S}[\Gamma]$, this is relevant for Waldhausen's algebraic K-theory $A(X) \simeq K(\mathbb{S}[\Gamma])$

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of the space $X = B\Gamma$. In the final section we discuss a program to analyze $K(\mathbb{S}[\Gamma])$ and $TC(\mathbb{S}[\Gamma])$ in terms of $K(Y^\bullet)$ and $TC(Y^\bullet)$ for $Y^q = \mathbb{S}[\Gamma] \wedge MU \wedge \cdots \wedge MU$, with $(q+1)$ copies of MU.

2. *Cubical descent*

2.1. *Cubical diagrams*

We use terminology similar to that in [Goo91, §1], including the notions of k-Cartesian and k-co-Cartesian cubes. For each integer $n \geq 1$, let P^n_η be the set of subsets $T \subseteq \{1, \dots, n\}$, partially ordered by inclusion, and let $P^n \subset P^n_\eta$ be the partially ordered subset consisting of the nonempty such T. A functor $X\colon P^n_\eta \to \mathscr{C}$ from P^n_η to any category $\mathscr{C}$ is called an n-dimensional cube, or an n-cube, in that category. The restriction of X to P^n is the subdiagram $X|P^n$ obtained by omitting the initial vertex $X(\varnothing)$ of the n-cube. Given any functor F from $\mathscr{C}$ to spectra, the composite functor $F \circ X$ is an n-cube of spectra, which we also denote as $F(X)$. There is a natural map

$$\eta_n\colon F(X(\varnothing)) \longrightarrow \operatorname*{holim}_{T \in P^n} F(X(T)) = \operatorname*{holim}_{P^n} F(X)$$

from the initial vertex of $F(X)$ to the homotopy limit [BK72, Ch. XI] of the remaining part of the n-cube. We simply write $F(X)$ in place of $F(X|P^n)$ when it is clear that the restriction over $P^n \subset P^n_\eta$ is intended. When forming the homotopy limit of a diagram of spectra we implicitly assume that each vertex has been functorially replaced by a fibrant spectrum, and dually for homotopy colimits. This requires that F takes values in a model category of spectra, such as that of [BF78] or any one of those discussed in [MMSS01], with homotopy category equivalent to the stable homotopy category. By definition, the n-cube $F(X)$ is called k-Cartesian if and only if η_n is a k-connected map. This is equivalent to the n-cube being $(n+k-1)$-co-Cartesian, since the iterated homotopy cofiber of an n-cube of spectra is equivalent to the n-fold suspension of its iterated homotopy fiber. Consider also the partially ordered set P_η of finite subsets T of $\mathbb{N} = \{1, 2, 3, \dots\}$, and let $P \subset P_\eta$ be the partially ordered subset of nonempty such T. A functor X from P_η is an infinite-dimensional cube, or ω-cube.

Definition 2.1. There is a natural map

$$\eta\colon F(X(\varnothing)) \longrightarrow \operatorname*{holim}_{T \in P} F(X(T)) = \operatorname*{holim}_{P} F(X)$$

and a natural equivalence $\operatorname{holim}_P F(X) \simeq \operatorname{holim}_n \operatorname{holim}_{P^n} F(X)$ that connects η to $\operatorname{holim}_n \eta_n$. We say that F satisfies *cubical descent* over X if η is an equivalence of spectra.

For example, if the connectivity of η_n grows to infinity with n, then η is an equivalence and F satisfies cubical descent over X. Cubical descent for F over X ensures that the homotopy type of the spectrum $F(X(\varnothing))$ is essentially determined by the homotopy types of the spectra $F(X(T))$ for nonempty finite subsets $T \subset \mathbb{N}$.

2.2. *Amitsur cubes*

Let R be a connective commutative $\mathbb{S}$-algebra, where $\mathbb{S}$ denotes the sphere spectrum. First, let A and B be connective R-modules, and let $\iota\colon R \to B$ be a map of R-modules. We can and will assume that A and B are flat, i.e., R-cofibrant as R-modules in the sense of [Shi04, Thm. 2.6(1)]. Let $n \geq 1$, and consider the n-cube $X^n = X^n_R(A, B)\colon T \mapsto X^n(T)$ of spectra given by

$$X^n(T) = A \wedge_R X_{1,T} \wedge_R \cdots \wedge_R X_{n,T}\,, \tag{2.1}$$

where $X_{i,T} = B$ for $i \in T$ and $X_{i,T} = R$ for $i \notin T$. For each inclusion $T' \subseteq T$ among subsets of $\{1, \dots, n\}$, the map $X^n(T') \to X^n(T)$ is the smash product over R of id_A with a copy of id_B for each $i \in T'$, a copy of $\iota\colon R \to B$ for each $i \in T \setminus T'$, and a copy of id_R for each $i \notin T$. Letting n vary, these definitions assemble to specify an ω-cube X^ω in spectra, whose restriction over $P^n_\eta \subset P_\eta$ is the n-cube X^n. These constructions are homotopy invariant, because of the assumption that A and B are flat as R-modules.

LEMMA 2.2. *Suppose that $\iota\colon R \to B$ is 1-connected. Then each d-dimensional subcube of the n-cube $X^n = X^n_R(A, B)$ is d-Cartesian and $(2d-1)$-co-Cartesian, for every $0 \le d \le n$.*

Proof. Let B/R denote the 1-connected homotopy cofiber of $\iota\colon R \to B$. The iterated homotopy cofiber of any d-dimensional subcube of X^n is equivalent to the smash product over R of A with d copies of B/R and $(n-d)$ copies of R or B. Hence it is at least $(2d-1)$-connected. Thus the d-dimensional subcube is $(2d-1)$-co-Cartesian, which, as we noted earlier, is equivalent to it being d-Cartesian. □

Next, suppose that A and B are connective R-algebras, and that $\iota\colon R \to B$ is the unit map of B. We can assume that A and B are R-cofibrant as R-algebras in the sense of [Shi04, Thm. 2.6(3)]. The underlying R-modules of A and B are then flat. We view R as a base, A as the object at which we wish to evaluate a functor, and $R \to B$ as a covering that induces a covering $A \to A \wedge_R B$. In this case the n-cube $X^n = X^n_R(A, B)\colon T \mapsto X^n(T)$, defined by the same expression as in (2.1), takes values in the category of connective R-algebras. For varying n, these assemble to an ω-cube $X^\omega = X^\omega_R(A, B)$.

DEFINITION 2.3. Let F be any functor from connective R-algebras to spectra. We call $F(X^\omega)$ the *Amitsur cube* for F at A along $\iota\colon R \to B$, by analogy with the algebraic construction in [Ami59]. When F satisfies cubical descent over X^ω we say that F satisfies *cubical descent at A along $R \to B$*.

Cubical descent for F at A along $R \to B$ ensures that $F(A)$ can be recovered from the diagram $T \mapsto F(X^\omega(T))$ for $T \in P$, having entries of the form $F(A \wedge_R B \wedge_R \cdots \wedge_R B)$ with one or more copies of B.

2.3. *Cubical descent for K, THH and TC*

Let $A \mapsto K(A)$ denote the algebraic K-theory functor from connective $\mathbb{S}$-algebras to spectra, see [BHM93, §5] and [EKMM97, Ch. VI].

THEOREM 2.4. *Let R be a connective commutative $\mathbb{S}$-algebra, let A and B be connective R-algebras, and suppose that the unit map $\iota\colon R \to B$ is 1-connected.*

(a) For each $n \ge 1$, the n-cube $K(X^n) = K(X^n_R(A, B))\colon T \mapsto K(X^n(T))$ is $(n+1)$-Cartesian.

(b) Algebraic K-theory satisfies cubical descent at A along $R \to B$.

Proof. By Lemma 2.2 the n-cube $X^n\colon T \mapsto X^n(T)$ has the property that every d-dimensional subcube is d-Cartesian. Hence the assertion that the n-cube $K(X^n)$ is $(n+1)$-Cartesian is the content of [Dun97, Prop. 5.1]. In other words, the natural map $\eta_n\colon K(A) \to$

$\operatorname{holim}_{P^n} K(X^n)$ is $(n+1)$-connected. Thus $\eta\colon K(A) \to \operatorname{holim}_P K(X^\omega)$ is an equivalence, and K satisfies cubical descent over X^ω. □

Let $A \mapsto THH(A)$ denote the topological Hochschild homology functor from connective $\mathbb{S}$-algebras to cyclotomic spectra, see [BHM93, §3] and [HM97, §2]. In particular, the circle group $\mathbb{T}$ acts naturally on the underlying spectrum of $THH(A)$, and for each subgroup $C = C_{p^m} \subset \mathbb{T}$ of order p^m, where p is a prime, there is a homotopy orbit functor $A \mapsto THH(A)_{hC}$ and a fixed point functor $A \mapsto THH(A)^C$. These are related by a natural homotopy cofiber sequence of spectra

$$THH(A)_{hC_{p^m}} \xrightarrow{N} THH(A)^{C_{p^m}} \xrightarrow{R} THH(A)^{C_{p^{m-1}}} \tag{2.2}$$

called the norm–restriction sequence. Let $TR(A;p) = \operatorname{holim}_{R,m} THH(A)^{C_{p^m}}$ be the sequential homotopy limit over the R-maps. Let $F\colon THH(A)^{C_{p^m}} \to THH(A)^{C_{p^{m-1}}}$ be the map forgetting part of the invariance, and let $TF(A;p) = \operatorname{holim}_{F,m} THH(A)^{C_{p^m}}$ be the sequential homotopy limit over the F-maps. The R-maps induce a self-map of $TF(A;p)$, also denoted R, and the topological cyclic homology functor $TC(A;p)$ can be defined as the homotopy equalizer of id and R:

$$TC(A;p) \xrightarrow{\pi} TF(A;p) \underset{R}{\overset{id}{\rightrightarrows}} TF(A;p)\,.$$

The (integral) topological cyclic homology of A, denoted $TC(A)$, is defined as the homotopy pullback of two maps

$$\prod_{p \text{ prime}} TC(A;p)_p \longrightarrow \prod_{p \text{ prime}} \operatorname*{holim}_{F,m} THH(A)_p^{hC_{p^m}} \longleftarrow THH(A)^{h\mathbb{T}}\,,$$

see [DGM13, Def. 6.4.3.1]. The left hand map is defined in terms of $\pi\colon TC(A;p) \to TF(A;p)$ and the comparison maps $\Gamma_m\colon THH(A)^{C_{p^m}} \to THH(A)^{hC_{p^m}}$ from fixed points to homotopy fixed points. The right hand map is induced by the forgetful maps $THH(A)^{h\mathbb{T}} \to THH(A)^{hC_{p^m}}$ associated to the inclusions $C_{p^m} \subset \mathbb{T}$, for varying p and m. The subscript "p" denotes p-completion. For each prime p, the projection $TC(A) \to TC(A;p)$ becomes an equivalence after p-completion.

Theorem 2.5. *Let R be a connective commutative $\mathbb{S}$-algebra, let A and B be connective R-algebras, and suppose that the unit map $\iota\colon R \to B$ is 1-connected. Consider the n-cube $X^n = X^n_R(A,B)$, as above.*

(a) Each d-dimensional subcube of $THH(X^n)$ is d-Cartesian and $(2d-1)$-co-Cartesian.

(b) Each d-dimensional subcube of $THH(X^n)_{hC}$, $THH(X^n)^C$ and $TR(X^n;p)$ is d-Cartesian and $(2d-1)$-co-Cartesian, for every $C = C_{p^m} \subset \mathbb{T}$.

(c) Each d-dimensional subcube of $TF(X^n;p)$ and $TC(X^n;p)$ is $(d-1)$-Cartesian and $(2d-2)$-co-Cartesian.

(d) Topological Hochschild homology, $THH(-)_{hC}$, $THH(-)^C$, $TR(-;p)$, $TF(-;p)$, $TC(-;p)$ and (integral) topological cyclic homology all satisfy cubical descent at A along $R \to B$.

Proof. (a) The underlying spectrum of $THH(A)$ is naturally equivalent to the realization $B^{cy}(A)$ of the cyclic bar construction $[q] \mapsto A \wedge \cdots \wedge A$, with $(q+1)$ copies of A. Hence there is a natural equivalence

$$\begin{aligned} THH(A \wedge_R B \wedge_R \cdots \wedge_R B) &\simeq B^{cy}(A \wedge_R B \wedge_R \cdots \wedge_R B) \\ &\cong B^{cy}(A) \wedge_{B^{cy}(R)} B^{cy}(B) \wedge_{B^{cy}(R)} \cdots \wedge_{B^{cy}(R)} B^{cy}(B)\,, \end{aligned}$$

where $B^{cy}(R)$ is a connective commutative $\mathbb{S}$-algebra, $B^{cy}(A)$ and $B^{cy}(B)$ are flat connective $B^{cy}(R)$-modules, and $B^{cy}(\iota)\colon B^{cy}(R) \to B^{cy}(B)$ is a 1-connected map of $B^{cy}(R)$-modules. By Lemma 2.2 each d-dimensional subcube of

$$THH(X^n_R(A,B)) \simeq X^n_{B^{cy}(R)}(B^{cy}(A), B^{cy}(B))$$

is $(2d-1)$-co-Cartesian.

(b) Each d-dimensional subcube of $THH(X^n)_{hC}$ is at least as co-Cartesian as the corresponding d-dimensional subcube of $THH(X^n)$, because homotopy orbits preserve connectivity. The analogous claim for the subcubes of $THH(X^n)^C$, with $C = C_{p^m}$, follows by induction on m from the norm–restriction homotopy cofiber sequence.

The equivalent Cartesian claims for the subcubes of $THH(X^n)_{hC}$ and $THH(X^n)^C$ follow. In other words, the iterated homotopy fiber of each d-dimensional subcube of $THH(X^n)_{hC}$ or $THH(X^n)^C$ is $(d-1)$-connected. By the norm-restriction sequence, it follows that the map of iterated homotopy fibers induced by the R-maps $THH(X^n)^{C_{p^m}} \to THH(X^n)^{C_{p^{m-1}}}$ is d-connected, for each m. Hence the iterated homotopy fiber of each d-dimensional subcube of $TR(X^n;p)$ is also $(d-1)$-connected, by the Milnor lim-lim^1 sequence. No connectivity is lost, because lim^1 vanishes on sequences of surjections, cf. [Dun97, Lem. 4.3].

(c) The Cartesian claims for $TF(X^n;p)$ and $TC(X^n;p)$ follow from those for the cubes $THH(X^n)^C$ and $TR(X^n;p)$, respectively, since the sequential homotopy limit over F-maps defining $TF(-;p)$, and the homotopy equalizer of id and F calculating $TC(-;p)$ in terms of $TR(-;p)$, both reduce connectivity of iterated homotopy fibers by at most one, cf. [Dun97, Prop. 4.4].

(d) For each of the functors $F = THH$, $THH(-)_{hC}$, $THH(-)^C$, $TR(-;p)$, $TF(-;p)$ and $TC(-;p)$, the natural map $\eta_n\colon F(A) \to \operatorname{holim}_{P^n} F(X^n)$ is n- or $(n-1)$-connected. Thus the natural map $\eta\colon F(A) \to \operatorname{holim}_P F(X^\omega)$ is an equivalence, and each of these functors satisfies cubical descent over X^ω. Finally, for integral topological cyclic homology we use that each vertex in the natural diagram defining it satisfies cubical descent, since holim_P commutes up to a natural chain of equivalences with other homotopy limits. □

2.4. *Two spectral sequences*

For each ω-cube $F(X)$ of spectra, the equivalent homotopy limits

$$\operatorname*{holim}_P F(X) \simeq \operatorname*{holim}_n \operatorname*{holim}_{P^n} F(X)$$

give rise to two spectral sequences. On the one hand, we have the homotopy spectral sequence

$$E_1^{s,t} = \pi_{t-s} \operatorname{hofib}(p_s) \Longrightarrow_s \pi_{t-s}(\operatorname*{holim}_n \operatorname*{holim}_{P^n} F(X))$$

with $s \geq 0$, associated to the tower of fibrations

$$\cdots \to \operatorname*{holim}_{P^{s+1}} F(X) \xrightarrow{p_s} \operatorname*{holim}_{P^s} F(X) \to \cdots \to \operatorname*{holim}_{P^1} F(X) \xrightarrow{p_0} *\,, \tag{2.3}$$

see [BK72, IX.4.2]. Here $\operatorname{hofib}(p_s)$ denotes the homotopy fiber of the map p_s, which is equivalent to the iterated homotopy fiber of the s-dimensional subcube of $F(X|P^{s+1})$ with vertices indexed by the $T \in P^{s+1}$ with $s+1 \in T$. The notation $\Longrightarrow_s$ indicates that s is the filtration degree.

This spectral sequence is conditionally convergent [Boa99, Def. 5.10] to the sequential limit $\lim_s \pi_*(\operatorname{holim}_{P^s} F(X))$ of the homotopy groups of that tower. For each $r \geq 1$, the bigrading of the d_r-differential is given by

$$d_r^{s,t}\colon E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}\,.$$

Following the usual conventions for Adams spectral sequences, d_r maps bidegree (x,y) in the $(t-s,s)$-plane to bidegree $(x-1, y+r)$. If the derived limit $RE_\infty = \lim_r{}^1 E_r$ of E_r-terms vanishes in each bidegree, then the spectral sequence converges strongly to

$\pi_*(\mathrm{holim}_n\, \mathrm{holim}_{P^n} F(X))$, see [Boa99, Thm. 7.4], which is isomorphic to $\pi_* F(X(\varnothing))$ when F satisfies cubical descent over X. We call this the *cubical descent spectral sequence*.

COROLLARY 2.6. *Let R be a connective commutative $\mathbb{S}$-algebra, let A and B be connective R-algebras, and suppose that the unit map $\iota\colon R \to B$ is 1-connected. Let $X = X^\omega_R(A, B)$, and let F be one of the functors*

- *K, TC (with $c = +1$),*
- *THH, $THH(-)_{hC}$, $THH(-)^C$, $TR(-;p)$ (with $c = 0$),*
- *$TF(-;p)$, $TC(-;p)$ (with $c = -1$).*

Then the cubical descent spectral sequence

$$E_1^{s,t} = \pi_{t-s}\,\mathrm{hofib}(p_s) \Longrightarrow_s \pi_{t-s} F(A)$$

vanishes above the line $t - s = s + c$ of slope $+1$ in the $s \geq 1$ part of the $(t - s, s)$-plane. Hence the spectral sequence collapses at a finite stage in each bidegree, $RE_\infty = 0$, and the spectral sequence is strongly convergent.

Proof. First consider the cases $F = K, THH, \dots, TC(-;p)$ (but not $F = TC$). By Theorems 2.4 and 2.5 the map $\eta_n\colon F(A) \to \mathrm{holim}_{P^n} F(X^n)$ is $(n + c)$-connected for each $n \geq 1$, so $p_s\colon \mathrm{holim}_{P^{s+1}} F(X^{s+1}) \to \mathrm{holim}_{P^s} F(X^s)$ is $(s + c)$-connected for each $s \geq 1$. Thus $E_1^{s,t} = 0$ for $t - s < s + c$ and $s \geq 1$.

The case $F = TC$ remains. For this we appeal to [DGM13, Thm. 7.0.0.2] (where $K(B)$ in the lower left-hand corner of the displayed square should be replaced with $K(A)$), to see that for each $s \geq 1$ the square

$$\begin{array}{ccc} \mathrm{holim}_{P^{s+1}} K(X^{s+1}) & \xrightarrow{p_s} & \mathrm{holim}_{P^s} K(X^s) \\ \downarrow{\scriptstyle \mathrm{trc}} & & \downarrow{\scriptstyle \mathrm{trc}} \\ \mathrm{holim}_{P^{s+1}} TC(X^{s+1}) & \xrightarrow{p_s} & \mathrm{holim}_{P^s} TC(X^s) \end{array}$$

is homotopy Cartesian. This uses that $\pi_0 X^{s+1}(T)$ is constant as a functor of T, by our assumptions on R, A, B and ι. Hence the spectral sequences for K and TC have the same groups $E_1^{s,t}$ for all $s \geq 1$, and from the case $F = K$ we know that these groups vanish for $t - s < s + 1$. □

On the other hand, we have the homotopy limit spectral sequence

$$E_2^{s,t} = \lim_P{}^s\, \pi_t F(X) \Longrightarrow_s \pi_{t-s}\, \mathrm{holim}_P F(X)$$

of [BK72, XI.7.1], associated to P-shaped diagrams of spectra. We call this the *cubical homotopy limit spectral sequence*. The abutment is isomorphic to $\pi_* F(X(\varnothing))$ when F satisfies cubical descent over X. We do not know whether the cubical descent spectral sequence and the cubical homotopy limit spectral sequence are isomorphic for all ω-cubical diagrams $F(X)$, but in Subsection 3.4 we will see that this is the case whenever the diagram $F(X|P)$ arises from a cosimplicial object $F(Y^\bullet)$ by composition with a specific functor $f\colon P \to \Delta$, which we will now introduce.

3. *Cosimplicial descent*

3.1. *Cosimplicial objects*

Let Δ_η be the category of finite totally ordered sets $[q] = \{0 < \cdots < q\}$ and order-preserving functions, where $q \geq -1$ is an integer. The object $[-1] = \varnothing$ is initial in this category. Let $\Delta \subset \Delta_\eta$ be the full subcategory generated by the objects $[q]$ with $q \geq 0$. For $n \geq 1$, let $\Delta_\eta^{<n} \subset \Delta_\eta$ and $\Delta^{<n} \subset \Delta$ be the respective full subcategories generated by the objects $[q]$ with $q < n$.

A coaugmented cosimplicial object in $\mathscr{C}$ is a functor $Y\colon \Delta_\eta \to \mathscr{C}$. We let $Y^q = Y([q])$, for each $q \geq -1$. We can also write Y as $\eta\colon Y^{-1} \to Y^\bullet$, where $Y^\bullet$ is the cosimplicial object given by the restriction of Y over $\Delta \subset \Delta_\eta$. For $n \geq 1$, a functor $Y_{<n}\colon \Delta_\eta^{<n} \to \mathscr{C}$ is called a coaugmented $(n-1)$-truncated cosimplicial object, also written as $\eta\colon Y^{-1} \to Y^\bullet_{<n}$, where $Y^\bullet_{<n} = Y_{<n}|\Delta^{<n}$.

Definition 3.1. Given any functor F from $\mathscr{C}$ to spectra, the composite functor $F \circ Y$ is a coaugmented cosimplicial spectrum $F(Y)$, which we can write as $\eta\colon F(Y^{-1}) \to F(Y^\bullet)$. We say that F satisfies *cosimplicial descent* over Y if the natural map

$$\eta\colon F(Y^{-1}) \longrightarrow \operatorname*{holim}_{[q]\in\Delta} F(Y^q) = \operatorname*{holim}_{\Delta} F(Y^\bullet)$$

is an equivalence of spectra. This ensures that the homotopy type of $F(Y^{-1})$ is essentially determined by the homotopy types of the spectra $F(Y^q)$ for $q \geq 0$.

3.2. *Comparison of cubical and cosimplicial objects*

There is a well-defined functor $f_\eta\colon P_\eta \longrightarrow \Delta_\eta$ that maps the element $T = \{t_0 < \cdots < t_q\} \subset \mathbb{N}$ to the object $[q]$, and maps the inclusion $T \setminus \{t_i\} \subset T$ to the i-th face operator $\delta^i\colon [q-1] \to [q]$, for each $0 \leq i \leq q$. By restricting f_η, one gets functors $f\colon P \to \Delta$, $f_\eta^n\colon P_\eta^n \to \Delta_\eta^{<n}$ and $f^n\colon P^n \to \Delta^{<n}$.

Composition with $f_\eta\colon P_\eta \to \Delta_\eta$ takes each coaugmented cosimplicial spectrum $F(Y)$ to an ω-cube $F(X) = F(Y) \circ f_\eta$ in the category of spectra. Likewise, composition with $f_\eta^n\colon P_\eta^n \to \Delta_\eta^{<n}$ takes each coaugmented $(n-1)$-truncated cosimplicial spectrum $F(Y_{<n})$ to an n-cube $F(X^n) = F(Y_{<n}) \circ f_\eta^n$ of spectra. If $F(Y_{<n})$ is given by restricting $F(Y)$ over $\Delta_\eta^{<n} \subset \Delta_\eta$ then $F(X^n)$ is given by restricting $F(X)$ over $P_\eta^n \subset P_\eta$.

Proposition 3.2. *The functors $f\colon P \to \Delta$ and $f^n\colon P^n \to \Delta^{<n}$ are left cofinal. Hence, for any cosimplicial spectrum $F(Y^\bullet)$ the canonical map*

$$f^*\colon \operatorname*{holim}_{\Delta} F(Y^\bullet) \xrightarrow{\simeq} \operatorname*{holim}_{P} (F(Y^\bullet) \circ f)$$

is an equivalence, and for any $(n-1)$-truncated cosimplicial spectrum $F(Y^\bullet_{<n})$ the canonical map

$$f_n^*\colon \operatorname*{holim}_{\Delta^{<n}} F(Y^\bullet_{<n}) \xrightarrow{\simeq} \operatorname*{holim}_{P^n} (F(Y^\bullet_{<n}) \circ f^n)$$

is an equivalence.

Proof. The assertion that f^n is left cofinal, i.e., that the left fiber $f_n/[q]$ has contractible nerve for each object $[q]$ of $\Delta^{<n}$, is proved in [Car08, §6]. Note that the argument offered at this point in [DGM13, A8.1.1] is flawed. The left fiber $f/[q]$ is the increasing union of the left fibers $f_n/[q]$, hence its nerve is also contractible. The equivalences of homotopy limits then follow from the cofinality theorem of [BK72, XI.9.2], applied level by level for diagrams of (implicitly) fibrant spectra. □

COROLLARY 3.3. *Let Y be a coaugmented cosimplicial object in $\mathscr{C}$, and let F be a functor from $\mathscr{C}$ to spectra. Then F satisfies cosimplicial descent over Y if and only if F satisfies cubical descent over $X = Y \circ f_\eta$.*

COROLLARY 3.4. *For any cosimplicial abelian group $A^\bullet$, the canonical homomorphisms*

$$f^*\colon \pi^s A^\bullet \cong \lim_\Delta{}^s A^\bullet \xrightarrow{\cong} \lim_P{}^s (A^\bullet \circ f) \qquad \text{and} \qquad f_n^*\colon \lim_{\Delta^{<n}}{}^s A^\bullet \xrightarrow{\cong} \lim_{P^n}{}^s (A^\bullet \circ f^n)$$

are isomorphisms, for each $s \geq 0$.

Proof. A direct algebraic proof is possible, but here is an argument based on the results of Bousfield–Kan [BK72]. Consider the cosimplicial Eilenberg–Mac Lane space $K(A^\bullet, s)$. By [BK72, XI.7.2] there are natural isomorphisms $\pi_0 \operatorname{holim}_\Delta K(A^\bullet, s) \cong \lim^s_\Delta A^\bullet$ and $\pi_0 \operatorname{holim}_P K(A^\bullet \circ f, s) \cong \lim^s_P (A^\bullet \circ f)$. By Proposition 3.2 and [BK72, XI.9.2] the map $f^*\colon \operatorname{holim}_\Delta K(A^\bullet, s) \to \operatorname{holim}_P K(A^\bullet \circ f, s)$ is a homotopy equivalence. This gives the isomorphism $f^*\colon \lim^s_\Delta A^\bullet \cong \lim^s_P (A^\bullet \circ f)$. The proof for $f_n\colon P^n \to \Delta^{<n}$ is practically identical. Finally, by [BK72, XI.7.3] there is a natural isomorphism $\pi^s A^\bullet \cong \lim_\Delta{}^s A^\bullet$. □

3.3. *Amitsur resolutions*

We now suppose that R is a connective commutative $\mathbb{S}$-algebra, that A is a connective R-algebra, and that B is a connective commutative R-algebra with unit map $\iota\colon R \to B$ and multiplication map $\mu\colon B \wedge_R B \to B$. The condition that B is commutative ensures that μ is a morphism of connective R-algebras. We can assume that A is R-cofibrant as an R-algebra, and that B is R-com-alg-cofibrant as a commutative R-algebra in the sense of [Shi04, Thm. 3.2]. (This follows if B is positive stable cofibrant in the sense of [MMSS01, Thm. 15.2(i)].) The underlying R-modules of A and B are then flat, by [Shi04, Cor. 4.3], so that the results of Section 2 carry over to this situation.

Consider the cosimplicial connective R-algebra $Y^\bullet = Y^\bullet_R(A, B)\colon [q] \mapsto Y^q$ given by

$$Y^q = A \wedge_R B \wedge_R \cdots \wedge_R B\,, \tag{3.1}$$

with $(q+1)$ copies of B. The coface maps

$$d^i = id_A \wedge (id_B)^{\wedge i} \wedge \iota \wedge (id_B)^{\wedge q-i}\colon Y^{q-1} \longrightarrow Y^q$$

for $0 \leq i \leq q$, and the codegeneracy maps

$$s^j = id_A \wedge (id_B)^{\wedge j} \wedge \mu \wedge (id_B)^{\wedge q-j}\colon Y^{q+1} \longrightarrow Y^q$$

for $0 \leq j \leq q$, are induced by the unit map ι and the multiplication map μ, respectively. The unit map ι also induces a coaugmentation $\eta\colon A \to Y^0 = A \wedge_R B$ that makes $A \to Y^\bullet$ a coaugmented cosimplicial connective R-algebra, i.e., a functor $Y = Y_R(A, B)$ from Δ_η to the category of connective R-algebras:

$$A \xrightarrow{\eta} A \wedge_R B \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; A \wedge_R B \wedge_R B \;\substack{\longrightarrow\\ \longleftarrow\\ \longleftarrow\\ \longrightarrow}\; \cdots\,.$$

DEFINITION 3.5. Given a functor F from connective R-algebras to spectra, we call the coaugmented cosimplicial spectrum $\eta\colon F(A) \to F(Y^\bullet) = F(Y^\bullet_R(A, B))$ the *Amitsur resolution* for F at A along $\iota\colon R \to B$. When F satisfies cosimplicial descent over $A \to Y^\bullet$ we say that F satisfies *cosimplicial descent at A along $R \to B$*.

Cosimplicial descent for F at A along $R \to B$ ensures that the coaugmented cosimplicial spectrum

$$F(A) \xrightarrow{\eta} F(A \wedge_R B) \rightleftarrows F(A \wedge_R B \wedge_R B) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots$$

induces an equivalence

$$\eta\colon F(A) \xrightarrow{\simeq} \operatorname*{holim}_{[q]\in\Delta} F(Y^q) = \operatorname*{holim}_{\Delta} F(Y^\bullet)$$

from $F(A)$ to the homotopy limit of the remainder of the diagram, having entries of the form $F(A \wedge_R B \wedge_R \cdots \wedge_R B)$ with one or more copies of B.

LEMMA 3.6. *There is a natural isomorphism $X^\omega_R(A,B) \cong Y_R(A,B) \circ f_\eta$.*

Proof. For $T = \{t_0 < \cdots < t_q\} \subset \mathbb{N}$, with $f_\eta(T) = [q]$, both $X^\omega_R(A,B)(T)$ and $Y_R(A,B)([q])$ are identified with

$$A \wedge_R B \wedge_R \cdots \wedge_R B\,,$$

where there are $(q+1)$ copies of B. For each $T' \subset T$, the induced morphisms in $X^\omega_R(A,B)$ and $Y_R(A,B) \circ f_\eta$ are evidently compatible with these identifications. □

THEOREM 3.7. *Let R be a connective commutative $\mathbb{S}$-algebra, A a connective R-algebra and B a connective commutative R-algebra, and suppose that the unit map $\iota\colon R \to B$ is 1-connected. Then algebraic K-theory, topological Hochschild homology, $THH(-)_{hC}$, $THH(-)^C$, $TR(-;p)$, $TF(-;p)$, $TC(-;p)$, and (integral) topological cyclic homology all satisfy cosimplicial descent at A along $R \to B$.*

Proof. Combine Theorems 2.4 and 2.5, Corollary 3.3 and Lemma 3.6. □

3.4. *More spectral sequences*

For each coaugmented cosimplicial spectrum $\eta\colon F(Y^{-1}) \to F(Y^\bullet)$, the equivalent homotopy limits

$$\operatorname*{holim}_{\Delta} F(Y^\bullet) \simeq \operatorname*{holim}_{n} \operatorname*{holim}_{\Delta^{<n}} F(Y^\bullet)$$

give rise to two spectral sequences. These turn out to be isomorphic to one another, as well as to the two spectral sequences of Subsection 2.4, when we consider cubical diagrams that arise from cosimplicial diagrams by composition with the left cofinal functor f.

On the one hand, we have the homotopy spectral sequence

$$E_1^{s,t} = \pi_{t-s}\operatorname{hofib}(\delta_s) \Longrightarrow_s \pi_{t-s}(\operatorname*{holim}_{n} \operatorname*{holim}_{\Delta^{<n}} F(Y^\bullet))$$

associated to the tower of fibrations

$$\cdots \to \operatorname*{holim}_{\Delta^{<s+1}} F(Y^\bullet) \xrightarrow{\delta_s} \operatorname*{holim}_{\Delta^{<s}} F(Y^\bullet) \to \cdots \to \operatorname*{holim}_{\Delta^{<1}} F(Y^\bullet) \xrightarrow{\delta_0} *\,, \tag{3.2}$$

which we call the *cosimplicial descent spectral sequence*.

By Proposition 3.2, the tower of fibrations (3.2) is equivalent to the tower (2.3) when $X = Y \circ f_\eta$, $F(X) = F(Y) \circ f_\eta$ and $F(X|P) = F(Y^\bullet) \circ f$. Hence in these cases the cosimplicial descent spectral sequence for $\eta\colon F(Y^{-1}) \to F(Y^\bullet)$ is isomorphic to the cubical descent spectral sequence for $F(X)$.

There is also the *Bousfield–Kan homotopy spectral sequence*

$$E_1^{s,t} = \pi_{t-s}\operatorname{hofib}(\tau_s) \Longrightarrow_s \pi_{t-s}\operatorname{Tot} F(Y^\bullet)$$

of the cosimplicial spectrum $F(Y^\bullet)$, associated to the tower of maps

$$\cdots \to \mathrm{Tot}_s\, F(Y^\bullet) \xrightarrow{\tau_s} \mathrm{Tot}_{s-1}\, F(Y^\bullet) \to \cdots \to \mathrm{Tot}_0\, F(Y^\bullet) \xrightarrow{\tau_0} *\,, \tag{3.3}$$

see [BK72, X.6.1]. To ensure that the maps τ_s are fibrations, we first implicitly replace $F(Y^\bullet)$ with an equivalent fibrant cosimplicial spectrum $F(Y^\bullet)'$ [BK72, X.4.6]. This fibrant replacement does not change the homotopy type of $F(X|P) = F(Y^\bullet) \circ f$ or any of the associated homotopy limits that we consider. The E_2-term of this homotopy spectral sequence can then be expressed as

$$E_2^{s,t} = \pi^s \pi_t F(Y^\bullet) \Longrightarrow_s \pi_{t-s}\, \mathrm{Tot}\, F(Y^\bullet)\,,$$

see [BK72, X.7.2]. We prove in Proposition 3.10 that there are natural equivalences

$$\mathrm{Tot}_n\, F(Y^\bullet) \xrightarrow{\simeq} \operatorname*{holim}_{\Delta^{<n+1}} F(Y^\bullet)$$

(after the implicit fibrant replacement of $F(Y^\bullet)$). Varying n, these equivalences are compatible with the maps τ_s and δ_s, and with the equivalence

$$\mathrm{Tot}\, F(Y^\bullet) \xrightarrow{\simeq} \operatorname*{holim}_{\Delta} F(Y^\bullet)$$

of [BK72, XI.4.4]. Hence the tower (3.3) is equivalent to the tower (3.2), and the cosimplicial descent spectral sequence is isomorphic to the Bousfield–Kan homotopy spectral sequence of $F(Y^\bullet)$, starting with the E_1-term.

On the other hand, we have the *cosimplicial homotopy limit spectral sequence*

$$E_2^{s,t} = \lim_{\Delta}{}^s\, \pi_t F(Y^\bullet) \Longrightarrow_s \pi_{t-s} \operatorname*{holim}_{\Delta} F(Y^\bullet)\,.$$

By [BK72, XI.7.5] there is a natural map from the Bousfield–Kan homotopy spectral sequence of $F(Y^\bullet)$ to the cosimplicial homotopy limit spectral sequence, and this map is an isomorphism at the E_2-term, hence also at all later terms.

Finally, the functor $f\colon P \to \Delta$ induces a map f^* of homotopy limit spectral sequences from

$$E_2^{s,t} = \lim_{\Delta}{}^s\, \pi_t F(Y^\bullet) \Longrightarrow_s \pi_{t-s} \operatorname*{holim}_{\Delta} F(Y^\bullet)$$

to

$$E_2^{s,t} = \lim_{P}{}^s\, \pi_t F(X) \Longrightarrow_s \pi_{t-s} \operatorname*{holim}_{P} F(X)\,,$$

where $F(X|P) = F(Y^\bullet) \circ f$. This is the map of Bousfield–Kan spectral sequences associated to a canonical map $f^*\colon \Pi^\bullet_\Delta F(Y^\bullet) \to \Pi^\bullet_P F(X|P)$ of cosimplicial spectra, see [BK72, XI.5.1], so the map of E_2-terms is the isomorphism f^* of Corollary 3.4. Thus, whenever $F(X|P)$ arises from $F(Y^\bullet)$ by composition with f, the cosimplicial homotopy limit spectral sequence for $F(Y^\bullet)$ is isomorphic to the cubical homotopy limit spectral sequence for $F(X|P)$.

Proposition 3.8. *(a) Let $F(Y^\bullet)$ be any cosimplicial spectrum, with fibrant replacement $F(Y^\bullet)'$, and let $F(X|P) = F(Y^\bullet) \circ f$. The first three of the spectral sequences*

$E_1^{s,t} = \pi_{t-s}\, \mathrm{hofib}(p_s) \Longrightarrow_s \pi_{t-s}(\operatorname*{holim}_n \operatorname*{holim}_{P^n} F(X))$ *(cubical descent)*

$E_1^{s,t} = \pi_{t-s}\, \mathrm{hofib}(\delta_s) \Longrightarrow_s \pi_{t-s}(\operatorname*{holim}_n \operatorname*{holim}_{\Delta^{<n}} F(Y^\bullet))$ *(cosimplicial descent)*

$E_1^{s,t} = \pi_{t-s}\, \mathrm{hofib}(\tau_s)\,,\ E_2^{s,t} = \pi^s\, \pi_t F(Y^\bullet) \Longrightarrow_s \pi_{t-s}\, \mathrm{Tot}\, F(Y^\bullet)'$ *(Bousfield–Kan)*

$E_2^{s,t} = \lim_{\Delta}{}^s\, \pi_t F(Y^\bullet) \Longrightarrow_s \pi_{t-s} \operatorname*{holim}_{\Delta} F(Y^\bullet)$ *(cosimplicial homotopy limit)*

$E_2^{s,t} = \lim_{P}{}^s\, \pi_t F(X) \Longrightarrow_s \pi_{t-s} \operatorname*{holim}_{P} F(X)$ *(cubical homotopy limit)*

are isomorphic at the E_1-term, and all five are isomorphic at the E_2-term, hence also at all later terms. If $RE_\infty = 0$ these spectral sequences all converge strongly to the indicated abutments.

(b) Let $F(Y)$ be any coaugmented cosimplicial spectrum, and let $F(X) = F(Y) \circ f_\eta$. If F satisfies cubical descent over X, or equivalently, if F satisfies cosimplicial descent over Y, then each abutment in (a) is isomorphic to $\pi_* F(Y^{-1}) = \pi_* F(X(\varnothing))$. If $F(Y)$ is the Amitsur resolution $\eta\colon F(A) \to F(Y_R^\bullet(A,B))$, so that $F(X)$ is the Amitsur cube $F(X_R^\omega(A,B))$, then this common abutment is $\pi_* F(A)$.

Proof. (a) The stated isomorphisms were all discussed before the statement of the proposition. Each spectral sequence is derived from a tower of fibrations, hence is conditionally convergent to the sequential limit of the homotopy groups of the terms in this tower. When RE_∞ vanishes, each homotopy group of the homotopy limit of the tower is isomorphic to that sequential limit, and the spectral sequence is strongly convergent, by [BK72, IX.5.4] or [Boa99, Thm. 7.4].

(b) Cosimplicial descent for F over Y ensures that $F(Y^{-1}) \simeq \operatorname{holim}_\Delta F(Y^\bullet)$. □

THEOREM 3.9. *Let R be a connective commutative $\mathbb{S}$-algebra, A a connective R-algebra, B a connective commutative R-algebra and suppose that the unit map $\iota\colon R \to B$ is 1-connected. Let F be one of the functors*

- *K, TC (with $c = +1$),*
- *THH, $THH(-)_{hC}$, $THH(-)^C$, $TR(-;p)$ (with $c = 0$),*
- *$TF(-;p)$, $TC(-;p)$ (with $c = -1$).*

Then the E_1-term of the (cubical/cosimplicial) descent spectral sequence for F at A along $\iota\colon R \to B$ vanishes in all bidegrees (s,t) with $t - s < s + c$ and $s \geq 1$. It is strongly convergent to $\pi_* F(A)$, with E_2-term given by

$$E_2^{s,t} = \pi^s\, \pi_t F(Y_R^\bullet(A,B)) \Longrightarrow_s \pi_{t-s} F(A)\,.$$

Proof. The description of the E_2-term is that of the Bousfield–Kan spectral sequence. The vanishing line for the E_1-term of the cubical descent spectral sequence is that of Corollary 2.6. It follows that in each bidegree (s,t) there is a finite r such that $E_r^{s,t} = E_\infty^{s,t}$, which implies that $RE_\infty^{s,t} = 0$. Hence each version of the spectral sequence converges strongly to the homotopy groups of $F(Y_R^{-1}(A,B)) = F(A)$. □

In the discussion above we used the following result, for which a proof does not seem to have appeared in the literature.

PROPOSITION 3.10. *Let $Z^\bullet$ be any fibrant cosimplicial space or spectrum. There are compatible natural equivalences*

$$z^*\colon \operatorname{Tot}_n Z^\bullet \xrightarrow{\simeq} \operatorname*{holim}_{\Delta^{<n+1}} Z^\bullet$$

for all $0 \leq n \leq \infty$.

Proof. When $Z^\bullet$ is a fibrant cosimplicial space and $n = \infty$, this result is [BK72, XI.4.4]. We indicate how to adapt their proof to the case of finite n.

Let $D = \Delta^{<n+1}$, let $i\colon D \to \Delta$ be the inclusion of the full subcategory, let $\mathscr{S}$ be the category of spaces (= simplicial sets), and let $\mathscr{S}^\Delta = c\mathscr{S}$ and $\mathscr{S}^D$ be the functor categories of cosimplicial spaces and n-truncated cosimplicial spaces, respectively. Composition with i defines the restriction functor $i^*\colon c\mathscr{S} \to \mathscr{S}^D$. Let $i_* = \operatorname{LKan}_i\colon \mathscr{S}^D \to c\mathscr{S}$ be the left Kan extension, left adjoint to i^*. For each $[k] \in D$, with $0 \leq k \leq n$, let $D/[k]$ be the over category

and let $N(D/[k])$ denote its nerve (also known as its underlying or classifying space). For $[k] \in D$, let $z\colon N(D/[k]) \to \Delta[k]$ be the "zeroth vertex map" sending each vertex $\alpha\colon [p] \to [k]$ in $N(D/[k])$ to the vertex $\alpha(0)$ in $\Delta[k]$, see [BK72, XI.2.6(ii)]. It is a weak equivalence for each $[k]$ in D, since both $N(D/[k])$ and $\Delta[k]$ are contractible. Composition with z defines a morphism of mapping spaces

$$\hom_{c\mathscr{S}}(i_*i^*\Delta, Z^\bullet) \cong \hom_{\mathscr{S}^D}(i^*\Delta, i^*Z^\bullet) \xrightarrow{z^*} \hom_{\mathscr{S}^D}(N(D/-), i^*Z^\bullet) = \operatorname*{holim}_D i^*Z^\bullet\,.$$

Here $(i_*i^*\Delta)[q] = \operatorname{colim}_{[k]\to[q]} \Delta[k] \cong \mathrm{sk}_n\,\Delta[q]$ for $[q] \in \Delta$, where $[k] \to [q]$ ranges over the left fiber category $i/[q]$ of i at $[q]$. Hence we can rewrite z^* as

$$z^*\colon\ \mathrm{Tot}_n\, Z^\bullet = \hom_{c\mathscr{S}}(\mathrm{sk}_n\,\Delta, Z^\bullet) \longrightarrow \operatorname*{holim}_D i^*Z^\bullet\,.$$

To prove that z^* is a weak equivalence, we will use that $\mathscr{S}^D$ with the Reedy model structure is a simplicial model category [Hir03, Thm. 15.3.4]. Here D has the evident Reedy category structure inherited from Δ. It suffices to prove that $z\colon N(D/-) \to i^*\Delta$ is a weak equivalence of cofibrant objects and that $i^*Z^\bullet$ is a fibrant object, in the Reedy model structure on $\mathscr{S}^D$. We have already observed that z is a Reedy weak equivalence. The cosimplicial space Δ is unaugmentable, hence cofibrant in the model structure on $c\mathscr{S}$, see [BK72, X.4.2]. The latching map of $i^*\Delta$ at each object $[k]$ in D is equal to the latching map of Δ at $[k] \in \Delta$, and therefore $i^*\Delta$ is Reedy cofibrant. By assumption, $Z^\bullet$ is fibrant in the model structure on $c\mathscr{S}$, see [BK72, X.4.6]. The matching map of $i^*Z^\bullet$ at each object $[k]$ in D is equal to the matching map of $Z^\bullet$ at $[k] \in \Delta$, hence $i^*Z^\bullet$ is Reedy fibrant. It remains to check that $N(D/-)$ is Reedy cofibrant. This follows immediately from the fact that it is cofibrant in the projective model structure on $\mathscr{S}^D$, see [Hir03, Prop. 14.8.9, Thm. 11.6.1].

Finally, the case of a fibrant cosimplicial spectrum follows by applying the unstable result level by level. □

3.5. *Tensored structure and topological André–Quillen homology*

In this subsection we specialize to the case when both A and B are commutative R-algebras. This category is tensored over spaces, taking any simplicial set $S\colon [p] \mapsto S_p$ and any commutative R-algebra A to the realization $S \otimes_R A$ of $[p] \mapsto S_p \otimes_R A = A \wedge_R \cdots \wedge_R A$, i.e., the coproduct of one copy of A for each element of S_p. (For infinite S_p we extend this construction by the filtered colimit over finite subsets.) In the case when $S = S^1 = \Delta[1]/\partial\Delta[1]$ is the simplicial circle and $R = \mathbb{S}$ is the sphere spectrum, $S^1 \otimes_{\mathbb{S}} A = B^{cy}(A)$ is the cyclic bar construction considered in the proof of Theorem 2.5, which is equivalent to $THH(A)$ when A is flat. Suppose now that A and B are R-com-alg-cofibrant, hence flat as R-modules. Our results about descent for THH generalize as follows.

Proposition 3.11. *If the unit map $R \to B$ is 1-connected, then tensoring with any simplicial set S satisfies cosimplicial descent at A along $R \to B$, meaning that*

$$\eta\colon S \otimes_R A \xrightarrow{\simeq} \operatorname*{holim}_\Delta S \otimes_R Y_R^\bullet(A, B)$$

is an equivalence.

Proof. Tensors commute with coproducts, so $S \otimes_R Y_R^\bullet(A, B)$ is isomorphic to $Y_R^\bullet(S \otimes_R A, S \otimes_R B)$. Lemma 2.2 shows that $X_R^n(S_p \otimes_R A, S_p \otimes_R B)$ is $(2n-1)$-co-Cartesian for each $p \geq 0$, which implies that $X_R^n(S \otimes_R A, S \otimes_R B)$ is $(2n-1)$-co-Cartesian and n-Cartesian, for each $n \geq 1$. Hence

$$\eta_n\colon S \otimes_R A \longrightarrow \operatorname*{holim}_{\Delta^{<n}} Y_R^\bullet(S \otimes_R A, S \otimes_R B) \simeq \operatorname*{holim}_{P^n} X_R^n(S \otimes_R A, S \otimes_R B)$$

is n-connected, and therefore η is an equivalence. □

Using either the Bökstedt type model as in [BCD10] (extended mutatis mutandis to connective orthogonal or symmetric spectra) or a model with more categorical control as in [BDS16], the fundamental homotopy cofiber sequence (2.2) leads to the equivariant extension given below. This generalizes our descent results for $TC(-;p)$ to the various forms of covering homology considered in [BCD10, Sec. 7]. In particular, the higher topological cyclic homology of [CDD11] satisfies cosimplicial descent at connective commutative $\mathbb{S}$-algebras along 1-connected unit maps.

Corollary 3.12. *If G is a finite group acting freely on S, and $R \to B$ is 1-connected, then tensoring with S satisfies equivariant cosimplicial descent at A along $R \to B$, meaning that*

$$\eta\colon S \otimes_R A \xrightarrow{\simeq} \operatorname*{holim}_{\Delta} S \otimes_R Y_R^\bullet(A,B)$$

is a G-equivariant equivalence.

Proof. We have to show that for every subgroup $H \subseteq G$, the map η^H of H-fixed points is an equivalence. This follows by essentially the same argument as for THH, e.g. by [BCD10, Lem. 5.1.3], using induction over the closed families of subgroups and the fact that homotopy orbits preserve connectivity. □

Coming back to the non-equivariant situation, cosimplicial descent is satisfied by topological André–Quillen homology, which we will denote by $TAQ^R(A)$.

Proposition 3.13. *If the unit map $R \to B$ is 1-connected, then topological André–Quillen homology satisfies cosimplicial descent at A along $R \to B$, in the sense that*

$$\eta\colon TAQ^R(A) \xrightarrow{\simeq} \operatorname*{holim}_{\Delta} TAQ^R(Y_R^\bullet(A,B))$$

is an equivalence.

Proof. By [BM05, Thm. 4], there is an equivalence

$$TAQ^R(A) \simeq \operatorname*{hocolim}_m \Sigma^{-m}(S^m \otimes_R A)/A$$

of A-module spectra, where $(S^m \otimes_R A)/A$ denotes the homotopy cofiber of the map $A \to S^m \otimes_R A$ associated to the base point in $S^m = (S^1)^{\wedge m}$. The map is at least $(m-1)$-connected, for each $m \geq 0$, and likewise for B in place of A. It follows that the map

$$A \wedge_R (B/R) \wedge_R \cdots \wedge_R (B/R) \longrightarrow (S^m \otimes_R A) \wedge_R (S^m \otimes_R B)/R \wedge_R \cdots \wedge_R (S^m \otimes_R B)/R$$

(with $n \geq 1$ copies of B/R and of $(S^m \otimes_R B)/R$) is at least $(m+2n-3)$-connected. Hence the n-cube

$$T \mapsto \Sigma^{-m}(S^m \otimes_R X^n(T))/X^n(T)$$

(with $X^n = X_R^n(A,B)$) is $(2n-3)$-co-Cartesian, for each $m \geq 0$. Thus the n-cube $T \mapsto TAQ^R(X^n(T))$ is $(2n-3)$-co-Cartesian and $(n-2)$-Cartesian, so that

$$\eta_n\colon TAQ^R(A) \longrightarrow \operatorname*{holim}_{P^n} TAQ^R(X_R^n(A,B))$$

is at least $(n-2)$-connected. Passing to the homotopy limit over n, it follows that η is an equivalence. □

3.6. *Less commutative examples*

Let us return to the situation where A is not necessarily commutative. In Subsection 3.3 we took B to be commutative to ensure that $\mu\colon B \wedge_R B \to B$ and the codegeneracy maps $s^j\colon Y^{q+1} \to Y^q$ are morphisms of R-algebras. For non-commutative B, we can still define what we might call a precosimplicial R-algebra $[q] \mapsto Y^q$, where now $[q]$ ranges over the subcategory $M \subset \Delta$ with morphisms the injective, order-preserving functions. The functor $f\colon P \to \Delta$ defined at the beginning of Subsection 3.2 factors as the composite of a functor $e\colon P \to M$ and the inclusion $i\colon M \to \Delta$. Here e is not left cofinal, so the analogue of Proposition 3.2 does not hold for general precosimplicial spectra $Z^\bullet$. On the other hand, $i\colon M \to \Delta$ is left cofinal, see [DD77, 3.17], so for cosimplicial $Y^\bullet$ the canonical map

$$\operatorname*{holim}_{\Delta} Y^\bullet \xrightarrow{\simeq} \operatorname*{holim}_{M} Y^\bullet \circ i$$

is an equivalence. Hence

$$\operatorname*{holim}_{M} Z^\bullet \xrightarrow{\simeq} \operatorname*{holim}_{P} Z^\bullet \circ e$$

is an equivalence for each precosimplicial spectrum $Z^\bullet = Y^\bullet \circ i$ that admits an extension to a cosimplicial spectrum.

By an $\mathscr{O}$ R-ring spectrum, for an operad $\mathscr{O}$, we mean an $\mathscr{O}$-algebra in the category of R-modules. We now relax the commutativity condition on B to only ask that it is an E_2 R-ring spectrum, i.e., an $\mathscr{O}$ R-ring spectrum for some E_2 operad $\mathscr{O}$. Then B is equivalent to a monoid in a category of A_∞ R-ring spectra, by [BFV07, Thm. C], since the tensor product of the associative operad and the little intervals operad $\mathscr{C}_1$ is an E_2 operad, and $\mathscr{C}_1$ is an A_∞ operad. In other words, we may assume that $\iota\colon R \to B$ and $\mu\colon B \wedge_R B \to B$ are morphisms of $\mathscr{C}_1$ R-ring spectra, so that the Amitsur resolution $A \to Y_R^\bullet(A, B)$ is a coaugmented cosimplicial object in the category of connective $\mathscr{C}_1$ R-ring spectra.

Using a monadic bar construction [May72, §9] to functorially turn $\mathscr{C}_1$-algebras into monoids, we may replace this Amitsur resolution with an equivalent coaugmented cosimplicial connective R-algebra $A \to \bar{Y}_R^\bullet(A, B)$, which we might denote by $\bar{Y}_R(A, B)$. Applying a homotopy functor F from connective R-algebras to spectra, we thus obtain a coaugmented cosimplicial spectrum $F(A) \to F(\bar{Y}_R^\bullet(A, B))$.

The ω-cubical diagram $\bar{Y}_R(A, B) \circ f_\eta$ remains equivalent to the Amitsur ω-cube associated to the R-module A and the unit map $\iota\colon R \to B$ of $\mathscr{C}_1$ R-ring spectra. Replacing B with an equivalent R-algebra $\bar{B}$, we obtain an equivalent ω-cube $X_R^\omega(A, \bar{B})$. Hence there is a chain of equivalences

$$X_R^\omega(A, \bar{B}) \simeq \bar{Y}_R(A, B) \circ f_\eta\,.$$

Substituting this for Lemma 3.6 in the proof of Theorem 3.7, we obtain the following generalization of that theorem.

THEOREM 3.14. *Let R be a connective commutative $\mathbb{S}$-algebra, let A be a connective R-algebra, and let B be a connective E_2 R-ring spectrum. Suppose that the unit map $\iota\colon R \to B$ is 1-connected. Then the functors $F = K$, THH and TC, as well as their intermediate variants, satisfy cosimplicial descent at A along $R \to B$, in the sense that*

$$\eta\colon F(A) \xrightarrow{\simeq} \operatorname*{holim}_{\Delta} F(\bar{Y}_R^\bullet(A, B))$$

is an equivalence. Here $\bar{Y}_R^q(A, B)$ and $Y_R^q(A, B)$ are equivalent as A_∞ R-ring spectra, for each $q \geq 0$.

4. *Applications*

4.1. *Algebraic K-theory of spaces*

For each topological group Γ, the spherical group ring $\mathbb{S}[\Gamma] = \Sigma^\infty \Gamma_+$ is a connective $\mathbb{S}$-algebra. When $X \simeq B\Gamma$, the algebraic K-theory of $\mathbb{S}[\Gamma]$ is a model for Waldhausen's algebraic K-theory $A(X)$ of the space X, see [Wal85]. When X is a high-dimensional compact (topological, piecewise-linear or differentiable) manifold, $A(X)$ is closely related to the space of h-cobordisms on X and the group of automorphisms of X [WJR13]. This motivates the interest in the algebraic K-theory of $\mathbb{S}$ and the associated spherical group rings.

Base change along the Hurewicz map $\mathbb{S} \to H\mathbb{Z}$ induces rational equivalences $\mathbb{S}[\Gamma] \to H\mathbb{Z}[\Gamma]$ and $K(\mathbb{S}[\Gamma]) \to K(H\mathbb{Z}[\Gamma])$. In particular, $A(*) = K(\mathbb{S}) \to K(H\mathbb{Z}) = K(\mathbb{Z})$ is a rational equivalence, and Borel's rational computation of the algebraic K-theory of the integers [Bor74] gives strong rational information about the h-cobordism spaces and automorphism groups of high-dimensional highly-connected manifolds [Igu88, p. 7].

4.2. *Descent along* $\mathbb{S} \to H\mathbb{Z}$

To obtain torsion information about $A(X) = K(\mathbb{S}[\Gamma])$ one can instead consider the cosimplicial resolution

$$\mathbb{S} \xrightarrow{\eta} H\mathbb{Z} \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; H\mathbb{Z} \wedge H\mathbb{Z} \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \cdots$$

in the category of connective $\mathbb{S}$-algebras, and the induced coaugmented cosimplicial spectrum

$$K(\mathbb{S}[\Gamma]) \xrightarrow{\eta} K(H\mathbb{Z}[\Gamma]) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; K((H\mathbb{Z} \wedge H\mathbb{Z})[\Gamma]) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \cdots .$$

By Theorem 3.7 the natural map from $K(\mathbb{S}[\Gamma])$ to the homotopy limit of this cosimplicial spectrum is an equivalence, and similarly for THH and TC. These cases of descent, along $\mathbb{S} \to H\mathbb{Z}$, are essentially those studied in [Dun97]. See also [Tsa00] for descent results in the context of commutative rings.

A computational drawback with this approach is the structure of the smash product

$$(H\mathbb{Z} \wedge \cdots \wedge H\mathbb{Z})[\Gamma] = \mathbb{S}[\Gamma] \wedge H\mathbb{Z} \wedge \cdots \wedge H\mathbb{Z}$$

with $(q+1)$ copies of $H\mathbb{Z}$. It is a connective $H\mathbb{Z}$-algebra, hence equivalent to a simplicial ring, but for $q \geq 1$ the algebraic K-theory and topological cyclic homology of this simplicial ring appear to be difficult to analyze.

4.3. *Descent along* $\mathbb{S} \to MU$

Experience from algebraic topology shows that the complex bordism spectrum MU is a convenient stopping point on the way from the sphere spectrum to the integers:

$$\mathbb{S} \longrightarrow MU \longrightarrow H\mathbb{Z}\,.$$

Here MU is a commutative $\mathbb{S}$-algebra with 1-connected unit map $\mathbb{S} \to MU$. The coefficient ring $MU_* = \pi_*(MU) = \mathbb{Z}[x_k \mid k \geq 1]$ and the homology algebra $H_*(MU) \cong H_*(BU) = \mathbb{Z}[b_k \mid k \geq 1]$ are explicitly known [Mil60], [Nov62], with $|x_k| = |b_k| = 2k$ for each k. The associated cosimplicial resolution

$$\mathbb{S} \xrightarrow{\eta} MU \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; MU \wedge MU \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \cdots$$

induces the coaugmented cosimplicial spectrum

$$K(\mathbb{S}[\Gamma]) \xrightarrow{\eta} K(MU[\Gamma]) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; K((MU \wedge MU)[\Gamma]) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \cdots .$$

By Theorem 3.7, applied with $R = \mathbb{S}$, $A = \mathbb{S}[\Gamma]$ and $B = MU$, the natural map from $K(\mathbb{S}[\Gamma])$ to the homotopy limit of this cosimplicial spectrum is an equivalence, and there are corresponding equivalences for THH and TC.

From its definition as an E_∞ Thom spectrum, MU comes equipped with a Thom equivalence $MU \wedge MU \simeq MU \wedge BU_+$. Hence the smash product

$$(MU \wedge \cdots \wedge MU)[\Gamma] \simeq \mathbb{S}[\Gamma] \wedge MU \wedge BU_+^q$$

that occurs in codegree q is not significantly more complicated for $q \geq 1$ than for $q = 0$. This means that it may be more realistic to study $K(\mathbb{S})$ by descent along $\mathbb{S} \to MU$ than by descent along $\mathbb{S} \to H\mathbb{Z}$. We propose that in order to understand $A(*) = K(\mathbb{S})$ from the chromatic point of view [MRW77], [Rav84], one should study this cosimplicial resolution, starting with $K(MU)$ and continuing with $K(MU \wedge BU_+^q)$ for each $q \geq 0$, together with the associated descent spectral sequence converging to $\pi_* K(\mathbb{S})$. Similar remarks apply for $A(X) = K(\mathbb{S}[\Gamma])$ and for the functors THH and TC, see Theorem 3.9.

4.4. *(Hopf-)Galois descent*

In the language of [Rog08, §12], the map $\mathbb{S} \to MU$ is a Hopf–Galois extension of commutative $\mathbb{S}$-algebras. Our result can be viewed as proving 1-connected Hopf–Galois descent for K, THH and TC, but the actual Hopf $\mathbb{S}$-algebra coaction plays no role in the proof. Analogously, consider a G-Galois extension $A \to B$ of commutative $\mathbb{S}$-algebras, in the sense of [Rog08, §4], with G a finite group. The equivalence $B \wedge_A B \simeq F(G_+, B) \cong F(G_+^2, B)^G$ induces a level equivalence of cosimplicial resolutions from

$$A \xrightarrow{\eta} B \rightrightarrows B \wedge_A B \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots$$

to

$$A \xrightarrow{\eta} F(G_+, B)^G \rightrightarrows F(G_+^2, B)^G \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots,$$

i.e., from the Amitsur resolution $Y^\bullet = Y_A^\bullet(A, B)$ to the cosimplicial commutative $\mathbb{S}$-algebra $F(E_\bullet G_+, B)^G$ obtained by mapping out of the free contractible simplicial G-set $E_\bullet G\colon [q] \mapsto E_q G = G^{q+1}$.

Algebraic K-theory preserves equivalences and commutes with finite products, so the cosimplicial spectra $K(Y^\bullet)$, $K(F(E_\bullet G_+, B)^G)$ and $F(E_\bullet G_+, K(B))^G$ are level equivalent. Hence the homotopy limit of $K(Y^\bullet)$ is equivalent to the homotopy fixed point spectrum $F(EG_+, K(B))^G = K(B)^{hG}$, where $EG = |E_\bullet G|$. (The homotopy type of $K(B)^{hG}$ only depends on the homotopy type of $K(B)$ as a spectrum with G-action, and the G-action is determined by functoriality.) The canonical map $\eta\colon K(A) \to K(B)^{hG}$ is not in general an equivalence, so algebraic K-theory does not in general satisfy descent along Galois extensions $A \to B$. However, a recent result of Clausen, Mathew, Naumann and Noel [CMNN17, Thm. 1.7] shows that, after any "periodic localization", algebraic K-theory satisfies descent along all maps $A \to B$ of commutative $\mathbb{S}$-algebras for which B is dualizable as an A-module and the restriction map $K_0(B) \to K_0(A)$ is rationally surjective. Again, the Galois condition plays no role for their proof. See also [Tho85] for the corresponding result for Bott localized algebraic K-theory of commutative rings (or schemes).

4.5. *Descent along* $\mathbb{S} \to X(n)$

There is a sequence of E_2 ring spectra $X(n)$ interpolating between $\mathbb{S}$ and MU, see [DHS88]. Recall that $MU = BU^\gamma$ is the Thom spectrum of a virtual vector bundle γ over BU. For each $n \geq 2$, let $X(n) = \Omega SU(n)^\gamma$ be the Thom spectrum of the pullback of γ over the double loop map $\Omega SU(n) \to \Omega SU \simeq BU$. Then $X(n)$ is an E_2 ring spectrum, by [LMSM86, Thm. IX.7.1],

with 1-connected unit map. There are natural maps of E_2 ring spectra

$$\mathbb{S} \longrightarrow \ldots \longrightarrow X(n) \longrightarrow \ldots \longrightarrow MU \longrightarrow H\mathbb{Z}$$

connecting these examples to those previously discussed. We identify $H_*(X(n)) \cong H_*(\Omega SU(n))$ with the subalgebra $\mathbb{Z}[b_1, \ldots, b_{n-1}]$ of $H_*(MU) \cong H_*(BU) = \mathbb{Z}[b_k \mid k \geq 1]$.

By Theorem 3.14, the functors K, THH and TC satisfy descent along $\mathbb{S} \to X(n)$ for each $n \geq 2$. There are Thom equivalences $X(n) \wedge X(n) \simeq X(n) \wedge \Omega SU(n)_+$, so the study of $K(\mathbb{S})$ by descent along $\mathbb{S} \to X(n)$ leads to the study of the algebraic K-theory of $X(n) \wedge (\Omega SU(n))^q_+$ for $q \geq 0$, and similarly for $K(\mathbb{S}[\Gamma])$, THH and TC. The E_2 ring spectra $X(n)$ are closer to $\mathbb{S}$ than MU, hence $K(X(n))$ can yield finer information about $K(\mathbb{S})$ than $K(MU)$ does. However, like in the case of $\mathbb{S}$, the homotopy groups of $X(n)$ are not explicitly known, so a direct analysis of $\pi_* THH(X(n))$ and $\pi_* TC(X(n))$ may be less feasible than in the case of MU.

4.6. *Trace methods*

The cyclotomic trace map trc: $K(B) \to TC(B; p)$ introduced in [BHM93], in conjunction with the relative equivalence theorem from [Dun97], is the main method available for calculating the algebraic K-groups of connective $\mathbb{S}$-algebras other than (simplicial) rings. See also [DGM13, Ch. 7]. In the case of the sphere spectrum, $TC(\mathbb{S}; p)$ is p-adically equivalent to $\mathbb{S} \vee \Sigma\mathbb{C}P^\infty_{-1}$, so calculations of $\pi_* K(\mathbb{S})$ are possible in a moderate range of degrees [Rog02], [Rog03] (see also more recent work of Blumberg–Mandell [BM16] at irregular primes). Nonetheless, complete calculations are at least as hard as those for $\pi_*(\mathbb{S})$, hence appear to be out of reach.

The difficulty of understanding the stable homotopy groups of spheres can be formulated as the difficulty of understanding the Adams–Novikov spectral sequence [Nov67]

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{MU_*MU}(MU_*, MU_*) \Longrightarrow_s \pi_{t-s}(\mathbb{S}), \tag{4.1}$$

i.e., to understand the descent spectral sequence

$$E_2^{s,t} = \pi^s \pi_t Y^\bullet \Longrightarrow_s \pi_{t-s} \operatorname*{holim}_\Delta Y^\bullet$$

associated with the cosimplicial commutative $\mathbb{S}$-algebra $Y^\bullet = Y^\bullet_{\mathbb{S}}(\mathbb{S}, MU)$, with $Y^q = MU \wedge \cdots \wedge MU \simeq MU \wedge BU^q_+$. An advantage of this approach is that chromatic phenomena in $\pi_*(\mathbb{S})$ are more readily visible at the E_2-term of the Adams–Novikov spectral sequence.

By analogy, the difficulty of understanding $\pi_* K(\mathbb{S})$ and $\pi_* TC(\mathbb{S}; p)$ can be separated into two parts: first that of understanding the cosimplicial objects $[q] \mapsto \pi_* K(Y^q)$ and $[q] \mapsto \pi_* TC(Y^q; p)$, and secondly that of understanding the behavior of the descent spectral sequences

$$E_2^{s,t} = \pi^s \pi_t K(Y^\bullet) \Longrightarrow_s \pi_{t-s} K(\mathbb{S})$$

and

$$E_2^{s,t} = \pi^s \pi_t TC(Y^\bullet; p) \Longrightarrow_s \pi_{t-s} TC(\mathbb{S}; p)\,.$$

The first aim of understanding $\pi_* K(MU)$ and $\pi_* TC(MU; p)$, corresponding to $q = 0$, then plays an analogous role to that of understanding $MU_* = \pi_*(MU)$. An optimist may seek to discern chromatic phenomena in $\pi_* K(\mathbb{S})$ and $\pi_* TC(\mathbb{S}; p)$ at the level of these E_2-terms.

4.7. *Descent for THH*

As an illustration, Theorem 3.9 for THH at $\mathbb{S}$ along $\mathbb{S} \to MU$ gives a strongly convergent descent spectral sequence

$$E_2^{s,t} = \pi^s \pi_t THH(Y^\bullet) \Longrightarrow_s \pi_{t-s} THH(\mathbb{S})\,.$$

There is an equivalence $THH(MU) \simeq MU \wedge SU_+$, by [BCS10, Cor. 1.1], and the Atiyah–Hirzebruch spectral sequence $E^2_{*,*} = H_*(SU; MU_*) \Longrightarrow \pi_*(MU \wedge SU_+)$ collapses at E^2 to give

the algebra isomorphism

$$\pi_* THH(MU) \cong MU_* \otimes E(e_k \mid k \geq 1) = LE$$

of [MS93, Rmk. 4.3]. Here $L = MU_* = \mathbb{Z}[x_k \mid k \geq 1]$ is the Lazard ring, $E = E(e_k \mid k \geq 1)$ is the exterior algebra over $\mathbb{Z}$ on a sequence of generators e_k, with $|e_k| = 2k+1$, and $LE = L \otimes E$ is shorthand notation for their tensor product. Similarly,

$$\begin{aligned} MU_* THH(MU) &\cong \pi_*(MU \wedge MU \wedge_{MU} THH(MU)) \\ &\cong LB \otimes_L LE \cong LBE \end{aligned}$$

is flat as a left MU_*-module, where $LB = MU_*MU$ and $B = H_*(MU) \cong \mathbb{Z}[b_k \mid k \geq 1]$, and

$$\begin{aligned} \pi_* THH(MU \wedge MU) &\cong \pi_*(THH(MU) \wedge_{MU} \wedge MU \wedge THH(MU)) \\ &\cong LE \otimes_L LBE \cong LE \otimes BE\,. \end{aligned}$$

In general, $\pi_* THH(Y^q) \cong LE \otimes (BE)^{\otimes q}$, and $[q] \mapsto \pi_* THH(Y^q)$ is the cobar construction associated to the split Hopf algebroid $(LE, LE \otimes BE)$, see [Rav86, App. A1]. Hence the descent spectral sequence for THH takes the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{LE \otimes BE}(LE, LE) \Longrightarrow_s \pi_{t-s} THH(\mathbb{S})\,. \tag{4.2}$$

The unit inclusion $\mathbb{Z} \to E$ induces an equivalence

$$(MU_*, MU_*MU) = (L, LB) \longrightarrow (LE, LE \otimes BE)$$

of Hopf algebroids, which induces an isomorphism from the Adams–Novikov spectral sequence (4.1) to the descent spectral sequence (4.2). This is of course compatible with the identity $\mathbb{S} = THH(\mathbb{S})$, and shows, somewhat tautologically, that the descent spectral sequence for THH at $\mathbb{S}$ along $\mathbb{S} \to MU$ has an E_2-term that is susceptible to chromatic analysis along the lines of [MRW77] and [Rav86, Ch. 5].

By contrast, descent for THH at $\mathbb{S}$ along $\mathbb{S} \to H\mathbb{Z}$ leads to the study of $\pi_* THH(H\mathbb{Z} \wedge \cdots \wedge H\mathbb{Z}) \cong \pi_*(THH(\mathbb{Z}) \wedge \cdots \wedge THH(\mathbb{Z}))$, with $(q+1)$ copies of $H\mathbb{Z}$ or $THH(\mathbb{Z})$, and while $\pi_* THH(\mathbb{Z})$ is explicitly known [BM94, §5], the term $\pi_*(THH(\mathbb{Z}) \wedge THH(\mathbb{Z}))$ is not flat over $\pi_* THH(\mathbb{Z})$, and there is no description of the resulting E_1-term as the cobar complex of a Hopf algebroid. If we instead were to study the functor $F(A) = THH(A) \wedge S/p$, where S/p is the mod p Moore spectrum, then $\pi_* F(H\mathbb{Z} \wedge H\mathbb{Z}) = \pi_*(THH(\mathbb{Z}) \wedge THH(\mathbb{Z}); \mathbb{Z}/p)$ is a flat Hopf algebroid over $\pi_* F(H\mathbb{Z}) = \pi_*(THH(\mathbb{Z}); \mathbb{Z}/p)$. The associated cobar complex is isomorphic to the E_1-term of the descent spectral sequence for F at $\mathbb{S}$ along $\mathbb{S} \to \mathbb{Z}$, which in turn is isomorphic to the $THH(\mathbb{Z})$-based Adams spectral sequence for S/p, see [MNN17, Prop. 2.14]. The canonical $THH(\mathbb{Z})$-based tower for S/p is also a mod p Adams tower for S/p, so these spectral sequences are therefore isomorphic to the mod p Adams spectral sequence for S/p, from the E_2-term and onwards.

4.8. *Fixed points of THH*

The next steps toward analyzing descent for TC along $\mathbb{S} \to MU$ would be to study descent for the fixed point functor $THH(-)^C$ along $\mathbb{S} \to MU$, for each $C = C_{p^m}$. Let $THH(A)^{tC} = [\widetilde{EC} \wedge F(EC_+, THH(A))]^C$ denote the C-Tate construction on $THH(A)$, i.e., the C-fixed points of the Tate C-spectrum of [GM95, pp. 3-4]. The comparison maps

$$\begin{aligned} \Gamma_m &\colon THH(MU)^{C_{p^m}} \longrightarrow THH(MU)^{hC_{p^m}} \\ \hat{\Gamma}_m &\colon THH(MU)^{C_{p^{m-1}}} \longrightarrow THH(MU)^{tC_{p^m}} \end{aligned}$$

are known to be equivalences after p-completion, by [LNR11, Thm. 1.1] in the case $m = 1$, hence also for the cases $m \geq 2$ by [Tsa98, Thm. 2.4] or [BBLNR14, Thm. 2.8]. It is very likely that the same methods will apply when $Y^0 = MU$ is replaced by $Y^q = MU \wedge \cdots \wedge MU$ for

$q \geq 1$. If so, the p-completed spectral sequence

$$E_2^{s,t} = \pi^s \pi_t THH(Y^\bullet)_p^{C_{p^m}} \Longrightarrow_s \pi_{t-s} THH(\mathbb{S})_p^{C_{p^m}}$$

can equally well be analyzed using C_{p^m}-homotopy fixed points or $C_{p^{m+1}}$-Tate constructions in place of C_{p^m}-fixed points. A first step in this direction would be to determine the differential structure of the C_p-Tate spectral sequence

$$\hat{E}^2_{s,t} = \hat{H}^{-s}(C_p; \pi_t THH(MU)) \Longrightarrow_s \pi_{s+t} THH(MU)^{tC_p} .$$

By truncation to the second quadrant, this would determine the C_p-homotopy fixed point spectral sequence converging to $\pi_* THH(MU)^{hC_p}$, hence also $\pi_* THH(MU)^{C_p}$ after p-completion.

The comparison maps $\Gamma_m \colon THH(\mathbb{S})^{C_{p^m}} \to THH(\mathbb{S})^{hC_{p^m}}$ and $\hat{\Gamma}_m \colon THH(\mathbb{S})^{C_{p^{m-1}}} \to THH(\mathbb{S})^{tC_{p^m}}$ are also p-adic equivalences, by the proven Segal conjecture [Car84], but even in the case $m = 1$ the differential structure of the C_p-Tate spectral sequence converging to $\pi_* THH(\mathbb{S})^{tC_p} \cong \pi_*(\mathbb{S})$ is not known [Ada74]. In the case of $THH(\mathbb{Z})$ the maps Γ_m and $\hat{\Gamma}_m$ are not quite equivalences, but they do induce isomorphisms in homotopy with mod p coefficients in non-negative degrees. The structure of the C_{p^m}-Tate spectral sequence converging to $\pi_*(THH(\mathbb{Z})^{tC_{p^m}}; \mathbb{Z}/p)$ is known [BM95], [Rog99a], [Rog99b], but these calculations appear to be difficult to extend to the case of $THH(H\mathbb{Z} \wedge \cdots \wedge H\mathbb{Z})$.

4.9. *Topological periodic homology*

The limiting maps

$$\Gamma \colon TF(MU; p) \longrightarrow \operatorname*{holim}_m THH(MU)^{hC_{p^m}} \longleftarrow THH(MU)^{h\mathbb{T}}$$
$$\hat{\Gamma} \colon TF(MU; p) \longrightarrow \operatorname*{holim}_m THH(MU)^{tC_{p^m}} \longleftarrow THH(MU)^{t\mathbb{T}}$$

are also equivalences after p-completion, cf. [BM95, (2.11)] and [AR02, Thm. 5.7]. Thus the E^2-terms of the $\mathbb{T}$-homotopy fixed point and $\mathbb{T}$-Tate spectral sequences

$$E^2_{*,*} = \mathbb{Z}[t] \otimes \pi_* THH(MU) \Longrightarrow \pi_* THH(MU)^{h\mathbb{T}}$$
$$\hat{E}^2_{*,*} = \mathbb{Z}[t^{\pm 1}] \otimes \pi_* THH(MU) \Longrightarrow \pi_* THH(MU)^{t\mathbb{T}}$$

are known, and both converge to $\pi_* TF(MU; p)$ after p-completion. Note that $TP(MU) = THH(MU)^{t\mathbb{T}}$ is the arithmetically interesting topological periodic homology studied by Hesselholt [Hes17], also known as periodic topological cyclic homology or topological de Rham homology. As in the case of $TF(\mathbb{S}; p)$, the inverse Frobenius operator $\varphi^{-1} = R \colon TF(MU; p) \to TF(MU; p)$ extends to $TP(MU)$ after p-completion, without any further localization.

4.10. *Rational analysis*

The rational algebraic K-groups of MU, i.e., $\pi_* K(MU) \otimes \mathbb{Q}$, were determined in [AR12, Thm. 4.2] by using Goodwillie's theorem [Goo86]. The Poincaré series is

$$\frac{x^5}{1-x^4} + \frac{1 + xh(x)}{1+x} = 1 + x^3 + 3x^5 + 3x^7 + x^8 + 6x^9 + 2x^{10} + \dots$$

where

$$h(x) = \prod_{k \geq 1} \frac{1 + x^{2k+1}}{1 - x^{2k}} .$$

A similar result holds for each $\pi_* K(Y^q) \otimes \mathbb{Q}$, where $Y^q = MU \wedge \cdots \wedge MU$, with $(q+1)$ copies of MU. Since $\pi_0 Y^q = \mathbb{Z}$ and Y^q is connective and of finite type, we know by an easy generalization of [Dwy80, Prop. 1.2] that $E_1^{s,t} = \pi_t K(Y^s)$ is a finitely generated abelian group in each bidegree (s, t). By the following result $E_2^{s,t} = \pi^s \pi_t K(Y^\bullet)$ is in fact finite in each bidegree, except at the edge $s = 0$, where $E_2^{0,t}$ is rationally isomorphic to $\pi_t K(\mathbb{Z})$.

PROPOSITION 4.1. *The rationalized descent spectral sequence*

$$E_2^{s,t} \otimes \mathbb{Q} = \pi^s \pi_t K(Y^\bullet) \otimes \mathbb{Q} \Longrightarrow_s \pi_{t-s} K(\mathbb{S}) \otimes \mathbb{Q}\,,$$

for algebraic K-theory at $\mathbb{S}$ along $\mathbb{S} \to MU$, collapses at the E_2-term to the edge $s = 0$.

Proof. The vanishing line and strong convergence of the descent spectral sequence $E_2^{s,t} \Longrightarrow_s \pi_{t-s}K(\mathbb{S})$ from Theorem 3.9 imply the same vanishing line and strong convergence for the rationalized spectral sequence.

The unit map $\mathbb{S} \to Y^q$ and the zeroth Postnikov section $Y^q \to H\mathbb{Z}$ induce maps $K(\mathbb{S}) \to K(Y^\bullet) \to K(\mathbb{Z})$ that, after rationalization, split off a copy of $K(\mathbb{S})$ from $K(Y^\bullet)$. It remains to prove that the remainder of the rationalized spectral sequence, associated to the cosimplicial spectrum with the homotopy fiber of $K(Y^q) \to K(\mathbb{Z})$ in codegree q, collapses to zero at the E_2-term.

The Hurewicz homomorphism $MU_* = \pi_*(MU) \to H_*(MU) \cong B$ is a rational equivalence, so $\pi_* THH(MU)$ is rationally isomorphic to $HH_*(B) = BE$, where $B = \mathbb{Z}[b_k \mid k \geq 1]$ and $E = E(e_k \mid k \geq 1)$. Connes' B-operator on $HH_*(B)$ corresponds to the suspension operator σ, which is the differential and derivation given by $\sigma(b_k) = e_k$ for each $k \geq 1$. Hence the rationalized de Rham homology $H_*^{dR}(B) \otimes \mathbb{Q} = H_*(BE, \sigma) \otimes \mathbb{Q} = \mathbb{Q}$ is trivial in positive degrees. By [AR12, Cor. 2.4], which is a consequence of [Goo86, II.3.4], it follows that the trace map $K(MU) \to THH(MU)$ identifies the kernel of $\pi_*K(MU) \to \pi_*K(\mathbb{Z})$ with the image of $\sigma\colon BE \to BE$, after rationalization.

Similarly, $\pi_* THH(Y^\bullet)$ is rationally isomorphic to the cosimplicial resolution $[q] \mapsto HH_*(B^{\otimes q+1}) \cong (BE)^{\otimes q+1}$ associated to $\iota\colon \mathbb{Z} \to BE$. By [AR12, Cor. 2.4] again, the trace map identifies the kernel of $\pi_*K(Y^q) \to \pi_*K(\mathbb{Z})$ with the image of $\sigma\colon (BE)^{\otimes q+1} \to (BE)^{\otimes q+1}$, after rationalization. Let (C^*, δ) and (D^*, δ) be the associated cochain complexes, with $C^q = (BE)^{\otimes q+1}$ and $D^q = \mathrm{im}(\sigma) \subset C^q$. The E_2-term of the remainder of the descent spectral sequence is given by the cohomology of (D^*, δ), after rationalization.

The augmentation $\epsilon\colon BE \to \mathbb{Z}$ induces a cochain contraction $\epsilon \otimes id_{BE}^{\otimes q}$ of $\eta\colon \mathbb{Z} \to C^*$, mapping $x_0 \otimes x_1 \otimes \cdots \otimes x_q \in C^q$ to $\epsilon(x_0)x_1 \otimes \cdots \otimes x_q \in C^{q-1}$. It restricts to a contraction of $0 \to D^*$, since $\epsilon \otimes id_{BE}^{\otimes q}$ commutes with σ. Hence the cohomology of (D^*, δ) is zero, as claimed. □

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