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A ROBUST NONCONFORMING H^2 -ELEMENT

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ABSTRACT. Finite element methods for some elliptic fourth order singular perturbation problems are discussed. We show that if such problems are discretized by the nonconforming Morley method, in a regime close to second order elliptic equations, then the error deteriorates. In fact, a counter example is given to show that the Morley method diverges for the reduced second order equation. As an alternative to the Morley element we propose to use a nonconforming H^2 -element which is H^1 -conforming. We show that the new element converges in the energy norm uniformly in the perturbation parameter under suitable regularity assumptions on the solution.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and let $\partial\Omega$ denote the boundary. The purpose of this paper is to discuss finite element methods for elliptic singular perturbation problems of the form

$$(1.1) \quad \begin{cases} \varepsilon^2 \Delta^2 u - \Delta u = f & \text{in } \Omega \\ u = 0, \quad \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, $\frac{\partial u}{\partial n}$ denotes the normal derivative on $\partial\Omega$ and ε is a real parameter such that $0 < \varepsilon \leq 1$. In particular, we are interested in the regime when ε is close to zero. We observe that if ε tends to zero the differential equation in (1.1) formally degenerates to Poisson's equation. Hence, we are studying a plate model which may degenerate towards an elastic membrane problem.

When fourth order problems like (1.1) are discretized by a finite element method the standard variational formulation will require function spaces which are subspaces of the Sobolev space $H^2(\Omega)$. Hence, we need piecewise smooth functions which are globally C^1 . However, it is well known that in order to construct C^1 -functions which are piecewise polynomials with respect to a given triangulation of Ω we are forced to use polynomials of degree five or higher, cf. e.g. [3]. Alternatively, we can use a macroelement technique like in the Hsieh-Clough-Tocher method [3]. In order to avoid high order polynomials or macroelements a common approach is to use nonconforming finite elements for such problems, i.e. the C^1 -continuity requirement is violated. The simplest

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nonconforming element for fourth order problems is the Morley element. The Morley space consists of piecewise quadratic functions with respect to a given triangulation of Ω . The elements of this function space are not even continuous functions, but still this space leads to a convergent nonconforming finite element method. For discussions on properties of the Morley method we refer for example to [1], [5], [7], and [9].

However, if the Morley method is applied to a nearly second order problem of the form (1.1) with ε close to zero, then the convergence rate of the method will deteriorate. In fact, if the Morley element is applied to a second order equation like Poisson's equation then the method will diverge. The main reason for this degeneracy of the Morley method is the fact that the finite element space is not a subspace of $H^1(\Omega)$, or more precisely, the Morley space is not a proper nonconforming finite element space for second order elliptic equations. This is of course in contrast to the conforming case, since any subspace of $H^2(\Omega)$ is also a subspace of $H^1(\Omega)$.

We will discuss the degeneracy described above for the Morley method in §2. Then, in §3, we will propose an alternative nonconforming finite element method which is robust with respect to the parameter ε . The new function space consists of continuous functions which locally belongs to a nine dimensional subspace of quadratic polynomials, constructed by the use of the "cubic bubble function." The space of C^0 -quadratic functions is a subspace of the new finite element space. The global dimension of the new space, corresponding to a fixed triangulation and the boundary conditions given in (1.1), is the sum of the number of interior vertices and twice the number of interior edges. As a comparison, the dimension of the Morley space is the the sum of interior vertices and edges.

2. PRELIMINARIES

The inner product on $L^2 = L^2(\Omega)$ will be denoted by (\cdot, \cdot) . For $m \geq 0$ we shall use $H^m = H^m(\Omega)$ to denote the usual Sobolev space of functions with partial derivatives of order less than or equal m in L^2 , and the corresponding norm by $\|\cdot\|_m$. Furthermore, the notation $\|\cdot\|_{m,K}$ is used to indicate that the norm is defined with respect to a domain K , different from Ω . The seminorm derived from the partial derivatives of order equal m is denoted by $|\cdot|_m$, i.e. $|\cdot|_m^2 = \|\cdot\|_m^2 - \|\cdot\|_{m-1}^2$. The space H_0^m is the closure in H^m of $C_0^\infty(\Omega)$. Alternatively, we have

$$H_0^1 = \{v \in H^1 : v|_{\partial\Omega} = 0\} \quad \text{and} \quad H_0^2 = \{v \in H^2 \cap H_0^1 : \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\},$$

where the restrictions to $\partial\Omega$ is taken in the sense of traces.

We let Du be the gradient of u and $D^2u = (\partial^2u/\partial x_i\partial x_j)_{i,j}$ the 2×2 -tensor of second order partials. In order to define weak solutions

of (1.1) we introduce the bilinear forms

$$(2.1) \quad a(u, v) = \int_{\Omega} D^2 u : D^2 v \, dx,$$

where the colon denotes the scalar product of tensors, and

$$(2.2) \quad b(u, v) = \int_{\Omega} Du \cdot Dv \, dx.$$

A function $u \in H_0^2$ is defined to be a weak solution of (1.1) if

$$(2.3) \quad \varepsilon^2 a(u, v) + b(u, v) = (f, v), \quad \forall v \in H_0^2.$$

In fact, in this weak formulation we may simplify the bilinear form a and use

$$a(u, v) = \int_{\Omega} \Delta u \Delta v \, dx$$

instead of the one given by (2.1), since

$$(2.4) \quad \int_{\Omega} \Delta u \Delta v \, dx = \int_{\Omega} (\text{trace} D^2 u)(\text{trace} D^2 v) \, dx = \int_{\Omega} D^2 u : D^2 v \, dx,$$

for all $u, v \in H_0^2$. However, since we will consider nonconforming finite element methods this identity may not hold on the proper finite element spaces, and therefore we use the form given by (2.1).

We recall from the regularity theory for elliptic problems in non-smooth domains, cf. [4] or [6], that if $f \in L^2$ and Ω is convex then $u \in H^3$, i.e. there is a constant c , independent of f , such that the corresponding weak solution u of (1.1) satisfies

$$\|u\|_3 \leq c \|f\|_0.$$

However, the constant c will in general depend on ε and will blow up as ε tends to zero.

3. THE MORLEY METHOD

Assume $\{\mathcal{T}_h\}$ is a quasi-uniform and shape regular family of triangulations of Ω , where the discretization parameter h is a characteristic diameter. We let \mathcal{X}_h be the set of vertices and \mathcal{E}_h the set of edges corresponding to \mathcal{T}_h . The corresponding Morley finite element space, M_h , consists of all piecewise quadratics which are continuous at each vertex of \mathcal{T}_h and such that the normal component of the gradient is continuous at the midpoint of each edge. Furthermore, in order to approximate the boundary conditions in (1.1) the functions in M_h are zero at boundary vertices and have zero normal derivatives at the midpoints of all boundary edges. A function $w \in M_h$ is uniquely determined by the value of w at each interior vertex and by the value of the normal component of Dw at the midpoint of each interior edge, cf. Figure 1.

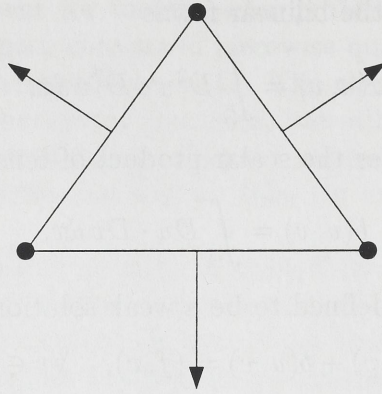


FIGURE 1. The six degrees of freedom of the Morley element

The finite element approximation $u_h \in M_h$ of u is now determined by the linear system

$$(3.1) \quad \varepsilon^2 a_h(u, v) + b_h(u, v) = (f, v), \quad \forall v \in M_h.$$

Here the bilinear forms a_h and b_h are obtained from a and b by summing over all triangles in \mathcal{T}_h , i.e.

$$a_h(u, v) = \sum_{T \in \mathcal{T}_h} \int_T D^2 u : D^2 v \, dx,$$

and

$$b_h(u, v) = \sum_{T \in \mathcal{T}_h} \int_T Du \cdot Dv \, dx.$$

Associated with the bilinear form $\varepsilon^2 a_h + b_h$ we define a seminorm $\|\cdot\|_{\varepsilon, h}$ by

$$\|w\|_{\varepsilon, h}^2 = \varepsilon^2 a_h(w, w) + b_h(w, w).$$

Observe that this seminorm is a norm on $H_0^2 + M_h$, and, as ε tends to zero, this energy norm approaches a piecewise H_0^1 -norm.

If $u \in H_0^2$ is the corresponding weak solution of (1.1) then the error $u - u_h$ can be estimated in the energy norm using the second Strang lemma, cf. [3], which states that

$$(3.2) \quad \|u - u_h\|_{\varepsilon, h} \leq \inf_{v \in M_h} \|u - v\|_{\varepsilon, h} + \sup_{w \in M_h} \frac{|E_{\varepsilon, h}(u, w)|}{\|w\|_{\varepsilon, h}},$$

where the consistency error $E_{\varepsilon, h}(u, w)$ is given by

$$E_{\varepsilon, h}(u, w) = \varepsilon^2 a_h(u, w) + b_h(u, w) - (f, w).$$

Assume that $f \in L^2$ and $u \in H^3 \cap H_0^2$. From the basic estimate (3.2) and by following the approach of [9] to estimate $E_{\varepsilon, h}(u, w)$ it is

straightforward to obtain an error estimate of the form

$$(3.3) \quad \|u - u_h\|_{\varepsilon,h} \leq C \left(\frac{h^2}{\varepsilon} \|f\|_0 + \frac{h}{\varepsilon} \|u\|_3 \right).$$

Note that if $\varepsilon = 1$ this estimate predicts linear convergence with respect to the mesh parameter h . However, if h is fixed and ε approaches zero then the estimate for the error behaves like $O(1/\varepsilon)$. The following numerical example indicates that this degeneracy is in fact real.

Example 3.1. We consider the problem (1.1) with Ω taken as the unit square and $f = \varepsilon^2 \Delta^2 u - \Delta u$, where $u(x) = (\sin(\pi x_1) \sin(\pi x_2))^2$. The domain is triangulated by first dividing it into $h \times h$ squares. Then, each square is divided into two triangles by the diagonal with a negative slope. In the table below we have computed the relative error in the energy norm, $\|u_h^I - u_h\|_{\varepsilon,h} / \|u_h^I\|_{\varepsilon,h}$, for different values of ε and h . Here u_h^I denote the interpolant of u on M_h defined from the values of u at each vertex and from the value of the normal component of Du on the midpoint of each edge. For a comparison we also consider the case $\varepsilon = 0$, i.e. the Poisson problem with Dirichlet boundary conditions, and the biharmonic problem $\Delta^2 u = f$.

$\varepsilon \backslash h$	2^{-3}	2^{-4}	2^{-5}	2^{-6}
2^0	0.3898	0.2008	0.1012	0.0507
2^{-2}	0.4016	0.2085	0.1053	0.0528
2^{-4}	0.5674	0.3262	0.1699	0.0858
2^{-6}	0.8937	0.7499	0.4981	0.2790
2^{-8}	0.9730	0.9934	0.9275	0.7487
2^{-10}	0.9791	1.0214	1.0265	1.0059
Poisson	0.9795	1.0234	1.0347	1.0376
Biharmonic	0.3891	0.2004	0.1009	0.0506

TABLE 1. The relative error measured by the energy norm

When ε is large the convergence appears to be linear with respect to h , while the convergence deteriorates as ε approaches zero. \square

3.1. The reduced problem. When ε tends to zero the problem (1.1) formally approaches a Poisson equation with Dirichlet boundary conditions. Below we shall give an analytical argument which shows that for such problems the Morley space will in fact lead to a divergent numerical method. Hence, this suggest once more that the Morley method is not suitable for problems of the form (1.1) when the parameter ε is sufficiently small.

In order to simplify some calculations below we shall modify the reduced problem slightly. Instead of the pure Dirichlet problem we

consider mixed boundary conditions. We assume that $\partial\Omega = \Gamma_D \cup \Gamma_N$, where Γ_D and Γ_N are disjoint subsets of $\partial\Omega$ and consider the problem

$$(3.4) \quad \begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N. \end{cases}$$

This problem can be considered to be the formal limit of the fourth order problems

$$(3.5) \quad \begin{cases} \varepsilon^2 \Delta^2 u - \Delta u = f, & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \frac{\partial}{\partial n}(u - \varepsilon^2 \Delta u) = g & \text{on } \Gamma_N \\ \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

As above let \mathcal{T}_h be a triangulation of Ω and let M_h be the Morley space corresponding to the boundary conditions of (3.4), i.e. we assume that the functions in M_h are zero on $\mathcal{X}_h \cap \Gamma_D$.

The approximation $u_h \in M_h$ of the solution u of problem (3.4) is determined by the linear system

$$(3.6) \quad b_h(u_h, w) = (f, w) + \langle g, w \rangle \quad \text{for all } w \in M_h.$$

Here

$$\langle g, w \rangle = \int_{\Gamma_N} gw \, ds,$$

where s denotes the arc length along $\partial\Omega$. Furthermore, the exact solution u of (3.4) satisfies

$$(3.7) \quad b_h(u, w) = (f, w) + \langle g, w \rangle + E_h(u, w) \quad \text{for all } w \in M_h,$$

where

$$E_h(u, w) = b_h(u, w) - (f, w) - \langle g, w \rangle.$$

Let $\|\cdot\|_h = \|\cdot\|_{0,h}$ be the energy norm for the reduced problem. From (3.6) and (3.7) we obtain the second Strang lemma

$$(3.8) \quad \|u - u_h\|_h \leq \inf_{v \in M_h} \|u - v\|_h + \sup_{w \in M_h} \frac{|E_h(u, w)|}{\|w\|_h}.$$

However, the lower bound

$$(3.9) \quad \|u - u_h\|_h \geq \sup_{w \in M_h} \frac{|E_h(u, w)|}{\|w\|_h}$$

is also valid, cf. [8]. This can be seen by introducing the auxiliary function $\tilde{u}_h \in M_h$ determined by

$$b_h(\tilde{u}_h, w) = (f, w) + \langle g, w \rangle + E_h(u, w) \quad \text{for all } w \in M_h.$$

Then

$$b_h(\tilde{u}_h - u_h, w) = E_h(u, w) \quad \text{for all } w \in M_h,$$

which implies that

$$(3.10) \quad \|\tilde{u}_h - u_h\|_h = \sup_{w \in M_h} \frac{|E_h(u, w)|}{\|w\|_h}.$$

Furthermore,

$$b_h(u - \tilde{u}_h, w) = 0 \quad \text{for all } w \in M_h.$$

Therefore, the decomposition

$$u - u_h = (u - \tilde{u}_h) + (\tilde{u}_h - u_h)$$

is b_h -orthogonal and as a consequence

$$\|u - u_h\|_h \geq \|\tilde{u}_h - u_h\|_h.$$

Hence, (3.9) follows from (3.10). This basic lower bound can be used to prove the divergence of the method if we can establish that the right hand side of (3.9) does not tend to zero with h . We will do this below for a suitable choice of the solution u .

However, first we will discuss a decomposition of the Morley space M_h . The function space M_h can naturally be written as a sum of two spaces

$$M_h = M_h^v + M_h^e$$

associated the vertex values and the edge values. More precisely,

$$M_h^v = \{w \in M_h : \int_e \frac{\partial w}{\partial n} ds = 0 \quad \text{for all } e \in \mathcal{E}_h\}$$

and

$$M_h^e = \{w \in M_h : w(x) = 0 \quad \text{for all } x \in \mathcal{X}_h\}.$$

For a given w , Let w_h^v be the corresponding interpolant of w on M_h^v . Since this interpolation process preserves constants locally, we have

$$(3.11) \quad \|w - w_h^v\|_{L^\infty} \leq ch \|Dw\|_{L^\infty},$$

where the constant c is independent of h .

Observe that since $\frac{\partial w}{\partial n}$ is linear on each side of an edge we must have that $\frac{\partial w}{\partial n}$ equal zero at the midpoint of each edge if $w \in M_h^v$. Similarly, if $w \in M_h^e$ then w restricted to a side of an edge is a quadratic function which is zero at the two endpoints. We can therefore conclude that the tangential derivative $\frac{\partial w}{\partial s}$ is zero at the midpoint. Also, for any $w \in M_h$ the function Δw is a constant on each triangle. In fact, if $w \in M_h^v$ then $\Delta w = 0$. This follows since

$$\int_T \Delta w dx = \int_{\partial T} \frac{\partial w}{\partial n} ds = 0.$$

Furthermore, the decomposition of M_h is in fact orthogonal with respect to the bilinear form b_h , i.e.

$$(3.12) \quad M_h = M_h^v \oplus M_h^e.$$

To see this note that if $w \in M_h^v$ and $\phi \in M_h^e$ then, since w is harmonic on each T ,

$$\int_T Dw \cdot D\phi \, dx = \int_{\partial T} \frac{\partial w}{\partial n} \phi \, ds.$$

However, on each edge the function $\frac{\partial w}{\partial n} \phi$ is a cubic function which is zero at the endpoints and the midpoint. Hence, all the boundary integrals are zero, which implies

$$b_h(w, \phi) = 0.$$

In fact, the decomposition is orthogonal locally on each triangle $T \in \mathcal{T}_h$.

Example 3.2. To construct the counter example, we let Ω be the unit square. The triangulations \mathcal{T}_h are constructed by dividing Ω into n^2 squares of size $h \times h$, where $h = 1/n$, and then dividing each square into two triangles using the negative sloped diagonals. We assume that Γ_D consists of the intersection of $\partial\Omega$ with the coordinate axis, while we assume Neumann boundary conditions on $x_1 = 1$ and $x_2 = 1$. Hence, the space M_h is assumed to consist of the Morley functions which are zero at the vertices on the coordinate axis.

We assume that the solution is given by $u(x) = x_1 x_2$. Hence, $g = x_1$ on $x_2 = 1$ and $g = x_2$ on $x_1 = 1$. Furthermore u is harmonic (i.e. $f = 0$). Therefore,

$$(3.13) \quad E_h(u, w) = b_h(u, w) - \langle g, w \rangle.$$

Note also that $u \in M_h$. Let u_h^v be the interpolant of u onto M_h^v . Hence,

$$u_h^v(x) = u(x) \quad \text{for all } x \in \mathcal{X}_h \quad \text{and} \quad \int_e \frac{\partial u_h^v}{\partial n} \, ds = 0 \quad \text{for all } e \in \mathcal{E}_h.$$

We shall show that $\lim_{h \rightarrow 0} |E_h(u, u_h^v)| / \|u_h^v\|_h$ is strictly positive. Due to the lower bound (3.9) this implies that the method diverges.

Let $u_h^e = u - u_h^v$. Then $u_h^e \in M_h^e$ and u_h^e is exactly the interpolant of u onto M_h^e . Furthermore, from (3.12) we have

$$(3.14) \quad b_h(u, u) = b_h(u_h^v, u_h^v) + b_h(u_h^e, u_h^e).$$

Since $|Du_h^v|^2$ is a piecewise quadratic it follows that for any $T \in \mathcal{T}_h$

$$\int_T |Du_h^v|^2 \, dx = \frac{|T|}{3} \sum_{m \in \mathcal{M}(T)} |Du_h^v(m)|^2,$$

where $|T|$ is the area of T and $\mathcal{M}(T)$ denote the set of the three edge midpoints, cf. [3]. However, the tangential derivative $\frac{\partial u_h^e}{\partial s}$ at each edge midpoint is zero, thus $\partial u_h^v / \partial s$ are exactly equal to the corresponding values for u . Observe that the normal component of Du_h^v at each edge midpoint is zero, we obtain

$$b_h(u_h^v, u_h^v) = \frac{h^2}{6} \sum_{T \in \mathcal{T}_h} \sum_{m \in \mathcal{M}(T)} \left| \frac{\partial u}{\partial s}(m) \right|^2.$$

However, since $u(x) = x_1x_2$, we can verify that

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{h^2}{6} \sum_{T \in \mathcal{T}_h} \sum_{m \in \mathcal{M}(T)} \left| \frac{\partial u}{\partial s}(m) \right|^2 \\ &= \lim_{h \rightarrow 0} \left\{ \frac{h^2}{6} \sum_{i=1}^n \sum_{j=1}^n (ih - jh)^2 + \frac{h^2}{3} \sum_{i=1}^n \sum_{j=1}^n ((ih)^2 + (jh)^2) \right\}. \end{aligned}$$

From this expression we obtain that

$$(3.15) \quad \lim_{h \rightarrow 0} b_h(u_h^v, u_h^v) = 1/4.$$

It is also straightforward to check that

$$b_h(u, u) = \int_{\Omega} |\nabla u|^2 dx = 2/3,$$

and hence (3.14) implies

$$(3.16) \quad \lim_{h \rightarrow 0} b_h(u_h^e, u_h^e) = 5/12.$$

From (3.13) we obtain

$$\begin{aligned} E_h(u, u_h^v) &= b_h(u, u_h^v) - \langle g, u_h^v \rangle = b_h(u_h^v, u_h^v) - \langle g, u_h^v \rangle \\ &= b(u, u) - \langle g, u_h^v \rangle - b_h(u_h^e, u_h^e) \\ (3.17) \quad &= \langle g, u - u_h^v \rangle - b_h(u_h^e, u_h^e). \end{aligned}$$

Hence, we derive from (3.11) and (3.17) that

$$\lim_{h \rightarrow 0} E_h(u, u_h^v) = - \lim_{h \rightarrow 0} b_h(u_h^e, u_h^e) = -5/12.$$

By using (3.15) this implies that

$$\lim_{h \rightarrow 0} \frac{|E_h(u, u_h^v)|}{\|u_h^v\|_h} = \lim_{h \rightarrow 0} \frac{b_h(u_h^e, u_h^e)}{b_h(u_h^v, u_h^v)^{1/2}} = 5/6.$$

The divergence of the method is therefore a consequence of the basic lower bound (3.9). \square

4. MODIFICATIONS OF THE MORLEY ELEMENT

As we have seen in the discussion above the Morley method is not well suited for solving problems of the form (1.1) when the positive parameter ε is small. The purpose of this section is to propose an alternative nonconforming finite element space which will lead to a numerical method which is robust with respect to the parameter ε . For a review of other conforming and nonconforming finite element methods for fourth order problems we refer to [3].

Let $T \subset \mathbb{R}^2$ be a triangle and consider the polynomial space on T given by

$$W(T) = \{w \in \mathbb{P}_4 : w|_e \in \mathbb{P}_2 \quad \forall e \in \mathcal{E}(T)\}.$$

Here \mathbb{P}_k denote the set of polynomials of degree k and $\mathcal{E}(T)$ denote the set of the edges of T . It is a consequence of Lemma 4.1 below that the space $W(T)$ can alternatively be defined as all functions

$$(4.1) \quad w = q + pb,$$

where $q \in \mathbb{P}_2$, $p \in \mathbb{P}_1$ and b is the cubic bubble function. We recall that the cubic bubble function b is defined by $b = \lambda_1\lambda_2\lambda_3$, where $\lambda_i(x)$ are the barycentric coordinates of x with respect to the three corners $\mathcal{X}(T)$ of T . Associated with an $x_i \in \mathcal{X}(T)$, the function $\lambda_i \in \mathbb{P}_1$ is uniquely determined by

$$\lambda_i(x_i) = 1, \quad \lambda_i(x) = 0 \quad \text{for } x \in \mathcal{X}(T), x \neq x_i.$$

Furthermore,

$$\sum_{i=1}^3 \lambda_i(x) \equiv 1.$$

A basis for the space $W(T)$, which is useful below, can be derived from the following result.

Lemma 4.1. *The space $W(T)$ is a linear space of dimension nine. Furthermore, an element $w \in W(T)$ is uniquely determined by the following degrees of freedom:*

- the values of w at the corners and edge midpoints
- $\int_e \frac{\partial w}{\partial n} ds$ for all $e \in \mathcal{E}(T)$

Proof. Since any function of the form (4.1) is in $W(T)$ we must have $\dim W(T) \geq 9$. Furthermore, it is consequence of the standard Lagrangian basis for $\mathbb{P}_4(T)$, cf. for example Chapter 3 of [2], that if $w \in \mathbb{P}_4(T)$, with $w|_{\partial T} \equiv 0$, then

$$(4.2) \quad w = pb$$

where $p \in \mathbb{P}_1$ and b is the cubic bubble function.

Assume that $w \in W(T)$ is such that the nine degrees of freedom specified in Lemma 4.1 are all zero. The proof will be completed if we can show that $w \equiv 0$. Since $w|_e \in \mathbb{P}_2$, with three roots, we must have $w|_{\partial T} = 0$. Therefore, w is of the form (4.2).

Let e be a fixed edge of T . Hence,

$$w = pb = p\lambda_+\lambda_-\lambda_e,$$

where $\lambda_e \in \mathbb{P}_1$ is the barycentric coordinate function such that $\lambda_e \equiv 0$ on e , and λ_+ and λ_- are the two other barycentric coordinates. Note that the ‘‘quadratic bubble’’, $b_e = \lambda_+\lambda_-$, is strictly positive in the interior of e . Furthermore,

$$(Dw)|_e = (pb_e D\lambda_e)|_e.$$

Note also that $\partial\lambda_e/\partial n < 0$, where n is the outward unit normal on e . Therefore, the condition

$$\int_e \frac{\partial w}{\partial n} ds = \int_e pb_e \frac{\partial \lambda_e}{\partial n} ds = 0$$

implies that p must have a root in the interior of e . Hence, $p \in \mathbb{P}_1$ has a root in the interior of all three edges. This implies that $p \equiv 0$, or equivalently, $w \equiv 0$. \square

As before let \mathcal{T}_h be a quasi-uniform and shape regular family of triangulations of Ω . The new finite element space, W_h , associated the triangulation \mathcal{T}_h will consist of continuous functions which are zero at the boundary, i.e. $W_h \subset C_0(\Omega)$. In addition:

- $w|_T \in W(T)$ for all $T \in \mathcal{T}_h$
- $\int_e \frac{\partial w}{\partial n} ds$ is continuous for all interior edges and zero for boundary edges

It follows from Lemma 4.1 that any function $w \in W_h$ is uniquely determined by the values of w at all interior vertices and edge midpoints and the mean value of $\partial w/\partial n$ for all interior edges, cf. Figure 2.

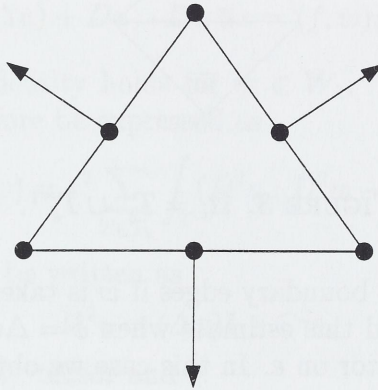


FIGURE 2. The nine degrees of freedom of the modified Morley element

These degrees of freedom also defines a local interpolation operator $I_h : H^2 \mapsto W_h$. Furthermore, since this operator preserves quadratics locally, it follows from a standard scaling argument, using the Bramble–Hilbert lemma, that

$$(4.3) \quad \sum_{T \in \mathcal{T}_h} \|v - I_h v\|_{j,T} \leq ch^{k-j} |v|_k \quad \text{for } v \in H_0^2 \cap H^k,$$

where $j = 0, 1, 2$ and $k = 2, 3$. Since the elements of W_h are continuous functions which vanish on the boundary the inclusion $W_h \subset H_0^1$ holds. However, W_h is not a subspace of H^2 . Therefore, the space W_h again leads to a nonconforming finite element method for the fourth order

problem (1.1). If $w \in W_h$ and $e \in \mathcal{E}_h$ is an edge we let $[\partial w / \partial n]_e$ denote the jump of the normal derivative on e . If e is a boundary edge then $[\partial w / \partial n]_e$ is defined by extending w by zero outside Ω . The continuity requirement on W_h implies that $[\partial w / \partial n]_e$ is a cubic function such that

$$(4.4) \quad \int_e [\frac{\partial w}{\partial n}]_e ds = 0.$$

Furthermore, by a standard scaling argument, cf. for example Section 8.3 of [2], this implies that there is a constant c independent of h such that

$$(4.5) \quad \int_e \phi [\frac{\partial w}{\partial n}]_e ds \leq ch |\phi|_{1, \Omega_e} (|w|_{2, T_e^+} + |w|_{2, T_e^-})$$

for all $\phi \in H^1$ and $w \in W_h$. Here we assume that e is an interior edge and Ω_e denote the union of the two triangles T_e^+ and T_e^- which have e as common edge, cf. Figure 3. The estimate can be generalized such

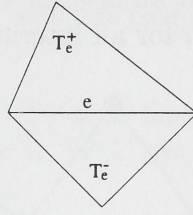


FIGURE 3. $\Omega_e = T_e^+ \cup T_e^-$.

that it is also valid for boundary edges if w is taken to be zero outside Ω . Below we shall need this estimate when $\phi = \Delta \psi - \partial^2 \psi / \partial s^2$, where s is a unit tangent vector on e . In this case we obtain from (4.5) that

$$\int_e (\Delta \psi - \partial^2 \psi / \partial s^2) [\frac{\partial w}{\partial n}]_e ds \leq ch |\psi|_{3, \Omega_e} (|w|_{2, T_e^+} + |w|_{2, T_e^-})$$

for all $\psi \in H^3$ and $w \in W_h$. By summing this estimate over all edges we derive

$$(4.6) \quad \sum_{e \in \mathcal{E}_h} \int_e (\Delta \psi - \partial^2 \psi / \partial s^2) [\frac{\partial w}{\partial n}]_e ds \leq c \frac{h}{\varepsilon} |\psi|_3 \|w\|_{\varepsilon, h}$$

for all $\psi \in H^3, w \in W_h$.

The finite element solution $u_h \in W_h$ is defined as the solution of

$$(4.7) \quad \varepsilon^2 a_h(u_h, w) + b(u_h, w) = (f, w), \quad \text{for all } w \in W_h.$$

The following theorem shows that the new nonconforming finite element method converges uniformly with respect to ε under the assumption that the solution u is uniformly bounded in H^3 .

Theorem 4.1. *Assume that the weak solution u of (1.1) is in $H_0^2 \cap H^3$ for a given $f \in L^2$. Furthermore, let $u_h \in W_h$ be the corresponding solution of (4.7). Then there is a constant c , independent of ε and h , such that*

$$\|u - u_h\|_{\varepsilon, h} \leq c(\varepsilon h + h^2)|u|_3.$$

Proof. The second Strang lemma (3.2) is still valid, i.e.

$$(4.8) \quad \|u - u_h\|_{\varepsilon, h} \leq \inf_{v \in W_h} \|u - v\|_{\varepsilon, h} + \sup_{w \in W_h} \frac{|E_{\varepsilon, h}(u, w)|}{\|w\|_{\varepsilon, h}},$$

where $E_{\varepsilon, h}(u, w)$ is given by

$$E_{\varepsilon, h}(u, w) = \varepsilon^2 a_h(u, w) + b(u, w) - (f, w).$$

Furthermore, the interpolation estimate (4.3) implies that

$$(4.9) \quad \inf_{v \in W_h} \|u - v\|_{\varepsilon, h} \leq c(\varepsilon h + h^2)|u|_3.$$

Hence, it remains to estimate $E_{\varepsilon, h}(u, w)$.

Since $u \in H^3$ it follows from the weak formulation (2.3) and the identity (2.4) that

$$\int_{\Omega} (-\varepsilon^2 D(\Delta u) + Du) \cdot Dw \, dx = (f, w), \quad \forall w \in H_0^1.$$

In particular, this identity holds for $w \in W_h$. The consistency error $E_{\varepsilon, h}(u, w)$ can therefore be expressed as

$$(4.10) \quad E_{\varepsilon, h}(u, w) = \varepsilon^2 \sum_{T \in \mathcal{T}_h} \int_T (D^2 u : D^2 w - D\Delta u \cdot Dw) \, dx$$

The tensor $D^2 u$ can be written as

$$D^2 u = (\Delta u)\mathbb{I} + C^2 u,$$

where \mathbb{I} is the identity tensor and

$$C^2 u = -\operatorname{curl} \operatorname{curl} u = \begin{pmatrix} -\frac{\partial^2 u}{\partial x_2^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} \\ \frac{\partial^2 u}{\partial x_1 \partial x_2} & -\frac{\partial^2 u}{\partial x_1^2} \end{pmatrix}$$

Furthermore,

$$(\Delta u)\mathbb{I} : D^2 w = (\Delta u)\Delta w$$

and therefore

$$(4.11) \quad \int_T (\Delta u)\mathbb{I} : D^2 w \, dx = \int_{\partial T} \Delta u \frac{\partial w}{\partial n} \, ds - \int_T D(\Delta u) \cdot Dw \, dx.$$

On the other hand, since each row of $C^2 u$ is divergence free, we have

$$\operatorname{div}(C^2 u \cdot Dw) = C^2 u : D^2 w,$$

and from this we obtain

$$\int_T C^2 u : D^2 w \, dx = \int_{\partial T} n \cdot C^2 u \cdot Dw \, ds.$$

However, by combining this identity with (4.10) and (4.11), and by using the fact that the tangential component of Dw is continuous on each edge we obtain

$$\begin{aligned} E_{\varepsilon,h}(u, w) &= \varepsilon^2 \sum_{e \in \mathcal{E}_h} \int_e (\Delta u + n \cdot C^2 u \cdot n) \left[\frac{\partial w}{\partial n} \right]_e ds \\ &= \varepsilon^2 \sum_{e \in \mathcal{E}_h} \int_e \left(\Delta u - \frac{\partial^2 u}{\partial s^2} \right) \left[\frac{\partial w}{\partial n} \right]_e ds. \end{aligned}$$

It therefore follows from (4.6) that

$$E_{\varepsilon,h}(u, w) \leq ch\varepsilon |u|_3 \|w\|_{\varepsilon,h}$$

and together with (4.8) and (4.9) this implies the desired estimate. \square

Note that in the limit as ε tends to zero the error estimate above gives the bound

$$\|u - u_h\|_1 \leq ch^2 |u|_3.$$

Since W_h is a subset of H_0^1 which contains C^0 -quadratics it is straightforward to derive this estimate for the reduced problem

$$(4.12) \quad \begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

However, if $f \in L^2$ and Ω is convex then we can only expect that the solution of the reduced problem (4.12) is in H^2 . The following modification of Theorem 4.1 is consistent with the proper result for the reduced problem in this case.

Corollary 4.1. *Let u and u_h be as in Theorem 4.1 above. There is a constant c , independent of ε and h , such that*

$$\|u - u_h\|_{\varepsilon,h} \leq ch(|u|_2 + \varepsilon|u|_3).$$

Proof. In the proof above we only need to modify the estimate (4.9). From (4.3) we can alternatively obtain

$$\inf_{v \in W_h} \|u - v\|_{\varepsilon,h} \leq ch(|u|_2 + \varepsilon|u|_3),$$

while as before

$$\sup_{w \in W_h} \frac{|E_{\varepsilon,h}(u, w)|}{\|w\|_{\varepsilon,h}} \leq ch\varepsilon |u|_3.$$

Hence, the desired estimate follows from (4.8). \square

$\varepsilon \backslash h$	2^{-3}	2^{-4}	2^{-5}	2^{-6}
2^0	0.3359	0.1790	0.09108	0.0457
2^{-2}	0.3016	0.1589	0.08061	0.0405
2^{-4}	0.1519	0.07627	0.03819	0.0190
2^{-6}	0.0564	0.0229	0.0107	0.0052
2^{-8}	0.0416	0.0113	0.0036	0.0014
2^{-10}	0.0406	0.0103	0.0026	0.0007
Poisson	0.0405	0.0101	0.0026	0.0006
Biharmonic	0.3386	0.1806	0.0919	0.0462

TABLE 2. The relative error measured by the energy norm

Example 4.1. We will redo the computations we did in Example 3.1, the only difference being that we use the new finite element space W_h instead of the Morley space M_h . The results for the relative error are given in Table 2.

These results clearly demonstrates the improved behaviour of the modified method when ε is close to zero. In particular, when ε is large the convergence appears to be linear with respect to h , while we observe nearly quadratic convergence when ε is small. This is in fact consistent with Theorem 4.1, since $|u|_3$ is independent of ε . \square

Let us compare the dimension of the space W_h with the dimension of the Morley space M_h . Let $|\mathcal{X}_h|$ be the number of interior vertices in the triangulation \mathcal{T}_h and let $|\mathcal{E}_h|$ be the corresponding number of interior edges. It follows from the discussion above that

$$\dim(W_h) = |\mathcal{X}_h| + 2|\mathcal{E}_h|,$$

while the dimension of the Morley space M_h is $|\mathcal{X}_h| + |\mathcal{E}_h|$. Hence, since $|\mathcal{E}_h| \approx 3|\mathcal{X}_h|$, the replacement of M_h by W_h leads to an increase in the number of unknowns of approximately 75%. Below we shall briefly discuss an alternative nonconforming finite element space \tilde{W}_h . This space has the same robustness property as W_h with respect to parameter ε , and the dimension of the space is $3|\mathcal{X}_h| + |\mathcal{E}_h|$, which represents an increase of approximately 50% as compared to the Morley space. On the other hand, the sparsity structure of the space W_h is more favourable than the sparsity structure of the new space.

If T is a triangle let

$$\tilde{W}(T) = \{w \in \mathbb{P}_4 : w|_e \in \mathbb{P}_3 \quad \forall e \in \mathcal{E}(T)\}.$$

It can easily be verified, using arguments as above, that $\tilde{W}(T)$ can be equivalently defined as all functions

$$(4.13) \quad w = q + pb,$$

where $q \in \mathbb{P}_3$, $p \in \mathbb{P}_1$ and b is the cubic bubble function. Hence, $\tilde{W}(T)$ is a linear space of dimension twelve. The basis given below will be used to define a proper finite element space.

Lemma 4.2. *The space $\tilde{W}(T)$ is a linear space of dimension twelve. Furthermore, an element $w \in \tilde{W}(T)$ is uniquely determined by the following degrees of freedom:*

- the values of w and Dw at the corners
- $\frac{\partial w}{\partial n}$ at the midpoint m_e of e , $\forall e \in \mathcal{E}(T)$

Proof. Since $\dim \tilde{W}(T) = 12$ it is enough to show that the twelve degrees of freedom determine elements of $\tilde{W}(T)$ uniquely. Assume $w \in \tilde{W}(T)$ such that the twelve degrees of freedom are all zero. Since $w|_e \in \mathbb{P}_3$, with a double zero at each endpoint we must have $w|_{\partial T} = 0$. From (4.13) we obtain that w is of the form

$$w = pb$$

where $p \in \mathbb{P}_1$ and b is the cubic bubble function. Note that

$$(Dw)|_e = (pD(b))|_e$$

for each edge e and that

$$(Db) \cdot n(m_e) \neq 0.$$

Therefore, the three conditions $(\partial w / \partial n)(m_e) = 0$ implies that $p(m_e) = 0$. Hence, $p \equiv 0$, or equivalently $w \equiv 0$. \square

As above the polynomial space $\tilde{W}(T)$ can be used to define a finite element space \tilde{W}_h consisting of functions which are locally in $\tilde{W}(T)$, with w and Dw continuous at each vertex, and with $\partial w / \partial n$ continuous at the midpoint of each edge. If $w \in \tilde{W}_h$ and $e \in \mathcal{E}_h$ then $[\partial w / \partial n]_e$ is a cubic function which has a root at the two endpoints and the midpoint. Therefore, the property (4.4) holds for all functions in \tilde{W}_h . Hence, by arguing almost exactly as above, we can also derive the result of Theorem 4.1 in this case.

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