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1 ANALYSIS AND NUMERICS OF STRONGLY DEGENERATE CONVECTION-DIFFUSION PROBLEMS MODELING SEDIMENTATION-CONSOLIDATION PROCESSES

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ABSTRACT

In one space dimension, the phenomenological sedimentation-consolidation model reduces to an initial-boundary value problem (IBVP) for a nonlinear strongly degenerate convection-diffusion equation. Due to the mixed hyperbolic-parabolic nature of the model, its solutions are discontinuous and entropy solutions must be sought. In this contribution, we review recent existence and uniqueness result for this and a related IBVP, and present numerical methods that can be used to correctly simulate this model, i.e. conservative methods satisfying an entropy principle. Included in our discussion are finite difference methods and methods based on operator splitting, which are employed to simulate the settling of flocculated suspensions.

Key words. Degenerate convection-diffusion equation, operator splitting, front tracking, sedimentation-consolidation processes.

1.1 INTRODUCTION

In this contribution, we consider the quasilinear strongly degenerate parabolic equation

$$\partial_t u + \partial_x g(u,t) = \partial_x^2 A(u), \quad A(u) := \textstyle\int_0^u a(s)\,ds, \quad a(u) \geq 0, \quad g(u,t) := q(t)u + f(u) \tag{1.1}$$

on a cylinder $Q_T := \Omega \times \mathcal{T}$, $\Omega := (0,1)$, $\mathcal{T} := (0,T)$, $T > 0$. We allow that $a(u) = 0$ on an interval $[0, u_c]$, where equation (1.1) is then of parabolic type, and that $a(u)$ may be discontinuous at $u = u_c$. The flux density function $f(u)$ is (for simplicity) assumed to be piecewise differentiable with $\operatorname{supp} f \subset [0,1]$ and $f(u) \leq 0$, and $q(t)$ is a nonpositive piecewise differentiable Lipschitz continuous function. These assumptions are motivated by the model of sedimentation-consolidation processes of flocculated suspensions presented in [5, 6], to which we come back in § 1.4. Moreover, we require that $\|f'\|_\infty \leq \infty$, $\mathrm{TV}_{\mathcal{T}}(q) < \infty$ and $\mathrm{TV}_{\mathcal{T}}(q') < \infty$. We consider two different IBVPs. Problem A consists of equation (1.1) together with the initial and boundary conditions

$$u(x,0) = u_0(x), \ x \in \Omega; \ u(1,t) = \varphi_1(t), \ (f(u) - \partial_x A(u))(0,t) = 0, \ t \in \mathcal{T}. \tag{1.2}$$

This problem has been studied previously by Bürger and Wendland [4]. The second IBVP, Problem B, is obtained from Problem A if the boundary condition $(1.2)_3$ is replaced by

$$(g(u,t) - \partial_x A(u))(1,t) = \Psi(t), \ t \in \mathcal{T}. \tag{1.3}$$

Let ω_ε be a standard C^∞ mollifier with $\operatorname{supp} \omega_\varepsilon \subset (-\varepsilon, \varepsilon)$ and define $a_\varepsilon(u) := ((a+\varepsilon) * \omega_\varepsilon)(u)$ and $A_\varepsilon(u) := \int_0^u a_\varepsilon(s)ds$ for $\varepsilon > 0$. For Problem A, the assumptions on the initial and boundary data can be stated as

$$\varphi_1(t) \in [0,1] \text{ for } t \in \overline{\mathcal{T}}, \ \varphi_1 \text{ has a finite number of local extrema;} \tag{1.4}$$

$$u_0 \in \{u \in BV(\Omega) : u(x) \in [0,1]; \exists M_0 > 0 : \forall \varepsilon > 0 : \mathrm{TV}_\Omega(\partial_x A_\varepsilon(u)) < M_0\}, \tag{1.5}$$

while for Problem B we require that (1.5) is valid and that either $\Psi \equiv 0$ or that there exist positive constants ξ and M_g such that $\xi a(u) - (q(t) + f'(u)) \geq M_g$ uniformly in ε.

Note that if $a(\cdot)$ is sufficiently smooth, then it is sufficient to require that $\mathrm{TV}_\Omega(\partial_x u_0)$ is finite. Multi-dimensional problems are treated in [2].

1.2 ENTROPY SOLUTIONS

It is well known that due to both the degeneracy of the diffusion coefficient $a(u)$ and to the nonlinearity of the flux density function $f(u)$, solutions of equation (1.1) are discontinuous and have to be considered as entropy solutions.

Definition 1 ([1]) *A function $u \in L^\infty(Q_T) \cap BV(Q_T)$ is an* entropy solution *of Problem A if the following conditions are satisfied:*

$$\partial_x A(u) \in L^2(Q_T); \tag{1.6}$$

$$\text{f. a. a. } t \in \mathcal{T}, \ \gamma_0(f(u) - \partial_x A(u)) = 0; \ \text{f. a. a. } x \in \overline{\Omega}, \ \lim_{t \downarrow 0} u(x,t) = u_0(x), \tag{1.7}$$

nd if $\forall\varphi \in C^\infty((0,1]\times\overline{\mathcal{T}})$, $\varphi \geq 0$, $\operatorname{supp}\varphi \subset (0,1]\times\mathcal{T}, \forall k \in \mathbb{R}$:

$$\begin{aligned}\iint_{Q_T}\Big\{|u-k|\partial_t\varphi + \operatorname{sgn}(u-k)[g(u,t)-g(k,t)-\partial_x A(u)]\partial_x\varphi\Big\}\,dtdx\\ +\int_0^T\Big\{-\operatorname{sgn}(\varphi_1(t)-k)[g(\gamma_1 u,t)-g(k,t)-\gamma_1\partial_x A(u)]\varphi(1,t)\\ +[\operatorname{sgn}(\gamma_1 u-k)-\operatorname{sgn}(\varphi_1(t)-k)][A(\gamma_1 u)-A(k)]\partial_x\varphi(1,t)\Big\}\,dt \;\geq 0. \qquad (1.8)\end{aligned}$$

Definition 2 ([1]) *A function* $u \in L^\infty(Q_T)\cap BV(Q_T)$ *is an* entropy solution *of Problem B if* (1.6) *and* (1.7) *are valid, if for all* $\varphi \in C_0^\infty(Q_T)$, $\varphi \geq 0$ *and* $k \in \mathbb{R}$ *the inequality*

$$\iint_{Q_T}\Big\{|u-k|\partial_t\varphi + \operatorname{sgn}(u-k)[g(u,t)-g(k,t)-\partial_x A(u)]\partial_x\varphi\Big\}\,dtdx \geq 0 \qquad (1.9)$$

holds, and if $\gamma_1(g(u,t)-\partial_x A(u)) = \Psi(t)$ for almost all $t \in \mathcal{T}$.

In these definitions, $\gamma_0 u := (\gamma u)(0,t)$ and $\gamma_1 u := (\gamma u)(1,t)$ denote the traces of u. Entropy inequalities like (1.8) go back to the pioneering papers of Kružkov [15] and Vol'pert [17] for first order equations and Vol'pert and Hudjaev [18] for second order equations.

We now briefly summarize some recent results on the existence and uniqueness of entropy solutions of Problems A and B, and state a new regularity result for the integrated diffusion coefficient for entropy solutions of Problem B. For details we refer to [1].

Theorem 1 ([1]) *Under the conditions stated in § 1.1, there exist entropy solutions to both problems A and B.*

Sketch of Proof. For both problems, existence of entropy solutions can be shown by the vanishing viscosity method. To this end, we consider the regularized parabolic IBVPs

$$\left.\begin{aligned}\partial_t u^\varepsilon + \partial_x(q_\varepsilon(t)u^\varepsilon + f_\varepsilon(u^\varepsilon)) = \partial_x^2 A_\varepsilon(u^\varepsilon),\ (x,t)\in Q_T;\ u^\varepsilon(x,0)=u_0^\varepsilon(x),\ x\in\Omega;\\ u^\varepsilon(1,t)=\varphi_1^\varepsilon(t),\ (f_\varepsilon(u^\varepsilon)-\partial_x A_\varepsilon(u^\varepsilon))(0,t)=0,\ t\in(0,T]\end{aligned}\right\}, \quad (1.10)$$

$$\left.\begin{aligned}\partial_t u^\varepsilon + \partial_x(q_\varepsilon(t)u^\varepsilon + f_\varepsilon(u^\varepsilon)) = \partial_x^2 A_\varepsilon(u^\varepsilon),\ (x,t)\in Q_T;\ u^\varepsilon(x,0)=u_0^\varepsilon(x),\ x\in\Omega;\\ (g_\varepsilon(u^\varepsilon,t)-\partial_x A_\varepsilon(u^\varepsilon))(1,t)=\Psi_\varepsilon(t),\quad (f_\varepsilon(u^\varepsilon)-\partial_x A_\varepsilon(u^\varepsilon))(0,t)=0,\ t\in(0,T]\end{aligned}\right\}, \quad (1.11)$$

where the functions q, f, u_0, φ_1 and Ψ have been replaced by particular smooth approximations for each problem that ensure compatibility conditions and existence of smooth solutions. It can then be shown that there exist constants M_1 to M_5 independent of ε such that the smooth solutions of Problem (1.10) satisfy

$$\|u^\varepsilon\|_{L^\infty(Q_T)} \leq M_1,\ \ \|\partial_x u^\varepsilon(\cdot,t)\|_{L^1(\Omega)} \leq M_2 \text{ for all } t\in\mathcal{T},\ \ \|\partial_t u^\varepsilon\|_{L^1(Q_T)} \leq M_3, \qquad (1.12)$$

while those of Problem (1.11) satisfy

$$\|u^\varepsilon\|_{L^\infty(Q_T)} \leq M_1,\ \ \|\partial_t u^\varepsilon(\cdot,t)\|_{L^1(\Omega)} \leq M_4 \text{ for all } t\in\mathcal{T}, \qquad (1.13)$$

and, in the case where $\Psi \equiv 0$,

$$\|\partial_x u^\varepsilon(\cdot,t)\|_{L^1(\Omega)} \leq M_5 \text{ for all } t\in\mathcal{T} \qquad (1.14)$$

and in the case where there exist constants $\xi, M_g > 0$ such that $\xi a(u)-(q(t)+f'(u)) \leq M_g$,

$$\|\partial_x u^\varepsilon\|_{L^1(Q_T)} \leq M_5. \tag{1.15}$$

Estimates (1.12) imply that the family $\{u^\varepsilon\}_{\varepsilon>0}$ of solutions of Problem (1.10) is bounded in $W^{1,1}(Q_T) \subset BV(Q_T)$. Hence there exists a sequence $\varepsilon = \varepsilon_n \downarrow 0$ such that $\{u^{\varepsilon_n}\}$ converges in $L^1(Q_T)$ to a function $u \in L^\infty(Q_T) \cap BV(Q_T)$. The same is true for the family of solutions of Problem B^ε. To prove that u is an entropy solution of Problem A or B, it has to be shown that the diffusion function $A(u)$ has the required regularity. In both cases, it is fairly easy to show that $\|\partial_x A_\varepsilon(u^\varepsilon)\|_{L^2(Q_T)}$ is uniformly bounded independently of ε. Therefore, passing if necessary to a subsequence, $A_\varepsilon(u^\varepsilon) \to A(u)$ in $L^2(Q_T)$ and $\partial_x A_\varepsilon(u^\varepsilon) \to \partial_x A(u)$ weakly in $L^2(Q_T)$ as $\varepsilon \downarrow 0$. It is now easy to show that the limit function u satisfies the remaining parts of Definitions 1 and 2, respectively. ■

For the case of Problem B, the regularity result $\partial_x A(u) \in L^2(Q_T)$ can be considerably improved; namely, we have that $A(u)$ is Hölder continuous on $\overline{Q_T}$:

Theorem 2 ([1]) *Assume that $u^\varepsilon \to u$ a.e. on Q_T as $\varepsilon \downarrow 0$. Then there exists a subsequence $\varepsilon_n \downarrow 0$ such that $A(u^{\varepsilon_n}) \to A(u)$ uniformly on $\overline{Q_T}$ and $A(u) \in C^{1,1/2}(\overline{Q_T})$.*

Sketch of Proof. The proof is essentially based on the observation that if u^ε is a smooth solution of Problem B^ε, then the quantity $V^\varepsilon := -g_\varepsilon(u^\varepsilon, t) - a_\varepsilon(u^\varepsilon)\partial_x u^\varepsilon$ satisfies a linear parabolic IBVP with Dirichlet boundary data that are uniformly bounded in ε. From the maximum principle, we obtain that $\partial_x A_\varepsilon(u^\varepsilon)$ is uniformly bounded on $\overline{Q_T}$. This and estimates (1.13) to (1.15) allow the application of Kružkov's interpolation lemma [15, Lemma 5] to the linear IBVP. Hence there exists a constant M_7 such that

$$|A_\varepsilon(u^\varepsilon(x,t_2)) - A_\varepsilon(u^\varepsilon(x,t_1))| \leq M_7\sqrt{|t_2-t_1|}, \quad \forall (x,t_1),(x,t_2) \in \overline{Q_T}.$$

The Ascoli-Arzelà compactness theorem then yields the existence of a subsequence of $\{A(u^{\varepsilon_n})\}$ converging uniformly on $\overline{Q_T}$ to $A(u) \in C^{1,1/2}(\overline{Q_T})$. ■

Theorem 3 ([1]) *Let u and v be two entropy solutions either of Problem A or of Problem B with initial data u_0 and v_0, respectively. Then $\|u(\cdot,t)-v(\cdot,t)\|_{L^1(\Omega)} \leq \|u_0 - v_0\|_{L^1(\Omega)}$ is valid. In particular, both problems have at most one entropy solution.*

Sketch of Proof. The proof is based on the technique known as "doubling of the variables" introduced by Kružkov [15] as a tool for proving the L^1 contraction principle for entropy solutions of scalar conservation laws. This technique was recently extended by Carrillo [7] to a class of degenerate parabolic equations. This recent extension is adopted here to Problems A and B and leads to the inequality

$$\iint_{Q_T} \Big\{|u-v|\partial_t\varphi + \operatorname{sgn}(u-v)[g(u,t)-g(v,t)-(\partial_x A(u)-\partial_x A(v))]\partial_x\varphi\Big\} dt dx \geq 0,$$

valid for two entropy solutions u and v either of Problem A or of Problem B and for all test functions $\varphi \in C_0^\infty(Q_T)$, from which stability and uniqueness can be obtained in a standard fashion. ■

Remark 1 *The proof of Theorem 3 (see [1]) is not based on a jump condition, in contrast to the uniqueness proof by Wu and Yin [20]. In fact, it is not clear whether a jump condition can be derived with integrated diffusion functions $A(u)$ that are only Lipschitz continuous. Moreover, it has been possible to derive jump conditions only in the 1-D case so far, while the new uniqueness proof can also be extended to multidimensions.*

1.3 NUMERICAL METHODS

This section provides the necessary background for the development and application of numerical methods for mixed hyperbolic-parabolic problems.

1.3.1 *Finite Difference Methods*

To focus on the main ideas, we consider here the simplified problem

$$\partial_t u + \partial_x f(u) = \partial_x^2 A(u), \qquad u(x,0) = u_0(x), \tag{1.16}$$

where $(x,t) \in Q_T = \mathbb{R} \times (0,T)$ and $f = f(u)$, $A = \int^u a$, $a = a(u) \geq 0$, $u_0 = u_0(x)$ are sufficiently smooth functions. The difference methods described here can be easily modified to solve the full sedimentation-consolidation model. The material presented here is based on the series of papers by Evje and Karlsen [10, 11, 12], see also [3].

Selecting a mesh size $\Delta x > 0$, a time step $\Delta t > 0$, and an integer N so that $N\Delta t = T$, the value of the difference approximation at $(x_j, t_n) = (j\Delta x, n\Delta t)$ will be denoted by u_j^n. There are special difficulties associated with equation (1.1) which must be dealt with in developing numerical methods. For example, numerical methods based on naive finite difference formulation of the diffusion term may be adequate for smooth solutions but can give wrong results when discontinuities are present, see [11, 12] for details. It turns out that it is preferable to use a conservative differencing of the second order term and upwind differencing of the convective flux and, i.e., a difference method of the form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F(u_j^n, u_{j+1}^n) - F(u_{j-1}^n, u_j^n)}{\Delta x} = \frac{A(u_{j-1}^n) - 2A(u_j^n) + A(u_{j+1}^n)}{(\Delta x)^2}, \tag{1.17}$$

where F is the upwind flux. For a monotone flux function f, the upwind flux is defined by $F(u_j^n, u_{j+1}^n) = f(u_j^n)$ if $f' > 0$ and $F(u_j^n, u_{j+1}^n) = f(u_{j+1}^n)$ if $f' < 0$. More generally, for a non-monotone flux function f, one needs the generalised upwind flux of Engquist and Osher defined by (see also see [11]) $F(u_j^n, u_{j+1}^n) = f^+(u_j^n) + f^-(u_{j+1}^n)$, where $f^+(u) = f(0) + \int_0^u \max(f'(s), 0)\, ds$ and $f^-(u) = \int_0^u \min(f'(s), 0)\, ds$. We assume that the following stability condition holds: $\max_u |f'(u)| \frac{\Delta t}{\Delta x} + 2 \max_u |a(u)| \frac{\Delta t}{(\Delta x)^2} \leq 1$.

As is well known, upwind differencing stabilizes profiles which are liable to undergo sudden changes, i.e., discontinuities and other large gradient profiles. Therefore upwind differencing is perfectly suited to the treatment of discontinuities (and thus of the sedimentation model). Let u_Δ, $\Delta = (\Delta x, \Delta t)$, be the interpolant of degree one associated with the discrete data points $\{u_j^n\}$. Regarding the sequence $\{u_\Delta\}$, we have:

Theorem 4 ([11]) *The sequence $\{u_\Delta\}$ built from (1.17) converges in $L^1_{\mathrm{loc}}(Q_T)$ to the unique entropy solution u of (1.16) as $\Delta \to 0$. Furthermore, $\{A(u_\Delta)\}$ converges uniformly on compact sets $\mathcal{K} \subset Q_T$ to $A(u) \in C^{1,1/2}(\bar{Q}_T)$ as $\Delta \to 0$.*

Sketch of Proof. An important part of the proof of this theorem is to establish the following three estimates for $\{u_j^n\}$: (a) a uniform L^∞ bound, (b) a uniform total variation bound, (c) L^1 Lipschitz continuity in the time variable, and the following two estimates for the discrete total flux $F(u^n;j) - \Delta_+ A(u_j^n)$: (d) a uniform L^∞ bound, (e) a uniform total variation bound. We refer to [11] for details concerning the derivation of these bounds. Then, using the three estimates (a)-(c), it is not difficult to show that there is a finite constant $C = C(T) > 0$ (independent of Δ) such that $\|u_\Delta\|_{L^\infty(Q_T)} + |u_\Delta|_{BV(Q_T)} \leq C$.

Hence, the sequence $\{u_\Delta\}$ is bounded in $BV(\mathcal{K})$ for any compact set $\mathcal{K} \subset Q_T$. It is thus possible to select a subsequence that converges in $L^1(\mathcal{K})$. Furthermore, using a standard diagonal process, we can construct a sequence that converges in $L^1_{\mathrm{loc}}(Q_T)$ to a limit $u \in L^\infty(Q_T) \cap BV(Q_T)$. It is possible to use, among other things, estimates (d) and (e) to prove that $A(u_\Delta)$ is Hölder continuous on $\overline{Q}_T$ independently of Δ. Then by repeating the proof of the Ascoli-Arzela compactness theorem, we deduce the existence of a subsequence of $\{A(u_\Delta)\}$ converging uniformly to $A(u) \in C^{1,1/2}(\bar{\Pi}_T)$.

Finally, convergence of $\{u_\Delta\}$ to the correct physical solution of (1.16) follows from the cell entropy inequality ($k \in \mathbb{R}$)

$$\frac{|u_j^{n+1} - k| - |u_j^n - k|}{\Delta t} + \Delta_-\Big(F(u^n \vee k; j) - F(u^n \wedge k; j) - \Delta_+|A(u_j^n) - A(k)|\Big) \leq 0,$$

where $u \vee v = \max(u, v)$ and $u \wedge v = \min(u, v)$. This discrete entropy inequality is in turn an easy consequence of the monotonicity of the scheme. The reader is referred to [11] for further details on the convergence analysis. ■

Remark 2 *In many applications it is desirable to avoid the explicit stability restriction associated with* (1.16). *One way to overcome the is of course to use an implicit version of* (1.17), *see [12] for details. Moreover, the upwind method and all other monotone methods are at most first order accurate, giving poor accuracy in smooth regions. To overcome these problems, Evje and Karlsen [10] used the generalized MUSCL (Variable Extrapolation) idea of van Leer to formally upgrade the upwind method* (1.17) *to second order accuracy. Although more difficult than in the monotone case, it can be shown that also the second order method satisfies a discrete entropy condition and that it converges to the unique generalized solution of the problem, see [10] for details.*

Finally, let us say a few words about the multi-dimensional case. For simplicity of notation, we consider only the two-dimensional problem

$$\partial_t u + \partial_x f(u) + \partial_y g(u) = \partial_x(d(u)\partial_x u) + \partial_y(d(u)\partial_y u), \quad u(x,y,0) = u_0(x,y). \tag{1.18}$$

Let $u_{j,k}^n$ denote the finite difference approximation at $(x, y, t) = (j\Delta x, k\Delta y, n\Delta t)$. A conservative finite difference method for (1.18) takes the form

$$\frac{u_{j,k}^{n+1} - u_{j,k}^n}{\Delta t} + \Delta_{x,-}(F(u^n; j, k) - \Delta_{x,+} D(u_{j,k}^n)) + \Delta_{y,-}(G(u^n; j, k) - \Delta_{y,+} D(u_{j,k}^n)) = 0, \tag{1.19}$$

where $\Delta_{\ell,-}, \Delta_{\ell,+}$ are the backward and forward differences, respectively, in direction ℓ, for $\ell = x, y$, and F, G are convective numerical fluxes that are consistent with f, g, respectively. Roughly speaking, one can choose F, G to be any reasonable numerical flux for

hyperbolic conservation laws. Let u_Δ, $\Delta = (\Delta x, \Delta y, \Delta t)$, be the interpolant of degree zero (piecewise constant) associated with the data points $\{u^n_{i,j}\}$. When (1.19) is monotone, one can establish the following convergence theorem:

Theorem 5 ([10, 11]) *Suppose that* (1.19) *is monotone. Then the sequence* $\{u_\Delta\}$ *built from* (1.19) *converges in* $L^1_{\text{loc}}(Q_T)$ *to the unique entropy solution* u *of (1.16) as* $\Delta \to 0$. *Furthermore,* $\{A(u_\Delta)\}$ *converges to* $A(u)$ *weakly in* $H^1(Q_T)$ *as* $\Delta \to 0$.

1.3.2 Operator Splitting Methods

There are essentially two ways of constructing methods for solving convection-diffusion problems. One approach attempts to preserve some coupling between the two processes involved (convection and diffusion). The finite difference methods considered in the previous section try to follow this approach. Another approach is to split the convection-diffusion problem into a convection problem and a diffusion problem, which are then solved sequentially to approximate the exact solution of the model. The main attraction of splitting methods lies, of course, in the fact that one can employ the optimal existing methods for each subproblem. The splitting methods presented here are similar to the splitting methods that have been used over the years to simulate multi-phase flow in oil reservoirs. We refer to the lecture notes by Espedal and Karlsen [9] for an overview of this activity and an introduction to operator splitting methods in general. For simplicity of presentation, we restrict ourselves to multi-dimensional Cauchy problems of the form

$$\partial_t u + \nabla_x \cdot f(u) = \Delta_x A(u), \quad u(x,0) = u_0(x), \quad (x,t) \in Q_T = \mathbb{R}^d \times (0,T). \qquad (1.20)$$

We emphasize that the numerical solution algorithms and their convergence analysis presented below carry over to more general convection-diffusion equations. To describe this operator splitting more precisely, we need the solution operator taking the initial data $v_0(x)$ to the entropy solution at time t of the hyperbolic problem

$$\partial_t v + \nabla_x \cdot f(v) = 0, \qquad v(x,0) = v_0(x). \qquad (1.21)$$

This solution operator we denote by $\mathcal{S}(t)$. Similarly, let $\mathcal{H}(t)$ be the solution operator (at time t) associated with the parabolic problem

$$\partial_t w = \Delta_x A(w), \qquad w(x,0) = w_0(x). \qquad (1.22)$$

Now choose a time step $\Delta t > 0$ and an integer N such that $N\Delta t = T$. Furthermore, let $t_n = n\Delta t$ for $n = 0, \dots, N$ and $t_{n+1/2} = (n + \frac{1}{2})\Delta t$ for $n = 0, \dots, N-1$. We then let the operator splitting solution $u_{\Delta t}$ be defined at the discrete times $t = t_n$ by

$$u_{\Delta t}(x, n\Delta t) = \Big[\mathcal{H}(\Delta t) \circ \mathcal{S}(\Delta t)\Big]^n u_0(x). \qquad (1.23)$$

Of course, the ordering of the operators in (1.23) can be changed as well as the so-called Godunov formula (1.23) can be replaced by the more accurate Strang formula. Note that we have only defined $u_{\Delta t}$ at the discrete times t_n. Between two consecutive discrete times, we use a suitable time interpolant (see [13, 14]). Regarding $u_{\Delta t}$ we have:

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hyperbolic conservation laws. Let e.g. $\Delta = (\Delta x, \Delta y, \Delta t)$ be the interpolant of degree zero (piecewise constant) associated with the data points $\{u^n_{ij}\}$. When (1.19) is monotone, one can establish the following convergence theorem:

Theorem 5 ([10, 11]) *Suppose that (1.19) is monotone. Then the sequence $\{u_\Delta\}$ built from (1.19) converges in $L^1_{loc}(Q_T)$ to the unique entropy solution u of (1.20) as $\Delta \to 0$. Furthermore, $\{A(u_\Delta)\}$ converges to $A(u)$ weakly in $L^2(Q_T)$ as $\Delta \to 0$.*

1.3.2 Operator Splitting Methods

There are essentially two ways of constructing methods for solving convection–diffusion problems. One approach attempts to preserve some coupling between the two processes involved (convection and diffusion). The finite difference methods considered in the previous section try to follow this approach. Another approach is to split the convection–diffusion problem into a convection problem and a diffusion problem, which are then solved sequentially to approximate the exact solution of the model. The main attraction of splitting methods lies, of course, in the fact that one can employ the optimal existing methods for each subproblem. The splitting methods presented here are similar to the splitting methods that have been used over the years to simulate multi-phase flow in oil reservoirs. We refer to the lecture notes by Espedal and Karlsen [6] for an overview of this activity and an introduction to operator splitting methods in general. For simplicity of presentation, we restrict ourselves to multi-dimensional Cauchy problems of the form

$$\partial_t u + \nabla \cdot (V f(u)) = \Delta A(u), \quad u(x,0) = u_0(x), \quad (x,t) \in Q_T = \mathbb{R}^d \times (0,T). \qquad (1.20)$$

We emphasize that the numerical solution algorithms and their convergence analysis presented below carry over to more general convection diffusion equations. To describe the operator splitting more precisely, we need the solution operator taking the initial data $u_0(x)$ to the entropy solution at time t of the hyperbolic problem

$$\partial_t u + \nabla \cdot (V f(u)) = 0, \qquad u(x,0) = u_0(x). \qquad (1.21)$$

This solution operator is denoted by $S(t)$. Similarly, let $H(t)$ be the solution operator (at time t) associated with the parabolic problem

$$\partial_t u = \Delta A(u), \qquad u(x,0) = u_0(x). \qquad (1.22)$$

Now choose a time step $\Delta t > 0$ and an integer N such that $N\Delta t = T$. Furthermore, let $t_n = n\Delta t$ for $n = 0, \ldots, N$ and $t_{n+1/2} = (n + \frac{1}{2})\Delta t$ for $n = 0, \ldots, N-1$. We then let the operator splitting solution $u_{\Delta t}$ be defined at the discrete times $t = t_n$ by

$$u_{\Delta t}(x, n\Delta t) = [H(\Delta t) \circ S(\Delta t)]^n u_0(x). \qquad (1.23)$$

Of course, the ordering of the operators in (1.23) can be changed as well as the so-called Godunov formula (1.23) can be replaced by the more accurate Strang formula. Note that we have only defined $u_{\Delta t}$ at the discrete times t_n. Between two consecutive discrete times, we use a suitable time interpolant (see [13, 14]). Regarding $u_{\Delta t}$ we have:

Lemma 1 ([13, 14]) *The following a priori estimates hold: (a)* $\|u_{\Delta t}(\cdot,t)\|_{L^\infty} \le \|u_0\|_{L^\infty}$, *(b)* $|u_{\Delta t}(\cdot,t)|_{BV} \le |u_0|_{BV}$, *(c)* $\|u_{\Delta t}(\cdot,t_2)-u_{\Delta t}(\cdot,t_1)\|_{L^1} \le \mathrm{Const}\cdot|t_2-t_1|^{1/2}$ *for all* $t_1, t_2 \ge 0$.

In view of estimates (a)-(c) in Lemma 1, there exists a subsequence $\{\Delta t_j\}$ and a limit function u such that $u_{\Delta t_j} \to u$ in $L^1_{\mathrm{loc}}(Q_T)$ as $j \to \infty$. In addition, one can prove via an energy type argument that this limit satisfies $\nabla_x A(u) \in L^2(Q_T; \mathbb{R}^d)$ (see [13]). Finally, one can prove that $u_{\Delta t}$ satisfies a discrete entropy condition and consequently that the limit u satisfies the entropy condition (see [13]). Summing up, we have:

Theorem 6 ([13]) *The operator splitting solution* $u_{\Delta t}$ *converges in* $L^1_{\mathrm{loc}}(Q_T)$ *to the unique entropy solution of the Cauchy problem* (1.20) *as* $\Delta t \to 0$.

So far we have assumed that the operators $\mathcal{S}^f(t)$ and $\mathcal{H}(t)$ determine exact solutions to their respective split problems and that discretization has been performed with respect to time only. In applications, the exact solution operators $\mathcal{S}^f(t)$ and $\mathcal{H}(t)$ are replaced by appropriate numerical approximations which involve discretization also with respect to space. For the split problem (1.21), one can choose from a diversity of methods for hyperbolic conservation laws. For the second split problem (1.22), one can also choose from a large collection of finite difference or element methods. Convergence results for fully discrete splitting methods can be found in, e.g., [13]. For a more complete overview of theoretical results for (fully discrete) operator splitting methods and references to papers dealing with such issues, we refer again to the lecture notes [9].

In what follows, we shall outline a fully discrete splitting method for the first sedimentation model (Problem A in §1.1), see [3] for a different one. This method has previously been employed by Bustos et al. [6] (see also [3]) and will be used in §1.4 below. This method splits the original Problem A into the second order problem

$$\partial_t w = \partial_x^2 A(w),\ (x,t) \in Q_T;\quad w(x,0) = w_0(x),\ z \in \Omega, \tag{1.24}$$

the linear convection problem

$$\partial_t u + q(t)\partial_x u = 0,\ (x,t) \in Q_T;\quad u(x,0) = u_0(x),\ x \in \Omega, \tag{1.25}$$

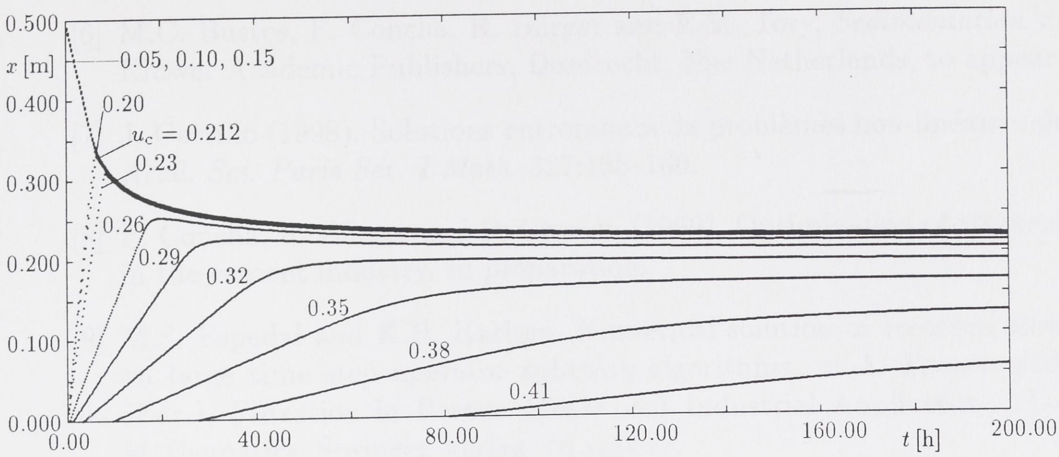
and the nonlinear hyperbolic IBVP

$$\left.\begin{aligned} &\partial_t v + \partial_x f(v) = 0,\ (x,t) \in Q_T;\quad v(x,0) = v_0(x),\ x \in \Omega;\\ &(f(v) - a(v)\partial_x v)(0,t) = 0,\ v(1,t) = \varphi_1(t),\ t \in \mathcal{T}.\end{aligned}\right\} \tag{1.26}$$

Note that the ordering of the operators in (1.24)–(1.26) is different from the ordering used in (1.23). The splitting (1.24)–(1.26) can be analysed using the techniques of [13] together with an appropriate treatment of the boundary conditions (details will be presented in future work).

1.4 APPLICATION TO THE SEDIMENTATION-CONSOLIDATION MODEL

To illustrate the application to the sedimentation-consolidation model, we employ the splitting (1.24)–(1.26) to simulate the batch settling of an initally homogeneous suspension



'igure 1.1: Numerical simulation of the settling of a flocculated suspension. The iso-oncentration lines correspond to the annotated values.

f concentration $u_0 = 0.18$ in a column of height $L = 0.5\,[\mathrm{m}]$. We use central differences o solve the second order problem (1.24), a first order upwind method to solve the linear convection problem (1.25), and, finally, a variant of Nessyahu and Tadmor's method [16] or the nonlinear convection problem (1.26), see [6] for details. Figure 1.1 shows the numerical solution calculated with $\Delta x/L = 1/400$, $\Delta t/\Delta z = 2000\,[\frac{\mathrm{s}}{\mathrm{m}}]$ and $\alpha = 1.3$, where α is the free parameter in Nessyahu and Tadmor's method [16]. The model functions (corresponding to a suspension of ground calcium carbonate in sea water, see [8]) are

$$f(u) = -1.87\times 10^{-4} u(1-u)^{16.4}\,[\tfrac{\mathrm{m}}{\mathrm{s}}], \quad a(u) = \begin{cases} 0, & u \le u_c := 0.212, \\ 0.0206u^6(1-u)^{16.4}\,[\tfrac{\mathrm{m}^2}{\mathrm{s}}], & u > u_c. \end{cases}$$

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