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On the mass transport induced by time-dependent oscillations of finite amplitude in a nonhomogeneous fluid

II General results for a liquid

by

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Report No. 44 August 1973

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Abstract

A general theory for the mass transport sustained by oscillations of finite amplitude in a stratified fluid in the field of gravity is presented. The mass transport is given by the Lagrangian mean velocity, calculated to the second order in the oscillatory perturbation. We compute the vertical component of this mean velocity, the divergence, and the vertical component of the vorticity. We assume an arbitrary equation of state, and take into account the effects of viscosity and thermal conductivity. The theory is applied to standing waves. The results seem to provide a qualitative explanation of experiments by Schaaffs and Haun [1] and others.

1. Introduction.

Schaaffs and Haun [1] have reported some observations with standing ultrasonic waves in a liquid in which there is a solution with a gradient (produced by diffusion) in the concentration. The waves are in vertical direction, i.e., along the concentration gradient. Periodic density stratifications (concentration zones) are found, the scale of which is the half wavelength, and they persist for a rather long time after the sound source has been switched off.

Hobæk [2] has observed similar concentration zones when the waves are in horizontal direction, perpendicular to the concentration gradient.

Schaaffs [3] has also observed periodic density stratifications, after the sound source is switched off, for the case of a homogeneous (one-component) fluid, and in mixtures with no concentration gradient. In these cases the effect was observed only when a temperature gradient was superimposed the wave motion.

Formation of bands of red cells in the blood vessels during ultrasonication has been observed by Dyson, Woodward and Pond [4], and has been associated to the standing wave by Vashon Baker [5], who has observed segregation and sedimentation of the cells in a standing ultrasonic wave.

All these observations indicate that mass transport is induced by the oscillations. The purpose of this paper is to present a general second order theory for mass transport

sustained by finite amplitude oscillations in a stratified fluid in the field of gravity. The mass transport is given by the Lagrangian mean velocity, calculated to the second order in the oscillatory perturbation. We compute the vertical component of this velocity, the divergence, and the vertical component of the vorticity, thereby determining the flow field. The theory is here worked out for a one-component liquid. We assume an arbitrary equation of state, and take into account the effects of viscosity and thermal conductivity. In a previous paper [6], we considered the case of a perfect gas, where also the radiative effects were accounted for. The motivation was there the possible applications in astrophysics and geophysics (upper atmosphere).

Applied to standing waves, the theory predicts a vertical drift within the wave zone, and a streaming system, the scale of which (in the wave direction) is the half wavelength. For a vertical directed beam the drift is always upward, and the flow outside the beam occurs in horizontal planes. A numerical example shows that the flow effect should be large enough to be observable in ultrasonic waves. One may expect that concentration zones are formed for instance in stagnation regions of the streaming. This seems to give a qualitative explanation of some features in the experiments referred to above.

2. General theory.

The basic equations are:

(1) $$\rho \frac{D\underline{V}}{Dt} + \nabla p + \underline{\lambda} g \rho = \underline{F}$$

(2) $$\frac{D\rho}{Dt} + \rho \nabla.\underline{V} = 0$$

(3) $$\frac{Ds}{Dt} = \varphi^V + \varphi^T ,$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{V}.\nabla$. The variables $\underline{V}, p, \rho, T, s$ are respectively the usual Eulerian velocity, pressure, density, temperature and entropy. $\underline{\lambda}$ is the unit vector in upward direction, g the gravitational acceleration; $\underline{F}$ is the viscous force, $\rho T \varphi^V$ the viscous dissipation, and $\rho T \varphi^T$ the accession of heat due to conduction. On has:

(4) $$\underline{F} = \nabla\left[\left(\frac{\mu}{3} + \mu_B\right)\nabla.\underline{V} + \underline{V}.\nabla\mu\right] + \mu\nabla^2\underline{V} - \underline{V}\nabla^2\mu + \nabla\times(\underline{V}\times\nabla\mu)$$

(5) $$\rho T \varphi^V = \mu\left[\nabla\underline{V}:\nabla\underline{V} + \nabla\underline{V}:(\nabla\underline{V})^* - \frac{2}{3}(\nabla.\underline{V})^2\right] + \mu_B(\nabla.\underline{V})^2$$

(6) $$\rho T \varphi^T = \nabla.(\sigma\nabla T) ,$$

where μ, μ_B are the two viscosity coefficients and σ the thermal conductivity. These coefficients are in general functions of ρ, T .

p, s, ρ, T are related by the following thermodynamical relations:

(7) $$dp = \frac{\beta c^2 \rho}{\gamma} dT + \frac{c^2}{\gamma} d\rho$$

(8) $$ds = \frac{c_v}{T} dT - \frac{\beta c^2}{\gamma \rho} d\rho \,.$$

Here β is the coefficient of thermal expansion, c the sound speed, γ the ratio of the specific heats at constant pressure and volume:

(9) $$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p , \quad c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s , \quad \gamma = \frac{c_p}{c_v} \,.$$

β, c, γ, c_p and c_v depend in general on ρ and T .

The following identities are consequences of (7)-(8) and the definitions (9):

(10) $$(\gamma - 1) c_p = \beta^2 c^2 T$$

(11) $$\frac{\partial}{\partial \rho}\left(\frac{\beta c^2 \rho}{\gamma}\right)_T = \frac{\partial}{\partial T}\left(\frac{c^2}{\gamma}\right)_\rho , \quad \frac{\partial}{\partial \rho}\left(\frac{c_v}{T}\right)_T = -\frac{\partial}{\partial T}\left(\frac{\beta c^2}{\gamma \rho}\right)_\rho \,.$$

The <u>state of equilibrium</u> is characterized by

$$\underline{V} = \underline{V}_0 = 0$$
$$\rho = \rho_0(z)$$
$$T = T_0(z)$$

where z denotes the altitude. It follows from (1), (3),

(7)-(8) that the other variables depend only on z and are such that

$$(12) \qquad p_0' = -g\rho_0 \quad , \quad (\sigma_0 T_0')' = 0$$

$$(13) \qquad \rho_0' = -\frac{\gamma_0 g \rho_0}{c_0^2} - \beta_0 \rho_0 T_0'$$

$$(14) \qquad s_0' = g\beta_0 + c_{p0}\frac{T_0'}{T_0} .$$

Subscript zero refers to equilibrium value, prime denotes the derivative with respect to z .

The special case of <u>isothermal equilibrium,</u> $T_0' = 0$, leads to $\rho_0'/\rho_0 = -1/H$, where H is the scale height defined by $H = c_0^2/\gamma_0 g$. If we suppose that H is independent of z , we obtain the exponential decay in density and pressure, $\rho_0 = \rho_s \exp(-z/H)$, $p_0 = Hg\rho_s[\exp(-z/H) - 1] + p_s$, where p_s , ρ_s are the values of p_0 , ρ_0 at $z = 0$.

We suppose that $\underline{V}$, p , ρ , T , s can be developed in powers of the oscillatory perturbation (or the acoustic Mach number) at least up to the second order

$$\underline{V} = \underline{V}_1 + \underline{V}_2 + \ldots$$
$$\rho = \rho_0 + \rho_1 + \rho_2 + \ldots \quad .$$
$$\vdots$$

Substituting these expansions in (1)-(3), we obtain to the first order

(15) $\rho_0 \dfrac{\partial \underline{V}_1}{\partial t} + \nabla p_1 + \underline{\lambda} g \rho_1 = \underline{F}_1$

(16) $\dfrac{\partial \rho_1}{\partial t} + \rho_0' w_1 + \rho_0 \nabla . \underline{V}_1 = 0$

(17) $\dfrac{\partial s_1}{\partial t} + s_0' w_1 = Q_1^T ,$

and to the second order

(18) $\rho_0 \dfrac{\partial \underline{V}_2}{\partial t} + \nabla p_2 + \underline{\lambda} g \rho_2 = \underline{F}_2 - \left[\rho_1 \dfrac{\partial \underline{V}_1}{\partial t} + \rho_0 \underline{V}_1 . \nabla \underline{V}_1 \right]$

(19) $\dfrac{\partial \rho_2}{\partial t} + \rho_0' w_2 + \rho_0 \nabla . \underline{V}_2 = - \nabla . (\rho_1 \underline{V}_1)$

(20) $\dfrac{\partial s_2}{\partial t} + s_0' w_2 = - \underline{V}_1 . \nabla s_1 + Q_2^V + Q_2^T .$

w_i denotes the vertical component of $\underline{V}_i$, $i = 1, 2$. The substitution of the same expansions in (7)-(8) gives to the first order

(21) $d p_1 = \dfrac{\beta_0 c_0^2 \rho_0}{\gamma_0} d T_1 + \dfrac{c_0^2}{\gamma_0} d\rho_1 + \left(\dfrac{\beta c^2 \rho}{\gamma} \right)_1 d T_0 + \left(\dfrac{c^2}{\gamma} \right)_1 d \rho_0 ,$

and a similar equation for $d s_1$. To the second order we obtain in the same way

(22) $d p_2 = \dfrac{\beta_0 c_0^2 \rho_0}{\gamma_0} d T_2 + \dfrac{c_0^2}{\gamma_0} d\rho_2 + \left(\dfrac{\beta c^2 \rho}{\gamma} \right)_1 d T_1 + \left(\dfrac{c^2}{\gamma} \right)_1 d\rho_1 + \left(\dfrac{\beta c^2 \rho}{\gamma} \right)_2 d T_0 + \left(\dfrac{c^2}{\gamma} \right)_2 d\rho_0$

and a similar equation for $d s_2$. Here a_i , $i = 1, 2$,

denotes the (i + 1)th term in the expansion of a in powers of the expansion parameter.

Let us now choose as first order solution a harmonic oscillatory motion, and take the average of the second order equations over one or several periods (the time of averaging must be small such that the mean drift during this time is small compared to the characteristic lengths). We assume that

$$(23) \qquad \overline{\frac{\partial \underline{V}_2}{\partial t}} = \overline{\frac{\partial \rho_2}{\partial t}} = \overline{\frac{\partial s_2}{\partial t}} = 0 ,$$

where bar denotes time-averaging.

This is a plausible assumption, due to the special form of the source terms in (18)-(20) (quadratic in the first order variables and their gradients), but to justify it, one has to define boundary conditions and prove that $\underline{V}_2$, p_2 and ρ_2 are non-secular in time. This problem will not be discussed here.

Further, we introduce the second order Lagrangian mean velocity,

$$(24) \qquad \overline{\underline{V}}_{L2} = \overline{\underline{V}}_2 + \overline{\left(\int_0^t \underline{V}_1\right) \cdot \nabla \underline{V}_1} \; .$$

From (19)-(20) we then obtain

$$(25) \qquad \nabla\cdot\left(\rho_0 \overline{\underline{V}_{L2}}\right) = \rho_0' \overline{w_{L2}} + \rho_0 \nabla\cdot\overline{\underline{V}_{L2}} = 0$$

$$(26) \qquad s_0' \overline{w_{L2}} = \overline{Q_2^T} + \overline{Q_2^V} + \overline{\left(\int_0^t \underline{V}_1\right)\cdot \nabla Q_1^T} .$$

For a two-dimensional flow, (25)-(26) determine the flow to an integration constant near, provided that $\overline{Q_2^T}$, i.e., $\overline{T_2}$, is known. The equation of motion (1) has not been used so far.

The three-dimensional flow is determined (to integration constants near) by $\overline{w_{L2}}$, $\nabla\cdot\overline{\underline{V}_{L2}}$ and the vertical component of the vorticity, $\overline{\Omega} = (\nabla\times\overline{\underline{V}_{L2}})_z$. From (15), (18) and (24) we derive

$$(27) \qquad \mu_0 \nabla^2 \overline{\Omega} + \mu_0' \frac{\partial}{\partial z}\overline{\Omega} = \underline{\lambda}\cdot\nabla_\perp \times \overline{\Big[\rho_1 \frac{\partial \underline{V}_{1\perp}}{\partial t} + \rho_0 \underline{V}_1\cdot\nabla\underline{V}_{1\perp} -} $$
$$\overline{- \nabla\cdot\left(\mu_1 \nabla \underline{V}_{1\perp} - \underline{V}_1 \nabla_\perp \mu_1 - \mu_0 \nabla\left(\left(\int_0^t \underline{V}_1\right)\cdot\nabla\underline{V}_{1\perp}\right)\right)\Big]} ,$$

where index $\perp$ denotes the horizontal component,

$$\underline{V} = \underline{V}_\perp + \underline{\lambda}\, w \quad , \quad \nabla = \nabla_\perp + \underline{\lambda}\frac{\partial}{\partial z} .$$

Remarks:

1. For the incompressible model, $\nabla\cdot\underline{V} = 0$, we have, [6], $\nabla\cdot\overline{\underline{V}_{L2}} = 0$, and it follows from (25) that $\overline{w_{L2}} = 0$.

Equation (26) then determines $\overline{T_2}$.

2. For the non-dissipative model, it follows from (25)-(26) that $\overline{w_{L2}} = \nabla \cdot \overline{\underline{V}_{L2}} = 0$.

We proceed now to the computation of $\overline{p_2}$, $\overline{\rho_2}$, $\overline{T_2}$. From (18) we obtain

$$(28) \qquad \nabla_\perp \overline{p_2} = - \underline{C}_\perp + \overline{F}_{2\perp} ,$$

where $\underline{C}$ is defined by

$$(29) \quad \underline{C} = \overline{\rho_1 \frac{\partial \underline{V}_1}{\partial t} + \rho_0 \underline{V}_1 \cdot \nabla \underline{V}_1} - \underline{\lambda} \frac{g\gamma_0}{c_0^2} \overline{\left[\frac{\partial}{\partial \rho}\left(\frac{c^2}{\gamma}\right)_{T,0} \frac{\rho_1^2}{2} + \frac{\partial}{\partial T}\left(\frac{c^2}{\gamma}\right)_{\rho,0} \rho_1 T_1 + \frac{\partial}{\partial T}\left(\frac{\beta \rho c^2}{\gamma}\right)_{\rho,0} \frac{T_1^2}{2} \right]}$$

The notation is the following: $\frac{\partial}{\partial \rho}(\cdot)_{T,0}$ (resp. $\frac{\partial}{\partial T}(\cdot)_{\rho,0}$) denotes the derivative with respect to ρ (resp. T) at constant T (resp. ρ), taken at $T = T_0$, $\rho = \rho_0$.

(28) gives $\overline{p_2}$ to an arbitrary function of z near. $\overline{p_2}$ being known, we obtain $\overline{\rho_2}$ by averaging the vertical component of (18):

$$(30) \qquad \overline{\rho_2} = \frac{1}{g}\left[- \frac{\partial \overline{p_2}}{\partial z} + \overline{F}_{2z} - \overline{\left(\rho_1 \frac{\partial w_1}{\partial t} + \rho_0 \underline{V}_1 \cdot \nabla w_1 \right)} \right] .$$

From (22) we obtain a relation between $\nabla_\perp \overline{T_2}$, $\nabla_\perp \overline{p_2}$, $\nabla_\perp \overline{\rho_2}$. Substituting the expressions (28) and (30) for $\nabla_\perp \overline{p_2}$ and $\overline{\rho_2}$ in this relation, we find

$$(31)\quad \nabla_\perp \overline{T}_2 = \frac{b_0}{g\beta_0\rho_0}\left[\nabla_\perp\left(\frac{C_z - \overline{F}_{2z}}{b_0}\right) - \frac{\partial}{\partial z}\left(\frac{C_\perp - \overline{F}_{2\perp}}{b_0}\right)\right],$$

which determines $\overline{T}_2$ to an arbitrary function of z near. Here b_0 is defined by $(\ln b_0)' = -g\gamma_0/c_0^2$ (for a perfect gas, $b_0 = p_0$).

By using the first order equations (15)-(17), (21) and the similar equation for ds_1, we compute $\underline{C}$, see appendix. We find that $\underline{C}$ is the average of quadratic forms of the first order quantities and their gradients. The coefficients of these forms depend on ρ_0, T_0, the thermodynamical coefficients (9), and their first order derivatives with respect to ρ and T taken at equilibrium.

Inserting the result for $\underline{C}$ into (28), we obtain

$$(32)\quad \overline{p_2} = \left(-\rho_0\frac{\overline{V_1^2}}{2} + \frac{\overline{p_1^2}}{2\rho_0 c_0^2} + \frac{g\rho_0\beta_0 T_0}{2s_0' c_{p0}}\overline{s_1^2}\right)\left(1 + O(S_M)\right) + \Pi(z),$$

where S_M is the modified Stokes number

$$(33)\quad S_M = \frac{3\delta|k|^2}{4\omega}, \quad \delta = \frac{1}{\rho_0}\left[\frac{\gamma_0 - 1}{c_{p0}}\sigma_0 + \frac{4}{3}\mu_0 + \mu_{B0}\right].$$

ω is the frequency of the given oscillation, $\underline{k}$ the complex wave number. (Indeed, we suppose here that μ, μ_B, σ vary little over a wavelenght, i.e., that $\frac{T}{\mu}\frac{\partial\mu}{\partial T}$ and other similar terms are of order one. If this was not the case, one should have to take such terms into account when defining δ). Π is an arbitrary function. For a procedure to obtain the general expressions for $\overline{\rho_2}$, $\overline{T_2}$, see appendix.

For the special case of <u>isothermal equilibrium</u>, we find

$$(34) \qquad \overline{T}_2 = -\overline{\left(\int_0^t \underline{V}_1\right)\cdot\nabla T_1}\left(1 + O(S_M)\right) + \theta(z)$$

$$(35) \qquad \overline{\rho}_2 = \Big\{ \beta_0\rho_0\overline{\left(\int_0^t \underline{V}_1\right)\cdot\nabla T_1} + \frac{\gamma_0\rho_0}{2c_0^2}\Big[-\overline{V_1^2} - \rho_0^2 T_0 \frac{\partial}{\partial\rho}\left(\frac{1}{\rho c_p}\right)_{T,0}\overline{s_1^2} + \frac{\partial}{\partial\rho}\left(\frac{1}{\rho c^2}\right)_{T,0}\overline{p_1^2} -$$

$$- 2\rho_0 T_0 \frac{\partial}{\partial\rho}\left(\frac{\beta}{\rho c_p}\right)_{T,0}\overline{p_1 s_1}\Big] - \frac{1}{g}\Pi'(z)\Big\}\left(1 + O(S_M)\right),$$

where Π an arbitrary function and $\theta = \gamma_0\left(\Pi + c_0^2\Pi'/\gamma_0 g\right)/\beta_0 c_0^2\rho_0 +$ $+$const. .

We have here supposed that S_M gives the order of magnitude of the dissipative terms (which may not be the case in boundary layers).

Inserting the result for $\underline{C}$ into (27), we obtain an equation for the vertical component of the vorticity

$$(36) \quad \mu_0\nabla^2\overline{\Omega} + \mu_0'\frac{\partial}{\partial z}\overline{\Omega} = \underline{\lambda}\cdot\nabla\times\Big[\overline{\frac{\rho_1}{\rho_0}\underline{F}_{1\perp}} - \rho_0\left(\overline{\underline{V}_1\times\nabla\times\int_0^t\frac{\underline{F}_1}{\rho_0}}\right)_\perp -$$

$$- \frac{\beta_0}{s_0'}\overline{\left(g\rho_0\nabla_\perp T_1 + T_0'\nabla_\perp p_1\right)\int_0^t \varphi_1^T} -$$

$$- \nabla\cdot\Big\{\overline{\mu_1\nabla_\perp\underline{V}_{1\perp} - \underline{V}_{1\perp}\nabla_\perp\mu_1 - \mu_0\nabla\left(\left(\int_0^t\underline{V}_1\right)\cdot\nabla\underline{V}_{1\perp}\right)}\Big\}\Big].$$

Substituting in the source terms of (36) the solution of

the non-dissipative first order equations, we obtain

$$(37)\quad \nabla^2 \bar{\Omega} + \frac{\mu_0'}{\mu_0}\frac{\partial}{\partial z}\bar{\Omega} = \underline{\lambda}\cdot\nabla_\perp \times \overline{\Big[-\Big(\int_0^t \nabla\cdot \underline{V}_1\Big)\nabla^2 \underline{V}_{1\perp} + \Big(\Big(\int_0^t \underline{V}_1\Big)\times\nabla^2\nabla\times\underline{V}_1\Big)_\perp + \nabla^2\Big(\Big(\int_0^t \underline{V}_1\Big)\cdot\nabla \underline{V}_{1\perp}}$$

$$- \Big(\frac{1}{3} + \frac{\mu_{B0}}{\mu_0}\Big)\underline{\lambda}\cdot\nabla_\perp \times \overline{\Big[\Big(\int_0^t \nabla\cdot\underline{V}_1\Big)\nabla_\perp \nabla\cdot\underline{V}_1\Big]} -$$

$$- \frac{\sigma_0 \beta_0}{\mu_0 s_0' \rho_0 T_0}\,\underline{\lambda}\cdot\nabla_\perp \times \overline{\Big[\Big(\int_0^t \nabla^2 T_1\Big)\big(g\rho_0 \nabla_\perp T_1 + T_0' \nabla_\perp p_1\big)\Big]} +$$

$$+ \underline{\lambda}\cdot\nabla_\perp \times \underline{A}_\perp$$

where $\underline{A}_\perp$ is the average of quadratic forms of the first order variables with coefficients depending linearly on $\frac{1}{\mu_0}\left(\frac{\partial\sigma}{\partial T}\right)_{\rho,0}$, $\frac{1}{\mu_0}\left(\frac{\partial\sigma}{\partial\rho}\right)_{T,0}$, $\frac{1}{\mu_0}\left(\frac{\partial\mu}{\partial T}\right)_{\rho,0}$ and $\frac{1}{\mu_0}\left(\frac{\partial\mu}{\partial\rho}\right)_{T,0}$. The general expression, which is rather long, will not be given here. If the equilibrium is isothermal and σ, μ depend only on T , $\underline{A}_\perp$ is reduced to

$$(38)\quad \underline{A}_\perp = \frac{1}{\mu_0}\left(\frac{\partial\mu}{\partial T}\right)_0 \overline{\nabla\cdot\big(\underline{V}_1 \nabla_\perp T_1 - T_1 \nabla \underline{V}_{1\perp}\big)}\,.$$

The vorticity equation contains source terms which are proportional to $\sigma_0 / c_{p0}\mu_0$, μ_{B0}/μ_0 and terms which are independent of the dissipative coefficients. If one of these terms is non-zero (which normally is the case for a laterally confined, horizontally propagating wave), the vertical component of the vorticity will be of zeroth order in S_M . The mass circulation in the horizontal plane will then dominate over the vertical drift, which is of the first order in S_M .

3. Standing waves.

Let us consider a flow in the (x, z)-plane, where z is the altitude and x the horizontal coordinate. The mass transport velocity is obtained from (25)-(26). If $\bar{u}_{L2}$ denotes the horizontal component of $\overline{\underline{V}}_{L2}$, we obtain

$$(39) \qquad \bar{w}_{L2} = \frac{g(\gamma_0 - 1)}{\beta_0 c_0^2 N^2} \overline{\left(Q_2^T + Q_2^V + \left(\int_0^t \underline{V}_1 \right) \cdot \nabla Q_1^T \right)}$$

$$(40) \qquad \bar{u}_{L2} = -\frac{1}{\rho_0} \int_0^x \frac{\partial}{\partial z} \left(\rho_0 \bar{w}_{L2} \right) dx + h(z),$$

where h is an arbitrary function, and N is the Väisälä frequency,

$$(41) \qquad N^2 = \frac{(\gamma_0 - 1) g}{\beta_0 c_0^2} s_0' .$$

To our order of approximation, where terms $O(S_M)$ have been neglected, we can now in (39) insert the wavefield in the non-dissipative approximation. First, we assume a beam in vertical direction, and put (in the non-dissipative approximation)

$$(42) \qquad \underline{V}_1 = F_A \, \underline{\lambda} \, W(z) \cos \omega t ,$$

where F_A is a non-zero constant A inside the beam and zero

outside. W satisfies

$$\frac{d}{dz}\left(\rho_0 c_0^2 \frac{dW}{dz}\right) + \rho_0 \omega^2 W = 0 . \tag{43}$$

We now assume that the equilibrium is isothermal and c_0 is constant. Then

$$W(z) = e^{z/2H} \sin kz \ , \quad k^2 + \frac{1}{4H^2} = \frac{\omega^2}{c_0^2} \tag{44}$$

is a solution of (43). With this, (8), (16), (17) and the expression (34) for $\overline{T_2}$, we determine the flow field given by (39) and (40), assuming that σ_0 is constant. A solution with $\overline{w_{L2}} = 0$ outside the beam, and $\overline{u_{L2}}$ symmetric in x, is obtained by putting the arbitrary functions in (34) and (40) equal to zero. Assuming for simplicity that β_0, γ_0 are constant and that σ, μ, μ_B are independent of ρ, T, we obtain in the limit $kH \gg 1$

$$\overline{w_{L2}}(x,z) \simeq \frac{F_A^2 k^2}{4 g \rho_s T_0 \beta_0} e^{2z/H} \left\{ \left(\frac{4}{3}\mu_0 + \mu_{B0}\right)(1 + \cos 2kz) + \right.$$
$$\left. + \frac{\sigma_0 \beta_0 T_0}{c_{p0}} \left[2 + \frac{\gamma_0 - 1}{\beta_0 T_0} (1 + \cos 2kz) \right] \right\} \tag{45}$$

$$(46)\quad \bar{u}_{L2}(x,z) \simeq \begin{cases} -x \dfrac{\partial \bar{w}_{L2}}{\partial z}(0,z) & \text{inside the beam} \\ -D \dfrac{\partial \bar{w}_{L2}}{\partial z}(0,z) & \text{outside the beam ,} \end{cases}$$

as $N^2 = (\gamma_0 - 1)\, g^2/c_0^2$ for isothermal equilibrium. $2D$ is the width of the beam. $\bar{w}_{L2}$ is positive inside the beam, and the flow picture is as shown in Figure 1.

To indicate the order of magnitude of the flow velocity, we choose water at about 20^0 C. Then $\rho_0 \simeq 1$ g/cm^3, $\beta_0 \simeq 2\times 10^{-4}$ 1/deg , $c_0 \simeq 1.5\times 10^5$ cm/sec, $\gamma_0 \simeq 1.006$, $(4/3)\mu_0 + \mu_{B0} \simeq 0.04$ g/cm sec and $\sigma_0 / c_{p0} \simeq \mu_0/\gamma_0 Pr$, where Pr is the Prandtl number, $Pr \simeq 6.75$. The terms containing σ_0 can here be neglected. With $g \simeq 980$ cm/sec^2 , $\omega = 2\pi \times 10^6$ 1/sec (i.e., $\lambda \simeq 0.15$ cm) and velocity amplitude 0.1 cm/sec, we then find as maximum value for $\bar{w}_{L2}$ 6×10^{-3} cm/sec, and for $\bar{u}_{L2}$ 10^{-1} cm/sec, with $D = 0.5$ cm. This shows that the flow should be observable at this high frequency for intensities in the mW region.

The stagnation regions of the flow are defined by $\cos 2kz = -1$, $\sin 2kz = 0$, i.e., $z = n\lambda/4$ ($n = 1, 3, 5, \ldots$). This may provide a qualitative explanation of the observations by Schaaffs and Haun [1]. The zones of higher concentrations and density in their experiments are probably formed in the stagnation regions of a flow system of this kind induced by the oscillations. However, the theory is here

worked out for a one-component fluid only and one should not stress too far this comparison with observations in miscible liquids. Also the plane wave model has its limitation, and a more realistic model for the oscillatory field may modify the details of this flow picture.

Schaaffs and Haun [1] explain the observed phenomena as caused by the fluctuating temperature gradient in the standing wave. Their estimate of the temperature fluctuation, however, seems to give a too high value. The general thermodynamic relation

$$\frac{T}{c_v}\frac{Ds}{Dt} = \frac{DT}{Dt} - \frac{\gamma-1}{\beta\rho}\frac{D\rho}{Dt}$$

gives for an adiabatic fluctuation, with $T_0' = 0$ and $kH >> 1$

$$T_1 \simeq \frac{\gamma_0 - 1}{\beta_0\rho_0}\rho_1 \simeq \frac{\gamma_0 - 1}{\beta_0}\frac{w_1}{c_0} .$$

If we use this formula, we find $T_1 \simeq 2 \times 10^{-5}$deg with the data above and $\gamma_0 = 1.006$. Even if the intensity is increased considerably, this seems to be too much below the order of magnitude required to produce the effect discussed in [1].

The general theory can be applied to non-isothermal equilibrium. Thus a similar flow probably occurs in the experiments by Schaaffs [3].

In order to compare the theoretical results with the other observations referred to in the introduction, we now consider the case with a <u>horizontally directed plane wave.</u>

We still suppose $T_0' = 0$, c_0 = constant. We have for the wave

$$(47) \quad \begin{cases} u_1 = B\, e^{(\gamma_0-1)gz/c_0^2} \cos kx \cos \omega t \ , & k = \dfrac{\omega}{c_0} \\ w_1 = 0 \ , & \end{cases}$$

where B is a constant and u_1 is the horizontal component of $\underline{V}_1$.

With this, and the expression (34) for $\overline{T}_2$, we find $\overline{w}_{L2}$ to the arbitrary function $\theta(z)$ near. This function is now determined such that the secularity in x , which occurs in the integral (40) is removed. We find

$$(48) \quad \theta'' = -\frac{B^2}{2} e^{2(\gamma_0-1)gz/c_0^2} \left\{ \frac{\mu_0}{\sigma_0}\left(\frac{4}{3}k^2 + \left(\frac{\gamma_0-1}{c_0^2}g\right)^2\right) + \frac{\mu_{B0}}{\sigma_0}k^2 - \frac{\beta_0 T_0}{c_{p0}}\left(\left(\frac{\gamma_0-1}{c_0^2}g\right)^2 - k^2\right)\left(3 + \frac{\gamma_0-1}{\beta_0 T_0}\right)\right\} + \text{const.} \ ,$$

which again leads to the following solution

$$(49) \quad \overline{w}_{L2} \simeq -\frac{B^2}{4g\beta_0 T_0 \rho_A} e^{(3\gamma_0-2)gz/c_0^2} \left\{ \left(\frac{4}{3}\mu_0 + \mu_{B0}\right)k^2 - \mu_0\left(\frac{\gamma_0-1}{c_0^2}g\right)^2 + \left[\left(\frac{\gamma_0-1}{c_0^2}g\right)^2 - k^2\right]\left(2 - \frac{\gamma_0-1}{\beta_0 T_0}\right)\frac{\beta_0 T_0}{c_{p0}}\sigma_0 \right\} \cos 2kx \ + \ \text{const.}\, e^{\gamma_0 g z/c_0^2}$$

$$\overline{u}_{L2} \simeq \frac{B^2}{4 g \beta_0 T_0 \rho_s} \frac{(\gamma_0 - 1) g}{k c_0^2} e^{(3\gamma_0 - 2) g z / c_0^2} \left\{ \left(\tfrac{4}{3}\mu_0 + \mu_{B0} \right) k^2 - \mu_0 \left(\frac{\gamma_0 - 1}{c_0^2} g \right)^2 + \right.$$

(50)

$$\left. + \left[\left(\frac{\gamma_0 - 1}{c_0^2} g \right)^2 - k^2 \right] \left(2 - \frac{\gamma_0 - 1}{\beta_0 T_0} \right) \frac{\beta_0 T_0}{c_{p0}} \sigma_0 \right\} \sin 2kx + h(z),$$

where the last term in $\overline{w}_{L2}$ can be put equal to zero. The arbitrary function h in (40) is zero due to symmetry requirements.

The viscous terms dominate also here for the case of a liquid. Further, $\overline{w}_{L2} = 0$ for $x = n\lambda/4\pi$ ($n = 1, 3, 5, \ldots$) and $|\overline{u}_{L2}| \ll |\overline{w}_{L2}|$. The flow is as shown in Figure 2. It explains, we believe, qualitatively some features of the observations by Hobæk [2], Dyson et al. [4], Vashon Baker [5].

4. Validity of the method.

The amplitude of the oscillation is assumed to be small compared to the characteristic lengths (wavelength, dimensions, scale height). Further, it is presumed that the Reynolds number $\overline{R}$ of the steady flow is small compared to unity. It gives an estimate of the order of magnitude of the neglected term $\overline{\underline{V}}_{L2} \cdot \nabla \overline{\underline{V}}_{L2}$ compared to the dominating dissipative terms, $(\mu/\rho) \nabla^2 \overline{\underline{V}}_{L2}$ or $\delta \nabla \nabla \cdot \overline{\underline{V}}_{L2}$. Having obtained the flow velocity, we may estimate $\overline{R}$. In our models with standing waves, a proper definition of $\overline{R}$ is $\overline{R} = |\overline{w}_{L2}| / k\delta$. It is verified that $\overline{R} \ll 1$ in the numerical example given above.

5. Concluding remarks.

Another possible field of application of the theory, is the studies of the mean ocean currents; for example the Cromwell current, which flows eastward beneath the surface in a narrow zone along the equator. Munk and Moore [7] have suggested that this current is driven by equatorial Rossby waves. Their model, however, being non-dissipative, leads to zero Lagrangian mean velocity and thus no mass transport. This has always been pointed out by Moore [8]. Including dissipation, our theory shows that internal waves may sustain a mean current in the ocean. It also predicts a smaller vertical drift if compressibility is taken into account.

The perfect gas model discussed previously [6], may be applied to study flow problems in astrophysics, and in the terrestrial atmosphere. For instance, it is known that guided acoustic-gravity waves in the upper atmosphere are influenced by viscous and thermal dissipation (see Francis [9]). As also non-linearity is likely to become important at high altitude, such waves may induce horizontal mass transport over long distances and sustain a vertical drift. The streaming Reynolds number will here stay small due to extremely high values of the kinematic viscosity and the diffusivity. The method may thus be used to calculate relatively large flow velocities.

In the solar atmosphere, where radiative loss is important, similar phenomenon may occur. The theory provides a mechanism,

we believe, for relating the observed steady velocity fields in the upper photosphere and lower chromosphere, to the oscillatory motion which is known to exist (for application of the theory to a simple model in this field, see Naze Tjøtta and Tjøtta [6]).

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Appendix.

From (7) - (8) we obtain

(A.1) $$p_1 = \frac{\beta_0 c_0^2 \rho_0}{\gamma_0} T_1 + \frac{c_0^2}{\gamma_0} \rho_1 \quad , \quad s_1 = \frac{c_{v0}}{T_0} T_1 - \frac{\beta_0 c_0^2}{\gamma_0 \rho_0} \rho_1 \; .$$

Taking the curl of (15) and using (16)-(17) and (A.1) we obtain

(A.2) $$\frac{\partial}{\partial t} \nabla \times \underline{V}_1 = \frac{\beta_0 c_0^2}{\gamma_0 \rho_0} \underline{\lambda} \times \nabla_{\perp} (\rho_0' T_1 - T_0' \rho_1) + \nabla \times \frac{\underline{F}_1}{\rho_0}$$

or

(A.3) $$\frac{\partial^2}{\partial t^2} \nabla \times \underline{V}_1 = \frac{\beta_0 c_0^2 T_0}{c_{p_0}} s_0' \underline{\lambda} \times \nabla_{\perp} \nabla \cdot \underline{V}_1 + \frac{\rho_0' \beta_0 c_0^2 T_0}{\rho_0 c_{p0}} Q_1^T + \nabla \times \frac{\partial}{\partial t} \frac{\underline{F}_1}{\rho_0} \; .$$

Using (A.1), (A.2) and (13) - (15) we compute $\overline{\rho_1 \frac{\partial \underline{V}_1}{\partial t} + \rho_0 \underline{V}_1 \cdot \nabla \underline{V}_1}$ and we obtain for $\underline{C}$:

(A.4)
$$
\begin{aligned}
\underline{C} = \sigma_0 \nabla \Bigg[\frac{1}{\sigma_0} \Bigg(& \frac{\rho_0 \overline{\underline{V}_1^2}}{2} - \frac{\overline{p_1^2}}{2\rho_0 c_0^2} - \frac{g\rho_0\beta_0 T_0 \overline{s_1^2}}{2 c_{p0} s_0'} \Bigg) \Bigg] + \\
& + \underline{\lambda} \Bigg[-g\beta_0\rho_0 \overline{\Big(\int_0^t \underline{V}_1\Big)\cdot \nabla T_1} - T_0'\rho_0\beta_0 \frac{\overline{\underline{V}_1^2}}{2} - \frac{T_0'}{2}\rho_0 c_0^4 \Big(\frac{\beta_0}{c_0^2} + \frac{\rho_0^2 c_{p0}}{T_0} \Big(\frac{\partial^2 T}{\partial p^2}\Big)_{s,0} \Big) \overline{\Big(\int_0^t \nabla\cdot \underline{V}_1 \\
& - T_0'' \frac{g\rho_0\beta_0 \overline{s_1^2}}{2 s_0'^2} \Bigg] + \Bigg[\overline{(\nabla\cdot\underline{V}_1)\int_0^t \underline{F}_1 - \underline{\lambda}\frac{g\gamma_0}{c_0^2}\underline{V}_1\cdot\int_0^t \underline{F}_1 - \underline{V}_1 \times \nabla \times \int_0^t \underline{F}_1} \Bigg] + \\
& + \beta_0 \overline{\Big[g\rho_0 \nabla T_1 + T_0' \nabla p_1 + \underline{\lambda} g T_0' \rho_1 \Big] \int_0^t \frac{\varphi_1^T}{s_0'}} - \\
& - \underline{\lambda} \frac{T_0' \rho_0' c_0^4}{2\rho_0} \Big(\frac{\beta_0}{c_0^2} + \frac{\rho_0^2 c_{p0}}{T_0} \Big(\frac{\partial^2 T}{\partial p^2}\Big)_{s,0} \Big) \overline{\Big[2\rho_0 \int_0^t \nabla\cdot\underline{V}_1 + \rho_0' \int_0^t \frac{\varphi_1^T}{s_0'} \Big] \int_0^t \frac{\varphi_1^T}{s_0'}} .
\end{aligned}
$$

Here σ_0 is defined by $(\ln \sigma_0)' = -g\frac{\gamma_0}{c_0^2}$. General expressions for $\overline{p_2}$, $\overline{\rho_2}$, $\overline{T_2}$ are then obtained by inserting (A.4) in (28), (30) and (31).

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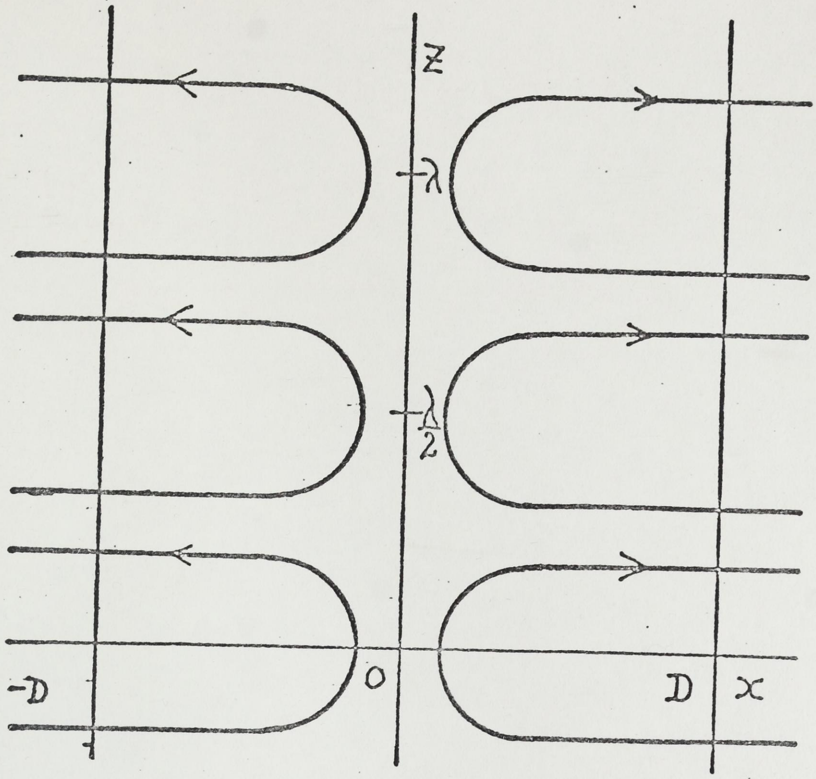

Figure 1.

Schematic diagram of the flow sustained by a standing wave in vertical direction.

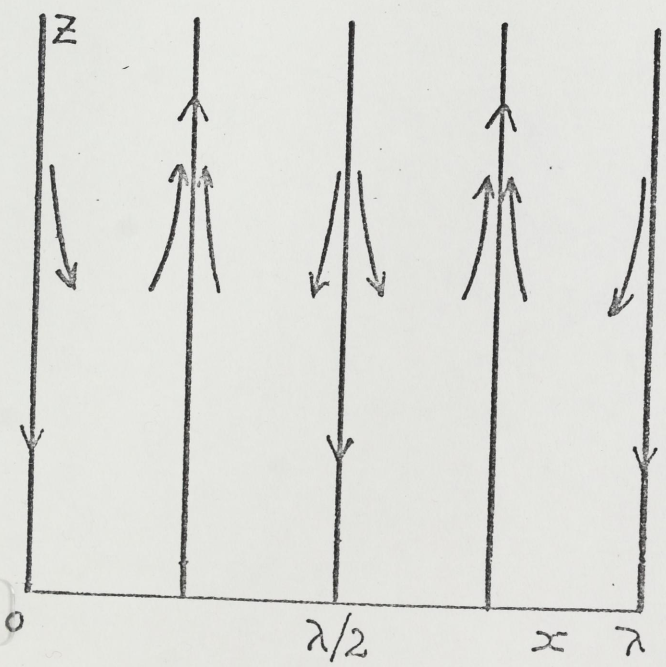

Figure 2.

Schematic diagram of the flow sustained by a standing wave in horizontal direction.