# Department of APPLIED MATHEMATICS

A Note on Portfolio oprimization in a levy Market with local Substitution and habit formation

by

Fred Espen Benth, Kenneth Hvistendahl Karlsen and Kristin Reikvam

Report no. 142

May 2000



# UNIVERSITY OF BERGEN Bergen, Norway



ISSN 0084-778x

Department of Mathematics University of Bergen 5008 Bergen Norway

# A Note on Portfolio oprimization in a levy Market with local Substitution and habit formation

by

# Fred Espen Benth, Kenneth Hvistendahl Karlsen and Kristin Reikvam

Report no. 142

May 2000





## A NOTE ON PORTFOLIO OPTIMIZATION IN A LEVY MARKET WITH LOCAL SUBSTITUTION AND HABIT FORMATION

### FRED ESPEN BENTH, KENNETH HVISTENDAHL KARLSEN, AND KRISTIN REIKVAM

ABSTRACT. We have in previous papers [2, 3] studied an optimal portfolio-consumption model which takes into account the notion of local substitution and allow the stock price to be governed by a general Lévy (jump-diffusion) process. In this note, we discuss a generalization of this model which includes the effect of habit formation. The resulting portfolio-consumption model is discussed within the framework of dynamic programming and the theory of viscosity solutions. The associated Hamilton-Jacobi-Bellman equation is a second order degenerate elliptic integro-differential variational inequality. We also review various economical interpretations as well as results given by Hindy, Huang, and Zhu [13, 14] for the portfolio-consumption model in the geometric Brownian case.

## 1. INTRODUCTION

In this paper we will present and discuss an optimal portfolio-consumption problem in a Lévy (jump-diffusion) market. A feature of this portfolio-consumption problem is the inclusion of local substitution and habit formation. More specifically, the utility of the investor will not be derived from present consumption directly but from averages over past consumption. The stochastic optimization problem is a generalization of the problem studied in Benth, Karlsen, and Reikvam [2], which does not take into account the effect of habit formation. In [2], we characterized the value function of the portfolioconsumption model as the unique constrained viscosity solution of the associated Hamilton-Jacobi-Bellman equation in the case of a pure-jump market. In the companion paper [3], we calculated explicit consumption and portfolio selection plans for power utility functions when the risky asset follows a geometric Lévy process (see also [5] for numerical examples in real markets). Although we will not discuss it here, a related portfolioconsumption model which also takes into account proportional transaction costs is analyzed in [4].

In this paper, we remark that the viscosity solution characterization of the value function proved in [2] is valid also if we include the effect of habit formation in our model. Although we state and discuss the results leading up to this characterization, the proofs are only sketched since the details will appear elsewhere in connection with numerical studies. In

Key words: Portfolio choice, local substitution, habit formation, singular stochastic control, dynamic programming, integro-differential variational inequality, viscosity solution.

Acknowledgements: FEB is partially supported by Centre for Mathematical Physics and Stochastics (MaPhySto). MaPhySto is funded by a grant from the Danish National Research Foundation. KR is supported by the Norwegian Research Council (NFR) under the grant 118868/410. The first author is grateful to the organizers of the Second Symposium on Mathematics of Finance in Gaborone, Botswana, for their kind invitation and warm hospitality.

# A NOTE ON PORTIOID OFTIMIZATION IN A LEVY MARKET

A market which takes in prevenue propriets? If somerial an optimum protection and allow the study price reached which takes into account the noncorrel form manifestion and allow the study price on or governed by a general Levy (thing different) process. In this takes we discuss a generalization of this model which informed the cloud of balls formation. The third portfolio-communities and it is discussed which are then of balls formations. The informaportfolio-communities and it is discussed which are then the formation of the formation and the theory of respects address The espected of and the formation below formation is a second order descent all display marginal results given below formation review various respondent interpretations are well as results given below formation and the interpretation and the start of a marginal at the generation formation of the review various respondent interpretations are well as results given believe and the interpretation of the contract of a start and the product of balls of the second of the contract of the start of the start of the start of the start of the second of the start of the start of the start of the start of the second of the start of the start of the start of the start of the second of the start of the start of the start of the start of the second of the start of the start of the start of the start of the second of the start of the start of the start of the start of the second of the start of the start of the start of the start of the second of the start of the start of the start of the start of the second of the start of the start of the start of the start of the second of the start of the start of the start of the start of the second of the start of the start of the start of the start of the second of the start of the start of the start of the start of the second of the start of the second of the start o

#### A0123UCOMENT

In this paper we will oreasing and discuss an optimal portionic consumption property in a Lery (arms-diffusion) matter. A feature of this particulated consumption problem is the inclusion of local anisaturation and habit fournation. More appendically the eventy of over past consumption. The stochastic optimization problem is a generalisation of the problem studied in Benth, Karlsen, and Fournization problem is a generalisation of the the effect of ashis formation. In particulation of the value inner to be and the problem studied in Benth, Karlsen, and Fournization problem is a generalisation of the the effect of ashis formation. In [2], we characterized the value innovation consumption model as the unique constrained viscosity educion of the associated flumitors when the risky asset follows a generating for generalization plane for momental when the risky asset follows a generating for generating for momental constraints in real matches). Although we will not discuss to base a relating for momental problem the risky asset follows a generating for generating plane for momental economies in real matches). Although we will not discuss to base, a relating possible constraints and the risky asset follows a generating for the states of the states of the momental economics of the states are a state of the states of the states of the states of the risky asset follows a generating for the states of the states of the momental economics of the states are been will not discuss to base a relating possible constraints of the relation of the states are been will not discuss to base a relating possible constraints of the states of the states of the states are states of the states of

in this papet, we result that the viscosity equilion characterization of the veloc simulan proved in [2] is valid also if we indicate the effect of helpit formation in out model. Although we state and discuss fits readily fouring up to this operatively the proofs are out statched when the details will appear elsewhere in counscise out with numerical sindles. In

Eey words: Contolia choke, Maai anistikation, inimi Jorn 1000, angedei stochash, control, dyngzan programming, maero-differential varianoisi iniquelip, viaceiry zoitzion.

Admentedgemente FEB is partially supported to Osme to Mathematical Porsies and Stochaster (MaFlySto), Salarsto is finded by a grant from the Daniel Matamia Research Domaalica, A.R. is supported by the Colverdan Research Connell (NFR) ender its grant 1188-8480. The first analog, is grateful to the crysnisened the broad Symposium on Mathematica Fluence in Haberone, Burrama, for their find invasced and were borned alternative. addition to a viscosity solution treatment of the portfolio-consumption model, we discuss various economical interpretations as well as reviewing the results given by Hindy, Huang, and Zhu [13, 14] for this model in the case of geometric Brownian motion. For an overview of papers dealing with control problems related to the one that we study herein, we refer to the discussions and citations in [2, 3, 4], see also the review paper by Zariphopoulou [19]. The reader is also encouraged to consult these papers for references to relevant papers dealing with the theory of viscosity solutions.

One of the main motivation for analyzing our portfolio-consumption model within the framework of viscosity solutions is that such analysis provides the first step in a numerical treatment of the model. When the notion of habit formation is included, it is hard (if possible?) to find explicit consumption and investment plans. If the risky asset follows a geometric Brownian motion, Hindy, Huang, and Zhu [13, 14] conclude from numerical solutions that, for instance, the optimal portfolio selection plan behaves quite differently from the case with no habit formation. It is of interest to generalize their analysis to assets that follow geometric Lévy processes, opening up for a much more realistic modeling of the stock price dynamics. Since we cannot expect to find solutions by analytical means, a natural approach is to attack the problem with a so-called Markov chain approximation method, as was done by Hindy, Huang, and Zhu [13, 14] in the geometric Brownian case. We refer to Kushner and Dupuis [16] for a general introduction to the Markov chain approximation method. The construction and analysis of numerical methods is outside the scope of this paper and will instead be the topic of future work. In fact, we will in future work present a Markov chain approximation method for computing the value function as well as the optimal policies (see [8] for preliminary work in this direction). As is well known by now, the viscosity solution theory provides a very flexible and powerful framework for proving convergence of numerical methods. However, to take advantage of this framework, the analytical results found in the present paper are necessary. In particular, the characterization of the value function as the unique constrained viscosity solution of an integro-differential variational inequality is of fundamental importance for the convergence analysis of a large class of (monotone, stable, and consistent) numerical methods for the portfolio-consumption model studied herein.

An outline of the paper is as follows: In Section 2, we formulate the portfolio-consumption problem and state the basic assumptions. In Section 3, we discuss the economical interpretations of the model, while in Section 4 we study the portfolio selection problem within a viscosity solution framework. The results of Hindy, Huang, and Zhu [13, 14] for geometric Brownian motion are briefly presented and discussed in Section 5.

### 2. The stochastic control problem

Let  $(\Omega, \mathcal{P}, \mathcal{F})$  be a probability space and  $(\mathcal{F}_t)$  a given filtration satisfying the usual hypotheses. Consider an investor operating in a financial market consisting of a risky asset (e.g., a stock) and a bond. The value of the risky asset is assumed to follow the stochastic process

$$(2.1) S_t = S_0 e^{L_t}$$

#### MANNER GARLSEN, AND REIKVAM

addition to a viscosity solation treatment of the portiono-consumption models we assons variants economical interpretations as well as reviewing the results gives by Hindy, Brang, and Zau [13, 14] for this model to the case of geometric Brownian metrics. For an overview of papers dealing with control problems minted to the one that we study became, we refut to the discussions and citations in [2, 3, 4], are also the series paget by fampleoutes [19]. The reader is also enclutaged to consult these map as for discusses to referent papers dealing with the theory of viscosity solutions.

One of the such motivation for analysing out possible array and the transfer model when a measure the transfer of the model. When the notice of a second weather is that and the model of the model. When the notice of a second weather is that and the notice of a second weather is the model. When the notice of a second weather is the model of the model of the model. When the notice of a second weather is the transfer of the notice of the notice of a second weather is the transfer of the notice of the not

An outline of the paper's se follows in Section 2, weightening the periodito-constantition problem and state the basic assurppions. In section 3, we discuss the contention margare tations of the model, while in Section 4 we study the periodic effection problem within a viscosity solution framework. The resultatof Higdy, Hump, and Zau L2, 14 for geometric Brownian motion are heldly presented and discussed in Section b.

#### THE STOCHASTIC CONTROL PROBLEM

Let  $(\Omega, \mathcal{P}, \mathcal{F})$  be a probability space and  $(\mathcal{F})$  a given birration satisfying the venal hypotheses. Considering investor operating in a mancial matter constraint of a risks asset (e.g., a stack) and a bond. The value of the taky easet is accommand to follow the stachastic process

S. - 200 -

In (2.1),  $L_t$  is a Lévy process with Lévy-Khintchine decomposition

$$L_t = \mu t + \sigma W_t + \int_0^t \int_{|\alpha| < 1} \alpha \, \tilde{N}(ds, d\alpha) + \int_0^t \int_{|\alpha| \ge 1} \alpha \, N(ds, d\alpha),$$

where  $\mu, \sigma$  are constants and  $W_t$  is a standard Brownian motion. Furthermore,  $N(dt, d\alpha)$  is a Poisson random measure on  $\mathbb{R}_+ \times \mathbb{R}$  with intensity  $dt \times n(d\alpha)$ ,  $n(d\alpha)$  is a  $\sigma$ -finite Borel measure on  $\mathbb{R} \setminus \{0\}$  called the Lévy measure, and  $\tilde{N}(dt, d\alpha) = N(dt, d\alpha) - dt \times n(d\alpha)$  is the compensated Poisson random measure. We assume that  $W_t$  and  $N(dt, d\alpha)$  are independent stochastic processes. From now on we shall use the unique càdlàg version of  $L_t$ , which is also denoted by  $L_t$ .

We recall that the Lévy measure has the property

(2.2) 
$$\int_{\mathbb{R}\setminus\{0\}} \min(1,\alpha^2) n(d\alpha) < \infty.$$

Under the following additional integrability condition on the Lévy measure

(2.3) 
$$\int_{|\alpha| \ge 1} \left| e^{\alpha} - 1 \right| n(d\alpha) < \infty,$$

we can write the differential of the stock price dynamics as (using Itô's formula for Lévy processes, see, e.g., [15])

(2.4) 
$$dS_t = \hat{\mu}S_t dt + \sigma S_t dW_t + S_{t-} \int_{\mathbb{R}\setminus\{0\}} \left(e^{\alpha} - 1\right) \tilde{N}(dt, d\alpha).$$

Here we have introduced the short-hand notation

(2.5) 
$$\hat{\mu} = \mu + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^\alpha - 1 - \alpha \mathbf{1}_{|\alpha|<1}\right) n(d\alpha).$$

The bond dynamics is

$$dB_t = rB_t \, dt,$$

with r > 0 being the interest rate. We make the basic assumption that  $r < \hat{\mu}$ . Hence, the expected rate of return from an investment in the risky asset is greater than the return of the bond, giving potential investors a risk premium  $\hat{\mu} - r$ .

The investor wants to allocate her wealth in the asset and the bond and consume so as to maximize her utility. Let  $\pi_t \in [0,1]$  be the fraction of wealth invested in the asset at time t. If we denote the cumulative consumption up to time t by  $C_t$ , we have the wealth process  $X_t^{\pi,C}$  given as

$$\begin{aligned} X_t^{\pi,C} &= x - C_t + \int_0^t \left( r + (\hat{\mu} - r)\pi_s \right) X_s^{\pi,C} \, ds + \int_0^t \sigma \pi_s X_s^{\pi,C} \, dW_s \\ &+ \int_0^t \pi_{s-} X_{s-}^{\pi,C} \int_{\mathbb{R} \setminus \{0\}} \left( e^\alpha - 1 \right) \tilde{N}(ds, d\alpha), \end{aligned}$$

where x is the investor's initial wealth. The market is supposed to be free of any transaction costs (see [4] for the case of transaction costs).

In (2.1), L is a Lavy process with Levy-Khintchine decomposition.

where  $\mu_i$   $\sigma$  are constants and W is a standard Brownin motion  $\tau$  under under  $M_i$   $\sigma_i$  and is a Poisson rand to repart con  $W_i$  r R with micrarry di r  $n/d\sigma_i$ ,  $r/d\sigma_i$  is  $\tau$   $r/d\sigma_i$  is measure on  $W_i$  (0) called () show measure and  $N_i$  (c),  $d\sigma_i = N_i dr_i$  for  $-dr_i$   $r/d\sigma_i$  is the compensated Poisson under method and set  $W_i$  are that  $W_i$  are  $N_i$  and  $N_i$  in the point of the stability denoted by  $L_i$ 

We recall that the LAVY mostant has the property

Under the following additional inferies shifty condition on the Leve niekance

we can write the differential of the stock price dramming as (using 116's formula for Lery, processes, see, e.r., [15])

$$(2.4) \qquad \qquad dS_{1} = \beta S_{2} dc + \sigma S_{3} dW_{3} + S_{4} \int_{S_{1}(d)} (d^{2} - 1) \beta^{2} (dc, d\alpha)$$

Here we have introduced the short-hand cotation

(2.5) 
$$B = B + \frac{1}{2}\sigma^2 + \int_{B_1(q)} (c^2 - 1 - A^2) q u_1(dq)$$

The bond dynamics is

with r > 0 being the interval rate. We make the basic assumption that r < 0. Hence, directed tate of return from an investment in the tisky asset is grouter than the rotana of the bond, giving potential avertors a tak paramum q + n.

The investor wants to allocate for weakle in the asset and the bond and constants to as to maximize her utility: Let  $z \in [0, 1]$  be the fraction of weakle invested in the asset at time t. If we denote the constant to construction up to denote the the vealth process  $X^{*,C}$  are as

where z is the investor's initial wealth. The market is supposed to be free of any transaction costs (see M for the case of gravation costs).

The investor derives utility from the two processes

(2.6)  

$$Y_t^{\pi,C} = ye^{-\beta t} + \beta e^{-\beta t} \int_{[0,t]} e^{\beta s} dC_s,$$

$$Z_t^{\pi,C} = ze^{-\lambda t} + \lambda e^{-\lambda t} \int_{[0,t]} e^{\lambda s} dC_s,$$

where y, z > 0 and  $\beta, \lambda$  are positive weighting factors. The integrals with respect to  $C_t$  are interpreted pathwise in a Lebesgue-Stieltjes sense. When there is no risk of confusion, we shall frequently write  $X_t, Y_t, Z_t$  instead of  $X_t^{\pi,C}, Y_t^{\pi,C}, Z_t^{\pi,C}$ , respectively. Note that the differential forms of  $Y_t$  and  $Z_t$  are

$$dY_t = -\beta Y_t \, dt + \beta \, dC_t,$$
  
$$dZ_t = -\lambda Z_t \, dt + \lambda \, dC_t.$$

The economical background for these two processes are discussed in the next section.

Denote by  $\mathcal{A}_{x,y,z}$  the set of all admissible controls and let

$$\mathcal{D} = \Big\{ (x, y, z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0 \Big\}.$$

We say that a pair of controls is admissible for  $(x, y, z) \in \overline{\mathcal{D}}$  and write  $\pi, C \in \mathcal{A}_{x,y,z}$  if:

- (c.1)  $C_t$  is an adapted process that is right continuous with left-hand limits (càdlàg), nondecreasing, with initial value  $C_{0-} = 0$  (to allow an initial jump when  $C_0 > 0$ ), and satisfies  $\mathbb{E}[C_t] < \infty$  for all  $t \ge 0$ .
- (c.2)  $\pi_t$  is an adapted càdlàg process with values in [0, 1].
- (c.3)  $X_t^{\pi,C} \ge 0$  almost surely for every  $t \ge 0$ .

Condition (c.3) is a state-space constraint, restricting the set of admissible consumption patterns to those avoiding negative wealth.

The objective of the investor is to find an allocation process  $\pi_t^*$  and a consumption pattern  $C_t^*$  which optimizes the expected discounted utility derived from  $Y_t^{\pi^*,C^*}$  and  $Z_t^{\pi^*,C^*}$  over an infinite investment horizon. The value function is defined as

(2.7) 
$$V(x, y, z) = \sup_{\pi, C \in \mathcal{A}_{x,y,z}} \mathbb{E} \Big[ \int_0^\infty e^{-\delta t} U(Y_t^{\pi, C}, Z_t^{\pi, C}) \, dt \Big],$$

where  $\delta > 0$  is the discount factor. The utility function  $U : [0, \infty)^2 \to [0, \infty)$  is assumed to have the following properties:

(u.1) U is nondecreasing, concave, and continuous in each variable.

(u.2) There exist constants K > 0 and  $\gamma \in (0, 1)$  such that  $\delta > k(\gamma)$  and

 $U(y,z) \le K(1+y+z)^{\gamma},$ 

The investor derives utility irons the two processes

$$\begin{aligned} & 2.6 \\ & 2.6 \end{aligned}$$

where  $y_i t > 0$  and  $B_i$ ,  $\lambda$  are possible wavelets; factors when the prospect much respect to  $D_i$ are interpreted pathwise in a Lebergue-Science colors, when there is no set of confusion, we shall frequently write  $X_i$ ,  $K_i$   $E_i$  hastead of  $X_i^{(n)}$ ,  $E_i^{(n)}$ ,  $E_i^{(n)}$ , respectively. Nonethal the differential forms of  $Y_i$  and  $B_i$  are

$$(21) = -33(22 + -32(22))$$

The economical background for these two processes are discussed in the next section. Denote by A... the set of all semissible controls and let

$$\mathcal{D} = \left\{ (z,y,z) \in \mathbb{R}^n : z > 0, y > 0, z > 0 \right\}$$

We say that a pair of controls is admissible for  $(x, y, z) \in \mathcal{D}$  and write  $x \in \mathcal{C}_{n,n,n}$  if  $(x, y, z) \in \mathcal{D}$  and write  $x \in \mathcal{C}_{n,n,n}$  if  $(x, y, z) \in \mathcal{D}$  and write  $x \in \mathcal{C}_{n,n}$  is right contributes with left-band functs (cadible), nondecreasing, with initial value  $C_{n,n} = 0$  (to allow an initial jump when  $C_{n,n} = 0$ ), and satisfies EIC.1 <  $\infty$  for all t > 0.

(c.2) re is an adapted cadlar proper with values in [0.1].

(c.3) 
$$X_1^{**} \ge 0$$
 almost surely for every  $t \ge 0$ .

Condition (c.3) is a state-space constraint, restricting the set of admissible consumption patterns to those avoiding regarded weaks.

The objective of the investor is to had an allocation process  $\pi_i$  and a consumption particle  $C_i$  which optimizes the expected disconsted withty darived from  $\Sigma^{n-1}$  and  $\Sigma^{n-1}$  over an infinite investment horizon. The value function is defined as

where  $\delta > 0$  is the discount factor. The utility function  $U > [0, \infty)^2 \rightarrow [0, \infty)$  is assumed to have the following properties:

(u.1) U is nondecreasing, concave, and continuous in each variable.

(u.2) There exist constants K > 0 and  $c \in (0, 1)$  such that  $\delta > h(c)$  and

$$U(x, x) \le V(x + y + 1)$$
  $X \ge (x, y)$ 

for all nonnegative y, z, where

(2.8) 
$$k(\gamma) = \max_{\pi \in [0,1]} \Big[ \gamma(r + (\hat{\mu} - r)\pi) - \gamma(1 - \gamma) \frac{\sigma^2}{2} \pi^2 \\ + \int_{\mathbb{R} \setminus \{0\}} \Big( \big(1 + \pi(e^{\alpha} - 1)\big)^{\gamma} - 1 - \gamma \pi(e^{\alpha} - 1) \Big) n(d\alpha) \Big].$$

A Taylor expansion shows that the integral term in  $k(\gamma)$  is finite since (2.2) and (2.3) hold.

We next recall a fundamental property of the value function that goes back to Bellman. Namely, we will assume throughout this paper that the dynamic programming principle holds, that is, for any stopping time  $\tau$  and  $t \geq 0$ ,

(2.9) 
$$V(x,y,z) = \sup_{\pi,C\in\mathcal{A}_{x,y,z}} \mathbb{E}\Big[\int_0^{t\wedge\tau} e^{-\delta s} U(Y_s^{\pi,C}, Z_s^{\pi,C}) \, ds + e^{-\delta(t\wedge\tau)} V(X_{t\wedge\tau}^{\pi,C}, Y_{t\wedge\tau}^{\pi,C}, Z_{t\wedge\tau}^{\pi,C})\Big],$$

where  $a \wedge b = \min(a, b)$ . The infinitesimal version of the dynamic programming principle (2.9) is the Hamilton-Jacobi-Bellman equation. In the our context, this equation is a nonlinear second order degenerate elliptic integro-differential equation subject to a gradient constraint (i.e., an integro-differential variational inequality). If we let  $\mathcal{A}$  denote the second order degenerate elliptic integro-differential operator defined as

$$\begin{aligned} \mathcal{A}v(x,y,z) &= -\beta y v_y - \lambda z v_z + \max_{\pi \in [0,1]} \left[ (r + (\hat{\mu} - r)\pi) x v_x + \frac{1}{2} \sigma^2 \pi^2 x^2 v_{xx} \right. \\ &+ \int_{\mathbb{R} \setminus \{0\}} \left( v(x + \pi x (e^{\alpha} - 1), y, z) - v(x, y, z) - \pi x v_x(x, y, z) (e^{\alpha} - 1) \right) n(d\alpha) \right], \end{aligned}$$

the Hamilton-Jacobi-Bellman equation takes the form

(2.10) 
$$\max\left\{\beta v_y + \lambda v_z - v_x; U(y, z) - \delta v + \mathcal{A}v\right\} = 0 \text{ in } \mathcal{D}.$$

Note that we have  $x + \pi x(e^{\alpha} - 1) \ge 0$  for all  $x \ge 0$  and  $\alpha \in \mathbb{R}$ . If v is  $C^2$  and sublinearly growing, it can be proven that (2.10) is well-defined (see, e.g., [2]). Moreover, if the value function V defined in (2.7) satisfies these conditions, then, by using the dynamic programming principle (2.9) and Itô's formula, one can easily prove that V solves (2.10). However, since it is hard in general to prove that V is sufficiently regular, we shall in Section 4 interpret (2.10) in the sense of viscosity solutions. More precisely, due to the state-space constraint (c.3), we shall consider constrained viscosity solutions of (2.10).

#### 3. Economical interpretations

In this section we discuss economical interpretations of the optimal portfolio-consumption problem described in Section 2. Contrary to most non-time-additive utility maximization problems, the investor does not derive her utility directly from present consumption but from averages over past consumption (through the processes  $Y_t$  and  $Z_t$  defined in (2.6)). This structure has many desirable interpretations from an economical point of view. Hindy, Huang, and Zhu [14] suggest three possible interpretations of the two processes  $Y_t$  and  $Z_t$ . In the first, they describe the notions of local substitution and habit formation. Secondly,

#### ON AN OFTENAL POSTPOLO CONFUNCTION MODE

for all nonnegative u.z. where

$$(2.8) \quad -k(\gamma) = \max_{\substack{n \in [0,1]\\ n \in [0,1]}} \left[ \gamma(r + (n - \gamma)n) - \gamma(1 - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(r + (n - \gamma)n) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma) - \gamma(n - \gamma) \sum_{\substack{n \in [0,1]\\ n \in [0,1]}} \gamma(n - \gamma) - \gamma(n - \gamma$$

A Taylor expansion shows that the integral term in  $k(\gamma)$  is mitteriere (3.2) and (2.3) hold. We next recall a fundamental property of the value function that gogs break to Britman Namely, we will assume throughout this paper that the dimanic programming principle holds, that is, for any storping time v and i > 0.

$$(2.9) \quad Y(z, y, z) = \sup_{u, v \in U_{u, v}} \mathbb{E}\left[\int_{\Omega} e^{-ivy(t) v_{u}} e^{-ivy(t) + e^{-ivy(v)} +$$

where a A b = min(a, b). The infinitesimal version of the dynamic programming principle (2.9) is the Hamilton-Jacobi-Beliman conston. In the our contrast, this equation is a nonlinear second order degenerate officies integro-differential equation subject to a gradient constraint (i.e., an integro-differential subjected members). If we let A denote the stocad order degenerate elliptic integro-differential operator definations

$$Av(\pi, g, z) = -\beta uv_{g} - \lambda vv_{g} + \max_{\substack{z \in [0,1]\\z \in [0,1]}} \left[ (z + (z - z)z) + z z z z (z + z) + z z z z (z + z) + z z z z (z + z) \right]$$

$$+ \int_{a_{1}(0)} \left( v(z + \pi z)(z^{2} - 1) + z (z) + v(z) + z z z z (z + z) + z z z (z + z) \right) n(dz) \right]$$

the Hamilton-Jacobi-Belinan equation takes the form

(2.10) 
$$\max\{\delta u_{1} + \lambda u_{2} - u_{2}(U(y,z) - \delta u + Au\} = 0.4n.2.$$

Note that we have  $x + mx[e^{\alpha} - 1) \ge 0$  for all  $x \ge 0$  and  $x \in \mathbb{R}$ . If x is C and sublinearly growing, it can be proven that (2.10) is well-delived (see, e.g., [2]). Moreover, if the value function V defined in (2.7) satisfies these conditions then 0 to what is break in (2.7) satisfies these conditions then 0 to what is break in (2.7) satisfies these conditions (see, e.g., [2]). Moreover, if the programming principle (2.9) and (5.7) satisfies these conditions (see, e.g., [2]). Moreover, if the programming principle (2.9) and (5.7 satisfies these conditions (see, e.g., [2]). Moreover, for the formula is programming principle (2.9) and (5.6 s formula, one can service prove that V moreover, [3.10]. Section 4 integrates of vascosity solutions, there provide the formula is break in the service of vascosity solutions, there provide, the formula is break in the service of vascosity solutions, there provide, the formula is break in the service constrained of the provide (2.10) is the service of vascosity solutions in the provide (2.10) is the service of vascosity solutions in the provide (2.10) is the service of vascosity solutions of the provide (2.10).

#### ECONOMICAL INTERFRETATIONS

In this section we discuss economical interpretations of the optimal portfolio-consumption problem described in Section 2. Contrary to most non-time-additive utility maximization problems, the investor does not derive her utility directly from protects construction of from averages over past consumption (through the processes 1, and 2, drifted in (2.6)). This structure has many desirable interpretations from an economical point of vario. Highly Huang, and 2ha [14] anggest three possible interpretations of the two processes 3, and 2, drifted in (2.6). they model the service flow from a durable good, and in the final interpretation,  $Y_t$  and  $Z_t$  describe the utility derived from a composite commodity. We next discuss the three interpretations in more detail.

The notion that consumption at one date reduces marginal utility at nearby dates and consumption at adjacent dates are complementary is called *local substitution*. If we have lunch at noon, the marginal utility of eating again shortly after will be lower since we are not hungry (provided we had a satisfactory lunch, of course). At dinner time we are again hungry (provided the lunch was not too satisfactory), so the marginal utility to eat then is complementary to lunch. If the mathematical model is able to catch the notion of local substitution, it should be optimal to consume (i.e., to eat in our example) periodically. or in gulps. Hindy and Huang [10] show that investors deriving utility from  $Y_t$  instead of  $C_t$  directly will consume in gulps (see discussion below). The process  $Z_t$  models the notion of habit formation. Agents develop habits from earlier consumption and a high standard of living increases the appetite for present consumption. If you are used to a delicious Botswana beef for supper, you will probably be very disappointed being offered a Norwegian beef as a substitute. When changing your old Mercedes car, you will probably want to buy a new and perhaps better Mercedes to keep up with your expectations of what a car should be like. DeTemple and Zapatero [7] suggest to model the mechanism of habit formation by  $dZ_t = \rho dC_t - \lambda Z_t dt^1$ . The constants  $\lambda$  and  $\rho$  describe the relative importance of consumption history to inherited standard of living. Furthermore,  $\rho$  is the intensity of consumption habitats, while  $\lambda$  is the persistence of past consumption. A low  $\lambda$  means a high persistence, while increasing  $\rho$  places more emphasis on the history of consumption. We choose to follow Hindy, Huang, and Zhu [13, 14] and let  $\rho = \lambda$ . Investigating this control problem is important in order to improve the understanding of the mechanisms driving security returns.

In the second interpretation of the model,  $C_t$  is the total purchase of a durable good up to time t. The durable good may be clothing, computers, cars, and even holidays. The process  $Y_t$  describes the service flow from the durable good. For instance, buying a car will provide the agent with a mean of transport. However, as long as you use the car, it will deteriorate, and after a while the service flow will start to decrease as long as you do not buy a new one. The standard of living of the agent is reflected through past consumption, and modeled by  $Z_t$ . Also  $Z_t$  will decrease as long as new goods are not purchased, however, at a slower rate. A natural condition from an economic point of view is to assume  $\beta > \lambda$ . Good quality and fashionable clothes will for instance provide you with a high standard of living (that is, high  $Z_t$ ), while the service of the clothes will be to keep you warm and dry (one may of course argue that fashion changes faster than deterioration of clothes, so perhaps  $\beta < \lambda$  instead).

The final interpretation mentioned by Hindy, Huang, and Zhu [14] is composite commodities. Many commodities may give the agent two (or more) utilities. The new portable computers from Macintosh provide you with a high quality computer, but at the same time with style (at least they try to advertise it like that). A bicycle gives you exercise

<sup>&</sup>lt;sup>1</sup>They use in fact absolute continuous consumption plans  $dC_t = c_t dt$ .

#### ARMENTER REALESEN AND REER VAN

they model the service flow from a durable good, and in the final interpretation, 16 and 24 describe the utility derived from a composite commodity. We next discuss the three interpretations in more dutail.

The notion that concerption at one data representative matrices interpret commutation at adjacent data are complementary is colled from attractions. If we have not hungry (provided the fanck we not no warrister, is obtain the server and we are a complementary to hunch if the ratificative matrix to the server and a time is complementary to hunch if the ratificative matrix to the server and a family abbetention, it should be optimal to constants [i.e. to server is are considered to be of C<sub>1</sub> directly will commute in galaxies (see direction below the server and the of C<sub>1</sub> directly will commute in galaxies (see direction below the server and the standard of living increases due optimal for an about the server of the standard of living increases due optimal the direction below the server standard of living increases due optimal the data to be very data particle in the standard of living increases due optimal the data to be very data particle in standard of living increases due optimal the data to be very data particle in standard of living increases due optimal the data to be very data particle in the model of the server of the standard of the server of the standard of living increases due optimate the data to be very data particle in the server of the server of the standard of the server of the server of constants by d2,  $= \rho dQ_{2} - \lambda Z_{1} d^{2}$ . The constants  $\lambda$  and  $\rho$  descent to be the server of constant particles have the indicative data to be very data particle in the server of constant by d2,  $= \rho dQ_{2} - \lambda Z_{1} d^{2}$ . The constants  $\lambda$  and  $\rho$  descent to be the server of constant problem history while increasing we describe the restriction of the high period to be interpreted to be server to be an optimation of the server of the problem instant while increasing we describe the restriction of an optimate problement in the server to be server of the server of the server of the server of the high period to be interpreted at a server to be served of the server of the server of the server of the server

In the second interpretation of the model, G is the total purchase of a durable pool up to time t. The durable good may be clatitine, computers, cars, and even indicars, like process Y, describes the service flow from the durable good. For instance, bayies non-valgrovide the agent with a mean of transport lifewayer, as long as you use the car, is will deteriorate, and after a while the service flow will start to detrease is long as you use the car, is will buy a new one. The standard of living of the agent is telested through participant on buy a new one. The standard of living of the agent is telested through participant on and modeled by Z. Also Z will detrease as long as new goods are not purchased, however, at a slower rate. A natural condition from an economic point of view is to standard is Good quality and fedinomiche clothes will for instance provide you with a high standard of bying (that is, high Z), while the service of the clothes will be to keep you with a high standard detrease durable condition from an economic point of view is to be to keep you would detrease the service that fastion changes faster than drafterinting and dry (one may of comes argue that fastion changes faster that drafterinting and dry (one may of comes argue that fastion changes faster that drafterinting of the start perturbes detrease is a start of the start of

The final interpretation methoned by Hindy, Huang, and Zhu (14) is composite conmodifies. Many commonifies may give the agent two (or maye) philities. The new postable computers from Machinesh provide you with a high spality computer, but it the value time with style (at least may my to advertice it Ele that). A firede grass you exercise

They use in fact absolute construction construction planar del at e de

#### ON AN OPTIMAL PORTFOLIO-CONSUMPTION MODEL

(increasing your health) as well as being a mean of transport. Food, for instance, provide you with vitamins and energy, both important for your well being. The utility derived from such dual purpose commodities are modeled through  $Y_t$  and  $Z_t$ . A natural condition of the utility function would be  $\partial^2 U/\partial y \partial z > 0$ , meaning that marginal utility of transport is increased at a higher level of health, if you think of bicycles. Of course we may think of commodities with more than two purposes. We will not include that generality here, since it is a straightforward extension mathematically.

### 4. VISCOSITY SOLUTIONS

Our analysis of the portfolio-consumption model described in Section 2 is based on the dynamic programming method and the newly developed theory of viscosity solutions of Hamilton-Jacobi-Bellman equations. For a general overview of the viscosity solution theory, we refer to the survey paper by Crandall, Ishii, and Lions [6] and the book by Fleming and Soner [9]. For an overview of the use of viscosity solutions in the area of portfolio management and derivative pricing, we refer to the review paper by Zariphopoulou [19].

As it turns out, the Hamilton-Jacobi-Bellman equation is a direct consequence of the dynamic programming principle and one expects the value function to satisfy this equation. However, due to degeneracy as well as market imperfections such as trading constraints (see (c.3)) and transaction costs, to mention only a few, the value function might not satisfy the Hamilton-Jacobi-Bellman equation in the classical sense, that is, the value function might not possess all the continuous derivatives occurring in the Hamilton-Jacobi-Bellman equation pointwise everywhere. It therefore becomes important to relax the notion of classical solution of Hamilton-Jacobi-Bellman equations so as to allow functions that are not necessarily smooth as (generalized) solutions. This has been achieved successfully by the introduction of the notion of viscosity solutions, which allows merely continuous functions to be solutions of fully nonlinear first and second order partial differential equations.

As already mentioned in (2.10), the Hamilton-Jacobi-Bellman equation associated with our singular control problem is a second order integro-differential variational inequality which contains a non-local (integral) operator with a highly singular Lévy measure  $n(d\alpha)$ . If we insist on interpreting (2.10) in the classical sense, we have to consider twice continuously differentiable functions because of the second order differential operator part of (2.10) as well as the (singular) Lévy measure  $n(d\alpha)$ . We point out that it is *not* easy to show directly that the value function (2.7) is twice continuously differentiable, although we can prove quite easily that it is continuous and sublinearly growing (see below). However, if we interpret (2.10) in the viscosity sense, it is sufficient to consider continuous functions, and one can indeed show that the value function (2.7) is a viscosity solution of (2.10) (see below). Moreover, one can prove that there exists only one viscosity solution (the value function!) of the integro-differential variational inequality (2.10) which is continuous and sublinearly growing (see below).

The (constrained) viscosity solution framework presented below is a straightforward adaption of the framework developed in [2, 3, 4] for first and second order integro-differential

#### DRIVEN SCILLERVICE DOUTBOTHOP SERVICES NO. WORLD'S NO. NO.

(increasing your health) as well as being a mean of transport. Food, for instance, provide you with vitamins and energy, bresh insportant for your well being. The utility derived from such deal purpose commodifies are motioled abrough 1, and 2, A matural condition of the utility function would be 2 0, by etc. > 0, maning that marginal athlets of transport is increased at a higher level of health, if you thus of measing that marginal athlet of transport commodifies with more than averpagness. We will not head points that generality here, since it is a straty built forward extension mutilien active and not be deal of the active it is a straty built forward extension mutilien active.

A VINCOSTLY SOLUTIONS

Our analysis of the portfolio-consumption model described in Section 2 is based on the dynamic programming method and the newly developed theory of viscosity colution in Hamilton-Jacobi-Beliman equations. For a sector is warness of the viscosity colution theory we refer to the curvey paper by Grandall, table and Linns 16 and the kook by Finnting and Soner [3]. For an overview of the use of viscosity colution is been according management and derivative pricing, we refer to the reserve paper by Camphopolden 19.

dynamic programming principle and one experts the value function to scalady this set actor. However, due to degeneracy as well as mather imperfectuous stall value thereich an analytic this set actor. (c.3)) and transaction costs to mention only a few, the wire thereich and the transition of setting, the Hamilton-Jacobi-Bellinna equation only a few, the destical years must be the transition of setting might not possess all the centeneous decreations or arrive to the framilton-Jacobi Bellinan equation and thus not satisfy this equation and interior or arrive to the framilton-Jacobi Bellinan setuation and thus not satisfy this equation and the transition-Jacobi Bellinan been achieved successfully by the introduction of framilton facebi Selfman equations been achieved successfully by the introduction of the notion of values the below whether allows merely continuous functions to be solutions of the notion of values to the arrivel differential equations.

As already mentioned in (2.10), the lisminen-lacobi-lishiman equation associated with our singular control problem is a accord order integer-differential variational acquably which contains a non-lecsi (integral) operator with a highly singular Levr measure n/40). If we insist on interpreting (2.10) is the elevated scient, we have to consider takes continuously differentiable functions because of the second order differential operator path of (2.10) is well as the (singular) Levr measure refus). We point out that it is not easy to show directly that the value functions (2.7) is twice continuously differentiable, affectuar we can we interpret (2.10) in the viscosity sense, it is sufficient to consider only of (2.20) (as we interpret (2.10) in the viscosity sense, it is sufficient to consider only (2.20) (as not one can indeed show that the value function (2.7) is twice coldineatly proving (see below). However, if below). Moreover, one can prove that the value function (2.7) is a viscosity solution of (2.20) (as and one can indeed show that the value function (2.7) is a viscosity solution of (2.20) (as below). Moreover, one can prove that there exists only one viscosity solution (1.20) (as and one can indeed show that the value function (2.7) is a viscosity solution of (2.20) (as below). Moreover, one can prove that there exists only one viscosity solution (1.20) (as sublinearly growing (sen below).

The (constrained) viscouty solution framework presented below is a straightforward aduption of the framework developed in [2, 3, 4] for first and second order fittegro-differential

S.

variational inequalities. Due to strong similarities with [2, 3, 4], we are very brief in this section and refer to [2, 3, 4] for details not found herein. Also, we refer to [2, 3, 4] for an overview of the literature dealing with viscosity solutions of integro-differential equations.

As already indicated, the ultimate goal of this section is to characterize the value function (2.7) as the unique constrained viscosity solution of the associated Hamilton-Jacobi-Bellman equation (2.10). To this end, we first verify as in [2, 3, 4] that the value function V is well defined, non-negative, non-decreasing, and concave. The arguments needed to establish these properties are standard (see, e.g., Zariphopoulou [18, 19]).

Next, one can show V is uniformly continuous on  $\overline{D}$  by following the arguments used in the proof of Theorem 3.1 in [2] (see also [1, 3, 4]). In fact, one can even show that V is Hölder continuous if U is Hölder continuous and some extra conditions on  $k(\gamma)$  are fulfilled. This was first observed by Alvarez [1] in the Brownian case and later generalized to the Lévy case in [2] (see also [3]). In addition, the value function has sublinear growth of the same order as the utility function, see [1, 2, 3, 4] different proofs of this fact. More precisely, there exists a positive constant K such that

(4.1) 
$$V(x, y, z) \le K(1 + x + y + z)^{\gamma} \quad \forall x, y, z \in \overline{\mathcal{D}}.$$

In view of (4.1) and for later use, we introduce the set

$$C_{\ell}(\overline{\mathcal{D}}) = \Big\{ \phi \in C(\overline{\mathcal{D}}) : \sup_{\overline{\mathcal{D}}} \frac{|\phi(x, y, z)|}{(1 + x + y + z)^{\ell}} < \infty \Big\}, \qquad \ell \ge 0.$$

In particular, we have

 $V \in C_{\gamma}(\overline{\mathcal{D}}).$ 

Later we prove that the characterization of V as a constrained viscosity solution is unique at least within the class of continuous and sublinearily ( $\gamma < 1$ ) growing solutions.

Before we introduce the notion of (constrained) viscosity solutions, let us introduce the following short-hand notations:  $X = (x, y, z) \in \mathbb{R}^3$ ,  $D_X v$  is the gradient of v with respect to X,  $D_X^2 v$  is the Hessian of v with respect to X, and  $G(D_X v) = \beta v_y + \lambda v_z - v_x$ . Furthermore, introduce the non-local operator

$$\mathcal{B}^{\pi}(X,v) = \int_{\mathbb{R}\setminus\{0\}} \left( v(x + \pi x(e^{\alpha} - 1), y, z) - v(X) - \pi x v_x(X)(e^{\alpha} - 1) \right) n(d\alpha),$$

and the operator

$$F(X, v, D_X v, D_X^2 v, \mathcal{B}^*(X, v))$$

$$= U(y,z) - \delta v - \beta y v_y - \lambda z v_z + \max_{\pi \in [0,1]} \Big[ (r + (\hat{\mu} - r)\pi) x v_x + \frac{1}{2} \sigma^2 \pi^2 x^2 v_{xx} + \mathcal{B}^{\pi}(X,v) \Big].$$

We can now write (2.10) as

(4.2) 
$$\max\left(G(D_X v); F(X, v, D_X v, D_X^2 v, \mathcal{B}^{\pi}(X, v))\right) = 0 \text{ in } \mathcal{D}.$$

A constrained viscosity solution of (4.2) is defined as follows:

#### BENTER, KARLERIN, AND REEKVAM

variational inequalities. Due to second analorus with [2, 3, 4], we are very brief in this section and refer to [2, 3, 4] for details not found berein. Also, we offer to [2, 5, 4] for an overview of the literature dealing with viscosity solutions of integro-differential eccations. As already indicated, the ultimate gall of this section is to characterize the value finetion (2.7) as the unique constrained viscosity solution of the resolution fine-line-line-Bellman equation [2,10]. To this draft, we find vertice as in [2, 6, 4] that the value function V is well defined, non-negative non-dectooling, and concerve. The arguments packed to be also the unique to access the test of the vertice of the resolution of the resolution of the resolution [2,10]. To this are non-dectooling, and concerve. The arguments packed to be also the resolution in the vertice of the concerve. The arguments packed to be also the resolution in the second test of the resolution of the resolution of the solution of the resolution of the second test of the resolution of

Next, one can show V is uniformly contantons on  $\overline{D}$  by following the arguments modin the proof of Therean 5.1 in [2] (see also [1], 3, 4]). In fact, can can even they that V is Hölder continuous if U is Hölder continue is and some extra conditions on k(r) are folfilled. This was first conserved by Alvarez [1] in the Browning case and time, generative to the Lévy case in [2] (see also [3]). In addition, the value function this sublinear growth of the same order as the utility function, see [1, 2, 3] shiften to proofs of one fact. More precisely, there exists a positive constant K such that

In view of (4.1) and for later use, we introduce the set

$$C_{i}(\overline{D}) = \left\{ \phi \in C(\overline{D}) : \sup_{i \in I} \frac{|\phi(x_{i}, y_{i}, z)|}{(1 + 2 + y_{i} + z)^{2}} \leq \infty \right\}, \quad i \geq 0$$

in particular, we have

$$V \in CL(D)$$
.

Later we prove that the characterization of N as a constrained viscosity solution is upique at least within the class of conclusions and sublinearly (n < 1) spowing solutions.

Before we introduce the zotion of (constrained) viscority sciences, let us introduce the following short-hand non-stations:  $X = (x, y, z) \in \mathbb{R}^{2}$ ,  $D_{X^{2}}$  is the gradient of r with respect to X,  $D_{Y}^{2}$  is the Hestian of r with respect to X, and  $O(D_{X} r) = \beta r_{y} + 3r_{y} + 7r_{z}$ . Furthermore, introduce the introduce the non-local operator.

$$B^{*}(X, v) = \int_{\mathbb{R}\setminus\{0\}} \left( v(x + \pi v(v) - 1), \pi(x) - v(X) - \pi \pi v_{x}(X)(v^{2} - 1) \right) v(dv),$$

and the operator

$$F(X, v, D_X v, D_Y v, B^*(X, v))$$

$$= U(y, z) - \delta v - \beta y v_0 - \lambda z v_0 + \max_{v \in [0,1]} (v + (y - v) \pi) z v_v + \frac{1}{2} \sigma^2 \pi^2 x^2 v_0 - \beta^2 (\lambda^2) x^2 v_0 + \beta^$$

We can now write (2.10) as

$$(4.2) = \min\{\Theta(D_X v), F(X, v, D_X v, D_Y^* v, S^*(X, v))\} = 0$$
 in

A constrained viscosity subtrian of [4.2] is defined as follows:

**Definition 4.1.** (i) Let  $\mathcal{O} \subset \overline{\mathcal{D}}$ . Any  $v \in C(\overline{\mathcal{D}})$  is a viscosity subsolution (supersolution) of (4.2) in  $\mathcal{O}$  if and only if we have, for every  $X \in \mathcal{O}$  and  $\phi \in C^2(\overline{\mathcal{D}}) \cap C_1(\overline{\mathcal{D}})$  such that X is a global maximum (minimum) relative to  $\mathcal{O}$  of  $v - \phi$ ,

$$\max\left(G(D_X\phi); F(X, v, D_X\phi, D_X^2\phi, \mathcal{B}^{\pi}(X, \phi))\right) \ge 0 \ (\le 0).$$

(ii) Any  $v \in C(\overline{\mathcal{D}})$  is a constrained viscosity solution of (4.2) if and only if v is a viscosity subsolution of (4.2) in  $\overline{\mathcal{D}}$  and v is a viscosity supersolution of (4.2) in  $\mathcal{D}$ .

Following closely the proof of Theorem 4.1 in [2], we can show that the constrained viscosity property of the value function holds.

**Theorem 4.1** (Existence). The value function V(x, y, z) defined in (2.7) is a constrained viscosity solution of the integro-differential variational inequality (2.10).

To prove this result, we first show that V is a viscosity supersolution directly by using the dynamic programming principle (2.9) and Itô's formula for Lévy processes. To prove the viscosity subsolution property, we argue by contradiction. Introducing stopping times such that we can control the jumps coming from the Lévy process and consumption, we are able to construct estimates on the value function V which contradict the dynamical programming principle. We refer to [2] for details.

To guarantee that the characterization in Theorem 4.1 is unique, a comparison result is needed. In a numerical treatment of the control problem, one approximates the state variables by Markov chains and consider instead the related discrete-time optimization problem. To ensure convergence of the discretized problem to the *correct* continuous-time problem, we need also in this context a comparison principle for (4.2) (see, e.g., [19] for this type of application).

We have the following theorem:

**Theorem 4.2** (Uniqueness). Let  $\gamma' > 0$  be such that  $\delta > k(\gamma')$ . Assume  $\underline{v} \in C_{\gamma'}(\overline{D})$  is a subsolution of (2.10) in  $\overline{D}$  and  $\overline{v} \in C_{\gamma'}(\overline{D})$  is a supersolution of (2.10) in D. Then  $\underline{v} \leq \overline{v}$  in  $\overline{D}$ . Consequently, in the class of sublinearly growing solutions, the Hamilton-Jacobi-Bellman equation (2.10) admits at most one constrained viscosity solution.

Theorem 4.2 can be proven in the same spirit as the comparison principles in [2, 3, 4]. The proof uses the classical "doubling of variables" device together with the maximum principle for semicontinuous functions (see Crandall, Ishii, and Lions [6]). Since our problem contains a second order differential operator, the proof requires that we use the maximum principle for semicontinuous functions and hence we need an alternative formulation of viscosity solutions based on the notion of sub- and superjets. We will not go into details about this formulation, but refer the reader instead to [3, 4]. We refer to [2] for the comparison proof in the case of a first order differential operator (pure-jump market), which does not require the jet formulation and the maximum principle for semicontinuous functions.

We mention that the treatment of the singular non-local operator  $\mathcal{B}^{\pi}$  is rather involved. Among other things, we need to distinguish the singularities at zero and infinity in the integral operator, which is thus split into two parts  $\mathcal{B}^{\pi,\kappa}$  and  $\mathcal{B}^{\pi}_{\kappa}$ . For any  $\kappa \in (0,1)$ , we

#### ON AN OPTIMAL PORTFOLO CONSUMPTION MODEL

Definition 4.1. (i) Let  $O \subset D$ : Any  $v \in O(D)$  is a viscosity subsolution (conservabilized) of (4.2) in O if and only if we have, for every  $X \in O$  and  $v \in C^{*}(D) \cap C_{1}(D)$  such that Xis a clobal maximum (minimum) relative to D of v = d.

# $\max\{O(D_X \phi) | F(X, u, D_X a, D_Y^* a, B^*(Y, \phi))\} \ge 0.(\le 0).$

(ii) Any  $v \in C(D)$  is a constrained viscosity solution of (4.2) If and only if where viscosity subsolution of (4.2) in D and v is a viscosity supersultation of (4.2) in D.

Following closely the proof of Theorem 4.1 in [2], we are play that the constrained views when the constrained

**Theorem 4.1** (Existence). The value function V (1.9.2) defined in (2.7) is a constrained interest solution of the interval differential correctional interval (2.10).

To prove this result, we that show that V is a visodely supersolution dirively by trang the dynamic programming principle (2.9) and hold formula for term brock and so prove the viscosity subsolution property, we argue by contradiction. Entroducing stopping transsuch that we can control the pumps coming from the Linty processed construction we are able to construct estimates on the value function V which contration to the space of processing ordered to be refer to [2] for the function V which contration to the space of are able to construct estimates on the value function V which contration to the space of processing ordered to be refer to [2] for details

To guarantee that the characterization in Theorem 4.1 is unique a comparison reader is needed. In a numerical treatment of the control problem, out approximates the state variables by Markov chains and consider instead the related discrete time optimization problem. To cansure convergence of the discretized problem to the covect continuous time problem, we need also in this context a comparison principle for (4.1) (see, e.g., [19] for this type of application).

We have the following theorem:

Theorem 4.2 (Uniqueness). Let  $\gamma' > 0$  be such that  $\delta > h(\gamma)$ . A stands  $\chi \in C_{-}(D)$  is a subsolution of (2.15) in  $\overline{D}$  and  $\overline{\eta} \in C_{+}(\overline{D})$  is a supervaluation of (2.15) in D. Then  $\eta \leq S_{+}(\overline{D})$  is a supervaluation of (2.16) in D. Then  $\eta \leq S_{+}(\overline{D})$  is a supervaluation of (2.16) in D. Then  $\eta \leq S_{+}(\overline{D})$  is a supervaluation of (2.16) in D. Then  $\beta \leq S_{+}(\overline{D})$  is a supervaluation of  $\beta \leq S_{+}(\overline{D})$  in  $\overline{D}$ . Consequently, in the class of subbiasering growing solutions, the Remains Jacobia Bellinan events on (2.10) elements of reset one constructed discours solution.

Theorem 4.2 can be proven in the trans quirit as the comparison principles in [2, 5, 4]. The proof uses the classical "doubling of variables", device together with the maximum principle for termicontinuous functions (see Qrandall, Ishii, and Licon [6]). Supervoir prohieus constants a second order differential operator, the preof requires that as use the maximum principle for semicontinuous functions and hence we need an autometry formulation of viscouple for semicontinuous functions and hence we need an autometry formulation of viscouple solutions based on the potent of sub- and superjets. We will not go into datally about this formulation, but refer the reader instead to [5, 4]. We refer to [2] for the comparison proof in the case of a first order differential operator (pare-jump firsteas), which does not require

We mention that the treatment of the singuise non-local operator is rather involved. Among other things, we need to distinguish the sing darkins at zero and joinally in the integral operator, which is thus split into two parts 3% and 5%. For any sec (0.1), we define

$$\mathcal{B}^{\pi,\kappa}(X,v,D_Xv) = \int_{|\alpha| > \kappa} \left( v(x + \pi x(e^{\alpha} - 1), y, z) - v(X) - \pi x v_x(X)(e^{\alpha} - 1) \right) n(d\alpha),$$
  
$$\mathcal{B}^{\pi}_{\kappa}(X,v) = \int_{|\alpha| \le \kappa} \left( v(x + \pi x(e^{\alpha} - 1), y, z) - v(X) - \pi x v_x(X)(e^{\alpha} - 1) \right) n(d\alpha).$$

It can be shown (see, e.g., [2]) that  $\mathcal{B}^{\pi,\kappa}(X, v, D_X v)$  is well defined for for  $v \in C^1(\overline{\mathcal{D}}) \cap C_1(\overline{\mathcal{D}})$ while  $\mathcal{B}^{\pi}_{\kappa}(X, v)$  is well defined for  $v \in C^2(\overline{\mathcal{D}})$ . The splitting of the integral operator  $\mathcal{B}^{\pi}$  is taken into account in the jet formulation of viscosity solutions and is essential for carrying out the comparison proof when the Lévy measure  $n(d\alpha)$  is singular (see [3, 4] for details).

Our problem involves a gradient constraint as well as a state constraint boundary condition. To treat the gradient constraint, we construct strict supersolutions that are close to the supersolution being compared. Following closely the proof of Lemma 4.3 in [2], by choosing K > 0 and  $\overline{\gamma} \in (0, 1)$  properly it is easily seen that

$$w = K + \chi^{\overline{\gamma}}, \qquad \chi(X) = \left(1 + x + \frac{y}{4\beta} + \frac{z}{4\lambda}\right)$$

is a strict supersolution of (2.10). When applying the maximum principle for semicontinuous functions, we choose a test function so that the minimum associated with the supersolution cannot be on the boundary (in the spirit of Soner [17]), we are able to handle the state constraint boundary condition. Similar treatments of gradient and state constraints have been given in [18, 19] (see also [13]) for a related portfolio-consumption model in a geometric Brownian market. Finally, let us mention that the strict supersolutions are also used to "localize" the proof of the comparison principle to a bounded domain (which is convenient). We refer to [2, 3, 4] for further details about the comparison principle.

### 5. Discussion of the geometric Brownian motion case

We will in this section recall the conclusions made by Hindy, Huang, and Zhu [14], which were based on a numerical treatment of portfolio-consumption model in the geometric Brownian motion case. Their results indicate the type of results that we may expect from a study of the Lévy case.

From the portfolio-consumption problem with utility of HARA (Hyperbolic Absolute Risk Aversion) type and local substitution (or durability) but without habit formation<sup>2</sup>, the investor optimally keeps a constant fraction of wealth in the stock. Consumption takes place only when the wealth reaches a certain barrier, leading to a periodic consumption pattern (or in more popular terms "consumption in gulps"). Reaching the optimal consumption barrier, the investor consumes a small amount only enough to prevent the state variables from leaving the barrier (i.e., increase  $Y_t$  while decreasing wealth  $X_t$  through consumption in a "local-time" fashion). Optimal consumption takes place only when the ratio between wealth and  $Y_t$  is equal to a constant  $k^*$ . Hence the optimal consumption boundary is linear in  $Y_t$ , as was proven by Hindy and Huang [10] when the stock price follows a Brownian motion. In [3], their conclusions were generalized to a Lévy market.

10

<sup>&</sup>lt;sup>2</sup>The investor derives utility only from  $Y_t$ .

define

$$B^{TA}(X, v, D, v) = \int_{|v|>n} \left(v(x + \pi x)(v' - 1), v, z) - v(X) - \pi x v_{a}(X)(v' - 1)\right) n(dv)$$

$$B_{a}^{TA}(X, v) = \int_{|v|>n} \left(v(x + \pi x)(v' - 1), v, z) - v(X) - \pi x v_{a}(X)(v'' - 1)\right) n(dv).$$

It can be shown (see, e.g. [2]) that  $B^{-1}(X_0, D_{X_0})$  is vell defined for form  $\in C^{+}(D)$  of  $X_0$  while  $B_{2}^{-}(X, v)$  is well defined for  $v \in C^{+}(D)$ . The splitting of the integral circulate  $E^{-}$  is called into account in the jet formalization of obscale remaining and its rescaled for called  $B^{-}(X, v)$  is well defined for  $v \in C^{+}(D)$ . The splitting of the integral circulate  $E^{-}$  is out the comparison proof when the integral  $D^{-}(D)$  while  $B_{2}^{-}(X, v)$  is well defined for  $v \in C^{+}(D)$ . The splitting of the integral  $D^{-}(D)$  is the proof of the integral  $D^{-}(D)$  is the proof of the integral  $D^{-}(D)$  out the comparison proof when the integral  $D^{-}(D)$  is the proof of the supersolution involves a gradient constraint as well as a state constraint boundary called  $D^{-}(D)$  of the supersolution boundary compared. Following closely the grad of the target  $D^{-}(D)$  is the proof of boundary  $D^{-}(D)$  is the proof of boundary  $D^{-}(D)$  is the proof of boundary  $D^{-}(D)$  in [2] by the supersolution boundary  $D^{-}(D)$  is the proof of boundary  $D^{-}(D)$  is [2] by the supersolution boundary  $D^{-}(D)$  is the proof of bound

is a strict supersolution of (2.10). When applying the standaum minimple for semication nous functions, we choose a test fitnement as that the minimum associated with the uppersolution cannot be on the boundary (in the spirit of Sonar (17)), we are able to fundin the state constraint boundary condition. Similar destrictions of gradient and state constraints have been given in [18, 19] (see also [15]) for a related portfolio-constraint in a geometric Brownian market. Finally, for us related the strict supersolution model in a used to "localize" the proof of the comparison philoty is a bounded formula (which is accurated to "localize" the proof of the comparison philoty is about the comparison provide the strict supersolution (which is a set of the strict supersolution as the state of the proof of the comparison philoty is a bounded formula (which is convenient). We refer to [2, 3, 4] for further states about the comparison provide

#### BE DISCUSSION OF THE CLONETERS BROWNAN MOTION CLEE

We will in this section recall the consistions made by Hindy, Smarg, and Zhu [14], which were based on a numerical treatment of portfolio-constrangtion readed in the prometric Brownian motion case. Their results indicate the provide of regults that he train expect brun a study of the Lévy case.

From the portfolio-constanction problem with utility of HATA (Expandence basedure) Risk Avertion) type and local substitution (or durability) but without habit formation, the investor optimally leages a constant framient of wallth in the stock. Constitution takes place only when the wealth reaches a certain basis is guize'), breaching the upstant on pattern (or in more popular terms' consumption in guize'), breaching the prevent the state sumption barrier (the investor consumer a small amount only manifi to prevent the state variables from leaving the barrier (i.e., increase N, while decreasing wealth X, thrach consumption in a 'local-time' fashion). Optimal consumption takes place only when the ratio between wealth and Y, is equal to a constant on the states place only when the boundary is linear in Y, as was proven by findly and thrang [10] when the states boundary is linear in Y, as was proven by findly and thrang [10] when the states boundary is linear in Y, as was proven by findly and thrang [10] when the states boundary is linear in Y, as was proven by findly and thrang [10] when the states boundary is linear in Y, as was proven by findly and thrang [10] when the states boundary is linear in Y, as was proven by findly and thrang [10] when the states place boundary is linear in Y. as was proven by findly and thrang [10] when the states place

The investor derives utility only irom if

10

Note the resemblence with the classical Merton problem, where the investor also keeps a constant fraction of the wealth in the risky asset.

For the stochastic control problem with both local substitution and habit formation, Hindy, Huang, and Zhu [14] compute the optimal consumption boundary  $X^*(y, z)$  using the Markov chain approximation method for a utility function on the form

$$U(y, z) = y^{\gamma_1} z^{\gamma_2}, \qquad \gamma_1, \gamma_2 \in (0, 1).$$

If the current wealth is less than the barrier, the investor refrains from consumption, waiting until the state variables hits the consumption barrier. During a period of no consumption, the standard of living and service flow from the good will decrease. When the current wealth is bigger than  $X^*(y, z)$ , the investor instantly consumes such that the wealth is reduced and y, z are increased to bring the state variabels to the boundary. This consumption pattern is in accordance with the model in [10], where the investor consumes in gulps, thereby introducing local substitution. However, the special feature of the current problem is that the optimal consumption barrier is cyclic as a function of y and z. For a fixed standard of living  $y, X^*(y, z)$  will increase as a function of z, then decrease and then increase again. A similar property holds for  $X^*(y, z)$  for fixed standard of living and varying z. Another striking feature is the suboptimality of keeping a constant fraction of wealth in the stock. The optimal investment policy  $\pi$  will be a cyclic function of wealth, standard of living, and service flow, i.e.,  $\pi^* = \pi^*(x, y, z)$ . The partial derivatives with respect to y and z will change sign periodically as y and z change, respectively.

The cyclic pattern in both consumption and investment is explained by Hindy, Huang, and Zhu [14] as coming from an interaction between durability and habit formation. An additional purchase of the durable good reduces the agent's appetite. This satiation effect is in conflict with the indirect stimulation of increasing the agent's appetite for a higher standard of living. When satiation dominates, the agent will tolerate high losses, thus investing a higher fraction of her wealth in the stock. When stimulation is dominating, the agent is more risk averse and protects her standard of living by reducing the fraction invested in the risky asset.

When generalizing to a more realistic Lévy market model, we expect the same qualitive conclusions to hold. However, the optimal consumption and investment policies will quantitatively look different. We remark that this is in accordance with the case of local substitution with no habit formation, where the optimal policies where qualitatively the same for the geometric model and the Lévy market (see [3]).

#### References

- O. Alvarez, A singular stochastic control problem in an unbounded domain, Comm. Partial Differential Equations 19 (1994), no. 11-12, 2075-2089.
- [2] F. E. Benth, K. H. Karlsen, and K. Reikvam, Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: A viscosity solution approach, *Finance* and Stochastics. To appear.
- [3] F. E. Benth, K. H. Karlsen, and K. Reikvam, Optimal portfolio management rules in a non-Gaussian market with durability and intertemporal substitution, Preprint, Department of Mathematics, University of Bergen, 2000.

#### ON AM OFFICE PORCESSION MODEL MANY MALE

Note the resemblence with the cleasical Merton problem, where the investor also keeps a constant fraction of the west'ri in the tisky asset.

For the stochastic control problem with both local substitution and habit formation, Hindy, Huang, and Zhu [14] compute the ordinal communition boundary  $X^*(y, z)$  using the Markov chain approximation method for a unifity from the term.

If the current weaks is less than the heavier, the averager revenue from consumption, waiting until the state variables hith the current for our the good will decrease. When consumption, this statement of living and service flow our the good will decrease. When the current wealth is bigger than X'(y, z), the investor matantic consumes such that the wealth is reduced and y, z are increased to bring the state unable to the boundary. This consumption pattern is in accordance with the model in [10], where this is reduced the current good will decrease of the current in goips, thereby introducing total substitution like the model in [10], where the investor consumes a fixed standard of living y, X'(y, z) will increase as a function of z, then decrease and then increase again. A similar property bolds for 1 (20) to fixed standard of living tak wealth in the stock. The optimal constance with the substitution of z, then decrease and then increase again. A similar property bolds for 1 (20) to fixed standard of living tak wealth in the stock. The optimal investment pulicy x will be a provided to a stock with standard of living, and service flow i.e., x' = x'(x, y, z'). The partial derivatives with standard of living, and service flow i.e., x' = x'(x, y, z').

The cyclic pattern in both concruption and intestructs is explained by Hindy, Huang, and Zhu [14] as coming from an interaction between durability and habit formation. An additional purchase of the durable good reduces the area a spartice. This estimion affect is in conflict with the indirect stremination of increasure the agent's appedite for a high or standard of living. When satistion dominates the agent will televate high iceses, thus investing a higher fraction of its works her standard of living he reducing the fraction the agent is more risk average and protects her standard of living he reducing the fraction invested in the risk average and protects her standard of living her reducing the fraction

When generalizing to a more realistic Levy market model, we expect the same qualitive conclusions to hold. However, the optimal consumption and inventment policies will quantitative look different. We remark that this is is accordance with the case of local substitution with no habit formation, where the optimal policies where qualitatively the same for the reometric model and the Levy market (see [3]).

#### REFERENCES

- [1] O. Alvarez, A singular stochastic control problem in an unbounded durants. Comm. Parist Differential Southers 19 (1994), pp. 13-12, 2075–2080.
- [2] F. E. Benth, K. H. Karizen, and K. Relivam, Optimal possible adaction with consumption and roalinear integro-differential equations with gradient constraints A viscosity solution approach, Pleamor and Stophartics. To approach.
- [3] P. E. Benth, K. H. Karleen, and K. Keleverr, Optiquel periodsho management rules in a near-transmismarket with dorability and intertemptical subscitutiza, Preprint, Department of Mathematics, University of Bergen, 2000.

11

- [4] F. E. Benth, K. H. Karlsen, and K. Reikvam, Portfolio management in a Lévy market with intertemporal substitution and proportional transaction costs. Preprint, MaPhySto, University of Aarhus, 2000.
- [5] F. E. Benth, K. H. Karlsen, and K. Reikvam, A note on portfolio management under non-Gaussian logreturns, Preprint, Department of Mathematics, University of Bergen, 2000.
- [6] M. G. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 1–67.
- [7] J. B. DeTemple and F. Zapatero, Asset prices in an exchange economy with habit formation. *Econometrica* **59**(6) (1991), 1633-1657.
- [8] S. Elganjoui, Diploma thesis, Department of Mathematics, University of Bergen, Norway, 2000.
- [9] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Applications of Mathematics 25, New York: Springer Verlag, (1993).
- [10] A. Hindy and C. Huang, Optimal consumption and portfolio rules with durability and local substitution, *Econometrica* **61** (1993), 85-122.
- [11] A. Hindy, C. Huang, and D. Kreps, On intertemporal preferences in continuous time: the case of certainty, J. Math. Econom. 21 (1992), no. 5, 401–440.
- [12] A. Hindy and C. Huang, Intertemporal preferences for uncertain consumption: a continuous time approach, *Econometrica* **60** (1992), no. 4, 781-801.
- [13] A. Hindy, C. Huang and H. Zhu, Numerical analysis of a free-boundary singular control problem in financial economics, J. Economic Dynamics and Control 21 (1997), 297-327.
- [14] A. Hindy, C. Huang and H. Zhu, Optimal consumption and portfolio rules with durability and habit formation, J. Economic Dynamics and Control 21 (1997), 525-550.
- [15] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, 2nd Edition, North-Holland/Kodansha, (1989).
- [16] H. J. Kushner and P. G Dupuis, Numerical methods for stochastic problems in continuous time, Springer-Verlag (1992).
- [17] H. M. Soner, Optimal control with state-space constraint. I, SIAM J. Control Optim. 24 (1986), no. 3, 552-561.
- [18] T. Zariphopoulou, Investment-consumption models with transaction fees and Markov-chain parameters, SIAM J. Control Optim. 30 (1992), no. 3, 613–636.
- [19] T. Zariphopoulou, Transaction costs in portfolio management and derivative pricing, in Introduction to Mathematical Finance, 101–163, Proc. Sympos. Appl. Math., 57, Amer. Math. Soc., Providence, RI, 1999.

#### SENTRA RAPIASSI AND REPAIR

- [4] F. E. Bench, K. H. Karlson, and K. Seilerare, Periodic management in a Livy moder, with investmeperal substitution and properlimed transaction costs. Preprint, Machaelto, Caircasty of Aarbus, 2000.
- [5] P. E. Banth, K. H. Karlson, and K. Beneven, A note on particlity minimum mater non-dimension for carries, Preprint, Department of Mathematics, Conversived Remea, 2000.
- [6] M. G. Crandall, H. Ishii and P.-L. Eines, Herris gride to structury solutions of second order partial differential constitute, Bull. Astron. Math. 766, 09 S.) 27 (2022) nov1, 3-87.
- [7] J. B. DeTampie and F. Zapataro, Aster prices in an evaluage atomorpy with holds formation. Francemetrics 53(6) (1941), 1633-1657.
  - [8] S. Elgungoul, Disherma Resia, Department of Modisanatics: (Indepartity of Barger, Nativary, 2000.
- [9] W. H. Fieming and H. M. Samer, Controlled Markon Processes and Famesing Dominants, Applications of Mathematics 25, New York: Springer Verlag, (1993).
- [10] A. Bindy and C. Huang, Optimal consumption and particle rules with databalay and local management tion. Sconometrics 61 (1993), 83-122.
- [11] A. Hindy, C. Huang, and D. Kreps, On Intertemperal preferences in functions time: the case of certainty, J. Math. Econom. 21 (1992), no. 5, 401–446
- [12] A. Hindy and C. Hueng, Intertamporal preferences for micrically commentates a constantous pane
- [13] A. Hindy, C. Huang and H. Shin, Numerical stories is of a free-boundary storage in quality control problem in fragment experiment. *L. Remarker, Description* and *Constructs* 1 (1917), 297-827.
- [14] A. Hindy, C.Henneg and H. Zher, Optimul constructed and polytolike with developing and habit. formation J. Sciencesic Demonstrate and Control 23 (1897), 503–560
- [15] N. Beda and S. Watanabe, Shochasha Differential Equations and Editors, End. Editors, March-Bolland (Ecdensis, 1993).
- [16] H. J. Kushuzz and P. G. Dopula, Numerical methods for stookastic problems in maticative stress. Socium Methods (1993).
- [17] H. M. Some, Optimal control with state-space constraint. I. StAM. J. Control Optima 24 (1986), no. 3, 552-561.
- [18] T. Zariphopoulou, Investment-Consumption models with triansaction two and Manager-chemmentumetone SIAN J. Control Optim. 20 (1992), pp. 3, 013–636.
- [10] T. Zaciphopoulon, Transaction costs in particle managements and discustive pairing, in minutacions to Mathematical Finance, 191-163, Proc. Sympos. Appl. Math., 57, Auntr. Math. Soc., Providuants R1, 1999.

(Fred Espen Benth) DEPARTMENT OF MATHEMATICS UNIVERSITY OF OSLO P.O. Box 1053, BLINDERN N-0316 OSLO, NORWAY AND MAPHYSTO - CENTRE FOR MATHEMATICAL PHYSICS AND STOCHASTICS UNIVERSITY OF AARHUS NY MUNKEGADE DK-8000 ÅRHUS, DENMARK *E-mail address*: fredb@math.uio.no *URL*: http://www.math.uio.no/~fredb/ (Kenneth Hvistendahl Karlsen) DEPARTMENT OF MATHEMATICS

DEPARIMENT OF MATHEMATICS UNIVERSITY OF BERGEN JOHS. BRUNSGT. 12 N-5008 BERGEN, NORWAY E-mail address: kenneth.karlsen@mi.uib.no URL: http://www.mi.uib.no/~kennethk/

(Kristin Reikvam) DEPARTMENT OF MATHEMATICS UNIVERSITY OF OSLO P.O. Box 1053, BLINDERN N-0316 Oslo, Norway *E-mail address*: kre@math.uio.no URL: http://www.math.uio.no/~kre/

#### ON AN OFTENAL FORTEOLOOPONCUEFTION MODEL

Der restrict of Osia Der Statt of Osia A. Osia Caso. Narma A. Osia Caso. Narma A. Osia Caso. Narma M. Munacaso M. Mu





