Department
of
APPLIED MATHEMATICS

On a stability problem in hydrodynamics.

Part I

by

Leif Engevik.

Report No. 11. November 1966.

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Abstract.

The problem to be studied in the present paper was previously investigated by A. Eliassen, E. Höiland, and E. Riis [1]. We find that in general both a discrete and a continuous spectrum of eigenvalues contribute to the solution. When, as in [1], the method of normal modes is applied care must be taken not to omit the continuous spectrum. Here we use the method of Laplace transform to solve the problem, and demonstrate how we can find the asymptotic behavior for large values of t (time) of that part of the solution which is due to the continuous sprectrum.

This work has been supported by the Royal Norwegian Council for Scientific and Industrial Research.

Introduction.

The problem to be studied in this paper is the one investigated by A. Eliassen, E. Høiland, and E. Riis [1]. In [1] the method of normal modes is applied to solve the problem. For $r > \frac{1}{4}$ (r = Richardson number) an infinite set of eigenvalues is found which in [1] is assumed to form a complete set. We will show that this assumption is not valid.

We use the Laplace transform to solve the problem. In general both a discrete and a continuous spectrum of eigenvalues will contribute to the solution. The contribution from the continuous spectrum is studied for large values of t in the two cases $r > \frac{1}{4}$ and $0 < r < \frac{1}{4}$. However the method used to study the asymptotic behavior of the contribution from the continuous spectrum may be applied to the cases $r = \frac{1}{4}$ and $r < 0$ as well.

If the method of normal modes is applied, we must take care not to omit the continuous spectrum of eigenvalues. If both the continuous and the discrete spectrum (when it exists) are taken into account, we find that the solution obtained by the method of normal modes is equivalent to the one found by using the Laplace transform. This equivalence will be demonstrated in a paper to appear later.

I. Formulation of the problem.

The system to be considered is the one studied in [1], viz: a horizontal flow with constant shear α of a stratified, incompressible and inviscid fluid confined between two rigid horizontal planes, situated at $z = \pm 1$ (see fig. 1). The velocity field and the density field in our basic motion are assumed to be given by:

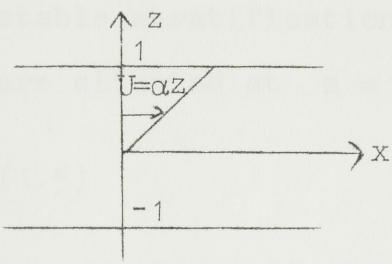

Fig. 1

$$(1.1)\quad \begin{cases} U = \alpha z & \text{(x-component of the velocity)} \\ \rho = \rho_o e^{-\beta z}, \end{cases}$$

where α, β and ρ_o are constants.

The equations governing this system, are the hydrodynamic equations for an incompressible, inviscid fluid, viz:

$$(1.2)\quad \begin{cases} \rho(\frac{\partial}{\partial t} + \underline{v}\cdot \nabla)\underline{v} = -\nabla p - \underline{k}\rho g \\ \nabla\cdot v = 0 \\ \frac{\partial \rho}{\partial t} + \underline{v}\cdot\nabla\rho = 0, \end{cases}$$

where $\underline{k}$ is the unit-vector in the z-direction, and g is the acceleration due to gravity. We assume the motion to be two-dimensional, and that it takes place in the xz-plane. Due to the second equation of (1.2), the pertubation velocity can be expressed by means of a stream function,

$$(1.3)\quad \underline{v}_1 = \nabla \times \Psi(x,z,t)\underline{j},$$

where $\underline{j}$ is the unit vector in the y-direction.

In order to find the equation for $\Psi(x,z,t)$, the eqs. (1.2) are linearized, and ρ_1 and p_1 *) are eliminated between the first and the third of eqs. (1.2). Taking α^{-1} as unit time, we

*) ρ_1 and p_1 are first order quantities.

obtain:

$$(1.4)\qquad (\frac{\partial}{\partial t} + z\frac{\partial}{\partial x})^2\nabla^2\Psi + r\frac{\partial^2}{\partial x^2}\Psi = 0 ,$$

where $r = \frac{\beta g}{\alpha^2}$ is the dimensionless Richardson number. We see that $r > 0$ when $\beta > 0$, which corresponds to a statically stable stratification. As mentioned earlier, the rigid planes are situated at $z = \pm 1$. The boundary conditions are therefore:

$$(1.5)\qquad \frac{\partial\Psi}{\partial x} = 0 \quad \text{at} \quad z = \pm 1$$

Let the field of vorticity and the field of rate of change of vorticity be assigned at $t = 0$, viz:

$$(1.6)\qquad \begin{cases} \nabla^2\Psi_{t=0} = F(x,z) \\ (\frac{\partial}{\partial t} + z\frac{\partial}{\partial x})\nabla^2\Psi_{t=0} = G(x,z) \end{cases}$$

We then have to solve a mixed boundary and initial value problem.

Let us assume that $\Psi(x,z,t)$ depends on x as:

$$(1.7)\qquad \Psi(x,z,t) = \psi(z,t)e^{ikx}$$

This is no restriction, since it is supposed that $\Psi(x,z,t)$ has a Fourier transform with respect to x, and then $\psi(z,t)$ represents the k^{th} Fourier component.

Introducing eq. (1.7) into eqs. (1.4), (1.5) and (1.6), we obtain:

$$(1.8)\qquad (\frac{\partial}{\partial t} + ikz)^2(\frac{\partial^2}{\partial z^2} - k^2)\psi - k^2r\psi = 0$$

$$(1.9)\qquad \psi(z,t) = 0 \quad \text{at} \quad z = \pm 1$$

$$(1.10)\quad \begin{cases} \left(\frac{\partial^2}{\partial z^2} - k^2\right)\psi = F_k(z) \\ \left(\frac{\partial}{\partial t} + ikz\right)\left(\frac{\partial^2}{\partial z^2} - k^2\right)\psi = G_k(z)\ , \end{cases}$$

where $F_k(z)$ and $G_k(z)$ are the Fourier transforms of $F(x,z)$ and $G(x,z)$.

II. Solution to the problem.

Multiplying the differential eq. (1.8) and the boundary conditions eq. (1.9) by the kernel e^{-pt} of the Laplace transform, integrating with respect to t between $0, \infty$, and using eqs. (1.10), we obtain:

$$(2.1)\quad (p + ikz)^2\left(\frac{\partial^2}{\partial z^2} - k^2\right)\overline{\psi} - rk^2\overline{\psi} = (p + ikz)F_k(z) + G_k(z)\ ,$$

$$(2.2)\qquad \overline{\psi} = 0 \quad \text{at} \quad z = \pm 1\ ,$$

where $\overline{\psi}(z,p) = \int_0^\infty \psi(z,t)e^{-pt}dt$.

From eq. (2.1) we obtain:

$$(2.3)\qquad \overline{\psi}'' - k^2\,\overline{\psi} + \frac{r\overline{\psi}}{(z-\zeta)^2} = I(z,\zeta)\ ,$$

where $\overline{\psi}' = \frac{d\overline{\psi}}{dz}$, and $\zeta = i\frac{p}{k}$.

$$(2.4)\qquad I(z,\zeta) = \frac{F_k(z)}{ik(z-\zeta)} - \frac{G_k(z)}{k^2(z-\zeta)^2}$$

Eq. (2.3) together with the boundary conditions is easily solved. We find :

$$(2.5)\qquad \overline{\psi}(z,\zeta) = \int_{-1}^{+1} I(u,\zeta)G(u - \zeta,\ z - \zeta)du\ ,$$

where:

$$(1.10)\quad \left\{\begin{aligned} &\left(\frac{\partial^2}{\partial z^2} - k^2\right)\psi = F_k(z) \\ &\left(\frac{\partial}{\partial t} + ikz\right)\left(\frac{\partial^2}{\partial z^2} - k^2\right)\psi = G_k(z) , \end{aligned}\right.$$

where $F_k(z)$ and $G_k(z)$ are the Fourier transforms of $F(x,z)$ and $G(x,z)$.

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$$(2.1)\quad (p + ikz)\left(\frac{\partial^2}{\partial z^2} - k^2\right)\bar{\psi} - nk^2\bar{\psi} = (p + ikz)F_k(z) + G_k(z) ;$$

$$(2.2)\quad \bar{\psi} = 0 \quad \text{at} \quad z = \pm 1 ,$$

where $\bar{\psi}(z;p) = \int_0^\infty \psi(z,t)e^{-pt}\,dt$.

From eq. (2.1) we obtain:

$$(2.3)\quad \bar{\psi}'' - k^2\bar{\psi} + \frac{n\bar{\psi}}{(z-\zeta)^2} = I(z,\zeta) ,$$

where $\bar{\psi}' = \frac{d\bar{\psi}}{dz}$, and $\zeta = \frac{ip}{k}$.

$$(2.4)\quad I(z,\zeta) = \frac{F_k(z)}{ik(z-\zeta)} - \frac{G_k(z)}{k^2(z-\zeta)^2}$$

Eq. (2.3) together with the boundary conditions is easily solved. We find :

$$(2.5)\quad \bar{\psi}(z,\zeta) = \int_{-1}^{+1} I(u,\zeta)\,G(u, z, \zeta)\,du ,$$

where:

$$(2.6)\quad G(u-\zeta, z-\zeta) = \frac{1}{W(\varphi_1,\varphi_2)} \cdot \begin{cases} \varphi_1(u-\zeta)\varphi_2(z-\zeta) & u < z \\ \varphi_2(u-\zeta)\varphi_1(z-\zeta) & u > z \end{cases},$$

$$(2.7)\quad \begin{cases} \varphi_1(z-\zeta) = g(-1-\zeta)f(z-\zeta) - f(-1-\zeta)g(z-\zeta) , \\ \varphi_2(z-\zeta) = g(1-\zeta)f(z-\zeta) - f(1-\zeta)g(z-\zeta) . \end{cases}$$

$$(2.8)\qquad f(v) = (ikv)^{\frac{1}{2}}J_{-\nu}(ikv), \quad g(v) = (ikv)^{\frac{1}{2}}J_{\nu}(ikv) .$$

J_ν and $J_{-\nu}$ are the Bessel functions of order ν and $-\nu$.

$$(2.9)\qquad \nu = \sqrt{1/4 - r} .$$

$\varphi_1(z-\zeta)$, $\varphi_2(z-\zeta)$, $f(z-\zeta)$ and $g(z-\zeta)$ are solutions of the homogenous equation, corresponding to eq. (2.3).

$W(\varphi_1,\varphi_2)$ is the Wronskian determinant of $\varphi_1(z-\zeta)$ and $\varphi_2(z-\zeta)$. It is easy to show that

$$(2.10)\qquad W(\varphi_1,\varphi_2) = D(\zeta)W(f,g) ,$$

where:

$$(2.11)\quad D(\zeta) = g(1-\zeta)f(-1-\zeta) - f(1-\zeta)g(-1-\zeta) .$$

$W(f,g) = f(z-\zeta)g'(z-\zeta) - f'(z-\zeta)g(z-\zeta)$ is the Wronskian of $f(z-\zeta)$ and $g(z-\zeta)$. Since in eq. (2.3) there is no first order derivative, $W(f,g) = W$ is a constant [2].

When $\overline{\Psi}(z,\zeta)$ is found, $\psi(z,t)$ is easily obtained by inversion.

$$\psi(z,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \overline{\Psi}(z,i\tfrac{p}{k})e^{tp}dp = \frac{k}{2\pi}\int_{i\zeta_o-\infty}^{i\zeta_o+\infty} \overline{\Psi}(z,\zeta)e^{-ik\zeta t}d\zeta =$$

$$(2.12) = \frac{k}{2\pi}\int_{i\zeta_o-\infty}^{i\zeta_o+\infty} e^{-ik\zeta t}\,\frac{d\zeta}{WD(\zeta)}\left\{\varphi_2(z-\zeta)\int_{-1}^{z} I(u,\zeta)\varphi_1(u-\zeta)du\, + \right.$$

$$\left. + \varphi_1(z-\zeta)\int_{z}^{1} I(u,\zeta)\varphi_2(u-\zeta)du\right\} .$$

In the complex p-plane γ must be greater than the real parts of all the singularities of $\overline{\Psi}(z,i\frac{p}{k})$, or equivalent: for $k > 0$ ζ_o must be greater than the imaginary parts of all the singularities of $\overline{\Psi}(z,\zeta)$. In order to evaluate the integral in eq. (2.12) we perform an integration around the closed contour c shown in fig. 2. $\overline{\Psi}(z,\zeta)$ is a many-valued function with branch-points at $\zeta = 1, z, -1$ $\{z \in [-1,1]\}$, so we have to make cuts in the complex ζ-plane. We choose the cuts as shown in fig. 2. We obtain:

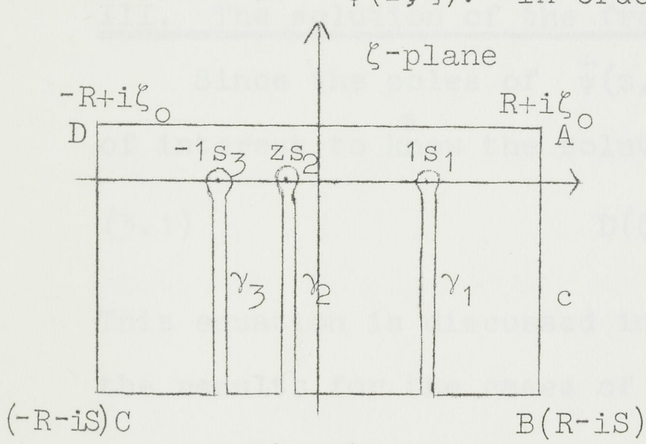

Fig. 2

$$(2.13)\ \frac{k}{2\pi}\oint_c \overline{\Psi}(z,\zeta)e^{-ik\zeta t}d\zeta = \int_{i\zeta_o-R}^{i\zeta_o+R} + \int_{ABCD} + \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{s_1} + \int_{s_2} + \int_{s_3} ,$$

where:

s_1, s_2 and s_3 are the small circles around $\zeta = 1, z, -1$,

$\int_{\gamma_1} = \int_{1-iS}^{1-i\rho_1} + \int_{1-i\rho_1}^{1-iS}$. We have analogous expressions for $\int_{\gamma_2}$ and $\int_{\gamma_3}$.

ρ_i is the radius of s_i, (i = 1, 2, 3).

Since $\overline{\psi}(z,\zeta)$ is a many-valued function, the integrals $\int_{\gamma_1}$, $\int_{\gamma_2}$ and $\int_{\gamma_3}$ will in general not vanish.

From Cauchy's residue theorem we have that

$$\oint_c \overline{\psi}(z,\zeta)e^{-ik\zeta t}d\zeta = -2\pi i\Sigma R .$$

ΣR is the sum of the residues of the integrand at its poles within c. The poles of $\overline{\psi}(z,\zeta)$ are the zeros of $D(\zeta)$.

From eq. (2.13) we can find $\psi(z,t)$, by letting $R,S \to \infty$, and $\rho_i \to 0$.

III. The solution of the frequency equation.

Since the poles of $\overline{\psi}(z,\zeta)$ are the zeros of $D(\zeta)$, it is of interest to know the solution of the frequency equation

(3.1) $$D(\zeta) = 0 .$$

This equation is discussed in [1], and we will here only give the results for the cases of interest to us.

$r > \frac{1}{4}$:

In this case the order ν of the Bessel function involved, is purely imaginary, see eq. (2.9). There is an infinite set of real eigenvalues, ζ_n, which satisfies eq. (3.1). Moreover $|\zeta_n| > 1$ and $\zeta_n \to \pm 1$ when $n \to \pm\infty$. There exist infinitely many eigenvalues within the regions $[-1-\varepsilon, 0]$ and $[1+\varepsilon, 0]$ $(\varepsilon > 0)$, regardless of how small ε may be.

$0 < r < \frac{1}{4}$:

The order ν is real. There are no eigenvalues.

IV. The case $r > \frac{1}{4}$.

As pointed out earlier, we can obtain $\psi(z,t)$ from eq. (2.13) by letting $R,S \to \infty$ and $\rho_i \to 0$, $(i = 1, 2, 3)$. But in order that $\int_{s_1}$ and $\int_{s_3}$ shall exist, we must choose the radii ρ_1 and ρ_3 in such a way that the circles do not pass through any of the zeros of $D(\zeta)$. If ρ_1 and $\rho_3 \to 0$ in this way, then $\int_{s_1}$ and $\int_{s_3} \to 0$. Also $\int_{s_2} \to 0$ when $\rho_2 \to 0$. It is also easy to show that $\int_{ABCD} \to 0$ when $R,S \to \infty$. Therefore

$$(4.1) \qquad \psi(z,t) = -2\pi i \Sigma R - \int_{\gamma_1} - \int_{\gamma_2} - \int_{\gamma_3},$$

where ΣR is a sum with infinitely many terms.

Th 1. $-2\pi i \Sigma R = \sum_{n=-\infty}^{n=+\infty} a_n \varphi_2(z - \zeta_n) e^{-ik\zeta_n t}$, where a_n $(n \neq 0)$ is given by: $a_n[W^2 - \varphi_2'^2(-1 - \zeta_n)] = -2a_n r \int_{-1}^{+1} \frac{\varphi_2^2(u - \zeta_n)}{(u - \zeta_n)^3} du =$

$$= \int_{-1}^{+1} \left\{ \frac{F_k(u)}{u - \zeta_n} + \frac{G_k(u)}{ik(u - \zeta_n)^2} \right\} \varphi_2(u - \zeta_n) du .$$

$a_o = 0$.

This is the solution found in [1].

Proof:

We will first show that the residue of $\frac{1}{D(\zeta)}$ at $\zeta = \zeta_n$

is given by:

$$\text{(4.2)} \qquad \operatorname{Res} \frac{1}{D(\zeta)}_{\zeta=\zeta_n} = \frac{1}{\left(\frac{\partial D}{\partial \zeta}\right)_{\zeta=\zeta_n}} = \frac{\varphi_2'(-1-\zeta_n)}{W^2 - \varphi_2'^2(-1-\zeta_n)}$$

We assume that $D(\zeta)$ has only zeros of first order. From eqs. (2.7) we find that $D(\zeta) = \varphi_2(-1-\zeta) = -\varphi_1(1-\zeta)$. Let us write:

$$\text{(4.3)} \qquad D(\zeta) = \tfrac{1}{2}[\varphi_2(-1-\zeta) - \varphi_1(1-\zeta)] \, .$$

From eq. (4.3) we obtain:

$$\text{(4.4)} \qquad \frac{\partial D}{\partial \zeta} = \frac{1}{2}\left[\frac{\partial \varphi_2(-1-\zeta)}{\partial \zeta} - \frac{\partial \varphi_1(1-\zeta)}{\partial \zeta}\right] .$$

From eqs. (2.7) we find:

$$\text{(4.5)} \quad \frac{\partial \varphi_1}{\partial \zeta} = -\varphi_1'(z-\zeta) + \frac{\partial g(-1-\zeta)}{\partial \zeta} f(z-\zeta) - \frac{\partial f(-1-\zeta)}{\partial \zeta} g(z-\zeta),$$

$$\text{(4.6)} \quad \frac{\partial \varphi_2}{\partial \zeta} = -\varphi_2'(z-\zeta) + \frac{\partial g(1-\zeta)}{\partial \zeta} f(z-\zeta) - \frac{\partial f(1-\zeta)}{\partial \zeta} g(z-\zeta),$$

where $\varphi_i'(z-\zeta) = \frac{\partial \varphi_i(z-\zeta)}{\partial z}$ as before, $(i = 1, 2)$. We put $z = 1$ into eq. (4.5) and $z = -1$ into eq. (4.6) and substract. We then obtain:

$$\text{(4.7)} \quad \frac{\partial \varphi_2(-1-\zeta)}{\partial \zeta} - \frac{\partial \varphi_1(1-\zeta)}{\partial \zeta} = \varphi_1'(1-\zeta) - \varphi_2'(-1-\zeta) + \frac{\partial D}{\partial \zeta}$$

Introducing eq. (4.4) into eq. (4.7), we find:

$$\text{(4.8)} \qquad \frac{\partial D}{\partial \zeta} = \varphi_1'(1-\zeta) - \varphi_2'(-1-\zeta)$$

When $\zeta = \zeta_n$ is an eigenvalue, then $\varphi_1(z-\zeta_n)$ and $\varphi_2(z-\zeta_n)$ are linearly dependent solutions, since the Wronskian $W(\varphi_1, \varphi_2) = D(\zeta_n)W = 0$. Then we also have:

$$\text{(4.9)} \qquad \varphi_1(x-\zeta_n)\varphi_2(z-\zeta_n) = \varphi_1(z-\zeta_n)\varphi_2(x-\zeta_n) \, .$$

Differentiating eq. (4.9) with respect to x and z and putting

$x = -1$ and $z = 1$, we obtain:

$$(4.10)\quad \varphi_1'(-1 - \zeta_n)\varphi_2'(1 - \zeta_n) = \varphi_1'(1 - \zeta_n)\varphi_2'(-1 - \zeta_n)$$

But

$$(4.11)\begin{cases}\varphi_1'(-1 - \zeta) = g(-1 - \zeta)f'(-1 - \zeta) - f(-1 - \zeta)g'(-1 - \zeta) = - W(f,g) \\ \varphi_2'(1 - \zeta) = g(1 - \zeta)f'(1 - \zeta) - f(1 - \zeta)g'(1 - \zeta) = - W(f,g)\end{cases}$$

Introducing eqs. (4.11) into eq. (4.10), we obtain:

$$(4.12)\qquad \varphi_1'(1 - \zeta_n) = \frac{W^2}{\varphi_2'(-1 - \zeta_n)} ,$$

where it is supposed that $\varphi_2'(-1 - \zeta_n) \neq 0$, which means that $\varphi_2(z - \zeta_n)$ has a simple zero at $z = -1$.

Combining eqs. (4.8) and (4.12), we obtain:

$$\left(\frac{\partial D}{\partial \zeta}\right)_{\zeta=\zeta_n} = \frac{W^2 - \varphi_2'^2(-1 - \zeta_n)}{\varphi_2'(-1 - \zeta_n)} ,$$

which is equivalent to eq. (4.2).

Differentiating eq. (4.9) with respect to z and putting $z = -1$, we obtain:

$$(4.13)\quad \varphi_1(x - \zeta_n)\varphi_2'(-1 - \zeta_n) = \varphi_1'(-1 - \zeta_n)\varphi_2(x - \zeta_n) \Rightarrow$$

$$\Rightarrow \varphi_1(x - \zeta_n) = \frac{-W}{\varphi_2'(-1 - \zeta_n)}\varphi_2(x - \zeta_n)$$

Using eqs. (4.2) and (4.13), we find that

$$- 2\pi i \Sigma R = \sum_{n=-\infty}^{n=+\infty} a_n \varphi_2(z - \zeta_n) \quad \text{with} \quad a_n[W^2 - \varphi_2'^2(-1 - \zeta_n)] =$$

$$\int_{-1}^{+1}\left\{\frac{F_k(u)}{u - \zeta_n} + \frac{G_k(u)}{ik(u - \zeta)^2}\right\}du .$$

It remains to prove that

$$(4.14) \qquad [W^2 - \varphi_2'^2(-1 - \zeta_n)] = -2r \int_{-1}^{+1} \frac{\varphi_2^2(u - \zeta_n)}{(u - \zeta_n)^3} du \quad ,$$

in order to show that $-2\pi i \Sigma R$ is the solution found in [1].
We have that

$$(4.15) \qquad \varphi_2''(z - \zeta_n) - k^2\varphi_2(z - \zeta_n) + \frac{r\varphi_2(z - \zeta_n)}{(z - \zeta_n)^2} = 0 \; .$$

We multiply eq. (4.15) with $\varphi_2'(z - \zeta_n)$ and integrate between -1 and $+1$. We obtain:

$$(4.16) \quad \frac{1}{2}\left[\varphi_2'^2(1 - \zeta_n) - \varphi_2'^2(-1 - \zeta_n)\right] = -r\int_{-1}^{+1} \frac{\varphi_2\varphi_2' dz}{(z - \zeta_n)^2} = -r\int_{-1}^{+1} \frac{\varphi_2^2}{(z - \zeta_n)^3} dz \; ,$$

where we have used the boundary conditions $\varphi_2(1 - \zeta_n) = \varphi_2(-1 - \zeta_n) = 0$. Eq. (4.16) is equivalent to eq. (4.14), and so Th. 1 is proved.

<u>Th. 2.</u> <u>The series $\sum_{n=-\infty}^{n=+\infty} a_n\varphi_2(z - \zeta_n)e^{-ik\zeta_n t}$ is uniform convergent with respect to z, and can be differentiated any time within every closed region $\Omega \subset \langle -1, 1\rangle$.</u>

<u>Proof</u>:

Let us first find an upper and a lower bound to ζ_n. The eigenvalues satisfy following relation, see [1]:

$$(4.17) \quad \mu \ln\left(\frac{\zeta + 1}{\zeta - 1}\right) + \left[\arg j_{i\mu}(k^2(\zeta + 1)^2) - \arg j_{i\mu}(k^2(\zeta - 1)^2)\right] = n\pi \quad ,$$

where $\mu = i\nu = \sqrt{r - \frac{1}{4}}$ has been used.

$j_{i\mu}$ is an analytic function of the argument, see [1].

Therefore the expression within the bracket will tend to finite limits when $n \to \pm\infty$, i.e. $\zeta_n \to \pm 1$, and for all values of n we may put

(4.18) $\mu\beta_1 < [\arg j_{i\mu}(k^2(\zeta + 1)^2) - \arg j_{-i\mu}(k^2(\zeta - 1)^2)] < \mu\beta_2$.

Introducing eq. (4.18) into eq. (4.17), we obtain:

(4.19) $\alpha n - \beta_2 < \ln\left(\frac{\zeta_n + 1}{\zeta_n - 1}\right) < \alpha n - \beta_1$, where $\alpha = \frac{\pi}{\mu}$.

Let $N_1 > 0$ be chosen so large that $\alpha n - \beta_1 > 0$, $\alpha n - \beta_2 > 0$ for $n > N_1$. When eq. (4.19) is solved, we obtain:

(4.20) $$\frac{e^{\alpha n-\beta_1}+1}{e^{\alpha n-\beta_1}-1} < \zeta_n < \frac{e^{\alpha n-\beta_2}+1}{e^{\alpha n-\beta_2}-1} \quad (n > N_1) .$$

If $N_2 > 0$ is chosen so large that $\alpha n - \beta_1 < 0$, $\alpha n - \beta_2 < 0$ for all $n < - N_2$, we find that

(4.21) $$\frac{e^{\alpha n-\beta_2}+1}{e^{\alpha n-\beta_2}-1} < \zeta_n < \frac{e^{\alpha n-\beta_1}+1}{e^{\alpha n-\beta_1}-1} \quad (n < - N_2) .$$

Let us examine a_n. Assume that $F_k(z)$ and the 1.order derivative of $G_k(z)$ are bounded for $z \in [-1,1]$.*) We have:

(4.22) $$\int_{-1}^{+1}\left\{\frac{F_k(u)}{(u-\zeta_n)} + \frac{G_k(u)}{ik(u-\zeta_n)^2}\right\}\varphi_2(u-\zeta_n)du = g(1-\zeta_n)\int_{-1}^{+1}\left[\frac{F_k(u)}{u - \zeta_n}f(u-\zeta_n) + \right.$$
$$+ \frac{1}{ikr}\{G_k'(u)f'(u-\zeta_n) + k^2G_k(u)f(u-\zeta_n)\}\Big]du -$$
$$- f(1-\zeta_n)\int_{-1}^{+1}\Big[\frac{F_k(u)}{u-\zeta_n} g(u-\zeta_n) + \frac{1}{ikr}\{G_k'(u)g'(u-\zeta_n) +$$
$$+ k^2G_k(u)g(u-\zeta_n)\}\Big]du - \frac{1}{ikr}\{G_k(1)W - G_k(-1)\varphi_2'(-1-\zeta_n)\} ,$$

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*) It is not necessary to assume that $G_k(z)$ has a 1. order derivative for $z \in [-1,1]$, see Appendix.

where we have used that

$$\varphi_2''(z - \zeta_n) - k^2\varphi_2(z - \zeta_n) + \frac{r\varphi_2(z - \zeta_n)}{(z - \zeta_n)^2} = 0 .$$

We observe that the last two integrals in eq. (4.22) exist when $n \to \pm\infty$, i.e. $\zeta_n \to \pm 1$. Let us first consider the case:

$n \to \infty$, i.e. $\zeta_n \to 1$.

Taking into account eq. (4.22), we find that

$$a_n \to - \frac{G_k(1)}{ikr\, W} .$$

Therefore we have:

(4.23) $$|a_n| \leq M \quad \text{for all} \quad n > 0.$$

If $G_k(1) = 0$, it follows from eq. (4.22) that $|a_n| \to 0$ as $(\zeta_n - 1)^{\frac{1}{2}}$ when $n \to \infty$.

We also have:

(4.24) $|\varphi_2(z-\zeta_n)| \leq (\zeta_n - 1)^{\frac{1}{2}} M_o$ for every $z \in [-1,1]$ and all $n>0$.

Using eq. (4.20), we find:

$$(\zeta_n - 1) < \frac{2}{e^{\alpha n - \beta_2} - 1} = \frac{2}{e^{\alpha n - \beta_2}} \cdot \frac{1}{1 - e^{-\alpha n + \beta_2}} \quad (n > N_1)$$

But $1 - e^{-\alpha n + \beta_2} > \frac{1}{2}$ for $n > N_3$.

That means:

(4.25) $$(\zeta_n - 1)^{\frac{1}{2}} < 2e^{\frac{\beta_2}{2} - \frac{\alpha n}{2}}$$

Taking into account eqs. (4.23), (4.24) and (4.25), we find that

$$\left|\sum_{n=p}^{\infty} a_n \varphi_2(z-\zeta_n) e^{-ik\zeta_n t}\right| \leq \sum_{n=p}^{\infty} |a_n||\varphi_2(z-\zeta_n)| < 2M\, M_o e^{\frac{\beta_2}{2}} \sum_{p}^{\infty} e^{-\frac{\alpha n}{2}} =$$

$$= 2MM_o \frac{e^{\frac{\beta_2}{2}}}{1 - e^{\frac{\alpha p}{2}}} ,$$

where $p = \max(N_1, N_3)$.

That means:

(4.26) $\sum_1^\infty a_n \varphi_2(z-\zeta_n)e^{-ik\zeta_n t}$ is uniform convergent for $z \in [-1,1]$.

It remains to show that the series in eq. (4.26) can be differentiated term by term for every $z \in \Omega \subset \langle -1,1 \rangle$. From the theory of series we know that we can differentiate a convergent series term by term if the resulting series also is uniform convergent. For every $z \in \Omega$ and all ζ_n, $\frac{\varphi_2(z-\zeta_n)}{(\zeta_n - 1)^{\frac{1}{2}}}$ is an analytic function. Therefore $\frac{|\varphi_2(z-\zeta_n)|}{(\zeta_n - 1)^{\frac{1}{2}}} \leq M_o$, where M_o can be taken to be the maximum of $|\varphi_2(z-\zeta_n)|/(\zeta_n - 1)^{\frac{1}{2}}$ within the closed region $\{z \in \Omega, \ \zeta_1 \geq \zeta \geq 1\}$ in the $x\zeta$-plane. But $\frac{\varphi_2(z-\zeta_n)}{(\zeta_n - 1)^{\frac{1}{2}}}$ can be differentiated with respect to z any time within this region, and the derivative is again an analytic function. Therefore:

$$|\varphi_2^{(m)}(z-\zeta_n)| \leq (\zeta_n - 1)^{\frac{1}{2}} M_m \quad \text{for } z \in \Omega, \text{ and all } n > 0.$$

This relation is equivalent to eq. (4.24), and the proof of uniform convergence for the series which is obtained by differentiating eq. (4.26) term by term, proceed as above.

In conclusion we can therefore say that the series in eq. (4.26) can be differentiated any time with respect to $z \in \Omega$.

Next let us consider the case:

<u>$n \to -\infty$, i.e. $\zeta_n \to -1$.</u>

From eq. (4.22) we obtain:

where $p = \max(N_1, N_2)$.

That means:

$$(4.26)\quad \sum_{1}^{\infty} a_n \varphi_2(z-\zeta_n) e^{-ik\zeta_n t} \text{ is uniform convergent for } z \in [-1,1].$$

It remains to show that the series in eq. (4.26) can be differentiated term by term for every $z \in \Omega = \langle -1,1 \rangle$. From the theory of series we know that we can differentiate a convergent series term by term if the resulting series also is uniform convergent. For every $z \in \Omega$ and all ζ_n, $\dfrac{\varphi_2(z - \zeta_n)}{(\zeta_n - 1)^{\frac{1}{2}}}$ is an analytic function. Therefore $\dfrac{|\varphi_2(z - \zeta_n)|}{(\zeta_n - 1)^{\frac{1}{2}}} \le M_0$, where M_0 can be taken to be the maximum of $|\varphi_2(z - \zeta_n)|/(\zeta_n - 1)^{\frac{1}{2}}$ within the closed region $|z| \le 1$, $\zeta_1 \ge \zeta \ge 1$ in the ζ-plane. But $\dfrac{\varphi_2(z - \zeta_n)}{(\zeta_n - 1)^{\frac{1}{2}}}$ can be differentiated with respect to z any time within this region, and the derivative is again an analytic function. Therefore:

$$|\varphi_2^{(m)}(z - \zeta_n)| \le (\zeta_n - 1)^{\frac{1}{2}} M_m \quad \text{for } z \in \Omega \text{ and all } m \ge 0.$$

This relation is equivalent to eq. (4.24), and the proof of uniform convergence for the series which is obtained by differentiating eq. (4.26) term by term, proceed as above.

In conclusion we can therefore say that the series in eq. (4.26) can be differentiated any time with respect to $z \in \Omega$.

Next let us consider the case:

$n \to \infty$, i.e. $\zeta_n \to 1$.

From eq. (4.16) we obtain:

$$a_n \cong - \frac{G_k(-1)}{ikr\varphi_2'(-1-\zeta_n)} \text{ , since } |\varphi_2'(-1-\zeta_n)| \to \infty \text{ when } \zeta_n \to -1.$$

Therefore:

$$(4.27) \qquad |a_n| < M(-1-\zeta_n)^{\frac{1}{2}} \quad \text{for all} \quad n < 0 \ .$$

Also:

$$(4.28) \quad |\varphi_2(z-\zeta_n)| \leq M_0 \quad \text{for all} \quad z \in [-1,1] \quad \text{and all} \quad n < 0 \ .$$

Taking into account eqs. (4.21), (4.27) and (4.28), we find that the series $\sum_{n=-\infty}^{1} a_n \varphi_2(z-\zeta_n) e^{-ik\zeta_n t}$ and all its derivatives are uniform convergent within every closed region $\Omega \subset \langle -1,1 \rangle$. This is proved as above. And so Th. 2 is proved.

It is to be noted that the series in Th. 2 cannot be differentiated term by term for all z in the region $[-1,1]$. This is shown by a simple example:

Let us differentiate the series in Th. 2 with respect to z, and let us put $z = 1$. We then observe that the series in eq. (4.26) is divergent, since $\varphi_2'(1-\zeta_n) = -W$.

<u>Th 3. The contribution to $\psi(z,t)$ from the integrals along γ_1 and γ_3 is damped out at least as fast as $t^{-1/2}$ when $t \to \infty$.</u>

<u>Proof</u>:

We have that $\int_{\gamma_1} = \int_{1-i\infty}^{1} + \int_{1}^{1-i\infty}$. Let us examine the integral $\int_{1}^{1-i\infty}$. Into this integral we introduce η , given by $\zeta = 1 - i\eta$, as a new variable. We obtain:

$$P(z,t) = -\frac{ik}{2\pi}\int_0^\infty e^{-ikt-k\eta t}\frac{d\eta}{WD(1-i\eta)}\Big[\varphi_2(z-1+i\eta)\int_{-2+i\eta}^{z-1+i\eta} I(v+1-i\eta, 1-i\eta)$$

(4.29)

$$\times\,\varphi_1(v)dv + \varphi_1(z-1+i\eta)\int_{z-1+i\eta}^{i\eta} I(v+1-i\eta, 1-i\eta)\varphi_2(v)dv\Big],$$

where

(4.30)
$$\begin{cases} D(1-i\eta) = g(i\eta)f(i\eta-2) - f(i\eta)g(i\eta-2) = (k\eta)^{\frac{1}{2}}(k\eta+2ik)^{\frac{1}{2}}D_1(\eta) \\ \varphi_1(z-1+i\eta) = g(i\eta-2)f(z-1+i\eta) - f(i\eta-2)g(z-1+i\eta) \\ \varphi_2(z-1+i\eta) = g(i\eta)f(z-1+i\eta) - f(i\eta)g(z-1+i\eta)\,. \end{cases}$$

$D_1(\eta)$ is a regular function of η except at $\eta = 0$, and it has no zero for $\eta \in [0,\infty]$. We have that

(4.31) $\quad |D_1(\eta)| \geq m(\delta) > 0 \quad \text{for} \quad \eta \in [0,\delta] \quad (\delta > 0)$.

Let us put $\quad P(z,t) = \int_0^\delta + \int_\delta^\infty$. We then have:

(4.32) $$|P(z,t)| \leq \Big|\int_0^\delta\Big| + \Big|\int_\delta^\infty\Big| .$$

Using eq. (4.31) we obtain:

(4.33) $$\Big|\int_0^\delta\Big| \leq \frac{k}{2\pi}\int_0^\delta e^{-k\eta t}\frac{d\eta}{m(\delta)(k\eta)^{\frac{1}{2}}|(k\eta+2ik)^{\frac{1}{2}}||W|}$$

$$\times\,\Big|\varphi_2\int_{-2+i\eta}^{z-1+i\eta}(\dots)dv + \varphi_1\int_{z-1+i\eta}^{i\eta}(\dots)dv\Big|$$

Moreover

(4.34) $$\Big|\int_\delta^\infty\Big| = O(e^{-\delta t})\,.$$

Let us find the asymptotic behavior of the integral in eq. (4.33) for large values of t.

We assume that $F_k(z)$ and $G_k(z)$ are analytic functions

of $z \in [-1,1]$.*)**) Then the integral

$\int_{-2+i\eta}^{z-1+i\eta} I(v+1-i\eta,\ 1-i\eta)\varphi_1(v)dv$ is an analytic function of η.

Moreover in the vicinity of $\eta = 0$ we have:

$$\int_{z-1+i\eta}^{i\eta} I(v+1-i\eta,1-i\eta)\varphi_2(v)dv = g(i\eta)\left[\sum_{n=0}^{\infty} a_n(z)\eta^n + \eta^{-\frac{1}{2}-\nu}\sum_{n=0}^{\infty} b_n\eta^n\right] -$$

(4.35)

$$- f(i\eta)\left[\sum_{n=0}^{\infty} c_n(z)\eta^n + \eta^{-\frac{1}{2}+\nu}\sum_{n=0}^{\infty} d_n\eta^n\right],$$

where $a_n(z)$ and $c_n(z)$ depend on $F_k(z)$ and $G_k(z)$ and their derivatives at z, and b_n and d_n depend on $F_k(z)$ and $G_k(z)$ and their derivatives at $z = 1$.

Taking into account eqs. (4.30) and (4.35), and using Watson's lemma [3], we find that the integral in eq. (4.33) is of order $t^{-1/2}$ when $t \to \infty$. It is the terms with b_o and d_o in eq. (4.35) that contribute to the term of order $t^{-1/2}$.

It is also easy to see that the contributions to the velocity field and the vorticity field from the integral along γ_1 are damped out at least as fast as $t^{-1/2}$.

The calculations are carried through only for the integral along γ_1. But the same result is obtained for the integral along γ_3. In this case we introduce η, given by $\zeta = 1 + i\eta$, as a new variable, and proceed as above.

---

*) This is of course no necessary condition in order to prove Th 3 - Th 6. But with this assumption we can obtain the asymptotic series for $\int_{\gamma_2}$ in the case $r > \frac{1}{4}$, and for $\int_{\gamma_1}$, $\int_{\gamma_2}$ and $\int_{\gamma_3}$ in the case $0 < r < \frac{1}{4}$.

**) We say that a function is analytic in a region if it can be expanded in a power series.

Th 4. The integral along γ_2 is of order $t^{-3/2}$ when $t \to \infty$. The contribution to the velocity field from this integral is of order $t^{-1/2}$.

The contribution to the vorticity field is of order $t^{1/2}$.

Proof:

We have $\int\limits_{\gamma_2} = \int\limits_{z-i\infty}^{z} + \int\limits_{z}^{z-i\infty}$, where z is assumed to be an interior point of $[-1,1]$. Let us consider the integral $\int\limits_{z}^{z-i\infty}$.

We introduce η, given by $\zeta = z - i\eta$, as a new variable into this integral and obtain:

$$P(z,t) = -\frac{ik}{2\pi}\int_0^\infty e^{-ikzt-k\eta t}\frac{d\eta}{WD(z-i\eta)}\Big\{\varphi_2(i\eta)\int_{-1-z+i\eta}^{i\eta} I(v+z-i\eta,z-i\eta) \tag{4.36}$$

$$\times\ \varphi_1(v)dv + \varphi_1(i\eta)\int_{i\eta}^{1-z+i\eta} I(v+z-i\eta,z-i\eta)\varphi_2(v)dv\Big\},$$

where

$$\begin{cases} D(z-i\eta) = g(1-z+i\eta)f(-1-z+i\eta) - f(1-z+i\eta)g(-1-z+i\eta) \\ \varphi_1(i\eta) = g(-1-z+i\eta)f(i\eta) - f(-1-z+i\eta)g(i\eta) \\ \varphi_2(i\eta) = g(1-z+i\eta)f(i\eta) - f(1-z+i\eta)g(i\eta)\ . \end{cases} \tag{4.37}$$

In the vicinity of $\eta = 0$ we have that

$$\int_{-1-z+i\eta}^{i\eta} I(v+z-i\eta,z-i\eta)\varphi_1(v)dv = g(-1-z+i\eta)\Big[\sum_{n=0}^{\infty}a_{1n}\eta^n + \eta^{-\frac{1}{2}-\nu}\sum_{n=0}^{\infty}b_{1n}\eta^n\Big] -$$

$$- f(-1-z+i\eta)\Big[\sum_{n=0}^{\infty}c_{1n}\eta^n + \eta^{-\frac{1}{2}+\nu}\sum_{n=0}^{\infty}d_{1n}\eta^n\Big] \tag{4.38}$$

$$\int_{i\eta}^{1-z+i\eta} I(v+z-i\eta,z-i\eta)\varphi_2(v)dv = g(1-z+i\eta)\Big[\sum_{n=0}^{\infty}a_{2n}\eta^n + \eta^{-\frac{1}{2}-\nu}\sum_{n=0}^{\infty}b_{2n}\eta^n\Big] -$$

$$- f(1-z+i\eta)\Big[\sum_{n=0}^{\infty}c_{2n}\eta^n + \eta^{-\frac{1}{2}+\nu}\sum_{n=0}^{\infty}d_{2n}\eta^n\Big],$$

where

$$(4.39)\quad \begin{cases} a_{10} = \text{Pf.} \displaystyle\int_{-1-z}^{0} \left(\frac{F_k(v+z)}{ikv} - \frac{G_k(v+z)}{k^2v^2}\right) f(v)dv\,, \\ c_{10} = \text{Pf.} \displaystyle\int_{-1-z}^{0} \left(\frac{F_k(v+z)}{ikv} - \frac{G_k(v+z)}{k^2v^2}\right) g(v)dv\,, \\ a_{20} = \text{Pf.} \displaystyle\int_{0}^{1-z} \left(\frac{F_k(v+z)}{ikv} - \frac{G_k(v+z)}{k^2v^2}\right) f(v)dv\,, \\ c_{20} = \text{Pf.} \displaystyle\int_{0}^{1-z} \left(\frac{F_k(v+z)}{ikv} - \frac{G_k(v+z)}{k^2v^2}\right) g(v)dv\,, \\ \ldots\ldots \end{cases}$$

where Pf. in front of the integral sign indicates the finite part of the integral, (see [4]).

Moreover

$$(4.40)\quad b_{1n} = - b_{2n}\,,\quad d_{1n} = - d_{2n} \quad (n = 0,\ 1,\ \ldots)\,.$$

Taking into account eqs. (4.40), we find that

$$(4.41)\quad \begin{cases} \varphi_2(i\eta)\left\{g(-1-z+i\eta)\eta^{-\frac{1}{2}-\nu}\displaystyle\sum_0^\infty b_{1n}\eta^n - f(-1-z+i\eta)\eta^{-\frac{1}{2}+\nu}\sum_0^\infty d_{1n}\eta^n\right\} + \\ \varphi_1(i\eta)\left\{g(1-z+i\eta)\eta^{-\frac{1}{2}-\nu}\displaystyle\sum_0^\infty b_{2n}\eta^n - f(1-z+i\eta)\eta^{-\frac{1}{2}+\nu}\sum_0^\infty d_{2n}\eta^n\right\} \end{cases}$$

is an analytic function of η in the vicinity of $\eta = 0$. Therefore this term in the integral in eq. (4.36) will not contribute to the asymptotic series for $\int_{\gamma_2}$, and is of no interest. The term that contributes to the asymptotic series for $\int_{\gamma_2}$, is the following one:

$$(4.42)\quad \begin{cases} \varphi_2(i\eta)\left\{g(-1-z+i\eta)\displaystyle\sum_0^\infty a_{1n}\eta^n - f(-1-z+i\eta)\sum_0^\infty c_{1n}\eta^n\right\} + \\ \varphi_1(i\eta)\left\{g(1-z+i\eta)\displaystyle\sum_0^\infty a_{2n}\eta^n - f(1-z+i\eta)\sum_0^\infty c_{2n}\eta^n\right\}\,. \end{cases}$$

We introduce the expression (4.42) into the integral in eq. (4.36) and find the asymptotic behavior for large values of t, using Watson's lemma. The first term in the asymptotic series is found to be:

$$
\begin{aligned}
&- \frac{ie^{-ikzt}}{2\pi WD(z)} \lim_{\eta\to 0}\Big\{[g(-1-z)a_{10}-f(-1-z)c_{10}] \\
&\times[g(1-z)\frac{f(i\eta)}{(k\eta)^{1/2-\nu}}\,\Gamma(\tfrac{3}{2}-\nu)t^{-3/2+\nu}-f(1-z)\frac{g(i\eta)}{(k\eta)^{1/2+\nu}}\,\Gamma(\tfrac{3}{2}+\nu)t^{-3/2-\nu}] \\
&+ [g(1-z)a_{20}-f(1-z)c_{20}][g(-1-z)\frac{f(i\eta)}{(k\eta)^{1/2-\nu}}\,\Gamma(\tfrac{3}{2}-\nu)t^{-3/2+\nu} - \\
&- f(-1-z)\frac{g(i\eta)}{(k\eta)^{1/2+\nu}}\,\Gamma(\tfrac{3}{2}+\nu)t^{-3/2-\nu}]\Big\} ,
\end{aligned}
\tag{4.43}
$$

where $\Gamma(x)$ is the Gamma function.

Expression (4.43) is only a part of the first term in the asymptotic series for $\int_{\gamma_2}$. The integral $\int_{z-i\infty}^{z}$ contributes with an analogous term. This term is easily found, but we must remember that $f(i\eta)$ and $g(i\eta)$ have branch-point at $\eta = 0$. Th 4 follows from expression (4.43), remembering that ν is purely imaginary in this case.

V. <u>The case $0 < r < \frac{1}{4}$, i.e. $0 < \nu < \frac{1}{2}$.</u>

In section III it is pointed out that there is no eigenvalue in this case. The stream function is therefore given by:

$$\psi(z,t) = - \int_{\gamma_1} - \int_{\gamma_2} - \int_{\gamma_3} . \tag{5.1}$$

Th 5. The integrals $\int_{\gamma_1}$ and $\int_{\gamma_3}$ are of order $t^{-1/2-\nu}$ when <u>$t \to \infty$. The contributions to the velocity field and vorticity field from these integrals are also of order $t^{-1/2-\nu}$.</u>

Proof:

We examine the integral in eq. (4.29). $D(1 - i\eta)$ is an analytic function of η, except at $\eta = 0$. But $D(1 - i\eta)/f(i\eta)$ tends to a finite value $\neq 0$ when $\eta \to 0$. Taking into account eqs. (4.30) and (4.35), we find that the first term in the asymptotic series for $\int_1^{1-i\infty}$ is given by:

$$(5.2) \qquad \frac{ie^{-ikt}}{2\pi W} \frac{\varphi_1(z-1)}{g(-2)} \frac{\Gamma(\frac{1}{2}+\nu)}{k^{-1/2+\nu}} \left\{ b_o \lim_{\eta \to 0} \frac{g(i\eta)}{f(i\eta)\eta^{2\nu}} - d_o \right\} t^{-1/2-\nu} .$$

The integral $\int_{1-i\infty}^{1}$ contributes with an analogous term to the asymptotic series for $\int_{\gamma_1}$. A corresponding result is obtained for $\int_{\gamma_3}$, and so Th 5 is proved.

If $G_k(1) = 0$ then $b_o = d_o = 0$, and the leading term in the asymptotic series will be of higher order in t.

<u>Th 6. The integral $\int_{\gamma_2}$ is of order $t^{-3/2+\nu}$ when $t \to \infty$. The contribution to the velocity field from this integral is of order $t^{-1/2+\nu}$, and the contribution to the vorticity field of order $t^{1/2+\nu}$.</u>

Proof:

The results follow from expression (4.43), remembering that ν is real and positive in this case.

Conclusion.

In addition to a discrete spectrum of eigenvalues there exists a continuous spectrum as well, which has to be taken into account. This is true also for the case $r > \frac{1}{4}$. (For $0 < r < \frac{1}{4}$ there is no discrete spectrum of eigenvalues.)

For $r > 0$ we have found that the part of the velocity

field which is due to the continuous spectrum, tends to zero. Therefore the energi associated with the perturbation is either finite (for $r > \frac{1}{4}$) or will tend to zero (for $0 < r < \frac{1}{4}$) when $t \to \infty$.

For $r < 0$ preliminary investigations have shown that this part of the velocity is of order $t^{-1/2+\nu}$, and will therefore become infinite with t.

Concerning the vorticity we have:

for $r > \frac{1}{4}$ $\quad \nabla^2\psi = O(t^{1/2})$ $\quad$ when $t \to \infty$,

for $r = \frac{1}{4}$ $\quad \nabla^2\psi = O(t^{1/2}\ln t)$ $\quad$ " ",

for $r < \frac{1}{4}$ $\quad \nabla^2\psi = O(t^{1/2+\nu})$ $\quad$ " ".

The vorticity becomes infinite with t for all values of r.

Acknowledgments.

The author is grateful to Professor J. Naze Tjötta for constructive discussions during this work.

The financial support given by the Royal Norwegian Council for Scientific and Industrial Research is acknowledged.

Appendix.

We will show that it is not necessary to assume that $G_k(z)$ has a first order derivative to obtain :

$$\text{(A1)}\quad \begin{cases} a_n \to -\dfrac{G_k(1)}{ikrW} & \text{when} \quad n \to \infty \;, \\[2ex] a_n \cong -\dfrac{G_k(-1)}{ikr\varphi_2'(-1-\zeta_n)} & \text{when} \quad n \to -\infty \;. \end{cases}$$

The following proof is suggested by Dr. J.N. Tjøtta.

We have:

$$\text{(A2)}\qquad a_n = \frac{\int_{-1}^{+1} \left\{\dfrac{F_k(u)}{(u-\zeta_n)} + \dfrac{G_k(u)}{ik(u-\zeta_n)^2}\right\}\varphi_2(u-\zeta_n)du}{W^2 - \varphi_2'^2(-1-\zeta_n)} \;.$$

Let us write:

$$\text{(A3)}\qquad \varphi_2(z-\zeta_n) = (\zeta_n - 1)^{1/2}(\zeta_n - z)^{1/2}\varphi_{21}(z-\zeta_n) \;,$$

where $|\varphi_{21}(z-\zeta_n)| < M$ for $z \in [-1,1]$ and all n. Let us assume that $F_k(z)$ and $G_k(z)$ are bounded for $z \in [-1,1]$. Moreover $G_k(z)$ is assumed to be continuous at $z = \pm 1$.

Let ε $(0 < \varepsilon < 1)$ be given, and let us write:

$$\text{(A4)}\qquad \int_{-1}^{+1} \frac{G_k(u)}{(u-\zeta_n)^2}\varphi_2(u-\zeta_n)du = \int_{-1}^{-1+\varepsilon} + \int_{-1+\varepsilon}^{1-\varepsilon} + \int_{1-\varepsilon}^{1}$$

Due to eq. (A3) we have that

$$\text{(A5)}\qquad \left|\int_{-1+\varepsilon}^{1-\varepsilon} \frac{G_k(u)}{(u-\zeta_n)^2}\varphi_2(u-\zeta_n)du\right| < \frac{M_1|\zeta_n - 1|^{1/2}}{\varepsilon^{1/2}} \quad \text{for all} \quad n.$$

M is a constant independent of ε and n.

We can find $N(\varepsilon) > 0$ so that $|\zeta_n - 1| < \varepsilon^2$ for all $n > N(\varepsilon)$.

Then

$$\text{(A6)} \qquad \left|\int_{-1+\varepsilon}^{1-\varepsilon}(\dots)\right| < M_1\varepsilon^{1/2} \quad \text{for} \quad n > N(\varepsilon).$$

Let δ be given. Since $G_k(z)$ is continuous at $+ z = \pm 1$, we can choose $\varepsilon(\delta)$ so that

$$\text{(A7)} \qquad |G_k(z) - G_k(1)| < \delta \quad \text{for} \quad |z - 1| < \varepsilon(\delta) .$$

We choose $\varepsilon < \delta^2$. The relation (A7) is still valid. From eq. (A6) we obtain:

$$\text{(A8)} \qquad \left|\int_{-1+\varepsilon}^{1-\varepsilon}(\dots)\right| < M_1\delta \quad \text{for} \quad n > N_1(\delta).$$

Also

$$\text{(A9)} \qquad \left|\int_{1-\varepsilon}^{1}\frac{G_k(u)}{(u-\zeta_n)^2}\varphi_2(u - \zeta_n)du - G_k(1)\int_{1-\varepsilon}^{1}\frac{\varphi_2(u-\zeta_n)}{(u-\zeta_n)^2}du\right| < M_2\delta \ .$$

Moreover

$$\text{(A10)} \qquad \int_{1-\varepsilon}^{1}\frac{\varphi_2(u-\zeta_n)}{(u-\zeta_n)^2}du = -\frac{1}{r}\int_{1-\varepsilon}^{1}\{\varphi_2''(u - \zeta_n) - k^2\varphi_2(u - \zeta_n)\}du =$$

$$- \frac{1}{r}\{W - \varphi_2'(1 - \zeta_n - \varepsilon)\} + \frac{k^2}{r}\int_{1-\varepsilon}^{1}\varphi_2(u - \zeta_n)du ,$$

where we have that

$$\text{(A11)} \qquad \begin{cases} \left|\int_{1-\varepsilon}^{1}\varphi_2(u - \zeta_n)du\right| < M_3\delta , \\ |\varphi_2'(1 - \zeta_n - \varepsilon)| < M_4\delta \quad \text{for} \quad n > N_1(\delta). \end{cases}$$

Also

$$\text{(A12)} \qquad \left| \int_{-1}^{-1+\varepsilon} \frac{G_k(u)}{(u-\zeta_n)^2} \varphi_2(u - \zeta_n)du \right| < M_5\delta \ ,$$

and

$$\text{(A13)} \qquad \left| \int_{-1}^{+1} \frac{F_k(u)}{(u - \zeta_n)} \varphi_2(u - \zeta_n)du \right| < M_6\delta \quad \text{for} \quad n > N_1(\delta) \ .$$

Taking into account eqs. (A4) - (A13), we obtain from eq. (A2) that

$$a_n \to - \frac{G_k(1)}{ikrW} \quad \text{when} \quad n \to \infty.$$

In order to investigate a_n when $n \to -\infty$, we study the expression

$$\text{(A14)} \qquad |\zeta_n+1|^{1/2} \int_{-1}^{+1} \frac{G_k(u)}{(u - \zeta_n)^2} \varphi_2(u-\zeta_n)du = |\zeta_n+1|^{1/2} \left(\int_{-1}^{-1+\varepsilon} + \int_{-1+\varepsilon}^{1-\varepsilon} + \int_{1-\varepsilon}^{1} \right).$$

We find that

$$\text{(A15)} \qquad a_n \cong - \frac{G_k(-1)}{ikr\varphi_2'(-1-\zeta_n)} \quad \text{when} \quad n \to -\infty.$$

The proof is analogous to that above.

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