Department
of
APPLIED MATHEMATICS

On Energy Conversion in a Sigma coordinate Oceaan Model

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Report no. 129 August 1999

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ISSN 0084-778x

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Abstract

Energy diagnostics are useful for understanding the transfer of energy through instabilities and between different scales. In this note the conservation equations for kinetic and potential energy, divided into suitable mean and eddy quantities, for a sigma coordinate ocean model are set up. By identifying the transfer terms responsible for the conservative conversions between the different energies, an energy flow diagram is suggested. The motivation for this is twofold. Firstly, the average operator required for dividing the quantities of the flow into mean and eddy parts is in general not well defined in Cartesian coordinates when the upper and lower boundaries are not at fixed vertical levels. This is overcome by introducing the "terrain-following" sigma as the vertical coordinate. Secondly, and most important, many of todays numerical ocean models have this as the vertical coordinate.

1 Introduction

The idea of understanding the energetics of geophysical flows by the separation of kinetic and potential energy into mean and eddy components goes back to Lorenz (1955). Mean quantities are defined from a suitable temporal or spatial average, and eddy quantities are the deviation from these. Orlanski and Cox (1973), Qiu *et al.* (1988), and Wood and Ikeda (1994), among others, have set up and utilized these quantities for the oceanic primitive equations in Cartesian coordinates with the model domain chosen to be a cyclic channel and the free surface replaced by a rigid lid. Mean flow quantities are then defined as along-channel averages. In the two latter papers, there is no bottom topography, while Orlanski and Cox (1973) account for cross-channel variations. Thus there are no temporal or spatial variations in the upper and lower boundaries of concern to the averaging process. Røed (1997) describes an energy diagnostics scheme for a reduced gravity, nonisopycnic ocean model. Bleck (1985), hereafter referred to as B85, provides a general framework. He finds energy conservation and conversion laws for the primitive equations for a general vertical coordinate, and no assumptions are made regarding the upper and lower boundaries. Neither is the Boussinesq approximation applied. The content of this note, where the vertical is resolved by the so-called sigma (σ) coordinate, may to an extent be said to be a special case of B85.

This note is organized as follows: in the rest of the introduction the well known energy equations in Cartesian coordinates, and the motivation for rephrasing them in sigma coordinates, are presented. In section 2 the sigma coordinate transformation is defined, and the equations governing the flow in sigma coordinates are set up. Section 3 consists of the kinetic, potential, and total energy equations in these coordinates, and their partition into mean an eddy quantities. In section 4, the associated energy flow is discussed, and compared with that of B85. A summary and concluding remarks are found in section 5.

The model ocean is described by the three-dimensional primitive equations. These are the horizontal equations of motion, with the Boussinesq approximation applied,

$$\frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{1}$$

$$\frac{\partial v}{\partial t} + \mathbf{u} \cdot \nabla v + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \tag{2}$$

hydrostatic balance

$$0 = -\frac{\partial p}{\partial z} - \rho g, \tag{3}$$

the continuity equation

$$\nabla \cdot \mathbf{u} = 0, \tag{4}$$

and the conservation equation for density

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0. \tag{5}$$

In the above Cartesian representation, ∇ is the three dimensional gradient operator, $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ is the three dimensional velocity field with x and y as the horizontal coordinates, respectively, and z as vertical coordinate, f is the Coriolis parameter, p is pressure, and ρ_0 is a constant reference density. The pressure at the free surface, $z = \eta(x, y, t)$, is set to zero.

Diffusion and dissipation have been neglected. There is no external forcing present. The inclusion of these processes has no influence on the reversible energy conversions which are the scope of this text.

Adding equations (1) and (2) multiplied by u and v, respectively, and using hydrostatic balance (3) and the continuity equation (4), gives the following conservation equation for the kinetic energy density, $k = \rho_0(u^2 + v^2)/2$,

$$\frac{\partial k}{\partial t} + \nabla \cdot (\mathbf{u}(k + p)) = -\rho w g. \tag{6}$$

In a hydrostatic fluid, only the horizontal velocities come into consideration, and k is proportional to ρ_0, not ρ, because of the Boussinesq approximation.

Potential energy density may be defined as $\phi = \rho z g$. The material derivative is $d\phi/dt = \rho w g$. Taking into account (4), this may be written on the flux form

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\mathbf{u}\phi) = \rho w g. \tag{7}$$

Thus, the equation governing the evolution of the total energy density is

$$\frac{\partial}{\partial t}(k + \phi) + \nabla \cdot (\mathbf{u}(k + \phi + p)) = 0. \tag{8}$$

The fluid is restricted by a rigid bottom at $z = -H(x, y)$, and the free surface $z = \eta(x, y, t)$. Let V denote the volume restricted by these surfaces, and some fixed lateral boundary S. The integral of equation (8) over V gives

$$\frac{\partial}{\partial t}\int_V (k + \phi) dV + \int_S (k + \phi + p)\mathbf{u} \cdot d\mathbf{S} = 0. \tag{9}$$

Thus, the total energy of the fluid contained in V may only be changed by energy fluxes through S. The upper and lower boundaries being material surfaces, give no contribution. For details concerning the above, see Gill (1982).

If there is no net flux of energy through S, for example if the lateral boundaries are either rigid or cyclic, then total energy is conserved

$$\frac{\partial}{\partial t}\int_V (k + \phi) dV = 0. \tag{10}$$

Herein, the focus will be on energy conversion processes internal to V, and the possible net contributions from flux divergence terms are not discussed.

Let $\overline{\psi}$ be a suitable (horizontal or temporal) averaging operator, where ψ may be any quantity of the flow. Then

$$\psi = \overline{\psi} + \psi'. \tag{11}$$

In the above $\psi' = \psi - \overline{\psi}$, $\overline{\psi'} = 0$, represents the residual, i.e., eddy or turbulent, part of ψ. Then, following for example Wood and Ikeda (1994), the conservation equations for kinetic and potential energy may be decomposed into

$$\frac{\partial k_m}{\partial t} + \nabla \cdot (\overline{\mathbf{u}}(k_m + \overline{p})) = c(\overline{\phi}, k_m) + c(k_e, k_m) \quad (12)$$

$$\frac{\partial k_e}{\partial t} + \nabla \cdot (\overline{\mathbf{u}} k_e + \overline{\mathbf{u}'(k' + p')}) = c(\overline{\phi}, k_e) + c(k_m, k_e) \quad (13)$$

$$\frac{\partial \overline{\phi}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}}\overline{\phi} + \overline{\mathbf{u}'\phi'}) = c(k_m, \overline{\phi}) + c(k_e, \overline{\phi}), \quad (14)$$

where $k_m = \rho_0(\overline{u}^2 + \overline{v}^2)/2$ and $k_e = \rho_0(\overline{u'^2} + \overline{v'^2})/2$, are the mean and eddy flow contributions to the average kinetic energy, i.e., $\overline{k} = k_m + k_e$. For $\overline{\phi} = \overline{\rho} z g$, there is no contribution from the eddy flow. The right hand sides of the above equations represent the possible conservative conversions between the different kinds of energy. The energy transfer terms, $c(\ldots)$, are such that $c(A, B) > 0$ represents the conversion of energy of type A into type B, and $c(B, A) = -c(A, B)$. The definition of individual transfer terms is not unique. The conservative system of energy equations (12)-(14) provides only two linearly independent equations to determine the three possible energy transfers. Additional physical arguments are required. The explicit expressions are then, still following Wood and Ikeda (1994), found to be

$$c(\overline{\phi}, k_m) = -\overline{\rho}\,\overline{w} g \quad (15)$$

$$c(\overline{\phi}, k_e) = -\overline{\rho' w'} g \quad (16)$$

$$c(k_m, k_e) = \rho_0 \overline{\mathbf{u}}_H \cdot (\nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'_H})). \quad (17)$$

Subscript H denotes the horizontal component of the vector quantity in question. Positive $c(\overline{\phi}, k_m)$ represents the average buoyant production of mean kinetic energy, and negative the transfer from k_m to $\overline{\phi}$. The equivalent for k_e is $c(\overline{\phi}, k_e)$. Baroclinic instabilities are characterized by positive $c(\overline{\phi}, k_e)$. The term $c(k_m, k_e)$ is recognized when positive (negative), as the production (loss) of eddy kinetic energy from (to) the mean flow through nonlinear advective interaction. Such shear production of eddy kinetic energy at the cost of mean kinetic energy constitutes barotropic instability. The commonly used term "shear production" is due to the alternative formulation

$$c(k_m, k_e) = \rho_0 \nabla \cdot (\overline{\mathbf{u}'(\mathbf{u}'_H \cdot \overline{\mathbf{u}}_H)}) - \rho_0 \overline{\mathbf{u}'_H \cdot (\mathbf{u}' \cdot \nabla)} \overline{\mathbf{u}}_H, \quad (18)$$

which shows that the net contribution to the internal barotropic conversion of energy is given by the interaction of the Reynolds stresses with the mean velocity shear. Both Orlanski and Cox (1973), and Qiu *et al.* (1988) in their numerical studies of predominantly baroclinic instabilities of ocean currents find that the integrated effect of $c(k_m, k_e)$ may be negative. That is, kinetic energy is fed back from the eddy field to the mean flow at certain stages of the instability. This is also found in the linear analysis of Shi and Røed (1999) of frontal instabilities in a two layer, primitive equation ocean model. In non-eddy-resolving simulations, such a process is not possible as energy is continuously dissipated from the flow through eddy viscosity for most parameterizations. Thus energy diagnostics as sketched above may give insight both into instability processes in eddy-resolving experiments, and a cue to how sub-grid scale effects should be parameterized in non-resolving experiments.

For the above Cartesian description to be correct, the averaging operator must be applicable to any position in the fluid at any time. A horizontal average is not applicable, at least not in a

mathematical sense, when either the upper or lower boundary is not at a fixed z-level. Neither is a temporal average applicable in the presence of a free surface, that is an upper boundary that vary both in space and time. These discrepancies may for many applications be of academic interest only. The upper boundary in most oceanic applications can be approximated by $z = 0$ when averaging, as long as the potential energy of the surface elevation is accounted for. If surface gravity waves may be neglected in the problem under consideration, the surface elevation present will correspond to the pressure anomalies at that boundary when replaced by a rigid lid. In both cases, a temporal average would indeed be applicable. If in addition, the lower boundary is chosen to be horizontal or the fluid assumed to be motionless below a certain z-level not intersecting the bottom topography, any horizontal average would be applicable as well.

By transforming the vertical coordinate to σ-levels, see next section, the potential problems of the averaging operator are overcome. In these new coordinates the fluid is contained between fixed upper and lower levels of σ at all times. Another obvious and more important reason for rephrasing the problem in these coordinates, is that this is the chosen coordinate representation for many of the numerical ocean models in use today, see for example Blumberg and Mellor (1987), and Berntsen *et al.* (1996).

2 The sigma coordinate model

The transformation from Cartesian coordinates (x, y, z, t) to the "terrain-following" sigma coordinate system $(\tilde{x}, \tilde{y}, \sigma, \tilde{t})$, originally due to Phillips (1957), is given by

$$\tilde{x} = x, \ \tilde{y} = y, \ \sigma = \frac{z - \eta}{D}, \ \tilde{t} = t. \tag{19}$$

The total depth of a fluid column is $D = H + \eta$, thus σ ranges from $\sigma = 0$ at the free surface $z = \eta(x, y, t)$, to $\sigma = -1$ at the bottom $z = -H(x, y)$. The unit directional vectors are those of the Cartesian system, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. The differential operators are transformed as follows:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}} - (1+\sigma)\frac{\partial D}{\partial \tilde{t}}\frac{1}{D}\frac{\partial}{\partial \sigma} \tag{20}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}} - \frac{1}{D}\frac{\partial \tilde{z}}{\partial \tilde{x}}\frac{\partial}{\partial \sigma} \tag{21}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{y}} - \frac{1}{D}\frac{\partial \tilde{z}}{\partial \tilde{y}}\frac{\partial}{\partial \sigma} \tag{22}$$

$$\frac{\partial}{\partial z} = \frac{1}{D}\frac{\partial}{\partial \sigma} \tag{23}$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla = \frac{\partial}{\partial \tilde{t}} + \tilde{\mathbf{u}}\cdot\tilde{\nabla} \equiv \frac{d}{d\tilde{t}}, \tag{24}$$

$\tilde{z}$ is the Cartesian vertical coordinate in sigma coordinates,

$$\tilde{z} = z(\tilde{x}, \tilde{y}, \sigma, \tilde{t}) = \eta(\tilde{x}, \tilde{y}, \tilde{t}) + \sigma D(\tilde{x}, \tilde{y}, \tilde{t}). \tag{25}$$

The $\tilde{\nabla}$ operator is defined as

$$\tilde{\nabla} = \frac{\partial}{\partial \tilde{x}}\mathbf{i} + \frac{\partial}{\partial \tilde{y}}\mathbf{j} + \frac{\partial}{\partial \sigma}\mathbf{k}, \tag{26}$$

and the "velocity" vector $\tilde{\mathbf{u}}$ as

$$\tilde{\mathbf{u}} = \tilde{u}\mathbf{i} + \tilde{v}\mathbf{j} + \omega\mathbf{k}, \tag{27}$$

where $\tilde{\mathbf{u}}_H = \mathbf{u}_H(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$, and ω is the material derivative of σ, ωD being the vertical velocity relative σ-levels, vanishing at the upper and lower boundaries, $\omega(0) = \omega(-1) = 0$. It may seem curious that the sigma coordinate gradient operator and velocity vector are defined such that their vertical component is not of the same dimension as the horizontal ones. This is done so that advective and flux terms may be put on a compact form in the upcoming energy analysis. Then $\tilde{\nabla}$ and $\tilde{\mathbf{u}}$ are both being present as parts of a dot product, the result being dimensionally consistent.

The relation between ω and w, the vertical velocity relative z-levels, in sigma coordinates is

$$\omega D = \tilde{w} - (\frac{\partial}{\partial \tilde{t}} + \tilde{\mathbf{u}} \cdot \tilde{\nabla})\eta - \sigma(\frac{\partial}{\partial \tilde{t}} + \tilde{\mathbf{u}} \cdot \tilde{\nabla})D \tag{28}$$

or

$$\tilde{w} \equiv \frac{d\tilde{z}}{d\tilde{t}} = (1 + \sigma)\frac{\partial D}{\partial \tilde{t}} + \tilde{\mathbf{u}} \cdot \tilde{\nabla}\tilde{z}, \tag{29}$$

where $\tilde{w} = w(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$.

Using the above relations, the continuity equation may be rewritten

$$0 = D\nabla \cdot \mathbf{u} = \frac{\partial D}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\mathbf{u}}D). \tag{30}$$

The primitive equations (1)-(5) transformed to the new coordinate system then, omitting tildes, on flux form readily follow:

$$\frac{\partial uD}{\partial t} + \nabla \cdot (\mathbf{u}uD) - fvD = -\frac{1}{\rho_0}(D\frac{\partial}{\partial x} - \frac{\partial z}{\partial x}\frac{\partial}{\partial \sigma})p \tag{31}$$

$$\frac{\partial vD}{\partial t} + \nabla \cdot (\mathbf{u}vD) + fuD = -\frac{1}{\rho_0}(D\frac{\partial}{\partial y} - \frac{\partial z}{\partial y}\frac{\partial}{\partial \sigma})p \tag{32}$$

$$0 = -\frac{\partial p}{\partial \sigma} - \rho D g \tag{33}$$

$$\frac{\partial D}{\partial t} + \nabla \cdot (\mathbf{u}D) = 0 \tag{34}$$

$$\frac{\partial \rho D}{\partial t} + \nabla \cdot (\mathbf{u}\rho D) = 0. \tag{35}$$

In (x, y, σ)-space, ρD is the mass in a unit volume, the σ-direction being dimensionless and of length D in physical space. In general, if ψ is a quantity per unit volume in Cartesian coordinates, then the corresponding density in sigma coordinates is $\Psi = \psi D$.

3 Energy partition and equations

The kinetic energy equation for this regime is found by multiplying equation (31) by u and (32) by v and adding them together taking hydrostatic balance (33) and the continuity equation (34) into account. Then

$$\frac{\partial K}{\partial t} + \nabla \cdot (\mathbf{u}(K + pD)) - \frac{\partial}{\partial \sigma}((1+\sigma)p\nabla \cdot (\mathbf{u}D)) = -\rho D w g, \tag{36}$$

$K = kD$, and $k = \rho_0(u^2 + v^2)/2$ as before. Note that the last term on the left hand side vanishes when integrating over a water column, thus having no net influence on the energetics of the flow. This term may also be understood as a flux divergence where $D^{-1}\partial z/\partial t\mathbf{k} = -D^{-1}(1+\sigma)\nabla \cdot (\mathbf{u}D)\mathbf{k}$ is the "velocity" causing the flux of pressure pD. The vertical velocity occurring in the second term is the vertical velocity relative a fixed z-level given by equation (29). Although the conversion terms and most of the equations presented herein are given on local form, it should be stressed that a description of the energetics from the conservative transfer terms $c(.,.)$ alone, is formally only valid when integrated over the volume in question. As stated earlier, the possible contribution from lateral fluxes into the domain is not considered.

The new definition of potential energy density is $\Phi = \phi D$, or equivalently $\Phi = \rho D z g$. Using the material derivative of ϕ and the continuity equation (34), potential energy is governed by

$$\frac{\partial \Phi}{\partial t} + \nabla \cdot (\mathbf{u}\Phi) = \rho D w g. \tag{37}$$

Thus total energy is conserved according to

$$\frac{\partial}{\partial t}(K + \Phi) + \nabla \cdot (\mathbf{u}(K + \Phi + pD)) - \frac{\partial}{\partial \sigma}((1+\sigma)p\nabla \cdot (\mathbf{u}D)) = 0. \tag{38}$$

Taking the integral over the geometric (in sigma coordinates) volume V, assuming no net fluxes through the lateral boundaries, states that the total energy within V is constant,

$$\frac{\partial}{\partial t}\int_V (K + \Phi) dV = 0. \tag{39}$$

Alternatively, the above energy conservation equations may be found by applying the transformations (20)-(24), and the continuity equation (34) to the Cartesian energy equations (6) and (7) directly.

Let $\overline{\phi}$ be the conventional averaging operator introduced previously. Averages are now to be evaluated at levels of constant σ, not z. This requires the introduction of an additional average. This depth weighted average is defined as

$$\hat{\psi} = \frac{\overline{\psi D}}{\overline{D}}, \tag{40}$$

thus the quantity ψ may also be written,

$$\psi = \hat{\psi} + \psi^*, \tag{41}$$

where the residual in the above sense, $\psi^* = \psi - \hat{\psi}$, obeys $\hat{\psi^*} = 0$, or equivalently $\overline{\psi^* D} = 0$.

The average of the equations (36) and (37), governing kinetic and potential energy gives the following relations:

$$\frac{\partial \overline{K}}{\partial t} + \nabla \cdot (\hat{\mathbf{u}}(\overline{K} + \overline{p}\overline{D}) + \overline{\mathbf{u}^* K} + \overline{\mathbf{u}p'D}) - \frac{\partial}{\partial \sigma}((1+\sigma)\overline{p\nabla \cdot (\mathbf{u}D)}) = c(\overline{\Phi}, \overline{K}) \tag{42}$$

$$\frac{\partial \overline{\Phi}}{\partial t} + \nabla \cdot (\hat{\mathbf{u}}\overline{\Phi} + \overline{\mathbf{u}^* \Phi}) = c(\overline{K}, \overline{\Phi}). \tag{43}$$

The conversion term giving the conservative transfer of energy between average kinetic and average potential energy is found to be

$$c(\overline{\Phi}, \overline{K}) = -\overline{\rho D w} g = -\hat{\rho}\overline{D}\overline{w}g - \overline{\rho D w'} g, \tag{44}$$

This is, as one would expect from the fact that the definitions of kinetic and potential energy densities have only been modified by the factor D, equivalent to the transfer found for the Cartesian representation.

The average kinetic and potential energy densities may be decomposed into contributions from the mean and eddy flow:

$$\overline{K} = \frac{1}{2}\rho_0 \overline{(u^2 + v^2)D} = K_\mathrm{m} + K_\mathrm{e} \tag{45}$$

$$K_\mathrm{m} = \frac{1}{2}\rho_0 (\hat{u}^2 + \hat{v}^2)\overline{D} \tag{46}$$

$$K_\mathrm{e} = \frac{1}{2}\rho_0 \overline{(u^{*2} + v^{*2})D} \tag{47}$$

$$\overline{\Phi} = \overline{\rho D z} g = \Phi_\mathrm{m} + \Phi_\mathrm{e} \tag{48}$$

$$\Phi_\mathrm{m} = \hat{\rho}\overline{D}\overline{z}g \tag{49}$$

$$\Phi_\mathrm{e} = \overline{\rho D z'} g \tag{50}$$

As opposed to the Cartesian representation, the eddy fields contribute to the average potential energy.

The sum of the averages of the momentum equations (31) and (32),

$$\frac{\partial \hat{u}\overline{D}}{\partial t} + \nabla \cdot (\hat{\mathbf{u}}\hat{u}\overline{D}) + \nabla \cdot (\overline{\mathbf{u}^* u^* D}) = -\frac{1}{\rho_0}\overline{(D\frac{\partial}{\partial x} - \frac{\partial z}{\partial x}\frac{\partial}{\partial \sigma})p} \tag{51}$$

$$\frac{\partial \hat{v}\overline{D}}{\partial t} + \nabla \cdot (\hat{\mathbf{u}}\hat{v}\overline{D}) + \nabla \cdot (\overline{\mathbf{u}^* v^* D}) = -\frac{1}{\rho_0}\overline{(D\frac{\partial}{\partial y} - \frac{\partial z}{\partial y}\frac{\partial}{\partial \sigma})p}, \tag{52}$$

multiplied by $\rho_0 \hat{u}$ and $\rho_0 \hat{v}$, respectively, when the averaged continuity equation,

$$\frac{\partial \overline{D}}{\partial t} + \nabla \cdot (\hat{\mathbf{u}}\overline{D}) = 0, \tag{53}$$

which states that the mean volume is conserved in the mean flow, multiplied by $\rho_0(\hat{u}^2 + \hat{v}^2)/2$ is subtracted, gives an evolution equation for the mean kinetic energy,

$$\frac{\partial K_m}{\partial t} + \nabla \cdot (\hat{\mathbf{u}}(K_m + \overline{p}\overline{D})) - \frac{\partial}{\partial \sigma}((1+\sigma)\overline{p}\nabla \cdot (\hat{\mathbf{u}}\overline{D}))$$
$$= -\hat{\rho}\overline{D}\overline{w}g + \hat{\rho}\overline{D}(\overline{\mathbf{u}_H^* \cdot \nabla_H z})g - \hat{\mathbf{u}}_H \cdot (\overline{D'\nabla_H p'} + (\overline{\rho D \nabla_H z'})g) - \rho_0 \hat{\mathbf{u}}_H \cdot (\nabla \cdot (\overline{\mathbf{u}^* \mathbf{u}_H^* D}))$$
$$= c(\Phi_m, K_m) + c(\Phi_e, K_m) + c(K_e, K_m). \tag{54}$$

The term mean/eddy kinetic (potential) energy is to be understood as the contribution to $\overline{K}$ ($\overline{\Phi}$) from the mean/eddy flow.

The difference between the averaged kinetic energy equation (42) and the above equation for the mean kinetic energy, provides an equation for the evolution of K_e,

$$\frac{\partial K_e}{\partial t} + \nabla \cdot (\hat{\mathbf{u}}K_e + \overline{\mathbf{u}^* K^*} + \overline{\mathbf{u}p'D}) - \frac{\partial}{\partial \sigma}((1+\sigma)\overline{p'\nabla \cdot (\mathbf{u}D)})$$
$$= -\overline{\rho D w'}g - \hat{\rho}\overline{D}(\overline{\mathbf{u}_H^* \cdot \nabla_H z})g + \hat{\mathbf{u}}_H \cdot (\overline{D'\nabla_H p'} + (\overline{\rho D \nabla_H z'})g) + \rho_0 \hat{\mathbf{u}}_H \cdot (\nabla \cdot (\overline{\mathbf{u}^* \mathbf{u}_H^* D}))$$
$$= c(\Phi_m, K_e) + c(\Phi_e, K_e) + c(K_m, K_e). \tag{55}$$

An evolution equation for the mean potential energy is found by using the average of the density equation (35),

$$\frac{\partial \hat{\rho}\overline{D}}{\partial t} + \nabla \cdot (\hat{\mathbf{u}}\hat{\rho}\overline{D}) + \nabla \cdot (\overline{\mathbf{u}^* \rho^* D}) = 0, \tag{56}$$

and the average of the vertical velocity w, that is the material derivative of z, given in equation (29),

$$\frac{\partial \Phi_m}{\partial t} + \nabla \cdot (\hat{\mathbf{u}}\Phi_m) = \hat{\rho}\overline{D}\overline{w}g - \hat{\rho}\overline{D}(\overline{\mathbf{u}_H^* \cdot \nabla_H z})g - \overline{z}\nabla \cdot (\overline{\mathbf{u}^* \rho^* D})g$$
$$= c(K_m, \Phi_m) + c(K_e, \Phi_m) + c(\Phi_e, \Phi_m). \tag{57}$$

Equation (43), subtracted the above expression for the mean potential energy, gives an equation governing the eddy potential energy,

$$\frac{\partial \Phi_e}{\partial t} + \nabla \cdot (\hat{\mathbf{u}}\Phi_e + \overline{\mathbf{u}^* \Phi^*}) = \overline{\rho D w'}g + \hat{\rho}\overline{D}(\overline{\mathbf{u}_H^* \cdot \nabla_H z})g + \overline{z}\nabla \cdot (\overline{\mathbf{u}^* \rho^* D})g$$
$$= c(K_m, \Phi_e) + c(K_e, \Phi_e) + c(\Phi_m, \Phi_e). \tag{58}$$

All possible conservative conversions of energy are represented by the $c(.,.)$s in equations (54), (55), (57), and (58).

4 The energy flow chart

In the previous section the four governing equations (54), (55), (57), and (58), for kinetic and potential energy, divided into suitable mean and eddy quantities, were set up. These equations provide three independent relations to determine the five unknown transfer terms.

$$c(\Phi_m, K_m) + c(\Phi_e, K_m)$$
$$= -\hat{\rho}\overline{D}\overline{w}g + \hat{\rho}\overline{D}(\overline{\mathbf{u}_H^* \cdot \nabla_H z})g - \hat{\mathbf{u}}_H \cdot (\overline{D'\nabla_H p'} + (\overline{\rho D \nabla_H z'})g) \tag{59}$$

$$c(\Phi_m, K_e) + c(\Phi_e, K_e)$$
$$= -\overline{\rho D w'}g - \hat{\rho}\overline{D}(\overline{\mathbf{u}_H^* \cdot \nabla_H z})g + \hat{\mathbf{u}}_H \cdot (\overline{D'\nabla_H p'} + (\overline{\rho D \nabla_H z'})g) \tag{60}$$

$$c(K_m, \Phi_m) + c(K_e, \Phi_m) + c(\Phi_e, \Phi_m)$$
$$= \hat{\rho}\overline{D}\overline{w}g - \hat{\rho}\overline{D}(\overline{\mathbf{u}_H^* \cdot \nabla_H z})g - \overline{z}\nabla \cdot (\overline{\mathbf{u}^* \rho^* D})g, \tag{61}$$

as the barotropic transfer of energy is recognized as

$$c(K_m, K_e) = \rho_0 \hat{\mathbf{u}}_H \cdot (\nabla \cdot (\overline{\mathbf{u}^* \mathbf{u}_H^* D})). \tag{62}$$

Additional arguments are required to close the system. Note that all the terms but the first ones on the right hand sides of the above system of equations are due to the change of coordinates. For the special case of a domain with rigid lid and flat bottom, where sigma coordinates are Cartesian, they all vanish.

Let $\hat{d}/dt = \partial/\partial t + \hat{\mathbf{u}} \cdot \nabla$. By the use of equation (29), the change of the mean geopotential level $g\overline{z}$ in the mean flow is

$$\frac{\hat{d}g\overline{z}}{dt} = (\overline{w} - \overline{\mathbf{u}_H^* \cdot \nabla_H z})g. \tag{63}$$

As $\overline{D}$ is conserved in the mean flow and K is insensitive to changes in ρ from the Boussinesq approximation, changes in Φ_m from changes in $\hat{\rho}\overline{D}$ should leave K_m unaffected. It is therefore suggested that the conversion between mean potential and mean kinetic energy is given by

$$c(\Phi_m, K_m) = -\hat{\rho}\overline{D}\overline{w}g + \hat{\rho}\overline{D}(\overline{\mathbf{u}_H^* \cdot \nabla_H z})g. \tag{64}$$

Negative (positive) $c(\Phi_m, K_m)$ gives an increase (decrease) in the mean potential energy through a rise (fall) in the mean geopotential level. This is done at the cost (gain) of mean kinetic energy.

Equation (59) then implies

$$c(\Phi_e, K_m) = -\hat{\mathbf{u}}_H \cdot (\overline{D'\nabla_H p'} + (\overline{\rho D \nabla_H z'})g). \tag{65}$$

The above definition is consistent as $c(\Phi_e, K_m)$ represents work done by mean kinetic energy against/with eddy potential energy.

With $c(\Phi_m, K_m)$ defined by equation (64), there is only one unresolved term at the right hand side of equation (61). From the averaged density equation (56), $-\nabla \cdot (\overline{\mathbf{u}^* \rho^* D})$ gives the rate of change of mean mass density from eddy flux of density. When negative (positive), Φ_m is being increased (decreased) at the cost (gain) of K_e. Thus

$$c(\Phi_m, K_e) = \overline{z}\nabla \cdot (\overline{\mathbf{u}^* \rho^* D})g, \tag{66}$$

and

$$c(\Phi_m, \Phi_e) = 0 \tag{67}$$

then follows.

The last conversion term must then be

$$c(\Phi_e, K_e) = -\overline{\rho D w' g} - \hat{\rho}\overline{D}(\overline{\mathbf{u}_H^* \cdot \nabla_H z}) g + \hat{\mathbf{u}}_H \cdot (\overline{D' \nabla_H p'} + (\overline{\rho D \nabla_H z'}) g) - \bar{z} \nabla \cdot (\overline{\mathbf{u}^* \rho^* D}) g. \quad (68)$$

At first sight, it is not obvious that this describes a process involving only the eddy contributions to $\overline{K}$ and $\overline{\Phi}$. The above expression may be rearranged, neglecting flux divergence terms, to give

$$c(\Phi_e, K_e) = -\overline{\mathbf{u}_H^* \cdot (D \nabla_H p' + (\rho D \nabla_H z') g)}. \quad (69)$$

It is then clear that $c(\Phi_e, K_e)$ describes the interaction between the eddy forms of kinetic and potential energy. This completes this sigma coordinate description of the energetics. The resulting energy flow diagram is displayed in figure 1.

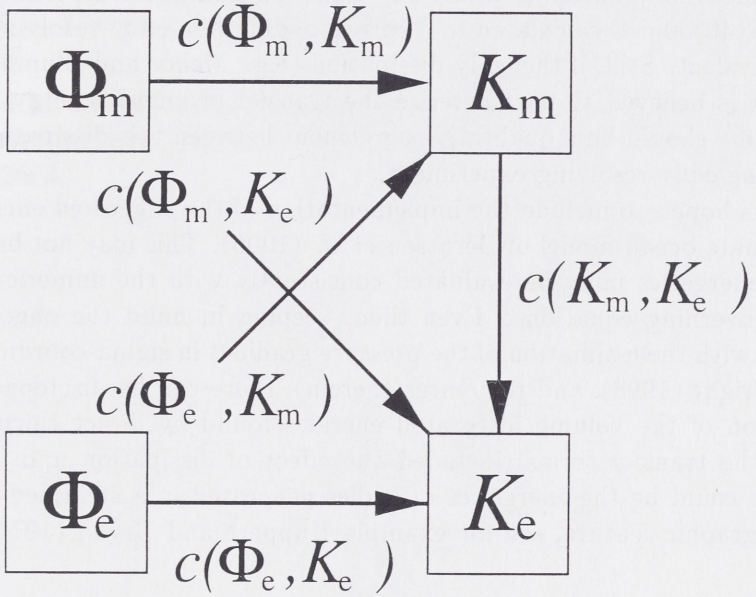

Figure 1: The energy flow chart. Arrows do not indicate any preferred flow direction of energy, only the direction for $c(.,.) > 0$.

Before concluding, a comparison with B85 is required. As B85 outlines the energetics for a general vertical coordinate, one should expect the findings herein to be consistent with his. The present definition of potential energy density is $\Phi = \rho D z g$, while the definition of B85 in sigma coordinates is pD. These two definitions are equivalent for describing the internal energetics as $\Phi = pD - \partial(pz)/\partial\sigma$, and the last term's net contribution to the flow is through work done at the lower boundary. The weighted average defined by (40) deviates slightly from that of B85. He uses the mass weighted average $\overline{\psi \rho D}/\overline{\rho D}$, which is not suitable in the present context because the inertial effects of varying density are neglected under the Boussinesq approximation. Another argument for choosing (40), is that when sigma coordinates are Cartesian, $\hat{\psi} \equiv \overline{\psi}$, and the transfer terms found in this section are equal to the Cartesian terms (multiplied by D) set up in the first section. This is not the case for the mass weighted average.

Taking the above into account, the conversion terms suggested in this section may, through manipulations similar to those leading to equation (69), be shown to be equivalent to those put forward by B85 (his equations (21)-(25)).

5 Concluding remarks

In the above, the conservation equations for kinetic and potential energy, divided into suitable mean and eddy quantities, for a sigma coordinate ocean model have been set up. The transfer terms, (62), (64), (65), (66), and (68), responsible for conservative transfers between the different energies and the resulting energy flow diagram, figure 1, have been suggested. This description is consistent with both B85, and the special case of sigma coordinates being Cartesian.

The generation of instabilities in the flow is diagnosed by

$$\frac{\partial}{\partial t}\int_V K_e dV = \int_V c(K_m, K_e)dV + \int_V (c(\Phi_m, K_e) + c(\Phi_e, K_e))dV. \tag{70}$$

The first integral on the right hand side contributes to barotropic instabilities, and the second to baroclinic instabilities.

The mean value of a quantity ψ in an eddy-resolving numerical experiment defined by the weighted average (40) and ψ calculated in a corresponding non-eddy-resolving experiment, are by no means equivalent. Still, if the eddy-dissipation (e.g., Mellor and Blumberg (1985)) of the latter experiment is believed to parameterize the transfer of kinetic energy between grid and sub-grid scale, there should be a qualitative agreement between this dissipation and $c(K_m, K_e)$ estimated from the eddy-resolving experiment.

Future work is hoped to include the implementation of the suggested energy diagnostics in the sigma coordinate ocean model of Berntsen *et al.* (1996). This may not be a trival exercise. First of all, the energetics must be evaluated consistently with the numerical scheme applied for solving the governing equations. Even then, keeping in mind the ongoing discussion on errors associated with the estimation of the pressure gradient in sigma-coordinate ocean models (see Song and Wright (1998), and references therein), there can be discrepancies between the temporal evolution of the volume integrated energies found by direct calculations, and that estimated from the transfer terms (included the effect of dissipation and lateral fluxes). A suitable test case could be the energetics of eddies generated in a stratified fluid flowing over an isolated topographic feature, see for example Huppert and Bryan (1976), restricted to a cyclic channel.

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