Department of APPLIED MATHEMATICS

A random exchange model with constant decrements.

by

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UNIVERSITY OF BERGEN Bergen, Norway



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Abstract.

Let d be a positive constant and let $\{U_n\}$ be a sequence of independent, identically distributed random variables. Define a new sequence of variables $\{X_n\}$ recursively by

$$X_{n} = \max(X_{n-1} - d, U_{n})$$
.

A crude model of physical exchange processes, based on the sequence $\{X_n\}$, is analyzed. The stationary distribution of X_n , of X_n at an epoch of exchange $(X_n = U_n)$ and of the time between consecutive exchanges are determined. For the case where U_n is $N(\mu, \sigma^2)$, an asymptotic expansion for the expected number of years between exchanges as $d \rightarrow 0$ is sought.

1. Introduction.

The model to be considered here was proposed by Herman G. Gade (Gade 1973) in connection with his investigations of deep water exchanges in sill fjords. Though I want to discuss the model abstractly (it may be used to describe other physical phenomena), it is useful to have the following physical process in mind.

The deep water masses of a sill fjord are characterized by a relatively high degree of uniformity. As a first approximation the density of the water may be considered homogeneous throughout the basin. Various diffusion processes causes this density to decrease approximately linearly with time. In this paper I will follow Gade and assume a constant annual density decrement d.

It is an empirical fact that influxes of coastal water into the basin tend to take place at the same time of the year, thus establishing a recurring phenomenon with time intervals being essentially multiples of a full year. The deep water renewals are relatively rapid events, often completed within the course of a few weeks. The influx will take place when the coastal water at sill depth is heavier than the resident water in the fjord basin. I assume that in this case all the resident water is replaced by water with the same density as that of the coastal water present at sill depth. The density of the coastal water in adjacent years are assumed

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1. Introduction.

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to be independent, identically distributed (i.i.d.) random variables.

In a forthcoming paper a more general model will be considered in which also the annual density decrements are assumed to be i.i.d. random variables. This will prove to be a generalization of existing models in the theory of queues. My reason for discussing the special case of constant decrement here is twofold. As far as I know, it is only this specialized model which up to now has been used to describe a physical process. Furthermore, in this special case it is possible to proceed by relatively elementary mathematical methods. Thus it should be possible to follow the discussion without being a specialist in complex integral equations.

2. The model.

Let $\{U_n; n = 1, 2, 3, ...\}$ be a sequence of i.i.d. random variables characterized by the distribution function

 $G(x) = P[U_n \le x]$; n = 1, 2, ...

(Here and in the following P denotes probability.) In the physical model, U_n is the density of coastal water at sill level in year number n, at the time of the year when exchanges tend to take place. We number the years consecutively. In most of the paper the U_n 's are assumed to be absolute

- 2 -

to be independent. ideasiesly distributed (1.1.8.) Endy variables.

In a forthcosting pages a more general padet will be considered in which blas the annual density desperance is assumed to be 1.1.0. ranges wartables. Whit will grove is be a generalization of existing models in the cheers of generalization by reason for discussing the special case of generally and by reason for discussing the special case of generali fields in there is twofold. As far so I know is is is is and the specify challed madel which up to now has been define and the shift of provide to second by reference, in this meeting case in it received to second by reference, in this meeting case in it mathed madel which we weather all of this meeting case in it received to second by reference, in this meeting case in it without tentus a substitute to weather a define of second of the second by reference, or define the shift of reference. The standa is substitute to second of the second of the second by reference, or define and the shift of reference.

2. The model.

Let $\{W_n : n = 1, 22, 3, \dots\}$ be a callence of $1 \le 1, 1$, $1 \le 1, 2$, $1 \le 1$

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(Mare and intervention control to where we dependent the prophetic tipe) (In the pay at call match and the set of dependent to of coastal water at all the set of a structure of the structu

continuous variables with probability density

$$g(x) = \frac{d}{dx} G(x) .$$

Let X_0 be independent of the sequence $\{U_n\}$ and define a new sequence $\{X_n\}$ of random variables by

(1)
$$X_n = max(U_n, X_{n-1} - d); n = 1, 2, ...$$

where d is a positive constant (the annual density decrement). It follows from our assumptions that X_n may be interpreted as the density of the deep water masses of the sill fjord in year number n, assuming an initial density X_0 . (X_0 may be constant).

Equation (1) may be written

$$X_{n} = \begin{cases} U_{n} & \text{if } U_{n} > X_{n-1} - d \\ \\ X_{n-1} - d & \text{if } X_{n-1} - d \ge U_{n} \end{cases}$$

Thus we are led to study the events

 $A_n : U_n > X_{n-1} - d$ (exchange at time n)

 A_n^c : $U_n \leq X_{n-1} - d$ (no exchange at time n).

Most of this paper is devoted to the study of the process { X_n ; n = 0,1,2,...}. Some information may be drawn directly from equation (1), e.g.

$$X_n \ge U_n$$

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bet $X_{\rm c}$ to intercentral of the secreties (3, 5) and the receiver (3, 5) and the receiver (3, 5) of r and variables by

Equation (1) may be written

$$\begin{cases} u_{n} & u_{n} > x_{n-1} = 6 \\ & & & & \\ & & & & \\$$

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$$\left(\begin{array}{ccc} 1 & 0 \\ 1 & 0 \end{array}\right) = \left(\begin{array}{ccc} 1 & 0 \\ 0 & 0 \end{array}\right) = \left(\begin{array}{ccc} 1 & 0 \\ 0 & 0 \end{array}\right) = \left(\begin{array}{ccc} 1 & 0 \\ 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc} \alpha & \alpha_{1,1} & \alpha_{2,2} &$$

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that is

$$E(X_n) \ge E(U_n)$$

provided both expectations exist. Here E denotes expectation, e.g.

$$E(U_n) = \int_{-\infty}^{\infty} x \cdot dG(x)$$

Thus the expected density of the resident water in the fjord is greater than or equal to the expected density of the coastal water.

Furthermore, $\{X_n\}$ is easily seen to be a Markov process. That is, the conditional distribution of X_n , given $X_k, X_{k+1}, \ldots, X_{n-1}$ (k < n-1) is independent of $X_k, X_{k+1}, \ldots, X_{n-2}$. This is immediate from equation (1). It is well known that all the information about a Markov process can in principle be obtained from the initial distribution and the transition probability function

 $P(x;y) = P[X_{n} \le y | X_{n-1} = x]$

In general this function depends on n, but here we have a so-called homogeneous Markov process where P(x;y) is independent of n. We find

that is

$$\mathbb{E}(x_{\alpha}) > \mathbb{E}(\overline{a}_{\alpha})$$

provided both expectations exists. Here Eukenergy argeorer, the state of the second st

$$\mathbb{E}\left(\mathbb{I}_{\mathbb{Q}}^{n}\right) = \sqrt{\sum_{\substack{n \in \mathbb{Q} \\ n \in \mathbb{Q}}} \mathbb{Q}\left(\mathbb{R}^{n}\right)}$$

innue the espected density of the resident seter to the diagon for the set of the 900000 for the the president for the strength of the set of the 900000 of the set o

Number and (X_{ij}) is each by norm the in a theritor procase. Such is, the constitutional all diribution of X_{ij} , O^{1} and V_{ij} . O^{1} and V_{ij} , O^{1} and V_{ij} , O^{1} and V_{ij} , O^{1} and V_{ij} . O^{1} and V_{ij} , O^{1} and V_{ij} , O^{1} and V_{ij} , O^{1} and V_{ij} . O^{1} and V_{ij} , O^{1} and V_{ij} , O^{1} and V_{ij} . O^{1} and V_{ij} , O^{1} and V_{ij} , O^{1} and V_{ij} . O^{1} and V_{ij} and V_{ij} and V_{ij} and V_{ij} . O^{1} and V_{ij} and V_{ij} and V_{ij} . O^{1} and V_{ij} and V_{ij} and V_{ij} and V_{ij} . O^{1} and V_{ij} and V_{ij} and V_{ij} and V_{ij} . O^{1} and V_{ij} and V_{ij} and V_{ij} and V_{ij} and V_{ij} . O^{1} and V_{ij} and

$$\mathbb{Y}_{1}(\mathbf{x}_{1}\mathbf{y}^{*}) = \mathbb{Y}_{1}[\mathbf{x}_{1}, \mathbf{x}_{2}\mathbf{y}_{1}] \mid \mathbb{Y}_{n-1} = \mathbf{x}_{1}]$$

Tragemental barde Tranoditon dispende no , batt Here we have A sto-reitike temperation at Mandov process shate Ph(xi)) is inder secondents of an (200 charles)

(2)

$$P(x;y) = P[\max(U_{n}, X_{n-1} - d) \le y | X_{n-1} = x]$$

$$= P[U_{n} \le y, X_{n-1} \le y + d | X_{n-1} = x]$$

$$= G(y)T(y + d - x)$$

where

$$f(3) I(t) = \begin{cases} 1 & \text{if } t \ge 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Here we have used the fact that X_{n-1} and U_n are independent, since X_{n-1} depends only on $X_0, U_1, U_2, \dots, U_{n-1}$ through equation (1).

In a corresponding way we find the n step transition probability function

$$P^{(n)}(x;y) = P[X_{k+n} \le y | X_k = x]$$
, $(n = 1, 2, ...)$,

which is independent of k .

$$P^{(n)}(x;y) = P[\max(U_{k+n}, X_{k+n-1}-d) \le y \mid X_{k} = x]$$

$$= P[\max(U_{k+n}, U_{k+n-1}-d, X_{k+n-2}-2d) \le y \mid X_{k} = x]$$

$$= \dots$$

(4)

$$= P[\max(U_{k+n}, U_{k+n-1}-d, \dots, U_{k+1}-(n-1)d, X_{k}-nd) \le y \mid X_{k} = x]$$

$$= P[X_{k} \le y + nd, U_{k+n-j} \le y + jd, (j=0, 1, \dots, n-1) \mid X_{k} = x]$$

$$= I(y + nd - x) \prod_{j=0}^{n-1} G(y + jd)$$

with I(t) given by (3).

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$$T((x_{i,y}) = P(abx(0_{i,y}, x_{i+1}, -0)) \geq y + \frac{1}{2} \sum_{i=1}^{n} \frac{1}{2} \sum_{i=1}$$

where

Here, we have used the Pact that k_{n-1} this up and the product, since k_{n-1} conduct, since k_{n-1} conduct, and k_{n-1} conduct. Shift, $k_n \in \mathbb{R}$ through equation (1):

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$$p^{(n)}(x,y) = P(x_{n+n} + y + x_n + z) \quad (n = n, 2, \dots, y)$$

which to independent of k

$$y^{(m)}(x,y) = F(max(0)_{k+m} \times k_{k+m+1} + d) \approx y + X_{k} = x^{1}$$

$$= F(max(0)_{k+m+2} + d) \times k_{k+m+2} + d \times X_{k+m+2} + d + X_{k+m+2} + d$$

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3. Stationary distribution.

When $n \to \infty$, $I(y + nd - x) \to 1$ for all x and y the sequence $\left[\prod_{j=0}^{n-1} G(y + jd), n=1,2,\ldots \right]$ decreases monotonically:

$$\prod_{j=0}^{n} G(y + jd) = G(y + nd) \prod_{j=0}^{n-1} G(y + jd) \leq \prod_{j=0}^{n-1} G(y + jd)$$

Therefore the following limit always exists and is independent of x

(5)
$$F(y) = \lim_{n \to \infty} P^{(n)}(x;y) = \prod_{j=0}^{\infty} G(y + jd)$$

It is easily seen that $0 \le F(y) \le 1$ for all x and $F(x) \le F(y)$ when x < y. Let X_0 have the distribution function $F_0(x)$. Then by dominated convergence

(6)
$$\lim_{n \to \infty} P[X_n \le y] = \lim_{n \to \infty} \int_{x=-\infty}^{\infty} P^{(n)}(x;y) dF_0(x) = F(y) .$$

The following theorem is the main result of this section. It may be generalized to the case of random decrements (see Helland, 1973).

and and brock of the provide second

where $n + 2 \cdots$, $1 - (y + 2d - x) \rightarrow 1$ for all x and y the sequence $\left(\prod_{i=1}^{n-1} (y_i + 2d), n + 1, 2, \dots \right)$ decreases monotonically the sequence $\left(\prod_{i=1}^{n-1} (y_i + 2d), n + 1, 2, \dots \right)$ decreases monotonically

$$\prod_{j=0}^{n} C((y+j)) = CC(y+j) = C$$

Therefore the jojloving limit siveys exists and is independent of x

(5)
$$T(v) = \frac{1+v}{n+v} P(E)(Ev) = \prod_{n=0}^{\infty} Q(E+2n) = \prod_{n=0}^{\infty} Q(E+2n) .$$

It is eachly seen that is right at for all x and T(x) = T(y) where $x < y < \frac{1}{2}$ is a problem of the distribution factorian $F_{y}(x)$. Where F_{y} is a factor of the distribution factorian $F_{y}(x)$. Where F_{y} is a factor of the distribution T

""herereslowing there is no main result of this section. The mathematical section is a section decrements (sec Helashder 975) Helashder 975) Theorem 1.

(i) If $E(U_1^+) < \infty$, then F(y) is a distribution function and $\{X_n\}$ converges in law to the distribution F.

(ii) If
$$E(U_1^+) = \infty$$
, then

 $F(y) = \lim_{n \to \infty} P[X_n \le y] = 0$ for all y.

Remark.

With U_n^+ is meant $\max(0, U_n)$. $E(U_n^+)$ is of course independent of n, since the distribution of U_n is. The term convergence in law is used in the usual probabilistic sense. It simply means that (6) holds (in general for all y where F(y) is continuous), where F is a distribution function.

Proof.

We have to show that F is a distribution function when $E(U_1^+)$ is finite. Now we have already remarked that F(y) is bounded by 0 and 1 and is monotonic in y. It is also easy to see that F(y) is continuous from the right. When G(y) = 0 it is trivial, otherwise show by monotone convergence

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Theorem

Reiducterations of (v) is not (v) is not (v) in mot. (1)

 $\mathbb{P}(\mathbf{y}) = \min_{1 \le i \le \infty} \mathbb{P}[\mathbb{Z}_{i_1} \models x_i = i_0 = i_2 : \underline{z}_i] = \underline{z}_i$

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The second of the country country ((0)) is (0), is a country of (0), where is $P_{1,2}$, is a country of (0), where is $P_{2,2}$, is a country of (0), where is $P_{2,2}$, is a country of (0), where is $P_{2,2}$, is a country of (0), where is $P_{2,2}$, is a country of (0), where is $P_{2,2}$, is a country of (0), where is $P_{2,2}$, is a country of (0), where is $P_{2,2}$, is a country of (0).

A State

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$$\lim_{z \downarrow y} \log F(z) = \lim_{z \downarrow y} \sum_{j=0}^{\infty} \log G(z+jd)$$
$$= \sum_{j=0}^{\infty} \lim_{z \downarrow y} \log G(z+jd)$$
$$= \sum_{j=0}^{\infty} \log G(y+jd) = \log F(y)$$

Furthermore $F(y) \leq G(y)$ so that $F(y) \rightarrow 0$ as $y \rightarrow -\infty$. Therefore F is a distribution function if and only if $\lim_{y \rightarrow +\infty} F(y) = \lim_{y \rightarrow +\infty} \prod_{j=0}^{\infty} G(y+jd) = 1.$

Now

$$F(y) = \prod_{j=0}^{\infty} G(y+jd)$$
$$= \prod_{j=0}^{\infty} (1 - P[U_1 > y + jd])$$
$$\ge 1 - \sum_{j=0}^{\infty} P[U_1 > y + jd]$$

and

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$$\sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{i=1}^{2} \sum_{j=1}^{2} (i(z+i))$$

x = x + x + (x) + (x)

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$$\begin{split} \sum_{j=0}^{\infty} P[U_{1} > y + jd] &= \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} P[kd < U_{1} - y \le (k+1)d] \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} P[kd < U_{1} - y \le (k+1)d] \\ &= \sum_{k=0}^{\infty} (k+1)P[kd < U_{1} - y \le (k+1)d] \\ &= \frac{1}{d} \sum_{k=0}^{\infty} kd P[kd < U_{1} - y \le (k+1)d] + P[U_{1} > y] \\ &\le \frac{1}{d} E((U_{1} - y)^{+}) + P[U_{1} > y] . \end{split}$$

When $E(U_1^+) < \infty$, the last expression tends to 0 as $y \rightarrow +\infty$, thus proving (i) of the theorem.

The proof of (ii) is similar.

$$0 \leq F(y) = \prod_{j=0}^{\infty} (1 - P[U_1 > y + jd])$$

$$\leq \exp\left\{-\sum_{j=0}^{\infty} \mathbb{P}[\mathbb{U}_{1} > y + jd]\right\}$$

where we have used the fact that $1 - t \leq \exp\{-t\}$ for all real t .

As in the first part of the proof we find

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 $\leq \frac{1}{6} \mathbb{E}\left(\left(\frac{1}{2}, -\frac{1}{2}\right)\right) = \frac{1}{2} \mathbb{E}\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}$

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The proof of (21) is front our

where we have a sold whe shap which $f = f \in Sin (-f)$ for all graph f .

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$$\sum_{j=0}^{\infty} P[U_1 > y + jd] = \frac{1}{d} \sum_{k=0}^{\infty} (k+1)d P[kd < U_1 - y \le (k+1)d]$$
$$\ge \frac{1}{d} E((U_1 - y)^+) = +\infty$$

when $E(U_1^+) = +\infty$. Thus $F(y) \equiv 0$ and the proof is complete. Assume from now on that $E(U_1^+)$ is finite. The following proposition shows that when we have an initial distribution $F_o(y) = F(y)$, all the variables X_n have the same distribution F(y). This may be deduced from more general theorems on Markov process, but the direct verification in the present case is simple.

Proposition.

When $E(U_1^+) < +\infty$, F(y) is a stationary distribution in the sense that

(7)
$$F(y) = \int_{-\infty}^{\infty} P(x;y) dF(x)$$

Proof.

The right-hand side of (7) is

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$$\sum_{j=0}^{\infty} P(U_{1} > y_{2} + jd) = \frac{1}{2} \sum_{j=0}^{\infty} (j+1)d P[kd < U_{1} - y_{1} \le (k+1)d]$$

when $E(0^{+}) = + \cdots + E(0^{+}) = 0$, and the proof is comblete. Assume from now on that $E(0^{+}) = 18$ (inite. The following

proposition shows that when we have an initial distribution $F_{(x)} = F(x)$ all the variables X_{n} bave the same directed bution F(x) which may be deduced from more general bicorement on Markov process, but the direct wanted and the threatent case is simple

When $\mathbb{E}(\mathbb{Q}^{+}) \leq \pm \infty : \mathbb{P}(\mathbb{P})$ is a particularly distribution with the sense that

$$\mathbb{E}\left(x\right) = \int_{-\infty}^{\infty} \mathbb{E}\left(x_{1}(x)\right) \, \mathrm{d}\mathbb{E}\left(x\right)$$

Proof.

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$$\int_{-\infty}^{\infty} P(x;y) dF(x) = \int_{-\infty}^{\infty} G(y)I(y+d-x) dF(x)$$
$$= G(y) \int_{-\infty}^{y+d} dF(x)$$
$$= G(y) F(y+d)$$
$$= G(y) \int_{j=0}^{\infty} G(y+d+jd)$$
$$= \int_{j=0}^{\infty} G(y+jd)$$

= F(y)

Example.

Usually it is difficult to get an explicit expression for F(y) from (5). However, the following example is easy to handle.

(8)
$$G(x) = \exp\{-a e^{-bx}\}, a, b > 0, -\infty < x < \infty$$

The probability density for U_1 is

(9)
$$g(x) = G'(x) = ab \exp \{-bx - a e^{-bx}\}$$

For later reference we need the expectation and variance of U_1 .

$$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2},$$

$$= \frac{\nabla \left(\left(\mathbf{v} \right) \right)}{\beta = 0} \quad \mathcal{C} \left(\left(\left(+ 0 + \frac{1}{2} \right) \right)$$
$$= \int_{\left\{ - 0 \right\}}^{\infty} \mathcal{C} \left(\left(\mathbf{v} + \frac{1}{2} \right) \right)$$

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The probability density for ofore U., is

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$$E(U_1) = ab \int_{-\infty}^{\infty} x \exp \{-bx - a e^{-bx}\} dx$$
$$= -\frac{1}{b} \int_{0}^{\infty} \ln \left(\frac{z}{a}\right) e^{-z} dz$$
$$= \frac{1}{b} (\ln a + C)$$

where C is the Euler-Mascheroni constant

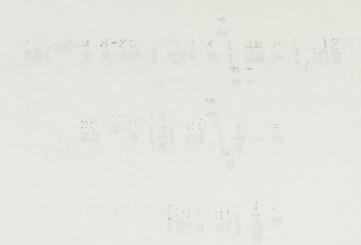
$$C = \lim_{m \to \infty} \left(\sum_{n=1}^{m} \frac{1}{n} - \ln m \right) = -\int_{0}^{\infty} \ln z e^{-z} dz \approx 0,5772 .$$

We have made the change of variable $x \rightarrow z = a e^{-bx}$. Similarly we find

$$E(U_{1}^{2}) = ab \int_{-\infty}^{\infty} x^{2} \exp \{-bx - a e^{-bx}\} dx$$
$$= \frac{1}{b^{2}} \int_{0}^{\infty} (\ln z - \ln a)^{2} e^{-z} dz$$
$$= \frac{1}{b^{2}} (\beta + 2C \ln a + (\ln a)^{2})$$

where

$$\beta = \int_{0}^{\infty} (\ln z)^{2} e^{-z} dz = \frac{\pi^{2}}{6} + c^{2} .$$



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We have made the change of variable $x \rightarrow z = 3$ when the state of variable $x \rightarrow z = 3$ when the state is the

 $= \frac{1}{r_2} \left(e^{-2\alpha} \ln a + (\ln a)^2 \right)$

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The relevant integrals may be found in any large table, for istance Erdélyi et al.(1954). The variance of U_1 is

$$\sigma^2 = Var(U_1) = \frac{1}{b^2}(\beta - C^2) = \frac{\pi^2}{6b^2}$$

which is independent of a .

The stationary distribution of $\{X_n\}$ is found from (5) and (8)

$$F(\mathbf{x}) = \prod_{k=0}^{\infty} \exp\left\{-a \ e^{-b(\mathbf{x}+kd)}\right\}$$

(10)
$$= \exp\left\{-a \ e^{-bx} \ \sum_{k=0}^{\infty} (e^{-bd})^k\right\}$$

$$= \exp\left\{\frac{-a e^{-bx}}{1 - e^{-bd}}\right\} = \exp\left\{-a_1 e^{-bx}\right\}$$

where $a_1 = a(1-e^{-bd})^{-1}$. Thus under stationary conditions

$$E(X_{n}) = \frac{1}{b} (\ln a_{1}-c) = \frac{1}{b} (\ln a-c-\ln(1-e^{-bd}))$$

> $E(U_{n})$
Var(X_n) = $\frac{\pi^{2}}{6b^{2}} = Var(U_{n})$

The stationary probability density
$$f(x)$$
 for $\{X_n\}$
is given by an expression similar to (9). In fig.1 and

The relevant integrals may be found in any large table, for istance Erdélyi et al. (1954). The variance of U, is

$$\sigma^{2} = \operatorname{Var}(v_{1}) = \frac{1}{b^{2}} (\beta - c^{2}) = \frac{\pi^{2}}{6b^{2}}$$

which is independent of a .

The stationary distribution of (X_n) is found from (5) and (8)

$$(10) = exp \left\{ -a e^{-Dx} \right\} \left\{ e^{-Dd} \right\}^{n} \left\{ e^{-Dd} \left\{ e^{-Dd} \right\}^{n} \left\{ e^{-Dd} \left\{ e^{-Dd} \right\}^{n} \left\{ e^{-Dd} \left\{$$

Thus under stationary conditions

$$B(X_{1}) = \frac{1}{5} (2n \ a - 0) = \frac{1}{5} (2n \ a - 0 - 2n (1 - 0^{-bd}))$$

$$B(X_{1}) = \frac{1}{5} (2n \ a - 0) = \frac{$$

The stationary probability density f(x) for $\{x_n\}$ to $\frac{\partial \theta}{\partial y}$ in expression similar to (9). In fig. 1 and ... by an expression similar to (9). In fig. 1 and

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fig.2 f and g are drawn for the two cases $\frac{\sigma}{d} = 1$ and $\frac{\sigma}{d} = 2$, a = 1 in both cases.

4. The exchanges.

From the model (1) we see that the process $\{X_n\}$ (density of resident water at time n) developes as follows

	X _o	
	$X_1 = X_0 - d$	(≥ U ₁)
	$X_2 = X_0 - 2d$	(≥ U ₂)
(11)		
	$X_{N_1-1} = X_0 - (N_1-1)d$	$(\geq U_{N_1-1})$
	$X_{N_1} = U_{N_1}$	(> X ₀ -N ₁ d)
	$X_{N_{1}+1} = U_{N_{1}} - d$	(≥U _{N1+1})

where N_1 is the year of the first exchange. Assume that the exchanges take place in the years N_1 , $N_1 + N_2$, $N_1 + N_2 + N_3$ and so on. We then have to study the associated process:

$\{N_k; k = 1, 2,\}$	(number of years between
	successive exchanges)
$\{S_k ; k = 1, 2,\}$	with $S_k = U_{N_1+N_2+\cdots+N_k}$
	(the density of water at the
	k'th exchange)

 $\frac{1}{2} = 2 \quad \text{and} \quad \gamma \quad \text{are draw for the set game (1.5.15)}$

The exchanges.

Arom the model (1) :* see that the process (2) (*6.1311 - dent - at the process (2) density of resident water at the process of feits and



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These two processes were also studied in Gade (1973). We shall attack the problem by different means, which give explicit formulae for the relevant stationary distributions.

First of all we want to find the probability of exchange in year n, given nothing but the initial density distribution (that is the distribution of X_0). This probability is

$$P[A_n] = P[U_n > X_{n-1} - d]$$

(12)
$$= P[X_{n-1} < U_n]$$

$$= \int_{-\infty}^{\infty} F_{n-1}^{(-)} (u+d) dG(u)$$

+ d]

where F_{n-1} is the distribution function of X_{n-1} and

$$F_{n-1}^{(-)}(x) = \lim_{h \downarrow 0} F_{n-1}(x-h)$$
.

We have utilized the independence of X_{n-1} and U_n .

From now on assume that $E(U_1^+) < \infty$ and that G is continuous. Then from (5) the stationary distribution function F is continuous and

$$\lim_{n\to\infty} F_{n-1}^{(-)}(x) = F(x) \quad \text{for all } x.$$

Therefore from (12)

These two processes with alloc studied in Gade (1973): We shall attack the problem by different means, which give explicit formulae for the relevant stationary distributions: First of all we want to find the probability of effe

change in year n, given nothing but the initial densities distribution (that is the distribution of X_0). This proof is the distribution of X_0). This proof

$$(\hat{c} + \alpha) \begin{pmatrix} \hat{c} \\ \hat{c} + \alpha \end{pmatrix} \begin{pmatrix} \hat{c} \\ \hat{c} \end{pmatrix} = \begin{pmatrix} \alpha \\ \hat{c} \end{pmatrix}$$

where First is the distribution duby the the of Xain and

$$\mathbb{P}_{n-1}^{(n)}(x) = \lim_{\substack{n \to 1 \\ n \to 1}} \mathbb{P}_{n-1}(x-n)$$

$$-x I(f = 10^{-1}) = (x) = (x) = 10^{-1}$$

We have been and

(13)
$$\pi_{A} = \lim_{n \to \infty} P[A_{n}] = \int_{-\infty}^{\infty} F(u+d) dG(u)$$

 π_A is the stationary probability of exchange in a given year.

Now turn to the density of water at exchange. It is easy to see that $\{S_k ; k = 1, 2, ...\}$ is a Markov process, but the transition probability function is cumbersome to handle (see Gade (1973), formula (8) which gives the derivative of this function). Therefore we will find the stationary distribution function T(x) of $\{S_k\}$ directly. Assume that stationary conditions have been reached, that is, all the variables X_n , n = 1, 2, ... have the same distribution F(x). Then

$$T(x) = P[S_k \leq x]$$

(14)

 $= P[X_n \leq x | A_n]$

$$= \frac{P[(X_n \le x) \cap (U_n > X_{n-1} - d)]}{P[A_n]}$$

$$= \pi_{A}^{-1} P[(U_{n} \le x) \cap (X_{n-1} < U_{n} + d)]$$
$$= \pi_{A}^{-1} \int_{-\infty}^{x} F(u+d) dG(u)$$

where again F is continuous and we have used the independence of X_{n-1} and U_n . By combining (5), (13) and (14)

(13) $\pi_{A'} = \lim_{n \to \infty} \mathbb{P}[\lambda_n]_{n} = \int \mathbb{P}(u+d) d\theta(u)$

mi is the stationary probability of exchange Inia gives years

where we show the formula is the standard of the terms is the formula (1973) where (1973), formula (8) without it is the second of the formula (1973), formula (8) without it is the second of the formula (1973), or (1973), formula (1973), so that is the second of th

2⁷(.) = 7[8, e =]

 $\frac{1(6+1+2) + (0) + (0) + (0)}{1+1+1} = (0)$

(a+, v > , _, x) n (x = +0) | i | i =

we can give T(x) directly in terms of the given distribution G(x):

(15)
$$T(x) = \frac{\int_{-\infty}^{x} \left\{ \prod_{j=1}^{\infty} G(u+jd) \right\} dG(u)}{\int_{-\infty}^{\infty} \left\{ \prod_{j=1}^{\infty} G(u+jd) \right\} dG(u)}$$

When T(x) is known, we can also find the probability distribution of the number of years between two successive exchanges. First assume that $X_0 = y$ is constant. Then we find by simple inspection of the scheme (11) :

$$P[N_1 = n | X_0 = y] = Q(n; y)$$

(16) = $P\left[U_1 \le y-d, U_2 \le y-2d, \dots, U_{n-1} \le y-(n-1)d, U_n > y-nd\right]$

=
$$\{1 - G(y-nd)\} \prod_{j=1}^{n-1} G(y-jd)$$

This formula is also given by Gade (1973) (formula(6)), and it is valid whether or not an exchange takes place in the year 0 .

Under stationary conditions the probability that two consecutive exchanges are separated by n years will be we can give T(x) directly in terms of the given distribution we can give T(x) due un tribute of the given distribupublich g(x)

$$P[N_1 = n|X_0 = y] = Q(n;y)$$

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. .

$$Q(n) = \int_{-\infty}^{\infty} Q(n;y) dT(y)$$

$$= \pi_{A}^{-1} \int_{-\infty}^{\infty} \left\{ 1 - G(y-nd) \right\} \left\{ \prod_{j=1}^{n-1} G(y-jd) \right\} F(y+d) dG(y)$$

$$= \pi_{A}^{-1} \int_{-\infty}^{\infty} \left\{ \frac{1}{G(y)} \prod_{j=-n+1}^{\infty} G(y+jd) - \frac{1}{G(y)} \prod_{j=-n}^{\infty} G(y+jd) \right\} dG(y)$$

$$= \pi_{A}^{-1} \int_{-\infty}^{\infty} \left\{ F(y-nd+d) - F(y-nd) \right\} \frac{dG(y)}{G(y)}$$

from (16) and (5).

(17)

Alternatively

(18)
$$P\left[N_{k} \ge n\right] = \sum_{m=n}^{\infty} Q(m) = \pi_{A}^{-1} \int_{-\infty}^{\infty} F(y-nd+d) \frac{dG(y)}{G(y)}$$

The (stationary) expectation of N_k is

$$\overline{n} = \sum_{n=1}^{\infty} n Q(n) = \sum_{n=1}^{\infty} P[N_k \ge n]$$

By a change of variable and partial integration in (18) we find

- 18 -

Trom (16) and (5).

A standard vare ly

$$\left[\frac{v_{1}}{2} \frac{\partial G}{\partial t} + \frac{\partial G}{\partial t} \right] = \frac{1}{2} \left[\frac{1}{2} + \frac{1}{2} \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial G}{\partial t$$

The (statomnary) expectation of N. 18

$$\overline{T} = = \sum_{n=1}^{\infty} \overline{T}_{n} Q(n) = \sum_{n=1}^{\infty} P[N_{n} \ge n]$$

""B? al Shango of wartable and gartial integration in (18)

$$P[N_{k} \ge n] = \pi_{A}^{-1} \int_{-\infty}^{\infty} F(y+d) d\{\ln G(y+nd)\}$$
$$= -\pi_{A}^{-1} \int_{-\infty}^{\infty} \ln G(y+nd) dF(y+d)$$

Therefore

$$\overline{n} = -\pi_{A}^{-1} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \ln G(y+nd) dF(y+d)$$
$$= -\pi_{A}^{-1} \int_{-\infty}^{\infty} \ln F(y+d) dF(y+d)$$
$$= -\pi_{A}^{-1} \int_{0}^{1} \ln x dx = \pi_{A}^{-1}$$

or by (13)

(19)
$$\overline{n} = \left(\int_{-\infty}^{\infty} F(u+d) \ dG(u)\right)^{-1}$$

The result $\overline{n} \pi_A = 1$ is also valid in the general model with random decrements. This was indicated in Helland (1973) and will be proved rigorously in a forthcoming paper.

By methods similar to those used above, we could find other quantities of interest, e.g. the (stationary) variance of N_k , the correlation between N_k and N_j (k \ddagger j), the probability distribution of the increase of the density of water by exchange and so on.

Therefore

or- by (13)

Example (continued).

When G(x) is given by (8) and F(x) by (10) we find by straightforward integration

(20)
$$\int_{-\infty}^{x} F(u+d) \, dG(u) = \int_{-\infty}^{x} \exp\left\{\frac{-a e^{-bu}}{1-e^{-bd}}\right\} \cdot ab e^{-bu} \, du$$
$$= \left(1 - e^{-bd}\right) \exp\left\{\frac{-a e^{-bx}}{1-e^{-bx}}\right\}$$

$$= \left(1 - e^{-bd}\right) \exp\left\{\frac{-a e^{-bx}}{1 - e^{-bd}}\right\}$$

Therefore from (13)

(21)
$$\pi_{A} = 1 - e^{-bd}$$

and from (19)

(22)
$$\overline{n} = \left(1 - e^{-bd}\right)^{-1} = \left(1 - \exp\left\{-\frac{\pi d}{\sigma\sqrt{6}}\right\}\right)^{-1}$$

where σ^2 is the variance of U_1 . In fig.3 \overline{n} is drawn as a function of $\frac{\sigma}{d}$.

From (14), (20) and (21) we get

$$T(x) = \exp\left\{\frac{-a e^{-bx}}{1 - e^{-bd}}\right\} = F(x)$$

by (10). That is, $\{X_n\}$ and $\{S_k\}$ have the same stationary distribution. In fact one can show that this property nearly characterizes the special distribution (8). More precisely,

Example (Solitinued).

when bits bits in given by (8) and Bits, (10) we that by he by etfolgheronward threegeletion

$$ub^{-\frac{1}{2}} = (u+d)^{-\frac{1}{2}} = (u+d)^{-\frac{1}{2$$

$$\left\{ \frac{20 - 2}{20 + 2} \right\} \operatorname{oko}^{1} \left(\frac{602}{2} - \frac{1}{2} \right) =$$

Therefore tron (13)

$$e^{-bd^2 - i} = i = A^2 \qquad (is)^2$$

and from (19)

$$= \left(1 - e^{\frac{1}{2}}\right)^{-1} = \left(1 - e^{\frac{1}{2}}\right)^{-1} = \left(1 - e^{\frac{1}{2}}\right)^{-1} = \frac{1}{2} \left(1 - e^{\frac{1}{2}}\right)^{-1} =$$

where o² the the vehicles of U, In Mg.3 in the the way

From ((14), (22)) and (21) we get

$$(\mathbf{x}) \mathbf{x}^{\mathrm{f}} = \left\{ \frac{\mathbf{x}^{\mathrm{f}^{\mathrm{f}}}}{\mathbf{x}^{\mathrm{f}^{\mathrm{f}}} - \mathbf{x}^{\mathrm{f}^{\mathrm{f}}}} \right\} \mathbf{x}^{\mathrm{f}^{\mathrm{f}}} = (\mathbf{x}) \mathbf{x}^{\mathrm{f}^{\mathrm{f}}}$$

ing (193) "Trinating (144) fond (134) Addre Huerdane Statilikary "Striffeligen, 4th Basic one Ban show Thist CHAT Persenenty nearly "DRA Stiffelige one books the show Thist CHAT Persenenty nearly if G(x) is such that T(x) = F(x), then either U₁ is concentrated on an interval of length less than d (in which case $\pi_A = 1$) or

$$G(x) = \exp \left\{ -a(x) e^{-bx} \right\}$$

where a(x) is periodic with periode d and such that G(x) is monotonic and continuous from the right. The proof is straightforward, but cumbersome, and will not be given.

The stationary distribution of $\{N_k\}$ is also easily found in this case. By (18), (8), (10) and (21) we find

$$P[N_{k} \ge n] = \pi_{A}^{-1} \int_{-\infty}^{\infty} F(y - (n-1)d) \frac{dG(y)}{G(y)}$$
$$= \pi_{A}^{-1} \int_{-\infty}^{\infty} exp \left\{ \frac{-a e^{-by} e^{(n-1)bd}}{1 - e^{-bd}} \right\} \text{ ab } e^{-by}dy$$
$$= e^{-(n-1)bd}$$

Thus

(23)
$$Q(n) = P[N_k = n] = (e^{bd} - 1) e^{-nbd}$$
, $n = 1, 2, ...$

That is, under stationary conditions N_k is geometrically distributed. From this we find again (22) and

(24)
$$Var(N_k) = \frac{e^{-bd}}{(1 - e^{-bd})^2}$$

and the second of the second of the second of

If $G(x)_{r,o}$ is such that T(x) = F(x), shen either G_{1}^{r} is concentrated on an interval of length less than G (in which

$$f(x) = exp \left\{ -a(x) e^{-bx} \right\}$$

 $\frac{1}{1+1} = \frac{1}{1+1} = \frac{1}$

 $2(n + n) = 2(1 + n) = (n^{1/2} + 1) = (n^{1/$

$$\operatorname{Var}(N_{R}) = \frac{1}{(1 - e^{-bd})^{2}}$$

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For other distributions than (8), the evaluation of F(x), T(x), π_A etc. is not so simple. Take for instance U_1 to be normally distributed. Without lack of generality we can take the mean of U_1 to be zero (translation off all the variables U_n , X_n , S_k will not alter the model). Then

(25)
$$G(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{t^2}{2\sigma^2}} dt = \Phi(\frac{x}{\sigma}),$$

where σ^2 is the variance of U_1 , Φ is the standard normal distribution. From (5) we find

(26)
$$F(x) = \prod_{k=0}^{\infty} \Phi\left(\frac{x}{\sigma} + k \frac{d}{\sigma}\right) .$$

(13) gives

(27)
$$\pi_{A} = \int_{-\infty}^{\infty} F(y+d)g(y)dy = \int_{-\infty}^{\infty} \left\{ \prod_{k=1}^{\infty} \Phi(z+k\frac{d}{\sigma}) \right\} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^{2}}{2}} dz$$

(14) gives

(28)
$$T(x) = \pi_{A}^{-1} \int_{-\infty}^{x} F(u+d)g(y)dy = \pi_{A}^{-1} \int_{-\infty}^{\infty} \left\{ \prod_{k=1}^{\infty} \Phi(z+k \ \frac{d}{\sigma}) \right\} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^{2}}{2}} dz$$

Finally, the corresponding probability densities f(x) = F'(x)and t(x) = T'(x) are

(29)
$$f(x) = \sum_{k=0}^{\infty} \frac{g(x+kd)}{G(x+kd)} F(x) = F(x) \sum_{k=0}^{\infty} e^{-\frac{(x+kd)^2}{2\sigma^2}} \left[\sqrt{2\pi\sigma} \Phi(\frac{x}{\sigma}+k\frac{d}{\sigma})\right]^{-1}$$

$$\left(\begin{pmatrix} c \\ c \end{pmatrix}\right) = \frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}} =$$

where M^{-1} is the variance of W_{1} , b is should nor-

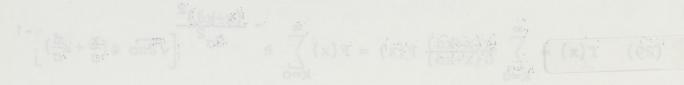
$$(38) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) \left(\frac{1}{2} + \frac{1}{2} \right) \left(\frac{1}{2} + \frac{1}{2} \right) \left(\frac{1}{2} + \frac{1}{2} \right)$$

$$\hat{e}_{i} = \int_{i}^{\infty} \frac{1}{2} \left\{ e_{i}^{2} + e_{i}^{2$$

aavia (tt)

$$\left(\hat{2}\hat{8} \right) \quad \hat{\Psi}(x) = \bar{\Psi}^{1}_{a} \left[\int_{-\infty}^{\infty} \left[(u+d)a(v) \partial v = \bar{\Psi}^{1}_{a} \right] \int_{-\infty}^{\infty} \left\{ \left[\int_{-\infty}^{0} u(u+x - b) \right] \int_{-\infty}^{\infty} e^{-\frac{2}{2}} dx \right]$$

Finally, the privesponding thought U_{i} sensities r(x) = F(x)and f(x) = F(x) at a



(30)
$$t(x) = \pi_A^{-1} F(x+d)g(x) = \frac{1}{\pi_A \sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}} \prod_{k=1}^{\infty} \Phi(\frac{x}{\sigma} + k\frac{d}{\sigma})$$

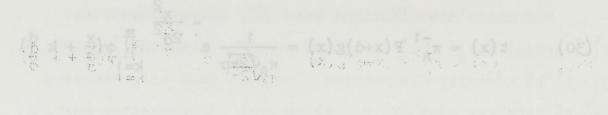
The last two expressions have been evaluated numerically for $\frac{\sigma}{d} = 1,2$ and 4 and the corresponding curves are shown in fig.4,5 and 6.

5. Asymptotic expansions.

This section is matematically a little more involved than the previous ones. However, it is simple to pose the problem: Is it possible to find simpler, approximate expressions for quantities characterizing the model when d is small? Physically we should expect the following when $d \rightarrow 0$. As the annual decrease of the density of the water becomes small, the expected number of years between successive exchanges should increase. Hence the probability of exchange in a given year should decrease. Finally, exchanges take place only when the densit; of the coastal water is extremely high, therefore the expected density of the exchanged water will increase.

First of all we must make precise what we mean by small d. The model is in general determined by d and the distribution function G(x) of U_1 , hence small d should

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5. Asymptotic expansions.

A. State of all we must make precise what we may be mailed by attractive of an in a state of a state of

mean that d is much less than some parameter characterizing the distribution G(x). It is easy to see that $E(U_1)$ is irrelevant in this connection $(E(U_1) \text{ may be changed arbitrarily by translating all the variables <math>U_n$, X_n etc). Usually by small d we will mean

$$\frac{\mathrm{d}}{\mathrm{\sigma}} \ll 1$$
 ,

where $\sigma^2 = \text{Var}(U_1)$. Note, however, that the asymptotic expressions which we are going to develop are not simply functions of d/σ , but depend on the detailed form of G(x).

Assume that $E(U_1^+)$ is finite and that G(x) is absolute continuous with probability density g(x). Then (see (5), (13) and (19))

(31)
$$F(x) = \prod_{k=0}^{\infty} G(x+kd)$$

(32)
$$\pi_{A} = (\overline{n})^{-1} = \int_{-\infty}^{\infty} \left\{ \prod_{k=1}^{\infty} G(x+kd) \right\} g(x) dx$$

The infinite products occuring here are difficult to handle analytically. By taking logarithm these may be expressed by means of infinite sums, which in turn may be approximated by integrals when d is small. In this way we get the following

 $\frac{1}{1} = \frac{1}{1} = \frac{1}$

Theorem 2.

Let $E(U_1^+) < \infty$ and let G(x) have continuous second derivative in $(-\infty,\infty)$. Assume that the density $g(y) \rightarrow 0$ as $y \rightarrow \infty$. Let $a \in [-\infty,\infty)$ be such that G(x) = 0 when $x \leq a$, G(x) > 0 when x > a (eventually $a = -\infty$). Then as $d \rightarrow 0$

(33)
$$F(x) = \sqrt{G(x)} \exp \left\{ d^{-1} \int_{x}^{\infty} [\ln G(y)] dy \right\} (1 + O(d))$$

uniformly in x when $x \ge c$ for all c > a, and

(34)
$$\pi_{A} = \int_{a}^{\infty} (G(x))^{-\frac{1}{2}} \exp\left\{d^{-1}\int_{x}^{\infty} [\ln G(y)] dy\right\} g(x) dx (1+0(d))$$

Proof.

From (31) F(x) = 0 when $x \le a$. When x > a

(35)
$$F(x) = \exp \left\{ \sum_{k=0}^{\infty} \ln G(x+kd) \right\}$$

and the sum converges because $E(U_1^+) < \infty$ (cf. Theorem 1). First we have to show

(36)
$$\sum_{k=0}^{\infty} \ln G(x+kd) = d^{-1} \int_{x}^{\infty} [\ln G(y)] dy + \frac{1}{2} \ln G(x) + O(d)$$

uniformly in x $(x \ge c)$ as $d \rightarrow 0$. To this end use the

S. 11. 10.31.1.

Let E(T) > c and H(T) < c and h(T) > c and

$$(33) \qquad F(x) = \sqrt{3}(x) \exp\left\{a^{-1} \int_{x}^{\infty} (\ln G(y)) \, dy\right\} (1 + O(0))$$

uniformity is x when x a o for all c > a . and

(34)
$$\pi_{A} = \int (0(x))^{-\frac{1}{2}} \exp \left\{ a^{-\frac{1}{2}} \int (2\pi - G(y)) dy \right\} g(x) dx (449(d))$$

10019

From (31) If (a) = 0 when it a a limen x > a

$$I(x) = \exp\left\{\frac{\sum_{i=0}^{\infty} 1 h(i)}{\sum_{i=0}^{\infty} 1 h(i)}\right\}$$

and the sum converges because (((()) < a ((cr. Theorem)) - c

$$\frac{1}{286} \sum_{x=1}^{N} \frac{1}{286} \sum_{x=1}^{N} \frac{1}{286} = \frac{1}{26} \sum_{x=1}^{N} \frac{1}{286} \sum_{x=1}^{N} \frac{1}{286}$$

trapezoidal formula

 $\int_{x+kd} \left[\ln G(y) \right] dy = \frac{d}{2} \left[\ln G(x+kd) + \ln G(x+(k+1)d) \right] - x+kd$

$$\frac{d^3}{12} \left[\frac{d^2}{dy^2} \ln G(y) \right]_{y=\eta_k}$$

where $x + kd < \eta_k < x + (k+1)d$. This may be shown by approximating the area corresponding to the integral by the area of a trapezoid and estimating the error. (Cf. e.g. Conte (1965), p.122). Now put

$$\varphi(y) = \frac{d}{dy} \ln G(y) = \frac{g(y)}{G(y)} \qquad (y \ge c) .$$

Then the trapezoidal formula may be written

(37)
$$\frac{1}{2} \ln G(x+kd) + \frac{1}{2} \ln G(x + (k+1)d) = d^{-1} \int_{x+kd}^{x+(k+1)d} [\ln G(y)] dy + \frac{1}{2} \ln G(x+kd) + \frac{1}{2} \ln G(x+kd) = d^{-1} \int_{x+kd}^{x+kd} [\ln G(y)] dy + \frac{1}{2} \ln G(x+kd) + \frac{1}{2} \ln G(x+kd) = d^{-1} \int_{x+kd}^{x+kd} [\ln G(y)] dy + \frac{1}{2} \ln G(x+kd) + \frac{1}{2} \ln G(x+kd) = d^{-1} \int_{x+kd}^{x+kd} [\ln G(y)] dy + \frac{1}{2} \ln G(x+kd) + \frac{1}{2} \ln G(x+kd) = d^{-1} \int_{x+kd}^{x+kd} [\ln G(y)] dy + \frac{1}{2} \ln G(x+kd) + \frac{1}{2} \ln G(x+kd) = d^{-1} \int_{x+kd}^{x+kd} [\ln G(y)] dy + \frac{1}{2} \ln G(x+kd) + \frac{1}{2} \ln G(x+kd) = d^{-1} \int_{x+kd}^{x+kd} [\ln G(y)] dy + \frac{1}{2} \ln G(x+kd) + \frac{1}{2} \ln G(x+kd) = d^{-1} \int_{x+kd}^{x+kd} [\ln G(y)] dy + \frac{1}{2} \ln G(x+kd) + \frac{1}{2} \ln G(x+kd) = d^{-1} \int_{x+kd}^{x+kd} [\ln G(y)] dy + \frac{1}{2} \ln G(x+kd) + \frac{1}{2} \ln G(x+kd) + \frac{1}{2} \ln G(x+kd) = d^{-1} \int_{x+kd}^{x+kd} [\ln G(x+kd)] dy + \frac{1}{2} \ln G(x+kd) + \frac{1}{2} \ln G(x+kd$$

$$+\frac{d^2}{12} \varphi'(\eta_k)$$

Since $g(y) \rightarrow 0$ as $y \rightarrow \infty$, also $\varphi(y) \rightarrow 0$ as $y \rightarrow \infty$ and $\varphi(y)$ will be uniformly bounded for $y \in [c,\infty)$. Therefore

$$d \sum_{k=0}^{\infty} \varphi'(\eta_k) = \int_{x}^{\infty} \varphi'(y) dy + o(1) = -\varphi(x) + o(1) = 0(1)$$

uniformly in x $(x \ge c)$. By adding the equations (37)

trapezoidal formula

where $x \pm kd < \eta_k < x + (k \pm 1)d$. This may be shown by approximating the area corresponding to the integral by the area of a trapezoid and equipating the error (cf. g.g. conte (1965), p.122). New put

 $-\left(x \in x\right) = \left\{\frac{x}{2}\right\} = \left(x\right) + \left(x \in x\right) + \left(x \in$

Then the trapezoidal formula may be written

 $(37) = 4^{-1}$ $(37) = 4^{-1}$ $(37) = 4^{-1}$ $(37) = 4^{-1}$ $(37) = 4^{-1}$ $(37) = 4^{-1}$

Since $f(y) \ge 0$ as $y \ge y$, $z_{1} = z_{1} \ge 0$, $f(y) \ge 0$ as $y \ge z_{2}$ and $\phi(y)$ will be uniformly bounded for $y \in [0,\infty)$. Therefore

$$a \sum_{k=0}^{\infty} \varphi^{1}(\eta_{k}) = \int_{0}^{\infty} \varphi^{1}(x) dx \pm \varphi(x) = - \varphi(x) \pm \varphi(x) \pm \varphi(x) = 0(x)$$

mailoraly in x (x = o) > By adding the equations (37)

for $k = 0, 1, 2, \dots$, (36) follows.

Put

$$r(x) = -\sum_{k=0}^{\infty} \ln G(x+kd)$$

$$s(x) = -d^{-1} \int_{x}^{\infty} [\ln G(y)] dy - \frac{1}{2} \ln G(x)$$

Then r and s are nonnegative and from (36)

$$\sup_{x \ge c} |r(x) - s(x)| = O(d)$$
.

Now

$$\sup_{x \ge c} \left| \frac{F(x) - \exp(-s(x))}{F(x)} \right|$$

$$= \sup_{x \ge c} \left| \frac{\exp(-r(x)) - \exp(-s(x))}{\exp(-r(x))} \right|$$

$$= \sup_{x \ge c} \left| 1 - \exp(r(x) - s(x)) \right|$$

$$\leq \max\left\{ 1 - \exp(-\sup_{x \ge c} |r(x) - s(x)|), \exp(\sup_{x \ge c} |r(x) - s(x)|) - 1 \right\}$$

$$= 0(d)$$

and this proves (33).

Finally (34) is proved by noting that

$$\pi_{A} = \int_{a}^{\infty} \frac{F(x)}{G(x)} g(x) dx = \int_{a}^{c} \frac{F(x)}{G(x)} g(x) dx + \int_{c}^{\infty} \frac{F(x)}{G(x)} g(x) dx$$

Since from (31) $0 \leq \frac{F(x)}{G(x)} \leq 1$, we can make the first integral as small we wish by taking c small enough. In the second integral we can replace F(x) by the uniform asymptotic expression (33). The proof of the theorem is completed by letting $c \rightarrow a$.

To analyse (34) further, we want to change the integration variable from x to

(38)
$$z = -\int_{x}^{\infty} [\ln G(y)] dy$$
, $(x > a)$

When x grows from a to ∞ , z will decrease monotonically from

(39)
$$h = -\int_{a}^{\infty} [\ln G(y)]dy$$

to zero. Let $b \in (a,\infty]$ be such that G(x) = 1 when $x \ge b$, G(x) < 1 when x < b (eventually $b = \infty$). Then the upper integration limits ∞ in (33), (34), (38) and (39) may be replaced by b. When a < x < b, $\ln G(y) < 0$ and the transformation (38) is one-to-one. The inverted transformation is

(40)
$$x = \psi(z)$$
, $0 < z < h$,

where ψ is differentiable. Straightforward change of variable

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When x grows from a to . z will decrease monotonically

$$(1(25))) = \frac{1}{2} \int_{0}^{\infty} \left[\frac{1}{2} + \frac{$$

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$$(40)^{(1)} = x = \psi(\vec{z})^{(1)}, \quad 0 < z < h$$

where V is differentiable. Straightforward change of variable

in (34) gives

(41)
$$\pi_{A} = -\int_{0}^{h} \frac{g(\psi(z))}{\sqrt{G(\psi(z))} \ln G(\psi(z))} e^{-d^{-1}z} dz(1+0(d))$$

By combining this with a well known result from the theory of asymptotic expansions we get the following

Theorem 3.

Let G(x) and g(x) satisfy the conditions in theorem 2. Let ψ be given by (38) and (40). Suppose that we can find an asymptotic expansion

(42)
$$-\frac{g(\psi(z))}{G(\psi(z)) \ln G(\psi(z))} \sim \sum_{n} a_{n} \Phi_{n}(z) , z \to 0$$

and that each Φ_n has a Laplace transform

$$\mathbf{r}_{n}(t) = \int_{0}^{\infty} \Phi_{n}(z) e^{-tz} dz , \quad t \ge t_{0}$$

for some $t_0 \ge 0$. Suppose that

$$e^{\varepsilon t}r_n(t) \to \infty$$
 as $t \to \infty$

for each $\varepsilon > 0$ and each n . Then

$$\pi_{\rm A} \sim \sum_{\rm n} a_{\rm n} r_{\rm n} (d^{-1}) (1 + O(d)) , d \to 0 .$$

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Theorem 3.

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and that each on has a Laplace transform

$$\mathcal{P}^{\mathcal{A}} = \left(\begin{array}{c} \mathbf{P} \\ \mathbf{P} \end{array} \right)_{\mathcal{A}} = \left(\begin{array}{c} \mathbf{P} \end{array} \right)_{\mathcal{$$

for some to s 0 s songous to

Proof.

The integral in (41) is the Laplace transform of

$$k(z) = \begin{cases} \frac{g(\psi(z))}{G(\psi(z)) \ln G(\psi(z))} & 0 < z < h \\ 0 & z \ge h \end{cases}$$

as a function of d^{-1} . Theorem 3 is therefore a simple consequence of (41) and the theorem on pp. 31-32 in Erdélyi (1956).

Example.

The last two theorems may easily be applied to the special distribution (8) discussed in section 3 and 4. However, here the results are of little value, as we already have closed expressions for π_A and F(x). Instead we will consider the case where U_1 is normally distributed. As before we can take the mean to be zero. Furthermore, from (27), π_A is a function of $d\sigma^{-1}$ only. Therefore, without lack of generality, we can take the variance σ^2 of U_1 to be unity. Then

$$G(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{u^2}{2}} du$$
$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



are a function of d in Theorem 3 is therefore a simple consequence of (41) and the theorem on pp. 31-32 in Brdelyi (1956).

The Anterspire ((A)) in the Laplace transform of

and these functions satisfy the conditions of theorem 2 with $a = -\infty$.

From this theorem we have

(43)
$$\pi_{A} \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\frac{x^{2}}{2}}}{\sqrt{\Phi(x)}} e^{-d^{-1}\theta(x)} dx (1+0(d))$$

where

(44)
$$\theta(\mathbf{x}) = -\int_{\mathbf{x}}^{\infty} \ln \phi(\mathbf{y}) \, d\mathbf{y} = \mathbf{x} \, \ln \phi(\mathbf{x}) + \frac{1}{\sqrt{2\pi}} \int_{\mathbf{x}}^{\infty} \frac{\mathbf{y} \, e^{\frac{\mathbf{y}^{2}}{2}}}{\phi(\mathbf{y})^{2}} \, d\mathbf{y}$$

by partial integration.

Now $\theta(x) > 0$ for all x and $\theta(x) \to 0$ as $x \to \infty$. Therefore, as $d \to 0$, the important contribution from the integrand in (43) comes for large positive values of x. We know that $\Phi(x) \to 1$ very fast as $x \to \infty$. Therefore we may safely neglect the factors $\Phi(x)$ and $\Phi(y)$ in the denominators in (43) and (44). Also we may neglect the integral from $-\infty$ to 0 in (43) This gives

(45)
$$\pi_{A} \sim \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{x^{2}}{2}} e^{-d^{-1}\theta_{1}(x)} dx$$

where

(46)
$$\theta_1(x) = x \ln \phi(x) + \frac{1}{\sqrt{2\pi}} \int_x^{\infty} y e^{\frac{y^2}{2}} dy = x \ln \phi(x) + \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}}$$

and this of a ferral a field the conditions of theorem 2 with

where

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To simplify (45) further, it is natural to utilize the well-known asymptotic expansion

(47)
$$\Phi(x) \sim 1 - \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{15}{x^7} \pm \cdots\right) \quad (x \to \infty)$$

(Cf. e.g. Dettman (1965) p.451-52). This gives

(48)
$$\theta_1(x) \sim \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(\frac{1}{x^2} - \frac{3}{x^4} + \frac{15}{x^6} \mp \cdots \right) \quad (x \to \infty)$$

Note that the approximation obtained by only taking a finite number of terms in (48) is coarser than the one used in going from (43) to (45).

From now on the rest is only technicalities and is defered to the appendix. By change of variable from x to $u = \theta_1(x)$ in (45) and by means similar to those used in proving theorem 3, we find an asymptotic expansion for π_A as $d \rightarrow 0$. I choose to state the result in terms of

$$\overline{n} = \pi_A^{-1} = E(N_k)$$

the expected number of years between consecutive exchanges under stationary conditions. Also, we allow U_1 to have a general $N(\mu,\sigma^2)$ -distribution, that is, we replace d by d σ^{-1} . To three terms the expansion is

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$$(2-1) \left(\dots + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} \right)^{\frac{1}{3}} = \frac{1}{13} - \frac{1}{13} - \frac{1}{13} + \frac{1}{13}$$

(Cr. c.g. Dettman (1965) p.451-52). This gives (Cr. c. sature (1965) (1.491-52). This of an (

$$\begin{pmatrix} a = x \\ a$$

Note that the approximation obtained by only taking a finite of the instance of the source in the second state of the second s

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(49)
$$\overline{n} = \frac{\sigma}{d\sqrt{\rho}} \left(1 + \frac{\ln\rho + a - 1}{\rho} + \frac{2(3 \ln\rho + 3a - 5)^2 + \pi^2 + 22}{12\rho^2} + O(\frac{\ln\rho}{\rho})^3 \right),$$

where

$$\rho = \ln \left(\frac{\sigma}{d}\right)^2$$

a = ln $\sqrt{2\pi} - C \approx 0,3417$
C $\approx 0,5772$ (Euler-Mascheroni constant).

In fig.7 and fig.8 the right-hand side of (49) with 1, 2 and 3 terms is drawn as a function of σd^{-1} . This is compared with a few exact values of \overline{n} calculated directly from (27) and with Gades "semiempirical" formula.

(50)
$$\overline{n} \approx 1 + 0,729 \left(\frac{\sigma}{d}\right)^2$$

6. Conclusions

The purpose of this paper was to analyze a concrete stochastical model. In general the model was determined by a constant d (the annual density decrement) and a distribution function G(x) (the distribution of the density of coastal water in the physical interpretation). We succeeded in finding explicit expressions for many quantities characterizing the model : the stationary distribution of the density of the resident water (5), the stationary density-distribution of the

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6. Conglustons

The puppe of the paper seart i analyter a condetter at ongetion, notate. Interperedatine models has detteteed by a ponetart of ((the series) forsity degreement) into a distantia by the strong time of ((the series) forsity degreement)) into a distantia atten strong time of ((the series)) (the strong tegeteen the strong time of the series of the series of the term water, and the strong state interpretent and is in the strong to be applied to any easi of the series of the series of the series of the applied to the state of the series of the state of the state and state of the series of the series of the series of the series of the applied to the state of the series of the series of the series of the series of the state of the series of the series of the series of the states of the state of the series of the series of the series of the series of the state of the series of the series of the series of the states of the series of the series of the series of the series of the states of the series of the series of the series of the series of the states of the series of the series of the series of the series of the states of the series of the series of the series of the series of the states of the series of the series of the series of the series of the states of the series of the water entering the basin (15), the expected number of years between exchanges (19) etc. Mostly these results depended on the detailed form of G(x) and were rather difficult to analyze. Some results were remarkably simple, however. For instance, when G(x) is absolutely continuous with probability density g(x), from (14) the probability density of the water entering the basin is

(51)
$$t(x) = \pi_A^{-1} g(x) \prod_{j=1}^{\infty} G(x+jd)$$

The normalizing factor $\overline{n} = \pi_A^{-1}$ may in principle be found by requiring the area under t(x) to be unity. This is to be compared to Gades procedure, finding t(x) by solving numerically a complicated integral equation.

Also, it is interesting to note how thing simplify when G(x) is the special distribution (8). This distribution resembles somewhat the normal one (see figs.1 and 2), and may be useful in applications.

When G(x) is the normal distribution, we do not find as simple answers. One main result here is the asymptotic expansion (49). To three terms it seems to give a better approximation to \overline{n} than (50) when $\sigma d^{-1} \ge 5$. Still more important, however, is that it indicates that the behavior of \overline{n} as $s = \sigma d^{-1} \rightarrow \infty$ is not like s^{α} ($\alpha < 1$), but rather like $s(\ln s)^{-\frac{1}{2}}$, that is, only a slight deflect from linearity. water entering the basis i(1), the expected matrice of yearbetween examples i(10), etc. (2000) rates modulity matrix 0 in the appended of the detailed dorm of (0(x)) and some matrix 0 if the difference 1 is lyre. Some real, the same manual side without the basis trates of the side 100 of 100, the example 100 of data to be shown in the same manual side 100 of data to be shown in the same manual side 100 of data to be shown in the same 100 of data to be shown in the same 100 of data to be shown 100 of 100 of data to be shown 100 of 100 of 100 of data to be shown 100 of 100 of

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The author is grateful to Herman G. Gade and Trygve S. Nilsen for proposing the model and for valuable discussions. Also he wants to thank The Norwegian Research Council for Science and the Humanities (NAVF) for financial support.

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Here I want to fill in the gaps in the derivation of (49) from (45) and (48).

Put

(52)
$$u = \theta_1(x)$$

where $\theta_1(x)$ is given by (46). Then

$$\frac{du}{dx} = \ln \phi(x) + \frac{x \frac{e^2}{2}}{\sqrt{2\pi} \phi(x)} - \frac{x \frac{e^2}{2}}{\sqrt{2\pi}}$$
$$\sim -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} \mp \cdots\right) + x \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \cdot 0 \left(e^{-\frac{x^2}{2}}\right)$$

as $x \to \infty$ from (47). Therefore

(53)
$$\frac{dx}{du} \sim -\sqrt{2\pi} e^{\frac{x^2}{2}} \left(x + \frac{1}{x} - \frac{2}{x^3} + 0(x^{-5})\right) \quad (x \to \infty)$$

Note that $u \downarrow 0$ as $x \to \infty$ and $\frac{du}{dx}$ is negative when $x \ge 0$. When x = 0, $u = (2\pi)^{-\frac{1}{2}}$. Therefore from (45) and (53)

(54)
$$\pi_{A} \sim \int_{0}^{(2\pi)^{-\frac{1}{2}}} \left[x(u) + \frac{1}{x(u)} - \frac{2}{x(u)^{3}} \right] e^{-d^{-1}u} du$$

The problem is to invert (52). From (48)

Here I want to fill in the gaps in the derivation of ((3)) from (45) and (48).

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$$u = \theta_{1}(x)$$

where $\theta_{1}(x)$ is given by (46). The

$$\frac{3u}{3x} = \ln \phi(x) + \frac{x}{\sqrt{2\pi}} - \frac{x}{\sqrt{2\pi}} - \frac{x}{\sqrt{2\pi}}^{2}$$

$$= \ln \phi(x) + \frac{x}{\sqrt{2\pi}} - \frac{x}{\sqrt{2\pi}} - \frac{x}{\sqrt{2\pi}}$$

$$= \frac{x}{\sqrt{2\pi}} - \frac{x}{\sqrt{2\pi}} + \frac{x}{\sqrt{2\pi}} + \frac{x}{\sqrt{2\pi}} - \frac{x}{\sqrt{2\pi}} - \frac{x}{\sqrt{2\pi}} - \frac{x}{\sqrt{2\pi}} + \frac{x}{\sqrt{2\pi}} - \frac{x}{\sqrt{2\pi}} - \frac{x}{\sqrt{2\pi}} - \frac{x}{\sqrt{2\pi}} - \frac{x}{\sqrt{2\pi}} + \frac{x}{\sqrt{2\pi}} - \frac{x}$$

as $x \to \infty$ from (47). Therefore

(53)
$$\frac{dx}{du} \sim -\sqrt{2\pi} e^{\frac{x^2}{2}} \left(x + \frac{1}{x} - \frac{2}{\sqrt{x}} + O(x^{-\frac{5}{2}})\right)$$
 (x + $\frac{1}{\sqrt{x}} - \frac{2}{\sqrt{x}} + O(x^{-\frac{5}{2}})$)

Note that $u \downarrow 0 0 38 x \rightarrow \infty$ and $\frac{du}{dx}$ is negative when $x \ge 0$. When x = 0 $\mu = (2\pi)^{-\frac{1}{2}}$. Therefore from (45) and (53)

(54)
$$\pi_{A} = \int_{0}^{1} \left[x(u) + \frac{1}{x(u)} - \frac{2}{x(u)} \right] e^{-d^{-1}u} du$$

The problem d.E. to dewert (52). From (48)

$$\ln u \sim -\frac{x^2}{2} - \ln x^2 - \ln \sqrt{2\pi} - \frac{3}{x^2} + \frac{6}{x^4} \qquad (x \to \infty)$$

or

(55)
$$x^2 \sim -2\ln u - 2\ln \sqrt{2\pi} - 2\ln x^2 - \frac{6}{x^2} + \frac{12}{x^4}$$

Put $w = -2 \ln u$ and take the logarithm in (55). This gives

$$\ln x^{2} \sim \ln w + \ln(1 - \frac{2 \ln x^{2}}{w} - \frac{2 \ln \sqrt{2\pi}}{w})$$
$$\ln x^{2} \sim \ln w - \frac{2 \ln(\sqrt{2\pi} w)}{w}$$

Insert this in (55) to get

$$x \sim \sqrt{w} - \frac{\ln(\sqrt{2\pi} w)}{\sqrt{w}} - \frac{(\ln(\sqrt{2\pi} w))^2 - 4 \ln(\sqrt{2\pi} w) + 6}{\frac{3}{2 w^2}}$$

Inserted in (54) this gives

(56)
$$\pi_{A} \sim \int_{0}^{(2\pi)^{-\frac{1}{2}}} \left\{ \sqrt{w} - \frac{\ln(\sqrt{2\pi} w) - 1}{\sqrt{w}} - \frac{1}{\sqrt{w}} \right\}$$

$$\frac{(\ln(\sqrt{2\pi} w))^2 - 6 \ln(\sqrt{2\pi} w) + 10}{\frac{3}{2} w^2} e^{-d^{-1}u} du$$

where

$$W = -2 \ln u$$
.

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(55))
$$x^{2} - 2x + 2 + \sqrt{3\pi} - 2 + \sqrt{3\pi} - 2 + x^{2} + \frac{1}{2} +$$

For $w_{n=n-2}$ in a and take whe logarithm in (55). This shifts a shift was shown in (55).

$$\lim_{N \to \infty} x^2 - \lim_{N \to \infty} \lim_{N \to \infty} \left(\frac{2 \lim_{N \to \infty} x^2}{N} - \frac{2 \lim_{N \to \infty} \sqrt{2\pi}}{N} \right)$$

In $x^2 - \lim_{N \to \infty} \frac{2 \lim_{N \to \infty} \sqrt{2\pi}}{N}$
Insert this in (55) to get

$$x \sim \sqrt{w} - \frac{\ln(\sqrt{2\pi} w)}{\sqrt{w}} - \frac{(\ln(\sqrt{2\pi} w))^2 - 4\ln(\sqrt{2\pi} w) + 6}{8w^2}$$

Inserted in (54) this gives

(56)
$$\pi_{A} \sim \int_{0}^{-\frac{1}{2}} \left\{ \sqrt{w} - \frac{\ln(\sqrt{2\pi} w) - 1}{\sqrt{w}} \right\}$$

$$(\ln (\sqrt{2\pi} n))^2 - 6 \ln (\sqrt{2\pi} n) + 10$$

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To perform the integrations in (56), we change variable from u to $t = d^{-1}u$. Then

$$w = -2\ln t - 2\ln d = \rho - 2\ln t$$

where

$$\rho = -2\ln d \rightarrow \infty$$
 as $d \rightarrow 0$.

The factor in curly brackets in (56) has the following asymptotic expansion as $\rho \rightarrow \infty$

(57)
$$\sqrt{\rho} - \frac{\ln(\sqrt{2\pi} \rho) + \ln t - 1}{\sqrt{\rho}}$$

 $\frac{(\ln(\sqrt{2\pi} \rho))^2 - 6\ln(\sqrt{2\pi} \rho) + 10 + (\ln t)^2 - 6\ln t + 2(\ln t)\ln(\sqrt{2\pi}\rho)}{2\rho^2}$

This expansion is valid when

$$|\ln t| < \frac{1}{2}\rho$$
,

that is

$$d < t < d^{-1}$$
): $d^2 < u < 1$

Now for each $\varepsilon > 0$ the factor in curly brackets in (56) is bounded in absolute value by $u^{-\varepsilon}$ as $u \to 0$. Therefore the part of the integral from 0 to d^2 is bounded by

The factor in curly brackets in (56) has the following asymptotic expansion as $\rho \rightarrow \infty$

(57)
$$\sqrt{p} - \frac{\ln(\sqrt{2\pi}, p) + \ln t - 1}{\sqrt{p}}$$

$$\frac{(\ln(\sqrt{2\pi} \circ))^2 - 6\ln(\sqrt{2\pi} \circ) + 10 + (\ln t)^2 - 6\ln t + 2(\ln t)\ln(\sqrt{2\pi} \circ)}{(\ln t)^2} + \frac{(\ln t)^2 - 6\ln t + 2(\ln t)}{(\ln t)^2}$$

This expansion is valid when It. a mission is version

that is

$$d < t < d^{-1}$$
): $d^2 < u < 1$

Now for each $\epsilon > 0$ the factor in ourly brackets in (56) if different is a set of the factor is an in the set of the factor is bounded in absolute value by u as in the factor is an in the part of the integral from 0 to d is bounded by the part of the integral from 0 to d is bounded by

$$\int_{0}^{d^{2}} u^{-\varepsilon} e^{-d^{-1}u} du \leq \int_{0}^{d^{2}} u^{-\varepsilon} du = \frac{d^{2-2\varepsilon}}{1-\varepsilon} = o\left(\frac{3}{d^{2}}\right)$$

when $\varepsilon < \frac{1}{4}$. Neglecting corrections of this order, we may multiply (57) by e^{-t} , integrate from 0 to ∞ to get

(58)
$$\pi_{A} \sim d \sqrt{\rho} \left(1 - \frac{\ln(\sqrt{2\pi\rho}) - C - 1}{\rho} - \frac{(\ln(\sqrt{2\pi\rho}))^{2} - 6\ln(\sqrt{2\pi\rho}) + 10 + C^{2} + \frac{\pi^{2}}{5} + 6C - 2C \ln(\sqrt{2\pi\rho})}{2\rho^{2}}\right)$$

Here we have again utilized the definite integrals which we used to find $E(U_1)$ and $E(U_1^2)$ in the first example in section 3. The order of the first neglected term may easily be checked. From (58) we find $\overline{n} = \pi_A^{-1}$ and replace d by $d\sigma^{-1}$ to obtain (49).

where $\vec{e} < \frac{1}{2}$. Magicalling corrections of this order, we may multiply (57)) by even the rate from 0 to a to get

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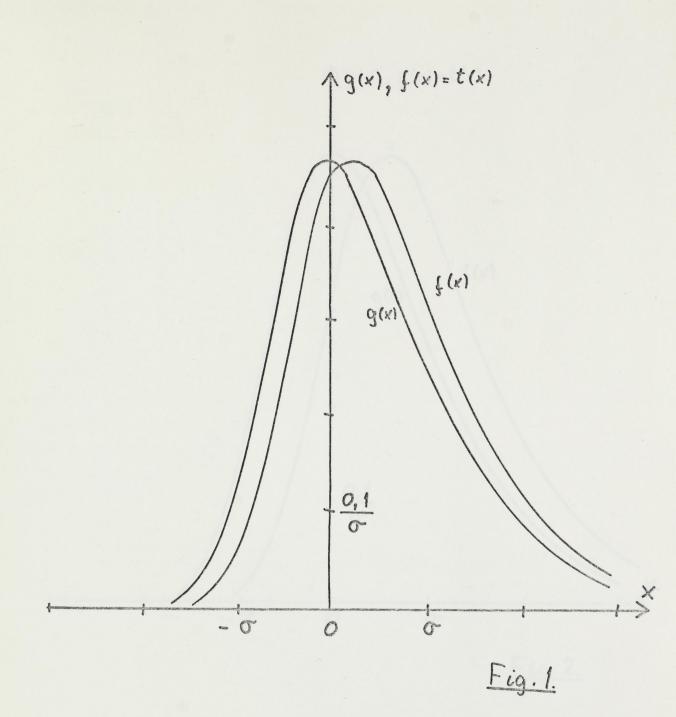
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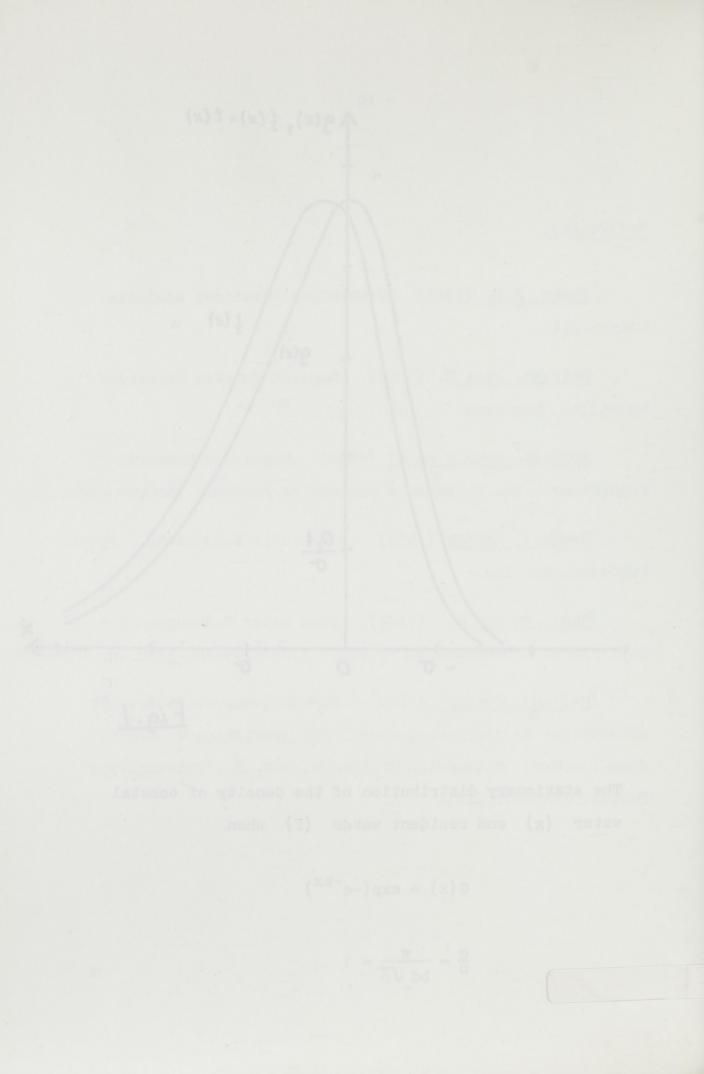
> Helland, Ince.S. (1975). Hovedfagsoppyave i envenit måtëmätikk: En Stokastisk modell for utskifting av vann i dype fjorffer. Måtemätisk institutt, savd. S., University of Bergen. (Unpublished).

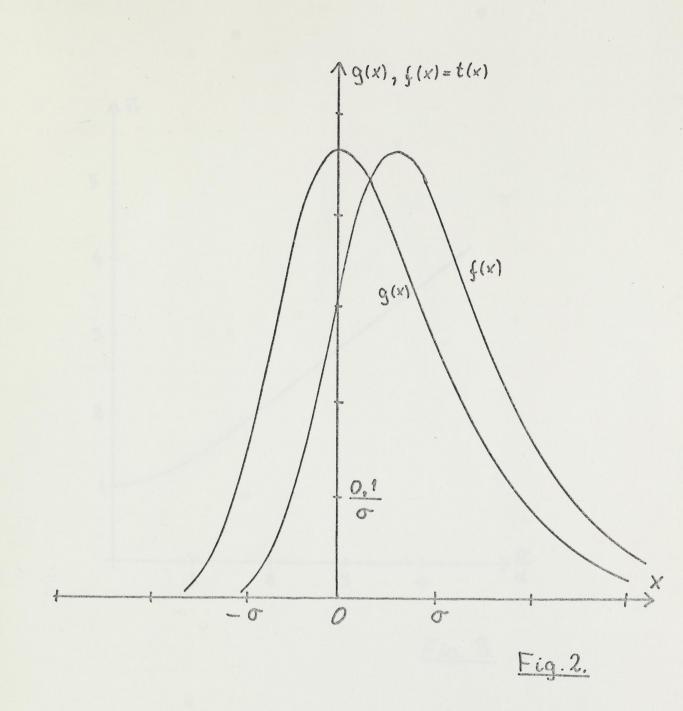


The stationary distribution of the density of coastal water (g) and resident water (f) when

$$G(x) = \exp(-e^{-bx})$$

$$\frac{\sigma}{d} = \frac{\pi}{bd\sqrt{6}} = 1$$

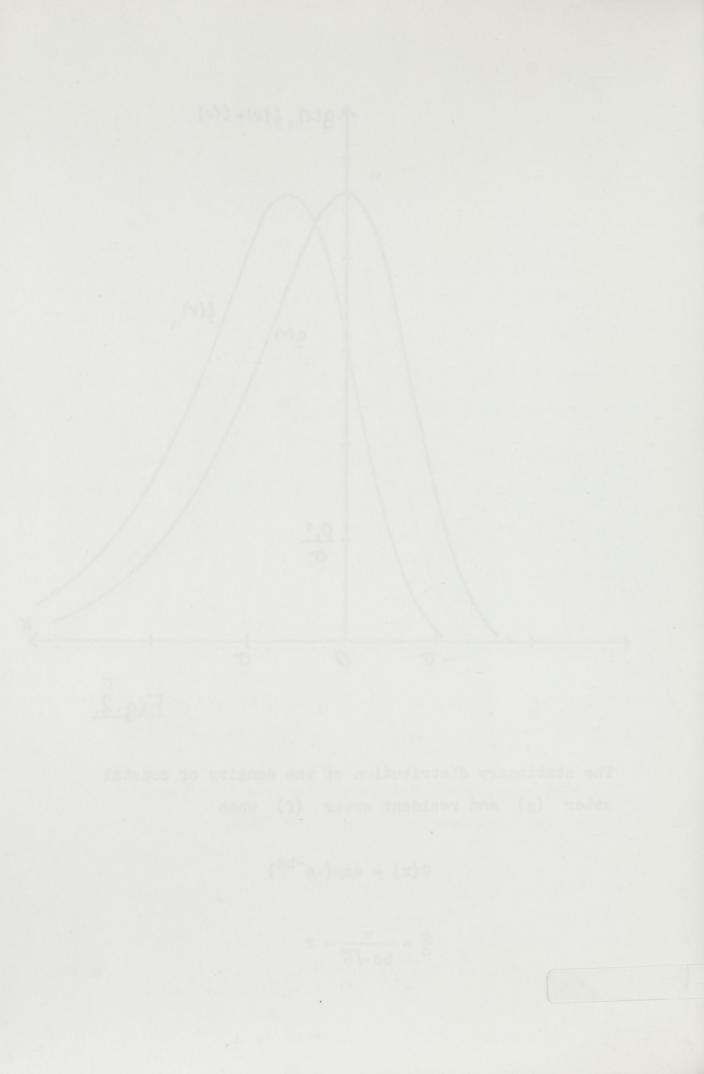


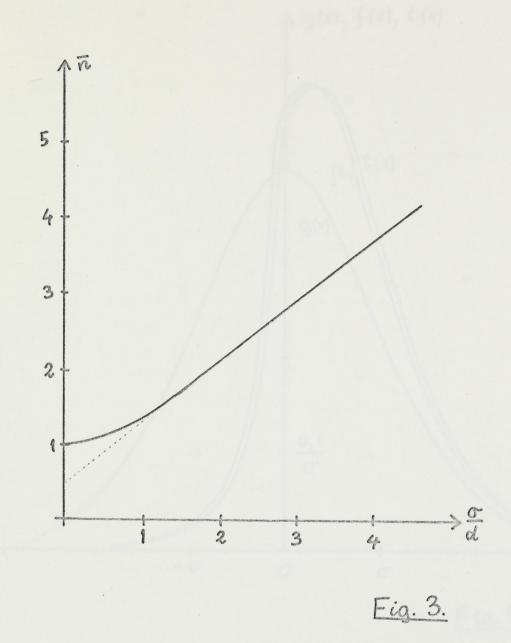


The stationary distribution of the density of coastal water (g) and resident water (f) when

$$G(x) = \exp(-e^{-bx})$$

$$\frac{\sigma}{d} = \frac{\pi}{bd\sqrt{6}} = 2$$



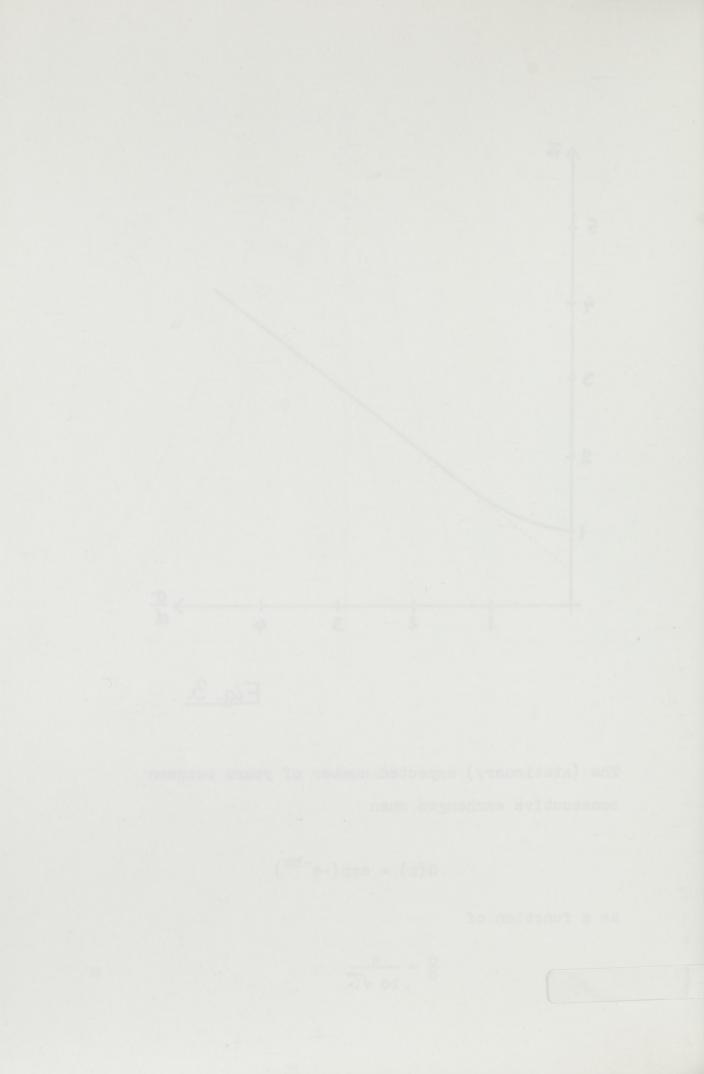


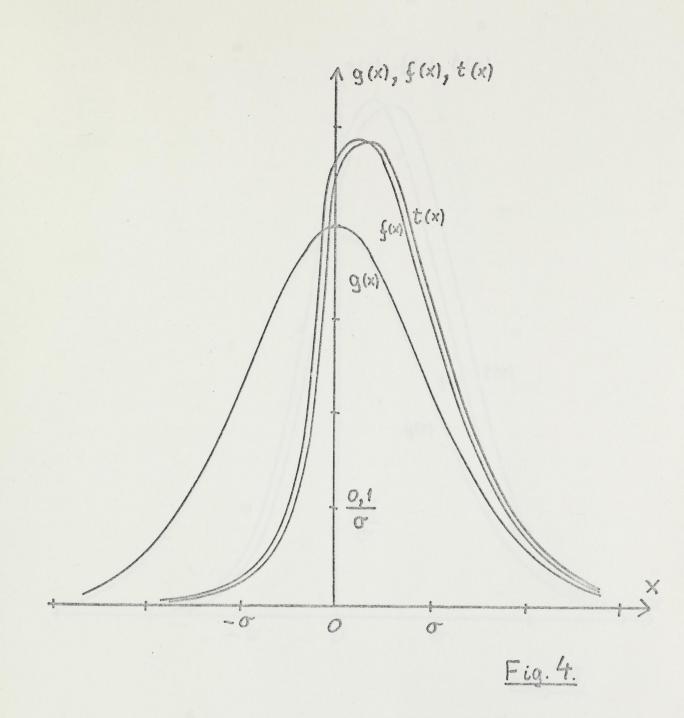
The (stationary) expected number of years between consecutive exchanges when

$$G(x) = \exp(-e^{-bx})$$

as a function of

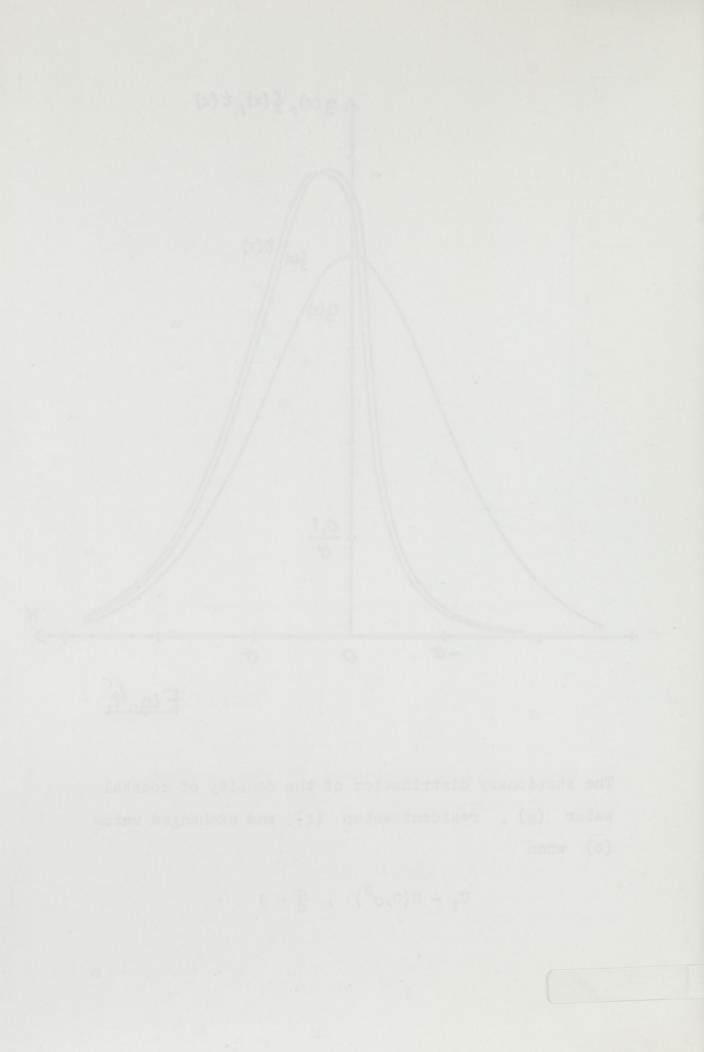
$$\frac{\sigma}{d} = \frac{\pi}{bd \sqrt{6}}$$

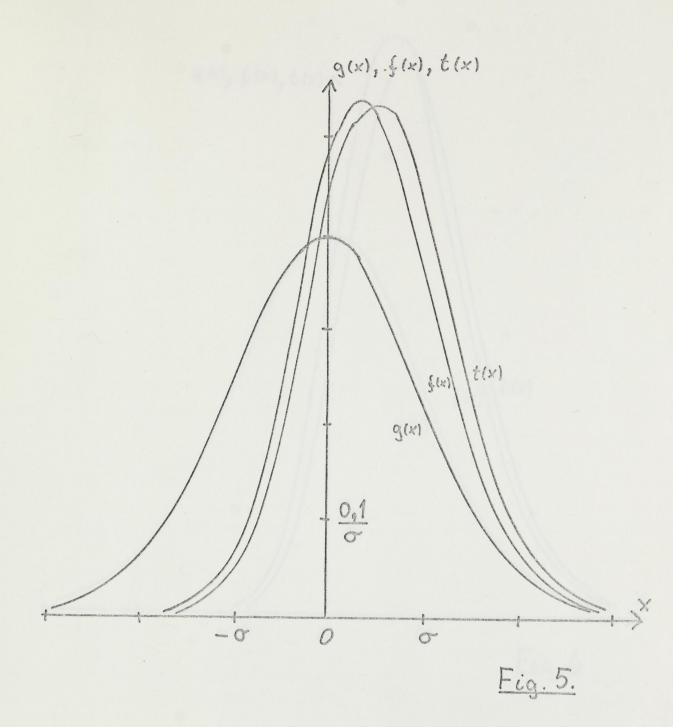




The stationary distribution of the density of coastal water (g), resident water (f) and exchanged water (t) when

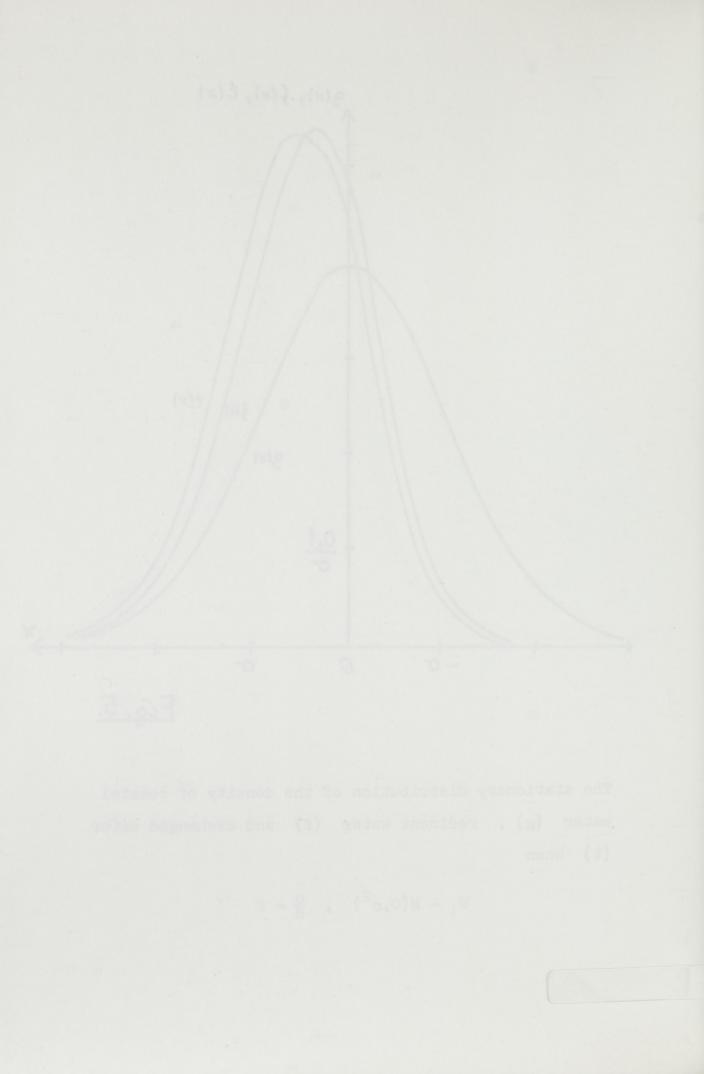
$$U_1 \sim N(0,\sigma^2)$$
, $\frac{\sigma}{d} = 1$

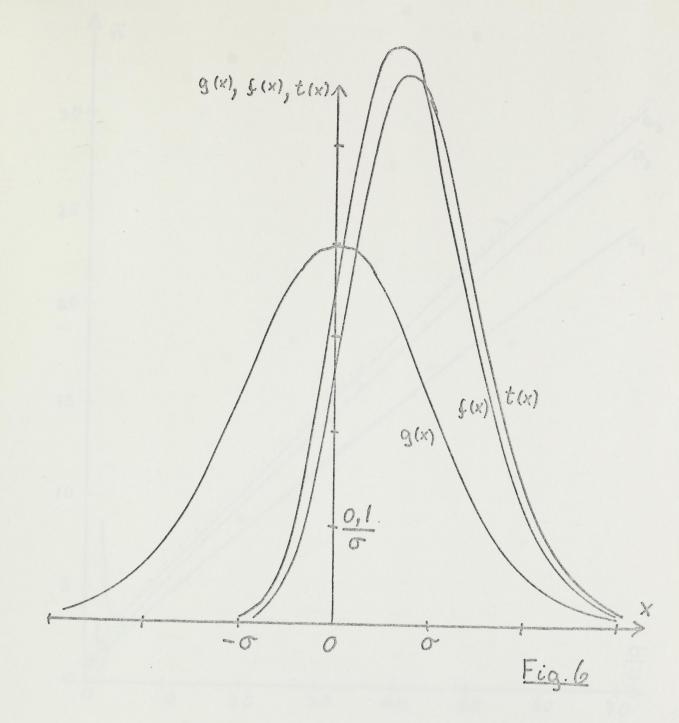




The stationary distribution of the density of coastal water (g), resident water (f) and exchanged water (t) when

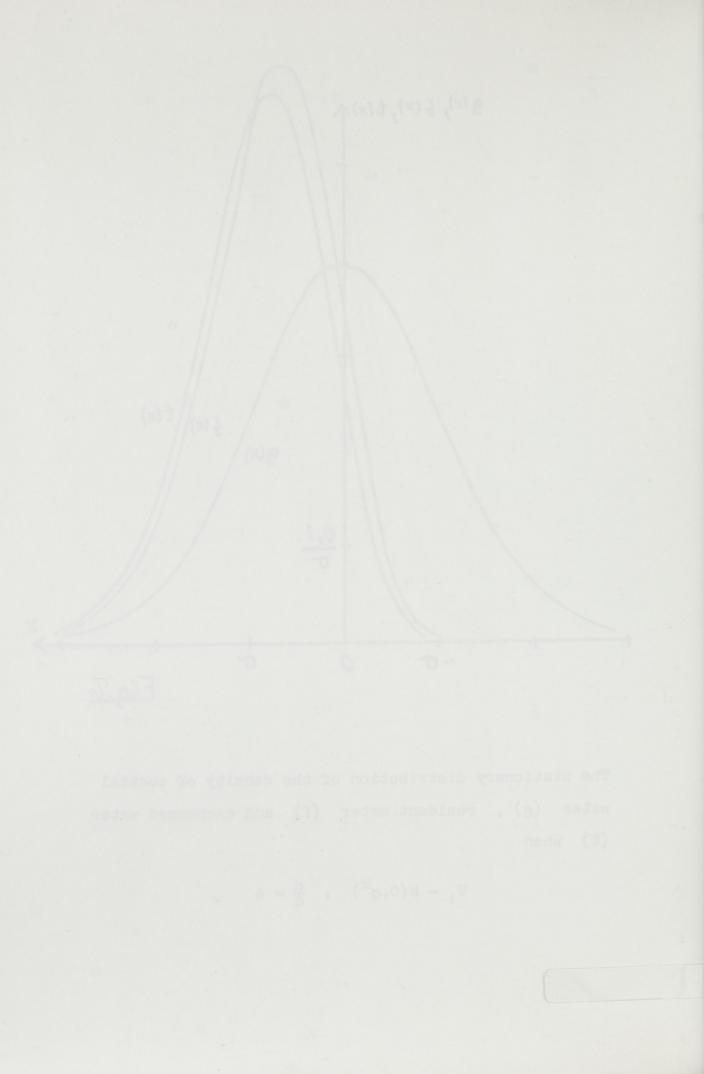
$$U_1 \sim N(0,\sigma^2)$$
, $\frac{\sigma}{d} = 2$

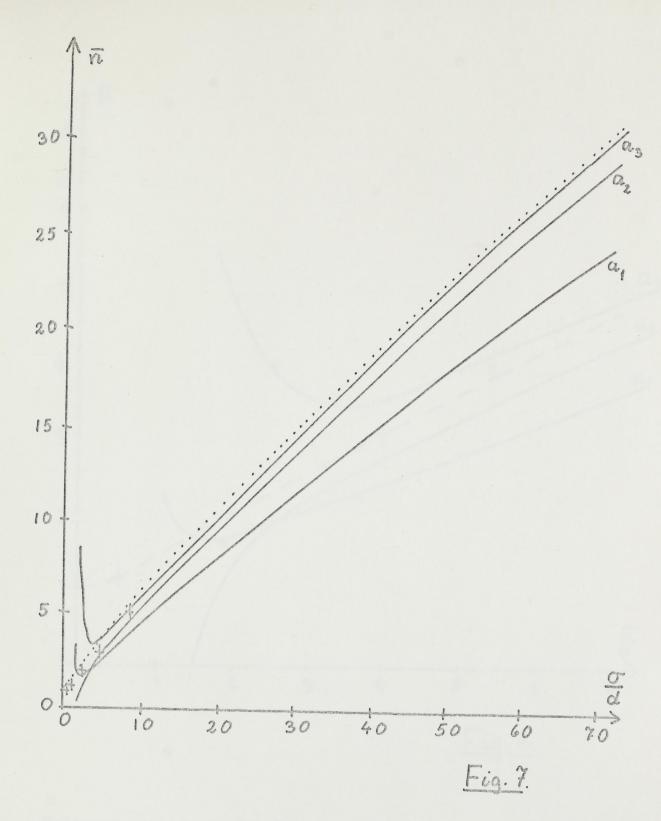




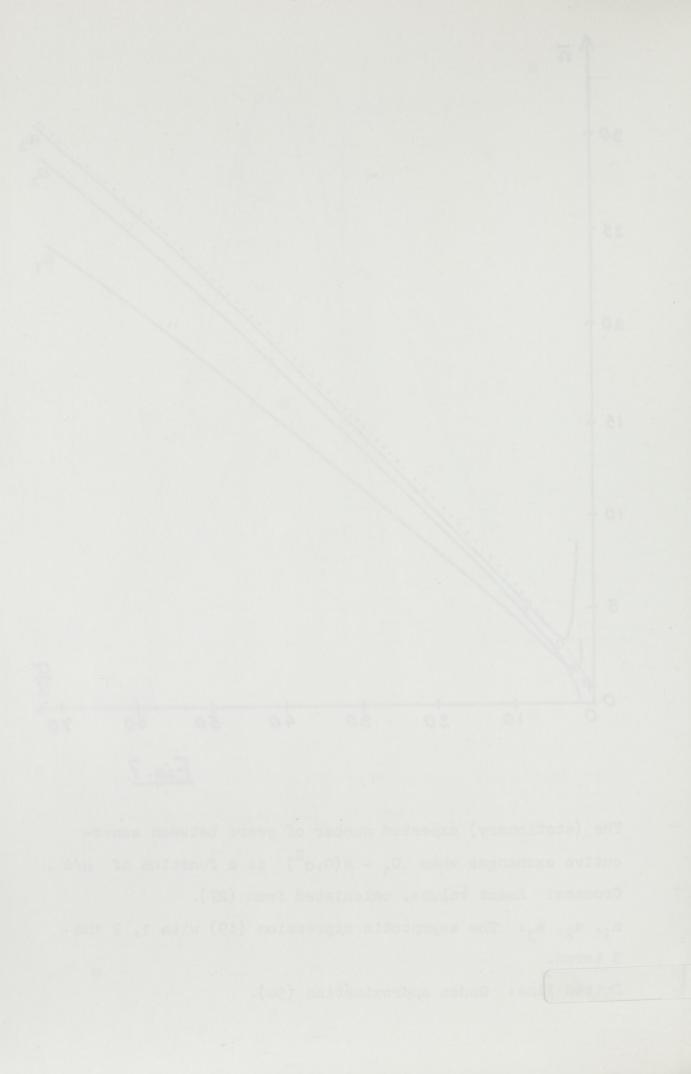
The stationary distribution of the density of coastal water (g), resident water (f) and exchanged water (t) when

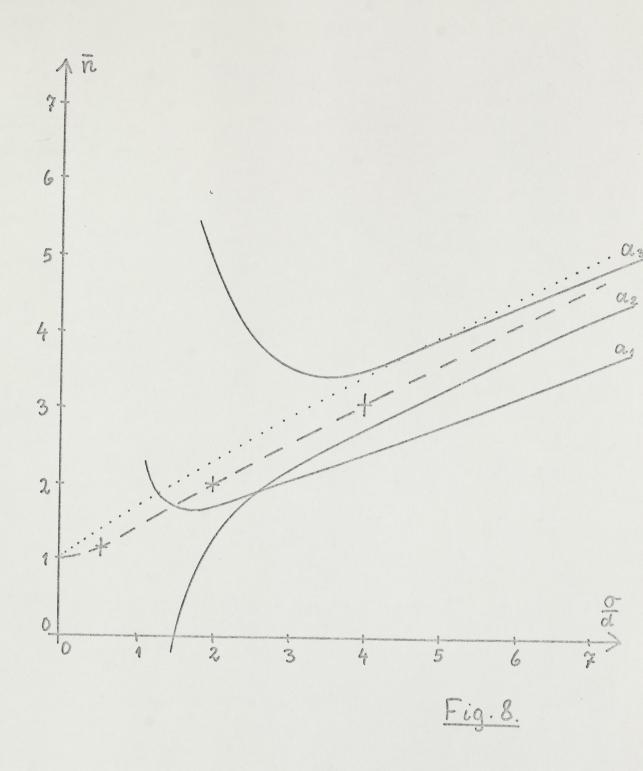
$$U_1 \sim N(0,\sigma^2)$$
, $\frac{\sigma}{d} = 4$





The (stationary) expected number of years between consecutive exchanges when $U_1 \sim N(0,\sigma^2)$ as a function of σ/d . Crosses: Exact values, calculated from (27). a_1, a_2, a_3 : The asymptotic expression (49) with 1, 2 and 3 terms. Dotted line: Gades approximation (50).





Enlarged portion of fig.7.

Dashed line: Drawn through exact, calculated values.

