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Rate of Convergence for Subspace
Correction Methods for nonlinear
Variational Inequalities

by

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Abstract

Some general subspace correction algorithms are proposed for a convex optimization problem over a convex constraint subset. One of the nontrivial applications of the algorithms is the solving of some obstacle problems by multilevel domain decomposition and multigrid methods. The essential new features of the algorithms when applied to domain decomposition and multigrid are the treatment of the coarse mesh problems. The subproblems over the coarser meshes are solved without any constraints. If the coarser mesh correction values are dragging the iterative solution out of the constraint set, we use the iterative solutions of the subproblems over the finest mesh to drag it back to the constraint set. The rate of convergence for the algorithms for the obstacle problems is of the same order as the rate of convergence for jump coefficient linear elliptic problems.

Keywords: Parallel, domain decomposition, multigrid, nonlinear, variational inequality, obstacle problems, space decomposition

1991 Mathematics Subject Classification: 65N55, 65Y05, 65J15, 65K10

1 Introduction

In this work, we extend the space decomposition and subspace correction algorithms of [54, 49] to solve convex optimization problems over a convex constraint subset. One of the main concerns of this work is the rate of convergence when multilevel domain decomposition and multigrid methods are used to solve some obstacle problems.

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From the time that multigrid and domain decomposition methods are getting the attention of numerical mathematicians and engineers, efforts have been continuously devoted to the study of using domain decomposition and multigrid methods for obstacle problems, see [1, 3, 2, 7, 14, 18, 23, 21, 22, 24, 25, 19, 20, 29, 27, 28, 30, 31, 32, 34, 33, 15, 35, 37, 39, 50, 51, 42, 43, 41, 40, 45, 48, 56]. In the book of McCormick [35, p.100], treatment of constraints for multilevel methods was listed as one of the open and challenging problems. For linear elliptic partial differential equations, it is known that the solution will be influenced globally if the boundary value or the right hand is perturbed around a point. This justifies the need for coarser meshes in using iterative solvers to solve the problems. However, this is not the case for obstacle type problems. A small perturbation of the input data may only influence a small part of the solution domain due to the appearance of the obstacles. This leads to the speculation that coarser meshes may not be necessary or should be handled differently for obstacle problems. Related to this difficulty, the algorithms in [18, 23, 25] are trying to use the active set strategy to separate the obstacle from the solving of the partial differential equations, i.e. during the iterative procedure, the algorithms are trying to identify the active regions of the obstacles and then solve a partial differential equation where the obstacle is not active. The algorithms proposed in [1, 2, 19, 30, 32, 50, 51, 56] are specified for domain decomposition methods. Due to the absence of the coarse mesh in the algorithms, the convergence of the algorithms depends on the number of subdomains. In Tai [45], the obstacle function is decomposed into a sum of obstacles from the subspaces, and the subproblems are solved with the subspace obstacles. The algorithms of [45] are applicable to multilevel methods, however, the convergence was only proved for overlapping domain decomposition methods without a coarser mesh. Comparing our algorithms to the ones of [7, 18, 23, 21, 22, 24, 25, 27, 28, 34, 33, 15, 37] for multigrid applications, the good point for our algorithm is that we do not impose any constraint on the coarser meshes. Our algorithms can be implemented in the same way as for linear problems. The only difference to the linear problems is that we need to project the solution of the one dimensional subproblems at the finest mesh into an one-dimensional obstacle constraint. In term of computational cost, the algorithms here are cheaper. Another contribution of this work is the convergence rate estimates. For the obstacle problem, it is shown that the algorithms have a convergence rate which is of the same order as the linear non-constrained elliptic problems when the diffusion coefficients have large jumps. Moreover, the convergence estimates are valid right from the first iteration. We do not need to assume that the obstacle problem is nondegenerate (c.f. [34, p.84]) and also do not need to assume that the active region of the obstacle has been identified, see [27, 28, 34, 15]. It seems that the only available earlier convergence rate estimates for obstacle problems with multilevel methods are the ones of [27, 28, 34, 15].

Even though our main concern is the obstacle problem, our algorithms are presented in a general setting for general space decomposition. The general algorithms as well as the assumptions are given in §2. The convergence analysis for the general algorithms under the given assumptions are stated in §3. The convergence rate depends essentially on two constants C_1 and C_2, see (18) and (19). In section §4, we show that domain decomposition and multigrid methods can be interpreted as space decompositions and be used for solving the obstacle problems. The constants C_1 and C_2 are estimated using some technical estimates of Bramble and Xu [6]. The rate of convergence for the obstacle problem is essentially the same as the non-constrained elliptic problems when the diffusion coefficients have large jumps, see [6].

2 The optimization problem and the algorithms

2.1 The optimization problem

Given a reflexive Banach space V and a convex functional $F : V \mapsto R$, we shall consider the following nonlinear optimization problem

$$\min_{v\in K} F(v), \quad K \subset V . \tag{1}$$

The nonempty convex subset K is assumed to be closed in the strong topology of V. We are interested in the case where the space V can be decomposed into a sum of subspaces V_i, i.e.

$$V = V_1 + V_2 + \cdots + V_m = \sum_{i=1}^{m} V_i . \tag{2}$$

This means that for any v, there exists $v_i \in V_i$ such that $v = \sum_{i=1}^m v_i$. Due to the appearance of the constraint K, we require that there exists an l ($1 \le l \le m$) and a nonempty convex subsets $K_i \subset V_i$, $i = l, l+1, \cdots, m$ such that

$$V \subset \sum_{i=l}^{m} V_i, \quad K = \sum_{i=l}^{m} K_i, \tag{3}$$

For reasons related to the existence of the subproblems, we require that K_i is closed in the strong topology of V. From (2), we can see that $l = 1$ is always a valid choice. In applications to domain decomposition and multigrid, due to the appearance of the coarser meshes, we can always choose an $l > 1$ such that (3) is valid.

We assume that the functional F is Gateaux differentiable (see [8]) and that there exists constants $\kappa, \ell > 0$, $p \ge q > 1$ such that

$$\begin{aligned} \langle F'(w) - F'(v), w - v\rangle &\ge \kappa \|w - v\|_V^p, \quad \forall w, v \in V , \\ \|F'(w) - F'(v)\|_{V'} &\le \ell \|w - v\|_V^{q-1}, \quad \forall w, v \in V . \end{aligned} \tag{4}$$

Here $\langle\cdot,\cdot\rangle$ to is the duality pairing between V and its dual space V'. Under the assumption (4), problem (1) has a unique solution, see [12, p. 35]. For some nonlinear problems, the constants κ and ℓ may depend on v and w.

For simplicity, we define

$$\sigma = \frac{p}{p-q+1}, \quad r = \frac{p(p-1)}{q(q-1)}.$$

When $p = q = 2$, we have $\sigma = 2, r = 1$. For a given function $w \in V$, we denote by $K - w$ the subset:

$$K - w = \{v - w |\; v \in K\}.$$

The general theory developed for (1) will be applied to the following obstacle problem in connection with finite element approximations:

$$\text{Find } u \in K, \quad \text{such that} \quad a(u, v-u) \geq f(v-u), \quad \forall v \in K, \tag{5}$$

with

$$a(v,w) = \int_\Omega \nabla v \cdot \nabla w + vw \, dx, \quad K = \{v \in H^1(\Omega) |\; v(x) \geq \psi(x) \text{ a.e. in } \Omega\}. \tag{6}$$

It is well known that the above problem is equivalent to the following minimization problem

$$\min_{v\in K} F(v), \quad F(v) = \frac{1}{2}a(v,v) - f(v), \tag{7}$$

assuming that $f(v)$ is a linear functional on $H^1(\Omega)$. For simplicity, the domain $\Omega \subset R^d$ is assumed to be bounded and to have a smooth boundary. Neumann boundary condition is imposed for the obstacle problem to ease some of the technical analysis.

For the obstacle problem (5), the minimization space $V = H^1(\Omega)$. Correspondingly, we have $p = q = 2$ and $\kappa = \ell = 1$ for assumption (4).

Standard notations for Sobolev spaces $H^1(\Omega), W^{k,p}(\Omega)$ will be used, i.e $\|\cdot\|_{k,p,D}$ denotes the $W^{k,p}$-norm on a domain D, $\|\cdot\|_{k,D}$ denotes the H^k-norm on a domain D. In the case $D = \Omega$, we will omit D. The generic positive constant C, which may differ from context to context, will be used to denote a constant that is independent of the variables appearing in the inequalities or equalities and the size of the finite element meshes.

Obstacle problems arise from many important applications. For some concrete examples, we refer to Baiocchi and Capelocite [4], Cottle et al. [10], Duvaut and Lions [11], Elliot and Ockendon [13], Glowinski [16], Glowinski et al. [17], Kinderlehrer and Stampaccia [26], Kornhuber [29], and Rodrigues [36].

2.2 The algorithms

The following two algorithms for general space decomposition can be regarded as a generalization of the Jacobi and Gauss-Seidel methods, see [54, 49]. In applications to domain decomposition methods for linear elliptic partial differential equations without constraints, Algorithm 2.1 is in fact the Additive Schwarz method and Algorithm 2.2 is the multiplicative Schwarz method. In applications to multigrid methods for linear elliptic partial differential equations without constraints, Algorithm 2.1 is essentially similar to the ideas used in the BPX preconditioner [5] and Algorithm 2.2 reduces to sequential multigrid methods. Algorithm 2.1 is sometimes called the additive space decomposition method and Algorithm 2.2 is sometimes called the multiplicative space decomposition method (c.f. [47]).

Algorithm 2.1 *[A parallel subspace correction method].*

1. *Choose initial values $u_i^0 \in V_i$ such that $\sum_{i=1}^m u_i^0 \in K$ and a relaxation parameter $\alpha \in (0, 1/m]$.*

2. *For $n \geq 1$, solve the following problems in parallel for different i:*

   2.1. Find $\hat{u}_i^{n+1} \in V_i$ for $i = 1, 2, \cdots, l-1$ such that

$$F\Big(\sum_{j=1, j\neq i}^{m} u_j^n + \hat{u}_i^{n+1}\Big) \leq F\Big(\sum_{j=1, j\neq i}^{m} u_j^n + v_i\Big), \quad \forall v_i \in V_i . \quad (8)$$

   2.2. Find convex closed subsets $K_i^n \subset V_i$ for $i = l, l+1, \cdots, m$ such that

$$K - \sum_{i=1}^{l-1} u_i^n = \sum_{i=l}^{m} K_i^n .$$

   2.3. For $i = l, l+1, \cdots, m$, find $\hat{u}_i^{n+1} \in K_i^n$ such that

$$F\Big(\sum_{j=1, j\neq i}^{m} u_j^n + \hat{u}_i^{n+1}\Big) \leq F\Big(\sum_{j=1, j\neq i}^{m} u_j^n + v_i\Big), \quad \forall v_i \in K_i^n . \quad (9)$$

   2.4. Set

$$u_i^{n+1} = u_i^n + \alpha(\hat{u}_i^{n+1} - u_i^n) , \quad (10)$$

   and go to the next iteration if not converged.

Algorithm 2.2 *[A successive subspace correction method].*

1. *Choose initial values $u_i^0 \in V_i$ such that $\sum_{i=1}^m u_i^0 \in K$ and a relaxation parameter $\alpha \in [0, 2]$.*

2. *For $n \geq 0$, solve the following problems sequentially for $i = 1, 2, \cdots, m$:*

2.1. Find $\hat{u}_i^{n+1} \in V_i$ for $i = 1, 2, \cdots, l-1$ such that

$$F\Big(\sum_{j=1}^{i-1} u_j^{n+1} + \hat{u}_i^{n+1} + \sum_{j=i+1}^{m} u_j^n\Big) \le F\Big(\sum_{j=1}^{i-1} u_j^{n+1} + v_i + \sum_{j=i+1}^{m} u_j^n\Big), \quad \forall v_i \in V_i, \tag{11}$$

and set

$$u_i^{n+1} = u_i^n + \alpha(\hat{u}_i^{n+1} - u_i^n) . \tag{12}$$

2.2. Find convex closed subsets $K_i^n \subset V_i$ for $i = l, l+1, \cdots, m$ such that

$$K - \sum_{i=1}^{l-1} u_i^{n+1} = \sum_{i=l}^{m} K_i^n .$$

2.3. For $i = l, l+1, \cdots, m$, find $\hat{u}_i^{n+1} \in K_i^n$ sequentially for $i = l, l+1, \cdots, m$ such that

$$F\Big(\sum_{j=i}^{i-1} u_j^{n+1} + \hat{u}_i^{n+1} + \sum_{j=i+1}^{m} u_j^n\Big) \le F\Big(\sum_{j=1}^{i-1} u_j^{n+1} + v_i + \sum_{j=i+1}^{m} u_j^n\Big), \quad \forall v_i \in K_i^n . \tag{13}$$

and set

$$u_i^{n+1} = u_i^n + \alpha(\hat{u}_i^{n+1} - u_i^n) . \tag{14}$$

2.4. Go to the next iteration if not converged.

Before we go any further, we shall remark on the existence of the subsets K_i^n. From assumption (3), we know that there must exist $w_i^n \in V_i$, $i = l, l+1, \cdots, m$ such that

$$\sum_{i=1}^{l-1} u_i^n = \sum_{i=l}^{m} w_i^n . \tag{15}$$

Thus for Algorithm 2.1, one of the choices of the subset K_i^n is

$$K_i^n = K_i - w_i^n . \tag{16}$$

This gives a K_i^n which is nonempty convex and closed in V. However, there may also exist other alternatives for K_i^n. In applications to domain decomposition, different decompositions K_i^n may give different iterative solutions $\hat{u}_i^{n+1}$, but the sum $\sum_{i=l}^{m} \hat{u}_i^{n+1}$ will always converge to the same solution, see Tai [45]. In applications to multigrid methods, we shall only impose constraint for the finest mesh and the decompositions K_i^n are in fact unique. The decomposition K_i^n for Algorithm 2.2 can be done similarly as for Algorithm 2.1.

We note that the above two algorithms are well-defined since the subspace problem (8), (9), (11) and (13) are uniquely solvable under the assumptions for F described earlier (see [12]). The above algorithms are proposed for general space decompositions. In real applications, the algorithms can be implemented in different ways depending on the structure of the decomposed subspaces, see Tai [44, p.39] and Tai and Espedal [46, p.725] for some implementation issues with the two-level domain decomposition methods. Note that the constraints are only imposed for some of the subspaces. For the subproblems on $V_i, i = 1, 2, \cdots, l-1$, we have no constraint.

Let u be the exact solution of (1) and u^n and e_i^{n+1} be defined by

$$u^n = \sum_{i=1}^{m} u_i^n, \quad e_i^{n+1} = \hat{u}_i^{n+1} - u_i^n, \quad \forall n \geq 0. \tag{17}$$

As in [47, 49, 48], we shall use two constants in the estimation of the rate of the convergence of the algorithms. First, we assume that there is a $C_1 > 0$ such that for any $n > 0$, we can find $u_i \in V_i$ to satisfy

$$\begin{cases} u_i + \hat{u}_i^{n+1} - u_i^{n+1} \in K_i^n, \ i = l, l+1, \cdots m, \\ u = \sum_{i=1}^{m} u_i, \quad \text{and} \quad \left(\sum_{i=1}^{m} \|u_i - u_i^{n+1}\|_V^\sigma \right)^{\frac{1}{\sigma}} \leq C_1 \|u - u^{n+1}\|_V \ . \end{cases} \tag{18}$$

Observe that u_i may depend on the iteration number n. In addition to the assumption of the existence of such a constant C_1, we also assume that the is a $C_2 > 0$, which is the least constant satisfying the following property: for any $w_{ij} \in V, u_i \in V_i$ and $v_j \in V_j$ the following inequality holds:

$$\sum_{i,j=1}^{m} \langle F'(w_{ij} + u_i) - F'(w_{ij}), v_j \rangle \leq C_2 \Big(\sum_{i=1}^{m} \|u_i\|_V^p \Big)^{\frac{q-1}{p}} \Big(\sum_{j=1}^{m} \|v_j\|_V^\sigma \Big)^{\frac{1}{\sigma}}. \tag{19}$$

The existence of C_2 is obvious by the assumption (4). A simple application of Hölder's inequality would give the following rough upper bound:

$$C_2 \leq Lm.$$

But better bounds may be obtained in applications.

3 Convergence analysis

We need to estimate the rate of reduction of the error $u - u^n$ for each iteration. As in Tai and Xu [49], we shall use

$$d_n = F(u^n) - F(u), \tag{20}$$

as a measurement of the error between u and u^n.

3.1 The convergence of the parallel subspace correction method

The convergence of Algorithm 2.1 is given in the following theorem.

Theorem 3.1 *Assuming that the space decomposition satisfies (18), (19) and that the functional F satisfies (4). Define*

$$C^* = \left[\frac{C_1 C_2 \left(\alpha^{\frac{(p-1)(q-1)}{p}} + \alpha^{-\frac{q-1}{p}} \right)}{\kappa} \right]^{\frac{p-1}{q}} \frac{p}{\kappa} \left(\frac{\ell}{q} \right)^r , \quad c_0 = \frac{(r-1)}{r d_0^{r-1} + C^*} .$$

Then for Algorithm 2.1 and d_n given by (17), we have

1. *If $r = 1$, the error satisfies*

$$d_{n+1} \le \frac{1}{1+C^*} d_n , \quad \forall n \ge 1 . \tag{21}$$

2. *If $r > 1$, the error satisfies*

$$\begin{aligned} d_n &\le \frac{d_{n-1}}{(1 + c_0 d_{n-1}^{r-1})^{\frac{1}{r-1}}} \\ &\le \frac{d_0}{(1 + c_0 d_0^{r-1} n)^{\frac{1}{r-1}}} , \quad \forall n \ge 1 . \end{aligned} \tag{22}$$

Proof. Define

$$u^{n+\frac{i}{m}} = \sum_{j=1, j\ne i}^{m} u_j^n + \hat{u}_i^{n+1} . \tag{23}$$

From (10) and (17), we see that $u^{n+\frac{i}{m}} = u^n + e_i^{n+1}$ and it is easy to calculate that

$$u^{n+1} = (1 - \alpha m) u^n + \sum_{i=1}^{m} \alpha u^{n+\frac{i}{m}} . \tag{24}$$

Using the notations of (17) and the fact that F is differentiable and convex, it is known (see Ekeland and Temam [12]) that (9) implies

$$\langle F'(u^n + e_i^{n+1}), v_i - \hat{u}_i^{n+1} \rangle \ge 0, \quad \forall v_i \in K_i^n, \ i = l, l+1, \cdots, m. \tag{25}$$

As there is no constraint for the subproblem for $i = 1, 2, \cdots, l$, it is true that

$$\langle F'(u^n + e_i^{n+1}), v_i \rangle = 0, \quad \forall v_i \in V_i, \quad i = 1, 2, \cdots, l. \tag{26}$$

Under the assumption of (4), it is known that (See Tai and Epsedal [47, Lemma 3.2])

$$F(w) - F(v) \ge \langle F'(v), w - v \rangle + \frac{\kappa}{p} \|w - v\|_V^p , \quad \forall v, w \in V . \tag{27}$$

Using (25), (24), the convexity of F and (4), and applying similar techniques as in [47, p.1563], it can be proved that

$$F(u^n) - F(u^{n+1}) \geq F(u^n) - \sum_{i=1}^{m} \alpha F\left(u^n + e_i^{n+1}\right) - (1 - \alpha m) F(u^n) \tag{28}$$

$$\geq \ -\sum_{i=1}^{m} \alpha \langle F'(u^n + e_i^{n+1}), e_i^n \rangle + \frac{\kappa}{p} \sum_{i=1}^{m} \alpha \|e_i^{n+1}\|_V^p \geq \frac{\kappa}{p} \sum_{i=1}^{m} \alpha \|e_i^{n+1}\|_V^p \,.$$

For simplicity, we define

$$\xi_j^n = \sum_{i=1}^{j} u_i^{n+1} + \sum_{i=j+1}^{m} u_i^n.$$

For $i = l, l+1, \cdots, m$, let u_i be the functions given in assumption (18). By (18) and (25), we see that

$$\langle F'(u^n + e_i^{n+1}), u_i^{n+1} - u_i \rangle = \langle F'(u^n + e_i^{n+1}), \hat{u}_i^{n+1} - (u_i + \hat{u}_i^{n+1} - u_i^{n+1}) \rangle \leq 0. \tag{29}$$

We shall use (18), (19), (10), (26) and (29) to estimate

$$\begin{aligned}
&\langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle \\
\leq\ & \langle F'(u^{n+1}), u^{n+1} - u \rangle = \sum_{i=1}^{m} \left\langle F'(u^{n+1}), u_i^{n+1} - u_i \right\rangle \\
\leq\ & \sum_{i=1}^{m} \left\langle F'(u^{n+1}) - F'(u^n + e_i^{n+1}), u_i^{n+1} - u_i \right\rangle \\
=\ & \sum_{i=1}^{m} \left\langle F'(u^{n+1}) - F'(u^n), u_i^{n+1} - u_i \right\rangle \\
-\ & \sum_{i=1}^{m} \left\langle F'(u^n + e_i^{n+1}) - F'(u^n), u_i^{n+1} - u_i \right\rangle \\
=\ & \sum_{i=1}^{m} \sum_{j=1}^{m} \left\langle F'(\xi_j^n) - F'(\xi_{j-1}^n), u_i^{n+1} - u_i \right\rangle + \sum_{i=1}^{m} \left\langle F'(u^n + e_i^{n+1}) - F'(u^n), u_i^{n+1} - u_i \right\rangle \\
\leq\ & C_2 \Big(\sum_{j=1}^{m} \|(\alpha e_j^n)\|_V^p \Big)^{\frac{q-1}{p}} \Big(\sum_{i=1}^{m} \|v_i\|_V^\sigma \Big)^{\frac{1}{\sigma}} + C_2 \sum_{i=1}^{m} \|e_i^n\|_V^{q-1} \|v_i\|_V \\
\leq\ & C_2 |\alpha|^{\frac{(p-1)(q-1)}{p}} \left(\sum_{i=1}^{m} \alpha \|e_i^n\|_V^p \right)^{\frac{q-1}{p}} \cdot C_1 \|u^{n+1} - u\|_V \\
& + C_2 \alpha^{-\frac{q-1}{p}} \left(\sum_{i=1}^{m} \alpha \|e_i^n\|_V^p \right)^{\frac{q-1}{p}} \cdot C_1 \ \|u^{n+1} - u\|_V
\end{aligned} \tag{30}$$

$$\leq C_1C_2\left(\alpha^{\frac{(p-1)(q-1)}{p}}+\alpha^{-\frac{q-1}{p}}\right)\left[\frac{p}{\kappa}\left(F(u^n)-F(u^{n+1})\right)\right]^{\frac{q-1}{p}}\cdot\|u^{n+1}-u\|_V .$$

The rest of the proof is the same as in [49]. □

3.2 The convergence of the successive subspace correction method

The convergence of Algorithm 2.2 is similar to Algorithm 2.1. The convergence is only proved for the case that the relaxation parameter α is taken to be 1, i.e. $\alpha = 1$ in (12) and (14). Note that $\hat{u}_i^{n+1} = u_i^{n+1}$ in this case.

Theorem 3.2 *Assuming that the space decomposition satisfies (18), (19) and that the functional F satisfies (4). Define*

$$C^* = \frac{\ell}{\kappa}\left(\frac{C_1C_2}{\kappa}\right)^2, \qquad c_0 = \frac{(r-1)}{rd_0^{r-1}+C^*}.$$

Taking the relaxation parameter $\alpha = 1$ for Algorithm 2.2, then we have

1. *If $r = 1$, the error satisfies*

$$d_{n+1} \leq \frac{1}{1+C^*}d_n, \quad \forall n \geq 1 . \tag{31}$$

2. *If $r > 1$, the error satisfies*

$$\begin{aligned} d_n &\leq \frac{d_{n-1}}{(1+c_0 d_{n-1}^{r-1})^{\frac{1}{r-1}}} \\ &\leq \frac{d_0}{(1+c_0 d_0^{r-1} n)^{\frac{1}{r-1}}}, \quad \forall n \geq 1 . \end{aligned} \tag{32}$$

Proof. Define

$$u^{n+\frac{i}{m}} = \sum_{j=1}^{i-1} u_j^{n+1} + \hat{u}_i^{n+1} + \sum_{j=i+1}^{m} u_j^n . \tag{33}$$

Since $u^{n+\frac{i}{m}}$ minimizes (11) or (13), it satisfies

$$\begin{aligned} &\langle F'(u^{n+\frac{i}{m}}), v_i\rangle = 0, \quad \forall v_i \in V_i, i = 1,2,\cdots,l-1. \\ &\langle F'(u^{n+\frac{i}{m}}), v_i - \hat{u}_i^{n+1}\rangle \geq 0, \quad \forall v_i \in K_i^n, i = l,l+1,\cdots,m. \end{aligned} \tag{34}$$

Using (27) and (34), we get that

$$F(u^{n+(i-1)/m}) - F(u^{n+i/m}) \geq \frac{\kappa}{p}\|e_i^{n+1}\|_V^p . \tag{35}$$

Thus, estimate (35) leads to

$$F(u^n) - F(u^{n+1}) = \sum_{i=1}^{m} \left[F(u^{n+(i-1)/m}) - F(u^{n+i/m}) \right] \ge \frac{\kappa}{p} \sum_{i=1}^{m} \|e_i^{n+1}\|_V^p \quad (36)$$

Note that $\hat{u}_i^{n+1} = u_i^{n+1}$ when $\alpha = 1$. Similar to the proof of (30), we use (18), (19) and (34) to get

$$\begin{aligned}
&\langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle \\
&= \sum_{i=1}^{m} \left\langle F'(u^{n+1}) - F'(u^{n+i/m}), u_i^{n+1} - u_i \right\rangle \\
&= \sum_{i=1}^{m} \sum_{j>i}^{m} \left\langle F'(u^{n+j/m}) - F'(u^{n+(j-1)/m}), u_i^{n+1} - u_i \right\rangle \qquad (37) \\
&\le C_2 \Big(\sum_{j=1}^{m} \|e_j^n\|_V^p \Big)^{\frac{q-1}{p}} \Big(\sum_{i=1}^{m} \|u_i^{n+1} - u_i\|_V^\sigma \Big)^{\frac{1}{\sigma}} \\
&\le C_1 C_2 \Big(\sum_{i=1}^{m} \|e_i^{n+1}\|_V^p \Big)^{\frac{q-1}{p}} \cdot \|u^{n+1} - u\|_V .
\end{aligned}$$

The rest of the proof is the same as in [49]. □

4 Space decomposition for $H^1(\Omega)$ and K

4.1 Overlapping domain decomposition

In this subsection, we show overlapping domain decomposition can be used to decompose a finite element space and the constraint set K.

4.1.1 Decomposition of $H^1(\Omega)$ by overlapping subdomains

Let $\{\Omega_i\}_{i=1}^M$ be a quasi-uniform finite element division, or a coarse mesh, of Ω where Ω_i has diameter of order H. We further divide each Ω_i into smaller simplices with diameter of order h. If Ω has a curved boundary, we shall also fill the area between $\partial\Omega$ and $\partial\Omega_H$, here $\bar{\Omega}_H = \cup_{i=1}^M \bar{\Omega}_i$, with finite elements with diameters of order h. We assume that the resulting elements form a shape regular finite element subdivision of Ω, see Ciarlet [9]. We call this the fine mesh or the h-level subdivision of Ω with mesh parameter h. We denote $\bar{\Omega}_h = \cup_{T\in\mathcal{T}_h} \bar{T}$ to be the fine mesh subdivision. Let $S^H \subset W^{1,\infty}(\Omega_H)$ and $S^h \subset W^{1,\infty}(\Omega_h)$ be the continuous, piecewise linear finite element spaces over the H-level and h-level subdivisions of Ω respectively. More specifically,

$$S^H = \left\{ v \in W^{1,\infty}(\Omega_H) \middle| \; v|_{\Omega_i} \in P_1(\Omega_i), \forall i \right\},$$

$$S^h = \left\{ v \in W^{1,\infty}(\Omega_h) \middle| \quad v|_{\mathcal{T}} \in P_1(\mathcal{T}), \forall \mathcal{T} \in \mathcal{T}_h \right\}.$$

For any $v \in S^h$, the following estimates are known from Bramble and Xu [6, Lemma 2.3] and also Xu and Zou [55, §4].

$$\|v\|_{0,\infty} \le \begin{cases} \|v\|_1, & \text{if} \quad d = 1; \\ |\log h| \, \|v\|_1, & \text{if} \quad d = 2; \\ h^{-\frac{1}{2}} \|v\|_1, & \text{if} \quad d = 3; \end{cases} \tag{38}$$

For each Ω_i, we consider an enlarged subdomain Ω_i^δ consisting of elements $\mathcal{T} \in \mathcal{T}_h$ with $dist(\mathcal{T}, \Omega_i) \le \delta$. The union of Ω_i^δ covers $\bar{\Omega}_h$ with overlaps of size δ. Let us denote the piecewise linear finite element space with zero traces on the boundaries $\partial\Omega_i^\delta \backslash \partial\Omega$ as $S^h(\Omega_i^\delta)$. Then one can show that

$$S^h = S^H + \sum S^h(\Omega_i^\delta). \tag{39}$$

For the overlapping subdomains, assume that there exist m colors such that each subdomain Ω_i^δ can be marked with one color, and the subdomains with the same color will not intersect with each other. For suitable overlaps, one can always choose $m = 2$ if $d = 1; m \le 4$ if $d = 2; m \le 8$ if $d = 3$. Let Ω_i' be the union of the subdomains with the i^{th} color, and

$$V_i = \{ v \in S^h | \quad v(x) = 0, \quad x \notin \Omega_i' \} \; i = 1, 2, \cdots, m.$$

By denoting subspaces $V_0 = S^H$, $V = S^h$, we find that decomposition (39) means

$$V = V_0 + \sum_{i=1}^{m} V_i, \tag{40}$$

and so the two level method is a way to decompose the finite element space.

4.1.2 Decomposition of K by overlapping subdomains

In using our algorithms, we choose to solve the coarse mesh problem without any constraints. The subdomain subproblems are solved with the constraints K_i^n. In order to get K_i^n, we first need to decompose K into a sum of $K_i \subset V_i$. Let $\psi \in V$, i.e. the obstacle has been replaced by a finite element obstacle which is often the interpolation of the continuous obstacle. Due to the overlaps between the subdomains, there must exist $\psi_i \in V_i$, $i = 1, 2, \cdots, m$, which may not be unique, such that

$$\psi = \sum_{i=1}^{m} \psi_i.$$

Correspondingly, by defining

$$K_i = \left\{ v_i \middle| \; v_i \ge \psi_i, \; v_i \in V_i \right\}, i = 1, 2, \cdots, m,$$

we find that (3) is satisfied. At each iteration, we also need to decompose the coarse mesh solution u_0^n (or u_0^{n+1}) into a sum of $w_i^n \in V_i$ (c.f. (15) and (16)) which can be done similarly to the decomposition of the obstacle ψ.

Due to the non-uniqueness of the decomposition of the functions, the constraint subsets K_i^n are also non-unique. Different decompositions K_i^n may give different iterative solutions $\hat{u}_i^{n+1}$. However, the sum $\sum_{i=l}^m \hat{u}_i^{n+1}$ will always converge to the same solution, see Tai [45].

4.1.3 Estimations for C_1 and C_2.

In order to verify the conditions concerning the constants C_1 and C_2, we need the following technical lemma:

Lemma 4.1 *Let S^H and S^h be defined as above. For any $v \in S^h$, there exists $v_0 \in S^H$ such that*

$$v_0 \le v, \quad \|v_0 - v\|_0 \le c_d H \|v\|_1, \qquad \|v_0\|_1 \le c_d \|v\|_1, \tag{41}$$

where $c_d = C$ if $d = 1$; $c_d = C\left(1 + \left|\log \frac{H}{h}\right|\right)$ if $d = 2$ and $c_d = C\left(\frac{H}{h}\right)^{\frac{1}{2}}$ if $d = 3$.

Proof. For a given $v \in S^h$, let v_0^I be the standard Lagrangian interpolation of v in the coarse mesh space S^H using the coarse mesh nodal values. Denote by $\{x_i^0\}_{i=1}^{n_0}$ the coarse mesh nodes. For a given x_i^0, we define η_i to be the union of the coarse mesh elements having x_i^0 as one of its nodes. We shall construct v_0 by defining its nodal values as

$$v_0(x_i^0) = v_0^I(x_i^0) - \max_{x \in \eta_i}\left(v_0^I(x) - v(x)\right), \quad \forall x_i^0.$$

For simplicity, we define $\rho_0(x) \in S^H$ to be the coarse mesh function having the nodal values

$$\rho_0(x_i^0) = \max_{x \in \eta_i}\left(v_0^I(x) - v(x)\right), \quad \forall x_i^0.$$

It is easy to see that $\rho_0(x) \ge v_0^I(x) - v(x)$, which implies

$$v_0(x) = v_0^I(x) - \rho_0(x) \le v_0^I(x) - \left(v_0^I(x) - v(x)\right) = v(x).$$

In addition,

$$\|v_0 - v\|_0 \le \|v_0^I - v\|_0 + \|\rho_0\|_0.$$

As $\rho_0 \in S^H$, it is known that the L^2-norm is equivalent to

$$\|\rho_0\|_0^2 = CH^d \sum_{i=1}^{n_0} |\rho_0(x_i^0)|^2.$$

Using a linear mapping to transform the domain η_i into a domain of unit size and applying the inequalities (38), we get that

$$\|\rho_0\|_0^2 \le CH^d \sum_{i=1}^{n_0} \|v_0^I - v\|_{0,\infty,\eta_i}^2 \le CH^2 c_d^2 \|v\|_1^2.$$

In the above inequality, we have used the regularity of the meshes, i.e. under the minimum angle condition, the number of elements around a nodal point is always less than a constant. Using the inverse inequality, we know that $\|\rho_0\|_1 \le CH^{-1}\|\rho_0\|_0$. Combining these estimates with standard estimates for $v - v_0^I$, we have proved the lemma. □

Following an argument in [52], let $\{\theta_i\}_{i=1}^m$ be a partition of unity with respect to $\{\Omega_i'\}_{i=1}^m$, i.e. $\theta_i \in C_0^\infty(\Omega_i' \cap \Omega)$, $\theta_i \ge 0$ and $\sum_{i=1}^m \theta_i = 1$. It can be chosen so that

$$|\nabla \theta_i| \le C/\delta, \quad \theta_i(x) = \begin{cases} 1 & \text{if distance } (x, \partial\Omega_i') \ge \delta \text{ and } x \in \Omega_i', \\ 0 & \text{on } \overline{\Omega \backslash \Omega_i'}. \end{cases}$$

Let I_h be an interpolation operator which uses the function values at the h-level nodes. For any $v \in V$, let $v_0 \in V_0$ be the coarse mesh function defined as in Lemma 4.1. Take $v = u - u^{n+1}$ and $v_i = I_h(\theta_i(v - v_0))$. They satisfy

$$v = \sum_{i=0}^m v_i = u - u^{n+1}. \tag{42}$$

In addition, we have

$$\left(\|v_0\|_1^2 + \sum_{i=1}^m \|v_i\|_1^2 \right)^{\frac{1}{2}} \le C(m+1)^{\frac{1}{2}} \left(1 + \left(\frac{Hc_d}{\delta} \right)^{\frac{1}{2}} \right) \|v\|_1. \tag{43}$$

The proof of the above inequality is essentially similar to the proofs for the non-constrained cases, c.f. [52], [53, 54] and [49].

At a given iteration n, we let

$$u_i = u_i^{n+1} + v_i.$$

Due to the fact that $v_0 \le v$, we have $v_i \ge 0$ for $i = 1, 2, \cdots, m$. Thus

$$u_i + \hat{u}_i^{n+1} - u_i^{n+1} = \hat{u}_i^{n+1} + v_i \ge \hat{u}_i^{n+1} \quad \text{and so} \quad u_i + \hat{u}_i^{n+1} - \hat{u}_i^{n+1} \in K_i^n.$$

As a consequence of (42) and (43), we see that

$$\sum_{i=1}^m u_i = u \quad \text{and} \quad \left(\sum_{i=1}^m \|u_i - u_i^{n+1}\|_1^2 \right)^{\frac{1}{2}} \le C(m+1)^{\frac{1}{2}} \left(1 + \left(\frac{Hc_d}{\delta} \right)^{\frac{1}{2}} \right) \|u - u^{n+1}\|_1. \tag{44}$$

Estimate (44) shows that for the overlapping domain decomposition methods, the constants in (18) and (19) are

$$C_1 = C(m)\left(1 + \left(\frac{Hc_d}{\delta}\right)^{\frac{1}{2}}\right), \qquad C_2 = Lm,$$

where m is the number of color for the subdomains. The estimate for C_2 follows from the standard Hölder's inequality.

4.2 Multigrid decomposition

In this subsection, we discuss the application of our theory to multigrid methods. From the space decomposition point of view, a multigrid algorithm is built upon the subspaces that are defined on a nested sequence of finite element partitions.

4.2.1 Decomposition of $H^1(\Omega)$ by multigrid

We assume that the finite element partition $\mathcal{T}$ is constructed by a successive refinement process. More precisely, $\mathcal{T} = \mathcal{T}_J$ for some $J > 1$, and $\mathcal{T}_j$ for $j \leq J$ is a nested sequence of quasi-uniform finite element partitions, i.e. $\mathcal{T}_j$ consist of finite elements $\mathcal{T}_j = \{\tau_j^i\}$ of size h_j such that $\Omega = \cup_i \tau_j^i$ for which the quasi-uniformity constants are independent of j (cf. [9]) and τ_{j-1}^l is a union of elements of $\{\tau_j^i\}$. We further assume that there is a constant $\gamma < 1$, independent of j, such that h_j is proportional to γ^{2j}.

As an example, in the two dimensional case, a finer grid is obtained by connecting the midpoints of the edges of the triangles of the coarser grid, with $\mathcal{T}_1$ being the given coarsest initial triangulation, which is quasi-uniform. In this example, $\gamma = 1/\sqrt{2}$. We can use much smaller γ in constructing the meshes, but the constant C_1 is getting larger when γ is becoming smaller, see (47).

Corresponding to each finite element partition $\mathcal{T}_j$, a finite element space $\mathcal{M}_j$ can be defined by

$$\mathcal{M}_j = \{v \in W^{1,\infty}(\Omega) : v|_\tau \in \mathcal{P}_1(\tau), \quad \forall\, \tau \in \mathcal{T}_j\}.$$

Each finite element space $\mathcal{M}_j$ is associated with a nodal basis, denoted by $\{\phi_j^i\}_{i=1}^{n_j}$ satisfying

$$\phi_j^i(x_j^k) = \delta_{ik}$$

where $\{x_j^k\}_{k=1}^{n_j}$ is the set of all nodes of the elements of $\mathcal{T}_j$. Associated with each such a nodal basis function, we define a one dimensional subspace as follows

$$\mathcal{M}_j^i = \text{span}\ (\phi_j^i).$$

It is easy to see that

$$\mathcal{M}_J = \sum_{j=1}^{J}\sum_{i=1}^{n_j}\mathcal{M}_j^i.$$

4.2.2 Decomposition of K by multigrid

In using our algorithms, only the one dimensional subproblems at the finest mesh, i.e. at level J, are solved with the constraints K_i^n. All the other one dimensional subproblems at the coarser meshes, i.e. at levels $j = 1, 2, \cdots, J-1$, are solved without any constraint.

We use $\hat{u}_{i,j}^{n+1}$ to denote the iterative solutions of the algorithms from the subspaces $\mathcal{M}_j^i$, c.f (8), (9), (11) and (13), and $u_{i,j}^{n+1}$ to denote the updated solutions of (10), (12) or (14). As we only have constraints at the finest mesh, the decomposition of K is in fact unique, i.e.

$$K_{i,J} = \left\{v | \ v \in \mathcal{M}_J^i, \ v(x_J^i) \geq \psi(x_J^i)\right\}, \ i = 1, 2, \cdots, n_J,$$

satisfy

$$K = \sum_{i=1}^{n_J} K_{i,J}.$$

The constraint subsets $K_{i,J}^n$ are also unique. For Algorithm 2.1, they are:

$$K_{i,J}^n = \left\{ v | \ v \in \mathcal{M}_J^i, \ v(x_J^i) \geq \psi(x_J^i) - \sum_{j=1}^{J-1} \sum_{k=1}^{n_j} u_{k,j}^n(x_J^i) \right\}.$$

For Algorithm 2.2, $K_{i,J}^n$ can be defined similarly, i.e. just replace $u_{k,j}^n$ by $u_{k,j}^{n+1}$. The subset $K_{i,J}^n$ is one dimensional which requires the function value at the node x_J^i to be bigger than or equal to a number. The functions from $K_{i,J}^n$ are zero at the other nodes, i.e. at nodes $x_J^k, k \neq i$.

To state it simply: the obstacle for the one dimensional subproblems at the finest mesh is ψ minus the sum of the solutions of all the one dimensional subproblems at the coarser meshes.

4.2.3 Estimations of C_1 and C_2

For any $j \leq J$, let Q_j be the constrained L^2 project operator to the finite element space $\mathcal{M}_j$ at level j, i.e. for any $v \in \mathcal{M}_J$, $Q_j v \in \mathcal{M}_j$ is the solution of

$$Q_j v \leq v, \quad (Q_j v - v, \phi - Q_j v) \geq 0, \quad \forall \phi \in \mathcal{M}_j \text{ satisfying } \phi \leq v.$$

The solution $Q_j v$ has the shortest distance to v in L^2 among all the functions of $\phi \in \mathcal{M}_j$ satisfying $\phi \leq v$, i.e.

$$\|Q_j v - v\|_0 \leq \|\phi - v\|_0, \quad \forall \phi \in \mathcal{M}_j \text{ satisfying } \phi \leq v.$$

With the help of Lemma 4.1 and the inverse inequalities for the finite element functions from $\mathcal{M}_j$, it is easy to show that

$$\|Q_j v - v\|_0 \leq \tilde{C} h_j \|v\|_1, \qquad \|Q_j v\|_1 \leq \tilde{C} \|v\|_1, \tag{45}$$

where

$$\tilde{C} = \begin{cases} C, & \text{if } d = 1; \\ C\left(1 + \left|\log \frac{h_j}{h}\right|\right), & \text{if } d = 2; \\ C\left(\frac{h_j}{h}\right)^{\frac{1}{2}}, & \text{if } d = 3. \end{cases}$$

Let u^{n+1} be the solution of Algorithms 2.2 and 2.1 at iteration $n+1$ and let u be the true finite element solution of the obstacle problem (5). We shall define $v_j = (Q_j - Q_{j-1})(u - u^{n+1}), j = 1, 2, \cdots, J$. It is clear that

$$v_J = (u - u^{n+1}) - Q_{J-1}(u - u^{n+1}) \geq 0.$$

A further decomposition of v_j is given by

$$v_j = \sum_{i=1}^{n_j} v_j^i \quad \text{with} \quad v_j^i = v_j(x_j^i)\phi_j^i.$$

Lemma 4.2 *For the decompositions v_j^i, we have*

$$u - u^{n+1} = \sum_{j=1}^{J}\sum_{i=1}^{n_j} v_j^i, \quad \textit{and} \quad \left(\sum_{j=1}^{J}\sum_{i=1}^{n_j} \|v_j^i\|_1^2\right)^{\frac{1}{2}} \leq \tilde{c}_d\|u - u^{n+1}\|_1, \tag{46}$$

where

$$\tilde{c}_d = \begin{cases} C\gamma^{-2}|\log\gamma|^{-1}|\log h|, & \textit{if } d = 1; \\ C\gamma^{-2}|\log h|, & \textit{if } d = 2; \\ C\gamma^{-2}h^{-\frac{1}{2}} & \textit{if } d = 3. \end{cases}$$

Proof. We first consider the case that $d = 2$. We estimate

$$\sum_{i=1}^{n_j} |v_j^i|_1^2 = \sum_{i=1}^{n_j} |v_j(x_j^i)|^2 |\phi_j^i|_1^2 \leq Ch_j^{(d-2)} \sum_{i=1}^{n_j} |v_j(x_j^i)|^2.$$

In the above, we have assumed that $\Omega \subset R^d, d = 1, 2, 3, \cdots$. Using the fact that, in the finite element space, an L^2 norm is equivalent to some discrete L^2 norm, namely

$$\|v_j\|_0^2 \cong h_j^d \sum_{i=1}^{n_j} |v_j(x_j^i)|^2,$$

we get that

$$\sum_{i=1}^{n_j} |v_j^i|_1^2 \leq Ch_j^{(d-2)} \sum_{i=1}^{n_j} |v_j(x_j^i)|^2 \leq Ch_j^{-2}\|v_j\|_0^2.$$

As a consequence,

$$\begin{aligned}
\sum_{j=1}^{J}\sum_{i=1}^{n_j}\|v_j^i\|_1^2 &\le C\sum_{j=1}^{J} h_j^{-2}\|v_j\|_0^2 \\
&\le C\sum_{j=1}^{J} h_j^{-2}\left\|\Big(Q_j - Q_{j-1}\Big)v\right\|_0^2 \le C\sum_{j=1}^{J} h_j^{-2}\left\|Q_j\Big(I - Q_{j-1}\Big)v\right\|_0^2 \\
&\le C\sum_{j=1}^{J} h_j^{-2}\left(1 + (J-j)|\log\gamma|\right)\left\|\Big(I - Q_{j-1}\Big)v\right\|_0^2 \\
&\le C\sum_{j=1}^{J} h_j^{-2}h_{j-1}^2\left[1 + (J-j+1)|\log\gamma|\right]^2\|v\|_1^2 \qquad (47)\\
&\le C\gamma^{-4}|\log\gamma|^2 J^2\,\|v\|_1^2 \le C\gamma^{-4}|\log h|^2\,\|v\|_1^2 .
\end{aligned}$$

The proof for $d = 1$ and $d = 3$ only differs in using different values for the constant $\tilde{C}$ of (45) and follows the same argument. For $d = 1$, we have used the relation $J = O(|\log h||\log\gamma|^{-1})$. □

With v_j^i as given above, we decompose u into

$$u_{i,j} = u_{i,j}^{n+1} + v_j^i.$$

As $v_J \ge 0$ and $\hat{u}_{i,J}^{n+1} \in K_{i,J}^n$, it is true that $v_J^i \ge 0$ and so

$$u_{i,J} + \hat{u}_{i,J}^{n+1} - u_{i,J}^{n+1} \ge \hat{u}_{i,J}^{n+1} \quad \text{which implies} \quad u_{i,J} + \hat{u}_{i,J}^{n+1} - u_{i,J}^{n+1} \in K_{i,J}^n,\ \forall i.$$

The relation $\sum_{j=1}^{J}\sum_{i=1}^{n_j} u_{i,j} = u$ is easy to deduce from (46). By the inequality of (46), we have that the inequality of (18) is true with $C_1 = \tilde{c}_d$. The estimation of C_2 is the same as for general second order elliptic equations, which is independent of the mesh size h and the number of levels J. Standard proofs can be found in [38], [54] and [49].

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