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On the mass transport induced by time-dependent oscillations of finite amplitude in a nonhomogeneous fluid

I General results for a perfect gas

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Report No. 42 May 1973

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Abstract

The mass transport induced by time-dependent oscillations of finite amplitude in a nonhomogeneous fluid, is considered. The mass transport is given by the Lagrangian mean velocity, calculated to the second order in the Mach number of the oscillations. We find no mass transport in a non-dissipative model. Taking into account dissipation, however, the theory leads to non-zero vertical drift and horizontal flow. The vertical drift becomes zero in an incompressible model.

Introduction.

In the present paper a general theory is outlined for the mass transport induced by time-dependent oscillations of finite amplitude in a nonhomogeneous, one-component fluid. The theory is motivated in the possible applications within astrophysics and geophysics, where non-linear effects like the one studied probably influence the mean currents and the flux of energy. (See, for example, Munk and Moore (1968), Longuet-Higgins (1970) for studies of ocean currents).

The mass transport is given by the Lagrangian mean velocity, calculated to the second order in the Mach number of the oscillations. We compute the vertical component of this mean velocity, the divergence and the vertical component of the vorticity, whereby determining the flow field. The fluid is a viscous, heat conductive and heat radiative perfect gas. Compressibility effects are fully accounted for. Different equilibrium models are considered.

There is no mass transport in a non-dissipative model. Taking into account dissipation, however, the theory leads in general to a non-zero effect. A vertical drift is obtained in addition to a flow in the horizontal plane. The vertical drift becomes zero in an incompressible model. *).

There are indications that flows of this type occur in experiments on standing acoustic waves (See Schaaffs and Haun (1968), Schaaffs (1973), Hobæk (1973) who have observed the formation of periodic density stratification, the scale of which is the half wave length).

*) This general result for an incompressible fluid explains why a zero mass transport is found by Kildal (1969), in his detailed analysis of a viscous boundary layer model.

2. Basic equations.

After elimination of the entropy, the equations of hydrodynamics can be written as follows:

$$(2.1) \quad \rho \frac{D\underline{V}}{Dt} + \nabla p + \underline{\lambda} g \rho = \underline{F}^V .$$

$$(2.2) \quad \frac{D\rho}{Dt} + \rho \nabla \cdot \underline{V} = 0$$

$$(2.3) \quad \frac{Dp}{Dt} + \rho c^2 \nabla \cdot \underline{V} = \Phi^T + \Phi^R + \Phi^V .$$

$\underline{V}, \rho, p, \underline{\lambda}, g, c$ are respectively the Eulerian velocity, density, pressure, unit vector in upward direction, acceleration of gravity and sound speed. Further, $\underline{F}^V$ is the viscous force, $\Phi^T/\gamma-1$ the accession of heat due to conduction, $\Phi^R/\gamma-1$ the accession of heat due to radiation, Φ^V the viscous dissipation function; γ is the ratio c_p/c_v of the specific heats.

$$(2.4) \quad \underline{F}^V = \nabla\left[\left(\frac{\mu}{3} + \mu_B\right)\nabla \cdot \underline{V} + \underline{V} \cdot \nabla\mu\right] + \mu \nabla^2 \underline{V} - \underline{V} \nabla^2 \mu + \nabla \times (\underline{V} \times \nabla\mu)$$

$$(2.5) \quad \Phi^T = (\gamma-1)\nabla \cdot (\sigma \nabla T)$$

$$(2.6) \quad \Phi^V = (\gamma-1)\left[\mu\left(\nabla\underline{V} : \nabla\underline{V} + \nabla\underline{V} : (\nabla\underline{V})^* - \frac{2}{3}(\nabla \cdot \underline{V})^2\right) + \mu_B (\nabla \cdot \underline{V})^2\right] ,$$

where $(\nabla \underline{V})^*$ denotes the conjugate of $\nabla \underline{V}$.

Further, we put

$$(2.7)a \quad \Phi^R = (\gamma-1)\nabla \cdot (\sigma_R \nabla T)$$

for the optically thick case, and

(2.7)b $\Phi^R = -(\gamma-1)c_v \rho q (T-T_o)$ (Newtons cooling law)

for the optically thin case.

The temperature is given by the equation of state

(2.8) $p = R\rho T \quad , \quad R = c_p - c_v \; .$

Thus the radiative effects are accounted for only in the energy equation (2.3).

γ and R are taken constant. The dissipative coefficients $\sigma, \mu, \mu_B, \sigma_R$ and q depend on the temperature (mainly) and the density. How, for example, σ and μ vary with these variables is known from classical kinetic theory. If the binary collision model with central force proportional to $r^{-\nu}$ is adopted, σ and μ are found to be independent of ρ but proportional to $T^{\frac{1}{2}+\frac{2}{\nu-1}}$, (Chapman & Cowling (1958)).

The following equation of evolution is obtained for T by eliminating p, ρ between (2.2), (2.3) and (2.8), and inserting $c^2 = \gamma p/\rho$:

(2.9) $$\frac{DT}{Dt} + (\gamma-1)T\nabla.\underline{V} = \frac{1}{R\rho}\left(\Phi^T + \Phi^R + \Phi^V\right) \; .$$

At equilibrium the velocity is supposed to be equal to zero. The other variables p_o, ρ_o, T_o must then satisfy the following equations:

(2.10) $$\begin{cases} \nabla p_o + \underline{\lambda} g \rho_o = 0 \\ \nabla.(\sigma_o \nabla T_o) = 0 \\ p_o = R\rho_o T_o \end{cases}$$

for the optically thin case, or the analogous equations where σ_o is replaced by $\sigma_o + \sigma_{Ro}$ for the optically thick case.

We suppose that p_0, ρ_0, T_0 depend only on the altitude z, so that we have

(2.11) $$\frac{\rho_0'}{\rho_0} = -\frac{g}{RT_0} - \frac{T_0'}{T_0} ,$$

where the prime denotes the z-derivative.

Of special interest is the <u>isothermal equilibrium</u>

(2.12) $$T_0 = \text{constant}, \quad p_0 = p_s e^{-z/H}, \quad \rho_0 = \rho_s e^{-z/H}$$

where p_s, ρ_s are the values of p, ρ at $z = 0$ and H is the scale heigth. Here $c_0^2 = \gamma p_0/\rho_0 = \gamma R T_0 = \gamma g H$.

Other possible equilibria are obtained by solving for instance, the equation $(\sigma(T_0)\, T_0')' = 0$. When σ is an increasing function of T, p_0, ρ_0, T_0 are power functions of z, and T_0 increases with altitude.

M being the Mach number of the perturbation, we suppose that $\underline{V}$, p, ρ, T can be developed in powers of M, at least up to the second order

(2.13) $$\begin{aligned} \underline{V} &= \underline{V}_1 + \underline{V}_2 + \dots \\ p &= p_0 + p_1 + p_2 + \dots \\ &\vdots \end{aligned}$$

Substituting these expressions in (2.1)-(2.3) and (2.8), we obtain to the first order the linearized equations, where $w = \underline{V}\cdot\underline{\lambda}$,

(2.14) $$\rho_0 \frac{\partial \underline{V}_1}{\partial t} + \nabla p_1 + \underline{\lambda}\, g \rho_1 = \underline{F}_1^V$$

(2.15) $$\frac{\partial \rho_1}{\partial t} + \rho_0' w_1 + \rho_0 \nabla\cdot\underline{V}_1 = 0$$

(2.16) $$\frac{\partial p_1}{\partial t} + p_0' w_1 + \gamma p_0 \nabla\cdot\underline{V}_1 = \Phi_1^T + \Phi_1^R$$

(2.17) $$p_1 = R(\rho_0 T_1 + \rho_1 T_0) .$$

From (2.9) (or from (2.15)-(2.17)) we obtain

(2.18) $$\frac{\partial T_1}{\partial t} + T_0' w_1 + (\gamma-1) T_0 \nabla . \underline{V}_1 = \frac{1}{R\rho_0} (\Phi_1^T + \Phi_1^R) .$$

Another consequence of the first order equations is obtained by taking the curl of (2.14)

(2.19) $$\frac{\partial}{\partial t} \nabla \times \underline{V}_1 = \underline{\lambda} \times \nabla \left(\frac{\rho_0' p_1 - p_0' \rho_1}{\rho_0^2} \right) + \nabla \times \frac{\underline{F}_1^V}{\rho_0} .$$

An alternative form is obtained by taking the time derivative of (2.19) and by using (2.15), (2.16)

(2.20) $$\frac{\partial^2}{\partial t^2} \nabla \times \underline{V}_1 = \frac{c_0^2 N^2}{g} \underline{\lambda} \times \nabla \nabla . \underline{V}_1 + \frac{\rho_0'}{\rho_0^2} \underline{\lambda} \times \nabla (\Phi_1^T + \Phi_1^R) + \frac{\partial}{\partial t} \nabla \times \frac{\underline{F}_1^V}{\rho_0} .$$

Here $N^2 = -g \left(\frac{\rho_0'}{\rho_0} + \frac{g}{c_0^2} \right)$ is the Väisälä frequency, see later, §.4.1.

Identifying terms of order M we obtain the second order equations

(2.21) $$\rho_0 \frac{\partial \underline{V}_2}{\partial t} + \nabla p_2 + \underline{\lambda} g \rho_2 = \underline{F}_2^V - \left(\rho_1 \frac{\partial \underline{V}_1}{\partial t} + \rho_0 \underline{V}_1 . \nabla \underline{V}_1 \right)$$

(2.22) $$\frac{\partial \rho_2}{\partial t} + \rho_0' w_2 + \rho_0 \nabla . \underline{V}_2 = - \nabla . (\rho_1 \underline{V}_1)$$

(2.23) $$\frac{\partial p_2}{\partial t} + p_0' w_2 + \gamma p_0 \nabla . \underline{V}_2 = \Phi_2^T + \Phi_2^R + \Phi_2^V - (\underline{V}_1 . \nabla p_1 + \gamma p_1 \nabla . \underline{V}_1)$$

(2.24) $$p_2 = R (\rho_0 T_2 + \rho_1 T_1 + \rho_2 T_0) .$$

Equation (2.9) (or (2.22)-(2.24)) gives now

(2.25) $$\frac{\partial T_2}{\partial t} + T_0' w_2 + (\gamma-1) T_0 \nabla . \underline{V}_2 = \frac{1}{R\rho_0} (\Phi_2^T + \Phi_2^R + \Phi_2^V) -$$
$$- (\underline{V}_1 . \nabla T_1 + (\gamma-1) T_1 \nabla . \underline{V}_1) - \frac{\rho_1}{R\rho_0^2} (\Phi_1^T + \Phi_1^R) .$$

$\underline{F}_1^V$, Φ_1^T, Φ_1^R are the first order terms in the expansion of $\underline{F}^V$, Φ^T, Φ^R in powers of M (Φ_1^V is equal to zero). They are linear combinations of first order variables and their gradients; the coefficients of these linear combinations depend on the dissipative coefficients and the first derivatives of these coefficients with respect to T, taken at $T = T_0(z)$. For example,

$$(2.26) \quad \Phi_1^T = (\gamma-1)\nabla\cdot\left(\sigma_0 \nabla T_1 + \left(\frac{d\sigma}{dT}\right)_0 T_1 \nabla T_0\right),$$

where σ_0, $\left(\frac{d\sigma}{dT}\right)_0$ stand for $\sigma(T_0)$, $\left(\frac{d\sigma}{dT}\right)(T_0)$ and vary with z in general. $\underline{F}_2^V$, Φ_2^T, Φ_2^R, Φ_2^V are the second order terms in the expansion of $\underline{F}^V$, Φ^T, Φ^R, Φ^V. Φ_2^V is a quadratic form of $\underline{V}_1$ and its gradient, with coefficients of the same form as above. In addition to similar quadratic forms of the first order variables, $\underline{F}_2^V$ contains terms in $\underline{V}_2$ and Φ_2^T, Φ_2^R terms in T_2. For example,

$$(2.27) \quad \Phi_2^T = (\gamma-1)\nabla\cdot\left(\sigma_0 \nabla T_2 + \sigma_1 \nabla T_1 + \sigma_2 \nabla T_0\right)$$

$$= (\gamma-1)\nabla\cdot\left[\left(\sigma_0 \nabla T_2 + \left(\frac{d\sigma}{dT}\right)_0 T_2 \nabla T_0\right) + \left(\left(\frac{d\sigma}{dT}\right)_0 T_1 \nabla T_1 + \left(\frac{d^2\sigma}{dT^2}\right)_0 \frac{T_1^2}{2} \nabla T_0\right)\right].$$

3. Lagrangian velocity. Time averages. Assumptions.

The Lagrangian velocity $\underline{V}_L(\underline{x}_o,t)$ is now introduced, $\underline{x}_o$ being the position at $t=0$ of a particle, the position of which is $\underline{x}$ at t :

$$(3.1) \quad \underline{V}_L(\underline{x}_o,t) = \underline{V}\left(\underline{x}_o + \int_0^t \underline{V}_L(\underline{x}_o,\tau)\,d\tau\,,\ t\right)$$

$$(3.2) \quad \underline{x}(\underline{x}_o,t) = \underline{x}_o + \int_0^t \underline{V}_L(\underline{x}_o,\tau)\,d\tau\ .$$

Assuming that $\underline{V}_L$ may be expanded in powers of M up to the second order, $\underline{V}_L = \underline{V}_{L1} + \underline{V}_{L2} + \cdots$, and that $\int_0^t \underline{V}_{Li}(\underline{x}_o,\tau)d\tau$ is of the same order in M as $\underline{V}_{Li}(\underline{x}_o,t)$, we obtain

$$\underline{V}_{L1}(\underline{x}_o,t) = \underline{V}_1(\underline{x}_o,t)$$

$$\underline{V}_{L2}(\underline{x}_o,t) = \underline{V}_2(\underline{x}_o,t) + \left(\int_0^t \underline{V}_1(\underline{x}_o,\tau)d\tau\right).\nabla\underline{V}_1(\underline{x}_o,t)\ .$$

We note that the last two equations hold for every $\underline{x}_o$, and therefore

$$(3.3) \quad \underline{V}_{L1}(\underline{x},t) = \underline{V}_1(\underline{x},t)$$

$$(3.4) \quad \underline{V}_{L2}(\underline{x},t) = \underline{V}_2(\underline{x},t) + \left(\int_0^t \underline{V}_1(\underline{x},\tau)d\tau\right).\nabla\underline{V}_1(\underline{x},t)$$

for every $\underline{x}$. (We build here on the assumption that every $\underline{x}$ in the volume occupied by the fluid may be regarded as an initial position, i.e. that the transformation (3.2) is invertible in this volume).

The last term in (3.4) is the Stokes drift velocity, (see Longuet-Higgins (1969) for its interpretation in geophysics). It is important to introduce the Lagrangian velocity when looking for mass transport; in fact, in the case of a finite amplitude oscillation, $\underline{V}_{L2}$ may have a non zero mean value, even if $\underline{V}_2$ has a zero mean value.

Let now $\underline{V}_1, p_1, \rho_1$ be a solution of (2.14) - (2.17) which is sinusoidal in time with period ω. The mass transport velocity may in principle be calculated by solving (2.21) - (2.24) and by using (3.4). As the solution of (2.21) - (2.24) is not readily obtained, we replace these equations by averaged equations. Here the time average $\overline{a}(\underline{x})$ of a function $a(\underline{x}, t)$ is defined by

$$(3.5) \qquad \overline{a}(\underline{x}) = \frac{\omega}{2\pi} \int_t^{t+\frac{2\pi}{\omega}} a(\underline{x}, \tau)\, d\tau \quad .$$

We then obtain a system of equations which contains the unknowns $\overline{\underline{V}_2}$ (or $\underline{V}_{L2}$), $\overline{p_2}$, $\overline{\rho_2}$, the averages of quadratic forms of the first order variables, and the averages of time derivatives of second order variables, as $\overline{\frac{\partial \rho_2}{\partial t}}$ for example. We suppose that the last type of averages are equal to zero, i.e.

$$(3.6) \qquad \overline{\frac{\partial \underline{V}_2}{\partial t}} = \overline{\frac{\partial p_2}{\partial t}} = \overline{\frac{\partial \rho_2}{\partial t}} = 0 \, .$$

This assumption is plausible since the source terms in (2.21) - (2.24) are periodic functions. However it has to be justified. To do this, we should prove, for instance, that $\underline{V}_2, p_2, \rho_2$ are not secular in time, which in turn would make

it necessary to define boundary conditions. This problem will not be treated here.

Further, we note the following:

(3.7) $\overline{\frac{\partial b}{\partial t}} = 0$ whenever b is periodic in time with period ω.

Taking the divergence and the average of (3.4), noting that

$$\left(\int_0^t \nabla \underline{V}_1\right) : \nabla \underline{V}_1 = \frac{1}{2}\frac{\partial}{\partial t}\left[\left(\int_0^t \nabla \underline{V}_1\right) : \int_0^t \nabla \underline{V}_1\right]$$

is periodic in time with period ω, we obtain the identity

(3.8) $$\overline{\nabla \underline{V}_{L2}} = \overline{\nabla \cdot \underline{V}_2} + \overline{\left(\int_0^t \underline{V}_1\right) \cdot \nabla \nabla \cdot \underline{V}_1}\,.$$

4. Study of the mass transport velocity.

We now proceed to the solution of the averaged equations (2.21)-(2.24), where $\overline{\underline{V}_{L2}}$, $\overline{p_2}$, $\overline{\rho_2}$ ($\overline{T_2}$) are the unknowns after $\underline{V}_1$ has been eliminated by using (3.4). Let w_{Li} be the vertical component of $\underline{V}_{Li}$, i=1,2, and let index $\perp$ indicate the horizontal projection, such that

$$\underline{V}_L = \underline{V}_{L\perp} + \underline{\lambda}\, w_L$$

$$\nabla = \nabla_\perp + \underline{\lambda}\frac{\partial}{\partial z} .$$

4.1. First results for $\overline{w_{L2}}$ and $\overline{\nabla\cdot\underline{V}_{L2}}$.

We take the average of the second order equation of continuity (2.22) and equation of energy (2.23) and obtain respectively (4.1) and (4.2). To get these equations we have used (3.4) and the identity (3.8), and have made the assumption (3.6).

$$(4.1)\qquad \rho_0'\,\overline{w_{L2}} + \rho_0\,\overline{\nabla\cdot\underline{V}_{L2}} = 0 = \nabla\cdot(\rho_0\,\overline{\underline{V}_{L2}})$$

$$(4.2)\qquad p_0'\,\overline{w_{L2}} + \gamma p_0\,\overline{\nabla\cdot\underline{V}_{L2}} = \overline{\Phi_2^T} + \overline{\Phi_2^R} + \overline{\Phi_2^V} - \overline{(\gamma-1)(\nabla\cdot\underline{V}_1)\int_0^t(\Phi_1^T+\Phi_1^R)} - \overline{\nabla\cdot\left(\underline{V}_1\int_0^t(\Phi_1^T+\Phi_1^R)\right)} .$$

(4.1)-(4.2) may be considered as a system of equations giving $\overline{w_{L2}}$ and $\overline{\nabla\cdot\underline{V}_{L2}}$. The determinant is

$$(4.3)\qquad \gamma p_0\rho_0' - p_0'\rho_0 = -\rho_0^2\left((\gamma-1)g + \gamma R T_0'\right) = -\frac{c_0^2\rho_0^2}{g}N^2 ,$$

where N is the Väisälä frequency, see Eckart (1960). For the

isothermal equilibrium $N^2 = (\gamma-1)g/\gamma H$. Solving (4.1)-(4.2) we obtain

$$(4.4)\quad \overline{w_{L2}} = \frac{g}{\rho_0 c_0^2 N^2}\Big[\overline{\Phi_2^T + \Phi_2^R + \Phi_2^V} - \overline{(\gamma-1)(\nabla\cdot \underline{V}_1)\int_0^t(\Phi_1^T+\Phi_1^R)} - \overline{\nabla\cdot\Big(\underline{V}_1\int_0^t(\Phi_1^T+\Phi_1^R)\Big)}\Big]$$

$$(4.5)\quad \overline{\nabla\cdot \underline{V}_{L2}} = \frac{g(g+RT_0')}{p_0 c_0^2 N^2}\Big[\overline{\Phi_2^T + \Phi_2^R + \Phi_2^V} - \overline{(\gamma-1)(\nabla\cdot \underline{V}_1)\int_0^t(\Phi_1^T+\Phi_1^R)} - \overline{\nabla\cdot\Big(\underline{V}_1\int_0^t(\Phi_1^T+\Phi_1^R)\Big)}\Big] .$$

In fact $\overline{w_{L2}}$ and $\overline{\nabla\cdot \underline{V}_{L2}}$ are completely determined only when $\overline{T_2}$, which appears through the second order thermal dissipation term $\overline{\Phi_2^T}$ and radiative dissipation term $\overline{\Phi_2^R}$, is computed. Before we show how this can be done, we make some comments.

(i) It follows from (4.1)-(4.2) that a vertical mass transport ($\overline{w_{L2}} \neq 0$) is impossible in absence of dissipation, i.e. when $\mu = \mu_B = \sigma = 0$, $\sigma_R = 0$ or $q = 0$. This result still holds in the case of a rotating but non-dissipative fluid, since the Coriolis force only appears in the equation of motion, which has not been used in deriving (4.1)-(4.2).

(ii) (4.1) and the identity (3.8) show that a vertical mass transport is impossible for the incompressible model ($\nabla\cdot \underline{V} = 0$). Again, the result still holds in the case of a rotating fluid.

4.2. <u>Computation of $\overline{p_2}$ and $\overline{T_2}$</u>.

Averaging the second order equation of motion (2.21) and equation of state (2.24) and eliminating $\overline{\rho_2}$ and $\overline{p_2}$ between these equations, we obtain successively

(4.6) $\nabla_\perp \overline{p_2} = -\underline{C}_\perp + \overline{F^v_{2\perp}}$

(4.7) $\nabla_\perp \overline{T_2} = \frac{R T_0^2}{g}\left[\nabla_\perp\left(\frac{C_z - \overline{F^v_{2z}}}{p_0}\right) - \frac{\partial}{\partial z}\left(\frac{C_\perp - \overline{F^v_{2\perp}}}{p_0}\right)\right]$.

Taking the curl of the averaged horizontal projection of the equation of motion and introducing for $\underline{F}_2^v$ an expansion similar to (2.27), we obtain an equation for the vertical component of $\nabla\times\overline{\underline{V}_2}$:

(4.8) $\nabla\cdot\left(\mu_0 \nabla(\nabla\times\overline{\underline{V}_2})_z\right) = \left\{\nabla\times\left[\underline{C} - \overline{\nabla\cdot(\mu_1\nabla\underline{V}_1 - \underline{V}_1\nabla\mu_1)}\right]\right\}_z$.

Here $\underline{C}$ is defined by

(4.9) $\underline{C} = \overline{\rho_1\frac{\partial \underline{V}_1}{\partial t} + \rho_0\,\underline{V}_1\cdot\nabla\underline{V}_1 - \underline{\lambda}\, g\,\frac{\rho_1 T_1}{T_0}}$.

By using the first order equations (2.14)-(2.19), we obtain after some computation (see Appendix I)

$$
\begin{aligned}
(4.10)\quad \underline{C} = {} & p_0\nabla\left\{\frac{1}{p_0}\overline{\left[\rho_0\frac{V_1^2}{2} - \frac{p_1^2}{2\gamma p_0} - \frac{g^2(p_1 - c_0^2\rho_1)^2}{2\rho_0 c_0^4 N^2}\right]}\right\} - \\
& -\underline{\lambda}\,\overline{\left[g\frac{\rho_0}{T_0}\left(\int_0^t \underline{V}_1\right)\cdot\nabla T_1 + \rho_0\frac{T_0'}{T_0}\frac{V_1^2}{2} + \frac{1}{2}\frac{T_0''}{T_0}\frac{g^3(p_1 - c_0^2\rho_1)^2}{\rho_0 c_0^4 N^4}\right]} + \\
& +\overline{\frac{g}{\rho_0 c_0^2 N^2}\left(\int_0^t(\phi_1^T + \phi_1^R)\right)\left[g\frac{\rho_0}{T_0}\nabla T_1 + \frac{T_0'}{T_0}(\nabla p_1 + \underline{\lambda}\, g\rho_1)\right]} + \\
& +\overline{\left[\frac{\rho_1}{\rho_0}\underline{F}_1^v + \rho_0\left(\int_0^t \underline{V}_1\right)\times\nabla\times\frac{\underline{F}_1^v}{\rho_0} - \underline{\lambda}\frac{T_0'}{T_0}\left(\int_0^t \underline{V}_1\right)\cdot\underline{F}_1^v\right]} .
\end{aligned}
$$

The first two terms in (4.10) do not depend explicitly on the dissipative coefficients, while the last two terms do.

Now let S_M be a number which gives the order of magnitude of the dissipative terms. For instance, when the scale heigth is H and the wave length λ , with $\lambda \lesssim H$, S_M is the modified Stokes number

$$(4.11)\quad S_M = \frac{3\delta\omega}{4c_o^2}\ , \quad \delta = \frac{1}{\rho_o}\left[\frac{\gamma-1}{c_p}(\sigma_o+\sigma_{Ro}) + \frac{\gamma-1}{\gamma}\left(\frac{\lambda}{2\pi}\right)^2\rho_o q_o + \frac{4}{3}\mu_o + \mu_{Bo}\right],$$

where δ is the diffusivity of sound.

If terms depending explicitly on S_M are neglected in (4.10), $\underline{C}$ reduces to the sum of the first two terms. We thus obtain from (4.6)-(4.7) the following approximated expressions for $\overline{p_2}$ and $\overline{T_2}$:

$$(4.12)\quad \overline{p_2} = \overline{\left[-\rho_o\frac{V_1^2}{2} + \frac{p_1^2}{2\rho_o c_o^2} + \frac{g^2(p_1 - c_o^2\rho_1)^2}{2\rho_o c_o^4 N^2}\right]}(1+O(S_M)) + \pi(z)$$

$$(4.13)\quad \overline{T_2} = -\overline{\left[\left(\int_o^t V_1\right)\cdot\nabla T_1 + \frac{T_o'}{g}\frac{V_1^2}{2} + \frac{g^2 T_o''(p_1 - c_o^2\rho_1)^2}{2\rho_o^2 c_o^4 N^4}\right]}(1+O(S_M)) + \theta(z)\,.$$

Here π and θ are arbitrary functions of z , which in general depend on the boundary conditions. Substituting (4.13) in (4.4) for example, we obtain $\overline{w_{L2}}$ correct to the first order in S_M. It should be noted, however, that this result may not be correct in other cases than the one mentioned above. In the boundary layer, for instance, it may be necessary to keep the dissipative terms in (4.10) in order to obtain $\underline{C}$, and therefore $\overline{w_{L2}}$, with the same accuracy.

Before we look further at the result obtained for $\overline{w_{L2}}$, we give some interpretations of the expression (4.12) for $\overline{p_2}$. $\rho_o\overline{V_1^2}/2$ is the second order kinetic energy, $\overline{p_1^2}/2\rho_o c_o^2$ is the elastic energy, while the third term may be written as

$$\frac{g\rho_0}{2c_p}\frac{\overline{\eta_1^2}}{\eta_0'} .$$

Here $\eta_1 = \dfrac{p_1 - c_0^2 \rho_1}{(\gamma-1)\rho_0 T_0}$ is the first order perturbation of entropy, and η_0 is the entropy at equilibrium, with

$$\eta_0' = -c_p\left(\frac{\rho_0'}{\rho_0} + \frac{g}{c_0^2}\right) = c_p\left(\frac{\gamma-1}{c_0^2}g + \frac{T_0'}{T_0}\right) = c_p\frac{N^2}{g} .$$

This term is discussed by Eckart (1960), p.53, and called the thermobaric energy. It is a part of what he defines as "external energy density" - a quadratic form he derives from the linearized equations.

4.3. Computation of $\overline{w}_{L2}$.

Substituting (4.13) in (4.4), we notice that the vertical mass transport velocity may be decomposed in the sum of three expressions

$$(4.14)\quad \overline{w_{L2}} = \overline{w_{L2}}^T + \overline{w_{L2}}^R + \overline{w_{L2}}^V + \ell(z)$$

which describe respectively the mean vertical drift due to thermal, radiative and viscous dissipation, correct to the first order in S_M . Here $\ell(z)$ is the sum of the terms depending on the arbitrary function $\theta(z)$ which appears in (4.13).

For the viscous vertical drift we get

$$(4.15)\quad \overline{w_{L2}}^V = \frac{(\gamma-1)g}{\rho_0 c_0^2 N^2}\left[\overline{\mu_0\left(\nabla V_1 : \nabla V_{-1} + \nabla V_1 : (\nabla V_{-1})^* - \tfrac{2}{3}(\nabla\cdot V_{-1})^2\right) + \mu_{B0}(\nabla\cdot V_{-1})^2}\right]\left(1 + O(S_M)\right) .$$

This expression is always positive.

In the same way we get

$$(4.16)\quad \overline{W}_{L2}^{T} = \frac{g}{\rho_0 c_0^2 N^2}\left[\phi_2^T - (\gamma-1)(\nabla\cdot \underline{V}_1)\int_0^t \phi_1^T - \nabla\cdot\left(\underline{V}_1 \int_0^t \phi_1^T\right)\right](1+O(S_M)),$$

where ϕ_1^T , ϕ_2^T are given by (2.26)-(2.27).

As an example, let us consider the special case of isothermal equilibrium, where we shall first neglect the variation of σ with temperature. Then ϕ_1^T , ϕ_2^T are simplified, and we obtain (see Naze Tjötta and Tjötta (1972))

$$(4.17)\quad \overline{W}_{L2}^{T} = \frac{(\gamma-1)g}{\rho_0 c_0^2 N^2}\,\sigma_0\left\{\frac{(\nabla T_1)^2}{T_0} + \nabla\cdot\left[\nabla\left(-\left(\int_0^t \underline{V}_1\right)\cdot\nabla T_1 - \frac{T_1^2}{2T_0}\right) + \left(\int_0^t \underline{V}_1\right)\nabla^2 T_1\right]\right\}(1+O(S_M)).$$

When the variation of σ is taken into account, the divergence term in (4.17) is modified by a term $\frac{1}{\sigma_0}\left(\frac{d\sigma}{dT}\right)_0 \overline{T_1 \nabla T_1}$.

For plane vertical progressive or standing waves where the first order variables only depend on z , t, we get

$$(4.18)\quad \overline{W}_{L2}^{T} = \frac{(\gamma-1)g}{\rho_0 c_0^2 N^2}\frac{\sigma_0}{T_0}\left[\left(\frac{\partial T_1}{\partial z}\right)^2 - \frac{\gamma-2}{\gamma-1}\frac{\partial}{\partial z}\left(T_1\frac{\partial T_1}{\partial z}\right)\right](1+O(S_M)).$$

Let us consider a vertical progressive beam

$$(4.19)\quad T_1 = F_A\, e^{-\alpha z}\sin(kz+\omega t),$$

where $F_A(x)$ is equal to a constant A inside the beam $\left(|x| \leqslant \frac{D}{2}\right)$ and is equal to zero outside the beam $\left(|x| > \frac{D}{2}\right)$. (α, k, ω) have to satisfy the dispersion relation associated with the linearized equations (2.14)-(2.17), i.e. in the non-dissipative approximation

$$(4.20)\quad \begin{cases} \alpha = -\dfrac{1}{2H}, & k^2 = \Delta, \quad \text{if} \quad \Delta \geqslant 0 \\ \alpha^2 + \dfrac{\alpha}{H} + \Delta + \dfrac{1}{4H^2} = 0, & k = 0, \quad \text{if} \quad \Delta < 0, \end{cases}$$

where $\Delta = (\omega^2/\gamma H H) - 1/4H^2$.

Equations (4.18) and (4.14) give

$$(4.21)\quad \overline{w}_{L2} \approx \frac{\sigma_0}{\rho_s g T_0} \frac{F_A^2}{2} \left(k^2 + \frac{3-\gamma}{\gamma - 1} \alpha^2 \right) e^{\left(\frac{1}{H} - 2\alpha\right) z} + \ell(z),$$

where for simplicity we have not taken into account the effects of radiation and viscosity.

If we suppose that the flow is two-dimensional, the horizontal component $\overline{u}_{L2}$ of the mass transport velocity is determined by (4.1)

$$(4.22)\quad \overline{u}_{L2} = - \int_0^x \frac{1}{\rho_0} \frac{\partial}{\partial z} \left(\rho_0 \overline{w}_{L2} \right) dx + m(z),$$

where m is an arbitrary function. If we suppose the mass transport velocity field to be non-secular in x, we must choose

$$\ell(z) = a\, e^{\frac{z}{H}},$$

where a is an arbitrary constant. Also, if we require $\overline{u}_{L2}$ to be antisymmetric in x, $m(z)$ has to be equal to zero.

Choosing for α the root of (4.19) which is not bigger than $-\frac{1}{2H}$, we may neglect the $a\,e^{z/H}$ term in $\overline{w}_{L2}$ inside the beam, thus obtaining

$$(4.23)\quad \overline{w_{L2}} \approx \frac{\sigma_0}{\rho_s g T_0}\,\frac{A^2}{2}\left(k^2+\frac{3-\gamma}{\gamma-1}\alpha^2\right) e^{\left(\frac{1}{H}-2\alpha\right)z} .$$

This shows that an upward directed material flow takes place inside the beam.

Now let us consider the case of a <u>vertical standing beam</u>

$$(4.24)\qquad T_1 = F_A\, e^{-\alpha z} \cos kz \cos \omega t ,$$

where F_A , α , k , ω are the same as above. Requiring for $\overline{u_{L2}}$ non-secularity and antisymmetry in x, we find for the vertical component of the mass transport velocity inside the beam

$$(4.25)\quad \overline{w_{L2}} \approx \frac{\sigma_0}{\rho_s g T_0}\,\frac{A^2}{2(\gamma-1)}\, e^{\left(\frac{1}{H}-2\alpha\right)z} \Big\{(3-\gamma)\alpha k \sin 2kz +$$

$$+(3-\gamma)\frac{\alpha^2-k^2}{2}\cos 2kz + (3-\gamma)\frac{\alpha^2}{2} + (\gamma-1)\frac{k^2}{2}\Big\} .$$

It can be shown that $\overline{w_{L2}}$ given by (4.25) changes sign twice on a half wave length. However, when integrating this expression of a half wave length $\left(n\frac{\pi}{k}, (n+1)\frac{\pi}{k}\right)$, we see that the first two terms do not contribute. The last term in (4.25) being positive, we conclude that this integrated vertical drift is directed upward inside the beam.

When taking into account the variation of σ with T , a term $\frac{T_0}{\sigma_0}\left(\frac{d\sigma}{dT}\right)_0 \frac{\partial}{\partial z}\left(\overline{T_1 \frac{\partial T_1}{\partial z}}\right)$ has to be added in the parenthesis

of (4.18). It is easily shown that this effect does not change qualitatively the above results for most of the interaction potentials mentioned in § 2. It is of special interest to note that in the case of a progressive wave, this effect always increases the value of $\overline{w_{L2}}$ given by (4.23).

Looking at the contribution from radiation in (4.4) and using (4.13), we obtain $\overline{w_{L2}}^R$. In the optically thick case, where ϕ^R is given by (2.7)a, $\overline{w_{L2}}^R$ has the same form as $\overline{w_{L2}}^T$ given by (4.10), only σ being replaced by σ_R. In the optically thin case, where ϕ^R is given by (2.7)b, we have

$$\phi_1^R = -(\gamma-1)\, c_v \rho_0 q_0 T_1$$

$$\phi_2^R = -(\gamma-1)\, c_v \left[\rho_0 q_0 T_2 + T_1 \left(q_0 \rho_1 + \rho_0 \left(\frac{dq}{dT}\right)_0 T_1 \right) \right] .$$

Substituting these expressions in the relevant part of (4.4), and using (2.15), (2.18), we obtain

$$(4.26) \quad \overline{w_{L2}}^R = -\frac{(\gamma-1)\, c_v g}{c_0^2 N^2}\, q_0 \overline{\left[T_2 + \left(\int_0^t V_{-1}\right) \cdot \nabla T_1 + (\gamma-1)\left(1 - \frac{T_0}{q_0}\left(\frac{dq}{dT}\right)_0\right) T_1 \int_0^t \nabla \cdot V_{-1} \right]} .$$

Using (4.13), we get finally

$$(4.27) \quad \overline{w_{L2}}^R = \frac{(\gamma-1)^2 c_v g}{c_0^2 N^2}\, q_0 \overline{\left[\left(\frac{T_0}{q_0}\left(\frac{dq}{dT}\right)_0 - 1 \right) T_1 \int_0^t \nabla \cdot V_{-1} + \frac{T_0' V_1^2}{2(\gamma-1) g} + \frac{T_0'' g^2 \eta_1^2}{2(\gamma-1) c_p^2 N^4} \right]} (1 + O(S_M)) .$$

The last two terms are equal to zero in the case of isothermal

equilibrium. For other equilibria mentioned in § 2, where T_0 increases with altitude as a power of z , these terms are positive and give an upward drift. In the case of isothermal equilibrium, $\overline{w}^R_{L2}$ is given by

$$(4.28) \quad \overline{w}^R_{L2} = \frac{(\gamma-1)c_v g}{c_0^2 N^2}\frac{q_0}{T_0}\left(1-\frac{T_0}{q_0}\left(\frac{dq}{dT}\right)_0\right)\overline{T_1^2}\,(1+O(S_M)),$$

showing that the direction of this vertical drift may be very sensitive to the variation of $q(T)$. For instance, if q is proportional to T^n , $\overline{w}^R_{L2}$ has the sigh of $1-n$.

It is easily seen that the results obtained for $\overline{w}^R_{L2}$ (as well as for $(\nabla\times\overline{\underline{V}}_{L2})_z$, see § 4.4) at the present order of approximation in S_M , are not modified if we add a radiative pressure term ∇p_R in the equation of motion.

4.4 Computation of $(\nabla\times\overline{\underline{V}}_{L2})_z$.

The mass transport velocity field is determined when we know $\overline{w}_{L2}$, $\nabla.\overline{\underline{V}}_{L2}$ and the vertical component of the vorticity, $\overline{\Omega} = (\nabla\times\overline{\underline{V}}_{L2})_z = \underline{\lambda}\cdot\nabla_\perp\times\overline{\underline{V}}_{L2\perp}$. We obtain an equation for $\overline{\Omega}$ by substituting (4.10) in (4.8):

$$(4.29) \quad \mu_0\nabla^2\overline{\Omega} + \mu_0'\frac{\partial}{\partial z}\overline{\Omega} = \underline{\lambda}\cdot\nabla_\perp\times\overline{\left[\underline{C}_\perp - \nabla.(\mu_1\nabla\underline{V}_{1\perp} - \underline{V}_1\nabla_\perp\mu_1 - \mu_0\nabla((\int_0^t \underline{V}_1)\cdot\nabla\underline{V}_{1\perp})\right]}.$$

The last term is the contribution of the Stokes velocity. Since this equation is linear, we may again decompose $\overline{\Omega}$ into the sum of three expressions

$$(4.30) \qquad \overline{\Omega} = \overline{\Omega}^T + \overline{\Omega}^R + \overline{\Omega}^V .$$

We have

$$(4.31) \quad \mu_0 \nabla^2 \overline{\Omega}^T + \mu_0' \frac{\partial}{\partial z} \overline{\Omega}^T = \frac{g}{\rho_0 c_0^2 N^2} \underline{\lambda} \cdot \nabla_\perp \times \overline{\left[\left(\frac{g \rho_0}{T_0} \nabla_\perp T_1 + \frac{T_0'}{T_0} \nabla_\perp p_1 \right) \int_0^t \Phi_1^T \right]}$$

and a similar equation for $\overline{\Omega}^R$, Φ_1^T being replaced by Φ_1^R. For $\overline{\Omega}^V$ we get

$$(4.32) \quad \mu_0 \nabla^2 \overline{\Omega}^V + \mu_0' \frac{\partial}{\partial z} \overline{\Omega}^V = \underline{\lambda} \cdot \nabla_\perp \times \overline{\left[\frac{\rho_1}{\rho_0} \underline{F}_{1\perp}^V + \rho_0 \left(\left(\int_0^t \underline{V}_1 \right) \times \nabla \times \frac{\underline{F}_1^V}{\rho_0} \right)_\perp - \nabla \cdot \left(\mu_1 \nabla \underline{V}_{1\perp} - \underline{V}_1 \nabla_\perp \mu_1 - \mu_0 \nabla \left(\left(\int_0^t \underline{V}_1 \right) \cdot \nabla \underline{V}_{1\perp} \right) \right) \right]} .$$

$\underline{F}_1^V$ is given by (2.4) where μ, μ_B are replaced by μ_0, μ_{B0}.

Expressions for $\overline{\Omega}^T$, $\overline{\Omega}^R$, $\overline{\Omega}^V$ correct to the zeroth order in S_M are obtained by substituting in the source terms of (4.31)-(4.32) the solution of the non-dissipative linearized equations (see §.4.2 for the validity of such an approximation).

For waves where the first order variables depend on the horizontal coordinates $\underline{r}_\perp = (x,y)$ only through $\underline{k}_\perp \cdot \underline{r}_\perp$, it is easily seen that the source terms of (4.31)-(4.32) are equal to zero. However, a lateral confinement in the horizontal plane of the waves, leads in general to non-zero source terms in (4.31)-(4.32), and thus to a strong horizontal circulation (see below).

For the sake of simplicity, let us now suppose that <u>the equilibrium is isothermal</u>. Using (4.31), (4.32) and the non-dissipative first order equations, we obtain

$$(4.33)\quad \nabla^2 \overline{\Omega}^T = \underline{\lambda}\cdot\nabla_\perp \times \overline{\left[\frac{1}{c_p} \frac{\sigma_0}{\mu_0} \frac{g^2}{T_0^2 N^2} \nabla_\perp T_1 \int_0^t \nabla^2 T_1 \right]} (1 + O(S_M))$$

$$(4.34)\quad \nabla^2 \overline{\Omega}^V = \underline{\lambda}\cdot\nabla_\perp \times \Big\{ -\overline{\left(\frac{1}{3}+\frac{\mu_{B0}}{\mu_0}\right)\left(\int_0^t \nabla\cdot\underline{V}_1\right)\nabla_\perp\nabla\cdot\underline{V}_1} + (\gamma-1)\frac{T_0}{\mu_0}\left(\frac{d\mu}{dT}\right)_0 \nabla\cdot\overline{\left[\left(\int_0^t \nabla\cdot\underline{V}_1\right)\nabla\underline{V}_{1\perp} - \right.}$$

$$\overline{\left. - \underline{V}_1 \nabla_\perp \int_0^t \nabla\cdot\underline{V}_1 \right]} - \overline{\left(\int_0^t \nabla\cdot\underline{V}_1\right)\nabla^2 \underline{V}_{1\perp}} + \overline{\left[\left(\int_0^t \underline{V}_1\right) \times \nabla^2 \nabla \times \underline{V}_1 + \nabla^2\left(\left(\int_0^t \underline{V}_1\right)\cdot\nabla\underline{V}_1\right) \right]_\perp} \Big\} (1 + O(S_M)).$$

In the optically thick case, the equation for $\overline{\Omega}^R$ is obtained by replacing σ_0 with σ_{R0} in (4.33). In the optically thin case, we obtain

$$(4.35)\quad \nabla^2 \overline{\Omega}^R = \frac{g^2 \rho_0}{\gamma T_0^2 N^2} \frac{q_0}{\mu_0} \underline{\lambda}\cdot \overline{\nabla_\perp T_1 \times \nabla_\perp \int_0^t T_1} \,(1 + O(S_M)).$$

It is easily seen that the source term in (4.35) is equal to zero in the case of a standing wave.

In the incompressible approximation, where $\nabla\cdot\underline{V}_1 \to 0$, the last two terms in (4.34) will give the dominant part of $\overline{\Omega}$.

In the non-dissipative case, (4.33)-(4.34) are no longer valid, since the hydrodynamical equations are singular at $\mu = 0$. We then obtain Ω by taking the curl of (2.21) and by taking into account the other second order equations. We obtain

$$\frac{\partial^2 \Omega}{\partial t^2} = 0 .$$

Because of the non secularity condition (see § .3), Ω is equal to its initial value. Together with the result obtained in §. 4.1, (i), this shows that a finite amplitude oscillation generates no mass transport in the non-dissipative case.

It is of interest to note that the source terms in the equations for $\overline{\Omega}^T$ and $\overline{\Omega}^R$ in the optically thick case, for example, contain terms which are proportional to $\frac{1}{c_p}\frac{\sigma_0}{\mu_0}$, $\frac{1}{c_p}\frac{\sigma_{R0}}{\mu_0}$

respectively. In the same way, the source term in the equation for $\bar{\Omega}^v$ contains terms proportional to $\frac{\mu_{Bo}}{\mu_o}$ and others independent of viscosity.

For a finite amplitude oscillation such that at least one of these source terms is non zero, the horizontal mass circulation is thus of zeroth order in S_M. The vertical mass transport, on the other hand, is of the first order in S_M. For such cases, the horizontal mass transport may dominate over the vertical drift, especially if $\frac{1}{c_p} \frac{\sigma_o + \sigma_{Ro}}{\mu_o}$ is large, which normally is the case in the lower solar atmosphere.

5. Concluding remarks.

We have shown that it is theoretically possible for finite amplitude oscillations in a dissipative, nonhomogeneous fluid to sustain a mass transport streaming. The fluid was a perfect gas, but generalisations to other models are possible. The theory, we believe , is applicable in the studies of transport phenomenon in the solar atmosphere. For instance, it provides a mechanism for associating vertical drift, and the observed large scale horizontal flow in the supergranules, with oscillatory motions in the atmosphere. The mass transport also influences the energy transport.

In oceanography, it has been suggested by Munk and Moore (1968) that the Cromwell current, which flows eastward in a narrow zone along the equator, with maximum velocities (in the Pacific) of the order 1 m/sec at 100 m depths, is associated with confined internal planetary waves. Their model, however, being non-dissipative, fails to predict the necessary transport of mass, a fact which was later pointed out by Moore (1969). It is suggested by our theory that mean currents in the ocean may be produced by such internal waves if dissipative effects are accounted for (cf. §.4.4). Further, including the effect of compressibility, it is also suggested that a vertical drift may be attributed to the same waves. An upward vertical drift up to order 1 m/day, i.e. of order 10^{-5} compared to the maximum horizontal flow, has been estimated at the equator (Knauss (1966)).

An experimental test of the results obtained should be possible. For instance, in a vertically directed standing

wave the theory predicts microstreaming, the scale of which in the vertical direction is the half wave length. We find that this streaming should be strong enough to be observable in an ultrasonic wave, and it may be the mechanism behind the formation of periodic density stratification observed by Schaaffs and Haun (1968), Schaaffs (1973) and Hobæk (1973). The last author's similar observation in a horizontally directed wave, can also be explained qualitatively by the vertical drift in our theory. However, further observations are needed before definite conclusion can be drawn.

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Appendix I

Let $\underline{C}$ be defined by (4.9). Using (2.14) and (2.17) together with the identity $\underline{V}_1\cdot\nabla\underline{V}_1 = \nabla\frac{V_1^2}{2} - \underline{V}_1\times\nabla\times\underline{V}_1$, we obtain

$$(A.1)\quad \underline{C} = \overline{\rho_0\nabla\frac{V_1^2}{2} - \frac{\rho_1}{\rho_0}\nabla p_1 - \underline{\lambda}\, g\,\frac{\rho_1 p_1}{p_0} - \rho_0\,\underline{V}_1\times\nabla\times\underline{V}_1 + \frac{\rho_1}{\rho_0}\,\underline{F}_1^V} .$$

The fourth term in the above expression is computed by integrating (2.19) with respect to t and by noting that the integration constant does not contribute to $\underline{C}$, ,since $\underline{V}_1$ is periodic in t. We obtain

$$(A.2)\quad -\rho_0\overline{\underline{V}_1\times\nabla\times\underline{V}_1} = \underline{\lambda}\,\overline{\left[-g\frac{\rho_0}{T_0}\nabla T_1\cdot\int_0^t\underline{V}_1 + \frac{T_0'}{T_0}\underline{V}_1\cdot\int_0^t\nabla p_1\right]} +$$

$$+\overline{\left[g\frac{\rho_0}{T_0}\nabla T_1 + \frac{T_0'}{T_0}\nabla p_1\right]\int_0^t w_1} + \overline{\rho_0\Big(\int_0^t\underline{V}_1\Big)\times\nabla\times\frac{\underline{F}_1^V}{\rho_0}} .$$

Integrating (2.14) with respect to t and multiplying by ∇p_1 we get in the same way

$$(A.3)\quad \overline{\underline{V}_1\cdot\int_0^t\nabla p_1} = -\rho_0\overline{V_1^2} + \overline{g\rho_1\int_0^t w_1} - \overline{\underline{F}_1^V\cdot\int_0^t\underline{V}_1} .$$

Elimination of $\nabla\cdot\underline{V}_1$ between (2.15) and (2.16), and integration with respect to t gives

$$(A.4)\quad \frac{c_0^2\rho_0^2N^2}{g}\int_0^t w_1 = \rho_0(c_0^2\rho_1 - p_1) + \rho_0\int_0^t(\Phi_1^T + \Phi_1^R) + \text{const.}$$

Using (A.2)-(A.4) to transform (A.1) we obtain finally

$$(A.5)\quad \underline{C} = p_0\nabla\Big(\overline{\frac{\rho_0 V_1^2}{2p_0}}\Big) + \underline{A} +$$

$$-\underline{\lambda}\,\overline{\left[g\frac{\rho_0}{T_0}\nabla T_1\cdot\int_0^t\underline{V}_1 + \rho_0\frac{T_0'}{T_0}V_1^2 + g\frac{\rho_1 p_1}{p_0} + \frac{g^2}{\rho_0 c_0^2 N^2}\frac{T_0'}{T_0}\rho_1(p_1 - c_0^2\rho_1)\right]} +$$

$$+ \overline{\frac{g}{\rho_0 c_0^2 N^2}\left[g\frac{\rho_0}{T_0}\nabla T_1 + \frac{T_0'}{T_0}\nabla p_1 + \underset{\sim}{\lambda}\, g\frac{T_0'}{T_0}\rho_1\right]\int_0^t(\phi_1^T + \phi_1^R)} \; +$$

$$+ \overline{\frac{\rho_1}{\rho_0}\underset{-}{F}_1^V} - \underset{\sim}{\lambda}\,\frac{T_0'}{T_0}\,\overline{\underset{-}{F}_1^V\cdot\int_0^t \underset{-}{V}_1} + \overline{\rho_0\left(\int_0^t \underset{-}{V}_1\right)\times\nabla\times\frac{\underset{-}{F}_1^V}{\rho_0}} \; ,$$

where $\underset{-}{A}$ is defined by

$$\underset{-}{A} = -\overline{\frac{\rho_1}{\rho_0}\nabla p_1} + \frac{g}{\rho_0 c_0^2 N^2}\overline{(c_0^2\rho_1 - p_1)\left(g\frac{\rho_0}{T_0}\nabla T_1 - \frac{T_0'}{T_0}\nabla p_1\right)} .$$

Using (2.17) we also have

$$\text{(A.6)} \quad \underset{-}{A} = -p_0\nabla\overline{\left[\frac{g^2(p_1 - c_0^2\rho_1)^2}{2p_0\rho_0 c_0^4 N^2} + \frac{p_1^2}{2\gamma p_0^2}\right]} \; +$$

$$+ \underset{\sim}{\lambda}\,\overline{\left[\frac{g^2}{c_0^2 N^2}\frac{T_0'}{T_0}(p_1 - c_0^2\rho_1)\frac{\rho_1}{\rho_0} + \frac{g}{p_0}p_1\rho_1 + \frac{T_0''}{T_0}\frac{g^3}{2\rho_0 c_0^4 N^4}(p_1 - c_0^2\rho_1)^2\right]} .$$

Substituting this expression for $\underset{-}{A}$ in (A.5) we obtain (4.10).

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