# Department of APPLIED MATHEMATICS

# SOLUTION OF A NON-STRICTLY HYPERBOLIC SYSTEM MODELLING NON-ISOTHERMAL TWO-PHASE FLOW IN A POROUS MEDIUM

by Tor Barkve

Report no.83

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$$ug_{\rm u} - vg_{\rm v} = 0 \tag{1.2}$$

defines two distinct curves in the (u,v)-space along which the system Eq.(1.1) has a parabolic degeneracy. As shown in the Appendix, such systems can be used modelling non-isothermal two-phase flow in a porous medium.

First, the Riemann problem associated to Eq.(1.1) is solved, i.e. the Cauchy problem defined by Eq.(1.1) and the initial data

$$(u, v)_{t=0} = \begin{bmatrix} (u^{L}, v^{L}) & x < 0 \\ \\ (u^{R}, v^{R}) & x > 0 \end{bmatrix}$$
 (1.3)

 $u^L$ ,  $u^R$  etc denote constant values. In the solution of the Riemann problem, entropy conditions valid independently of local linear degeneracies of the system Eq.(1.1) are defined. The system allows for an additional conservation law, an entropy equation, and this equation is solved explicitly. Opposite to strictly hyperbolic systems, it is not possible to construct locally a convex entropy at all points in the phase space. Results from application of the Riemann solver in the Random Choice Method for numerical solution of Eq.(1.1) is presented.

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Hyperbolic systems with parabolic degeneracies have been studied by several authors [4,7,11-16,26,27]. The solution of the Riemann problem for Eq.(1.1) involving a single transition curve in phase space where the eigenvalues are equal, has been given by Keyfitz and Kranzer [14], assuming one of the wave families to be genuinely nonlinear. A specific application to a reservoir modelling problem, where a single local degeneracy exists, was discused by Isaacson [12]. Johansen and Winther [13] solved the Riemann problem for a system closely related to Eq.(1.1), also involving a single transition curve. Parts of the general solution of Eq.(1.1), with special significance to reservoir modelling, was first given by Hovdan [11] and by Pope [26]. The general solution presented in this report has also beed found independently by Da Mota [4], with a slightly more restrictive definition of the function g.

Useed on the Riemann solution. It is proven that the ganeral Cauchy problem for Eq.(1.1) has a solution. This is a generalization of a proof given by [emple [27] in the case of a tingle transition curve.

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$$\begin{bmatrix} S \\ T \end{bmatrix}_{t} + \begin{bmatrix} f & f \\ g & 0 \end{bmatrix} \begin{bmatrix} S \\ T \end{bmatrix}_{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
(2.1)

The relationship between (S,T) and (u,v) is given by

$$S = u - \beta$$

$$T = \frac{v}{u - \beta}$$
(2.2)

and the functions f and g are related as

$$a = \frac{f + \alpha}{S + \beta}$$
(2.3)

 $\alpha$  and  $\beta$  are positive constants representing thermodynamic parameters. In the physical model  $\alpha < \beta$ , but this restriction is not imposed in the following. Note that g will be used both as function of (u,v) and of (S,T).

A polar-coordinate form of the equations is written as

$$r_{t} + (rg) = 0$$
  
 $r_{t} = 0$   
 $r_{t} = 0$   
 $r_{t} = 0$   
(2.4)

The dependent variables are then defined by

$$\theta = \operatorname{Arctg}(\frac{v}{u}) = \operatorname{Arctg}(T)$$

$$r = \sqrt{u^{2} + v^{2}} = (S + \beta) \sqrt{1 + T^{2}}$$
(2.5)

2.11 ALTERNATIVE FORMS OF THE MODEL BOURTIONS

It will be conveqient to operate with several different forms of the system Eq.[1.1]. As shown in the Appendix, when modelling nonisothermal flow in a porous mudia, the system originated through the physical vertables 5 and 7, ceptesenting saturation and tabgerators form as form as

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A polar-coordinate form of the equations is written as

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3. DESCRIPTION OF THE FUNCTION f = f(S,T)

The function f is defined on the (S,T)-domain  $[0,1]\times[0,1]$ . For constant T, f is assumed to be the S-shaped fractional-flow function well-known from isothermal flow in porous media:

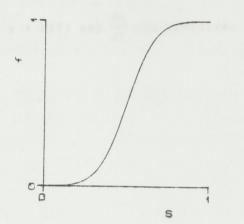


Fig 3.1 : Example of f(S,T=const)

Further, f will be assumed to have the following properties:

$$f_{T} < 0 \qquad T \in (0,1) \qquad (3.1)$$

$$f_{S} < \frac{\alpha}{\beta} \qquad S = 0 \qquad (3.2)$$

$$f_{S} < \frac{\alpha+1}{\beta+1} \qquad S = 1 \qquad (3.3)$$

Eqs.(3.2-3) together assure the existence of two distinct transition curves in phase space, i.e. in (S,T)-space, where the eigenvalues of the system-matrix are equal. These properties are only possible if for each Te[0,1], there exist at least one point where  $f_{SS} = 0$ . As long as two and only two transition curves exist, no restriction will be made on the number of inflextion points of f(T=const). Hence, gravity may be included in the model equations, as described in the Appendix. It will however be assumed that f(T=const) is convex in the vicinety of the transition curve  $S_1$ , concave in the vicinety of  $S_2$ ,  $S_1(T) < S_2(T)$ .

1. RESERVETION OF THE FUNCTION F + FISTER

The function f is defined on the (5.71-domain [2.114(2.1]. For constant T. f. is assumed to be the S-shaped fractional-flow function well-known f on isothermal flow in porous media:



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Equil2.2.31 together assume the existence of two distinct transition curves in phase space, i.e. in [5,1]-space, where the enjouvalues of the system matrix are equal. These properties are only possible if (or each TriD.11 there exist at least one paint where f<sub>15</sub> · 8. As long as two and only two transition curves exist, no restriction will be made on the number of inflextion points of fiftconsil. Mande, stavity may be included in the model equations, as described in the stavity of the cransition curve field in the constraints of starts in the stavity of the cransition curve field in the constraints of the starts in the starts of the cransition curve field of the starts of starts of starts in the starts of curve field of the starts of the starts of starts in the starts of starts

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A simple example of a function f with the desired properties is given by

$$f = \frac{s^2}{s^2 + \kappa(1-s)^2}$$

where both  $\kappa = \kappa(T)$  and  $\frac{d\kappa}{dT}$  are positive.

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(3.4)

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A simple example of a function f with the desired properties is

given by

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where both n = n(T) and  $\frac{dn}{dT}$  are positive.

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# 4. STRUCTURE OF THE FUNDAMENTAL WAVES

The structure of the fundamental waves is independent of the values of the constants  $\alpha$  and  $\beta$ , and has previously been discussed by Keyfitz & Kranzer [14] and by Isaacson [12]. The main properties will be reviewed here for the sake of completeness.

The eigenvalues and eigenvectors of the system-matrix given in Eq.(2.1) are given by

$$\lambda^{1} = f_{S} \qquad \lambda^{2} = g$$

$$r^{1} = (1,0)^{T} \qquad r^{2} = (f_{T},g - f_{S})^{T} \qquad (4.1)$$

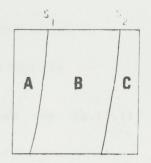
$$l^{1} = (f_{S} - g,f_{T})^{T} \qquad l^{2} = (0,1)^{T}$$

Consequently, the two transition-curves  $S_1$  and  $S_2$  where the eigenvalues are equal are defined implicitly by the following equation, equivalent to Eq.(1.2):

 $f_{c} - g = 0$  (4.2)

The system is not diagonalizable along  $S_1$  and  $S_2$ . Note that in the (f,S)-space, the transition curves are easily determined as points where straight lines through the point P =  $(-\alpha, -\beta)$  are tangents to the curves f = f(S,T=const). The transition curves divide the phase space into three separate regions, defined as:

 $A = \{ (S,T) | T \in [0,1] \quad 0 \le S \le S_1(T) \}$  $B = \{ (S,T) | T \in [0,1] \quad S_1(T) \le S \le S_2(T) \}$  $C = \{ (S,T) | T \in [0,1] \quad S_2(T) \le S \le 1 \}$ 



# 4. STRUCTURE OF THE FUNDAMENTAL MAYES

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From Eqs.(3.2-3), it follows that  $f_S = g$  is non-positive in A C, nonnegativ in B. A point in the phase space will be denoted U = (S,T).

The characteristic family belonging to the eigenvalue  $\lambda^2$  is linearly degenerate, whereas the other family has a local degeneracy at each reflextion point of f(T=const):

$$r^{1} \cdot \nabla \lambda^{1} = f_{SS}$$

$$r^{2} \cdot \nabla \lambda^{2} \equiv 0$$
(4.3)

The last equation also shows that  $\lambda^2 = g$  is a Riemann invariant for the system, the other invariant is given by T. Due to the lack of strict hyperbolicity, no unique transformation between the Riemann invariants and the original variables (S,T) exists:

$$\frac{\partial (g,T)}{\partial (S,T)} = \begin{vmatrix} g_{S} & g_{T} \\ 0 & 1 \end{vmatrix}$$

$$= g_{S} = \frac{1}{S + \beta} (f_{S} - g)$$

$$(4.4)$$

The discontinuities are described by the Rankine-Hugoniot conditions:

$$\sigma = \frac{[f]}{[S]} = \frac{[(f+\alpha)]}{[(S+\beta)T]}$$
(4.5)

a denotes the speed of the discontinuity and the symbol [x] the jump in x across the discontinuity. The last equation may easily be transformed to a form showing that one of the Riemann invariants has to be constant across a discontinuity. Hence, a discontinuity belongs to one of the two following types:

 $\sigma = \frac{[f]}{[S]} \qquad T = const. \qquad "Buckley-Leverett shock"$  $\sigma = g \qquad g = const. \qquad Contact discontinuity$ 

In a summary, the solution of the Riemann problem for Eq.(1.1) consists of a sequence of the follwing waves: .

From Eqs. (3.2-3), it follows that  $f_{S}^{-1}$  g is non-positive in A. C. nonnegative in S. A point in the phase space will be denoted G = 15.71.

The characteristic family belonging to the eigenvalue A is linearly degenerate, whereas the other family has a local degeneracy at each reflextion point of fif-constly

The last equation also shows that h = 0 is a Riemann invariant for the system, the other invariant is given by T. Due to the lack of strict hyperbolicity, no unique transformation between the Riemann invariants and the original variables (3.7) exists:

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The discontinues are described by the Sentine-Huscond's conditions:

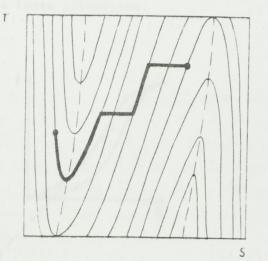
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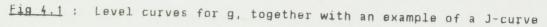
in a summary, the solution of the Hiemann mobiled for Eq. (4.11) consists of a sequence of the following waves:

- 1) Rarefaction waves where T is constant and the wave speed given by  ${\rm f}_{\rm S}^{-}$
- Shocks where T is constant across the discontinuity, and the shock speed is given by the Rankine-Hugoniot condition.
- Contact-discontinuities where g is constant across the discontinuity, and the speed of the discontinuity equals g.

The two first types will be treated together and denoted as a <u>S-wave</u>, the last type will be denoted as a <u>T-wave</u>. Consequently, if a fundamental wave is allowed to be "degenerate" in the way that the left state equals the right state, the general form of the Riemann solution is  $S_1T_1S_2...T_n$ .  $S_1$  is a wave with left state  $U^L$ ,  $T_n$  a wave with right state  $U^R$ . Let a <u>J-curve</u> denote a general contineous, piecewise smooth curve in phase space, connecting two states and consisting of a union of sements of level curves for T and g. In the phase space, the solution of the Riemann problem obviously is a Jcurve. Fig 4.1 shows level curves for g, together with an example of a J-curve.

Note that changes in T can only occur through a T-wave, i.e through a contact discontinuity.





- is sacesaction waves where I is constant and the wave sneed given by
- Shocks where I is constant across the discontinuity, and the shock speed is given by the Ranking-Hugonist condition
  - It contact-discontinuities where g is constant across the discontinuity, and the spaud of the discontinuity equals g.

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Note that changes in I can only otcur through a T-wave, i.a



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# 5. ENTROPY CONDITIONS

An entropy condition for a non-strictly hyperbolic system was formulated by Keyfitz and Kranzer [14], generalizing the well-known Lax entropy condition [18]. For a 2x2 system, the Keyfitz-and-Kranzer entropy condition states that for a discontinuity in the solution to be admissible, either

- 1) 3 characteristics enter the discontinuity and 1 leaves, or
- 2) 2 characteristics are tangents to the discontinuity and at least one of the remaining enters the discontinuity, - or
- 3) the shock may be regarded as a limit of a sequence of shocks satisfying 1) or 2)

As an example of a shock admissible according to condition 3), take a situation where one of the shock values is situated on a transition curve; then 3 characteristics are tangents, while the remaining enters or leaves the discontinuity. By a careful choice of the function f, it is also possible to construct discontinuities where all the characteristics are tangents to the line of discontinuity, confer Fig 5.1. The condition (E) given by Temple [27] does not adequately describe these situations.

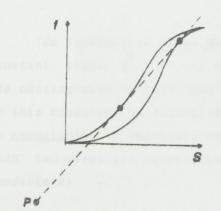


Fig.5.1 : Example of a discontinuity where all characteristics are tangents to the line of discontinuity

It is well-known that for a strictly hyperbolic system with local linear degeneracies, the Lax criterion is not restrictive enough

#### 5. ENTROPY CONDITIONS

An antropy condition for a non-strictly hyperbolic system was formulated by Keyfetz and Kranzer (141, generalizing the well-known Lax entropy condition [18]. For a 2x2 system, the Keyfitz-and-Kranzer entropy condition states that for a discontinuity in the solution to be admissible, either

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As an example of a shock admissible seconding to condition 2), take a situation where one of the shock values is situated on a transition curve; then 3 characteristics are tangents, while the remaining enters or leaves the discontinuity. By a careful chairs of the function f, it is also possible to construct discontinuities where all the characteristics are tangents to the line of discontinuity, confer Fig 5.1. The condition (6) given by femple [27] does not adequately



Lig.b.1 1 Example of a discontinuity where all characteristics are tangents to the line of discontinuity

to resolve a unique solution [20,23]. As the function f(T=const) may have more than one inflection point, we will not use the generalized Lax criterion directly. Combining the Rankine-Hugoniot condition, Eq.(4.5), and the relation between f and g, Eq.(2.3), we have

$$g^{\mathsf{R}} - \sigma = \frac{s^{\mathsf{L}} + \beta}{s^{\mathsf{R}} + \beta} (g^{\mathsf{L}} - \sigma)$$
 (5.1)

As (S +  $\beta$ ) is always positive, it follows that if  $\sigma \neq g^L$ , one of the characteristics belonging to the second family leaves, while the other enters the discontinuity. Hence, for a "Buckley-Leverett" shock it is sufficient to study the behaviour as if only one family of waves is present. The following shock-admissibility criterion is eqivalent to the generalized Lax criterion if no local linear degeneracies is present:

# Shock-admissibility criterion

A T-wave is admissible if it does not cross a transition curve. A Swave is admissible if it satisfies the Oleinik condition [23] for constant T:

 $\frac{f - f^{R}}{s - s^{R}} \leq \frac{f^{L} - f^{R}}{s^{L} - s^{R}} \qquad T = T^{L} = T^{R}$ 

Two fundamental waves may be combined into one wave through a constant state  $U^{M}$  ( $U^{L}$  -->  $U^{M}$  -->  $U^{R}$ ) if the speed of the front of the chasing wave is less than the speed of the tail of the chased, and if this condition is fullfilled, the two fundamental waves are said to be <u>compatible</u>. A necessary and sufficient criterion for determing when two waves are compatible is given by the following compatibility conditions:

to resolve a unique solution (20.23) As the function (IT-const) may nave more than one inflection point, we will not use the generalized [as criterion directly. Combining the Sanking-Sugoniol condition, Eq.(4.3), and the relation between 4 and o. Eq.(2.3), we have

As  $[5 + \beta]$  is simple positive, it follows that if  $\sigma \in G'$ , one of the characteristics belonging to the second family lasves, while the other enters the discontinuity. Hence, for a "Buckley-Leverst" shock it is sufficient to study the behaviour is if only one family of waves is present. The following shock-admissibility criterion is equivalent to the generalized tex criterion if no local linear degeneracies is present:

## Shock-admirsibility critering

A T-wave is admissible if it does not cross a transition curve. A 5wave is admissible if it satisfies the Ulminsk concltien [23] for constant T:

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# Compatibility conditions (CC)

The two fundamental waves in a TS-wave are compatible if and only if one of the following conditions is satisfied:

- 1) The S-wave contains a rarefaction part close to  $\mathrm{U}^{M},$  and  $\mathrm{U}^{M}$   $\varepsilon$  B.
- 2) The S-wave contains a shock close to  ${\rm U}^{\rm M}$  and, if supscript R denotes the right shock value,

$$\frac{g^{M} - g^{R}}{s^{M} - s^{R}} \ge 0$$

The two fundamental waves in a ST-wave are compatible if and only if one of the following conditions is satisfied:

- 1) The S-wave contains a rarefaction part close to  $U^{M},$  and  $U^{M} \in \Lambda C.$
- 2) The S-wave contains a shock close to  $\text{U}^{\text{M}}$  and, if supscript L denotes the left shock value,

$$\frac{g^{M}-g^{L}}{s^{M}-s^{L}} \leq 0$$

#### Proof:

0 4

Only the first part concerning the TS-wave will be shown, as the second case is analogous. If S has a rarefaction part close to  $U^M$ , the waves are compatible iff  $g^M \leq f_S^M$ , i.e iff  $U^M \in B$ . If S has a shock close to  $U^M$ , the waves are compatible iff

$$\frac{f^{M} - f^{R}}{s^{M} - s^{R}} - g^{M} = \frac{g^{M} - g^{R}}{s^{M} - s^{R}} (s^{R} + \beta)$$
(5.2)

Here, the Rankine-Hugoniot expression for the shock speed has been used. As ( S +  $\beta$  ) is always positive, the result follows.

The shock-admissibility criterion is valid independently of the number of transition curves and the number of inflextion points of f. Also, the CC is easily extended to be valid for a general number of transition curves. Together, the shock-admissibility criterion and the CC will be used to construct a solution of the Riemann problem, unique in the (x,t)-space, termed the entropy solution.

#### Compartabality conditions icc

ine imeriendamental waves in a Fi-wave are competible if and only if

- I) The 5-wave contains a rarefaction part close to U<sup>M</sup>
- 21 The S-wave contains a shock close to U" and, if supacript R denotes the right snock value.

The two fundamental waves in a ST-wave are competible if and only if

- 1) The 5-wave contains a ratefaction part close to U<sup>H</sup>, and U<sup>H</sup> e A C.
- 2) The 5-wave contains a shuck close to U" and, if supacript t

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Hora, the Sentine-Support expression for the shock speed has been used. An ( 5 - 8 ) is slupps positive, the result follows.

The shock-admissionity contents is valid independently of the number of transition curves and the number of inflexiton points of f. Also, the CC is easily unterded to be valid for a general number of transition curves. Together, the shock-admissionity criterion and the CC will be used to construct a colution of the Sismann problem,

# 6. CONSTRUCTION OF AN ENTROPY FUNCTION

An entropy function E for the a hyperbolic system is a scalar function, having the following properties

E is convex, i.e the Hessian d<sup>2</sup>E is positive definite
 A scalar function F exists such that VE = A VF where A is the system matrix and V is the gradient operator in the unknown functions

-

For a strictly hyperbolic, genuine non-linear system, if an entropy exists, a viscous regularization must satisfy the following inequality in the topology of distributions [19]:

$$E_{t} + F \leq 0$$
(6.1)

If E is a known function, Eq.(6.1) can be used as a shockadmissibility criterion. Using Eq.(1.1), is it easy to to show that a function E = E(u,v) can have the property 2) only if it satisfies the following equation:

$$ug_{V}E + (vg_{V} - ug_{L})E - vg_{L}E = 0$$
 (6.2)

For a general  $2\times2$  system, the entropy-equation is of equal type as the original 1.order system, i.e. Eq.(6.2) is hyperbolic everywhere except along the transition curves. The equation may be integrated by writing the equation on the form

$$(g_{v}, -g_{l}) * \nabla [uE_{l} + vE_{l} - E] = 0$$
 (6.3)

As  $(g_v, -g_u)$  is the tangent-vector to level-curves of g, it follows that E must satisfy Clairaut's differential equation with an inhomogeneity term  $\varphi_i - \varphi$  being an arbitrary function of g:

$$uE + vE - E = \phi(g)$$
 (6.4)

By transforming to polar coordinates as defined in Eq.(2.5), the solution for E is written:

# & CONSTRUCTION OF AN ENTROPY FUNCTION

An entropy function E for the a hyperbolic system is a scalar function, having the following progerties

1) E is convex, i.e the Hassian d E is positive definite 21 A scalar function F exists such that 91 = 4 99 where A is the system matrix and V is the gradient operator in the enthewn

For a strictly hyperiolic, genuine non-lineer system, if an entropy exists, a viscous regularization must satisfy the following inequality in the topology of distributions [19]:

If E is a known function. Eq.(6.1) can be used as a shockadmissibility criterian. Using Eq.(1.1), is it easy to to show that a function E = E(u,v) can have the property P( only IV it satisfies the following equation:

For a general 2x2 system, the entropy equation is of upual type as the original 1.order system, 1.x. Eq. (6.2) is hyperbolic everywhere except along the transition curves. The equation may be integrated by writting the equation on the form

As  $(q_{u_1} - q_{u_2})$  is the tangent-vector to level-curves of  $q_1$ . It follows that E must satisfy Clairant's differential squation with an inhomogeneity term  $q_2$ .  $q_1$  being an arbitrary function of  $q_2$ .

By transforming to colar coordinates as covined in Eq. (1.51, the

12.3

$$E = r \int \frac{\varphi}{r^2} dr + r\psi \qquad (6.5)$$

 $\psi = \psi(\theta)$  is a new arbitrary function. The entropy-flux F is given by  $F = gE + \int \phi \, dg$ (6.6)

A necessary condition for the given entropy function to be convex in (r, 0) is that  $E_{rr}$  is positive:

$$E_{rr} = \frac{\varphi}{2} g_{r} \qquad (6.7)$$

 $\varphi'$  denotes the derivative of  $\varphi$  with respect to g. As  $g_r$  changes sign across the transition curves, it follows that  $E_{rr}$  also changes sign in the phase space. Hence, it is not possible to construct a contineous entropy that is globally convex, and at the transition curves it is not possible to construct a locally convex entropy. Note that for a strictly hyperbolic 2x2 system, it was shown by Lax [19] that it is always possible to define a convex entropy locally.

If [x] denotes a change in the quantity x through a discontinuity,  $[x] = x^{L} - x^{R}$ , then the entropy production caused by a discontinuity is given by

$$\sigma[E] - [F] = -r^{L}r^{R} \frac{g^{R} - g^{L}}{r^{R} - r^{L}} \int \frac{\phi}{r^{2}} dr + \int \phi dg \qquad (6.8)$$

It follows easily that for a contact discontinuity - and also for a "Buckley-Leverett" shock having  $g^{L} = g^{R}$  - the entropy change across the discontinuity is zero. Expanding the right hand side of Eq.(6.8) in a Taylor series in the "shock-strenght" ( $r^{R} - r^{L}$ ), we have

$$\sigma[E] - [F] = \left\{ \frac{\phi' g}{r^2} (r_r^2 g) \right\}^L (r^R - r^L)^3 + \dots \quad (6.9)$$

Now assume that a given wave is fully contained in <u>one</u> of the regions A, B or C and satisfies the shock-admissibility criterion given in 11

the is a new arbitrary function. The entropy-flux f is given by

A necessary condition for the given entropy function to be convex in (r.0) is that E\_\_ is positive:

 $q^2$  denotes the derivative of a with respect to q. As  $q_p$  changes sign across the transition cuives, it follows that  $f_{pr}$  also changes sign in the phase space. Hence, it is not possible to construct a contineous entropy that is globally convex, and at the transition curves it is not possible to construct a locally convex antropy. Mole that for a strictly hyperbolic 2n2 system, it was shown by lax [19] that it is a sinays possible to define a convex entropy locally.

If [x] dendies a change in the quantity x through a discontinuity [x] at a seven by a discontinuity is given by

It follows easily that for a contact discontinuity - and also for a "Buckley-Leversts" shock naving  $g^L - g^2 - the entropy change scross the discontinuity is zero. Expending the right hand alde of Eq.(2.3) in a laylot series in the "shock strenght" <math>(r^R - r^L)$ , we have

Now assume that a given wave is fully contained in gos of the regions

Chapter 5. As  $(r^2g_r)_r = (S + \beta)f_{SS}$ , Eq.(6.9) shows that for a sufficiently weak shock where  $f_{SS} \neq 0$ , the shock-admissibility criterion is equivalent to an increase or decrease in entropy, according to whether the wave is contained in AUC or in B. However, waves crossing a transition curve may cause an increase, a decrease or no effect in the entropy.

When deriving the property Eq.(6.1) for a strictly hyperbolic system, the choice of a convex or concave entropy is just a matter of convenience, - the inequality sign in Eq.(6.1) must however be reversed if a concave entropy is chosen [6]. Hence, for a weak shock the given entropy could be used to resolve the entropy solution if and only if it could be guaranteed that the solution does not cross a transition curve.

It will be shown in the next section that in some cases where no entropy-change is produced, the solution of the Riemann problem is not unique in the phase space. Chapter 5. As  $(r_{1})_{r}^{-1} = (1 + \beta)_{55}^{-1}$ , Eq.(6.0) shows that for a sufficiently weak shock where  $r_{55}^{-1} = 0$ , the shock-admissibility criterion is equivalent to an increase of ducrease in entropy, according to whather the wave is contained in AUC or in 8. However, waves crossing a transition curve may cause an increase, a decrease or no effect in the entropy.

When deriving the property Eq.(5.1) for a strictly hyperbolic system, the choice of a convex or concave entropy is just a hatter of convenience, the inequality sign in Eq.(5.1) must nowever be reversed if a concave entropy is cnoten [5]. Hence, for a weak shock the given entropy could be used to resolve the entropy solution if and only if it could be querenteed that the solution does not cross a transition curve.

It will be shown to the next section that in tome cates where no entropy-change is produced, the solution of the Riemann problem is not unique in the phase space. In the following, let  $R[U^L, U^R]$  denote the solution of the Riemann problem Eqs.(1.1) and (1.3). As already shown, the solution is composed of a sequence of fundamental waves  $J = S_1 T_1 S_2 \dots T_n$ , where  $S_1$  is a wave with left state  $U^L$ . The goal of this chapter is to show the following theorem

THEOREM 1

The Riemann problem for Eqs.(1.1) has an entropy solution of the form  $S_1 TS_2$ . The solution is unique in (x,t)-space, but not in the phase space.

The proof is based on a study of cases, where the solution is shown to belong to one of the following classes of J-curves:

Class 1 : J = TS where  $T : U^{L} - \rightarrow U^{M}$ S :  $U^{M} - \rightarrow U^{R}$ 

Each fundamental wave satisfies the shock-admissibility criterion. If  $U^L \in S_1 \cup S_2$ , then  $U^M \in B$ .

Class 2 : J = ST where S:  $U^{L} \rightarrow U^{M}$ T:  $U^{M} \rightarrow U^{R}$ 

Each fundamental wave satisfies the shock-admissibility criterion. If  $U^R$   $\epsilon$   $S_1 \cup S_2$ , then  $U^M$   $\epsilon$   $A \cup C.$ 

Class 3 :  $J = S_1 TS_2$  where  $S_1 : U^L \longrightarrow U^M$   $U^L \in A \cup C$   $T : U^M \longrightarrow U^R$  $S_2 : U^N \longrightarrow U^R$ 

Each fundamental wave satisfies the shock-admissibility criterion. One of the states  $U^{M}$  or  $U^{N}$  is lying on the transition curve near  $U^{L}$ , according to the following rule:

## CONSTRUCTION OF THE RIEHAMM SOLUTION

In the following, lat  $R(U^2, U^2)$  denote the folution of the foundament problem Eq.(1.1) and (1.3). Is strandy shown, the follotion is compared of a sequence of fundamental waves 1 = 2, 1, 1, 2, 1, where 2, is a wave with left state  $U^2$ . The goal of this chapter is to show the following theorem

## THEOREM

The Riemann problem for Eqs. (1, (1, has an entropy solution of the form  $S_1 r S_2$ . The solution is unique in (x, t)-space, but not in the phase space.

The proof is based on a shudy of cases, where the solution is shown to belong to one of the following classes of J-curves:

Each fundamental wave satisfies the shock admissibility criterion. If  $U^{L} = S_{L}US_{L}$  then  $U^{M} = B$ .

Class 2 : 2 · 57 Where 5: 2 · -- 2 " T: 4<sup>H</sup> -- 2 U<sup>2</sup> Each fundamental wave satisfies the shock-admissibility criterion. If U<sup>2</sup> c 5, U5, then U<sup>2</sup> c 6 UC.

fach fundamental wave satisfies the shock-admissibility criterion. One of the states U<sup>R</sup> or U<sup>R</sup> is lying on the framething curve near U<sup>R</sup>, according to the following rule:

$$U^{L} \in C \text{ and } T^{L} < T^{R} \Rightarrow U^{N} \in S_{1} \cup S_{2}$$

$$U^{L} \in A \text{ and } T^{L} > T^{R} \Rightarrow U^{N} \in S_{1} \cup S_{2}$$
Else
$$U^{M} \in S_{1} \cup S_{2}$$

It is easily seen from Fig 4.1 that between two arbitrary states, infinitely many J-curves not belonging to one of the classes 1, 2 or 3 exist. However, at least one curve belonging to one of the classes does exist, - and in each class, if it exists, the J-curve is unique.

In three following lemmas, criterions for when the Riemann solution belongs to one of the three classes will be given. Finally, it will be shown that the given criterions covers all possible combinations of  $U^L$  and  $U^R$ . To facilitate the statement of the criterions, define the quantities  $g_1$  and  $g_2$  by:

$$g_{1} = g(S_{1}(T^{R}), T^{R})$$

$$g_{2} = g(S_{2}(T^{R}), T^{R})$$
(7.1)

Also, let  $T_1$  and  $T_2$  be defined implicitly as solutions of the equations

$$g^{R} = g(S_{1}(T_{1}), T_{1})$$

$$g^{R} = g(S_{2}(T_{2}), T_{2})$$
(7.2)

If no solution for  $T_1$  exists, define  $T_1 = 0$ . If no solution for  $T_2$  exists, define  $T_2 = 1$ .

If a solution of Class 1-3 exists, it remains to show that also the compatibility criterion CC is satisfied, as each of the fundamental waves satisfies the shock-admissibility criterion. To show that a given J-curve consists of compatible waves, the following corollaries of the CC will be used:

## COROLLARY 1

The fundamental waves in a Class-1 wave are compatible if both  $U^L$  and  $U^R$  are contained in B. The fundamental waves in a Class-2 wave are compatible if both  $U^L$  and  $U^R$  are contained in A or if both are contained in C.

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It is easily seen from Fig 4.1 that netween two arbitrary states, infinitely many 1-curves not belonging to one of the classes 1, 2 dr 3 wist. However, at least one curve belonging to one of the classes dose exist. - and in each class, if it exists; the 1-curve is unique.

In three following lemmas, criterions for when the Riemann solution belongs to one of the three classes will be given. Finally, it will be shown that the given criterions revers all possible combinations of U and U<sup>A</sup>. To facilitate the statement of the criterions, define the quantities g, and g, by:

Also, ist 1, and T<sub>2</sub> he defined implicitly as solutions of the equations

If no solution for  $T_{i}$  exists, define  $T_{i}$ . If no solution for  $T_{i}$  exists, define  $T_{i}$  = 1.

If a solution of class 1-3 exists, it romains to snow that also the compatibility criterion GC is satisfied, as each of the fundamental waves satisfies the snock-admissibility criterion. To show that a given 1-cutve consists of compatible waves, the following corollaries of the CC will be issue

## COROLLARY

The fundamental waves in a Class I wave are compatible if both  $U^2$  and  $U^2$  are contained in B. The fundamental waves in a Class 2 wave are compatible if both  $U^2$  and  $U^2$  are contained in a or if both are contained in a or if both are

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## COROLLARY 2

The fundamental waves in a J-curve of Class 1 or Class 2 are compatible if  $U^M \in S_1 \cup S_2$ .

# Proof of Corollary 1:

When a given wave J is of Class 1 and  $U^L$ ,  $U^R \in B$ , the whole J-curve is contained in B. As  $g_S \ge 0$  in B, the CC is satisfied. Analogous for a Class-3 wave.

# Proof of Corollary 2:

The proof is shown for  $U^M \in S_1$ . If the S-wave has a rarefaction part close to  $U^M$ , the result follows immediately from the CC. As f(T=const.) is convex in the vicinety of  $S_1$ , the S-wave contains a shock close to  $U^M$  only if  $U^R \in A$ . This gives two possibilities for a shock, according to whether the S-wave or the T-wave is contained in A Taking each case separately, it is easily checked that the CC is satisfied.

## LEMMA 1

 $R[U^{L}, U^{R}]$  is of Class 1 if  $U^{L} \in B$ , and in addition, one of the following conditions is satisfied:

1)  $U^{R} \in A$  and  $g^{R} \leq g^{L} \leq g_{2}$ 2)  $U^{R} \in B$  and  $g_{1} \leq g^{L} \leq g_{2}$ 3)  $U^{R} \in C$  and  $g_{1} \leq g^{L} \leq g^{R}$ 

### Proof

From Fig 4.1 showing the level curves for g, it is obvious that when  $U^{L} \in B$ ,  $g_{1} \leq g^{L} \leq g_{2}$  guarantees the existence of a J-curve of Class 1 connecting  $U^{L}$  and  $U^{R}$ . It remains to show that the fundamental waves are compatible.

If the S-wave has a rarefaction wave close to  $U^M$ , the CC is immediately satisfied, as  $U^M \in B$  when  $U^L \in B$ . Hence, in the following, it is sufficient to study the situations where the S-wave contains a shock close to  $U^M$ :

### COROLLARY

The fundamental waves in a lourve of Class 1 or Class 2 are compatible if U<sup>H</sup> e 5, U 5,

### Proof of Corollary J:

When a given wave j is of Glass 1 and 0 , 0 e B, the whole jcurve is contained in B. As  $g_2 \neq 0$  in B, the CC is satisfied.

### Prant of Corollary 2

The proof is shown for B' = 5, . If the 5 wave has a rerefaction part close to U<sup>H</sup>, the result follows immediately from the CC. As filecenst.1 is convex in the vicinaty of 5, the 5-wave contains a shock close to U<sup>H</sup> only if U<sup>H</sup> c A. This gives two possibilities for a shock, according to whother the 5-wave or the T-wave is contained in A Taking each case separately. It is essily checked that the CC is astisfied.

LEMMA I Alu<sup>1</sup>, u<sup>2</sup>l is of Class I if u<sup>1</sup> s.S. and in addition, one of the following conditions is satisfied: 11 u<sup>2</sup> c.A and g<sup>2</sup> c.g<sup>1</sup> c.g 21 u<sup>2</sup> c.H and g. c.g<sup>1</sup> c.g

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from Fig 6.1 showing the level curves for g. it is obvious that when U i S. g. s.g. i g.guarantees the existence of a J-curve of Class I connecting G and U<sup>R</sup>. It remains to show that the fundamental waves are compatible.

If the 2-wave has a restruction wave class to  $U^{0}$ , the CC is immediately satisfied, as  $U^{0} \neq B$  when  $U^{1} \in B$ . Hence, in the failowing, it is sufficient to cludy the silestions whuse the 5-wave contains a shock close to  $U^{0}$ .

When  $U^R \in A$ , the S-wave contains a shock crossing the transition curve  $S_1$ , as f(T=const) is convex in the vicinety of  $S_1$ . If  $U^*$ denotes the right shock value, then  $S^* \ge S^R => g^* \le g^R$ , and the condition 1) in the lemma guarantees that the CC is satisfied. The argument is identical when  $U^R \in C$ . When  $U^R \in B$ , the lemma follows immediately from Corollary 1.

## Lemma 2

 $R[U^L, U^R]$  is of Class 2 if one of the following conditions are satisfied:

1)  $U^{R} \in A$ ,  $U^{L} \in A$  and  $T^{L} \ge T_{1}$ 2)  $U^{R} \in A$ ,  $U^{L} \in B$  and  $g^{L} \le g^{R}$ 3)  $U^{R} \in C$ ,  $U^{L} \in C$  and  $T^{L} \le T_{2}$ 4)  $U^{R} \in C$ ,  $U^{L} \in B$  and  $g^{L} \ge g^{R}$ 

# Proof :

The lemma is shown for  $U^R \in A$ , as the proof for  $U^R \in C$  is analogous. When  $U^R \in A$ , the condition  $T^L \ge T_1$  guarantees the existence of a J-curve of Class 2 connecting  $U^L$  and  $U^R$ .

When  $U^{L} \in A$ , the CC is satisfied by Corollary 1. If  $U^{L} \in B$ , the S-wave must cross  $S_{1}$  by a shock, as f(T=const) is convex in the vicinety of  $S_{1}$ . The S-wave may still contain a rarefaction part close to  $U^{M}$ , and the CC is then straightforward satisfied. If the S-wave contains a shock close to  $U^{M}$ , the S-wave must be a pure shock wave. As  $S^{M} \leq S^{L}$  and as  $g^{R} = g^{M} \geq g^{L}$ , the CC is satisfied.

terms 2 R[u<sup>1</sup>, u<sup>2</sup>] 111 of e.x. 1 free of the following conditions si activited 1 0<sup>2</sup> e.x. 0<sup>1</sup> e.a and 1<sup>2</sup> a 1 21 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 31 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>2</sup> a 0<sup>2</sup> 41 u<sup>2</sup> e.c. 0<sup>1</sup> e.a and 0<sup>1</sup> a 0<sup>2</sup> 41 u<sup>2</sup> 41 u<sup>2</sup>

The looms is shown for  $U^{R} \in A$ , is the proof for  $U^{R} \in C$  is analogous. When  $U^{R} \in A$ , the condition  $T^{L} \ge T_{1}$  guarantees the

When  $U^* \in A$ , the CC is satisfied by Corollary 1. If  $U^* \in B$ , the S-wave must cross S, by a shock, as fill-const1 is convex in the vicinety of S. The S-wave may still contain a rerefection part class to  $U^{m}$ , and the CC is then straightforward satisfied. If the S-wave contains a shock close to  $U^{m}$ , the S-wave must be a pure shock wave.

Lemma 3

 $R[u^L, u^R]$  is of Class 3 if one of the following conditions are satisfied:

1)  $U^{R} \in A \cup B$  and  $U^{L} \in C$ 2)  $U^{R} \in B \cup C$  and  $U^{L} \in A$ 3)  $U^{R} \in A$ ,  $U^{L} \in A$  and  $T^{L} < T_{1}$ 3)  $U^{R} \in C$ ,  $U^{L} \in C$  and  $T^{L} > T_{2}$ 4)  $U^{R} \in A \cup B$ ,  $U^{L} \in B$  and  $g_{2} < g^{L}$ 5)  $U^{R} \in B \cup C$ ,  $U^{L} \in B$  and  $g^{L} < g_{1}$ 

## Proof :

The lemma is proven by in each case combining Lemma 1 or Lemma 2 together with Corollary 2. Only one case will be shown, the case  $U^R \in A$ ,  $U^L \in C$ : In this case, it is obviously possible to construct a Class-3 J-curve with  $U^N \in S_2$ . Lemma 2 then gives that the fundamental waves in the  $S_1$ T-wave are compatible. Corollary 2 gives that the combination  $TS_2$  is compatible.

Fig 7.1-7, pp 21-22 show all the combinations of  $U^{L}$  and  $U^{R}$  covered by the Lemmas 1-3. In each figure,  $U^{R}$  is specified, and the phase space is divided into different regions showing the solution type if  $U^{L}$  is contained in the region. It is easily seen that all possible combinations have been covered by the lemmas.

If U<sup>L</sup> is situated on the boundary between two regions, the lemmas may state that solutions of two different classes exist. The reason for this may be that the J-curve belongs to two of the classes simultaneously, as when a S-wave in a Class-3 curve is "degenerate", i.e has equal left and right states. However, in certain cases two different solutions exist in phase space, - this is when

$$U^{L} \varepsilon B \qquad U^{R} \varepsilon A \cup C$$

$$g^{L} = g^{R}$$
(7.1)

setisfied:

11 U<sup>P</sup> E A U B and U<sup>P</sup> E C 21 U<sup>P</sup> E B U C and U<sup>P</sup> E A 31 U<sup>P</sup> E A U<sup>P</sup> E A and T<sup>P</sup> C T 31 U<sup>P</sup> E C U<sup>P</sup> E C and T<sup>P</sup> A T 41 U<sup>P</sup> E A U B U<sup>P</sup> E B and B 51 U<sup>P</sup> E B U C B<sup>P</sup> E B and B 51 U<sup>P</sup> E B U C B<sup>P</sup> E B and B

Proof

The lamma is proven by in each case combinend Lemma 1 of Lemma 2 together with Corollary 2. Only one case will be shown, the case  $U^R \in A$ ,  $U^L \in C$ : In this case, it is obviously possible to construct the Clare-3 J-curve with  $U^R \in S_2$ . Lemma 2 then gives that the fundamental waves in the  $S_2$  is compatible. Corollary 2 gives that the the combination IS, is compatible.

Fig 7.1-7, pp 21-22 them all the combinations of U<sup>4</sup> and U<sup>8</sup> covered by the Lemmas 1-3. In each figure U<sup>8</sup> is spectrad, and the phase space is divided into different regions showing the solution type if U<sup>4</sup> is contained in the region. It is easily seen that all possible combinations have been covered by the lemmas.

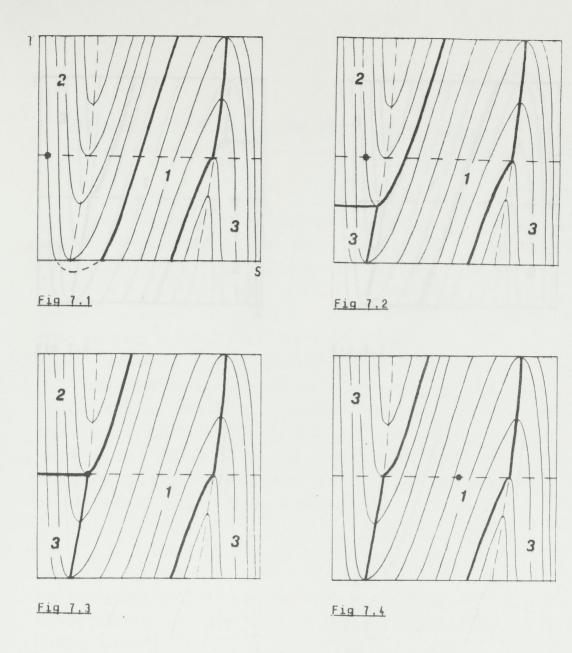
if U is situated on the boundary between two regions, the lemmas may state that solutions of two different classes exist. The resson for this may be that the J-corve belongs to two of the classes simultaneously, as when a 5-wave in a Class-3 curve is "degenerate", i.e has equal left and right states. However, in contain cases two different solutions exist in phase space, this is when It is easily checked that in (x,t)-space, the solution is unique, being a single discontinuity with speed  $g^L$ . The solution could be regarded as a pure T-wave, but this would however not satisfy the shock-admissibility criterion. The non-uniqueness in the phase-space was first pointed out by Isaacson [12] in the case when the system has a single transition curve. It follows from Eq.(7.1) and Eq.(6.8) that the change in entropy caused by the non-unique waves is zero.

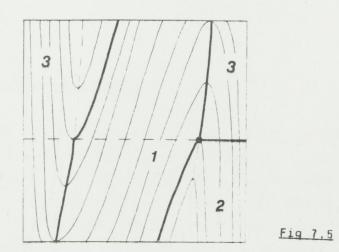
Note that in some cases, a small perturbation of the initial states may change the solution drasticly. This is the case for instance if the initial states are close to a situation as described by Eq.(7.1); a perturbation may cause the solution to change between a Class-1 wave and a Class-2 wave.

It is easily checked that in (a.t)-anace, the solution is unique, being a single discontinuity with spaced g<sup>1</sup>. The solution could be regarded as a pure T-wave, but this would nowever not satisfy the shock-admissibility criterion. The non-uniqueness in the phase-space was first pointed out by isaacson [12] in the case when the system has a single transition curve. It follows from Eq.(7.1) and Eq.(6.8] that the change in antropy caused by the non-unique waves is zero.

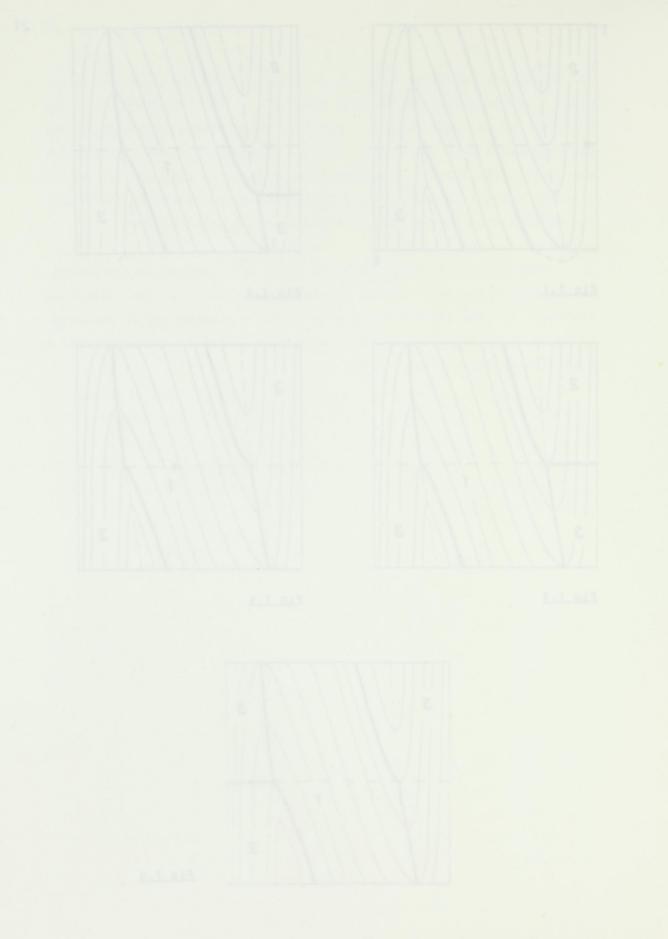
Auto that in some cases, a small perturbation of the initial states may change the solution drasticly. This is the case for instance if the initial states are close to a situation as described by Eq. (7.1); a perturbation may cause the solution to change between a Class. Wayo and a Class-2 wayo.

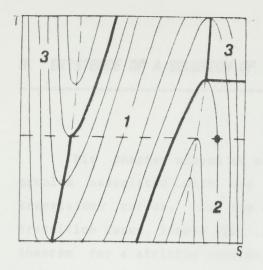
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<u>Fig 7.1-7</u> : Solution regions in phase space. The position of  $U^R$  is shown by a black dot





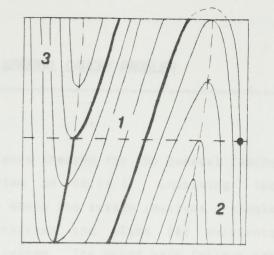




Fig 7.6

construction of the planet for the break form the proof minimized by the saves of the break form functional certain property of the provide the proof bay, he eremaned to penerel number of the provide the break bay, he eremaned to

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# 8. EXISTENCE OF A SOLUTION OF THE GENERAL CAUCHY PROBLEM

This chapter proves an existence theorem for the general Cauchy problem associated with the system in Eq.(1.1), involving two transisiton curves. For the case where the system involves a single transition curve, Temple [27] generalized the Glimm [8] existence theorem for a strictly hyperbolic system. The proof here follows the steps outlined by Temple, defining a transformation  $\Psi$ : (S,T) -> (2,T), regular everywhere except along the transisition curves, and using this transformation to construct the Glimm functional. As only the construction of the Glimm functional deviates from the proof of Temple, it is sufficient to prove that the Glimm functional is minimized by the waves of the Riemann solution R[U<sup>L</sup>,U<sup>R</sup>]. Given a certain property of the function g, the proof may be extended to a general number of distinct transition curves.

### THEOREM 2

The Cauchy problem for the system Eq.(1.1), involving two transisition curves, has a global weak solution for arbitrary initial data of bounded variation in Z and T.

## Definition of the Glimm functional

The variable Z is defined by first extending the definition of g into the domain  $(S,T) \in [0,1]\times[-1,2]$ . This is done by extending the level curves for g in a non-intersecting differentiable manner, as shown on Fig 8.1. The level curves are monotone everywhere except at the transisiton curves, and the extension of the level curves also involves an extension of the transition curves in a smooth way.

Note that to each point U in the original phase space it is possible to associate one point on each of the extended transition curves lying on the same g-level curve as U. These two point will be

# 8. EXISTENCE OF A SOLUTION OF THE GENERAL CAUCHY PROBLEM

ints chapter proves in existence theorem for the general fauchy problem exoctated with the system in Eq.(1.1) involving two transition curves. For the case where the system involves a single transition curve. Temple [27] generalized the Climm [8] existence theorem for a strictly hyperbolic system. The moor here follows the steps outlined by Temple, defining a transformation V: (5,7) -> (2,7), regular everywhere except along the transformation V: (5,7) -> (2,7), this transformation to construct the blimm functions. As ealy the construction of the Glimm functional deviates from the proof of minimized by the waves of the Riemann solution 211 . Gentain property of the function g, the proof may be extended to a general number of distinct transition curves and a size a

### S MEGOREM 2

The Cauchy problem for the system Eq.(1.1), involving two transisition curves, has a global weak solution, for arbitrary initial data of bounded variation in 2 and T.

### Definition of the Glimm functional

The variable 2 is defined by first extending the definition of g into the domain (5,7) a [d,1]a(-1,2]. This is done by extending the lavel curves for g in a num-intersecting differentiable manner, as shown on Fig 5.1. The lowel curves are monotone everywhere except at the transistion curves, and the extension of the lavel curves also involves an extension of the transition curves in a smooth way.

Note that to each pount U 10 the original phase apace it is possible to associate one point on each of the extended transition curves lying on the same grievel curve is U. These two coint will be termed the <u>associated points</u> for U and denoted U<sub>1</sub> and U<sub>2</sub> respectively,  $U_i = (S_i, T_i)$ .

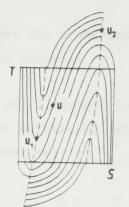


Fig 8.1 : Extension of the function g

The extension of the level curves is not unique, and to facilitate the analysis, an extension will be chosen such that if A and B are two arbitrary states,

$$|T_2^A - T_2^B| \le 2|T_1^A - T_1^B|$$
 (8.1)

This is always possible if the following condition is satisfied:

$$g[S_1(0), 0] \leq g[S_2(1), 1]$$
 (8.2)

If it is not possible to define a extension satisfying Eq.(8.1), the significance played by the to transition curves in the following must be interchanged. In the case of n transition curves, one must assume the following condtion can be satisfied:

$$|T_{n}^{A} - T_{n}^{B}| \le 2|T_{n-1}^{A} - T_{n-1}^{B}| \le \dots \le 2(n-1)|T_{1}^{A} - T_{1}^{B}|$$
 (8.3)

Now define the variable 2 in the following manner:

$$Z(S,T) = \begin{cases} T_2 - T & U \in C \\ T - T_2 & U \in B \\ - (T - T_1) - (T_2 - T_1) & U \in A \end{cases}$$
 (8.4)

The Jacobian of the transformation  $\Psi$  equals  $\boldsymbol{Z}_{\boldsymbol{S}}$  , and from the relation

termed the associated nointh for U and denoted U, and U, respectively.



Ein 8.3 : Extension of the function g

The extension of the level curves is not unique, and to facilitate the analysis, an extension will be chowen such that if A and B are two arbitrary states.

this is always possible if the following condition is waterfield

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if it is not possible to define a extension satisfying [q\_(8.1), the significance played by the to transition curves in the following must be interchanged. In the case of n transition curves, one must assume the following condition can be satisfied:

the variable I in the fallowing manner:

he lacobian of the transformation Y equals [ ... and from the relation ...

PS.

$$\frac{\partial T_{i}}{\partial S} = \frac{g_{S}}{[g_{S} \frac{dS_{i}}{dT} + g_{T}]} \qquad i=1,2 \quad (8.5)$$

it follows that the transformation is regular everywhere, except along the transition curves.

Let  $T^{\dagger}$  and  $T^{-}$  denote T-waves where S, from left to right, is increasing or decreasing respectively. Then define the <u>wave strength</u> in the following way:

$$|S| = |Z_R - Z_|$$

$$= \begin{bmatrix} 8 | T_R - T_L | & T \in A \text{ and } T = T \end{bmatrix}$$

$$= \begin{bmatrix} 8 | T_R - T_L | & T \in A \text{ and } T = T \end{bmatrix}$$

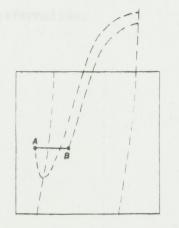
$$= \begin{bmatrix} 2 | T_R - T_L | & T \in B \text{ and } T = T \end{bmatrix}$$

$$= \begin{bmatrix} 4 | T_R - T_L | & T \in B \text{ and } T = T \end{bmatrix}$$

$$= \begin{bmatrix} 4 | T_R - T_L | & T \in B \text{ and } T = T \end{bmatrix}$$

$$= \begin{bmatrix} 4 | T_R - T_L | & T \in C \text{ and } T = T \end{bmatrix}$$

Note that for a S-wave in A B, the definition of wave strenght may create two different situations, demonstrated in Fig 8.2:



T

$$|S| = 2(T^{A} - T_{1}) + (T_{2}^{A} - T_{2}^{B})$$

Fig 8.2A : Wave strength for waves crossing  $S_1$ .

$$\frac{6T_{1}}{65} = \frac{9_{2}}{1 \cdot 9_{2}} + \frac{9_{2}}{9_{1}} + \frac{1}{15_{1} \cdot T_{1}}$$
(6.5)

it follows that the transformation is regular averywhere, except slong the transition curves.

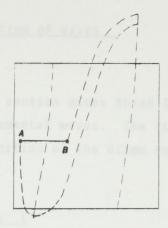
Let T and T denote I-waves where S, from left to right, it increasing or decreasing respectively. Then define the wave strength in the following way:

Note that for a 5-wave in A B, the definition of wave strenght may





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$$|S| = 2(T - T_1^A) - (T_2^B - T_2^A)$$

Fig 8.28 : Wave strength for waves crossing S.

The significance of the extension condition Eq.(8.1) is demonstrated in the latter possibility, as Eq(8.1) ensures that the expression given for |S| is positive.

Finally, if J denotes a general J-curve as defined in Chapter 4, the Glimm functional is defined by

$$F(J) = \Sigma (|S| + |T|)$$
 (8.7)

To prove that F is minimized by  $J = R[U^L, U^R]$ , the concepts of <u>addition</u> and <u>interchange</u> of waves will be used, as introduced by Temple. Additionally, a concept of <u>reduction</u> of waves will be defined. The purpose is to use these operations successively to transform an arbitrary J-curve into the Riemann solution, and and the same time ensure that the Glimm functional decreases through each step of transformation.



Eig 1.28 : Nave strength for waves crossing 5 .

The significance of the extension condition Eq.(2.1) is demonstrated in the latter possibility, as Eq.(2.1) ensures that the explanation given for [5] is positive.

Finally, if J denotes a general 1-curve as defined in Chapter 4, the Glimm functional is defined by

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To prove that F is manimized by J = ALU<sup>2</sup>, U<sup>2</sup> ; the concepts of addition and interchands of waves will be used, as infroduced by femole. Additionally, a concept of reduction of waves will be defined. The purpose is to use these operations successively to transform an arbitrary Leurva into the Riemann solution, and and the same time ensure that the Glimm functional decreases through each step of transformation.

## ADDITION OF WAVES

This section gives three lemmas stating the behaviour of F when adding fundamental waves. The first is a straight-forward consequence of the definition of the Glimm functional, Eq.(8.7):

# LEMMA 1

F(ST) = F(S) + F(T)

The next lemma concerns addition of S-waves. If  $J = S_1 S_2$  takes  $U^L$  to  $U^R$ , define the sum  $S = S_1 + S_2$  as the unique wave  $S = R[U^L, U^R]$ .

# LEMMA 2

 $F(S_1 + S_2) \leq F(S_1) + F(S_2)$ . If  $S_1$  and  $S_2$  have only one state in common, then  $F(S) = F(S_1) + F(S_2)$ .

## Proof

A complete proof involves a study of all possible combinations of  $U^{L}$  and  $U^{R}$ , and only a few will be shown here to verify the lemma. Let A,B and C be three states such that  $T = T^{A} = T^{B} = T^{C}$ . Let  $S_{1}$  be a S-wave from A to B,  $S_{2}$  a S-wave from B to C, and let S =  $S_{1} + S_{2}$ . The next page shows some typical cases. Note that the extension condition Eq.(8.1) is used several times.

ADDITION OF MAVES

Thus section gives three lemmas stating the mehaviour of F when adding fundamental waves. The first is a straight forward consequence of the definition of the Glimm functional, Eq. (8.7):

LEMMA\_1

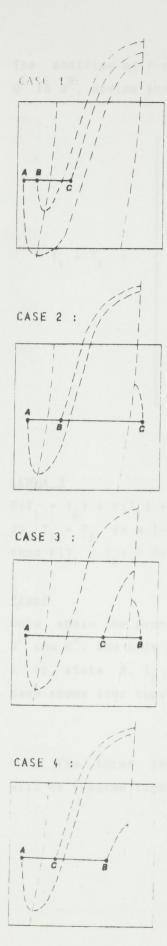
[T]] + [2]] + [T]]

The next lemma concerns addition of 2-waves: if J = 5, 5 takes  $U^{2}$  to  $U^{2}$ , define the sum  $S = S_{1} + S_{2}$  as the unique wave  $S = R[U^{2}, U^{2}]$ .

LEMMA 2 F12 - 2,1 + F15, 1 + F(5,1. 16 5, and 5, have only one state in common, then F151 - F15,1 + F15,1.

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A complete proof involves a study of all paretble comministions of  $u^{2}$ and  $u^{2}$ , and only a few mult be shown here to verify the lemma. Let A, B and C be three states such that  $I = T^{2} = T^{2}$ . Let 5, be a 5ways from A to 8, 3; a 5-ways from 8 to 6, and let 5 = 5,  $r_{2}$ . The next page shows some typical cases. Mole that the extension condition Eq.(3.1) is used several times.



$$F(S_1) = 2(T_1^B - T_1^A) - (T_2^B - T_2^A)$$
  

$$F(S_2) = 2(T_2^B - T_2^C) - (T - T_1^B)$$
  

$$F(S) = 2(T - T_1^A) - (T_2^C - T_2^A)$$

 $F(S_1) + F(S_2) - F(S) = 0$ 

$$F(S_1) = 2(T - T_1^A) - (T_2^B - T_2^A)$$

$$F(S_2) = T_2^B + T_2^C - 2T$$

$$F(S) = T_2^A + T_2^C - 2T_1^A$$

 $F(S_{1}) + F(S_{2}) - F(S) = 0$ 

F

F

$$F(S_{1}) = T_{2}^{A} + T_{2}^{B} - 2T_{1}^{C}$$

$$F(S_{2}) = (T_{2}^{B} - T_{2}^{C}) + 2(T - T_{1}^{B})$$

$$F(S) = T_{2}^{A} + T_{2}^{C} - T$$

$$(S_{1}) + F(S_{2}) - F(S) = 2(T_{2}^{B} - T_{2}^{C}) + 4(T - T_{2}^{B}) \ge 0$$

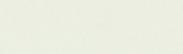
$$F(S_{1}) = (T_{2}^{A} - T_{2}^{B}) + 2(T - T_{1}^{A})$$

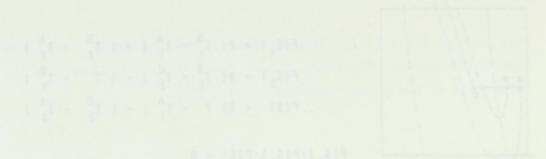
$$F(S_{2}) = T_{2}^{C} - T_{2}^{B}$$

$$F(S) = 2(T - T_{1}^{A}) - (T_{2}^{C} - T_{2}^{A})$$

$$(S_{1}) + F(S_{2}) - F(S) = 2(T_{2}^{C} - T_{2}^{B}) \ge 0$$

28

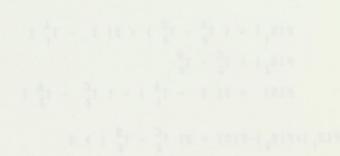














The addition of T-waves is a bit more complicated. If  $J = T_1 T_2$  takes  $U^L$  to  $U^R$ , define the sum as following:

 $R[U^{L}, U^{R}]$  when  $T_{1}$  and  $T_{2}$  both are contained in the same domain A, B or C.

The unique TS-wave taking U<sup>L</sup> to U<sup>R</sup> when the two states are separated by S<sub>1</sub> and T<sup>L</sup>  $\ge$  T<sup>R</sup>

The unique TS-wave taking U<sup>L</sup> to U<sup>R</sup> when the two states are separated by S<sub>2</sub> and T<sup>L</sup>  $\leq$  T<sup>R</sup>

The unique ST-wave taking U to U when the two states are separated by S<sub>1</sub> and T  $\leq$  T

The unique ST-wave taking U<sup>L</sup> to U<sup>R</sup> when the two states are separated by  $S_2$  and T<sup>L</sup>  $\geqslant$  T<sup>R</sup>

## LEMMA 3

 $T_1 + T_2 =$ 

# Proof

Once again the proof involves a study of all possible combinations of  $U^L$  and  $U^R$ , and only a few will be shown. Let  $T_1$  be a wave from state A to state B,  $T_2$  a wave from B to C, and let  $J = T_1 + T_2$ . The next page shows four typical cases.

The three lemmas stating the behaviour of F when adding waves will be treated together and termed <u>the addition lemma</u>.

The addition of T-waves is a hit more complicated. If  $J = T_1 J_2$  takes  $U^L$  to  $U^R$ , define the sum as following:

HIU . U J when J, and J, both are contained in the same domain A, 8 or C. The unique 15-wave taxing U to U when the two states are separated by 5, and T i T two states are reparated by 5, and T i T two states are reparated by 5, and T i T two states are reparated by 5, and T i T two states are reparated by 5, and T i T two states are reparated by 5, and T i T two states are reparated by 5, and T i T two states are reparated by 5, and T i T two states are reparated by 5, and T i T

### LEMMA 3

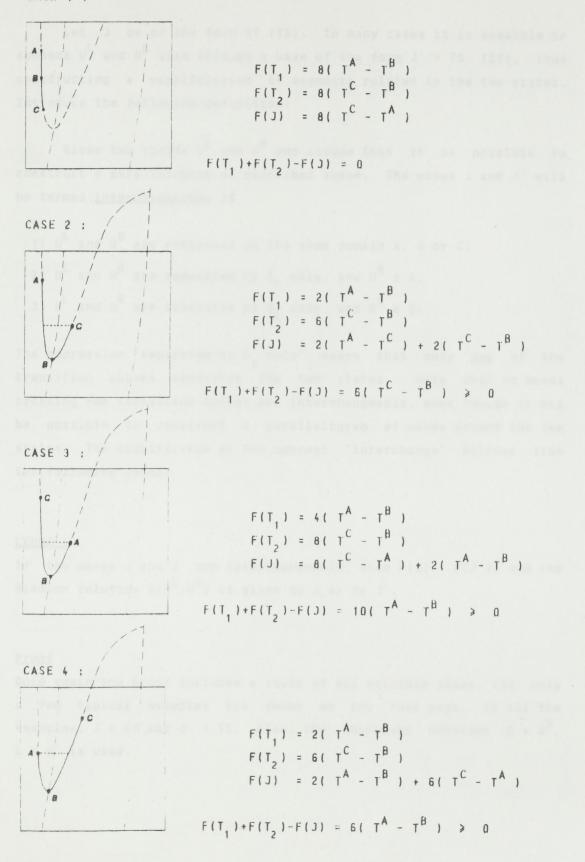
If T a s T use a T-wave, and T and T have one point in common only, then Fit a T, t - Fit t - Fit T.

### 10019

Unce spain the proof involves a study of all possible combinations of  $U^{\mu}$  and  $U^{\mu}$ , and only a few will be shown. Let  $\Gamma_{\mu}$  be a wave from state A to state H.  $\Gamma_{2}$  a wave from 8 to C, and bet J =  $\Gamma_{\mu}$  +  $\Gamma_{2}$ . The next pape shows four tvotcal cases.

The three lemmas stating the behaviour of F when adding waves will be trasted together and termed the addition lemma.

CASE 1 :



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1 10 10 10



## INTERCHANGE OF WAVES

Let J be of the form ST (TS). In many cases it is possible to connect  $U^L$  and  $U^R$  also through a wave of the form J' = TS (ST), thus constructing a parallelogram of segments related to the two states. Introduce the following definition:

Given two states  $U^L$  and  $U^R$  and assume that it is possible to construct a parallelogram as described above. The waves J and J' will be termed <u>interchangeable</u> if

- 1)  $U^{L}$  and  $U^{R}$  are contained in the same domain A, B or C.
- 2)  $U^{L}$  and  $U^{R}$  are separated by S<sub>1</sub> only, and  $U^{R} \in A$ .
- 3)  $U^{L}$  and  $U^{R}$  are separated by S<sub>2</sub> only, and  $U^{R} \in C$ .

The expression "separated by S<sub>i</sub> only" means that only <u>one</u> of the transition curves separates the two states. Note that no waves crossing two transition curves are interchangeable, even though it may be possible to construct a parallelogram of waves around the two states. The significance of the concept "interchange" follows from the following lemma:

## LEMMA 4

If the waves J and J' are interchangeable, then F(J) = F(J'), and the Riemann solution  $R[U^L, U^R]$  is given by J or by J'.

## Proof

Once again the proof includes a study of all possible cases, and only a few typical examples are shown on the next page. In all the examples, J = ST and J' = TS. Also, the shorthand notation  $R = U^R$ , L = U<sup>L</sup> is used.

31

### INFERCHANGE OF MAYES

Let 3 be of the form ST [15]. In many cases it is possible to connect H<sup>1</sup> and H<sup>2</sup> also through a wave of the form J = 75 (17). Thus constructing a parallelogram of segments related to the two states. Introduce the following definition:

biven two states U" and U" and assume that it is possible to construct a parallelogram as described above. - The waves J and J' will be termed interpretable if

> 11 U<sup>4</sup> and U<sup>8</sup> are contained in the same domain  $\lambda$ , U or C. 21 U<sup>4</sup> and U<sup>8</sup> are separated by S<sub>1</sub> only, and U<sup>8</sup> e A. 31 U<sup>4</sup> and U<sup>9</sup> are separated by S<sub>2</sub> only, and U<sup>8</sup> e C.

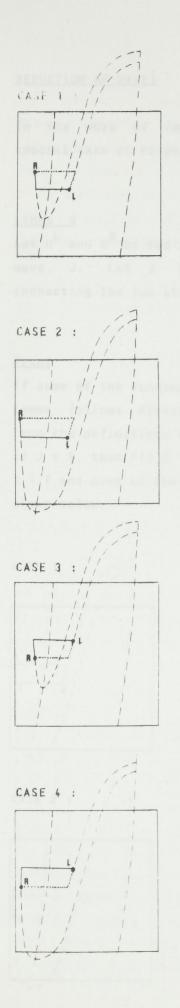
The expression 'separated by 5, only' means that only <u>one</u> of the transition curves separates the two states. Note that no waves crossing two transition curves are interchangeable, even though it may be possible to construct a parallelogram of waves around the two states. The significance of the concept 'interchange' follows from the following lemma:

### A KIMAL

if the waves ) and J' are interchangeaule, band Fill = F(1'), and the Riemann colution R[U<sup>1</sup>, 0<sup>2</sup>] is given by J or by J'.

### 10079

Once again the proof includes a study of all negatiple cases, and only a few typical examples are shown on the next page. In all the exemples, J = 51 and J = 15, Also, the shorthand notation  $B = \mu^{B}$ ,  $L = \mu^{L}$  is used.



$$F(J) = 8(T^{R} - T^{L}) + (T^{R}_{2} - T^{L}_{2}) + 2(T^{L} - T^{R}_{1})$$

$$F(J') = 6(T^{R} - T^{L}) + (T^{R}_{2} - T^{L}_{2}) + 2(T^{R} - T^{R}_{1})$$

$$F(J) = F(J')$$

$$R[U^{L}, U^{R}] = J'$$

$$F(J) = 8(T^{R} - T^{L}) - (T^{R}_{2} - T^{L}_{2}) + 2(T^{L} - T^{R}_{1})$$

$$F(J') = 6(T^{R} - T^{L}) - (T^{R}_{2} - T^{L}_{2}) + 2(T^{R} - T^{R}_{1})$$

$$F(J) = F(J')$$

$$R[U^{L}, U^{R}] = J$$

$$\begin{split} \mathsf{F}(\mathsf{J}) &= 2\,(\mathsf{T}^{\mathsf{L}} - \mathsf{T}^{\mathsf{R}}) + (\mathsf{T}^{\mathsf{R}}_{2} - \mathsf{T}^{\mathsf{L}}_{2}) + 2\,(\mathsf{T}^{\mathsf{L}} - \mathsf{T}^{\mathsf{R}}_{1}) \\ \mathsf{F}(\mathsf{J}') &= 4\,(\mathsf{T}^{\mathsf{L}} - \mathsf{T}^{\mathsf{R}}) + (\mathsf{T}^{\mathsf{R}}_{2} - \mathsf{T}^{\mathsf{L}}_{2}) + 2\,(\mathsf{T}^{\mathsf{R}} - \mathsf{T}^{\mathsf{R}}_{1}) \\ \mathsf{F}(\mathsf{J}) &= \mathsf{F}(\mathsf{J}') \\ \mathsf{R}[\mathsf{U}^{\mathsf{L}}, \mathsf{U}^{\mathsf{R}}] = \mathsf{J}' \end{split}$$

$$\begin{split} F(J) &= 2(T^{L} - T^{R}) - (T^{R}_{2} - T^{L}_{2}) + 2(T^{L} - T^{R}_{1}) \\ F(J') &= 4(T^{L} - T^{R}) - (T^{R}_{2} - T^{L}_{2}) + 2(T^{R} - T^{R}_{1}) \\ F(J) &= F(J') \\ R[U^{L}, U^{R}] &= J \end{split}$$

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CASE 2 1





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# REDUCTION OF WAVES

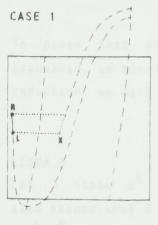
In the work of Temple [27], the following lemma is included as a special case of Proposition 5.1:

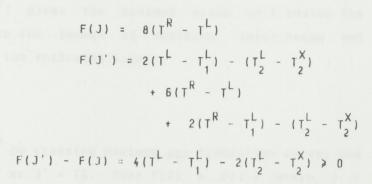
## LEMMA 5

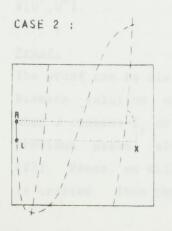
Let U<sup>L</sup> and U<sup>R</sup> be two states that may be joined by a single T- or Swave, J. Let J' be another J-curve of the form  $S_1 TS_2$  or  $T_1 ST_2$ connecting the two states. Then F(J)  $\leq$  F(J').

# Proof

If some of the fundamental waves in J' are interchangeable, then the lemma follows directly from the lemmas of addition and interchange. From the definitions of wave strenghts, it also follows readily that if J = S, then  $F(J') - F(J) \ge 4|T^R - T^L|$ . Some typical examples when J = T and none of the fundamental waves in J' are interchangeable are shown below:







FIJ

$$F(J) = \Theta(T^{R} - T^{L})$$

$$F(J') = (T_{2}^{R} + T_{2}^{X} - 2T_{1}^{L})$$

$$+ 4(T^{R} - T^{L})$$

$$(T_{2}^{R} + T_{2}^{X} - 2T_{1}^{L})$$

$$') - F(J) = 2(T_{2}^{L} - T^{L}) + 2(T_{2}^{L} - T^{X})$$

$$+ 4(T^{L} - T_{1}^{L}) \ge 0$$

### REDUCTION OF MAVES

in the work of Temple [27], the following lemma is included as a a special date of Proposition 6.1:

## ABH31

Let U and U be two states that may be joined by a single I- or Swave. J. Let J be another J-curve of the form 5,15; or 1,57; connecting the two states. Then F(J) ( F(J)).

### 10019

If some of the fundamental waves in 1 are interchangeable. then the limma follows directly from the lemma of addition and interchange. From the definitions of wave strenghts, it size follows readily that if J = 5, then F12 - F13 > 417<sup>2</sup> -  $1^{1}$ ; from typical examples when J = 1 and hone of the fundamental waves in J are interchangeable are shown below:

CASE 2 1 TANK

It is not possible to transform an arbitrary J-curve J into the Riemann solution through the operations addition, interchange and reduction only. However, by a successive application of the three lemmas, J may be transformed into a new curve J',  $F(J') \leq F(J)$ , where J' is contained on or inside a certain "trapezoid" around  $U^L$  and  $U^R$ . J' can also be restricted to follow certain "main routes" inside the trapezoid, confer Fig 8.3:

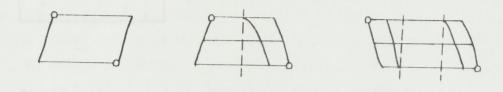






Fig 8.3 : "Main routes" after a transformation of J

To prove that  $R[U^L, U^R]$  gives the minimum value of F inside the trapezoid, in addition to the lemmas of addition, interchange and reduction, we will need the following lemma:

### LEMMA 6

Let J' take  $U^L$  to  $U^R$  by crossing maximum <u>one</u> transition curve, and also assume that J' = ST or J' = TS. Then F(J)  $\leq$  F(J') where J = R[ $U^L$ ,  $U^R$ ].

#### Proof:

The proof can be divided into three parts, according to whether the Riemann solution crosses  $S_1$ ,  $S_2$  or no transition curve. The proof when J crosses  $S_1$  or  $S_2$  follows the same lines as the proofs on the previous pages, also confer the proof of Proposition 5.0 by Temple [27]. Hence, we will only study the situation when <u>no</u> transiton curve is crossed. Then three possibilities exist: Either J = J', J and J'

It is not possible to transform an arbitrary 1-curve 1 into the Riemann solution through the operations addition, interchange and reduction only. However, by a successive application of the three lemmas, 1 may be transformed into a new curve 1'. F(1') ( F(1), where 1' is contained on or inside a certain 'trapazoid' around 0' and 0' 1' can also be restricted to follow certain 'main routes' inside the trapazoid, confer Fig 3.3:



Lin half "Main routes" after a transformation of J

To prove that REU<sup>6</sup>, U<sup>6</sup>] gives the minimum value of Finalde the trapezoid. In addition to the lemmae of addition, interchange and reduction, we will need the following lemma:

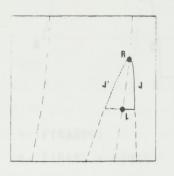
#### B AMMAL

Let J' take U' to U' by crossing maximum one transition curve, and also assume that J = 51 or J' = 12. Thun Fill 6. Fil') where  $J = 310^{-1}$ ,  $U^{R}$ .

#### 110019

The proof can be divided into three parts, according to whather the Atemann colution crosses 5, 5, or no transition curve. The proof when J crosses 5, or 5, follows the same lines as the proofs on the provious pages, also conter the proof of Proposition 5.0 by female [21]. Hence, we will only study the situation when an transiton curve is crossed. Then three presentities exist: Sither 1 = J , J and J

are interchangeable, or  $U^L$  and  $U^R$  are both contained on  $S_1 = S_2$ . In the two first cases, obviously F(J) = F(J'). An example of the last case is shown by the following:



$$R[U^{L}, U^{R}] = J$$

$$F(J') = 6(T^{R} - T^{L}) + (T^{R} - T^{L})$$

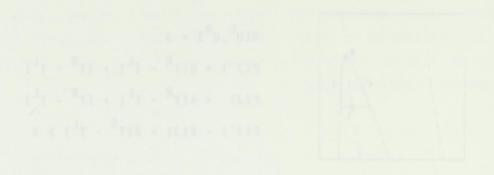
$$F(J) = 4(T^{R} - T^{L}) + (T^{R} - T^{L})$$

$$F(J') - F(J) = 2(T^{R} - T^{L}) \ge 0$$

Finally, the next page show some examples on how the lemmas 1-6 can be combined to successively transforming a J-curve following one of the "main routes" of Fig 8.3 into the Riemann solution. In the examples shown, the following abbreviation indicate the lemma used during an operation:

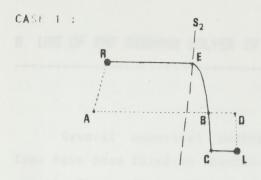
- (A) Lemma of addition
- (I) Lemma of interchange
- (X) Lemma 6

are interchangeable, or U' and U' are both contained on 5, 5, in the two first cases, obviously F(J) = F(J'). An example of the last case is shown by the following:



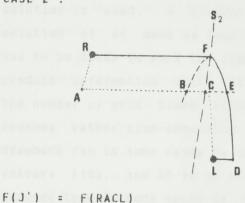
Finally, the next page thew some examples on how the lemmas 1-5 can be combined to successively transforming a 1-turve following one of the "main routes" of Fig 6.1 into the Rismann solution. In the examples shown, the following abbreviation indicate the lemma used during an operation:

- holtibbs to ammal (A)
- [1] Lamma of interchange
  - (X) Lomma 5



F(J')	=	F(RABDL)
	=	F(RAB) + F(BDL)
	*	F(REB) + F(BDL)
	=	F(REB) + F(BCL)
	=	F(RECL)
	::	F(J)

CASE 2 :



1	-	TIRALL	1					
	I	F(RAB)	+	F(BC)	+	F(CL)	(A)	
	3	F(RFB)	+	F(BC)	+	F(CL)	(I)	
	1	F(RF)	+	F(FE)	+	F(EC) + F(CL)	(A)	(X)
	=	F(RF)	+	F(FE)	+	F(EDL)	(A)	(1)
	E	F(RFDL)					(A)	

= F(J)

(A) (X)

(I) (A)

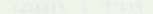












# 9. USE OF THE RIEMANN SOLVER IN NUMERICAL APPLICATIONS

Several numerical methods for solving hyperbolic conservation laws have been based on solution of local Riemann problems, examples are the Random Choice Method [2,3], Godunov-type methods [10], and the front-tracking method of Glimm et al [9]. As noted in Chapter 7, the Riemann problem of the Eqs.(1.1) is "unstable" in the sense that small perturbations in the initial states may change the solution drasticly, and this indicates that use of the Riemann solver in numerical solution of the Eqs.(1.1) is not advantageous. This chapter shows examples of the use of the solver in the Random Choice Method (RCM), also named the Uniform Sampling Method.

A common feature of methods based on exact Riemann solvers is that only a small part of the information contained in the Riemann solution is used. A Riemann problem for Eqs.(1.1) may involve solution of as much as four non-linear equations, and as great care has to be taken to pick the right solution, much time is used to produce information which is later disregarded in the solution. As the number of grid block increases, the RCM applied to Eq.(1.1) becomes rather time-consuming. For strictly hyperbolic systems, this drawback can in some cases be compensated by using <u>approximate</u> Riemann solvers [10], and it is an interesting question whether for instance Godunov-type methods could be constructed with approximate Riemann solvers in the case of non-strictly hyperbolic systems. Godunov-type methods are obviously less sensitive to the instability in the Riemann solution than the RCM, as the first tend to average the Riemann solution.

In practical applications, the function f may be represented by a table only, and as the problem is not structural stable in the same way as for a single hyperbolic equation 15,25, the method chosen for interpolating f could highly influence the solution. In all examples shown in this chapter, the function f is represented analytically, using Eq.(3.4) together with the definition 3. USE OF THE RIEHAWN SOLVER IN NUMERICAL APPLICATION

Several numerical methods for solving hyperbolic convervation inws have been based on solution of local Hismann problems, examples are the Sandom Choice Method [2,3]. Godunov-type methods [184, and the front-tracking pethod of Glimm et al [9]. As noted in Chepter 7, the Riemann problem of the Eqs.(1.1) is 'unstable' in the example that small perturbations in the initial states may change the solution drasticly. and this indicates that use of the Sigmann solver in numerical solution of the San.(1.1) is not advantageous. This chapter shows axamples of the use of the solver in the statement (SCM).

A common fasture of methods based on exect finmenn solvers is that only a small part of the information contained in the Niemann solution is used. A Riemann problem for Eqs.(1.1) may involve restition of as much as four non-linear equations, and as great care bas to be taken to nick the right solution, much time is used to produce information which is later disregarded in the solution. As becomes 'rather time-consuming. Yor exercisy hyperbolic systems, this drawback can in some cases be compensated by using amoraimate Sismann solution the case of non-structed with approximate Sismann solvers in the case of non-structed with approximate Riemann solvers in the case of non-structed with approximate Riemann solvers in the case of non-structed with approximate Riemann solvers in the case of non-structed with approximate Riemann solvers in the case of non-structed with approximate Riemann solvers in the case of non-structed with approximate Riemann solvers in the case of non-structed with approximate Riemann solvers in the case of non-structed with approximate Riemann solution than the ACM, as the first tond to sverage the Riemann solution.

In practical applications, the function f may be represented by a table only, and at the problem is not structural stable in the same way at for a single hyperbolic equation [5,25], the method chosen for interpolating f could highly influence the solution. In all exemples shown in this chapter, the function f is represented ensistically.

$$\kappa(T) = \frac{1}{2 - T}$$
 (9.1)

All non-linear equations are solved by the Newton-Raphson method. Also, in all the examples shown,  $\alpha = 0.5$ ,  $\beta = 3.5$  and a Courant number of 0.8 is used. If nothing else is specified, the number of grid blocks is 200. The solutions are shown for t = 1.

As g is a slowly varying function of T, the initial T-profile is convected with a minor deformation only, and in most of the examples, only the S-profile is shown. With the given function f, both the eigenvalues of the system matrix are positive, and Godunov's method is equivalent to standard upstream differencing; use of the Riemann solver is not necessarry. For comparison, results using this method is shown together with the results from the RCM.

In cases where the solution of the Eqs.(1,1) does not posess a transition state in contineous parts of the solution, the RCM behaves as for strictly hyperbolic systems. It is well-known that the method then resolves discontinuities without dispersion, but has rather low accuracy in smooth parts of the solution. Fig 9.1 shows the solution of a Riemann problem modelling injection of cold water into a hot oil reservoir:  $U^{L} = (1,0)$ ,  $U^{R} = (0,1)$ . Godunov's method needs a very high number of grid blocks to resolve the plateau with constant S sufficiently.

Also, Fig 9.2-4 all show solutions of Riemann problems, these satisfying or close to satisfying the conditions of Eq.(7.1) producing non-uniqueness in phase space. The intital states are given in Table 9.1. Obviously, the upstream differencing method are not capable of resolving the abrubt changes in the solution and also reflects the non-uniqueness of the solution in the case where Eq.(7.1) is exactly satisfied. Also note that the numerical solution in this case is nonmonotone, even though Godunov's method is monotone for strictly hyperbolic equations.

All non-linear equations are solved by the Newton-Raphson mathod. Also, in all the examples shown,  $\alpha = 0.0$ ,  $\beta = 3.3$  and a Courant number of 0.4 is used. (f nothing also is specified, the number of grid blocks is 200. The solutions are shown for t = 1.

As g is a slowly verying function of 7, the initial 7-profile is convected with a minor deformation only, and in most of the examples, only the S-profile is shown. With the given function f, both the eigenvalues of the system matrix are positive, and Godunov's method is equivalent to standard upstream differencing; use of the Riemann solver is not necessivy. For comparison, results using this method is shown together with the results from the SCM.

In faces where the solution of the Eqs.(1.1) deer not posess a transition state in continence parts of the solution. The ACM behaves as for strictly hyperbolic cystems. It is well-known that the method then resolves discontinuities without dispersion, but has rather low action for a flammin problem modelling injection of cold weter into a hot oil reservoir: U - (1.0). U - (0.1). Sodunov's method needs a very high number of grid blacks to resolve the plates u with constant of solutions in sectors.

Also, 119 3.2-1 all show solutions of Rismann problems, these satisfying or close to satisfying the conditions of Eq.(7.1) producing non-uniqueness to phase space. The intital states are given in Table 9.1. Obviously, the upstress differencing method are not carable of needlying the solution in the case where Eq.(7.1) is exactly autisfied. Also note that the numerical solution in this case is nonmonotone, even though Sedunov's method is monotone for strictly hyperbolic coustions.

Fig.	sL	τ <sup>L</sup>	sL	тL	g	g R
9.2	0.78	0,7			0.3369940	
9.3	0.7938178	0.7	0.9319771	0.0	0.3378500	0.3378500
9.4	0.81	0.7	-		0.3386065	

<u>Table 9.1</u> : Initial states for the solution shown in Fig 9.2-4. U is identical in all three cases.

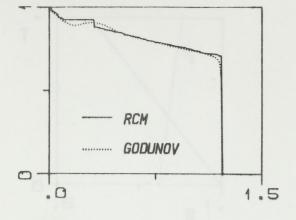
In cases where the solution has a transition state in smooth parts of the solution, large instabilities may occur in the RCM, and the method converges slowly as the block lenght of the grid goes to zero. This is demonstrated in Fig 9.7-10, using the initial condition shown in Fig 9.5-6. Note the difference in the solution produced by merely changing the random-number generator involved. Except from the results of Fig 9.9, the random-number generator described in [24] is used.

Finally, two examples when all the initial states are situated on the transition curve  $S_2$  is shown in Fig 9.11-12. T is chosen to vary linearly with x initially. Both when T increases and decreases with x, the solution seem to "avoid" the transition curve, and the solution does not involve any specific problems.

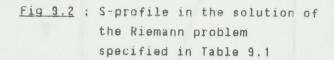
Table 9.1 : Initial states for the colution shows in Fig 9.2-4.

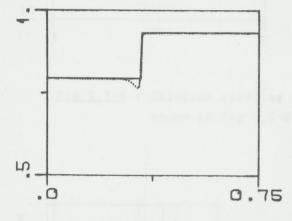
In cases where the volution has a francition state in endoth parts of the solution, large instabilities may occur in the RCH, and the method converges slowly as the block langht of the grid goes to zero. This is demonstrated in Fig 9 1-10, using the initial condition shown in Fig 9.5-5. Note the difference in the solution produced by mersiv changing the random number generator involved. Except from the results of Fig 9.9, the random number generator involved in [24] is used.

Finally, two examples when all the initial states are situated on the transition curve's is shown in Fig 3.11-12. I is chosen to vary linearly with x initially. Both when I increases and decreases with x, the solution seem to "evoid" the transition curve, and the solution does not involve any specific problems.



<u>Fig 9.1</u> : S-profile in the solution of the Riemann Problem U = (1,0), U = (0,1)



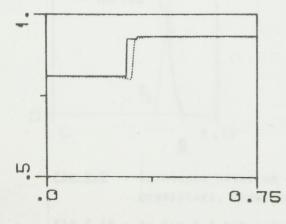


0.75

L

.0

Fig 9.3 : S-profile in the solution of the Riemann problem specified in Table 9.1



Eig 9.4 : S-profile in the solution of the Riemann problem specified in Table 9.1









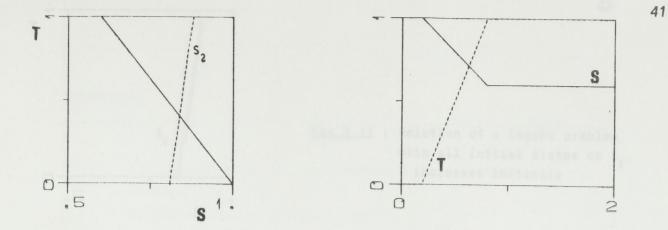


Eig<u>1.1</u>; S-profile in the solution of the Bisminn Problem U = (1.0), H = (0,1)

# Fig. 3.2 : 5-profile in the solution of the Riemann problem specified in Table 3.1

<u>Fig 9.3</u> : 5-profile in the solution of the Riemann problem specified in Table 5.1

18.3.1 : 5-profile in the solution of the Rinmann problem spacified in Table 5.1





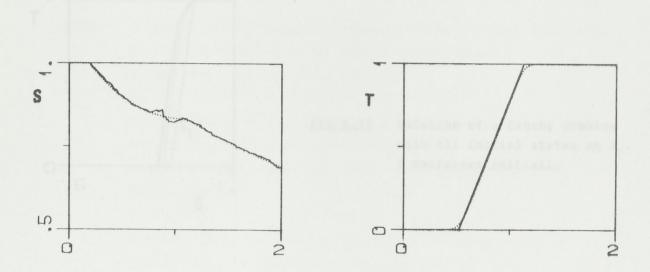


Fig 9.7-8 : Solution profiles using the initial condition shown in Fig 9.5-6

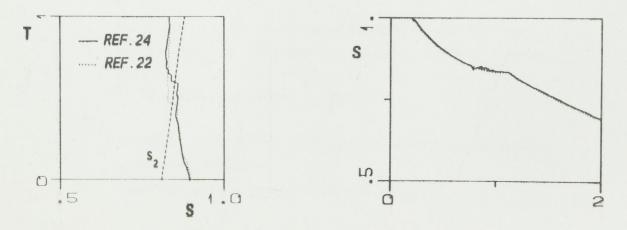


Fig 9.9 : Solutions based on two different random-number generators, shown in the phase space

Fig 9.10 : As Fig 9.7 but using 2000 grid blocks



Eig 1.8-5 ; Initial 5- and T-profile crossing 5,



110 2.1-3 ) Solution profiles using the initial condition



and a Solutions based on two different random-number generators, whown in the phase space

A I At E18 4.7 but using 2000 grid blocks

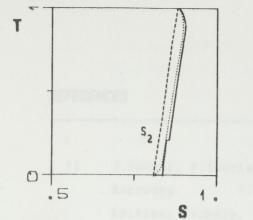
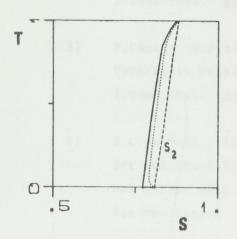


Fig 9.11: Solution of a Cauchy problem with all initial states on S<sub>2</sub>. T increases initially



<u>Fig 9.12</u> : Solution of a Cauchy problem with all initial states on  $S_2$ . T decreases initially <u>lo 4.11</u> : Solution of a Cauchy problem with all initial states on 5 T increases initially

a 5.12 : Salution of a Cauchy problem

with all initial states on S<sub>2</sub>. I downeases initially

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# APPENDIX: DERIVATION OF THE MODEL EQUATIONS

To correct a small flaw and to supplement a derivation first given by Fayers [7], this Appendix shows the derivation of the model equations describing non-isothermal two-phase flow in a porous medium.

In the following, the subscribts w,o and r will be used to denote parameters characterizing water, oil and rock respectively. Let  $\lambda_q$  (q=w,o) be the mobility of fluid q, i.e. the relative permeability divided by the viscosity, and let  $\kappa_q$  (q=w,o,r) denote the thermal capacity per unit mass.  $p_c$  is the capillary pressure, and  $\Lambda$  is the total thermal conductivity,- both are functions of water saturation S and temperature T.  $\phi$  is used for porosity, k for absolute permeability and u for the total volume flux. Also introduce the notation

$$f = \frac{\lambda_w}{\lambda_w + \lambda_o}$$
$$a = -\frac{k}{u} \frac{\lambda_w \lambda_o}{\lambda_w + \lambda_o} \frac{\partial p_c}{\partial S}$$

 $b = -\frac{k}{u} \frac{\lambda_{w}\lambda_{o}}{\lambda_{w} + \lambda_{o}} \frac{\partial p_{c}}{\partial T}$ 

 $F = f - aS_x - bT_x$ 

Both the functions a and b are normally positive [1], and f<sub>T</sub> is normally negative [17]. Assuming incompressibility and neglecting gravity, conservation of mass is expressed through the equation

$$p \frac{\partial S}{\partial t} + u \frac{\partial f}{\partial x} = u \frac{\partial}{\partial x} \left[ a \frac{\partial S}{\partial x} + b \frac{\partial T}{\partial x} \right]$$
(A2)

Gravity is easily included by a redefinition of the function f [21].

46

(A1)

APPENDIX: DERIVATION OF THE MODEL EQUATIONS

To correct a small flaw and to supplement a serivation first siven by Fayers [7], thus Appendix shows the derivation of the model equations describing non-isothermal two-phase flow in a porous medium.

In the following, the subscribts w.o and r will be used to denote parameters characterizing water, oil and rock respectively. Let  $A_{\rm q}$  (q-w,o) bo the mobility of fluid q. i.e. the relative parmeability divided by the viscosity, and let  $a_{\rm q}$  (q-w.o.r) denote the thermal conductively, and let  $a_{\rm q}$  (q-w.o.r) denote the thermal total the viscosity, and let  $a_{\rm q}$  (q-w.o.r) denote the thermal conductively. The capillary pressure, and A is the total thermal conductively, both are functions of water isturation for and total thermal conductively, both are functions of water isturation for and total the mobility and u for the capillary parameter is the shealute for a conductive of the the total the shealute for a construct of the total total

Both the functions a and b are normally positive [1]. and fr is normally negative [17]. Assuming incompressibility and peglecting gravity, conservation of mass is expressed through the equation

Gravity is easily included by a redefinition of the function 6 [31].

Conservation of energy is expressed as

$$\frac{\partial}{\partial t} \{ [\phi \kappa_{W}^{S} + \phi \kappa_{Q}(1-S) + (1-\phi)\kappa_{T}]T \}$$

$$+ u \frac{\partial}{\partial x} \{ [\kappa_{W}^{F} + \kappa_{Q}(1-F)]T \} = \frac{\partial}{\partial x} \{ \Lambda \frac{\partial T}{\partial x} \}$$
(A3)

Introduce the thermodynamic functions

$$A = \kappa_{w} - \kappa_{0}$$

$$B = \kappa_{0} + \frac{1 - \varphi}{\varphi} \kappa_{s}$$

$$C = \kappa_{0}$$

$$(A4)$$

$$\alpha = \frac{C + C'T}{A + A'T}$$

$$A + A'T \neq 0$$

$$\beta = \frac{B + B'T}{A + A'T}$$

In general, the thermal capacities per unit mass are functions of temperature, and the sign "'" is used to denote derivation with respect to T. After a scaling of the equations, and after a substitution of Eq.(A2) into Eq.(A3), the system of conservation laws simplifies to:

$$S_{t} + f_{x} = (aS_{t} + bT_{t})_{x}$$
(A5)
$$T_{t} + \frac{f + \alpha}{S + \beta} T_{x} = \frac{1}{A + A'T} [AT_{x}]_{x} + (aS_{t} + bT_{t})T_{x}$$

The functions  $a,b,\Lambda$  etc are now redefined as dimensionless. In his derivation, Fayers [7] seem to neglect A' B' and C' even when A, B and C are allowed to vary as function of temperature. Following Fayers, the function (f+ $\alpha/S+\beta$ ) is termed the <u>thermal advance function</u>.

As a model for high-rate conditions, the terms representing capillary pressure and thermal conduction will now be neglected and the system reduced to a first order system. If  $\alpha$  and  $\beta$  are assumed constant, the equations may be written in the form given in Eq.(2.1). se passaidka st Abieus jo uorgenissuoj

$$\frac{6}{26} \left( \frac{16}{16} \times \frac{6}{16} \right) = \left( \frac{11}{11} \times \frac{6}{16} \times$$

Introduce the thermodynamic functions

In general, the thermal capacities per unit mass are functions of temperature, and the sign "" is used to danote derivation with respect to T. After a scaling of the equations, and after a substitution of Eq.(A2) into Eq.(A3), the system of conservation laws

The functions a,b,A etc are now rederined as dimensionless. In his derivation, fayers [7] seem to neglect A' B' and C' even when A, B and C are allowed to vary as function of temperature, following favers, the function (feq/S+B) is termed the thermal advance function.

At a model for high-rate conditions, the terms representing capillary pressure and thermal conduction will now he neglected and the system reduced to a first order system. If  $\alpha$  and  $\beta$  are present constant, the equations may be written in the form given in to.[2.1].

This is achieved if A,B and C have the following functional form:

 $AT = A_0 T + A_1$  $A_0$ ,  $A_1$ ,  $B_0$ , ..., consts.  $BT = B_0 T + B_1$  $CT = C_0 T + C_1$ C # 0

This is obviously satisfied if the thermal capacities are independent of temperature, and from the definition of A, B and C, it then follows that  $\alpha < \beta$ . Note that if CT = const., the system reduces to the form studied by Johansen and Winther [13].

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(A6)

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 $AT : A_{0}T : A_{1}$   $AT : B_{0}T : B_{1}$   $AT : C_{0}T : C_{0}T$ 

This is obviously satisfied if the thermal capacities are independent of temperature, and from the definition of A. B and C. it then follows that  $a \in \beta$ . Note that if CI = const., the system reduces to the form studied by Johanson and Winther [13].

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