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A streamline front tracking method for two- and three-phase flow including capillary forces.

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1. Introduction/Outline

Currently, there seems to be a trend within the petroleum industry to use fast and accurate methods to simulate reduced sets of equations. Typically streamline and/or front tracking methods are used to solve the hyperbolic Buckley-Leverett equation for two-phase flow [**8, 12, 14**]. This allows for estimates of reservoir performance on more complete geological data, possibly without using upscaling. However, important physical phenomena are neglected which may greatly influence the overall fluid distribution.

In this work we consider models of multi-phase flow which do include capillary forces. We also allow for three phases. In particular we shall investigate a streamline front tracking method (SFTM) [**1**]. This method is based on calculating streamlines locally around grid points. The hyperbolic part of the problem is then calculated using front tracking. Since the front tracking method requires a Riemann solver, we do currently only consider a triangular model for three phase flow, i.e., the advective flux of the gas phase does only depend on gas-saturation and not on the other saturations. To account for capillary effects we use operator splitting [**6**]. The hyperbolic part of the solution is projected onto a regular grid, and a complete solution in each time step is obtained by solving a parabolic equation.

The paper is organized as follows: In Section 2 we state a standard model for two- and three-phase flow. In Section 3 the streamline front tracking method (SFTM) [**1**] is developed. A modified method of characteristics (MMOC) [**5**] is also briefly discussed and we we give an outline of a fast marching method (FMM) [**7**]. These methods are used in the two-phase case as comparisons with the SFTM-method proposed here. In Section 4 we discuss a possible extension of the SFTM-method to three phase flow. This extension is based on a triangular approximation of the fully coupled three-phase problem, and the so-called H-set method is used to solve associated Riemann problems. Some consequences of the triangular approximation are also discussed. Finally, in Section 5 we present some numerical

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experiments for two- and three-phase flow to demonstrate the methodology, and a summary and conclusions are given in Section 6.

2. Governing equations

The basic equations describing multi-phase immiscible flow in a porous medium, say water (w), gas (g) and oil (o), is given by mass balance equations and Darcy's law. Assuming that the flow is incompressible, and that gravity can be neglected, the equations can be written in a global pressure/total velocity formulation, see [**3, 4**], as follows:

$$\nabla \cdot \mathbf{v} = q(\mathbf{x}, t), \tag{1}$$

$$\mathbf{v} = -\lambda_T(\mathbf{x}, S_\alpha)\mathbf{K}(\mathbf{x}) \cdot \nabla p, \tag{2}$$

$$\phi \frac{\partial S_\alpha}{\partial t} + \nabla \cdot [F_\alpha \mathbf{v} - \epsilon_\alpha \nabla \cdot D_\alpha(\mathbf{x}, S_\alpha) \nabla S_\alpha] = q_\alpha(\mathbf{x}, t). \tag{3}$$

Here ϕ and $\mathbf{K}$ are the porosity and absolute permeability of the porous medium; S_α, $\mathbf{v}_\alpha$, $k_{r\alpha}$ and μ_α are, respectively, the reduced saturation, Darcy velocity, relative permeability, and viscosity of phase α; p is a global pressure derived from the phase pressures and the capillary pressures, see [**4**]; $\lambda_\alpha = k_{r\alpha}/\mu_\alpha$ denote the mobility ratio, $\lambda_T = \sum_\alpha \lambda_\alpha$ is the total mobility, and $F_\alpha = \lambda_\alpha/\lambda_T$ is the fractional flow of phase α; $\mathbf{v} = \sum_\alpha \mathbf{v}_\alpha$ is the total velocity; D_α is a simplified capillary diffusion/dispersion term and ϵ_α is (typically) a small parameter which gives the relative importance of advective and capillary/dispersive forces; finally q, q_α account for injection and production wells. An additional constraint is given by $\sum_\alpha S_\alpha = 1$, and we note that $\sum_\alpha F_\alpha = 1$. To close the above system we need constitutive relationships for the capillary pressures and relative permeabilities. We shall assume that the concentration of gas in oil phase is constant (no phase transfer), which means that do not have to calculate phase pressures. Thus, it suffices to specify D_α directly rather than capillary pressures. In the experiments reported below we shall use $D_\alpha(\mathbf{x}, S_\alpha) = \mathbf{K}$. For the relative permeabilities use quadratic forms $k_{r\alpha} = S_\alpha^2$, combined with Stone models for the relative permeability of oil in the three-phase case, see Section 5. In the numerical experiments, we consider a quarter-of-a-five-spot problem. Thus, no-flow conditions are imposed on the boundaries.

3. Solution Strategy

To decouple the Pressure/Velocity equations (1), (2), from the saturation equations (3), we use sequential time stepping. Thus, for a given saturation-field, say at time t^n, we calculate a new velocity field. The saturation field is then advanced to a new time-step t^{n+1} by solving (3), using the most recent velocity field. This is continued sequentially up to a predetermined time $t = T$. To recover conservative and accurate fluxes we have used a control volume finite element method [**5**]. From these fluxes we obtain a velocity field in the lowest order Raviart-Thomas space RT_0, on a regular Cartesian grid.

To solve the parabolic saturation Equation (3) with a given velocity field, we again use operator splitting. First the purely hyperbolic part of Equation (3) is solved to advect the solution up to a new time level, then diffusion/dispersion is accounted for by solving a heat type equation on a regular grid, see [**6**] and references therein.

In previous work a Modified-Method-of-Characteristics (MMOC) has been used to solve the two-phase flow problem based on the ideas outlined above, see [5]. This method works excellent when the wave structure of the solution is known *a priori*. For an established front it has been shown that when the front is resolved this method gives the correct front-width of $O(D/b)$, where D and b is respectively an estimate of the diffusion and the residual of the linearized fractional flow function. Below we present two alternative methods to the MMOC method, which both preserve the shape of self-sharpening fronts and are more flexible than the MMOC method in the sense that no a priori knowledge of the wave structure is required. However, the fast marching method presented below is based on the restriction that the initial data are monotone and that level sets propagate outwards from their source.

3.1. A streamline front tracking method (SFTM). For simplicity we assume that the computational domain is discretized by a regular Cartesian grid such that the velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}) \in RT_0$ is given. Furthermore, assume that approximate saturation values S^n is known at the cell centers at time-level t^n. Saturation values refer to water-saturation S_w in case of two-phase flow, and water- and gas-saturation S_w, S_g in case of three-phase flow. To obtain saturation values S^{n+1} at time-level t^{n+1}, we split Equation (3) into a hyperbolic part:

$$S_t + \mathbf{v} \cdot \nabla F(S) = 0, \tag{4}$$

and a parabolic heat type equation

$$S_t = \epsilon \nabla \cdot (D \nabla S). \tag{5}$$

In a standard operator-splitting fashion, equation (3) is solved in two steps: First, Equation (4) is solved with S^n as initial condition to obtain a first approximation, say $\bar{S}^{n+1}$ at the next time-level. Then, saturation values at the next time-level are obtained by solving (5), with $\bar{S}^{n+1}$ as initial condition. Now, consider the solution of Equation (4): Observe that on streamlines $\mathbf{r} = \mathbf{r}(\xi)$ such that

$$\frac{d\mathbf{r}}{d\xi} = \mathbf{v}, \tag{6}$$

Equation (4) becomes one-dimensional:

$$S_t + F_\xi(S) = 0. \tag{7}$$

We exploit this to obtain new saturation-values at cell centers $\mathbf{x}_I$, in the following way: First, trace streamlines (6) analytically for $\mathbf{r}(0) = \mathbf{x}_I$ and $-\xi_{\max} < \xi < \xi_{\max}$. Here $\xi_{\max} = |\lambda_{\max}|(t^{n+1} - t^n)$ with $\lambda_{\max}$ being an estimate of the maximum wave speed of the system, such that the streamline cover the domain of dependence for $(\mathbf{x}_I, t^{n+1})$. The streamline is only traced in the upstream direction if all the wave speeds are positive. The piecewise constant cell values of the saturations are then projected onto these local streamlines, thus defining piecewise constant initial conditions for Equation (7). This conveniently arrange for Equation (7) to be solved by the front tracking method: For a two-phase (scalar) problem the flux function F, is simply replaced by a piecewise linear approximation F^δ. We then solve the Riemann problem associated with each of the initial discontinuities. Since F^δ is piecewise linear, the rarefaction part of the solution is approximated by discontinuities. Thus, the solutions will always consist of a set of fronts traveling with distinct speeds. We then have to keep track of the position of each such

front, and solve new Riemann problems whenever fronts collide, thereby obtaining a global solution $\bar{S}$ for any time $t > t^n$ on the streamline. In particular we will obtain approximate saturation values $\bar{S}^{n+1}$ at the cell centers. We refer to [**11**] for more details about the front tracking method.

The main advantages of the front tracking method, is that the method is *super fast* [**11**], and preserves the frontal structure of the solutions extremely well. On the other hand, since the method heavily depends on solving Riemann problems, it is not easy to extend the method to three-phase flow. In Section 4, we will discuss a solution strategy for so-called triangular systems which may be a step towards this end. Equation (5) may be solved by finite element or finite difference type of methods. In this work we have for convenience used a standard explicit central finite difference scheme. Note that a local time step, $\Delta t_{diff} \leq t^{n+1} - t^n$, is required to satisfy the stability constraint inherent in the explicit finite difference method.

3.2. A fast marching method (FMM). An alternative to the front-tracking approach is the fast marching method for reservoir simulation developed in [**7**]. This methodology has shown to give very fast and accurate results for Buckley-Leverett type problems. Here we extend the work in [**7**] by including diffusion/dispersion-effects using operator splitting as described above. To outline this procedure we consider the hyperbolic part of the transport equation (3):

$$\frac{\partial S_w}{\partial t} + \mathbf{v} \cdot \nabla F_w(S_w) = 0, \qquad S(\mathbf{x}, 0) = S_0(\mathbf{x}).$$

We will assume that the initial function S_0 is monotone, and that the velocity field is so that the level-sets of the saturation function never pass a point twice. This is not a restrictive assumption for the reservoir problem, since the flow will be directed from injection wells towards production wells. Following [**7**], we consider a level set $S(x,t) = k$ of the saturation function. Let the front be defined by $\Xi_k(t) = \{\mathbf{x} : S(\mathbf{x}, t) = k\}$. According to the classical method of characteristics, the front will move with a characteristic speed given by $\mathbf{V}(\mathbf{x}) = \frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x})F'(k)$. Let $T_k(\mathbf{x})$ be the time $\Xi_k(t)$ crosses the point $\mathbf{x}$. The arrival time satisfies the Eikonal equation

$$\nabla T_k \cdot \mathbf{v}(\mathbf{x}) = \frac{1}{F'(k)}, \qquad F'(k) > 0. \tag{8}$$

The evolution problem for the propagating front is hence reformulated in an Eulerian framework.

The equation (8) can be solved rapidly by means of the fast marching method (see, e.g., [**17, 16**]). Since the flow is directed from injection towards production wells, information will flow in one direction, that is, from regions with smaller to regions with larger arrival times. Hence, the arrival time at a grid point can only depend on points with smaller values. This fact motivates an upwind discretization for the Eikonal equation. For the comparison results presented later we have applied a first order 9-point scheme based on linear approximation of streamlines locally through each grid point. When the Eikonal equation is solved to a prescribed final time, fronts corresponding to larger values of S may have crossed fronts corresponding to smaller values of S. This corresponds to the development of shocks. However, since the level sets have been transported according to the characteristic speed, an approximation to the correct entropy weak solution can be obtained from the multivalued solution by means of the collapse transform, i.e., a proper vertical

averaging of the multivalued solution, see [2], $S(\mathbf{x}) = \sum_{i=0}^{2p(\mathbf{x})}(-1)^i \hat{S}_i(\mathbf{x})$. Here the multivalued solution $\hat{S}_i$ is obtained by considering the positions of the level sets. The resulting method is not subject to CFL-conditions. The method described in this section will be described in detail in a forthcoming paper.

4. Triangular systems

A possible extension of the SFTM approach to three-phase flow, is to use an approximate Riemann solver to generate front speeds. However, to our knowledge, the construction of accurate and reliable Riemann solvers for fully coupled three-phase flow is not a trivial task, and we have chosen a different approach here: Observe that the viscosity of the gas phase is usually at least an order less than the viscosities of the liquid phases. Motivated by this fact it seems reasonable to assume that the fractional flow of the gas-phase can be approximated by a flux function which only depends on the saturation of the gas phase. Thus, we may consider the following 2×2-triangular hyperbolic system, as an approximation to the hyperbolic part of the fully coupled system (3):

$$\frac{\partial S_g}{\partial t} + \frac{\partial}{\partial x} F_g(S_g) = 0, \tag{T1}$$

$$\frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x} F_w(S_g, S_w) = 0. \tag{T2}$$

Systems of this type has been investigated in [**9, 10, 13**], and existence and uniqueness of the solution to the Riemann problem is shown under general conditions.

4.1. A Riemann solver for triangular systems. The numerical construction of a solution for a Riemann problem associated with equations (T1)-(T2) was developed by Gimse [**9**], and is rather involved. The idea is to solve (T1) first using the front tracking method. Thus, the approximate solution to (T1) consists of a set of constant states, say $S_g^L = S_g^1 < S_g^2 < \ldots < S_g^{N+1} = S_g^R$, separated by jump discontinuities traveling with the Rankine-Hugoniot shock speed:

$$s_i = \frac{F_g(S_g^{i+1}) - F_g(S_g^i)}{S_g^{i+1} - S_g^i}, \quad i = 1, 2, \ldots, N. \tag{9}$$

Within each wedge of the solution fan, the fractional flow of the water phase depends only on the water saturation and is given as:

$$F_w^i(S_w) \stackrel{def}{=} F_w(S_g^i, S_w), \quad i = 1, 2, \ldots, N+1.$$

Thus, we may easily solve for the water saturation within each wedge once we know the left- and right-hand state of the water saturation within the wedge. Obviously, successive left- and right-hand states over the discontinuities in the gas-phase must also satisfy the jump condition:

$$\frac{F_w^{i+1}(S_w^{i+1}) - F_w^i(S_w^i)}{S_w^{i+1} - S_w^i} = s_i, \quad i = 1, 2, \ldots, N, \tag{10}$$

There are infinitely many states S_w^i, which satisfy conditions (10), leading to the definition of the so-called H-sets:

$H_{1,in}$: The set of S_w-values in region S_g^L, that can be reached from S_w^L with speed $\sigma \leq s_1$.

$H_{i,in}$: The set of S_w-values in region S_g^i, $i = 2, 3, \ldots N$, that can be reached from $H_{i,out}$ with speed σ such that $s_{i-1} \leq \sigma \leq s_i$.

$H_{i+1,out}$: The set of S_w-values in region S_g^{i+1}, $i = 1, 2, \ldots N+1$, that can be reached from $H_{i,in}$ by a shock with speed s_i.

The solution can now be assembled from these sets by connecting S_w^R to S_w^L by tracking through admissible waves given by the H-sets. This procedure is described schematically in the following diagram:

$$\begin{aligned} S_w^R \rightarrow H_{n+1,out} \rightarrow H_{n,in} \rightarrow \cdots \\ \rightarrow H_{i+1,out} \rightarrow H_{i,in} \rightarrow \cdots \\ \rightarrow H_{2,out} \rightarrow H_{1,in} \rightarrow S_w^L \end{aligned}$$

Note that a jump from one set to the next always happens at the first possible value in the current H-set.

Gimse has shown [**9**] that the above construction gives a *unique* solution of equation (T2) if the following conditions hold:

$$(A)\ F_g(0) = 0, \quad (B)\ F_g(1) = 1, \quad (C)\ F_w(S_g, 1 - S_g) = 1 - F_g(S_g)$$

$$(D)\ \tfrac{\partial F_w}{\partial S_w} \geq 0, \quad (E)\ \tfrac{\partial F_w}{\partial S_g} < 0, \quad (F)\ F_g'(S_g) \geq 0.$$

Conditions (A) and (B) are always satisfied if the transport equations are properly scaled. Conditions (D) and (F) are usually satisfied if gravity is neglected. Condition (E) guarantees that successive F_w^i do not cross each other. This eventually simplifies the construction of the H-sets. However, it has been shown that condition (E) is not necessary for the existence and uniqueness of the solution , and can therfore be removed as a necessary condition. Finally, condition (C) guarantees that the endpoints of successive F_w^i, are connected by correct slopes given by the shock speeds s_i. Endpoints are here defined as points where oil is at residual saturation $S_o = 0$, or $S_g + S_w = 1$. Note that this condition is always satisfied for *genuinely* triangular systems (i.e, $F_g \equiv F_g(S_g)$), since $F_o = 0$ at residual oil-saturation $S_o = 0$.

4.2. Can physical three-phase transport equations be approximated by a triangular system? Since the fractional-flow of the gas-phase is nearly independent of the water saturation it seems natural to decouple the gas-phase from the other phases by simply plug in a value for the water-saturation, S_w^0, in the gas fractional-flow:

$$F_g(S_w, S_g) \approx F_g(S_w^0, S_g) \stackrel{\text{def}}{=} F_g(S_g). \tag{11}$$

Possibly, we can choose S_w^0 so that the decoupled fractional-flow function is as close as possible to the complete function in some norm over the admissible section of state space. However, S_w^0 is chosen more or less arbitrary to be $S_w^0 = 0$ in the following experiments. In Figures 3 and 4 we compare the solution of the triangular system, with the solution of the full system in two different one dimensional cases. In Case 1 we get excellent agreement between the two solutions. In Case 2 the full solution for the gas-phase consists of two rarefaction waves connected by an intermediate shock wave. This wave structure cannot be captured by the triangular system with a simplified fractional-flow function as given by (11). A possible solution to this problem can be to let S_w^0 be defined locally for each Riemann problem to be solved, which will allow for some feedback. This can be combined with the fact that the total mobility λ_T act as an approximate invariant for the transport.

Thus $\lambda_T(S_w, S_g) \approx \lambda_T(S_w^0, S_g)$ may be used to eliminate S_w from F_g in a more accurate way. However, we will not pursue the idea further here.

Another, difficulty that arises from the approximation (11), is that condition (C) is violated since

$$F_w(1 - S_g, S_g) = 1 - F_g(1 - S_g, S_g) \neq 1 - F_g(S_w^0, S_g).$$

If condition (C) is not satisfied the construction of the solution may fail in two ways close to values where $S_o = 0$: Either the construction of appropriate H-sets will fail, or the tracking of admissible waves becomes impossible. Since $S_o \approx 0 \implies S_w \approx 1 - S_g$, we may circumvent the problem by replacing S_w^0 with $1 - S_g$ in (11) when the oil phase is close to residual. Again this require a local definition of S_w^0 which has not been implemented in our codes yet. For the robustness of the method this needs to be done.

5. Numerical experiments

5.1. Two-phase flow: To obtain a comparison between the different methods we first ran a series of experiments with a two-phase flow problem for a quarter-of-a-five-spot problem with noflow conditions at the boundaries. The implementation has been done in two space dimensions. The permeability field in these experiments was generated from a (log-normal?) distribution and is plotted in Figure 1, together with typical flow directions. The permeability contrast in this figure is a factor of 10. To avoid difficulties at the corners the initial saturation is given by:

$$S_0(r) = \begin{cases} 1 - (r/0.3)(1 - \sqrt{1/2}), & r \leq 0.3, \\ 0, & r > 0.3, \end{cases} \tag{12}$$

where r denote the radial distance from the lower left corner and $\sqrt{1/2}$ is the shock saturation of the Buckley-Leverett profile. In Figure 2, saturation contours are plotted after four time steps for one of the experiments performed. Observe that the results are in excellent agreement. Also note that the three methods allow for very long time steps.

The complexity of the MMOC method is comparable with the SFTM approach, whereas the FMM method gives a much faster advection solver. In Table 1, we have attempted to compare the efficiency of these methods. To do so we fixed a velocity field and did a pure advection of the saturation front up to four time steps. The results in the table indicates that when the number of nodes is increased by a factor 4, the run time increases by a factor 4 for the fast marching method and a factor of 6.5 for the front tracking method. Thus, the fast marching method is optimal in this sense, whereas the front tracking method becomes sub optimal because the number of fronts increase on each local streamline, in addition to the increase of local streamlines.

5.2. Three-phase flow: In figures 3 and 4 we have compared solutions of triangular approximations with the solutions of corresponding full three phase flow problems in one space dimension. Below follows mobilities, fractional flows for the triangular approximations and initial conditions:

Case 1:

$$\lambda_g(S_g) = S_g^2, \qquad \lambda_w(S_w) = S_w^2/10, \qquad \lambda_o(S_o) = S_o^2/10.$$

$$F_g(S_g) = \frac{\lambda_g(S_g)}{\lambda_{\text{tot}}(S_g, 0)}, \qquad F_w(S_g, S_w) = \frac{\lambda_{\text{tot}}(1 - S_g, 0)}{\lambda_{\text{tot}}(S_g, 1 - S_g)} \cdot \frac{\lambda_w(S_w)}{\lambda_{\text{tot}}(S_w, 0)},$$

$$S_g^0 = \begin{cases} 0.6 & \text{if } x < 0.4, \\ 0.2 & \text{if } x > 0.4, \end{cases} \quad \text{and } S_w^0 = \begin{cases} 0.3 & \text{if } x < 0.4, \\ 0.7 & \text{if } x > 0.4, \end{cases}$$

Case 2:

$$\lambda_g(S_g) = 50 S_g, \qquad \lambda_w(S_w) = S_w^3, \qquad \lambda_o(S_o) = 3.14 S_o (1 - S_w)^2 (1 - S_g)^2.$$

$$F_g(S_g) = \frac{\lambda_g(S_g)}{\lambda_{\text{tot}}(S_g, 0)}, \qquad F_w(S_g, S_w) = \frac{\lambda_w(S_w)}{\lambda_{\text{tot}}(S_g, S_w)}.$$

$$S_g^0 = \begin{cases} 0.25 & \text{if } x < 0.4, \\ 0.1 & \text{if } x > 0.4, \end{cases} \quad \text{and } S_w^0 = \begin{cases} 0.75 & \text{if } x < 0.4, \\ 0.0 & \text{if } x > 0.4. \end{cases}$$

A standard upwind difference scheme is used to generate the reference solutions for the fully coupled system. In Case 1 the triangular flow functions are carefully constructed to satisfy the conditions (A-F), see [**15**] for further details. In Case 2, only the flow functions for gas is changed and as a result condition (C) is not satisfied. We are generally not guaranteed that a solution exists, but for the chosen initial value problem it did. We note that the shape of these functions differ significantly from the full three-phase model, especially for Case 1. From these one dimensional examples we conclude that the triangular model may and may not give a good approximation to a fully coupled three-phase flow problem.

To investigate the robustness of the SFTM method we have constructed a two dimensional test example. To make sure that the conditions (A-F) are satisfied the fractional-flows of Case 1 have been used, and we use the same permeability field and boundary conditions as for the two-phase problem of Section 5.1. Initially the saturation for the water is given by Equation (12), and the remaining volume at any point is partitioned into 10% gas and 90% oil. The result is shown in Figure 5 and appears to be much more diffused than the solutions of the two-phase problem. However, as seen from Figure 3, the shape of the solution is really due to the shape of the fractional flows. Again we note long and stable time steps are used.

6. Summary/conclusions

We have presented a front tracking streamline method (SFTM) for multi-phase flow in porous media. The main advantage of this method is to handle the advective part of nonlinear transport using streamline information, and still be able to solve for diffusive/dispersive effects on a regular grid. The method has been compared with a Modified Method of Characteristics (MMOC) and a Fast Marching (FMM) approach for two-phase flow problems. The solutions obtained seems to be equally accurate. The SFTM and the MMOC are comparable in computational efficiency, whereas the FMM has a much faster advection solver. However, compared to the MMOC and the FMM, the SFTM is more flexible and has less restrictions with respect to the complexity of the problems that may be solved.

Using the H-set method the SFTM approach has been extended to solve three-phase flow problems which are triangular. A 2×2 system is triangular if one of the fractional-flows only depend on one of the saturations, whereas the other depends on both saturations. Since most three-phase flow problems are fully coupled in both saturations, triangular systems can only be approximate. However, because

	Grid	Time		Scaling	
		FMM	SFTM	FMM	SFTM
Homogeneous	33×33	0.09	0.14	–	–
	65×65	0.38	0.89	4.22	6.36
	129×129	1.59	5.93	4.18	6.66
	257×257	6.98	43.30	4.39	7.30
Heterogeneous	33×33	0.10	0.17	–	–
	65×65	0.39	1.08	3.90	6.35
	129×129	1.56	7.52	4.00	6.96

TABLE 1. Comparison of run times for the advection solver using the FMM and the SFTM, with $\Delta t = 0.5$ and $T = 2.0$.

the viscosity of the gas is much smaller than the viscosities of the liquid phases the gas-phase is often nearly decoupled from the other phases. We have shown that a *naive* triangular approximation of a fully coupled system may and may not reproduce good approximations to the solutions of the complete system. We have also suggested ways to improve the triangular approximation when this gives poor comparisons with a full three-phase flow system.

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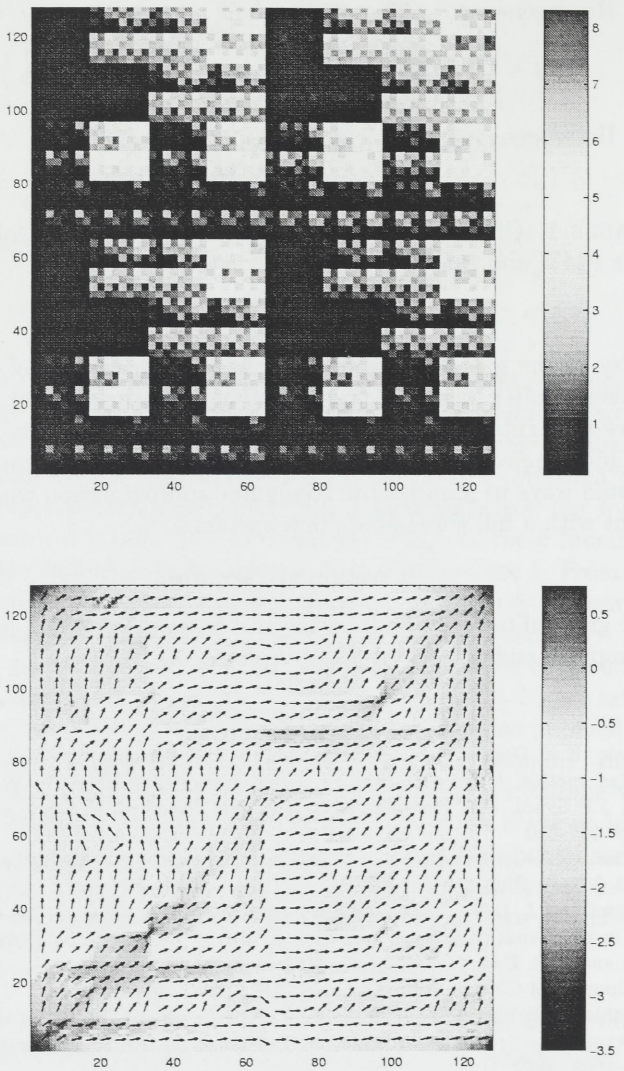

FIGURE 1. Permeability and velocity fields.

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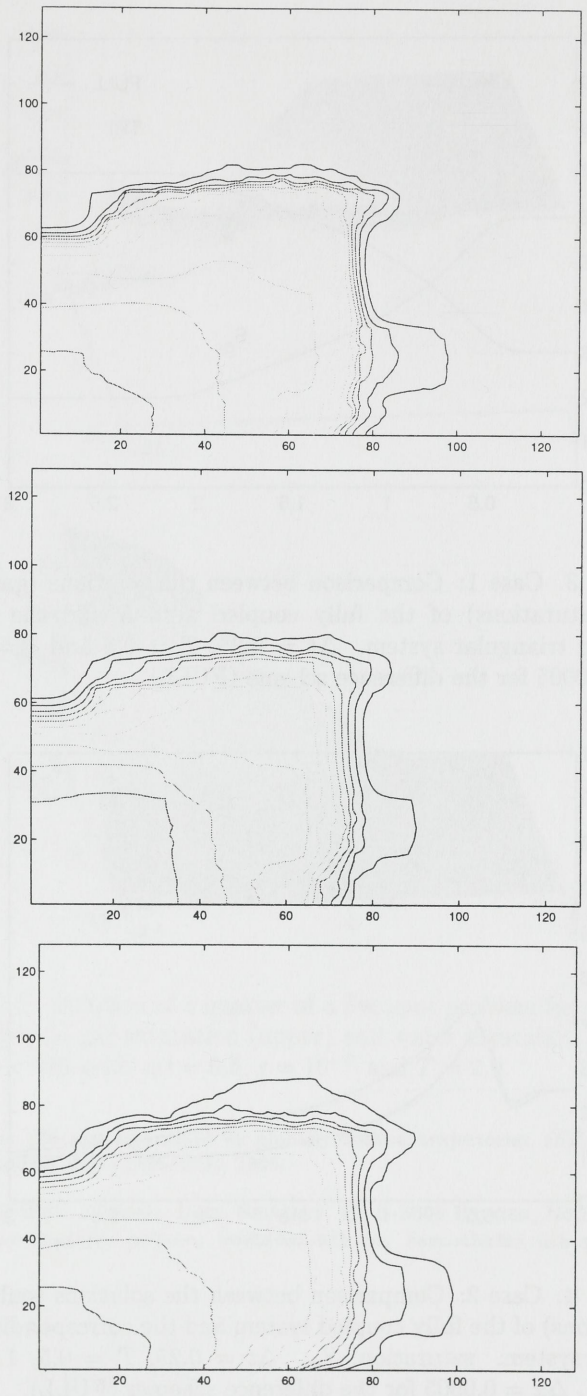

FIGURE 2. Comparisons between the MMOC (upper), the FMM (middle) and the SFTM (lower) on 129 × 129 grid. $\Delta t = 0.5$, $\epsilon = 10^{-2}$ and $T = 2.0$.

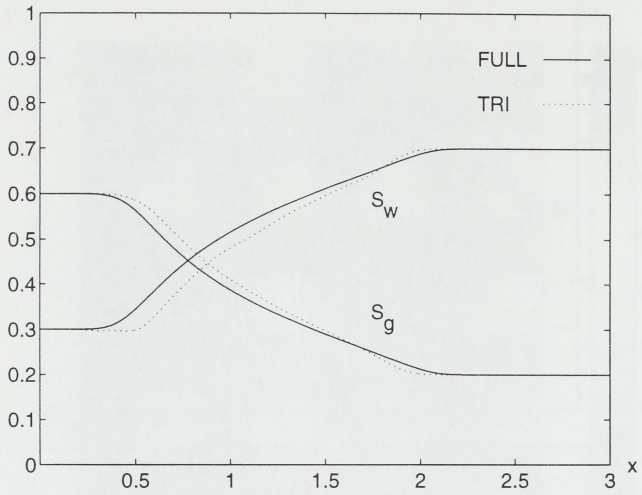

FIGURE 3. Case 1: Comparison between the solutions (gas- and water saturations) of the fully coupled system and the corresponding triangular system. $\Delta t = 0.25$, $T = 0.5$ and $\epsilon = 0.01$. $\Delta x = 0.0005$ for the difference scheme (FULL).

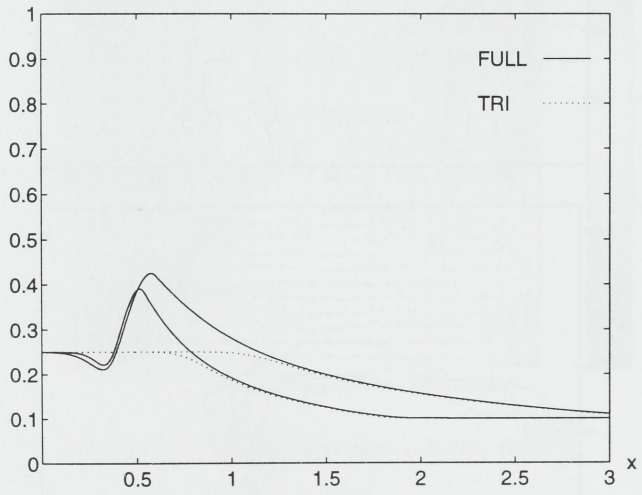

FIGURE 4. Case 2: Comparison between the solutions (only gas-saturations) of the fully coupled system and the corresponding triangular system. saturation, S_w. $\Delta t = 0.25$, $T = 0.5$, 1.0 and $\epsilon = 0.01$. $\Delta x = 0.0005$ for the difference schemes (FULL).

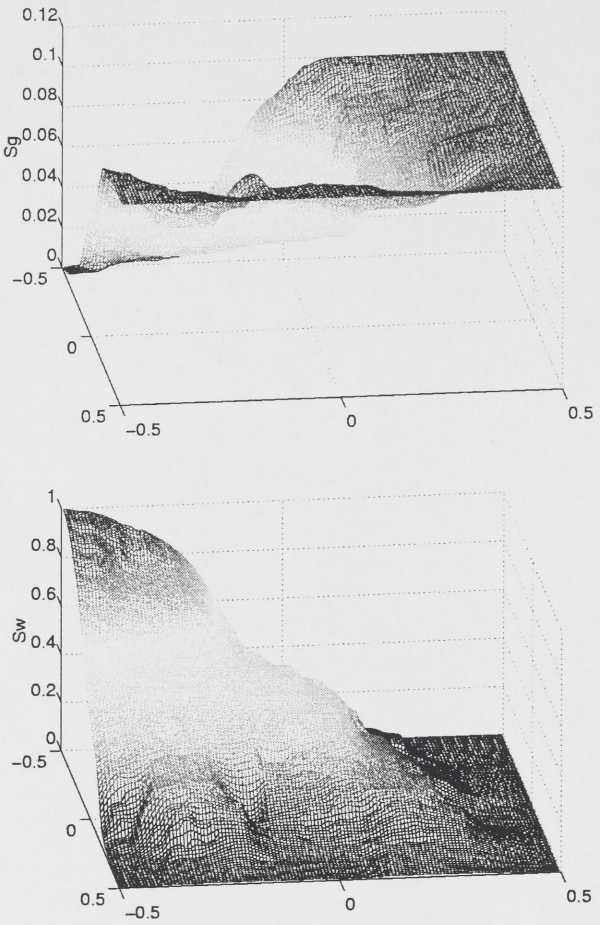

FIGURE 5. Solution of a quarter of a five spot problem for a triangular system, gas saturation (upper) and water saturation (lower) on 129 × 129 grid. $\Delta t = 0.5$, $\epsilon = 10^{-2}$ and $T = 2.0$.

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