Department of APPLIED MATHEMATICS

On the Uniqueness and Stability of Entropy Solutions of nonlinear degenerate parabolic Equations with rough Coefficients

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KENNETH HVISTENDAHL KARLSEN AND NILS HENRIK RISEBRO

ABSTRACT. We study nonlinear degenerate parabolic equations where the flux function f(x, t, u) does not depend Lipschitz continuously on the spatial location x. By properly adapting the "doubling of variables" device due to Kružkov [23] and Carrillo [12], we prove a uniqueness result within the class of entropy solutions for the initial value problem. We also prove a result concerning the continuous dependence on the initial data and the flux function for degenerate parabolic equations with flux function of the form k(x)f(u), where k(x) is a vector-valued function and f(u) is a scalar function.

1. INTRODUCTION

The main subject of this paper is uniqueness and stability properties of entropy solutions of nonlinear degenerate parabolic equations where the flux function depends explicitly on the spatial location. In particular, this paper is concerned with the case where the flux function does not depend Lipschitz continuously on the spatial variable. Our study is motivated by applications where one frequently encounters flux functions possessing minimal smoothness in the spatial variable.

The problems that we study are initial value problems of the form

(1.1)
$$u_t + \operatorname{div} f(x, t, u) = \Delta A(u) + q(x, t, u), \quad (x, t) \in \Pi_T = \mathbb{R}^d \times (0, T), \\ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d.$$

where T > 0 is fixed, u(x,t) is the scalar unknown function that is sought, f = f(x,t,u) is called the flux function, A = A(u) the diffusion function, and q = (x,t,u) the source term. The coefficients f, A, q of problem (1.1) are given functions satisfying certain regularity assumptions. The regularity assumptions on f, q will be given later.

For the initial value problem (1.1) to be well-posed, we must require that $A : \mathbb{R} \to \mathbb{R}$ satisfies

(1.2)
$$A \in \operatorname{Lip}_{\operatorname{loc}}(\mathbb{R}) \text{ and } A(\cdot) \text{ is nondecreasing with } A(0) = 0.$$

Notice that (1.2) implies that the nonlinear operator $u \mapsto \Delta A(u)$ is of *degenerate elliptic* type, and hence many well known nonlinear and linear partial differential equations are special cases of (1.1). In particular, the scalar conservation law $(A' \equiv 0)$ is a "simple" special case. Included is also the heat equation, porous medium type equations characterized by one-point degeneracy, two-phase reservoir flow equations characterized by the two-point degeneracy, as well as *strongly* degenerate convection-diffusion equations where $A'(s) \equiv 0$ for all s in some interval $[\alpha, \beta]$. Consequently, partial differential equations of the type (1.1) model a wide variety of phenomena, ranging from porous media flow [31], via flow of glaciers [18] and sedimentation processes [9], to traffic flow [34].

We recall that if the problem (1.1) is non-degenerate (uniformly parabolic), it is well known that it admits a unique classical solution. This contrasts with the case where (1.1) is allowed to degenerate at certain points, that is, A'(s) = 0 for some values of s. Then solutions are not necessarily smooth (but typically continuous) and weak solutions must be sought. On the other hand, if A'(s) is zero on an interval $[\alpha, \beta]$, (weak) solutions may be discontinuous and are not uniquely determined by their initial data. Consequently, an entropy condition must be imposed to single out the physically correct solution.

Date: April 28, 2000.

Key words and phrases. degenerate parabolic equation, rough coefficient, entropy solution, uniqueness, stability.

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Roughly speaking, we call a function $u \in L^1 \cap L^\infty$ an *entropy solution* of the initial value problem (1.1) if

(1.3)
$$\begin{cases} \text{(i)} \quad |u-c|_t + \operatorname{div}\left[\operatorname{sign}\left(u-c\right)\left(f(k,u) - f(k,c)\right)\right] \\ +\operatorname{sign}\left(u-c\right)\left(\operatorname{div}f(k,c) - q(x,t,u)\right) - \Delta \left|A(u) - A(c)\right| \le 0 \text{ in } \mathcal{D}' \,\,\forall c \in \mathbb{R}, \\ \text{(ii)} \quad \nabla A(u) \text{ belongs to } L^2. \end{cases}$$

In addition, we require that the initial function u_0 is assumed in the strong L^1 sense. We refer to §2 for a precise definition of an entropy solution.

The mathematical (L^1/BV) theory of parabolic equations was initiated by Oleĭnik [26]. She proved well-posedness of the initial value problem in the non-degenerate case with A(u) = u, and showed that weak solutions are in this case classical.

In the hyperbolic case $(A' \equiv 0)$ with the flux f = f(x, t, u) depending (smoothly) on x and t, the notion of entropy solution was introduced independently by Kružkov [23] and Vol'pert [32] (the latter author considered the smaller BV class). These authors also proved general existence, uniqueness, and stability results for the entropy solution, see also Oleĭnik [26] for similar results in the convex case $f_{uu} \geq 0$.

In the mixed hyperbolic-parabolic case $(A' \ge 0)$, the notion of entropy solution goes back to Vol'pert and Hudjaev [33], who were the first to study strongly degenerate parabolic equations. These authors also showed existence of a BV entropy solution using the viscosity method and obtained some partial uniqueness results in the BV class (i.e., when the first order partial derivatives of u are finite measures). In the one-dimensional case, Wu and Yin [35] later provided a complete uniqueness proof in the BV class. Further results in the one-dimensional case were obtained by Bénilan and Touré [3, 4] using nonlinear semigroup theory.

As for the uniqueness issue in the multi-dimensional case, Brézis and Crandall [6] established uniqueness of weak solutions when $f \equiv 0$. Later, under the assumption that A(s) is strictly increasing, Yin [36] showed uniqueness of weak solutions in the BV class. Bénilan and Gariepy [2] showed that the BV weak solution studied in [36] is actually a strong solution. The assumption that u_t should be a finite measure was removed in [37, 38].

An important step forward in the general case of $A(\cdot)$ being merely nondecreasing was made recently by Carrillo [12], who showed uniqueness of the entropy solution for a particular boundary value problem with the boundary condition "A(u) = 0". His method of proof is an elegant extension of the by now famous "doubling of variables" device introduced by Kružkov [23]. In [12], the author also showed existence of an entropy solution using the semigroup method.

In [7] (see also [28]), the uniqueness proof of Carrillo was adopted to several initial-boundary value problems arising the theory of sedimentation-consolidation processes [9], which in some cases call for the notion of an entropy boundary condition (see also [8] for the BV approach).

In the present paper we generalize Carrillo's uniqueness result [12] by showing that it holds for the Cauchy problem with a flux function f = f(x, t, u) where the spatial dependence is nonsmooth (non-Lipschitz). Only the case f = f(u) was studied in [12]. Moreover, we also establish continuous dependence on the flux function in the case f(x, t, u) = k(x)f(u).

With the assumptions on the diffusion function A already given (see (1.2)), we now present the (regularity) assumptions that are needed on the flux function f and the source term q, with the those on f being the most important ones. Concerning the source term $q : \mathbb{R}^d \times (0,T) \times \mathbb{R} \to \mathbb{R}$, we assume that $q(x,t,0) = 0 \ \forall x, t$ and

(1.4)
$$q(\cdot, \cdot, u) \in L^1(0, T; L^\infty(\mathbb{R}^d)) \ \forall u; \quad q(x, t, \cdot) \in \operatorname{Lip}_{\operatorname{loc}}(\mathbb{R}) \text{ uniformly in } x, t.$$

With the phrase "uniform in x, t" in (1.4), we mean

$$|q(x,t,v) - q(x,t,u)| \le C|v-u|, \quad \forall x,t,v,u,$$

for some constant C > 0 (independently of x, t, v, u).

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in addition, we require that the initial function as to assume in the strong 1. Seese, No rate to Of for a precise definition of an entropy of hubble

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Concerning the flux function $f : \mathbb{R}^d \times [0,T] \times \mathbb{R} \to \mathbb{R}^d$, we assume without loss of generality that $f(x,t,0) = f_x(x,t,0) = 0$. Moreover, we assume that

(1.5)
$$f(\cdot, \cdot, u) \in L^1(0, T; W^{1,1}_{loc}(\mathbb{R}^d)) \ \forall u; \quad f(x, t, \cdot) \in \operatorname{Lip}_{loc}(\mathbb{R}) \text{ uniformly in } x, t;$$

(1.6)
$$f_x(\cdot, \cdot, u) \in L^1(0, T; L^\infty(\mathbb{R}^d)) \ \forall u; \quad f_x(x, t, \cdot) \in \operatorname{Lip}_{\operatorname{loc}}(\mathbb{R}) \text{ uniformly in } x, t,$$

where $f_x = f_x(x, t, u)$ in (1.6) denotes the function obtained by taking the divergence of the flux f = f(x, t, u) with respect to the first variable. With the phrase "uniformly in x, t" in (1.5) and (1.6), we mean

$$|f(x,t,v) - f(x,t,u)|, |f_x(x,t,v) - f_x(x,t,u)| \le C|v-u|, \quad \forall x,t,v,u,$$

for some constant C > 0 (independently of x, t, v, u).

The conditions in (1.4)-(1.6) are sufficient to make sense to the notion of entropy solution (see §2). In the general case, however, we need one additional regularity assumption on the x dependency of f to get uniqueness of the entropy solution. Inspired by Capuzzo-Dolcetta and Perthame [10], we assume that

(1.7)
$$(F(x,t,v,u) - F(y,s,v,u)) \cdot (x-y) \ge -\gamma |v-u| |x-y|^2, \quad \forall x,y,t,v,u,$$

for some constant $\gamma > 0$ (independent of x, t, v, u), where

(1.8)
$$F(x,t,v,u) := \text{sign}(v-u) \left[f(x,t,v) - f(x,t,u) \right].$$

Note that condition (1.7) does not imply that f is Lipschitz continuous in the spatial variable x. We remark that if f = f(x, u) is of the form

$$f = k(x)h(u),$$

for some vector-valued function $k : \mathbb{R}^d \to \mathbb{R}^d$, and a Lipschitz continuous function h, then (1.7) reduces to

(1.9)
$$(k(x) - k(y)) \cdot (x - y) \ge -\gamma |x - y|^2, \quad \forall x, y, t, v, u,$$

for some constant $\gamma > 0$ (depending also on the Lipschitz constant of h). As pointed out in [10], this condition requires a bound only on the matrix $\nabla_x k + (\nabla_x k)^{\mathrm{T}}$ (the symmetric part of the Jacobian $\nabla_x k$) and k itself need not belong to any Sobolev space. To see this, let z = x - y and rewrite the left-hand side of (1.9) as follows

$$(k(x) - k(y)) \cdot (x - y) = \int_0^1 \frac{d}{d\xi} \left[(k(y + \xi z) - k(y)) \cdot z \right] d\xi$$
$$= \int_0^1 \nabla_x k(y + \xi z) z \cdot z \, d\xi$$
$$= \frac{1}{2} \int_0^1 \left(\nabla_x k + (\nabla_x k)^{\mathrm{T}} \right) (y + \xi z) z \cdot z \, d\xi,$$

since $\frac{1}{2} \left(\nabla_x k - (\nabla_x k)^{\mathrm{T}} \right) (y + \xi z) z \cdot z \equiv 0.$

In [10], the authors showed the universality of (1.7) by proving that under this condition, uniqueness holds for the Kružkov-Vol'pert entropy solution of hyperbolic equations, the Crandall-Lions viscosity solution of Hamilton-Jacobi equations, and the DiPerna-Lions regularized solution of transport equations. With the present paper, we add to that list uniqueness of the entropy solution of degenerate parabolic equations. More precisely, we prove the following theorem:

Theorem 1.1 (Uniqueness). Assume that (1.2) and (1.4)-(1.7) hold. Let v, u be two entropy solutions of (1.1) with initial data $u_0 \in L^1(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d)$. Then v = u a.e. in $\Pi_T = \mathbb{R}^d \times (0,T)$.

By combining the arguments used in the present paper by those used in [16], Theorem 1.1 can be proved even for a large class of weakly coupled systems of degenerate parabolic equations.

We next restrict our attention to problems of the form

(1.10)
$$\begin{aligned} u_t + \operatorname{div}(k(x)f(u)) &= \Delta A(u), \quad (x,t) \in \Pi_T, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}^d, \end{aligned}$$

Concerning the flux function $f \in \mathbb{R}^n$ is $[0, k] \times \mathbb{R}^n \to \mathbb{R}^n$, we assume without loss of generality that $f(x, t, 0) = f_*(x, t, 0) = 0$. Mapping we assume that

$$(1.5) \qquad f(r,r,q) \in L^1([0,T]; W_{ch}^{-1}(\mathbb{R}^n)), \forall rq \in F(h,t_r) \in Lq_{n,n}(\mathbb{R}) \text{ and mady in 2.3};$$

where $f_{i} = f_{i}(x, t, u)$ is (1.5) denotes the function obtained by relating the divergence of the this f = f(x, t, u) with respect to the first variable. With the physics "attained by $x \in [1, 2]$ and (1.5), we mean

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Theorem 1.1 (Hundrenser): Assume that (1.2) and (1.4) (1.1). Add. Let u. a be for entropy solutions of (1.1) with unitarial data as $C_1(\mathbb{R}^4)$, $C_2(\mathbb{R}^4)$. Then $\mu = \mu$ we say $\Pi_1 = M^2 \times (1,2)$.

By combinent the accounters used in the present paper of these luid in [16]. Theorem 1.1 conbe proted even for a bitre date of weath fourpled avarages of degenerate parabolic equations, we

where $k : \mathbb{R}^d \to \mathbb{R}^d$, $f : \mathbb{R} \to \mathbb{R}$, and f(0) = 0. Problems of the form (1.10) occur in several important applications. Our first result for (1.10) states that in the $L^{\infty}(0,T;BV(\mathbb{R}^d))$ class of entropy solutions, an L^1 contraction principle actually holds provided

(1.11)
$$f \in \operatorname{Lip}_{\operatorname{loc}}(\mathbb{R}); \quad k \in W^{1,1}_{\operatorname{loc}}(\mathbb{R}^d); \quad k, \operatorname{div} k \in L^{\infty}(\mathbb{R}^d).$$

More precisely, we prove the following theorem:

Theorem 1.2 (L^1 contraction). Assume that (1.2) and (1.11) hold. Let $v, u \in L^{\infty}(0, T; BV(\mathbb{R}^d))$ be entropy solutions of (1.10) with initial data $v_0, u_0 \in L^1(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$, respectively. Then for almost all $t \in (0, T)$,

$$||v(\cdot,t) - u(\cdot,t)||_{L^1(\mathbb{R}^d)} \le ||v_0 - u_0||_{L^1(\mathbb{R}^d)}$$

In particular, there exists at most one entropy solution of the initial value problem (1.10).

We remark that the existence of an $L^{\infty}(0,T;BV(\mathbb{R}^d))$ entropy solution of (1.10) is guaranteed if div $k \in BV(\mathbb{R}^d)$. This follows from the results obtained by Karlsen and Risebro [19], who prove convergence (within the entropy solution framework) of finite difference schemes for degenerate parabolic equations with rough coefficients. For an overview of the literature on numerical methods for approximating entropy solutions of degenerate parabolic equations, we refer to the first section of [19] and the lecture notes [14] (see also the references given therein).

Let us mention that Theorem 1.2 includes the L^1 contraction property proved by Klausen and Risebro [20] for the one-dimensional scalar conservation law with a discontinuous coefficient k(x). Throughout this paper the coefficient k(x) is not allowed to be discontinuous. In the onedimensional hyperbolic case $(A' \equiv 0)$ with k(x) depending discontinuously on x, the equation (1.1) is often written as the following 2×2 system:

(1.12)
$$u_t + f(k, u)_x = 0, \quad k_t = 0.$$

If $\partial f/\partial u$ changes sign, then this system is non-strictly hyperbolic. This complicates the analysis, and in order to prove compactness of approximated solutions a singular transformation $\Psi(k, u)$ has been used by several authors [29, 15, 22, 21]. In these works convergence of the Glimm scheme and of front tracking was established in the case where k may be discontinuous. If $k \in C^2(\mathbb{R}^d)$, then convergence of the Lax-Friedrichs scheme and the upwind scheme was proved in [26]. Under weaker conditions on k ($k' \in BV$) and for f convex in u, convergence of the one-dimensional Godunov method for (1.12) (not for (1.1)) was shown by Isaacson and Temple in [17]. Recently, convergence of the one-dimensional Godunov method for (1.1) was shown by Towers [30] in the case where k is piecewise continuous. In this case, the Kružkov entropy condition (1.3) no longer applies, and in [22] a wave entropy condition analogous to the Oleĭnik entropy condition introduced in [26] was used to obtain uniqueness, see also [21]. Klausen and Risebro [20] analyzed the case of discontinuous k by "smoothing out" the coefficient k and then passing to the limit as the smoothing parameter tends to zero. In particular, they showed that the limit "entropy" solution satisfied the L^1 contraction property. We intend to study the degenerate parabolic problem (1.10) when k(x) is discontinuous in future work.

Theorem 1.2 gives the desired continuous dependence on the initial data in degenerate parabolic problems of the type (1.10). Next we will establish continuous dependence also on the flux function. To this end, let us also introduce the problem

(1.13)
$$v_t + \operatorname{div}(l(x)g(u)) = \Delta A(u), \quad (x,t) \in \Pi_T, \\ v(x,0) = v_0(x), \quad x \in \mathbb{R}^d,$$

where $l : \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R} \to \mathbb{R}$, and g(0) = 0. We are interested in estimating the L^1 difference between the entropy solution v of (1.13) and the entropy solution u of (1.10). To this end, we assume

(1.14)
$$f, g \in \operatorname{Lip}_{\operatorname{loc}}(\mathbb{R}); \quad k, l \in W^{1,1}(\mathbb{R}^d); \quad k, l, \operatorname{div} k, \operatorname{div} l \in L^{\infty}(\mathbb{R}^d).$$

Under these assumptions, we prove the following continuous dependence result:

where k : $\mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$, $f \in \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ and $f(0) \approx 0$. Problems of the focus (1.10) secure in several interportant spin them. (1.10) respective (r, 10) is a provided in the $L^{n}(0, 1, N(2^{n}))$ class of entroity solutions, as L^{n} coversaction principle, actually holds provided

More precisely, we preve the following the branch

Liberren 1.2 (f. contraction): Association (f.2) and (f.1) fold. Leta, e = (2-(0, 2, 37, 72)) le carroph adations of (1.10) with annal data Socia E (2 (11) 1. (2 (2) 1. (2) (2) (2) (2) (2) (2) (2) (Then for almost all t = (0, 2)

is particular, there exists to provide the minerary solution of the rate of value providers (9.30).

We remark that can existence of an $\Sigma^{\infty}(\alpha, \beta, \beta)$ [20] (and one contains of al. (5) is guaranteed if divise β $B'(R^2)$. These follows from the sector obtained on K states and 5 control 8, whereas a convergence (within the entropy controls from a memory of 18,122, 64 forests statement for organization percebolic equals as whereas reaction is for an overcover of the left of forests statement for organization for approximation entropy solutions of the above entropy of 18,122, 64 forests statement for organization for approximation of the entropy solution of the left of the entropy of the left of the entropy of the entropy of [16] and the left reaction of the case the the reference of the entropy of

Let us multion this 'incorest' (1) incluses the L' multiplies property moved by Element and Roebro (20) for the my dimensional scalar conservation lay with a denominative porfliqued k(x). Throughout this paper the coefficient k(x) is not allowed to be dimensional do has one dimensional hyperbolic man (L' = 0 with k(x) representing deconstructions, in the one is often written as the following $T \times 2$ system:

) i bearena 1.2 givesene ekse og sonkannna dependerte en rou artige dutarne dagsoche partigolde. problems of die 1994 (1110) "Next we will en ablieh continuéns dependence also ar she tras tupelien. To takis een, fat us else introducer he rechteo.

$$(1.15) = (1.15) + ($$

where I : M == 20, g = 0 = 2, and a(0 = 0. We are interested to compare the L' difference between the entropy solution o of (1.13) module response solution wit (1.10). To this end, we

Under three assistantions, we prove the following canetanous distendence result:

Theorem 1.3 (Continuous dependence). Assume that the regularity conditions (1.2) and (1.14) hold. Let $v, u \in L^{\infty}(0,T; BV(\mathbb{R}^d))$ be entropy solutions of (1.13), (1.10) with initial data $v_0, u_0 \in L^1(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$, respectively. For definiteness, let us assume that v, u take values in in the closed interval $I \subset \mathbb{R}$ and that there are constants $V_v, V_u > 0$ such that

$$|v(\cdot,t)|_{BV(\mathbb{R}^d)} \le V_v \ \forall t \in (0,T), \qquad |u(\cdot,t)|_{BV(\mathbb{R}^d)} \le V_u \ \forall t \in (0,T)$$

Then for almost all t > 0,

$$\begin{aligned} \|v(\cdot,t) - u(\cdot,t)\|_{L^{1}(\mathbb{R}^{d})} &\leq \|v_{0} - u_{0}\|_{L^{1}(\mathbb{R}^{d})} \\ &+ t \bigg[\Big(C_{1}^{g,v} \|l - k\|_{L^{\infty}(\mathbb{R}^{d})} + C_{2}^{g} |l - k|_{BV(\mathbb{R}^{d})} + C_{3}^{k} \|g - f\|_{L^{\infty}(I)} + C_{4}^{k,v} \|g - f\|_{\operatorname{Lip}(I)} \Big) \\ &\wedge \Big(C_{1}^{f,u} \|l - k\|_{L^{\infty}(\mathbb{R}^{d})} + C_{2}^{f} |l - k|_{BV(\mathbb{R}^{d})} + C_{3}^{l} \|g - f\|_{L^{\infty}(I)} + C_{4}^{l,u} \|g - f\|_{\operatorname{Lip}(I)} \Big) \bigg], \end{aligned}$$

where $C_1^{g,v} = ||g||_{\operatorname{Lip}(I)} V_v$, $C_1^{f,u} = ||f||_{\operatorname{Lip}(I)} V_u$, $C_2^g = ||g||_{L^{\infty}(I)}$, $C_2^f = ||f||_{L^{\infty}(I)}$, $C_3^k = |k|_{BV(\mathbb{R}^d)}$, $C_3^l = |l|_{BV(\mathbb{R}^d)}$, $C_4^{k,v} = ||k||_{L^{\infty}(\mathbb{R}^d)} V_v$, $C_4^{l,u} = ||l||_{L^{\infty}(\mathbb{R}^d)} V_u$, and $a \wedge b = \min(a, b)$.

We remark that Theorem 1.3 includes the continuous dependence result obtained in Klausen and Risebro [20] for the one-dimensional scalar conservation law with a discontinuous coefficient k(x). Results regarding continuous dependence on the flux function in scalar conservation laws with $k(x) \equiv 1$ have been obtained by Lucier [25] and Bouchut and Perthame [5]. Finally, we mention that Cockburn and Gripenberg [13] have obtained a result regarding continuous dependence on both the flux function and the diffusion function in (1.10) when k(x) = 1. Their result does *not*, however, imply uniqueness of the entropy solution since their "doubling of variables" argument requires that one works with (smooth) approximate solutions. By properly combining the ideas in the present paper with those in [13], one can prove a version of Theorem 1.3 which also includes continuous dependence on the diffusion function A. We will present the details elsewhere.

The rest of this paper is organized as follows: In the next section we introduce (precisely) the notion of entropy solution as well as stating and proving a version of an important lemma due to Carrillo [12]. Equipped with our version of Carrillo's lemma, Theorems 1.1, 1.2, and 1.3 are proved in §3, §4, and §5, respectively. Finally, in §6 (an appendix) we provide a proof of the weak chain rule needed in the proof of Carrillo's lemma.

2. Preliminaries

We shall use the following definition of an entropy solution of (1.1):

Definition 2.1. An entropy solution of (1.1) is a measurable function u = u(x,t) satisfying: D.1 $u \in L^1(\Pi_T) \cap L^{\infty}(\Pi_T) \cap C(0,T; L^1(\mathbb{R}^d)).$

D.2 For all $c \in \mathbb{R}$ and all non-negative test functions in $C_0^{\infty}(\Pi_T)$, the following entropy inequality holds:

$$\iint_{\Pi_T} \left(|u - c|\phi_t + \operatorname{sign} (u - c) \left(f(x, t, u) - f(x, t, c) \right) \cdot \nabla \phi + |A(u) - A(c)| \Delta \phi \right)$$

$$- \operatorname{sign} (u - c) \left(\operatorname{div} f(x, t, c) - q(x, t, u) \right) \phi \right) dt \, dx \ge 0.$$

D.3 $A(u) \in L^2(0,T; H^1(\mathbb{R}^d)).$ D.4 Essentially as $t \downarrow 0$,

(2.1)

$$\int_{\mathbb{R}^d} \left| u(x,t) - u_0(x) \right| dx \to 0.$$

Remark 2.1. (i) Observe that when $A' \equiv 0$, (2.1) reduces to the well known entropy inequality for scalar conservation laws introduced by Kružkov [23] and Vol'pert [32].

(ii) Condition (**D**.4), i.e., that the initial datum u_0 should be taken by continuity, motivates the requirement of continuity with respect to t in condition (**D**.1).

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Then for element all t > 0

 $||u_{n}(x)| + |u_{n}(x)| \geq ||u_{n}(x)| + |u_{n}(x)| + ||u_{n}(x)|$

where $Q^{(n)} = [letternecht Q^{(n)} = M(letternecht Q^{(n)} = letternecht Q^{(n)} = [M(letternecht Q^{(n)} = M(letternecht Q^{(n)} = M(letternecht$

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We shall use the fullowing definition of an entropy solution of (1.1).

Definitiviti 2.1. A a collegen solution of (1.1) is a measurable femalies is a man of value female. D to a difference and an order to the statements

D.2. For all d. and all accordence test functions in C. (Red) the following currence tempolic indise

$$(a_{1},a_{2}) = (a_{1},a_{2}) + (a_{2},a_{2}) + (a_{2},a_{2}$$

D.3 $A(u) \in L^{2}(0,T; H^{2}(\mathbb{R}^{4}))$

$$\int_{\Omega} \left[u(x,t) - u(x) \right] dx = v 0,$$

Hernest's 2.1. (1) Observe that alwar A = 0, (2:1) reduces to dark well known, extremy internativy for scalar conservetions have internative to diverse and the formation and the formation of the second statement of the sec

(ii) Condition (D.31) het, that der instell detuin ag should be taken by continuity indrivutes the requirement of continuity world respect to the toted block (D.4).

Let u be an entropy solution. Then, since $A(u) \in H^1(\mathbb{R}^d)$ for a.e. $t \in (0,T)$, it follows from general theory of Sobolev spaces that $\nabla |A(u) - A(c)| = \operatorname{sign} (A(u) - A(c)) \nabla A(u)$ a.e. in Π_T . Also, sign $(A(u) - A(c)) = \operatorname{sign} (u - c)$ provided $A(u) \neq A(c)$. Again since $A(u) \in H^1(\mathbb{R}^d)$ a.e. for $t \in (0,T)$, it follows that $\nabla A(u) = 0$ a.e. (w.r.t. $dt \, dx$) in $\{(x,t) \in \Pi_T : A(u(x,t)) = A(c)\}$. We therefore conclude that

$$\nabla |A(u) - A(c)| = \operatorname{sign}(u - c) \nabla A(u)$$
 a.e. in Π_T

and the entropy inequality (2.1) can be written equivalently as

(2.2)
$$\iint_{\Pi_T} \left(|u-c|\phi_t + \operatorname{sign}(u-c) \left[f(x,t,u) - f(x,t,c) - \nabla A(u) \right] \cdot \nabla \phi - \operatorname{sign}(u-c) \left(\operatorname{div} f(x,c) - q(x,t,u) \right) \phi \right) dt \, dx \ge 0, \qquad \forall \phi \in C_0^\infty(\Pi_T).$$

If we take $c > \operatorname{ess sup} u(x, t)$ and $c < \operatorname{ess inf} u(x, t)$ in (2.1), then we deduce that u satisfies

(2.3)
$$\iint_{\Pi_T} \left(u\phi_t + f(x,t,u) \cdot \nabla\phi + A(u)\Delta\phi + q(x,t,u)\phi \right) dt \, dx = 0, \quad \forall \phi \in C_0^\infty(\Pi_T)$$

Note that (1.5) implies

(2.4)
$$\|f(x,t,u)\|_{L^{2}(\Pi_{T})}^{2} \leq \operatorname{Const} \|u\|_{L^{\infty}(\Pi_{T})} \|u\|_{L^{1}(\Pi_{T})} < \infty,$$

so that $f(x,t,u) - \nabla A(u) \in L^2(\Pi_T; \mathbb{R}^d)$. Similarly, (1.4) implies q(x,t,u) belongs to $L^2(\Pi_T)$. An integration by parts in (2.3) followed by an approximation argument will then show that the equality

(2.5)
$$\iint_{\Pi_T} \left(u\phi_t + \left[f(x,t,u) - \nabla A(u) \right] \cdot \nabla \phi + q(x,t,u)\phi \right) dt \, dx = 0$$

holds for all $\phi \in L^2(0,T; H^1_0(\mathbb{R}^d)) \cap W^{1,1}(0,T; L^{\infty}(\mathbb{R}^d)).$

We can even go one step further. To this end, let $\langle \cdot, \cdot \rangle$ denote the usual pairing between $H^{-1}(\mathbb{R}^d)$ and $H^1_0(\mathbb{R}^d)$. From (2.5), we conclude that

$$\partial_t u \in L^2(0,T; H^{-1}(\mathbb{R}^d)),$$

so that the equality

(2.6)
$$-\int_0^T \left\langle \partial_t u, \phi \right\rangle dt + \iint_{\Pi_T} \left(\left[f(x, t, u) - \nabla A(u) \right] \cdot \nabla \phi + q(x, t, u) \phi \right) dt \, dx = 0$$

holds for all $\phi \in L^2(0,T; H^1_0(\mathbb{R}^d)) \cap W^{1,1}(0,T; L^{\infty}(\mathbb{R}^d))$. The fact that an entropy solution u satisfies (2.6) will be important for the uniqueness proof.

We now set

(2.7)
$$\mathcal{A}_{\psi}(z) = \int_{z_0}^{z} \psi(A(r)) dr,$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing and Lipschitz continuous function and $z_0 \in \mathbb{R}$. Concerning the function \mathcal{A}_{ψ} , we shall need the following associated "weak chain rule":

Lemma 2.1. Let $u: \Pi_T \to \mathbb{R}$ be a measurable function satisfying the following four conditions:

- (a): $u \in L^1(\Pi_T) \cap L^{\infty}(\Pi_T) \cap C(0,T; L^1(\mathbb{R}^d)).$
- (b): $u(0, \cdot) = u_0 \in L^{\infty}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d).$
- (c): $\partial_t u \in L^2(0,T; H^{-1}(\mathbb{R}^d)).$
- (d): $A(u) \in L^2(0,T; H^1(\mathbb{R}^d))$.

Let a be an entropy solution. Then, some $A(n) \in M(\mathbb{R}^n)$ for a.e. $t \in \{0, T\}$, it follows from granted theory of Sobolev spaces (but $\nabla_{1,4}(n) - A(n)$) $\nabla_{1,4}(n)$ a.e. in Res Also, e.g. (A(n) - A(n)) - A(n) $\nabla_{1,4}(n)$ a.e. in Res Also, e.g. (A(n) - A(n)) - A(n) = start - charter - charter Also + A(n) + A(n), spain store $A(n) \in K^n(\mathbb{R}^n)$ and for $t \in \{0, T\}$, it follows that $\nabla_{1,4}(n) = 0$ and (n + 1) and (n + 1), it follows that $\nabla_{1,4}(n) = 0$ and (n + 1). We therefore conclude that $\nabla_{1,4}(n) = 0$ and (n + 1), (n + 1),

and the entropy increality? 2.1) can be written equivalently as

 $-aga(n-ac)(dar)(a,a) - g(a,b,a)(a)(ab,ba) \geq 0, \qquad y \in G(2(10n))$

If we take z > 666600 $\pi/c(z)$ and z < 66166 π/cz , β in (2.1), then we initialized that is suitable.

$$(2.3) \qquad \int \left(u \phi_1 + f(x, \xi_1) + f(x, \xi_2) + f(x, \xi_2) + f(x, \xi_2) \right) dt dt = 0, \quad \forall t \in G^{(1)}_{2}(\mathbb{R}^{n})$$

Note that (1.5) implies

$$(2.4)$$

so that f(c.t.u) - 7340) 6 L⁵(Br(2)). Similaria, (b3) making f(c.(.u) belongs to 62662). An unogration by periods (24) followed by an approximation argument will then show that the equality

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We raw new go een stepduring. To this and let (...) denote the usual pointing between N. (...) and N(T) From (2.5), we conclude that

$$A_{ii} \in L^{2}(0, \mathcal{L}, H^{-1}(\mathbb{R}^{d}))$$

so that the equality?

$$(2.6) \qquad -\int \left[(\partial_t u, \phi) \, dt + \iint \left([f(u, v, u) - \nabla A(u)] f(\nabla u + q(u, \phi, u) \phi) \, dt \, dv = 0 \right] \right]$$

holds for all $\phi \in L^2(0, \mathbb{Z})$ $H_{0}^{1}(\mathbb{R}^{n})$ $\cap W^{1,1}(0, \mathbb{T}; L^{\infty}(\mathbb{R}^{n}))$. This fact that an entropy solution $v \in \mathbb{R}^{n}$ satisfies (2.6) yeth for important for the uniqueness proof.

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$$f(x, y) = \int dx dx = \int dx dx dy dx dy dx dy dy dx$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a solutorreading and Lipschitz for lifetoys function and $z_0 \in \mathbb{R}$. Containing the function $A_{0,0}$ we black the file 0 determines "weak analy role".

Lemma 2.1. Let u : By -> 8, be avaneare white from correspond the following from environment

Then, for a.e. $s \in (0,T)$ and every nonnegative $\phi \in C_0^{\infty}(\mathbb{R}^d \times [0,T])$, we have

$$-\int_0^s \langle \partial_t u, \psi(A(u))\phi \rangle dt$$

= $\int_0^s \int_{\mathbb{R}^d} \mathcal{A}_{\psi}(u)\phi_t dt dx + \int_{\mathbb{R}^d} \mathcal{A}_{\psi}(u_0)\phi(x,0) dx - \int_{\mathbb{R}^d} \mathcal{A}_{\psi}(u(x,s))\phi(x,s) dx.$

Lemma 2.1 can proved more or less in the same way as the "weak chain rule" in Carrillo [12], see also Alt and Luckhaus [1] and Otto [27]. For the sake of completeness, a proof of Lemma 2.1 is given in $\S6$ (the appendix).

In what follows, we shall frequently need a continuous approximation of sign (·). For $\varepsilon > 0$, set

$$\operatorname{sign}_{\varepsilon} (\tau) = \begin{cases} -1, & \tau < \varepsilon, \\ \tau/\varepsilon, & \varepsilon \le \tau \le \varepsilon, \\ 1 & \tau > \varepsilon. \end{cases}$$

Note that $\operatorname{sign}_{\varepsilon}(-r) = -\operatorname{sign}_{\varepsilon}(r)$ and $\operatorname{sign}_{\varepsilon}'(-r) = \operatorname{sign}_{\varepsilon}'(r)$ a.e.

We let $A^{-1}: \mathbb{R} \to \mathbb{R}$ denote the unique left-continuous function satisfying $A^{-1}(A(u)) = u$ for all $u \in \mathbb{R}$, and by E we denote the set

$$E = \left\{ r : A^{-1}(\cdot) \text{ discontinuous at } r \right\}.$$

Note that E is associated with the set of points $\{u : A'(u) = 0\}$ at which the operator $u \mapsto \Delta A(u)$ is degenerate elliptic.

We are now ready to state and prove the following version of an important observation made by Carrillo [12]:

Lemma 2.2 (Entropy dissipation term). Let u be an entropy solution of (1.1). Then, for any non-negative $\phi \in C_0^{\infty}(\Pi_T)$ and $c \in \mathbb{R}$ such that $A(c) \notin E$, we have

$$2.8) \qquad \qquad \int_{\Pi_T} \int \left(|u - c|\phi_t + \operatorname{sign}(u - c) \left[f(x, t, u) - f(x, t, c) - \nabla A(u) \right] \cdot \nabla \phi \right. \\ \left. - \operatorname{sign}(u - c) \left(\operatorname{div} f(x, t, c) - q(x, t, u) \right) \phi \right) dt \, dx \\ \left. = \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \left| \nabla A(u) \right|^2 \operatorname{sign}'_{\varepsilon} \left(A(u) - A(c) \right) \phi \, dt \, dx.$$

Proof. The proof is similar to the proof of the corresponding result in [12]. In (2.7), introduce the function $\psi_{\varepsilon}(z) = \operatorname{sign}_{\varepsilon}(z - A(c))$ and set $z_0 = c$. Notice that the conditions of Lemma 2.1 are satisfied and hence

$$-\int_0^T \left\langle \partial_t u, \operatorname{sign}_{\varepsilon} \left(A(u) - A(c) \right) \phi \right\rangle dt = \iint_{\Pi_T} \mathcal{A}_{\psi_{\varepsilon}}(u) \phi_t \, dt \, dx.$$

Since u satisfies (2.6) and $[\operatorname{sign}_{\epsilon}(A(u) - A(c))\phi] \in L^2(0,T; H^1_0(\mathbb{R}))$ is a test function, we have

$$-\int_{0}^{T} \left\langle \partial_{t} u, \operatorname{sign}_{\varepsilon} \left(A(u) - A(c) \right) \phi \right\rangle dt \\ + \iint_{\Pi_{T}} \left(\left[f(x, t, u) - f(x, t, c) - \nabla A(u) \right] \cdot \nabla (\operatorname{sign}_{\varepsilon} \left(A(u) - A(c) \right) \phi \right) \\ - \left(\operatorname{div} f(x, t, c) - q(x, t, u) \right) (\operatorname{sign}_{\varepsilon} \left(A(u) - A(c) \right) \phi \right) dt \, dx = 0$$

(2

Then, for a.e. $a \in \{0,T\}$ and above normalizative $a \in C^\infty_{\mathbb{C}}(\mathbb{R}^n \times [0,T])$, we have

$$(a_{a,b}(A_{a})) = \int_{a} \int_{a} A_{b}(a_{b}) (a_{b}) (a_{b})$$

Lemma 2.1 can proved provid relax in the same wey on the "weak chein feld" in Carifle (12), sue also Alt and Doktrons [1] and Oro [21]. For the sales of complexents is proof of Lemma 2.1 is given in §6. One appendix

in what follows, he shall frequently need a continuent, and restriction of size (-). For c'h Q, set

We let A ' : i A -- E denote the introve bit contranents indchon satisfying A ' (Aba)) - a lot all to E E and by E we denote the set

$$\left\{ i : semicurrent \cap f^{(1)} : s \right\} = \mathbb{Z}_{i \in \mathbb{Z}}$$

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We are now ready to state and provin the following version of an imperiant observation reads by Gaudio [12]:

Laurana 2.2 (Entropy dasapation terrs). Let b is an entropy solution of (L1). Then for any advances of C (R)-) and ct R secondar Hel 4 E, secondaria

Freq. The proof is singler to any fixed of the consuporating ready $z_1(z_1)$ is (2.7) in quites (5) formation $\psi_1(z) = sign_1(z - d(z))$ and set $z_0 = c$. Notice that the readitions of Levin 2.1 form satisfied and have

$$\int_{0}^{\infty} \left(\partial_{t} u_{s} g_{t} g_{t} g_{s} \left(A(u) - A(v) \right) \phi \right) dv = \int_{0}^{\infty} \int_{0}^{\infty} A_{t} g_{s} g_{s} dv dv$$

Since a settation (2,4) and (size (A(a) -- A(c)) at (c (P(a) 2) (2) (P(a) iou non-femerical taxes)

$$= \int \left(\frac{\partial e_{11}}{\partial e_{12}} \left(\frac{\partial e_{12}}{\partial e_{12}} - \frac{\partial e_{12}}{\partial e_{12}} \right) e_{12} \right) e_{12} \left(\frac{\partial e_{12}}{\partial e_{12}} + \frac{\partial e_{12}}{\partial e_{12}} \right) - \frac{\partial e_{12}}{\partial e_{12}} + \frac{\partial e_{12}}{\partial e_{12}$$

(d(r)(t, t, t, n) - q(t, t, n))(d(t, t, t, n)) = (t, t, t, t)(t) = 0

which implies that

(2.9)
$$\iint_{\Pi_T} \mathcal{A}_{\psi_{\varepsilon}}(u)\phi_t \, dt \, dx + \iint_{\Pi_T} \left(\left[f(x,t,u) - f(x,t,c) - \nabla A(u) \right] \cdot \nabla(\operatorname{sign}_{\varepsilon} \left(A(u) - A(c) \right) \phi \right) \\ - \operatorname{sign}_{\varepsilon} \left(A(u) - A(c) \right) \left(\operatorname{div} f(x,t,c) - q(x,t,u) \right) \phi \right) dt \, dx = 0.$$

Since A(r) > A(c) if and only if r > c, $\operatorname{sign}_{\varepsilon} (A(r) - A(c)) \to 1$ as $\varepsilon \downarrow 0$ for any r > c. Similarly, $\operatorname{sign}_{\varepsilon} (A(r) - A(c)) \to -1$ as $\varepsilon \downarrow 0$ for any r < c. Consequently, whenever $A(c) \notin E$,

 $\mathcal{A}_{\psi_{\varepsilon}}(u) \to |u-c|$ a.e. in Π_T as $\varepsilon \downarrow 0$.

Moreover, we have $|\mathcal{A}_{\psi_{\varepsilon}}(u)| \leq |u - c| \in L^{1}_{loc}(\Pi_{T})$, so by the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \mathcal{A}_{\psi_\varepsilon}(u) \phi_t \, dt \, dx = \iint_{\Pi_T} |u - c| \phi_t \, dt \, dx.$$

For c such that $A(c) \notin E$, we have

$$\begin{split} \lim_{\varepsilon \downarrow 0} \iint_{\Pi_{T}} \left[f(x,t,u) - f(x,t,c) - \nabla A(u) \right] \cdot \nabla \left[\operatorname{sign}_{\varepsilon} \left(A(u) - A(c) \right) \phi \right] dt \, dx \\ &= \lim_{\varepsilon \downarrow 0} \iint_{\Pi_{T}} \left[f(x,t,u) - f(x,t,c) - \nabla A(u) \right] \cdot \nabla \operatorname{sign}_{\varepsilon} \left(A(u) - A(c) \right) \phi \, dt \, dx \\ &+ \lim_{\varepsilon \downarrow 0} \iint_{\Pi_{T}} \operatorname{sign}_{\varepsilon} \left(A(u) - A(c) \right) \left[f(x,t,u) - f(x,t,c) - \nabla A(u) \right] \cdot \nabla \phi \, dt \, dx \\ &= \underbrace{\lim_{\varepsilon \downarrow 0} \iint_{\Pi_{T}} \operatorname{sign}_{\varepsilon}' \left(A(u) - A(c) \right) \left(f(x,t,u) - f(x,t,c) \right) \cdot \nabla A(u) \phi \, dt \, dx \\ &- \underbrace{\lim_{\varepsilon \downarrow 0} \iint_{\Pi_{T}} \left| \nabla A(u) \right|^{2} \operatorname{sign}_{\varepsilon}' \left(A(u) - A(c) \right) \phi \, dt \, dx \\ &+ \underbrace{\lim_{\varepsilon \downarrow 0} \iint_{\Pi_{T}} \operatorname{sign}_{\varepsilon} \left(A(u) - A(c) \right) \left[f(x,t,u) - f(x,t,c) - \nabla A(u) \right] \cdot \nabla \phi \, dt \, dx \\ &+ \underbrace{\lim_{\varepsilon \downarrow 0} \iint_{\Pi_{T}} \operatorname{sign}_{\varepsilon} \left(A(u) - A(c) \right) \left[f(x,t,u) - f(x,t,c) - \nabla A(u) \right] \cdot \nabla \phi \, dt \, dx \\ &- \underbrace{I_{2}} \\ \end{array}$$

One can check that

$$I_1 = \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \operatorname{div} \mathcal{Q}_{\varepsilon}(A(u)) \phi \, dt \, dx,$$

where Q_{ε} is defined as

$$\begin{aligned} \mathcal{Q}_{\varepsilon}(z) &= \int_{0}^{z} \operatorname{sign}_{\varepsilon}'\left(r - A(c)\right) \left(f(x, t, A^{-1}(r)) - f(x, t, c)\right) dr \\ &= \frac{1}{\varepsilon} \int_{\min(z, A(c) - \varepsilon)}^{\min(z, A(c) + \varepsilon)} \left(f(x, t, A^{-1}(r)) - f(x, t, A^{-1}(A(c)))\right) dr \end{aligned}$$

Since f = f(x, t, u) is locally Lipschitz continuous with respect to u, $Q_{\varepsilon}(z)$ tends to zero as $\varepsilon \downarrow 0$ for all z in the image of A. Consequently, by the Lebesgue dominated convergence theorem,

$$I_1 = -\lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \mathcal{Q}_{\varepsilon}(A(u)) \nabla \phi \, dt \, dx = 0.$$

Observe that for each $c \in \mathbb{R}$ such that $A(c) \notin E$,

$$\operatorname{sign}(u-c) = \operatorname{sign}(A(u) - A(c))$$
 a.e. in Π_T .

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$$\int_{\Omega} \int_{\Omega} A_{0,1}(a) da da + \iint_{\Omega} \left(\left[B(a_{1}, t_{1}a) - f(a_{1}, t_{2}a) - \nabla(a_{1}a) \right] \cdot \nabla(a_{2}a_{2}, (A(a) - A(a)) a) \right]$$
(2.9)

$$-\alpha(\alpha, \beta, \beta) = \alpha(\alpha) - \alpha(\alpha) + \alpha(\alpha, \beta) - \alpha(\alpha, \beta, \alpha) + \beta(\alpha, \beta) = \alpha(\alpha, \beta) + \alpha(\alpha$$

Share stat > A(c) if and only if t > c sign (A(t) - A(t)) = 0 as $t \downarrow 0$ for any t > c. Similarly, t sign (A(t) - A(t)) = 0 for any t > c. Similarly, $t \in [0, t] = 0$.

Moreover, we have $bd_{n}(a) \leq |a - a| \in L_{0,a}(\mathbb{R}_{2})$, so by the behavior dominated convergence theorem

$$\lim_{n \to 0} \iint A_{n,n}(n) dn, dn dx = \iint_{\Omega_n} n n - d\Omega_n dn dn$$

For e such that A(c) & S, we have

$$\sum_{i=1}^{n} \int \left[f(i_{i_{1}}, i_{1}, u_{2}) - f(i_{2}, i_{2}) - \nabla A(u_{1}) + \nabla A(u_{1}) - A(u_{1}) A(u_{2}) A(u_{2}) + A(u_$$

- ng ff nam, (And - And) (Marand - Karing) - Valen). Velena

$$= \lim_{t \to 0} \left\{ \int_{0}^{t} |\nabla A(u)|^{2} u(u) | \Delta (u) - \Delta (u) | \Delta A(u) - \Delta (u) | \Delta (u) - \lambda (u) | \Delta (u)$$

One cent directo rado

$$\label{eq:static_state} \int dv dt (A (u)) \phi dt dd,$$

where Q. Is dollard as

$$S(a) = \int_{a}^{b} \frac{dan}{dr} (r - A(n)) \left(f(a) h A^{-1}(r)) - f(a, h, a)\right) dr,$$

$$= \int_{a}^{b} \frac{dan}{dr} \frac{dan}{dr} (f(a, h, A^{-1}(r)) - f(r), h, A^{-1}(A(a))) dr,$$

Since f = f(x, t, v) is locally Lipsciets continuous with respect to u. $Q_t(x)$, built to zero as $u \downarrow 0$ for all x in the integer of A. Consequently, by the behavior dominined convergence chooses,

Observe that for each c 6 R such that A(c) 6 B

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DEGENERATE PARABOLIC EQUATIONS WITH ROUGH COEFFICIENTS

Therefore, from the Lebesgue bounded convergence theorem, it follows that

$$I_2 = \iint_{\Pi_T} \operatorname{sign} \left(u - c \right) \left[f(x, t, u) - f(x, t, c) - \nabla A(u) \right] \cdot \nabla \phi \, dt \, dx$$

and

$$\lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \operatorname{sign}_{\varepsilon} \left(A(u) - A(c) \right) \left(\operatorname{div} f(x, t, c) - q(x, t, u) \right) \phi \, dt \, dx$$
$$= \iint_{\Pi_T} \operatorname{sign} \left(u - c \right) \left(\operatorname{div} f(x, t, c) - q(x, t, u) \right) \phi \, dt \, dx,$$

Therefore, letting $\varepsilon \downarrow 0$ in (2.9), we obtain the desired equality (2.8).

3. PROOF OF THEOREM 1.1

Equipped with the results derived in §2 (in particular Lemma 2.2), we now set out to prove Theorem 1.1 using the "doubling of variables" device, which was introduced by Kružkov [23] as a tool for proving the uniqueness (L^1 contraction property) of the entropy solution of first order hyperbolic equations. We refer to Carrillo [11, 12], Otto [27], and Cockburn and Gripenberg [13] for applications of the "doubling" device in the context of second order parabolic equations. The presentation that follows below is inspired by Carrillo [12].

Let $\phi \in C^{\infty}(\Pi_T \times \Pi_T)$, $\phi \ge 0$, $\phi = \phi(x, t, y, s)$, v = v(x, t), and u = u(y, s). We shall also need to introduce the "hyperbolic" sets

$$\mathcal{E}_v = \Big\{ (x,t) \in \Pi_T : A(v(x,t)) \in E \Big\}, \qquad \mathcal{E}_u = \Big\{ (y,s) \in \Pi_T : A(u(y,s)) \in E \Big\}.$$

Observe that we have

(3.1)
$$\operatorname{sign}(v-u) = \operatorname{sign}(A(v) - A(u))$$

a.e. (w.r.t. $dt \, dx \, ds \, dy$) in $\left[\Pi_T \times (\Pi_T \setminus \mathcal{E}_v)\right] \bigcup \left[(\Pi_T \setminus \mathcal{E}_u) \times \Pi_T\right]$ and

(3.2)
$$\nabla_x A(v) = 0$$
 a.e. (w.r.t. $dt dx$) in \mathcal{E}_v , $\nabla_y A(u) = 0$ a.e. (w.r.t. $ds dy$) in \mathcal{E}_u .

From the definition of entropy solution, Lemma 2.2, and the first part of (3.2), we have

$$- \iiint_{\Pi_T \times \Pi_T} \left(|v - u| \phi_t + \operatorname{sign} \left(v - u \right) \left[f(x, t, v) - f(x, t, u) - \nabla_x A(v) \right] \cdot \nabla_x \phi \right)$$

(3.3)
$$-\operatorname{sign}(v-u)\left(\operatorname{div}_{x}f(x,t,u)-q(x,t,v)\right)\phi\right)dt\,dx\,ds\,dy$$

(3.4) $\leq -\lim_{\varepsilon \downarrow 0} \iiint_{(\Pi_T \setminus \mathcal{E}_u) \times \Pi_T} |\nabla_x A(v)|^2 \operatorname{sign}_{\varepsilon}' (A(v) - A(u)) \phi \, dt \, dx \, ds \, dy$ $= -\lim_{\varepsilon \downarrow 0} \iiint_{(\Pi_T \setminus \mathcal{E}_u) \times (\Pi_T \setminus \mathcal{E}_v)} |\nabla_x A(v)|^2 \operatorname{sign}_{\varepsilon}' (A(v) - A(u)) \phi \, dt \, dx \, ds \, dy.$

The inequality (3.3) is obtained by using Lemma 2.2 with v(x,t) where (x,t) is not in the hyperbolic set \mathcal{E}_u , noting that the integral over $\Pi_T \setminus \mathcal{E}_u$ is less than the integral over Π_T . Finally, (3.4) follows from (3.2).

Therefore from the Lebesgie bounded convergence therears in follows that

Therefore, lotting r 10 m (201 we obtain the desired equidity (2.8

The Manager Theorem 1.1

Equipped with the results derived in [2 (in particular Leanna [2)), we now an out do prove Theorem 1.1 value the "doubling of variables" device, where we relationed by Erretice [23] as a tool for proving the uniqueness (L² contraction property) of the autropy valuation of dust order hyperbolic aquations. We refer to Cartillo [1], 22], Orta [27], and Cortabely and Granesberg [13], for applications of the "doubling" device in the contract of soccess order parabolic equivalence. The presentation that follows below is lumated by Cartillo [1].

Let $d \in C^{\infty}(\Pi_{T} \times \Pi_{T})$, $\phi \geq 0$, $\phi \in \mathcal{O}(x, x, y, x)$, $y \in \mathcal{O}(x, t)$, and $u \in \mathcal{O}(y, y)$. We shall allocated to introduce the "bipperbolic" sets

Observe that we have,

$$(\{u\}) = \{u\} \land \{u\} = \{u \rightarrow u\}$$
 on the second second

a. (were and even in [2 + × (2 + 2)] () [The / 2. + 2 +] and

$$(3.2) \quad \nabla_{a} A(b) = 0 \text{ and } (w_{a}, b, d) \text{ in } S_{a}, \quad \nabla_{b} A(b) = 0 \text{ and } (w_{a}, a, b) \text{ in } S_{a}$$

From the definition of mitropy solution; foruma 2.2, and the first part of (3.2), we have

The inequality (3.3) is obtained by using Legarm 7.2 with u(x, t) where (1.3) is not in the hyperbolic set \mathcal{E}_{i} , noting that the integral area $\Pi_{i} \setminus \mathcal{E}_{i}$ is less than the integral over Π_{i} . Finally, (3.4) follows tron (3.2).

KARLSEN AND RISEBRO

Similarly, using Lemma 2.2 for u = u(y, s), and the second part of (3.2), we find the inequality

$$-\iiint_{\Pi_T \times \Pi_T} \left(|u - v| \phi_t + \operatorname{sign} (u - v) \left[f(y, s, u) - f(y, s, v) - \nabla_y A(u) \right] \cdot \nabla_y \phi \right.$$
$$- \operatorname{sign} (u - v) \left(\operatorname{div}_y f(y, s, v) - q(y, s, u) \right) \phi \right) dt \, dx \, ds \, dy$$
$$\leq - \lim_{\varepsilon \downarrow 0} \iiint_{(\Pi_T \setminus \mathcal{E}_u) \times (\Pi_T \setminus \mathcal{E}_v)} \left| \nabla_y A(u) \right|^2 \operatorname{sign}'_{\varepsilon} \left(A(u) - A(v) \right) \phi \, dt \, dx \, ds \, dy.$$

Observe that whenever $\nabla_x A(v)$ is defined,

$$\iint_{\Pi_T} \nabla_x A(v) \cdot \nabla_y (\operatorname{sign}_{\varepsilon} \left(A(v) - A(u) \right) \phi) \, ds \, dy = \nabla_x A(v) \cdot \iint_{\Pi_T} \nabla_y (\operatorname{sign}_{\varepsilon} \left(A(v) - A(u) \right) \phi) \, ds \, dy = 0,$$

or more conveniently,

$$(3.6) - \iint_{\Pi_T} \operatorname{sign}_{\varepsilon} \left(A(v) - A(u) \right) \nabla_x A(v) \cdot \nabla_y \phi \, ds \, dy = \iint_{\Pi_T} \nabla_y \operatorname{sign}_{\varepsilon} \left(A(v) - A(u) \right) \cdot \nabla_x A(v) \phi \, ds \, dy.$$

Similarly, for a.e. $(y, s) \in \Pi_T$,

$$(3.7) - \iint_{\Pi_T} \operatorname{sign}_{\varepsilon} \left(A(u) - A(v) \right) \nabla_y A(u) \cdot \nabla_x \phi \, dt \, dx = \iint_{\Pi_T} \nabla_x \operatorname{sign}_{\varepsilon} \left(A(u) - A(v) \right) \cdot \nabla_y A(u) \phi \, dt \, dx.$$

Now using integrating (3.6), (3.1), and (3.2), we find that

$$(3.8) \qquad -\iiint_{\Pi_{T}\times\Pi_{T}} \operatorname{sign} (v-u) \nabla_{x} A(v) \cdot \nabla_{y} \phi \, dt \, dx \, ds \, dy$$
$$= -\iiint_{\Pi_{T}\times(\Pi_{T}\setminus\mathcal{E}_{v})} \operatorname{sign} (A(v) - A(u)) \nabla_{x} A(v) \cdot \nabla_{y} \phi \, dt \, dx \, ds \, dy$$
$$= -\lim_{\varepsilon \downarrow 0} \iiint_{(\Pi_{T}\setminus\mathcal{E}_{v})\times(\Pi_{T}\setminus\mathcal{E}_{v})} \nabla_{y} A(u) \cdot \nabla_{x} A(v) \operatorname{sign}_{\varepsilon}' (A(v) - A(u)) \phi \, dt \, dx \, ds \, dy.$$

Similarly, using (3.7), (3.1), and (3.2), we find that

(3.9)
$$- \iiint_{\Pi_T \times \Pi_T} \operatorname{sign} \left(A(u) - A(v) \right) \nabla_y A(u) \cdot \nabla_x \phi \, dt \, dx \, ds \, dy \\ = - \lim_{\varepsilon \downarrow 0} \iiint_{(\Pi_T \setminus \mathcal{E}_u) \times (\Pi_T \setminus \mathcal{E}_v)} \nabla_x A(u) \cdot \nabla_y A(v) \operatorname{sign}'_{\varepsilon} \left(A(v) - A(u) \right) \phi \, dt \, dx \, ds \, dy.$$

Adding (3.3) and (3.8) yields

$$(3.10) - \iiint_{\Pi_{T} \times \Pi_{T}} \left(|v - u| \phi_{t} + \operatorname{sign} (v - u) \left[\left(f(x, t, v) - f(x, t, u) \right) \cdot \nabla_{x} \phi \right. \\ \left. - \nabla_{x} A(v) \cdot \left(\nabla_{x} \phi + \nabla_{y} \phi \right) \right] - \operatorname{sign} (v - u) \left(\operatorname{div}_{x} f(x, t, u) - q(x, t, v) \right) \phi \right) dt \, dx \, ds \, dy \\ \left. \leq - \lim_{\varepsilon \downarrow 0} \iiint_{(\Pi_{T} \setminus \mathcal{E}_{u}) \times (\Pi_{T} \setminus \mathcal{E}_{v})} \left(\left| \nabla_{x} A(v) \right|^{2} - \nabla_{y} A(u) \cdot \nabla_{x} A(v) \right) \operatorname{sign}_{\varepsilon}' \left(A(v) - A(u) \right) \phi \, dt \, dx \, ds \, dy.$$

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(3.5)

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Similarly, adding (3.5) and (3.9) yields

$$(3.11) - \iiint_{\Pi_{T} \times \Pi_{T}} \left(|u - v| \phi_{s} + \operatorname{sign} (u - v) \left[\left(f(y, s, u) - f(y, s, v) \right) \cdot \nabla_{y} \phi \right. \\ \left. - \nabla_{y} A(u) \cdot \left(\nabla_{y} \phi + \nabla_{x} \phi \right) \right] - \operatorname{sign} (u - v) \left(\operatorname{div}_{y} f(y, s, v) - q(y, s, u) \right) \phi \right) dt \, dx \, ds \, dy \\ \leq - \lim_{\varepsilon \downarrow 0} \iiint_{(\Pi_{T} \setminus \mathcal{E}_{u}) \times (\Pi_{T} \setminus \mathcal{E}_{v})} \left(\left| \nabla_{y} A(u) \right|^{2} - \nabla_{x} A(u) \cdot \nabla_{y} A(u) \right) \operatorname{sign}_{\varepsilon}' \left(A(u) - A(v) \right) \phi \, dt \, dx \, ds \, dy.$$

Note that we can write

$$\operatorname{sign}(v-u)\left(f(x,t,v) - f(x,t,u)\right) \cdot \nabla_x \phi - \operatorname{sign}(v-u)\operatorname{div}_x f(x,t,u)\phi$$

= sign (v-u) $\left(f(x,t,v) - f(y,s,u)\right) \cdot \nabla_x \phi + \operatorname{sign}(v-u)\operatorname{div}_x \left[\left(f(y,s,u) - f(x,t,u)\right)\phi\right]$

and

$$\begin{aligned} \operatorname{sign}\left(u-v\right)\left(f(y,s,u)-f(y,s,v)\right)\cdot\nabla_{y}\phi-\operatorname{sign}\left(u-v\right)\operatorname{div}_{y}f(y,s,v)\phi\\ &=\operatorname{sign}\left(v-u\right)\left(f(x,t,v)-f(y,s,u)\right)\cdot\nabla_{y}\phi-\operatorname{sign}\left(v-u\right)\operatorname{div}_{y}\left[\left(f(x,t,v)-f(y,s,v)\right)\phi\right]\end{aligned}$$

Taking these identities into account when adding (3.10) and (3.11), we get

$$(3.12) \qquad - \iiint_{\Pi_T \times \Pi_T} \left(|v - u| \left(\phi_t + \phi_s \right) + I_1 + I_2 + I_3 \right) dt \, dx \, ds \, dy$$
$$\leq - \lim_{\varepsilon \downarrow 0} \iiint_{(\Pi_T \setminus \mathcal{E}_u) \times (\Pi_T \setminus \mathcal{E}_v)} \left| \nabla_x A(v) - \nabla_y A(u) \right|^2 \operatorname{sign}_{\varepsilon}' \left(A(v) - A(u) \right) \phi \, dt \, dx \, ds \, dy$$
$$\leq 0$$

where

$$I_1 = \operatorname{sign} (v - u) \left[f(x, t, v) - f(y, s, u) - (\nabla_x A(v) - \nabla_y A(u)) \right] \cdot (\nabla_x \phi + \nabla_y \phi)$$

$$I_2 = \operatorname{sign} (v - u) \left[\operatorname{div}_x \left[\left(f(y, s, u) - f(x, t, u) \right) \phi \right] - \operatorname{div}_y \left[\left(f(x, t, v) - f(y, s, v) \right) \phi \right] \right],$$

$$I_3 = \operatorname{sign} (v - u) \left(q(x, t, v) - q(y, s, u) \right) \phi.$$

We are now on familiar ground [23, 24] and introduce a nonnegative function $\delta \in C_0^{\infty}(\mathbb{R})$ which satisfies

$$\delta(\sigma) = \delta(-\sigma), \quad \delta(\sigma) \equiv 0 \text{ for } |\sigma| \ge 1, \quad \int_{\mathbb{R}} \delta(\sigma) \, d\sigma = 1.$$

For $\rho_0 > 0$, let

$$\delta_{\rho_0}(\sigma) = \frac{1}{\rho_0} \delta\left(\frac{1}{\rho_0}\right).$$

Pick two (arbitrary but fixed) Lebesgue points $\nu, \tau \in (0,T)$ of $||v(\cdot,t) - u(\cdot,t)||_{L^1(\mathbb{R}^d)}$. For any $\alpha_0 \in (0,\min(\nu, T - \tau))$, let

$$W_{\alpha_0}(t) = H_{\alpha_0}(t-\nu) - H_{\alpha_0}(t-\tau), \quad H_{\alpha_0}(t) = \int_{-\infty}^t \delta_{\alpha_0}(s) \, ds.$$

Inspired by [10], we introduce a nonnegative function $\omega \in C^{\infty}(\mathbb{R}_+)$ which satisfies

(3.13)
$$\omega(z) = 0 \text{ for } z \ge 1, \quad \omega'(z) \le 0 \text{ for } z \in (0,1), \quad \int_{\mathbb{R}^d} \omega(|z|^2) dz = 1.$$

For $\rho > 0$ and $x \in \mathbb{R}^d$, let

$$\omega_{\rho}(x) = \frac{1}{2\rho^d} \omega\left(\frac{|x|^2}{\rho^2}\right).$$

Similarly, adding (3.5) and (3.9) yields

$$-\int \int \int \int (|u - u| \phi_0 + u|_{2^2} (u - u) \int [f(y_1, z, u) - f(y_1, u, u)) \cdot \nabla_y \phi_1$$

Note that we can write

 $\sup_{n \in \mathbb{N}} (u - u) \left(f(x, t, u) - f(x, t, u) \right) \sum_{n \in \mathbb{N}} \sup_{n \in \mathbb{N}} (u - u) g(t_{n}, f(u), t, u) \right)$ $= \sup_{n \in \mathbb{N}} (u - u) \left(f(x, t, u) - f(x, t, u) \right) \sum_{n \in \mathbb{N}} f(u + u) g(t_{n}, f(u), t, u) - f(x, t, u) \right)$

bus

 $\sup_{i \in \mathcal{D}} (u - u) \left(f(u, a, b) - f(u, a, v) \right) - \nabla_{i} \phi - i f(u (u - u) d(v_{i}) f(u, a, b) \phi \right)$ $f(u, a, b) - f(u, a, v) f(v, c, u) - f(u, a, v) f(v, c, v) + f(u, a, i) d(v_{i}) f(v, c, b) + f(u, a, v) f(v) + f(v) +$

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$$I_{0} = alga (u - u) \left[f(u, t, u) - f(u, z, u) - (u, z, u) - \nabla_{u} f(u) - \nabla_{u} f(u) \right] \cdot \left[\nabla_{u} \mu + \nabla_{u} \eta \right]$$

$$I_{0} = alga (u - u) \left[auv_{u} \left[(f(u, u, u) - f(u, t, u)) \right] - d(u_{u}) \left[(f(u, t, u) - f(u, z, u)) \right] \right]$$

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We are now on funding ground [23, 24] and introduce a nonnegative fundines of C(21(20) which satisfies

fick two (arbitrary-bot fixed) intervals points $r \in (0, T)$ of $[m_1, d(r-u), t)|_{L^2(B_1, T)}$. For any $\in (0, \min(r, T - r))$, its

Inspired by [10], we infroduce a acase gotive function are C* (2.,) which estimate

$$1 = 2b \left(2 \ln |u|_{1/2} \int_{-\infty}^{\infty} (2 \ln |u|_{1/2} + 2 \ln |u|_$$

For $\mu > 0$ and $x \in \mathbb{R}^d$, let

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \int_{-\infty}^{\infty} d x \int_{-\infty}^{\infty} d x$$

Observe that

$$\nabla_x \omega_\rho(x-y) = \frac{1}{\rho^{d+2}} \,\omega\left(\frac{|x-y|^2}{\rho^2}\right)(x-y) = -\nabla_y \omega_\rho(x-y).$$

Moreover, we introduce $R \in C_0^{\infty}(\mathbb{R}_+)$ such that

$$R(z) = \begin{cases} 1, & z \in \left[0, \frac{1}{2}\right], \\ 0, & z \ge 1, \end{cases}$$

and $R'(z) \leq 0$ for all $z \geq 0$. For $\alpha > 0$ and $x \in \mathbb{R}^d$, let

$$R_{\alpha}(x) = R(\alpha |x|^2).$$

Observe that

$$\nabla_x R_\alpha \left(\frac{x+y}{2}\right) = \frac{\alpha(x+y)}{2} R' \left(\alpha \left|\frac{x+y}{2}\right|^2\right) = \nabla_y R_\alpha(x+y).$$

We now take ϕ to be of the form

(3.14)
$$\phi(x,t,y,s) = R_{\alpha}\left(\frac{x+y}{2}\right) W_{\alpha_0}(t) \,\omega_{\rho}(x-y) \delta_{\rho_0}(t-s) \in C_0^{\infty}(\Pi_T \times \Pi_T),$$

so that the derivatives of ϕ which are singular in the limit $\rho, \rho_0 \downarrow 0$ cancel:

(3.15)
$$\phi_t + \phi_s = R_\alpha \left(\frac{x+y}{2}\right) \left[\delta_{\alpha_0}(t-\nu) - \delta_{\alpha_0}(t-\tau)\right] \omega_\rho(x-y) \delta_{\rho_0}(t-s),$$
$$\nabla_x \phi + \nabla_y \phi = \left[\frac{\alpha(x+y)}{2} R' \left(\alpha \left|\frac{x+y}{2}\right|^2\right)\right] W_{\alpha_0}(t) \,\omega_\rho(x-y) \delta_{\rho_0}(t-s).$$

With ϕ as defined in (3.14), inequality (3.12) now takes the form

$$(3.16) \qquad -\iiint_{\Pi_T \times \Pi_T} |v(x,t) - u(y,s)| (\phi_t + \phi_s) dt dx ds dy \leq \iiint_{\Pi_T \times \Pi_T} (I_1 + I_2 + I_3) dt dx ds dy.$$

Sending $\alpha_* \alpha_0, \rho, \rho_0 \downarrow 0$ in (3.16) by an L^1 continuity argument, we get

(3.17)
$$\int_{\mathbb{R}^d} |v(x,\tau) - u(x,\tau)| \, dx$$
$$\leq \int_{\mathbb{R}^d} |v(x,\nu) - u(x,\nu)| \, dx + \lim_{\alpha,\alpha_0,\rho,\rho_0\downarrow 0} \iiint_{\Pi_T \times \Pi_T} \left(I_1 + I_2 + I_3 \right) \, dt \, dx \, ds \, dy.$$

Before we continue, let us write $I_2 = I_{2,1} + I_{2,2}$, where

$$I_{2,1} = \operatorname{sign} \left(v - u \right) \left[\left(f(y, s, u) - f(x, t, u) \right) \cdot \nabla_x \phi - \left(f(x, t, v) - f(y, s, v) \right) \cdot \nabla_y \phi \right],$$

$$I_{2,2} = \operatorname{sign} \left(v - u \right) \left(\operatorname{div}_y f(y, s, v) - \operatorname{div}_x f(x, t, u) \right) \phi.$$

Inserting this into (3.16), we get

(3.18)
$$\int_{\mathbb{R}^d} |v(x,\tau) - u(x,\tau)| \, dx \le \int_{\mathbb{R}^d} |v(x,\nu) - u(x,\nu)| \, dx + \lim_{\alpha,\alpha_0,\rho,\rho_0\downarrow 0} \Big(E_1 + E_2 + E_3 + E_4\Big),$$
where

where

$$E_{1} = \iiint_{\Pi_{T} \times \Pi_{T}} I_{1} dt dx ds dy \qquad E_{2} = \iiint_{\Pi_{T} \times \Pi_{T}} I_{2,1} dt dx ds dy,$$
$$E_{3} = \iiint_{\Pi_{T} \times \Pi_{T}} I_{2,2} dt dx ds dy, \qquad E_{4} = \iiint_{\Pi_{T} \times \Pi_{T}} I_{3} dt dx ds dy.$$

Olsorve share

$$\nabla_{\mu\nu}(x-x) = \frac{1}{p^{2}x^{2}} \omega \left(\frac{|x-y|^{2}}{p^{2}}\right) (x-y) = -\nabla_{\mu}(y) (x-y)$$

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and $R'(z) \leq 0$ for all $z \geq 0$. For $\alpha > 0$ and $\alpha \in \mathbb{R}^n$ let

$$P_{\alpha}(a) = P(a|a|^{2})$$

Observe that

$$(u+z)_{2} = \frac{(z+z)}{2} \approx \left(u \left(\frac{z+z}{2} \right) \right) = \frac{(u+z)}{2} \approx \left(u \left(\frac{z+z}{2} \right) \right)$$

We now take o to be of the form

$$(3.14) \qquad \phi(x,t,y,z) = S_{0}\left(\frac{z+1}{2}\right) W_{0}(t) + v(t) - v(t_{0}(t) + v) \in C_{0}^{\infty}(\Pi_{0}^{*} \in \Pi_{0}^{*}).$$

so that the derivatives of a which are singular to the limit p. as 1.0 cancel

$$(3.15)$$

$$n + n = R_{n} \left(\frac{2 + n}{2}\right) [n_{n}(t - n) - n_{n}(t) = r_{n}(t_{n} - n)_{n}(t_{n} -$$

With d us defined in (3.14), inequality (3.12) now tables due form

$$(3.16) = - \iiint \left[\ln(\alpha, \beta) - u(\alpha, \beta) \ln(\alpha + \alpha_0) \ln(\alpha + \alpha_0) \ln(\alpha + \alpha_0) + \ln(\alpha, \beta_0) \ln(\alpha, \beta_0) + \ln(\alpha, \beta_0)$$

Sending of an p. o. 1 in (3.16) by an E¹ continuuty approximately we get

$$(3.17) = \int_{\mathbb{R}^{2}} \left[h(a, r) - u(a, r) \right] dr + \int_{a,a} \lim_{n \to \infty} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} h(a, n) - u(a, n) \int_{\mathbb{R}^{2}} dr + \int_{a,a} \lim_{n \to \infty} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} h(a, n) - u(a, n) \int_{\mathbb{R}^{2}} dr + \int_{a,a} \lim_{n \to \infty} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} h(a, n) - u(a, n) \int_{\mathbb{R}^{2}} dr + \int_{a,a} \lim_{n \to \infty} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} h(a, n) - \int_{\mathbb{R}^{2}} h(a, n) \int_{\mathbb{R}^{2}} h$$

before we continue, let us write by a high when

$$I_{2,2} = \operatorname{sign} \left(u - u \right) \left[\left(f(y, x, u) - f(z, z, u) \right) \cdot \nabla_{-u} - \left(f(z, z, u) - f(y, x, u) \right) \cdot \nabla_{+u} \right) \\ I_{2,2} = \operatorname{sign} \left(u - u \right) \left(\operatorname{sin}_{+} f(y, u, v) - \operatorname{sin}_{+} f(z, z, u) \right) u_{-} \right]$$

Inserting this into (3.16), we get

$$(3.18) \int_{\mathbb{R}^{d}} |u(x,r)| - u(x,r)| dx \leq \int_{\mathbb{R}^{d}} |u(x,r)| - u(x,r)| dx| + \lim_{n \to \infty} \int_{\mathbb{R}^{d}} |x| + B_{2,2} E_{n} + B_{n} \\$$

Sending $\rho, \rho_0 \downarrow 0$ in E_1 using (1.5) and an L^1 continuity argument, we get

$$\lim_{\rho,\rho_0\downarrow 0} E_1 = \iint_{\Pi_T} \operatorname{sign} \left(v(x,t) - u(x,t) \right) \left(f(x,t,v(x,t)) - f(x,t,u(x,t)) \right) \cdot x \, \alpha R'(\alpha|x|^2) W_{\alpha_0}(t) \, dt \, dx$$

$$\xrightarrow{\to 0 \text{ as } \alpha \downarrow 0 \text{ due to } (1.5) \text{ (see also } (2.4))} \int_{\Pi_T} \int_{\Pi_T}$$

$$- \underbrace{\iint_{T_T} \operatorname{sign} \left(v(x,t) - u(x,t) \right) \left(\nabla_x A(v(x,t)) - \nabla_x A(u(x,t)) \right) \cdot x \, \alpha R'(\alpha |x|^2) W_{\alpha_0}(t) \, dt \, dx}_{\to 0 \text{ as } \alpha \downarrow 0 \text{ due to } (\mathbf{D}.3)}$$

Equipped with (1.5), we can subsequently send $\alpha, \alpha_0 \downarrow 0$ to obtain

(3.19)
$$\lim_{\alpha,\alpha_0,\rho,\rho_0\downarrow 0,} E_1 = 0.$$

Next, using (1.6), (1.4), and an L^1 continuity argument, we get

$$\lim_{\alpha,\alpha_0,\rho,\rho_0\downarrow 0} \left(E_3 + E_4 \right) \le \operatorname{Const} \int_{\nu}^{\tau} \int_{\mathbb{R}^d} |v(x,t) - u(x,t)| \, dt \, dx.$$

It remains to pass to the limit in E_2 . To this end, we introduce the shorthand notation Ψ_1, Ψ_2 :

(3.20)

$$\Psi_{1} = R_{\alpha} \left(\frac{x+y}{2}\right) (x-y) \frac{1}{\rho^{d+2}} \omega' \left(\frac{|x-y|^{2}}{\rho^{2}}\right)$$

$$\Psi_{2} = \frac{\alpha(x+y)}{2} R' \left(\alpha \left|\frac{x+y}{2}\right|^{2}\right) \omega_{\rho}(x-y),$$

so that $\nabla_x \left[R_\alpha \left(\frac{x+y}{2} \right) \omega_\rho(x-y) \right] = \Psi_1 + \Psi_2$. If we take into account the second part of (3.15), then $I_{2,1}$ can be rewritten as

$$I_{2,1} = (F(x, t, v, u) - F(y, s, v, u)) \cdot \nabla_x \phi - \operatorname{sign} (v - u) (f(x, t, v) - f(y, s, v)) \cdot \Psi_2 W_{\alpha_0}(t) \delta_{\rho_0}(t - s),$$

where F is defined in (1.8). Sending $\alpha_0, \rho_0 \downarrow 0$ in E_2 (again using (1.5) and an L^1 continuity argument), we obtain

$$\lim_{t_{0},\rho_{0}\downarrow 0} E_{2} = \int_{\nu}^{\tau} \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \left(F(x,t,v(x,t),u(y,t)) - F(y,t,v(x,t),u(y,t)) \right) \cdot \Psi_{1} \, dy \, dx \, dt + \underbrace{\int_{\nu}^{\tau} \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \operatorname{sign} \left(v(x,t) - u(y,t) \right) \left(f(x,t,u) - f(y,t,u) \right) \cdot \Psi_{2} \, dy \, dx \, dt}_{\rightarrow 0 \text{ as } \alpha \downarrow 0 \text{ due to } (1.5)}$$

Taking (1.7) into account, we have

$$\left(F(x,t,v(x,t),u(y,t)) - F(y,t,v(x,t),u(y,t)) \right) \cdot (x-y) \frac{1}{\rho^{d+2}} \omega' \left(\frac{|x-y|^2}{\rho^2} \right)$$

 $\leq \gamma |v(x,t) - u(y,t)| \frac{|x-y|^2}{\rho^2} \frac{1}{\rho^d} \left| \omega' \left(\frac{|x-y|^2}{\rho^2} \right) \right| \leq \gamma |v(x,t) - u(y,t)| \max |\omega'| \frac{1}{\rho^d} \mathbf{1}_{|x-y| < \rho}.$

From this we obtain the following estimate

$$\begin{split} \lim_{x,\rho\downarrow} \int_{\nu}^{\tau} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\left(F(x,t,v(x,t),u(y,t)) - F(y,t,v(x,t),u(y,t)) \right) \cdot \Psi \right) \\ &\leq \lim_{\rho\downarrow 0} \frac{\operatorname{Const}}{\rho^d} \int_{\nu}^{\tau} \int_{\mathbb{R}^d} \int_{|x-y|<\rho} |v(x,t) - u(y,t)| \, dy \, dx \, dt. \\ &= \operatorname{Const} \int_{\nu}^{\tau} \int_{\mathbb{R}^d} |v(x,t) - u(x,t)| \, dx \, dt. \end{split}$$

leading $\rho, \rho_0 \downarrow 0$ in Eq. many (1.5) and on L^1 constrainty argument, we get

$$\lim_{n \to \infty} E_1 = \iint \max \{n(x, b) - n(x, b)\} \{f(x, b, b(x, b)) - f(x, b, a(x, b))\} + a \, dt^2 \{a|x|^2\} W_{ab}(t) \, dt \, dx$$

 $\operatorname{det} (\mathfrak{g}) = \mathfrak{W}(\operatorname{Subol}^{\mathcal{S}}) = \mathfrak{V}(\mathfrak{g}(\mathfrak{g},\mathfrak{g}) = \mathfrak{V}(\mathfrak{g}(\mathfrak{g},\mathfrak{g})) = \mathfrak{V}(\mathfrak{g}(\mathfrak{g})) = \mathfrak{V}(\mathfrak{g})$ = \mathfrak{V}(\mathfrak{g}(\mathfrak{g})) = \mathfrak{V}(\mathfrak{g}) = \mathfrak{V}(\mathfrak{g})) = \mathfrak{V}(\mathfrak{g}) = \mathfrak{

Equipped with (1.5), we can subsequently east a, no 1.0 to eodals

Next, print (1.5), (1.4), and an L' continuity regularate we get

$$\lim_{n \to \infty} \left(E_1 + E_n \right) \leq \operatorname{Const} \left(\int_{-\infty} \left[e(x, t) - u(x, t) \right] dt dt,$$

It remains to pass to the limit in $E_{\rm e}$. Relativity we introduce the shorthand botation M_{\odot} , Ψ_{2} ,

so that $\nabla_{\mathbf{r}} \left[R_{\alpha} \left(\frac{\alpha+\mu}{2} \right) \omega_{\alpha} (\alpha+\mu) \right] = \Psi_{\beta} + \Psi_{\beta}$. If we take into account the second pair of (3.1.5), then $I_{\alpha,\beta}$ can be rewritten as

$$g = (R(z, t, u, u) - R(z, s, t, u)), R_{ud}$$

$$= (R(z, t, u, u) - R(z, s, t, u)), R_{ud} = R_{ud} (0.5_{ub} (t - u))$$

where F is defined in (1.3). Sending ∞ , α , β , β in E_{2} (approximating (1.5) and α , b^{1} contrologily argument), we obtain

Taking (1.7) into account, we have

$$\leq \operatorname{vis}(x,t) - \operatorname{vis}(x)t \frac{1}{2} \sum_{i=1}^{n} \left[\operatorname{vis}\left(\frac{1}{2} \sum_{i=1}^{n} \left[\frac{1}{2} \sum_{i=1}^{n} \left[\frac{vis}\left(\frac{1}{2} \sum_{i=1}^{n} \left[\frac{vis}\left(\frac{1}{2} \sum_{i=$$

From this we obtain the following estimate.

$$\frac{1}{2} \frac{1}{2} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}}$$

Summing up, we have proved that

$$\int_{\mathbb{R}^d} |v(x,\tau) - u(x,\tau)| \, dx$$

$$\leq \int_{\mathbb{R}^d} |v(x,\nu) - u(x,\nu)| \, dx + C \int_{\nu}^{\tau} \int_{\mathbb{R}^d} |v(x,t) - u(x,t)| \, dx \, dt,$$

for some constant C > 0 depending on f, q and the test function. Sending $\nu \downarrow 0$ and then using Gronwall's lemma, we get

(3.21)
$$\int_{\mathbb{R}^d} |v(x,\tau) - u(x,\tau)| \, dx \le e^{C\tau} \int_{\mathbb{R}^d} |v(x,0) - u(x,0)| \, dx \equiv 0.$$

Since this inequality holds for almost all $\tau \in (0,T)$, we can conclude that v = u a.e. in Π_T .

4. PROOF OF THEOREM 1.2

In this section, we restrict ourselves to problems of the form (1.10), i.e., f(x,t,u) = k(x)f(u)and $q(x,t,u) \equiv 0$. Let $u, v \in L^{\infty}(0,T; BV(\mathbb{R}^d))$ be two entropy solutions of (1.10) with initial data $u_0, v_0 \in L^1(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$, respectively. As before, we are interested in estimating the L^1 distance between v and u. In what follows, the test function $\phi = \phi(x, t, y, s)$ is still the one defined in (3.14). Repeating everything up to (3.17), we find that

(4.1)
$$\int_{\mathbb{R}^d} |v(x,\tau) - u(x,\tau)| \, dx \le \int_{\mathbb{R}^d} |v(x,\nu) - u(x,\nu)| \, dx + \lim_{\alpha,\alpha_0,\rho,\rho_0\downarrow 0} \Big(E_1 + E_2 + E_3 \Big),$$

where

$$E_{1} = \iiint_{\Pi_{T} \times \Pi_{T}} \operatorname{sign} (v - u) \left[k(x)f(v) - k(y)f(u) - (\nabla_{x}A(v) - \nabla_{y}A(u)) \right] \cdot (\nabla_{x}\phi + \nabla_{y}\phi) \, dt \, dx \, ds \, dy$$
$$E_{2} = \iiint_{\Pi_{T} \times \Pi_{T}} \operatorname{sign} (v - u) \left[\left(k(y)f(u) - k(x)f(u) \right) \cdot \nabla_{x}\phi - \left(k(x)f(v) - k(y)f(v) \right) \cdot \nabla_{y}\phi \right] dt \, dx \, ds \, dy,$$
$$E_{3} = \iiint_{\Pi_{T} \times \Pi_{T}} \operatorname{sign} (v - u) \left(\operatorname{div}_{y}k(y)f(v) - \operatorname{div}_{x}k(x)f(u) \right)\phi.$$

As before (3.19), it is not difficult to show that

$$\lim_{\alpha \to \alpha} E_1 = 0$$

Next we estimate E_2 . To this end, introduce the function

(4.3)
$$F(v,u) := \operatorname{sign} (v-u) \left[f(v) - f(u) \right]$$

and observe that from the identity (3.15) we have

$$E_{2} = \iiint_{\Pi_{T} \times \Pi_{T}} \left(\left(k(x) - k(y) \right) F(v, u) \cdot \nabla_{x} \phi - \operatorname{sign} \left(v - u \right) f(v) \left(k(x) - k(y) \right) \cdot \Psi_{2} W_{\alpha_{0}}(t) \delta_{\rho_{0}}(t - s) \right) dt \, dx \, ds \, dy,$$

where Ψ_2 is defined in (3.20). To continue, we need the following simple lemma (whose easy proof can be found in, e.g., [5]):

Lemma 4.1. Consider a function z = z(x) belonging to $L^{\infty}(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ and let $h \in \text{Lip}(I_z)$. Then h(z) belongs to $L^{\infty}(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ and

$$\left|\frac{\partial}{\partial x_j}h(z)\right| \le ||h||_{\operatorname{Lip}(I_z)} \left|\frac{\partial}{\partial x_j}z\right| \text{ in the sense of measures, } j=1,\ldots,d,$$

14

Summing up, we have proved that

$$\int \left\{ \int_{\mathbb{R}^{2}} \left[h(x, r) - u(x, r) \right] dx + O \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \left[h(x, r) - u(x, r) \right] dx dr.$$

for some constant C > 0 depending on f_1 q and the test humanon. Sending $\nu \downarrow 0$ and then using Gronwall's lateria, we get

Since this inequality holds for alchest all $r \in \{0, T\}$, we can equivale that $v \in u$ i.e. Π_{T}

A. PROOF OF TRANSMAN 1.2.

In this section, we restrict outselves to problems of the dots (1,10) i.e., (levels) = when ((1,10) and $q(x,t,u) \equiv 0$. Let $u \neq 1$. (0,7), $B^*(0,7)$, $B^$

$$(4.1) \qquad \int_{a_{1}} [n(z,\tau) - n(z,\tau)] dz \leq \int_{a_{1}} [u(z,v) - u(z,v)] dz + \lim_{u,v,v \neq a_{1},v} (E_{1}v + E_{1}v + E_{2})]$$

stad a

$$E_{i} = \iiint_{i=1}^{i} \left[f(x) + u \right] \left[h(x) f(x) - h(y) f(u) \right]$$

$$(0, A(u) = \nabla_{\mu}A(u))$$
, $(\nabla_{\mu}\phi + \nabla_{\mu}\phi) dv dv dv$

$$\phi_{1} \nabla \cdot \left((u) \left\{ (u) \right\} - (u) \left\{ (u) \right\} \right\} \left\{ (u - u) \right\} = c$$

 $-(h(s))f(t) - h(t)f(s) - \nabla_{t} \phi(t) dt dt ds dt,$

$$E_{0} = \iint \int \int d\mathbf{x} n \left(u - u \right) \left(dh v_{\mu} h(\mu) f(v) - dh v_{\mu} h(u) f(u) \right) dx$$

As before (0.19), it is not difficult to show that

(4.2)
$$E_1 = 0$$

vert we estimate if. To this end, mundere the institut

$$F(u,u) := sign(v - u) f(u) - f(u)$$

and observe that from the identify (3.15) we have

$$E_{2} = \iiint_{B_{2}\times G_{2}} \left(\left(k(x) - k(y) \right) F(y,y) \cdot \nabla_{x} y \right)$$

(a) = a(a) = (a) = (a)

where Ψ_2 is defined in (3.20). To combine, we need the following simple lemma (where easy score can be found in , e.g., [5]):

Lemma 4.1. Consider a function z = v(z) belonging to $L^{\infty}(\mathbb{R}^{n})$ if $EV(\mathbb{R}^{n})$ und let $u \in Lip(L)$. Thus u(z) belongs to $L^{\infty}(\mathbb{R}^{n})$ if $EV(\mathbb{R}^{n})$ and

$$\left|\frac{\partial}{\partial t_{0}}h(z)\right| \leq \left|\left|h\right|\right|_{L^{\infty}(z)}$$
 on the sums of metasters, $y = 1, \dots, \infty$

where I_z denotes the interval $\left[-||z||_{L^{\infty}(\mathbb{R}^d)}, ||z||_{L^{\infty}(\mathbb{R}^d)}\right]$.

Note that the function F(v, u) defined in (4.3) is locally Lipschitz continuous in v and u with Lipschitz constant that of f. Now since $v(\cdot, t) \in L^{\infty}(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ for each t, by Lemma 4.1 $\nabla_x F(v, u)$ is a finite measure. After an integration by parts, we thus get

$$E_{2} = -\iiint_{\Pi_{T} \times \Pi_{T}} \left(\operatorname{div}_{x} k(x) F(v, u) \phi \, dt \, dx \, ds \, dy + \left(k(x) - k(y) \right) \cdot \nabla_{x} F(v, u) \phi \right. \\ \left. + \operatorname{sign} \left(v - u \right) f(v) \left(k(x) - k(y) \right) \cdot \Psi_{2} W_{\alpha_{0}}(t) \delta_{\rho_{0}}(t - s) \right) dt \, dx \, ds \, dy.$$

Since $k \in L^1_{loc}(\mathbb{R}^d)$ and $\nabla_x F(v, u)$ is a finite measure, it follows that

$$\iiint_{\Pi_T \times \Pi_T} \left(\left(k(x) - k(y) \right) \cdot \nabla_x F(v, u) \phi + \operatorname{sign} \left(v - u \right) f(v) \left(k(x) - k(y) \right) \cdot \Psi_2 W_{\alpha_0}(t) \delta_{\rho_0}(t - s) \right) dt \, dx \, ds \, dy \to 0 \text{ as } \rho \downarrow 0.$$

Consequently, we end up with

$$\lim_{\alpha_0,\rho,\rho_0\downarrow 0} E_2 = -\int_{\nu}^T \int_{\mathbb{R}^d} \operatorname{div} k(x) F(v(x,t), u(x,t)) R_{\alpha}(x) \, dx \, dt.$$

Finally, since $k \in W^{1,1}_{loc}(\mathbb{R}^d)$, the usual L^1 continuity argument gives

(4.4)
$$\lim_{\alpha_0,\rho,\rho_0\downarrow 0} E_3 = \int_{\nu}^{\tau} \int_{\mathbb{R}^d} \operatorname{div} k(x) F(v(x,t), u(x,t)) R_{\alpha}(x) \, dt \, dx \equiv -\lim_{\alpha_0\rho,\rho_0\downarrow 0} E_2.$$

From (4.1), (4.2), and (4.4), we get

$$\int_{\mathbb{R}^d} |v(x,\tau) - u(x,\tau)| \, dx \le \int_{\mathbb{R}^d} |v(x,\nu) - u(x,\nu)| \, dx \to \int_{\mathbb{R}^d} |v(x,0) - u(x,0)| \, dx \text{ as } \nu \downarrow 0.$$

Since $\tau \in (0,T)$ was an arbitrary Lebesgue point of $||v(\cdot,t) - u(\cdot,t)||_{L^1(\mathbb{R}^d)}$, we immediately obtain the L^1 contraction property claimed in Theorem 1.2.

5. Proof of Theorem 1.3

In this section, we are going to estimate the L^1 difference between the entropy solution v of (1.13) and the entropy solution u of (1.10). To do this, we proceed exactly as in the proof of Theorem 1.1. In what follows, we let $\phi = \phi(x, t, y, s)$ be an arbitrary test function on $\Pi_T \times \Pi_T$.

Similarly to (3.10), we can derive the following integral inequality for the entropy solution v = v(x, t) of (1.13):

(5.1)

$$-\iiint_{\Pi_{T}\times\Pi_{T}}\left(|v-u|\phi_{t}+\operatorname{sign}(v-u)\left[l(x)\left(g(v)-g(u)\right)\cdot\nabla_{x}\phi\right.\right.\right.$$

$$\left.-\nabla_{x}A(v)\cdot\left(\nabla_{x}\phi+\nabla_{y}\phi\right)\right]-\operatorname{sign}(v-u)\operatorname{div}_{x}l(x)g(u)\phi\right)dt\,dx\,ds\,dy$$

$$\leq -\lim_{\varepsilon\downarrow 0}\iiint_{(\Pi_{T}\setminus\mathcal{E}_{u})\times(\Pi_{T}\setminus\mathcal{E}_{v})}\left(\left|\nabla_{x}A(v)\right|^{2}-\nabla_{y}A(u)\cdot\nabla_{x}A(v)\right)\operatorname{sign}_{\varepsilon}'\left(A(v)-A(u)\right)\phi\,dt\,dx\,ds\,dy.$$

where I, denotes the interval - lat second it haven

Note that the function $\mathcal{F}(u, u)$ defined in (4.3) is locally Lipschitz continuous in a and a with Lipschitz constant that of f. Now since $u(\cdot, t) \in L^{\infty}(\mathbb{R}^{n})$ is $\mathbb{R}^{n}(\mathbb{R}^{n})$ for each t, by Lemma 4.4 $\nabla_{\tau} F(u, u)$ is a finite measure. After an integration by parts, are thus get

$$E_{1} = - \iint_{\mathbb{R}^{2}} \left((Invit(a))^{p}(u, a) a dt dx as dy + (b(a) - b(y))^{-1} \nabla_{u} E(y, a) d \right)$$

$$+ \sin ab + (a - a) f(a) (b(a) - b(a)) < 0 = (a - a) = (a - a) + (a - a) = (a$$

Since $k \in L_{i,n}(\mathbb{R}^n)$ and $\nabla_{\mathbf{e}} f(\mathbf{v}, \mathbf{u})$ is a finite measure, it follows that

$$\iiint_{i=1}^{n} \left(\left(k(a) - k(a) \right) \cdot \nabla_{a} F(a, a) \phi_{i}(a, a) \phi_{$$

的现在分子的是他们的第三人称单数。他们的。他们的这个人的是我们的是你的。

Consequently, as ead up with

$$\lim_{x \to 0} |B| = -\int \int_{\mathbb{R}^d} dx h(x) F(x(x, \Omega, u(x, t)) |B_0(x) dx dt)$$

Finally, since $k \in W_{i,j}^{\infty}(\mathbb{R}^{n})$, the usual L^{1} establishing a symmetry gives

$$(4,4) \qquad \lim_{n \to \infty} B_n = \int_{\mathbb{R}^n} div \mathbf{x}(n) \mathcal{P}(n(n,1), u(n,2)) \mathcal{P}_n(n) dn dn dn = \lim_{n \to \infty} B_n dn$$

From (4.1), (4.2), and (4.4), we get

$$\int_{\mathbb{R}^{N}} |v(x,\tau) - u(x,\tau)| \, dx \leq \int_{\mathbb{R}^{N}} |v(x,x) - v(x,x)| \, dx \rightarrow \int_{\mathbb{R}^{N}} |v(x,0) - v(x,0)| \, dx \, ux \, v \geq 0.$$

Since $r \in (0, T)$ was an arbitrary industry plant of $[n(\cdot, t) - n(\cdot, t)]_{E^1(\alpha, t)}$ we immediately obtain the L^1 contraction property distance in Theorem 1.2.

PROOF OF TREAMEN 1.3

In this section, we are point to selvate the E^2 difference between the encouve volution u of (1.13) and the encouve solution u of (1.13). In this, we protocol exactly us in the wood of Theorem 1.1. In what follows, we have a set u, v, v?) be an arbitrary test function of H_1 . Similarly to (3.10), we can denote the following integral inequality for the entropy solution v = v(u, v, v) be an arbitrary test function of H_1 .

Similarly to (3.11), we can derive the following inequality for the entropy solution u = u(y, s) of (1.10): (5.2)

$$-\iiint_{\Pi_{T}\times\Pi_{T}}\left(|u-v|\phi_{s}+\operatorname{sign}(u-v)\left[k(y)(f(u)-f(v))\cdot\nabla_{y}\phi\right.\right.\\\left.\left.-\nabla_{y}A(u)\cdot(\nabla_{y}\phi+\nabla_{x}\phi)\right]-\operatorname{sign}(u-v)\operatorname{div}_{y}k(y)f(v)\phi\right)dt\,dx\,ds\,dy\\\leq-\lim_{\varepsilon\downarrow0}\iiint_{(\Pi_{T}\setminus\mathcal{E}_{u})\times(\Pi_{T}\setminus\mathcal{E}_{v})}\left(\left|\nabla_{y}A(u)\right|^{2}-\nabla_{x}A(v)\cdot\nabla_{y}A(u)\right)\operatorname{sign}_{\varepsilon}'\left(A(u)-A(v)\right)\phi\,dt\,dx\,ds\,dy.$$

Next we write

$$\operatorname{sign}(v-u) l(x) (g(v) - g(u)) \cdot \nabla_x \phi - \operatorname{sign}(v-u) \operatorname{div}_x l(x) g(u) \phi$$

= sign (v-u) (l(x)g(v) - k(y)f(u)) \cdot \nabla_x \phi + sign (v-u) \operatorname{div}_x [(k(y)f(u) - l(x)g(u)) \phi]

and

$$\begin{aligned} \operatorname{sign} (u-v) \, k(y) \big(f(u) - f(v) \big) \cdot \nabla_y \phi &- \operatorname{sign} (u-v) \operatorname{div}_y k(y) g(v) \phi \\ &= \operatorname{sign} (v-u) \, \big(l(x) g(v) - k(y) f(u) \big) \cdot \nabla_y \phi - \operatorname{sign} (v-u) \operatorname{div}_y \big[\big(l(x) g(v) - k(y) g(v) \big) \phi \big]. \end{aligned}$$

Similarly to (3.12), adding (3.10) and (3.11) we obtain

(5.3)
$$-\iiint_{\Pi_T \times \Pi_T} \left(|v - u| \left(\phi_t + \phi_s \right) + I_1 + I_2 \right) dt \, dx \, ds \, dy \le 0,$$

where

$$I_1 = \operatorname{sign} (v - u) \left[l(x)g(v) - k(y)f(u) - (\nabla_x A(v) - \nabla_y A(u)) \right] \cdot (\nabla_x \phi + \nabla_y \phi)$$

$$I_2 = \operatorname{sign} (v - u) \left[\operatorname{div}_x \left[\left(k(y)f(u) - l(x)g(u) \right) \phi \right] - \operatorname{div}_y \left[\left(l(x)g(v) - k(y)f(v) \right) \phi \right] \right].$$

At this stage, we need to choose a suitable test function ϕ . In view of (1.14), we will not use the test function defined (3.14), but the simpler one

(5.4)
$$\phi(x,t,y,s) = W_{\alpha_0}(t)\delta_{\rho}(x-y)\delta_{\rho_0}(t-s), \qquad \rho, \rho_0 > 0,$$

so that (ν and τ are as before arbitrary but fixed Lebesgue points in (0,T))

$$\phi_t + \phi_s = \left[\delta_{\alpha_0}(t-\nu) - \delta_{\alpha_0}(t-\tau)\right]\delta_{\rho}(x-y)\delta_{\rho_0}(t-s), \qquad \nabla_x \phi + \nabla_y \phi \equiv 0.$$

Before we continue, let us write $I_2 = I_{2,1} + I_{2,2}$ with

$$I_{2,1} = \operatorname{sign} (v-u) \left[\left(k(y)f(u) - l(x)g(u) \right) \cdot \nabla_x \phi - \left(l(x)g(v) - k(y)f(v) \right) \cdot \nabla_y \phi \right],$$

$$I_{2,2} = \operatorname{sign} (v-u) \left(\operatorname{div}_y k(y)f(v) - \operatorname{div}_x l(x)g(u) \right) \phi.$$

With the test function ϕ defined in (5.4), we can send $\alpha_0, \rho, \rho_0 \downarrow 0$ as usual and get

(5.5)
$$\int_{\mathbb{R}^d} |v(x,\tau) - u(x,\tau)| \, dx \le \int_{\mathbb{R}^d} |v(x,\nu) - u(x,\nu)| \, dx + \lim_{\alpha_0,\rho,\rho_0\downarrow 0} \left(E_1 + E_2 \right),$$

where

$$E_1 = \iiint_{\Pi_T \times \Pi_T} I_{2,1} dt dx ds dy, \quad E_2 = \iiint_{\Pi_T \times \Pi_T} I_{2,2} dt dx ds dy.$$

Taking into account the identity $\nabla_y \phi = -\nabla_x \phi$, we get

$$I_{2,1} = \left(l(x)G(v,u) - k(y)F(v,u) \right) \cdot \nabla_x \phi,$$

where F is defined in (4.3) and G is defined by the same formula but with f replaced by g. Since $v(\cdot,t) \in L^{\infty}(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ for each t and F, G are locally Lipschitz continuous, $\nabla_x F(v,u)$ and

minute to (3.11), we can desire the following manuality for the astrony solution v = v

Next we write

 $= \operatorname{dgn}(u + u) \left(l(u) d(u) - b(u) f(u) \right) \cdot \nabla_{u} d + \operatorname{dgn}(u + u) \operatorname{div}\left(b(u) f(u) - b(u) d(u) \right) d = 0$

 $\sup_{x \in \mathcal{X}} (u - v) \lambda(y)(f(u) - f(v)) \cdot \nabla_y g - \sup_{x \in \mathcal{X}} (u - v) d(v_y h(y)y(v)g)$ $= \sup_{x \in \mathcal{X}} (u - v) (f(v) h(v) - h(y)f(v)) \cdot \nabla_y g - \sup_{x \in \mathcal{X}} (v - v) d(v_y)(y(v)) - h(y)g$

Similarly to (3.12), adding (3.10) and (3.11) we obtain

where

$$I_{1} = a_{0}n(v - u) \left[(a_{0})r(v) - h(y)f(u) - (\nabla_{x}A(v) - \nabla_{y}A(v)) \right] - (\nabla_{x}A + \nabla_{y}A(v)) \\ I_{2} = a_{0}n(v - u) \left[a_{0}r_{x} \left[(h(v)f(v) - h(x)g(v)) \right] \right] + dhv_{0} \left[(h(v)f(v) - h(x))f(v) \right] \\ A_{1} = a_{0}n(v - u) \left[a_{0}r_{x} \left[(h(v)f(v) - h(x)g(v)) \right] \right] + dhv_{0} \left[(h(v)f(v) - h(x))f(v) \right] \\ A_{2} = a_{0}n(v - u) \left[a_{0}r_{x} \left[(h(v)f(v) - h(x)g(v)) \right] \right] + dhv_{0} \left[(h(v)f(v) - h(x))f(v) \right] \\ A_{2} = a_{0}n(v - u) \left[a_{0}r_{x} \left[(h(v)f(v) - h(x)g(v)) \right] \right] + dhv_{0} \left[(h(v)f(v) - h(x))f(v) \right] \\ A_{2} = a_{0}n(v - u) \left[a_{0}r_{x} \left[(h(v)f(v) - h(x)g(v)) \right] \right] + dhv_{0} \left[(h(v)f(v) - h(x)g(v) \right] \right] \\ A_{3} = a_{0}n(v - u) \left[a_{0}r_{x} \left[(h(v)f(v) - h(x)g(v) \right] \right] + dhv_{0} \left[(h(v)f(v) - h(x)g(v) \right] \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(x)g(v) \right] + dhv_{0} \left[(h(v)f(v) - h(x)g(v) \right] \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] + dhv_{0} \left[(h(v)f(v) - h(v)f(v) \right] \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] + dhv_{0} \left[(h(v)f(v) - h(v)f(v) \right] \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] + dhv_{0} \left[(h(v)f(v) - h(v)f(v) \right] \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] + dhv_{0} \left[(h(v)f(v) - h(v)f(v) \right] \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] + dhv_{0}n(v - u) \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] + dhv_{0}n(v - u) \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] + dhv_{0}n(v - u) \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] + dhv_{0}n(v - u) \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] \\ A_{3} = a_{0}n(v - u) \left[(h(v)f(v) - h(v)f(v) \right] \\ A_{3} = h(v)f(v) \left[(h(v)f(v) - h(v)f(v) \right]$$

At this stage, we need to choose a mitable test function \$. In view of (1.14), we will not use the test function defined (3.14), but the simpler one

$$a < a_1 a_1 \cdots a_{n-1} a_n (a - a_{n-1}) a_n (1) a_n (1 - (a - a_{n-1}) a_n) a_n (1 - (a - a_{n-1}) a_n) a_n (a - a_{n-1}) a_n (a - a_{n-$$

 $\phi_{2} + \phi_{2} = \{\overline{a}_{2n}(t-t) - \delta_{2n}(t-t)\}\delta_{2n}(t-t), \quad \nabla_{2n} + \nabla_{2n} \equiv 0.$

Before we could up, let us write $i_2 = i_2 + i_3 g$ with

$$I_{2,2} = \operatorname{sign}(v - v) \left[\left(h(v) f(v) - f(v) g(v) \right) \right] \nabla_v u - \left(f(v) g(v) - h(v) f(v) \right) \cdot \nabla_v u \\ I_{2,2} = \operatorname{sign}(v - v) \left[\operatorname{sign}_v f(v) f(v) - \operatorname{sign}_v f(v) f(v) \right]$$

With the test firstlike, d'defined in (5.4), we can send do y, 6. 10 as usablend

$$\int_{\mathbb{R}^{d}} \left[u(a, a) - u(a, a) \right] da \leq \int_{\mathbb{R}^{d}} \left[u(a, a) - u(a, a) \right] da + \int_{\mathbb{R}^{d}} u(a, a) \int_{\mathbb{R}^{d}} u(a, a) da + \int_{\mathbb{R}^{d}} u(a, a) \int_{\mathbb{R}^{d}} u(a, a) da + \int_{\mathbb{R}^{d}} u(a, a) \int_{\mathbb{R}^{d}} u(a,$$

where F is defined in (4.3) and G is defined as the same termula but with f replaced by g. Since $u(\cdot, t) \in L^{\infty}(\mathbb{R}^{d}) \cap B^{1}(\mathbb{R}^{d})$ but each t and $F \subseteq u_{0}$ are boally langthing continuous, $\nabla_{\tau} F(u, u)$ and

$$\begin{split} E_1 &= \iiint_{\Pi_T \times \Pi_T} \left(-\operatorname{div}_x l(x) G(v, u) - l(x) \cdot \nabla_x G(v, u) + k(y) \cdot \nabla_x F(v, u) \right) \phi \, dt \, dx \, ds \, dy \\ &= \iiint_{\Pi_T \times \Pi_T} \left(-\operatorname{div}_x l(x) G(v, u) + \left(k(y) - l(x) \right) \cdot \nabla_x G(v, u) \right. \\ &+ k(y) \cdot \nabla_x \left(F(v, u) - G(v, u) \right) \right) \phi \, dt \, dx \, ds \, dy. \end{split}$$

By adding and subtracting identical terms, we obtain

$$-\operatorname{div}_{x}l(x)G(v,u)\phi + I_{2,2}$$

= sign $(v-u)\operatorname{div}_{y}k(y)f(v)$ - sign $(v-u)\operatorname{div}_{x}l(x)g(v)\phi$
= sign $(v-u)\left[\operatorname{div}_{y}k(y)(f(v) - g(v)) - (\operatorname{div}_{y}k(y) - \operatorname{div}_{x}l(x))g(v)\right]\phi$.

Adding E_1 and E_2 , we thus get

(5.6)

$$E_{1} + E_{2} = \iiint_{\Pi_{T} \times \Pi_{T}} \left(\operatorname{sign} \left(v - u \right) \left[\operatorname{div}_{y} k(y) \left(f(v) - g(v) \right) - \left(\operatorname{div}_{y} k(y) - \operatorname{div}_{x} l(x) \right) g(v) \right] \right. \\ \left. + \left(k(y) - l(x) \right) \cdot \nabla_{x} G(v, u) + k(y) \cdot \nabla_{x} \left(F(v, u) - G(v, u) \right) \right) \phi \, dt \, dx \, ds \, dy.$$

Observe that by Lemma 4.1 we have

(5.7)
$$\left| \frac{\partial}{\partial x_j} G(v, u) \right| \leq ||g||_{\text{Lip}} \left| \frac{\partial}{\partial x_j} v(x, t) \right|, \ j = 1, \dots, d, \\ \left| \frac{\partial}{\partial x_j} \left(F(v, u) - G(v, u) \right) \right| \leq ||f - g||_{\text{Lip}} \left| \frac{\partial}{\partial x_j} v(x, t) \right|, \ j = 1, \dots, d.$$

Equipped with (5.7) and (1.14), we send $\alpha_0, \rho, \rho_0 \downarrow 0$ in (5.6) to obtain

$$\begin{split} \lim_{\alpha_{0},\rho,\rho_{0}\downarrow0} & \left(E_{1}+E_{2}\right) \\ & \leq \int_{\nu}^{\tau} \int_{\mathbb{R}^{d}} \left(\left|\operatorname{div}k(x)\right| \|f-g\|_{L^{\infty}} + \left|\operatorname{div}k(x) - \operatorname{div}l(x)\right| \|g\|_{L^{\infty}} \right. \\ & \left. + \left|k(x) - l(x)\right| \|g\|_{\operatorname{Lip}} \, \sum_{j=1}^{d} \left|\frac{\partial}{\partial x_{j}} v(x,t)\right| \\ & \left. + \|k\|_{L^{\infty}} \|f-g\|_{\operatorname{Lip}} \, \sum_{j=1}^{d} \left|\frac{\partial}{\partial x_{j}} v(x,t)\right| \right) dx \, dt. \end{split}$$

In view of (5.5), the following continuous dependence estimate now follows

$$\begin{split} \int_{\mathbb{R}^d} |v(x,\tau) - u(x,\tau)| \, dx &\leq \int_{\mathbb{R}^d} |v(x,\nu) - u(x,\nu)| \, dx \\ &+ \tau \left(||g||_{\operatorname{Lip}} \sup_{t \in (0,T)} |v(\cdot,t)|_{BV(\mathbb{R}^d)} ||k - l||_{L^{\infty}(\mathbb{R}^d)} + ||g||_{L^{\infty}} |k - l|_{BV(\mathbb{R}^d)} \right. \\ &+ |k|_{BV(\mathbb{R}^d)} ||f - g||_{L^{\infty}} + ||k||_{L^{\infty}} \sup_{t \in (0,T)} |v(\cdot,t)|_{BV(\mathbb{R}^d)} ||f - g||_{\operatorname{Lip}} \Big). \end{split}$$

Sending $\nu \downarrow 0$ and using symmetry, we finally conclude that Theorem 1.3 holds.

V. G(n, a) are flatte mensares. Therefore, after an integration by parts followed by adding and subtracting identical terms, we get,

By adding and enbranking industrial terms, we obtain

$$dr_{n}I(z) - (n \cdot n) dr_{n}h(y) f(u) - h(z)(n - u) dr_{n}I(h)(u) + h(z)(u) = h(z)(h(y)(y))$$

$$= h(z)(u - u) (dr_{n}h(y) f(u) - h(z)(u) - u(u)) - (dr_{n}h(y) - dr_{n}I(z)) g(u))$$

Adding E, and E, we thus get

$$E_{i} + E_{2} = \iiint_{i \neq x = i} \left[(\sin_{i}(v - u) [\sin_{i}(v)(v) + (v)) - (\sin_{i}(v)) - \sin_{i}(v)] \right]$$

Observe that by Lemma 4.1 we have

$$(z_1) = \frac{\theta}{2\pi_1} G(u, u) \leq \|u\|_{L^2} \int_{\partial Z_1}^{\infty} u(u, 0) \cdot f = 1, \dots, d,$$

$$(z_1) = \frac{\theta}{2\pi_1} (z_1(u, u) - G(u, u)) \leq \|f - u\|_{L^2} \int_{\partial Z_1}^{\infty} u(u, 0) \cdot f + f + 1, \dots, d.$$

Equipped with (5.7) and (1.14), we with an e o 10 in (5.5) to obtain

in view of (5.5), the following continuous dependence estimate now follows:

$$\int_{\mathbb{R}^{d}} |u(x, \tau) - u(x, \tau)| dx \leq \int_{\mathbb{R}^{d}} |u(x, t) - u(x, t)| dx$$

$$+ \tau (|to||_{t, 0} = tr |to||_{t, 0} |to||_{t$$

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KARLSEN AND RISEBRO

6. Appendix (proof of Lemma 2.1)

In this appendix, we give a proof of Lemma 2.1. The proof follows Carrillo [12], but see also Alt and Luckhaus [1] and Otto [27]. Note that \mathcal{A}_{ψ} is a nonnegative and convex function. Convexity implies that for a.e. $(x, t) \in \Pi_T$, we have

$$\mathcal{A}_{\psi}(u(x,t)) - \mathcal{A}_{\psi}(u(x,t-\tau)) \leq (u(x,t) - u(x,t-\tau))\psi(A(u(x,t)))$$

where we define $u(t) = u_0$ for $t \in (-\tau, 0)$. In the sequel let $\phi \in C_0^{\infty}(\mathbb{R}^d \times [0, T])$. Multiplying the above inequality by $\phi(x, t)$ yields

$$\begin{aligned}
\mathcal{A}_{\psi}\big(u(x,t)\big)\phi(x,t) - \mathcal{A}_{\psi}\big(u(x,t-\tau)\big)\phi(x,t-\tau) + \mathcal{A}_{\psi}\big(u(x,t-\tau)\big)\big(\phi(x,t-\tau) - \phi(x,t)\big) \\
(6.1) &= \mathcal{A}_{\psi}\big(u(x,t)\big)\phi(x,t) - \mathcal{A}_{\psi}\big(u(x,t-\tau)\big)\phi(x,t) \\
&\leq \big(u(x,t) - u(x,t-\tau)\big)\psi\big(A(u(x,t))\big)\phi(x,t),
\end{aligned}$$

where we define $\phi(x,t) = \phi(x,0)$ for t < 0. Note that $\mathcal{A}_{\psi}(u_0) \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\mathcal{A}_{\psi}(u) \in L^1_{\text{loc}}(\mathbb{R}^d)$ $L^{\infty}(0,T; L^{1}_{loc}(\mathbb{R}^{d}))$. Dividing (6.1) by τ and integrating over $\mathbb{R}^{d} \times (0,s)$, we get

(6.2)
$$\frac{1}{\tau} \int_{s-\tau}^{s} \int_{\mathbb{R}^{d}} \mathcal{A}_{\psi} \left(u(x,t) \right) \phi(x,t) \, dx \, dt - \frac{1}{\tau} \int_{0}^{\tau} \int_{\mathbb{R}^{d}} \mathcal{A}_{\psi} \left(u_{0}(x) \right) \phi(x,0) \, dx \, dt$$
$$+ \frac{1}{\tau} \int_{0}^{s} \int_{\mathbb{R}^{d}} \mathcal{A}_{\psi} \left(u(x,t-\tau) \right) \left(\phi(x,t-\tau) - \phi(x,t) \right) \, dx \, dt$$
$$\leq \frac{1}{\tau} \int_{0}^{s} \int_{\mathbb{R}^{d}} \left(u(x,t) - u(x,t-\tau) \right) \psi \left(\mathcal{A}(u(x,t)) \right) \phi(x,t) \, dx \, dt.$$

Since $\phi \in C_0^\infty(\mathbb{R}^d \times [0,T])$ and $A(u) \in L^2(0,T; H^1(\mathbb{R}^d))$, we have $\psi(A(u))\phi \in L^2(0,T; H^1_0(\mathbb{R}^d))$. Therefore, exploiting that $u \in C(0,T; L^1(\mathbb{R}^d))$ and $\partial_t u \in L^2(0,T; H^{-1}(\mathbb{R}^d))$, we can let $\tau \downarrow 0$ in (6.2) and obtain

$$\int_{\mathbb{R}^d} \mathcal{A}_{\psi}(u(x,s)) \phi(x,s) \, dx - \int_{\mathbb{R}^d} \mathcal{A}_{\psi}(u_0) \phi(x,0) \, dx \\ - \int_0^s \int_{\mathbb{R}^d} \mathcal{A}_{\psi}(u) \phi_t \, dx \, dt \le \int_0^s \langle \partial_t u, \psi(A(u)) \phi \rangle \, dt$$

for a.e. $s \in (0,T)$. Convexity implies also that for a.e. $(x,t) \in \Pi_T$ and $t > \tau$, we have

$$\mathcal{A}_{\psi}(u(x,t)) - \mathcal{A}_{\psi}(u(x,t-\tau)) \ge (u(x,t) - u(x,t-\tau))\psi(A(u(x,t-\tau))).$$

Multiplying this inequality by $\phi(x, t - \tau)$ yields

$$\begin{aligned}
\mathcal{A}_{\psi}(u(x,t))\phi(x,t) - \mathcal{A}_{\psi}(u(x,t-\tau))\phi(x,t-\tau) + \mathcal{A}_{\psi}(u(x,t))(\phi(x,t-\tau) - \phi(x,t)) \\
= \mathcal{A}_{\psi}(u(x,t))\phi(x,t-\tau) - \mathcal{A}_{\psi}(u(x,t-\tau))\phi(x,t-\tau) \\
\geq (u(x,t) - u(x,t-\tau))\psi(\mathcal{A}(u(x,t-\tau)))\phi(x,t-\tau).
\end{aligned}$$

After dividing (6.3) by τ and integrating over $\mathbb{R}^d \times (\tau, s)$, we obtain

$$\frac{1}{\tau} \int_{s-\tau}^{s} \int_{\mathbb{R}^{d}} \mathcal{A}_{\psi} \left(u(x,t) \right) \phi(x,t) \, dx \, dt - \frac{1}{\tau} \int_{0}^{\tau} \int_{\mathbb{R}^{d}} \mathcal{A}_{\psi} \left(u(x,t) \right) \phi(x,t) \, dx \, dt \\
+ \frac{1}{\tau} \int_{\mathbb{R}^{d}} \int_{\tau}^{s} \mathcal{A}_{\psi} \left(u(x,t) \right) \left(\phi(x,t-\tau) - \phi(x,t) \right) \, dx \, dt \\
\geq \frac{1}{\tau} \int_{\tau}^{s} \int_{\mathbb{R}^{d}} \left(u(x,t) - u(x,t-\tau) \right) \psi \left(\mathcal{A}(u(x,t-\tau)) \right) \phi(x,t-\tau) \, dx \, dt.$$
Finally, similarly to (6.2), letting $\tau \downarrow 0$ in (6.4), we get, for a.e. $s \in (0,T)$

$$\int_{\mathbb{R}^d} \mathcal{A}_{\psi}(u(x,s))\phi(x,s) \, dx - \int_{\mathbb{R}^d} \mathcal{A}_{\psi}(u_0)\phi(x,0) \, dx$$
$$- \int_0^s \int_{\mathbb{R}^d} \mathcal{A}_{\psi}(u)\phi_t \, dx \, dt \ge \int_0^s \langle \partial_t u, \psi(A(u))\phi \rangle \, dt$$

This concludes the proof of the Lemma 2.1.

18

In this appendix, we give a proof of Lemma 2.1. The proof follows Cardila [12], but see also Alt and Luckhaus [1] and Otto [27]. Note there A_i is a nonnegative and convex function. Convolved implies that for a.e. $(\pi^i) \in \Pi_2$, we have

$$A_{0}(u(x,t)) = A_{0}(u(x,t-x)) \leq (u(x,t) - u(x,t-\tau))u(A(u(x,t))).$$

where we define $u(t) = u_0$ for $t \in (-\tau, 0)$. In the sequel let $v \in C_0^{\infty}(\mathbb{R}^d \times [0, T_1])$, Multiplying the above increality by $\phi(x, t)$ yields

$$(a, a)a - (-, i, a)b(a - i, a)a(a, i - i, a)b(a - i, a)b(a - i, a)b(a, i - (i, a)b(a, i, a)b(a$$

$$(1.1) = (1, x)\phi((x, t))\phi(x, t) - A_{\psi}(u(x, t-x))\phi(x, t)$$

$$\leq (u(x,t) - u(x,t-\tau))\psi(A(u(x,t))) du(x))$$

where we define $\phi(x,t) = \phi(x,0)$ for t < 0. Note that $A_0(u_0) \in L_{0,t}(\mathbb{R}^n)$ and $A_0(u) \in L_{0,t}(\mathbb{R}^n)$. Let $L^{\infty}(0,T; L_{0,t}(\mathbb{R}^n))$. Dividing (6.1) by τ and integrating and $\mathbb{R}^1 \times (0,T; L_{0,t}(\mathbb{R}^n))$.

$$(0, 0) = \frac{1}{2} \int_{\mathbb{R}^{2}} A_{0}(u(x, t)) \phi(x, t) dx dt - \frac{1}{2} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} A_{0}(u(x, t)) \phi(x, t) dx dt - \frac{1}{2} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} A_{0}(u(x, t) - \tau) (h(x, t - \tau) - h(x, t)) dx dt$$

$$(0, 0) = \frac{1}{2} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} (u(x, t) - u(x, t - \tau)) \phi(A(u(x, t))) \phi(x, t) dx dt$$

Since $\phi \in C_{c}^{\infty}(\mathbb{R}^{d} \times [0, T])$ and $A(u) \in L^{2}(0, T; R^{2}(\mathbb{R}^{d}))$, we have $u(0, 0) \phi \in L^{2}(0, T; R_{0}^{2}(\mathbb{R}^{d}))$. Therefore, exploiting that $u \in C(0, T; L^{2}(\mathbb{R}^{d}))$ and $\partial_{t} u \in L^{2}(0, 2^{n}; R^{-1}(\mathbb{R}^{d}))$, we can let $v \in (0, 2^{n}; R^{-1}(\mathbb{R}^{d}))$.

$$\int_{\mathbb{R}^{d}} A_{\Phi}(u(x, s)) \rho(u, s) du = \int_{\mathbb{R}^{d}} A_{\Phi}(u(u)) \rho(x, s) du = \int_{\mathbb{R}^{d}} A_{\Phi}(u) \rho(x, s) du = \int_{\mathbb{R}^{d}} A_{\Phi}(u) \rho(x, s) \rho(x,$$

for a.e. $e \in (0, T)$. Convertivinglies deer that for a.e. $(x, t) \in \mathbb{R}^n$ and t, x, r, we have <math>A for a.e. (x, t) = A e (a(x, t) - A) = (a(x, t) - A) = (a(x, t) - A)

Manipiying this becaulity by start every youth

$$(n - 1, 2) \phi \{(n - 1, 2) \phi \} (n - (n - 1, 2) \phi) (n - (n - 1, 2) \phi \} (n - (n - 1, 2) \phi \} (n - (n - 1, 2) \phi) (n - (n - 1, 2)$$

After dividing (C.H. in a and interesting over 24 M (r. a) has ablain

$$= \int_{A_{n-1}} \int_{A_{n-1}} du (u(u, v)) u(u, v) dv dv = \frac{1}{2} \int_{A_{n-1}} \int_{A_{n-1}} du (u(u, v)) v(v, v) dv dv$$

$$det ab (x - 1) \int \left\{ (u(x, z) - u(x, z - \tau)) u(z) (u(u, z - \tau)) \right\} dz dz$$

Finally, similarly to (0.2), letting $r \downarrow 0$ in (6.4), we get for all $a \in (0, T)$.

$$= \frac{1}{2} \left(h\left(x, z \right) \right) \phi(x, z) dx = \int_{\partial B} A_{x} \left(x, z \right) \phi(x, 0) \phi(x, 0) dx = \int_{\partial B} A_{x} \left(x, z \right) \phi(x, 0) dx dx$$

$$-\int_{0}\int_{\mathbb{R}^{d}}A_{0}(n)n(dxd)\geq\int_{0}(\partial_{t}n,n(d,d)n)$$

This concludes the proof of the bergman 212

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