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APPLICATION OF THE GALERKIN'S METHOD ON THE PROBLEM OF CELLULAR CONVECTION INDUCED BY SURFACE TENSION GRADIENTS.

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Abstract.

We consider the problem of cellular convection induced by surface tension gradients. The solutions are expanded in Fourier series, and we use a modification of the Galerkin's method for the case of natural boundary conditions to determine the Fourier coefficients. The problem is reduced to a system of ordinary differential equations, and some explicit calculations are carried out to illustrate the non-linear effects.

1. Introduction.

We study the cellular convection in a horizontal fluid layer heated from below, and allow for surface tension gradients resulting from temperature variations at a free surface. The fluid is assumed to be infinite in horizontal extent.

We assume that the Fourier decompositions of the convective motions in the horizontal (x,y) directions can be represented by the wave numbers ia_1 and $ja_2, i, j = 1,2,3,\ldots$. Choosing the decomposition in the vertical (z) direction at will, the solutions are sought as (truncated) Fourier series. We use a modification of the Galerkin's method for the case of natural boundary conditions to determine the Fourier coefficients.
- Since the explicit calculations required for a non-linear analysis of this problem can be very long, we have used as our model the limiting case of an infinite Prandtl number.

In this way we obtain the equations (14a,b,c) which govern the Fourier coefficients (the amplitude equations).
- In what follows, we use a decomposition of the convective motions in the z-direction given by the eigenvalue problems (15a,b,c). We consider a state which differs slightly from the onset of convection, and carry out some preliminary calculations to illustrate the non-linear problem. We obtain the amplitude equations(20a,b) which have the same form as those discussed by Segel and Stuart (1962). For the situation to which our analysis applies, hexagonal convection cells <u>may</u> be attributed to surface tension gradients.

2. Formulation of the problem.

Referring to Cartesian coordinates (x,y,z) with the z-axis taken to be in the vertical direction, we consider a horizontal fluid layer of infinite horizontal extent bounded by a rigid plane at $z = -h$ and a non-deformable free surface at $z = 0$. The lower boundary is taken to be a solid body at rest which has a constant temperature T_1. At the free surface we consider the fluid to be in contact with an inviscid atmosphere which has a constant atmospheric pressure p_a. It is assumed that the fluid density and the surface tension are the only physical properties which vary with temperature.

Neglecting the dissipation of energy due to viscosity, the governing equations in the Boussinesq approximation are

$$(1a) \qquad \rho_m(\partial \underline{u}/\partial t + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \mu\nabla^2\underline{u} - \rho_m(1-\alpha(T-T_m))g\underline{k}$$

$$(1b) \qquad \rho_m c_p(\partial T/\partial t + \underline{u}\cdot\nabla T) = \lambda\nabla^2 T$$

$$(1c) \qquad \nabla\cdot\underline{u} = 0$$

where $\underline{u} = (u,v,w)$ is the velocity, p is the pressure, T is the temperature, g is the acceleration of gravity, $\underline{k}$ is the vertical unit vector, ρ_m is a constant reference density and T_m a constant reference temperature. α, c_p, λ and μ are the coefficient of thermal expansion, the specific heat at constant pressure, the coefficient of heat conduction and the coefficient of viscosity, respectively. - (The thermal diffusivity $\lambda/\rho_m c_p$ and the kinematic viscosity μ/ρ_m ,

will be denoted by χ and ν, respectively).

The kinematic boundary condition at the free surface is that the normal component of the velocity vanishes

$$\text{(2a)} \qquad \underline{u}\cdot\underline{n} = 0 \ , \ (\underline{n} \equiv \underline{k}) \quad \text{at} \quad z = 0$$

The dynamical free-surface condition is imposed by the requirement that the viscous stress on the two sides of the surface can differ only as a result of surface tension

$$\text{(2b)} \qquad -(p-p_a)\underline{n} + \underset{\sim}{P}\cdot\underline{n} = \nabla_s\sigma \quad \text{at} \quad z = 0$$

where $\underset{\sim}{P}$ is the viscous stress tensor in the fluid, σ is the surface tension and ∇_s is the surface gradient operator.

The transport of heat across the free surface is supposed to be proportional to the temperature difference between the boundary and the adjacent medium. Denoting the temperature in the adjacent medium by T_a, this can be written

$$\text{(2c)} \qquad -\lambda\underline{n}\cdot\nabla T = \kappa(T-T_a) \quad \text{at} \quad z = 0$$

where κ is the heat-transfer coefficient, assumed constant.

At the lower boundary, the boundary conditions are

$$\text{(2d,e)} \qquad \underline{u} = \underline{0} \ , \ T = T_1 \quad \text{at} \quad z = -h$$

where T_1 is a prescribed temperature.

The difficulties in formulating proper boundary

conditions as $x,y \to \pm\infty$ are circumvented by limiting the discussion to solutions which are spatially periodic in these directions. The interval over which this periodicity takes place is, however, not known apriori.

When no motion is present, $\underline{u} \equiv \underline{0}$, the temperature distribution which satisfies (1b) and (2c,e) is

$$T = T_s(z) = T_0 - \beta z$$

where $T_0 = T_s(0) = T_a + \beta\lambda/\kappa$ and $\beta = (T_1 - T_0)/h$. - The pressure distribution $p = p_s(z)$ is not explicitly required, we note, however, that $p_s(0) = p_a$.

The stability problem is now formulated in the usual manner by substituting

$$\underline{u} = \underline{u}' \ , \quad T = T_s(z) + T' \ , \quad p = p_s(z) + p'$$

into the governing equations (1a-c) and boundary conditions (2a-e). - Assuming that the surface tension can be regarded as a linear function of temperature, this can be written

$$\sigma(T) = \sigma(T_0) + \gamma(T - T_0)$$

where T_0 is the undisturbed temperature at $z = 0$, and $\gamma = d\sigma/dT$ evaluated at T_0. For most fluids the surface tension decreases with increasing temperature, i.e. $\gamma < 0$.

Measuring the velocity $\underline{u}'$, temperature T', pressure p' and the time and length in units of χ/h, βh, $\mu\chi/h^2$,

h^2/χ and h , respectively, and dropping the primes, the governing equations are:

(3a) $$P^{-1}(\partial\underline{u}/\partial t+\underline{u}\cdot\nabla\underline{u}) = -\nabla p + \nabla^2\underline{u} + RT\underline{k}$$

(3b) $$\partial T/\partial t + \underline{u}\cdot\nabla T = \nabla^2 T + \underline{u}\cdot\underline{k}$$

(3c) $$\nabla\cdot\underline{u} = 0$$

with the boundary conditions

(4a,b) $$\underline{u}\cdot\underline{n} = 0 \ , \quad -p\underline{n} + \underset{\sim}{P}\cdot\underline{n} = -M\nabla_s T \qquad \text{at} \quad z = 0$$

(4c) $$\underline{n}\cdot\nabla T + LT = 0 \qquad \text{at} \quad z = 0$$

(4d,e) $$\underline{u} = 0 \ , \quad T = 0 \qquad \text{at} \quad z = -1$$

where R, P, M and L are the Rayleigh-, the Prandtl-, the Marangoni- and the Nusselt number, respectively,

$$R = \alpha\beta g h^4/\nu\chi \ , \quad P = \nu/\chi \ , \quad M = -\beta\gamma h^2/\mu\chi \ , \quad L = \kappa h/\lambda \ .$$

3. Method of solution.

The method of solving our problem will be as follows. We ask for approximate solutions $\underline{u}_n$ and T_n which have the form

$$\underline{u}_n(\underline{x},t) = \sum_{k=1}^{n} A_{kn}(t)\underline{\Phi}_k(\underline{x}) \tag{5a}$$

$$T_n(\underline{x},t) = \sum_{k=1}^{n} B_{kn}(t)\varphi_k(\underline{x}) \tag{5b}$$

where $\underline{\Phi}_k$ and φ_k are some functions chosen beforehand, and $\nabla\cdot\underline{\Phi}_k=0$, $k=1,2,3,\ldots,n$. Due to the constraint $\nabla\cdot\underline{\Phi}_k=0$ the pressure $p_n(\underline{x},t)$ is not explicitly required. - We can always consider the functions $\underline{\Phi}_k$ to be linearly independent and to represent the first n functions of some set of functions $\{\underline{\Phi}_k\}$, $k=1,2,3,\ldots$, which is complete.*) The same assumptions apply to the functions φ_k. The functions $\underline{\Phi}_k$ and φ_k are moreover taken to be orthonormal

$$(\underline{\Phi}_i,\underline{\Phi}_j) = \delta_{ij} \ , \quad (\varphi_i,\varphi_j) = \delta_{ij}$$

where $(\ ,\ )$ denotes the spatial average over the layer, or inner product. - Our aim is to determine the functions $A_{kn}(t)$ and $B_{kn}(t)$ so that (5a,b) satisfies the governing equations (3a,b) and the boundary conditions (4a-e) as accurately as possible. To do so, we shall use a modification of Galerkin's method for the case of natural boundary conditions, Mikhlin (1964). In applying this method to our problem, we need not subject the functions $\underline{\Phi}_k$ and φ_k to any boundary conditions beforehand, but can choose them at will.

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*) In a function space, a set $\{\Phi_k\}$ is complete if any function in the space can be expanded in terms of the $\{\Phi_k\}$. - In this paper, we shall be concerned with the set of infinitely differentiable functions.

The modified Galerkin's method takes the form:

$$(6a)\quad (P^{-1}\partial \underline{u}_n/\partial t + P^{-1}\underline{u}_n \cdot \nabla \underline{u}_n, \underline{\Phi}_\gamma) = R(T_n \underline{k}, \underline{\Phi}_\gamma) - (\underset{\sim}{P}_n, \nabla \underline{\Phi}_\gamma) - M(\nabla_s T_n, \underline{\Phi}_\gamma)_s$$

$$(6b)\quad (\partial T_n/\partial t + \underline{u}_n \cdot \nabla T_n, \varphi_\gamma) = (w_n, \varphi_\gamma) - (\nabla T_n, \nabla \varphi_\gamma) - L(T_n, \varphi_\gamma)_s$$

where (,)s denote the horizontal average at the free surface and $\gamma = 1,2,\ldots,n$. Substituting the expressions (5a,b) into (6a,b), we obtain a system of ordinary differential equations for the amplitudes $A_{kn}(t)$ and $B_{kn}(t)$, which can be solved when given suitable initial conditions, say

$$A_{kn}(0) = (\underline{u}(\underline{x},0), \underline{\Phi}_k(\underline{x})) , \quad B_{kn}(0) = (T(\underline{x},0), \varphi_k(\underline{x})) .$$

There is a valuable physical interpretation of the approximate method given by (6a,b). If we multiply (6a) by $A_{\gamma n}(t)$ and sum over γ from 1 to n, we obtain the equation which is the balance equation for the kinetic energy of the approximate solution. An analogous relation is obtained if we multiply (6b) by $B_{\gamma n}(t)$ and sum over γ. This relation is often interpreted as an entropy balance equation. - If we take the inner product of (3a) and $\underline{u}$, of (3b) and T, and use (3c) and (4a-e), we obtain the same integral properties for the exact solutions $\underline{u}$ and T. Then, even though $\underline{u}$ and T are approximated by the forms $\underline{u}_n$ and T_n, the fundamental integral properties are satisfied.

The most important and most difficult step is the selection of the functions $\underline{\Phi}_k$ and φ_k , $k = 1,2,3,...,$ In selecting these, we should carefully insure that the functions are linearly independent and members of a complete set. Violation of these requirements can lead to gross error if n (the degree of approximation) is successively increased.*) - Besides, we should insure that the functions incorporate the most important physical characteristics of the problem, e.g. some (or all) of the boundary conditions.

4. The amplitude equations.

When the solutions are expanded in orthogonal functions, the analysis is in general very long and complicated. To simplify the analysis, we will consider the limiting case of an infinite Prandtl number. It is believed that this simplification gives a good description for fluids which have large Prandtl numbers and provides at least a qualitatively

*) Some convergence proofs are available for certain problems in hydrodynamics. Ladyzhenskaya (1964) uses Galerkin's method to prove the existence of a "generalized solution" to the incompressible Navier-Stokes equation. Chernyakov (1966a) considered the problem of thermal convection in a bounded region, and a related problem with a free surface was considered by Chernyakov (1966b), but he did not allow for surface tension. It is worth noting that the modified Galerkin's method is in agreement with the definition of a "generalized solution".

correct description of fluids with Prandtl numbers greater than unity, Scanlon and Segel (1967).

An important consequence in the limit of infinite Prandtl number is that the vertical component of the curl of the velocity, $(\nabla x \underline{u}) \cdot \underline{k} = \zeta$, vanishes. This can be seen from the equations of motion and the boundary conditions, which take the form (cf. Chandrasekhar (1961), Chap. II)

$$\nabla^2 \zeta = 0, \quad \partial\zeta/\partial z = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad \zeta = 0 \quad \text{at} \quad z = -1$$

which admits only the solution $\zeta \equiv 0$ (assuming boundedness of ζ as $x, y \to \pm \infty$).

It can be shown from the identity $\nabla x(\nabla x \underline{u}) \equiv \nabla(\nabla \cdot \underline{u}) - \nabla^2 \underline{u}$, that in general

$$\hat{\nabla}^2 \hat{\underline{u}} = \underline{k} x \hat{\nabla} \zeta - \hat{\nabla}(\partial w/\partial z) - \underline{k} \partial(\nabla \cdot \underline{u})/\partial z$$

where $\hat{\underline{u}} = (u, v, 0)$, $\hat{\nabla} = (\partial/\partial x, \partial/\partial y, 0)$ and $\hat{\nabla}^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$. When $\zeta = 0$ and $\nabla \cdot \underline{u} = 0$, it follows that

$$\hat{\nabla}^2 \hat{\underline{u}} = -\hat{\nabla}(\partial w/\partial z) \ . \tag{7}$$

When explicit calculations are required, it is convenient to divide each dependent variable into two parts: one, which depends on z and t alone, is the horizontal average of that variable, the remaining part is then periodic in the horizontal directions. We shall use a horizontal bar to denote the horizontal average,

and write

$$T(x,y,z,t) \equiv \overline{T}(z,t) + \theta(x,y,z,t) \;, \quad p(x,y,z,t) \equiv \overline{p}(z,t) + \omega(x,y,z,t)$$

By averaging (3c) and (4a,d), it follows that $\underline{\overline{u}}(z,t) \equiv \underline{0}$. Then, by averaging (3a,b) directly ($P^{-1} = 0$!), we obtain

$$\text{(8a,b)} \qquad -\nabla\omega + \nabla^2\underline{u} + R\theta\underline{k} = 0 \;, \quad \overline{p}_z = R\overline{T}$$

$$\text{(8c)} \qquad \theta_t + \underline{u}\cdot\nabla\theta - (\overline{w\theta})_z + w\overline{T}_z = \nabla^2\theta + w$$

$$\text{(8d)} \qquad \overline{T}_t + (\overline{w\theta})_z = \overline{T}_{zz}$$

(subscripts z and t denote partial derivatives). By means of the constraint $\nabla\cdot\underline{u} = 0$, the boundary conditions (4a-e) take the form

$$\text{(9a,b)} \qquad w = 0, \quad w_{zz} - \hat{\nabla}^2 w = +M\hat{\nabla}^2\theta \quad \text{at} \quad z = 0$$

$$\text{(9c,d)} \qquad \overline{T}_z + L\overline{T} = 0 \;, \quad \theta_z + L\theta = 0 \quad \text{at} \quad z = 0$$

$$\text{(9e-h)} \qquad w = 0, \; w_z = 0, \; \overline{T} = 0, \; \theta = 0 \text{ at } \; z = -1$$

Note that $\overline{p}$ does only appear in the equation (8b). Hence this equation is used only to evaluate $\overline{p}$ after the other equations have been solved.

We expand the solutions in Fourier series, and assume that the x,y- and z-dependencies (in each term of the

series) are separable. The x,y dependence which is periodic and wave-like, will be denoted by $f_{ij}(x,y)$, $i,j = 1,2,3,\ldots$.

$$\hat{\nabla}^2 f_{ij} + a^2_{ij} f_{ij} = 0 \tag{10}$$

where $a^2_{ij} = (ia_1)^2 + (ja_2)^2$ is the overall wave number of the periodic structure and defines the size (though not the shape) of the cellular pattern. ia_1 and ja_2, $i,j = 1,2,3,\ldots,$ are the wave numbers in the x and y directions, respectively.

If the velocity $w(x,y,z,t)$ is represented by a truncated Fourier expansion, this can be written

$$w_n(x,y,z,t) = \sum_{ij} w_{ij}(z,t) f_{ij}(x,y)$$

(on the cell-walls certain boundary conditions must be satisfied, Appendix A). For the functions $w_{ij}(z,t)$, $i,j = 1,2,3,\ldots,$ we write

$$w_{ij}(z,t) = \sum_k A_{ijk}(t) W_k(z)$$

where the sequence $\{W_k(z)\}$, $k = 1,2,3,\ldots,$ is complete on the interval: $-1 \leqq z \leqq 0$. In this way we obtain

$$w_n(x,y,z,t) = \sum_{ijk} A_{ijk}(t) f_{ij}(x,y) W_k(z). \tag{11a}$$

In the same way we obtain the following expressions for $\theta_n(x,y,z,t)$ and $\overline{T}_n(z,t)$:

$$\theta_n(x,y,z,t) = \sum_{ijk} B_{ijk}(t) f_{ij}(x,y) F_k(z) \tag{11b}$$

$$\text{(11c)} \qquad \bar{T}_n(z,t) = \sum_k C_k(t) T_k(z)$$

and it follows from equations (7), (10) and (11a) that

$$\text{(11d)} \qquad \hat{\underline{u}}_n = (u_n, v_n, 0) = \sum A_{ijk} a_{ij}^{-2} \hat{\nabla} f_{ij} \partial W_k / \partial z$$

The expressions (11a-d) should be compared with (5a,b). It was necessary to introduce the functions $f_{ij}(x,y)$ to account for the horizontal structure. For reasons of convenience we have omitted the index n in the amplitudes . - We have pointed out that we should choose the sequences $\{W_k\}$, $\{F_k\}$ and $\{T_k\}$ to form complete sets. While any complete sets may be used it is often convenient to choose the functions W_k, F_k and T_k as eigenfunctions of one or another simple eigenvalue problem, but at the same time related to the problem. For the moment we shall only assume that the following boundary conditions are satisfied

$$\text{(12a)} \qquad W_k(0) = W_k(-1) = DW_k(-1) = 0$$

$$\text{(12b)} \qquad DF_k(0) + LF_k(0) = 0, \quad F_k(-1) = 0$$

$$\text{(12c)} \qquad DT_k(0) + LT_k(0) = 0, \quad T_k(-1) = 0$$

for $k = 1,2,3,\ldots$, where $D \equiv \partial/\partial z$.

By means of the boundary conditions (12a,b,c), it is convenient to rewrite the equations which govern the

amplitudes A_{ijk}, B_{ijk} and C_k as, cf. (6a,b) and (8a,c,d)

$$(13a)\quad (\nabla^2\underline{\hat{u}}_n, a_{\alpha\beta}^{-2}\hat{\nabla}f_{\alpha\beta}DW_\gamma)+(\nabla^2 w_n+R\theta_n, f_{\alpha\beta}W_\gamma)-$$

$$- (\frac{\partial}{\partial z}\underline{\hat{u}}_n + M\hat{\nabla}\theta_n, a_{\alpha\beta}^{-2}\hat{\nabla}f_{\alpha\beta}DW_\gamma)_s = 0$$

$$(13b)\quad \left(\frac{\partial\theta_n}{\partial t} + \underline{u}_n\cdot\nabla\theta_n - \frac{\partial}{\partial z}(\overline{w_n\theta_n}) + w_n\frac{\partial\overline{T}_n}{\partial z} - \nabla^2\theta_n - w_n, f_{\alpha\beta}F_\gamma\right) = 0$$

$$(13c)\quad \left(\frac{\partial\overline{T}_n}{\partial t} + \frac{\partial}{\partial z}(\overline{w_n\theta_n}) - \frac{\partial^2\overline{T}_n}{\partial z^2}, T_\gamma\right) = 0$$

for $\alpha,\beta,\gamma = 1,2,\ldots,n$. Substituting the expressions (11a-d) into (13a,b,c) the relations between $A_{ijk}(t)$, $B_{ijk}(t)$ and $C_k(t)$ are explicitly obtained. In doing so, we use the integral relations given in Appendix A, besides, some other simplifications are made by means of the boundary condition (12a).

Suppressing the summation sign, the summation now being indicated by the repeated indices i,j,k,ℓ,m and n, we obtain the following amplitude equations:

$$(14a)\quad A_{\alpha\beta k}((D^2-a_{\alpha\beta}^2)^2W_k, W_\gamma) = a_{\alpha\beta}^2 RB_{\alpha\beta k}(F_k, W_\gamma) -$$

$$- (A_{\alpha\beta k}D^2W_k(0) + a_{\alpha\beta}^2 MB_{\alpha\beta k}F_k(0))DW_\gamma(0)$$

$$\frac{dB_{\alpha\beta k}}{dt}(F_k,F_\gamma) = B_{\alpha\beta k}((D^2-a^2_{\alpha\beta})F_k,F_\gamma) +$$

$$+ A_{\alpha\beta k}(W_k,F_\gamma) - A_{\alpha\beta k}C_n(W_kDT_n,F_\gamma) +$$

(14b)

$$+ \tfrac{1}{2}a^{-2}_{ij}A_{ijk}B_{\ell mn}(a^2_{ij} + a^2_{\ell m} - a^2_{\alpha\beta})(\overline{f_{ij}f_{\ell m}f_{\alpha\beta}})(W_kF_n,DF_\gamma) -$$

$$- \tfrac{1}{2}a^{-2}_{ij}A_{ijk}B_{\ell mn}(a^2_{ij} - a^2_{\ell m} + a^2_{\alpha\beta})(\overline{f_{ij}f_{\ell m}f_{\alpha\beta}})(W_kDF_n,F_\gamma).$$

(14c) $$\frac{dC_k}{dt}(T_k,T_\gamma) = C_k(D^2T_k,T_\gamma) + A_{ijk}B_{ijn}(W_kDT_\gamma,F_n)$$

for $\alpha,\beta,\gamma = 1,2,\ldots,n$.

Usually, it is not possible to choose some approximating functions as the "best", but the following eigenvalue problems turn out to be suitable and will be used in the next section.

(15a) $$(D^2 - a^2_{\alpha\beta})^2W_k + \lambda^2_k(D^2 - a^2_{\alpha\beta})W_k = 0$$

(15b,c) $$(D^2 - a^2_{\alpha\beta})F_k + \mu^2_kF_k = 0 , \quad D^2T_k + \nu^2_kT_k = 0$$

$k = 1,2,3,\ldots$, with the boundary conditions (12a,b,c) together with $D^2W_k(0) = 0$. We verify that (15a,b,c) define denumerable infinite sequences of eigenfunctions, Appendix B. Due to the boundary conditions, the following orthogonality relations are obtained

$$((D^2-a^2_{\alpha\beta})^2W_k,W_\gamma) = -\lambda^2_k((D^2-a^2_{\alpha\beta})W_k,W_\gamma) = \lambda^2_k\delta_{k\gamma}$$

$$((D^2-a_{\alpha\beta}^2)F_k,F_\gamma) = -\mu_k^2(F_k,F_\gamma) = -\mu_k^2\delta_{k\gamma}$$

$$(D^2T_k,T_\gamma) = -\nu_k^2(T_k,T_\gamma) = -\nu_k^2\delta_{k\gamma}$$

It is worth noting that eigenvalue problems which are not selfadjoint can also be used. The adjoint problem generally differs from the original, but has the same eigenvalues, and each of its eigenfunctions is orthogonal to every eigenfunction of the original problem except the one belonging to the same eigenvalue. For that reason, we replace W_γ, F_γ and T_γ in (13a,b,c) and (14a,b,c) by the adjoint functions.

5. Solutions of the amplitude equations.

In the present section we will investigate the factors which govern the wave numbers a_1 and a_2 by studying some simplified systems of the amplitude equations. We consider first the linearized problem which gives the growth or decay of small perturbations. At the onset of convection, we assume that stable modes are divided from unstable modes by curves of marginal stability for which $\partial/\partial t \equiv 0$ *).

*) The validity of the principle of the exchange of stabilities is not exactly known for this problem. The validity can be verified rigorously if $M = 0$, Chandrasekhar (1961), but it seems impossible to prove or disprove it analytically if $M \neq 0$. Numerical computations of Vidal and Acrivos (1966) indicate, however, that oscillatory instability does not occur when $R = 0$.

Assuming that only one overall wave number is present, $(\alpha a_1)^2 + (\beta a_2)^2 = a^2$, say, we obtain in the simplest case $\gamma = 1$

$$(16a) \qquad \lambda_1^2 A_{\alpha\beta 1} = a^2 B_{\alpha\beta 1}\{R(F_1,W_1)-MF_1(0)DW_1(0)\}$$

$$(16b) \qquad \mu_1^2 B_{\alpha\beta 1} = A_{\alpha\beta 1}(W_1,F_1)$$

This is an eigenvalue problem from which M can be found in terms of a^2, L and R; alternatively, R in terms of a^2, L and M. Solving the eigenvalue problem (16a,b) and setting $M = M_1$ in the case $R = 0$, and $R = R_1$ in the case $M = 0$, we obtain

$$\frac{M}{M_1} + \frac{R}{R_1} = 1$$

where

$$M_1 = \frac{-\lambda_1^2 \mu_1^2}{a^2(W_1,F_1)F_1(0)DW_1(0)} , \quad R_1 = \frac{\lambda_1^2 \mu_1^2}{a^2(W_1,F_1)^2}$$

If we minimise M or R as functions of the wave number a, we obtain the critical values M_c or R_c. Both M and R are proportional to the (static) temperature gradient, and the trivial solutions $\underline{u} \equiv \underline{0}$ and $T \equiv 0$ are stable solutions only if the temperature gradient is so small that $M < M_c$ and $R < R_c$.

In the general case the minimasation process must certainly be done by numerical methods, but some preliminary

calculations can easily be made in the simple cases $R = 0$ or $M = 0$ to get some idea of the accuracy which can be attained. *) Although expansion in other functions may give greater accuracy, we believe that the functions W_γ and F_γ defined by (15a,b) are useful in solving the problem.

When the temperature gradient becomes large enough to make $M > M_c$ or $R > R_c$, the neglect of the non-linear terms in the amplitude equations (14a,b,c) is no longer justified. We consider the case in which either M_c or R_c are slightly exceeded. We may then assume that the wave numbers that are amplified most according to linear theory, dominate in the non-linear problem. (The experimental results do not indicate motions which have a continous range of wave numbers.) If so, the linear theory can be useful in the prediction of cell size (that is a^2), since it provides realistic initial conditions for the non-linear problem. Formulated in this way, a non-linear analysis can specify the wave numbers a_1 and a_2.

In the non-linear equation (14b) it is to be expected that the horizontal average $(f_{ij} f_{\ell m} f_{\alpha\beta})$ deserves particular interest, because when this average vanishes, the modes $f_{\alpha\beta}(x,y)$ do not interact except through the mean temperature profile $T_\gamma(z)$. If only modes with wave numbers $a^2 = (ia_1)^2+(ja_2)^2$ are present, we write

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*) The "exact" values of M_c and R_c in these cases are computed by other methods and tabulated by Nield (1964).

calculations can easily be made in the simple cases $R = 0$ or $M = 0$ to get some idea of the accuracy which can be attained. *) Although expansion in other functions may give greater accuracy, we believe that the functions W_γ and F_γ defined by (15a,b) are useful in solving the problem.

When the temperature gradient becomes large enough to make $M > M_c$ or $R > R_c$, the neglect of the non-linear terms in the amplitude equations (14a,b,c) is no longer justified. We consider the case in which either M_c or R_c are slightly exceeded. We may then assume that the wave numbers that are amplified most according to linear theory, dominate in the non-linear problem. (The experimental results do not indicate motions which have a continuous range of wave numbers.) If so, the linear theory can be useful in the prediction of cell size (that is a^2), since it provides realistic initial conditions for the non-linear problem. Formulated in this way, a non-linear analysis can specify the wave numbers a_1 and a_2.

In the non-linear equation (14b) it is to be expected that the horizontal average $(\overline{\zeta_{ij}\zeta_{mn}\zeta_{ap}})$ deserves particular interest, because when this average vanishes, the modes $\zeta_{ap}(x,y)$ do not interact except through the mean temperature profile $T_1'(z)$. If only modes with wave numbers $a^2 = (\pi a_1)^2 + (\pi a_2)^2$ are present, we write

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*) The "exact" values of M_c and R_c in these cases are computed by other methods and tabulated by Nield (1964).

$$ia_1 = a \sin \varphi_k \ , \quad ja_2 = a \cos \varphi_k \ , \quad k = 1,2,3,\ldots$$

and we consider a particular solution of (10), for instance

(17) $$f_{ij}(x,y) = \cos\{ia_1 x + ja_2 y\}$$

Setting $x = r \cos \theta$ and $y = r \sin \theta$, this can be written

$$f_{ij}(x,y) \equiv f_k(r,\theta) = \cos\{ar \sin(\varphi_k + \theta)\} \ , \ k = 1,2,3,\ldots$$

It can be shown that the average $(\overline{f_1 f_2 f_3})$ is nonzero only if

$$\varphi_2 = \varphi_1 \pm \frac{\pi}{3} \quad \text{and} \quad \varphi_3 = \varphi_1 \mp \frac{\pi}{3} \ .$$

or: $$\varphi_2 = \varphi_1 \pm \frac{2\pi}{3} \quad \text{and} \quad \varphi_3 = \varphi_1 \pm \frac{\pi}{3}$$

The results follow from the analysis given by Segel (1965b)*).

For our purpose, it is convenient to consider the modes f_1 and $f_2 - f_3$. Then, by rotation of the frame of reference we can choose $\varphi_1 = 0$, and reverting to our earlier notation, we can write the modes as

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*) Segel (1965b) not only allows for cosines in (17), but also sines. The result is the same: modes interact with each other only if they are associated with the wave-number angles φ, $\varphi - \frac{\pi}{3}$ and $\varphi + \frac{\pi}{3}$.

$$(18a,b) \quad f_{02}(x,y) = \sqrt{2}\cos 2a_2y \ , \quad f_{11}(x,y) = 2\cos a_1x \cos a_2y$$

where $a_1 = a \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}a$ and $a_2 = a\cos\frac{\pi}{3} = \frac{1}{2}a$. These modes are of particular interest in the investigations of cellular convection, since

$$\sqrt{2}\, f_{11}(x,y) \pm f_{02}(x,y) = \sqrt{2}(2\cos a_1x\cos a_2y \pm \cos 2a_2y)$$

is the analytical expression for a hexagonal cell pattern, cf. Chandrasekhar (1961), besides, they are of interest since $\overline{(f_{11}f_{11}f_{02})} = \frac{\sqrt{2}}{2} \neq 0$.

Since $\overline{(f_{11}f_{11}f_{02})} \neq 0$, we can carry out some preliminary calculations to illustrate the non-linear effects by studying the amplitude equations for $\gamma = 1,2$ and $(\alpha,\beta) = (0,2)$ and $(1,1)$. However, even this simplification results in extremely complicated sets of equations for the amplitudes, and it seems likely that we have to make some approximations if our aim is to solve the equations in a closed form. To obtain a closed form, we will follow the arguments given by Segel and Stuart (1962) in a related case. It is valid to neglect the time differentials in the equations for the mean field C_γ, and all components of the disturbances $B_{\alpha\beta\gamma}$, except in those for the fundamentals $B_{\alpha\beta 1}$, provided the purpose is to obtain the dominant part of the non-linear problem.

Then, if we eliminate $A_{\alpha\beta\gamma}$, we believe that and adequate approximation to the problem is:

$$(18a,b) \quad f_{02}(x,y) = \sqrt{2}\cos 2a_2 y \; , \quad f_{11}(x,y) = 2\cos a_1 x \cos a_2 y$$

where $a_1 = a \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}a$ and $a_2 = a\cos\frac{\pi}{3} = \frac{1}{2}a$. These modes are of particular interest in the investigations of cellular convection, since

$$\sqrt{2}\, f_{11}(x,y) \pm f_{02}(x,y) = \sqrt{2}(2\cos a_1 x \cos a_2 y \pm \cos 2a_2 y)$$

is the analytical expression for a hexagonal cell pattern, cf. Chandrasekhar (1961); besides, they are of interest since $\overline{(f_{11}f_{11}f_{02})} = \frac{\sqrt{2}}{2} \neq 0$.

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Then, if we eliminate $A_{\alpha\beta\gamma}$, we believe that an adequate approximation to the problem is:

(19a) $dB_{021}/dt = \varepsilon_1 B_{021} + \alpha_1 B_{022} + \beta_1 B_{111} B_{112} + \gamma_1 B_{021} C_1$

(19b) $dB_{111}/dt = \varepsilon_1 B_{111} + \alpha_1 B_{112} + \beta_1 (B_{111} B_{022} + B_{021} B_{112}) + \gamma_1 B_{111} C_1$

(19c) $0 = \varepsilon_2 B_{022} + \alpha_2 B_{021} + \beta_2 B_{111}^2$

(19d) $0 = \varepsilon_2 B_{112} + \alpha_2 B_{111} + 2\beta_2 B_{111} B_{021}$

(19e) $0 = C_1 + \gamma_2 (B^2_{111} + B^2_{021})$

where the coefficients in these equations are functions of a^2, L, M and R, and are given in Appendix C.

By elimination of B_{022}, B_{112} and C_1 we obtain

(20a) $dB_{021}/dt = \epsilon B_{021} - \Gamma B_{111}^2 - Q_1 B_{021}^3 - Q_3 B_{111}^2 B_{021}$

(20b) $dB_{111}/dt = \epsilon B_{111} - 2\Gamma B_{111} B_{021} - Q_2 B_{111}^3 - Q_3 B_{021}^2 B_{111}$

where

$$\epsilon = \frac{\varepsilon_1 \varepsilon_2 - \alpha_1 \alpha_2}{\varepsilon_2}, \quad \Gamma = \frac{\alpha_1 \beta_2 + \alpha_2 \beta_1}{\varepsilon_2}$$

$$Q_1 = \gamma_1 \gamma_2, \quad Q_2 = \gamma_1 \gamma_2 + \frac{\beta_1 \beta_2}{\varepsilon_2}, \quad Q_3 = \gamma_1 \gamma_2 + \frac{2\beta_1 \beta_1}{\varepsilon_2}$$

Apart from a slightly different notation (due to the normalized modes $f_{ij}(x,y)$), the equations (20a,b) are identical with those discussed by Segel and Stuart (1962).

We do not quote the general results given by Segel

and Stuart. However, the most "interesting" steady state solutions of (20a,b) is $B_{111} = \pm\sqrt{2}B_{021}$(note that $Q_3 = 2Q_2 - Q_1$) which characterizes the (hexagonal) convection cells

$$\{B_{111}f_{11} + B_{021}f_{02}\} = B_{111}\{2\cos a_1 x \cos a_2 \pm \cos 2a_2 y\}$$

This particular solution may be a stable equilibrium state only if $\Gamma \neq 0$, cf. Segel and Stuart. The coefficient Γ is given explicitly in Appendix C, and we observe that $\Gamma = 0$ if $M = 0$. The hexagonal convection cells may then be attributed to surface tension gradients (and not to buoyancy). - A different approach to this problem is given by Scanlon and Segel (1967).

6. Final Remarks.

We can of course proceed to study other disturbances $B_{\alpha\beta\gamma}$ than those discussed in the previous section, but the complexity of the amplitude equations becomes even greater, and they may lose their attraction. However, the behaviour of any finite number of modes which have the same overall wave number can be deduced with little further work. The reader is referred to the analysis by Segel (1965a,b) in his analysis of buoyancy driven flow.

Originally, our prupose was to suggest an approximate method to solve the problem of cellular convection subject to natural boundary conditions. The problem is reduced to

a system of ordinary differential equations, which is the most attractive result. Selecting approximating functions may, however, be crucial in the applications, and is often regarded as a major disadvantage of the method. As far as we know, no rational methods for selecting approximating functions are known, and it remains somewhat dependent on the user's intuition and experience. Nevertheless, we believe that, if only the most important physical characteristics are incorporated in the approximating functions, the qualitative description of the present problem should not be affected.

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Appendix A.

Previously we described the solutions of the equation (10) as periodic cell pattern. However, the precise definition requires that on the cell-walls the normal gradient of the vertical velocity vanishes, that is $\underline{\hat{n}}\cdot\hat{\nabla}f_{ij} = 0$, where $\underline{\hat{n}}$ is a unit vector normal to the cell-walls, Chandrasekhar (1961).

By means of Green's first identity in the plane

$$\iint \hat{\nabla}\varphi\cdot\hat{\nabla}\psi dxdy + \iint \varphi\hat{\nabla}^2\psi dxdy = \oint \varphi\underline{\hat{n}}\cdot\hat{\nabla}\psi ds$$

where ds is a line element on the boundary enclosing the region $-\pi/a_1 \leqq x \leqq \pi/a_1$ and $-\pi/a_2 \leqq y \leqq \pi/a_2$, we obtain the following integral relations when φ and ψ are any (orthonormalized) solutions of (10) subject to the boundary conditions given above

$$\overline{\hat{\nabla} f_{ij} \cdot \hat{\nabla} f_{\alpha\beta}} = a^2_{ij} \overline{f_{ij} f_{\alpha\beta}} = a^2_{ij} \delta_{i\alpha} \delta_{j\beta}$$

and, setting $\varphi = f_{ij} f_{\ell m}$, $\psi = f_{\alpha\beta}$, permutating the indices and taking sum and difference

$$2\overline{(f_{\alpha\beta} \hat{\nabla} f_{ij} \cdot \hat{\nabla} f_{\ell m})} = (a^2_{ij} + a^2_{\ell m} - a^2_{\alpha\beta}) \overline{(f_{ij} f_{\ell m} f_{\alpha\beta})}$$

Appendix B.

The solutions of the eigenvalue problems (15a,b,c) with the boundary conditions (12a,b,c) and $D^2 W_\gamma(0) = 0$, are

$$W (z) = C_1 \left\{ \frac{\sinh \; a_{\alpha\beta} z}{\sinh \; a_{\alpha\beta}} - \frac{\sin\sqrt{\lambda^2_\gamma - a^2_{\alpha\beta}} z}{\sin\sqrt{\lambda^2_\gamma - a^2_{\alpha\beta}}} \right\}$$

$$F_\gamma(z) = C_2 \sin\sqrt{\mu^2_\gamma - a^2_{\alpha\beta}} (z+1)$$

$$T_\gamma(z) = C_3 \sin \; \nu_\gamma (z+1)$$

where C_1, C_2 and C_3 are normalization constants, while the eigenvalues are solutions of

$$\sqrt{\lambda^2_\gamma - a^2_{\alpha\beta}} \cdot \tanh\, a_{\alpha\beta} = a_{\alpha\beta} \tan\sqrt{\lambda^2_\gamma - a^2_{\alpha\beta}}$$

$$\sqrt{\mu^2_\gamma - a^2_{\alpha\beta}} = -L \tan\sqrt{\mu^2_\gamma - a^2_{\alpha\beta}}$$

$$\nu_\gamma = -L \tan\nu_\gamma$$

which yields $\lambda_\gamma = \lambda_\gamma(a^2_{\alpha\beta})$, $\mu_\gamma = \mu_\gamma(a^2_{\alpha\beta}, L)$ and $\nu_\gamma = \nu_\gamma(L)$

Appendix C.

For reasons of convenience, we use the notations

$p_{k\gamma} \equiv (F_k, W_\gamma)$, $q_{k\gamma} \equiv F_k(0)DW_\gamma(0)$, $r_{kn\gamma} \equiv (W_k DF_n, F_\gamma)$,

$s_{kn\gamma} \equiv (W_k DT_n, F_\gamma)$.

The coefficients in the equations (19a-e) are

$$\varepsilon_1 = a^2R(p^2_{11}\lambda^{-2}_1 + p^2_{12}\lambda^{-2}_2) - a^2M(p_{11}q_{11}\lambda^{-2}_1 + p_{12}q_{12}\lambda^{-2}_2) - \mu^2_1$$

$$\varepsilon_2 = a^2R(p^2_{21}\lambda^{-2}_1 + p^2_{22}\lambda^{-2}_2) - a^2M(p_{21}q_{21}\lambda^{-2}_1 + p_{22}q_{22}\lambda^{-2}_2) - \mu^2_2$$

$$\alpha_1 = a^2R(p_{11}p_{21}\lambda^{-2}_1 + p_{12}p_{22}\lambda^{-2}_2) - a^2M(p_{11}q_{21}\lambda^{-2}_1 + p_{12}q_{22}\lambda^{-2}_2)$$

$$\alpha_2 = a^2 R(p_{21}p_{11}\lambda_1^{-2} + p_{22}p_{12}\lambda_2^{-2}) - a^2 M(p_{21}q_{11}\lambda_1^{-2} + p_{22}q_{12}\lambda_2^{-2})$$

$$\beta_1 = -\beta_2 = \tfrac{1}{2}a^2\lambda_1^{-2}\,(Rp_{11} - Mq_{11})(r_{112} - r_{121})(\overline{f_{11}f_{11}f_{02}})$$

$$\gamma_1 = \nu_1^2\gamma_2 = -a^2\lambda_1^{-2}\, s_{111}(Rp_{11} - Mq_{11})$$

where λ_1^2, λ_2^2, μ_1^2, μ_2^2 and ν_1^2 are the eigenvalues given in Appendix B.

The coefficient Γ in the equations (20a,b) can now be written,

$$\Gamma = \frac{\alpha_1\beta_2 + \alpha_2\beta_1}{\varepsilon_2} = \frac{\beta_2}{\varepsilon_2}(\alpha_1 - \alpha_2)$$

and by the elimination of α_1 and α_2 we obtain

$$\Gamma = a^2 M\,\frac{\beta_2}{\varepsilon_2}\left\{(p_{21}q_{11} - p_{11}q_{21})\lambda_1^{-2} + (p_{22}q_{12} - p_{12}q_{22})\lambda_2^{-2}\right\}$$

References.

Chandrasekhar, S. (1961), Hydrodynamic and Hydromagnetic Stability. Oxford Clarendon Press.

Chernyakov, P.S. (1966a,b), U.S.S.R. Comp. Math. and Math. Phys., (a) 6, no.2, 128; (b) 6, no.6, 135.

Ladyzhenskaya, O.A. (1964), The Mathematical Theory of Viscous Incompressible Flow. Gordon and Breach.

Mikhlin, S.G. (1964), Variational Methods in Mathematical Physics. Macmillan.

Nield, D.A. (1964), J. Fluid Mech., 19, 341.

Scanlon, J.W. and Segel, L.A. (1967), J. Fluid Mech., 30, 149.

Segel, L.A. (1965a,b), J. Fluid Mech., (a) 21, 345; (b) 21, 359.

Segel, L.A. and Stuart, J.T. (1962), J. Fluid Mech., 13, 289.

Vidal, A. and Acrivos, A. (1966), Phys. Fluids, 9, 615.