Department of APPLIED MATHEMATICS

STELLARATOR - TOKAMAK CONFIGURATIONS

by

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Abstract

The stellarator configuration and tokamak configuration with helical fields are studied both from an equilibrium and stability point of view. The model is resricted to a surface current model wiht a sharp boundary between plasma and vacuum. A general derivation of equilibrium and stability based on the Energy Principle is given. Physically the unstable modes are identifyed as external global modes. Detailed numerical results in different parameter regimes are presented and discussed. Critical β- limits for equilibrium and stability are obtained and in particular we show that in ceartain parameter ranges there exist a high-β as well as a low β- region of stability.

Chapter 1

Analytic Derivation

1.1 Introduction

This work is primarily motivated by the promising aspects of the stellarator configuration. Recent results suggest that a toroidal device with helical fields may have some advantages compared to axisymmetric devices, i.e., tokamaks, with regard to controlling disruptions. In this context one could think of two classes of systems: (a) pure stellarators (no ohmic heating current) or (b) a tokamak with superimposed helical windings. The present work is generalization of previous work [1] which was restricted to one type of helical field (one ℓ-number, ℓ being the poloidal multipolarity) and no vertical field. We have now been able to generalize this to include any combination of helical fields and also an arbitrary vertical field (which is essential for having an average magnetic well). This applies both to the equilibrium and the stability analysis.

The study is restricted to the surface current model, where we assume all the current to be flowing in a thin sheath forming the boundary between plasma and vacuum. Previous experience suggest that such a model gives a reasonable description of the equilibrium and the stability properties of the global modes in such systems.

The main part of this work is analytic, and we resort to numerical solutions only at the final step both in the equilibrium and the stability analyses.

The equilibrium is established by the Princeton stellarator expansion in the inverse aspect ratio (ε) . We can solve for the critical β for any net current, including the pure stellarator case with zero net current.

The stability part is based upon the MHD-energy principle. We are able to write this in a concise form suitable for numerical evaluation.

We discuss critical β-limits from equilibrium and stability for systems with different combinations of helical fields.

A brief preliminary account of this work was presented elsewhere[2].

1.2 Equilibrium

We consider the equilibrium and stability of a toroidal stellarator/tokamak hybrid system as described by the sharp boundary surface current model. Although the analytic as well as the numerical work permits the study of hybrid systems, we shall here consider the case of a pure stellarator configuration (no net current). The geometry is illustrated in fig. 1. The cylindrical coordinates (R, θ, z) are related to toroidal coordinates by $R = R_0 + r\cos\theta$, $Z = r\sin\theta$, $\phi = -z/R_0$.

As stated earlier this class of configurations are characterized by having arbitrary helical fields, i.e. combinations of several helisities simultaneously as well as a vertical field. The fields are written as

$$\mathbf{B}_{plasma} = B_i\mathbf{b} \ , \quad \mathbf{B}_{vacuum} = B_0(\mathbf{b} + \hat{\mathbf{b}}) \, , \tag{1.1}$$

$$\mathbf{b} \equiv \frac{R_0}{R}\mathbf{e}_z + \frac{1}{h}\nabla(\psi + \chi) \, , \tag{1.2}$$

where h is the helical wavenumber and ψ and χ represents the helical and vertical fields respectively. The inverse aspect ratio is $a/R_0 \equiv \varepsilon$ where a is the average plasma radius. Our expansion parameter is δ, the measure of the amplitudes of the helical fields, and the following ordering is assumed,

$$\varepsilon \approx \delta^2 \approx \frac{1}{N} \approx \beta \ , \tag{1.3}$$

where $\beta = p/\frac{1}{2}\mathbf{B}_0^2$ (p is plasma pressure) and $N = hR_0$, the number of helical periods.

We introduce new variables by $x = hr$, $s = hz$, and take the plasma surface to be given by $x = x(\theta, s)$. By solving the problem order by order in δ we obtain

$$x(\theta, s) = x_0(\theta) + x_1(\theta, s) + \dots , \tag{1.4}$$

$$B_r/B_i = \frac{\partial \psi_1}{\partial x_0} + x_1 \frac{\partial^2 \psi_1}{\partial x_0^2} + \frac{\partial \chi_2}{\partial x_0} + O(\delta^3) , \tag{1.5}$$

$$B_\theta/B_i = \frac{1}{x_0}\frac{\partial \psi_1}{\partial \theta} + \frac{x_1}{x_0}\frac{\partial^2 \psi_1}{\partial \theta \partial x_0} + \frac{1}{x_0}\frac{\partial \chi_2}{\partial \theta} - \frac{x_1}{x_0^2}\frac{\partial \psi_1}{\partial \theta} + O(\delta^3) , \tag{1.6}$$

$$B_z/B_i = 1 + \frac{\partial \psi_1}{\partial s} + \frac{\partial \chi_2}{\partial s} + x_1 \frac{\partial^2 \psi_1}{\partial s \partial x_0} - \frac{x_0}{hR_0}\cos\theta + O(\delta^3) , \tag{1.7}$$

$$\hat{\mathbf{b}} = \frac{1}{h}\nabla\{\psi_2(x,\theta) + \psi_3(x,\theta,s)\} + O(\delta^4) . \tag{1.8}$$

For convenience we write

$$\psi_1 = \frac{1}{2}\hat{\psi}_1 e^{is} + c.c. , \quad x_1 = \frac{i}{2}\hat{x}_1 e^{is} + c.c. , \tag{1.9}$$

$$\psi_3 = \frac{1}{2i}\hat{\psi}_3 e^{is} + c.c. , \quad \chi_2 = \frac{1}{2}\hat{\chi}_2 e^{i\theta} + c.c. , \tag{1.10}$$

$$\frac{d}{dn} \equiv \frac{\partial}{\partial x_0} - \frac{\dot{x}_0}{x_0^2}\frac{\partial}{\partial \theta} , \quad \frac{d}{d\theta} \equiv \frac{\partial}{\partial \theta} + \dot{x}_0 \frac{\partial}{\partial x_0} , \quad \dot{x}_0 = \frac{dx_0}{d\theta} . \tag{1.11}$$

We define the following important quantity

$$F \equiv i\left\{\nabla\hat{\psi}_1^* \times \nabla\hat{\psi}_1 \cdot \mathbf{e}_z + 2(\hat{\chi}_2 - \hat{\chi}_2^*)\right\} = F(\theta, x_0) . \tag{1.12}$$

where * means complex conjugate (c.c.). From now on $\nabla \equiv \nabla_0 = \mathbf{e}_x \frac{\partial}{\partial x_0} + \mathbf{e}_\theta \frac{1}{x_0}\frac{\partial}{\partial \theta}$. The solution to the problem can be written as

$$\frac{dF}{d\theta} = 0, \qquad \text{(determines } x_0(\theta)) , \tag{1.13}$$

$$\hat{x}_1 = -\frac{d\hat{\psi}_1}{dn}, \qquad \frac{d\psi_2}{dn} = 0, \tag{1.14}$$

$$\frac{1}{2i}\frac{d}{dn}(\hat{\psi}_3 e^{is}) + c.c. = \frac{1}{x_0}\frac{d}{d\theta}(\frac{x_1}{x_0}\frac{\partial\psi_2}{\partial\theta}), \tag{1.15}$$

$$b^2 + 2\hat{\iota}_h b - (\beta/\varepsilon)(w+\lambda) = 0, \tag{1.16}$$

where

$$b = \frac{1}{\varepsilon x_0 Q_0}\frac{d\psi_2}{d\theta}, \quad Q_0 = \sqrt{1+\frac{\dot{x}_0^2}{x_0^2}} \tag{1.17}$$

$$\boldsymbol{\hat{\iota}}_h = \frac{1}{4\varepsilon}\nabla F, \tag{1.18}$$

$$\hat{\iota}_h = |\boldsymbol{\hat{\iota}}_h| \,\mathrm{sgn}(F_{x_0}) = \frac{1}{4\varepsilon}Q_0 F_{x_0} \qquad \text{(helical transform)}, \tag{1.19}$$

$$w = \frac{1}{2\varepsilon}|\nabla\hat{\psi}_1|^2 + \frac{2x_0}{x_{av}}\cos\theta \qquad \text{(magnetic well)}, \tag{1.20}$$

$x_{av} = ha$ and λ is a constant related to net current.

From eq.(1.16) we notice that b is determined by a quadratic equation and this equation has two branches of solutions. However, the stellarator case with no net current is obtained only from one of these branches, which we shall discuss here. We also notice that unless β/ε is below a certain value there is no solution, this condition for solution determines the critical equilibrium β-limit. The problem is solved numerically by first integrating eq.(1.13) to find the surface, and then determine λ and the critical β for a given net current. Some typical crossections for different configurations are given in figs. 2a and 2b. More details about the equilibrium derivation is provided in appendix A. In the following figures case C and case D refer to table 1 on page 13.

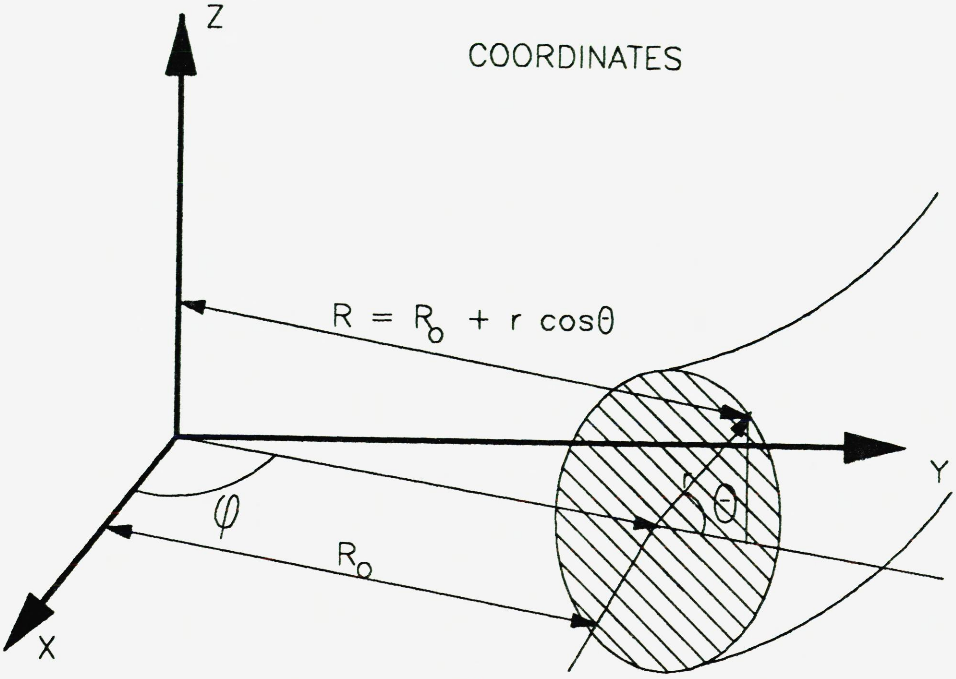
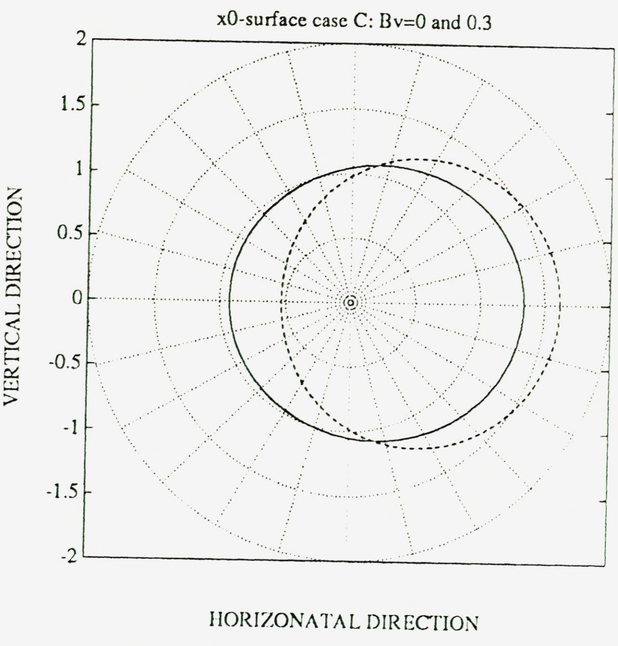
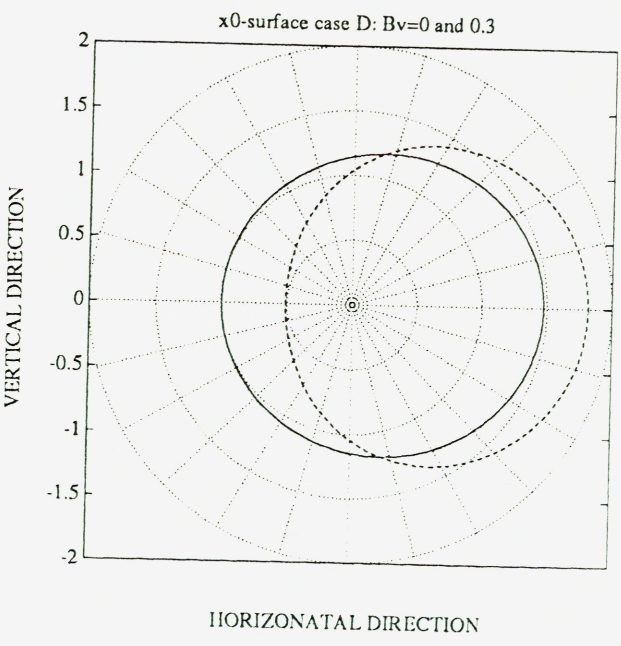

COORDINATES
Z
$R = R_o + r\cos\theta$
Y
φ
R_o
X

Fig. 1


x0-surface case C: Bv=0 and 0.3
VERTICAL DIRECTION
HORIZONATAL DIRECTION

(a)


x0-surface case D: Bv=0 and 0.3
VERTICAL DIRECTION
HORIZONATAL DIRECTION

(b)

Fig. 2

1.3 Stability Analysis

We investigate the stability of this configuration by means of the Energy Principle [3]. From previous experience the surface current model provides a reasonable description of long wavelength (low toroidal mode number, low poloidal mode number) instabilities. A simplifying feature of the analysis follows from the fact that in minimizing δW the most unstable modes comes out to be incompressible to leading order i.e. $\nabla\cdot\boldsymbol{\xi}_0 = 0$ with $\boldsymbol{\xi}_0$ the leading order plasma displacement. We notice that in general δW can be reduced to a form which only depend on a single scaler quantity, $\xi_\perp = \mathbf{n}\cdot\boldsymbol{\xi}$ evaluated on the plasma surface.

For the surface current model, the potential energy δW is conveniently written as a plasma-, surface- and vacuum-contribution

$$\delta W = \delta W_p + \delta W_s + \delta W_v \,, \tag{1.21}$$

where

$$\delta W_p = \frac{1}{2}\int_p |\mathbf{B}_1|^2 d\tau,\ \delta W_s = \frac{1}{2}\int_s |\xi|^2 \mathbf{n}\cdot\nabla \left[\!\left[(p + B^2/2)\right]\!\right] ds,\ \delta W_v = \frac{1}{2}\int_v |\hat{\mathbf{B}}_1|^2 d\tau \tag{1.22}$$

The simplified expression for δW_p, reflects the fact that the most unstable modes are almost incompressible, and $\mathbf{B}_1$ and $\hat{\mathbf{B}}_1$ are the perturbations in the magnetic field in the plasma and vacuum regions respectively, $\xi = \xi(\theta, z) \equiv \mathbf{n}\cdot\boldsymbol{\xi}|_{r_p}$ is the normal component of plasma displacement evaluated on the plasma surface $(r = r_p)$. The notation $[\![A]\!]$ denotes the jump in A across the sharp boundary from vacuum to plasma.

1.3.1 The Perturbation

The first step in the stability analysis is the specification of the perturbation ξ. The most general form of ξ can be quite complicated in an arbitrary three- dimensional geometry. However, if we restrict the attention to long wavelength modes and make use of the stellarator expansion, then the most general form of ξ can be written as

$$\xi(\theta, s) = \left\{\xi_0(\theta) + \delta\{\xi_+ e^{is} + \xi_- e^{-is}\} + \delta^2 \xi_2(\theta, s)\right\} e^{iks} \tag{1.23}$$

here, $k \equiv n/hR_0 = n/N$, n is the toroidal wavenumber of the perturbation and "the long wavelength assumption" implies $k \approx \varepsilon n \approx \delta^2$. The quantities ξ_0 , ξ_+ , ξ_- and ξ_2 are each of order unity, and it is these functions which must be varied to minimize δW. (Note that a slightly different definition of ξ is used in eq.(C.3). Eventually, by analytically minimizing δW , ξ_2 is eliminated and ξ_+ and ξ_- are expressed in terms of ξ_0. Therefore the final minimization requires the variation of only one single scaler quantity of one variable $\xi_0(\theta)$.

Physically ξ_0 represents the basic "flute like" contribution to the perturbation and ξ_+ and ξ_- represents helical sideband distortions induced by the helical field.

1.3.2 Surface Energy

The second step in the analysis is the evaluation of δW_s, which is the only term that can give rise to an instability. A straight forward calculation shows that the surface element $\mathbf{n}dS$ can be expressed as

$$\mathbf{n}dS = \mathbf{n}_s(r_pR/hR_0)d\theta ds, \quad 0 \le \theta < 2\pi, \; 0 \le s < 2\pi N , \tag{1.24}$$

where

$$\mathbf{n} = \mathbf{n}_s/|\mathbf{n}_s|, \quad \mathbf{n}_s = \mathbf{e}_r - \frac{1}{r_p}\frac{\partial r_p}{\partial\theta}\mathbf{e}_\theta - \frac{R_0}{R}\frac{\partial r_p}{\partial z}\mathbf{e}_z\,. \tag{1.25}$$

This formula is valid fo an arbitrary, (unexpanded) three-dimensional surface, $r = r_p(\theta, z)$. We note that $\nabla \times \mathbf{B} = 0$ in both the plasma and vacuum region and that $\mathbf{n}\cdot\mathbf{B}|_{r_p} = 0$. Consequently we may write $\mathbf{n}\cdot\nabla(p + \frac{B^2}{2}) = -\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)\mathbf{n}$. After a lengthy calculation, this term can be evaluated and substituted into δW_s. The result is

$$\begin{aligned}\delta W_s \;=\; & \frac{1}{2}\int d\theta ds |\mathbf{n}\cdot\boldsymbol{\xi}|^2 \frac{r_pR}{hR_0}\left\{\frac{[\![B_\theta^2]\!]}{r_p}\{1 - r_p\frac{\partial}{\partial\theta}(\frac{1}{r_p^2}\frac{\partial r_p}{\partial\theta})\}\right.\\ & -2[\![B_\theta B_z]\!]\frac{\partial}{\partial\theta}\{\frac{R_0}{r_pR}\frac{\partial r_p}{\partial z}\}\\ & \left. -[\![B_z^2]\!][\frac{\partial}{\partial z}(\frac{R_0^2}{R^2}\frac{\partial r_p}{\partial z}) - \frac{1}{r_pR}\frac{\partial}{\partial\theta}(r_p\sin\theta)]\right\}. \end{aligned}\tag{1.26}$$

Eq.(1.26) is valid for an arbitrary surface given by $r = r_p(\theta, z)$.

We now substitute the expanded form of the equilibrium and the perturbation into the expression for δW_s. After a considerable amount of algebra the first non-vanishing terms are of order δ^4 and can be written as

$$\frac{\delta W_s}{2\pi R_0} = A\int_0^1 |\xi_0|^2 \left\{ b\nabla \cdot \boldsymbol{\imath}_h - \frac{g(x_0)}{x_0 Q_0^3}[b^2 + 2\hat{\imath}_h b] - \frac{\beta}{2\varepsilon}\frac{dw}{d\hat{n}} \right\} dv\,, \qquad (1.27)$$

where

$$A = \frac{1}{2}\varepsilon^2 \mathbf{B}_0^2 C\,, \qquad \boldsymbol{\imath}_h = \frac{1}{4\varepsilon}\nabla F\,, \qquad \frac{d}{d\hat{n}} = \frac{1}{Q_0}\frac{d}{dn}\,, \qquad (1.28)$$

and C is the circumference of the plasma boundary.

$$\hat{\imath}_h = |\hat{\boldsymbol{\imath}}_h| \,\mathrm{sgn} F_{x_0} = \frac{1}{4\varepsilon} Q_0 F_{x_0}\,, \qquad \text{(helical transform)}\,, \qquad (1.29)$$

$$w = \frac{1}{2\varepsilon}|\nabla\hat{\psi}_1|^2 + \frac{2x_0}{x_{av}}\cos\theta\,, \qquad \text{(magnetic well)}\,, \qquad (1.30)$$

$$b^2 + 2\hat{\imath}_h b - (\beta/\varepsilon)(w + \lambda) = 0\,, \qquad (1.31)$$

the last equation determines b for a given current (λ) and

$$g(x_0) = 1 + 2\frac{\dot{x}_0^2}{x_0^2} - \frac{\ddot{x}_0}{x_0}\,, \quad Q_0 = \sqrt{1 + \frac{\dot{x}_0^2}{x_0^2}}\,, \quad \dot{x}_0 = \frac{dx_0}{d\theta}\,. \qquad (1.32)$$

Notice that in order to arrive at this form ξ_+ and ξ_- has to be determined from the plasma and vacuum energies, which is discussed in the following section, and arclenth variable v is replacing θ, where

$$\frac{d\theta}{dv} = \frac{C}{x_0 Q_0}, \qquad 0 \leq v < 1\,. \qquad (1.33)$$

Details of this derivation are given in appendix B.

1.3.3 Plasma and Vacuum Energies

The perturbations which minimize δW_p and δW_v, subject to the constraints $\nabla \cdot \mathbf{B}_1 = 0$, $\nabla \cdot \hat{\mathbf{B}}_1 = 0$ has $\nabla \times \mathbf{B}_1 = 0$ and $\nabla \times \hat{\mathbf{B}}_1 = 0$. Therefore the magnetic fields that minimizes δW_p and δW_v have all the currents flowing on the plasma surface. As a result we can write

$$\mathbf{B}_1 = \nabla V\,,\ \ \hat{\mathbf{B}}_1 = \nabla \hat{V}\,,\ \ \text{with } \nabla^2 V = 0 \text{ and } \nabla^2 \hat{V} = 0\,. \tag{1.34}$$

We require V regular at the origin and $\hat{V}$ regular at infinity (no conducting walls which gives a pessimistic estimate on stability). Under these conditions, the plasma and vacuum terms can be converted to surface integrals in the usual way

$$\delta W_p = \frac{1}{2}\int \frac{r_p R}{h R_0} V^* \mathbf{n}_s \cdot \nabla V d\theta ds\,,\ \ \delta W_v = -\frac{1}{2}\int \frac{r_p R}{h R_0} \hat{V}^* \mathbf{n}_s \cdot \nabla \hat{V} d\theta ds\,. \tag{1.35}$$

The problem now is to express V, $\hat{V}$ and $\mathbf{n}_s \cdot \nabla V$, $\mathbf{n}_s \cdot \nabla \hat{V}$ in terms of $\boldsymbol{\xi} = \boldsymbol{\xi}(\theta, z)$. This is accomplished in two steps. First we observe that $\mathbf{n}_s \cdot \nabla V$, and $\mathbf{n}_s \cdot \nabla \hat{V}$ are related to $\boldsymbol{\xi}$ by using the boundary conditions $\mathbf{n} \cdot \mathbf{B}_1 = \mathbf{n} \cdot \hat{\mathbf{B}} = 0$ at the plasma- surface. Secondly, after some algebra and analytic minimization using the freedom to choose ξ_2 we obtain

$$\frac{\delta W_p}{2\pi R_0} = \frac{A}{C}\int_0^1 \left\{\frac{d}{dv}(\xi \hat{\imath}_h) + i k_T \xi\right\} V^*(v) dv\,, \tag{1.36}$$

$$\frac{\delta W_v}{2\pi R_0} = -\frac{A}{C}\int_0^1 \left\{\frac{d}{dv}[\xi(\hat{\imath}_h + b)] + i k_T \xi\right\} \hat{V}^*(v) dv\,. \tag{1.37}$$

Details are given in appendix C.

Here k_T is the toroidal mode number. We are now left with the problem of determining V^* and $\hat{V}^*$ at the boundary (V^* is the complex conjugate of V). We use a Greens function technique as described in[4] for doing this. We also use truncated Fourier expansion in v to represent all physical quantities. The perturbation $\xi(v)$ is represented as a vector in Fourier space and δW can be conveniently written in matrix form

$$\delta W \approx \boldsymbol{\xi}^* \cdot \mathbf{W} \cdot \boldsymbol{\xi}. \tag{1.38}$$

Let λ_{min} be the smallest eigenvalue of $\mathbf{W}$. We then have that $\lambda_{min} > 0$ is a necessary and sufficient condition for $\delta W > 0$, i.e. stability is determined by the sign of λ_{min}. A numerical procedure is used to determine λ_{min} and a scan in β is used to determine where λ_{min} changes sign, which correspond to the critical value of stable β. The critical stability β curves in figs. 3 - 7, 11 - 14 are determined by this procedure.

The details of the derivation of the $\mathbf{W}$- matrix is given in appendix D, where $\mathbf{W}$ is given by eq.(D.45).

Chapter 2

Numerical Procedure

The final steps in both the equilibrium and stability analysis must be done numerically. The numerical code has two main elements, the equilibrium part and the stability part. The equilibrium code has the following input parameters: The helical field amplitudes, the average radius of plasma crossection, and the vertical field. (One can also have a non zero net toroidal current as input parameter.) The helical field amplitudes are normalized so that each amplitude given, corresponds to the transform of the actual helical field, provided there is only one helical field component present and with the vertical field set to zero. Notice that the net transform produced, when there are several helical fields of different helisities as well as a vertical field, has a rather complicated dependency on the field amplitudes. This total transform can, however, easily be determined numerically. The equilibrium code provides the necessary information for the stability analysis. The following quantities as functions of θ are provided: $x_0(\theta)$ $\dot{x}_0(\theta)$, $\hat{\iota}_h(\theta)$, $\nabla \cdot \hat{\boldsymbol{\iota}}_h(\theta)$, $W(\theta)$, $dW(\theta)/dn$ as well as related quantities. In figs. 8, 9 and 10, we have plotted "shear", "well" and "transform" versus vertical field. These quantities are in the general case a function of θ. We have made the following simplification when these quantities are plotted. As representative of "shear" we have taken the maximum value of $\nabla \cdot \hat{\boldsymbol{\iota}}_h$ and for "well" we have taken the minimum value of $dW(\theta)/dn$ with respect to θ. For the transform $\hat{\iota}_h$ we have plotted the total average value. The equilibrium code also compute a maximum value of β for obtaining equilibrium for a given net toroidal current, β_{ecrit}. One can at this point also give a value of $\beta < \beta_{ecrit}$ and compute the corresponding equilibrium.

The stability code uses this part of the equilibrium code when iterating to find the critical β for stability. The stability code uses the information from the equilibrium code to test for stability, based on matrix manipulations leading to the δW-matrix. First the Greens function problem is solved numerically to account for the plasma and vacuum contributions to δW. Then the plasma-surface contribution is computed. These three contributions add up to give the full δW- matrix. The next step is to minimize δW numerically. The δW-matrix is symmetric, and the lowest eigenvalue corresponds to the minimum value of δW. Since we are not concerned with growth rates, we need not normalize the eigenfunctions. Stability changeover occurs when the lowest eigenvalue of the δW-matrix changes sign. A negative eigenvalue corresponds to an unstable system. When all eigenvalues are positive it means that the system is stable. The normal procedure is as follows: First compute the equilibrium and check whether this equilibrium is stable or unstable for the actual value of β_{ecrit}. If it is unstable, then decrease β to find the critical β for stability. If it is stable, it could still be unstable for lower β- values, and then become stable again for even lower β- values, as is shown in fig. 5 and fig. 6. To run a typical case, i.e., compute the equilibrium and test for stability requires approximately 3 sec. of cpu time on a Cray X-MP computer.

2.1 Results and Discussion

When presenting these results one should keep in mind that there is a difficult problem of optimization as far as finding the best regime of operation in parameter space. Basically we have a six parameter problem: four helisities, a vertical field, and the plasma crossection. If a net current is included this adds one more parameter. The plasma crossection is scaled with the helical wavelength. Concerning β and ε the critical parameter is the ratio β/ε. The results will be given in terms of this parameter, and will apply to any value of ε within the limits of validity of the expansion. Given the six basic parameters the ratio β/ε is determined as a critical value for obtaining equilibrium and stability.

Thus, notice that in the following figures the BETA-axis is scaled as BETA $= \beta/\varepsilon$ and the B - vertical axis is scaled as $\mathrm{B} - \text{vertical} = B_v/\varepsilon B_0$.

2.1.1 Stellarator

We first consider the following cases according to table 1.

Cases with Different Parameters					
Case	$\hat{\iota}_1$	$\hat{\iota}_2$	$\hat{\iota}_3$	$\hat{\iota}_4$	x_0
A	.01	.4	0	0	.8
B	0	.9	0	0	1.4
C	-.05	.8	.03	0	1.1
D	-.05	.7	.02	0	1.2
E	0	.9	.1	0	1.5

Table 1

In case A we see from fig. 3 that we have a stable plasma all the way to the equilibrium β-limit for a vertical field $B_V > .052 \times \varepsilon B_0$. The equilibrium and stability β-limit is rather low in this case amounting to approximately 0.085 in the parameter β/ε. This is in contrast to case B fig. 4, where the equilibrium β-limit is more than three times larger, but in this case there is virtually no stable β-region. This demonstrates that a high equilibrium β-limit does not necessarily mean anything in terms of confinement, because the configuration may be unstable as this case clearly shows.

From fig. 8 and 9 we see that the main difference between case A and case B is that there is a pronounced difference in the "well"-effect for the two cases, whereas "shear" is not so different. We therefore conclude that the main reason for the improved stability for case A as compared to case B is due to an average magnetic "well"-effect.

Turning now to cases C and D, figs. 5 and 6, we notice that the stability boundary turns around giving a range in the vertical field where there is a low as well as a high β - region of stability. This effect is most pronounced in case D. Referring to the magnetic "well"-effect these cases are similar, see fig. 9. There is, however, somewhat higher shear in case C than in case D, which is the likely explanation fore the overall better stability characteristics for case C. Case D has, however, a larger "second region" of stability, for moderate values of the vertical field. We notice that for higher values of the vertical field $B_V/\varepsilon B_0 > .45$ case C has the highest equilibrium β-limit as well as stability limit with $\beta/\varepsilon \sim .33$. We also notice that in agreement with most cases the equilibrium β-limit decreases as the vertical field is applied in a direction so that it pushes the plasma outward. If the plasma is pushed inward one can theoretically obtain very high equilibrium β-limits as shown in figs. 4 and 7, but this is at the expense of a very low critical β-limit for stability. This appears to be in agreement with the conclusion reached by Mikhailov and Shafranov[5]. We notice also that figs. 5 and 6 show that as the plasma is pushed outward the equilibrium β passes through a minimum and then start increasing again. This effect can be explained by fig. 10 which shows a increase in the net helical transform induced by the increasing vertical field for these cases. Most of the cases discussed so far has relevance to the ATF- stellarator [6] being constructed at ORNL. ATF is basically an $\ell = 2$ system with high shear, and with $\iota_2 \sim .35$ on axis and $\iota_2 \sim .9$ at the edge. The aspect ratio is 1/7 and x_0 at the plasma edge is 1.7. The Wendelstein VII-A stellarator [7] operates in a different regime as it is a low shear system with helical transform $\iota_h \sim .45$ and large aspect ratio, $x_0 \sim .25$. Results relevant to this configuration are presented in figs. 11 - 14. The figures reveal the same general behavior as for the ATF-regime of parameters We notice, however, that there is in general a lower critical β-limit in this regime of parameters.

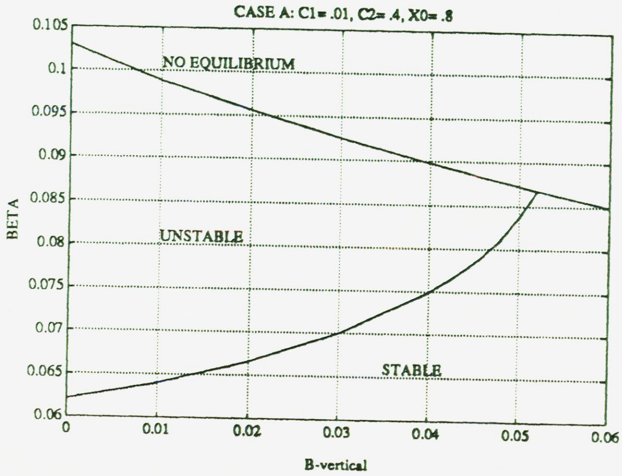


Fig. 3

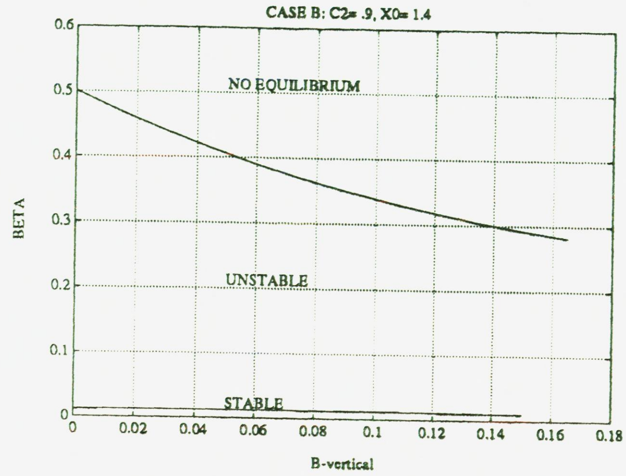


Fig. 4

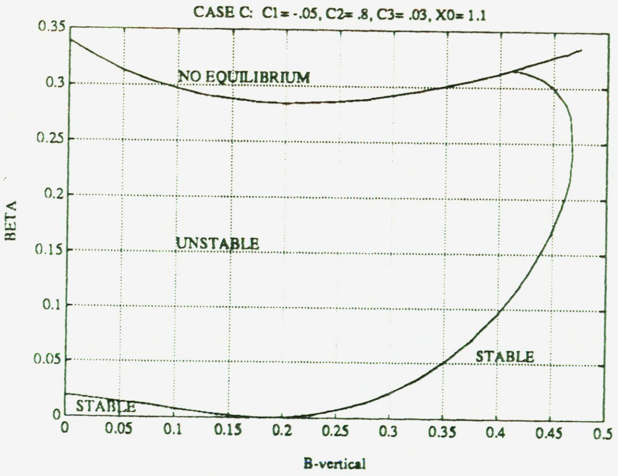


Fig. 5

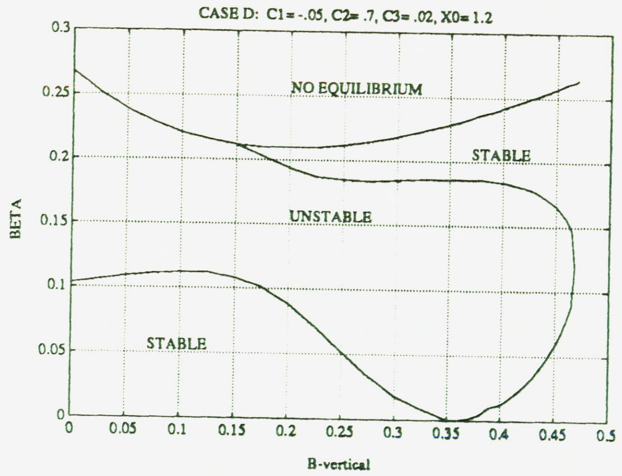


Fig. 6

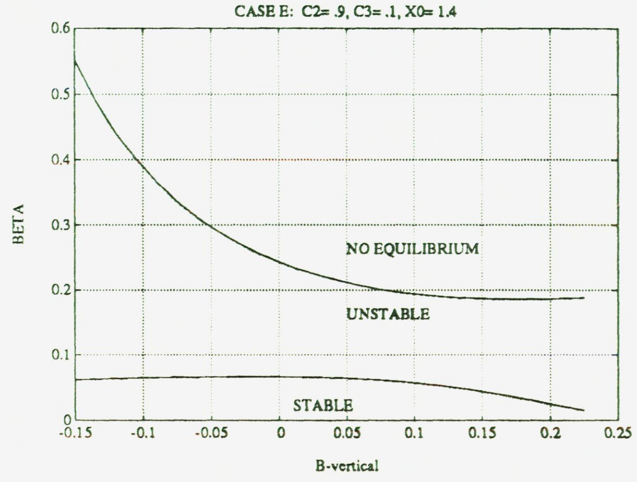


Fig. 7

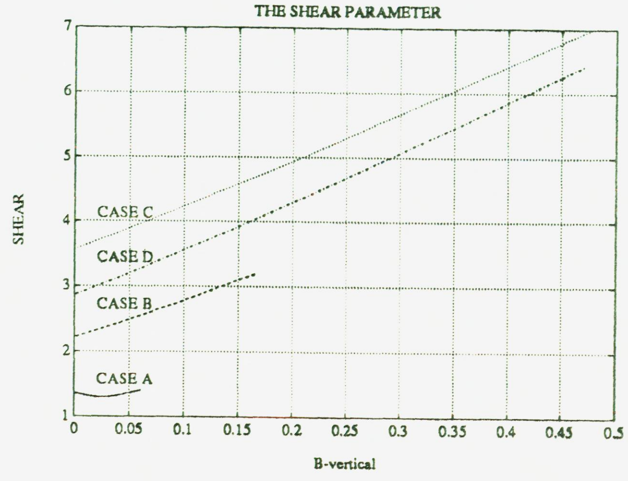


Fig. 8

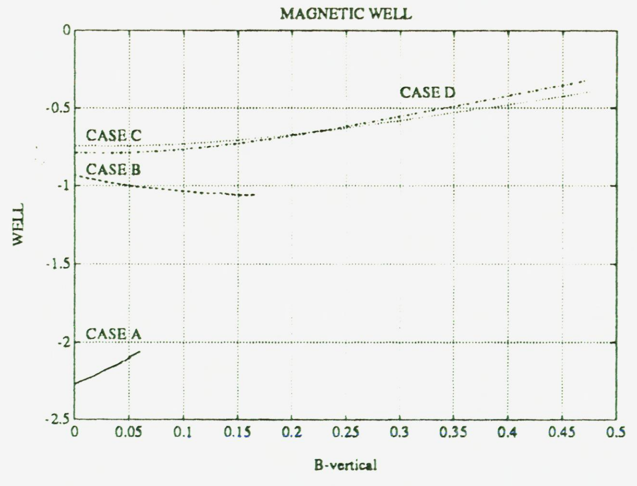


Fig. 9

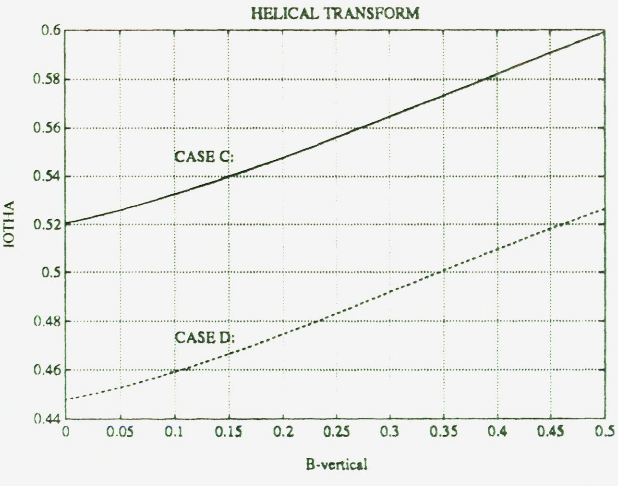


Fig. 10

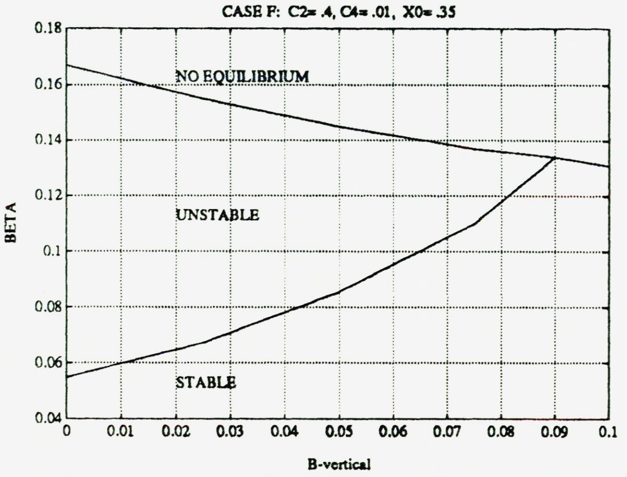


Fig. 11

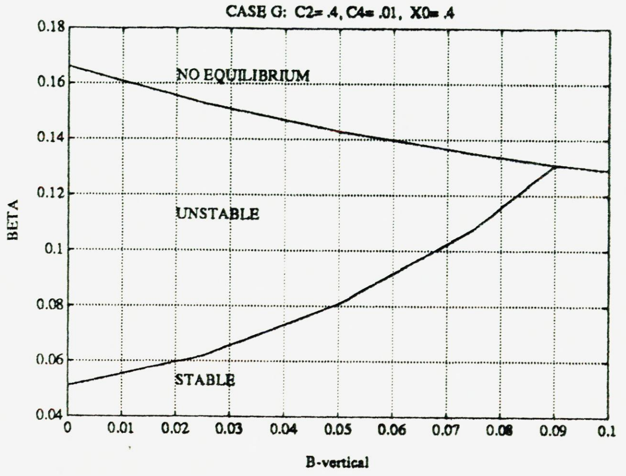


Fig. 12

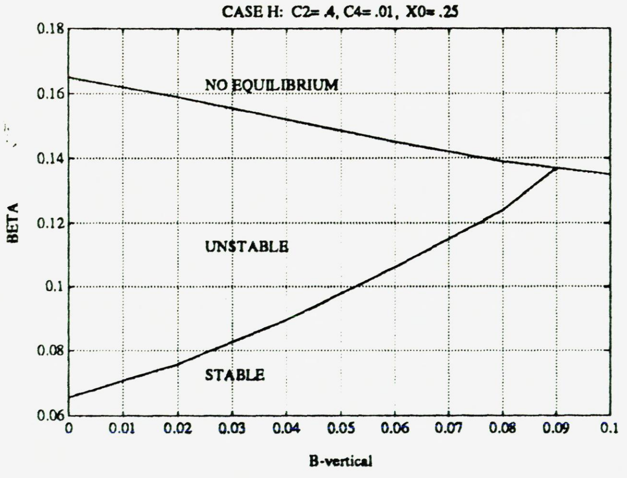


Fig. 13

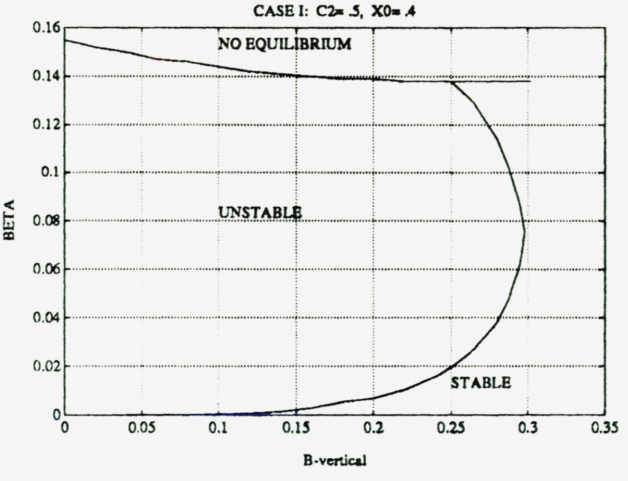


Fig. 14

2.2 Summary and Conclusions

We have, largely by analytic means, established a sharp boundary toroidal equilibrium with an arbitrary harmonic content of helical fields, as well as a vertical field. The helical fields are ordered to be small as $\sqrt{\varepsilon}$ and the vertical field is small of order ε (ε is the inverse aspect ratio, a/R_0). This class of equilibria is tested for stability for parameter regimes relevant to the ATF-experiment at ORNL U.S.A. and the W-VII A experiment at Garching, F.R.G. It is found that there exists stable equilibria in both regimes. The critical parameter is β which is relatively low for stable confinement. The critical β for stable confinement appears to be lower for the W-VII A regime of parameters than for the ATF-regime. This is in agreement with other independent investigations on this subject. For the ATF-parameter regime the highest β-limit found is about 5%. (the sharp boundary model has some uncertainty concerning the interpretation of the β-limit).

In a previous paper[1] a similar problem with a single harmonic field and no vertical field was studied. The present investigation clearly shows the effectiveness of a vertical field in terms of creating an effective magnetic well which permits stable confinement for the pure stellarator case (no net current). The vertical field also has a positive influence on the equilibrium β-limit in some cases. We believe this is due to an increase in the effective transform as demonstrated in fig. 10.

Another interesting feature is the presence of a second region of stability around the equilibrium β-limit for some parameter regimes relevant to the ATF. We want to point out that usually the second region of stability is associated with ballooning mode theory with $n \to \infty$ (n being the toroidal wavenumber). In our case, however, this stability regime is associated with the $n = 1$ mode. Note that this analysis is based on a low n-mode expansion, or equivalently weak z-dependency, where z is the toroidal coordinate. It then turns out that in this regime there are strong indications that $n = 1$ is the most unstable mode.

When we compute critical β-limits for equilibrium and stability it becomes clear that one can easily find parameter- regions where there is a very high critical equilibrium β-limit. However, it also turns out that all these regions have a very low stability β-limit. One therefore has to make a tradeoff in equilibrium β in order to gain in stability-β. As already mentioned, in some parameter-regions it is possible to push up the equilibrium

β in a good stability region by increasing the vertical field, an effect which is due to the influence of a vertical field on the total transform.

Finally we conclude that stable β-limits exists for current free stellarators, and by carefully optimizing in parameter space, these β-limits may be sufficiently high to provide the basis for a steady-state fusion reactor.

There is another regime of parameters which could easily be explored by the present code, and that is the hybrid systems. This could be in parameter-regimes ranging from a pure tokamak to a pure stellarator. One way of determining the current would be to look at flux conserving equilibria, in which case there is no equilibrium β-limit. However, since the most attractive feature of stellarators is associated with current free operation we do not include any results from this regime of operation in this presentation.

In summary we have found:

- The magnetic well effect produced by the vertical field is apparent.
- The stable β-regimes are sensitive to the harmonic content of the helical fields.
- Shear has a positive influence on stability.
- In cases where a second region of stability exists, this region can be accessed from low β by operating at higher vertical fields.
- There is a noticeable difference in the maximum β-values for ATF-like parameter values and Wendelstein-like parameter values. The difference being that the latter regime has lower maximum β-values.
- It has also been shown that it is easy to find parameter values giving high equilibrium β-limits, but always at the expense of very low stable β-limits.
- A systematic optimization in parameter space is difficult due to the large dimension of this space. Therefor the specific results presented constitute examples and not optimal values, even though some effort was spent in searching parameter space for good values.

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Appendix A

Equilibrium

A.1 Plasma Vacuum Interface

The plasma-vacuum interface is given by

$$x = x(\theta, s)\,. \tag{A.1}$$

A surface normal is given by

$$\begin{aligned}
\mathbf{n}' &= \mathbf{n}_0 + \mathbf{n}_1 + \mathbf{n}_2 + \dots\,, \\
\mathbf{n}_0 &= \mathbf{e}_r - \frac{1}{x_0}\frac{dx_0}{d\theta}\mathbf{e}_\theta\,, \\
\mathbf{n}_1 &= \left\{\frac{x_1}{x_0^2}\frac{dx_0}{d\theta} - \frac{1}{x_0}\frac{\partial}{\partial\theta}x_1\right\}\mathbf{e}_\theta - \frac{\partial}{\partial s}x_1\mathbf{e}_z\,, \\
\mathbf{n}_2 &= \{\}\mathbf{e}_r + \{\}\mathbf{e}_\theta - \frac{\partial x_2}{\partial s}\mathbf{e}_z\,,
\end{aligned}$$

(The prime on $\mathbf{n}'$ indicates that $|\,\mathbf{n}'\,| \neq 1$).

A.2 The Interface is a Flux Surface

The boundary condition $\mathbf{n}\cdot\mathbf{B} = 0$ is trivially satisfied to leading order. To first order one obtains

$$\frac{\partial x_1}{\partial s} = \frac{d\psi_1}{dn}\,. \tag{A.2}$$

After some algebra the second order condition yields

$$\frac{d}{d\theta}\left\{\nabla\hat{\psi}_1^* \times \nabla\hat{\psi}_1 \cdot \mathbf{e}_z + 2(\hat{\chi}_2 - \hat{\chi}_2^*)\right\} = 0\,. \tag{A.3}$$

or

$$\frac{d}{d\theta}F(\theta, x_0) = 0\,, \tag{A.4}$$

where we define $F(\theta, x_0)$ as the real quantity

$$F(\theta, x_0) \equiv i\left\{\nabla\hat{\psi}_1^* \times \nabla\hat{\psi}_1 \cdot \mathbf{e}_z + 2(\hat{\chi}_2 - \hat{\chi}_2^*)\right\}\,. \tag{A.5}$$

Notice that $\frac{1}{Q_0}\frac{d}{dn} = \mathbf{n}\cdot\nabla$ and $\frac{1}{x_0 Q_0}\frac{d}{d\theta} = \mathbf{t}\cdot\nabla$, where $\mathbf{n}$ and $\mathbf{t}$ are the unit normal and tangent vectors to the plasma vacuum- interface, to leading order respectively. The tangent vector is taken in a plane s=constant. Similarly for the vacuum field we obtain to second order

$$\frac{d\psi_2}{dn} = 0\,, \tag{A.6}$$

and to third order

$$\frac{1}{2i}\frac{d}{dn}\hat{\psi}_3 e^{is} + c.c. = \frac{1}{x_0}\frac{d}{d\theta}\left(\frac{x_1}{x_0}\frac{\partial\psi_2}{\partial\theta}\right)\,. \tag{A.7}$$

This is all the information we need from the condition $\mathbf{n}\cdot\mathbf{B} = 0$ at the interface. We then proceed to look at the pressure balance condition at the interface.

A.3 Pressure Balance

We expand the relation

$$2p + B_i^2 b^2 = B_0^2\left\{b^2 + 2\mathbf{b}\cdot\hat{\mathbf{b}} + \hat{b}^2\right\} \tag{A.8}$$

and obtain an equation to each order. We define

$$\beta \stackrel{\text{def}}{=} \frac{2p}{B_0^2}. \tag{A.9}$$

The significant information we shall need is contained in the third and fourth order equations. To third order a straight forward calculation gives

$$\hat{\psi}_3 = -i\beta\hat{\psi}_1 - \left\{\frac{\partial\hat{\psi}_1}{\partial x_0}\frac{\partial\psi_2}{\partial x_0} + \frac{1}{x_0^2}\frac{\partial\hat{\psi}_1}{\partial\theta}\frac{\partial\psi_2}{\partial\theta}\right\} = -i\beta\hat{\psi}_1 - \frac{1}{x_0^2}\frac{d\hat{\psi}_1}{d\theta}\frac{\partial\psi_2}{\partial\theta}. \tag{A.10}$$

To fourth order a more elaborate calculation results in the following equation

$$\begin{aligned}
&- \quad \beta\left\{\lambda + \frac{2x_0}{N}\cos\theta + \frac{1}{2}\nabla\hat{\psi}\cdot\nabla\hat{\psi}^*\right\} \\
&+ \quad \frac{1}{x_0^2}\left[\frac{i}{2}\left\{\nabla\hat{\psi}_1^*\frac{d}{d\theta}\nabla\hat{\psi}_1 - \nabla\hat{\psi}_1\frac{d}{d\theta}\nabla\hat{\psi}_1^*\right\} - \frac{i}{x_0}\nabla\hat{\psi}_1^*\times\nabla\hat{\psi}_1\cdot\mathbf{e}_z\right. \\
&+ \quad \left.\frac{d}{d\theta}\left\{\hat{\chi}_2 + \hat{\chi}_2{}^*\right\}\right]\frac{\partial\psi_2}{\partial\theta} + \frac{Q_0^2}{x_0^2}\left(\frac{\partial\psi_2}{\partial\theta}\right)^2 = 0\,.
\end{aligned}$$

Now we have

$$\frac{d\psi_2}{d\theta} = \frac{\partial\psi_2}{\partial\theta} + \dot{x}_0\frac{\partial\psi_2}{\partial x_0} = \left(1 + \frac{\dot{x}_0^2}{x_0^2}\right)\frac{\partial\psi_2}{\partial\theta} = Q_0^2\,\frac{\partial\psi_2}{\partial\theta} \tag{A.11}$$

(Notice that $\frac{d\psi_2}{dn} = 0$, see eq.(A.6)).
We may now write the fourth order equation as

$$\frac{Q_0^2}{x_0^2}\left(\frac{\partial\psi_2}{\partial\theta}\right)^2 + A\frac{1}{x_0^2}\frac{\partial\psi_2}{\partial\theta} + B = 0 \tag{A.12}$$

where

$$A = \frac{1}{2i}\left\{\nabla\psi_1\frac{d}{d\theta}\nabla\psi_1^* - \nabla\hat{\psi}_1^*\frac{d}{d\theta}\nabla\hat{\psi}_1\right\} + \frac{i}{x_0}\nabla\hat{\psi}_1 \times \nabla\hat{\psi}_1 \cdot \mathbf{e}_z + \frac{d}{d\theta}(\hat{\chi}_2 + \hat{\chi}_2^*)$$

$$B = -\beta\left\{\lambda + \frac{2x_0}{N}\cos\theta + \frac{1}{2}\mid\nabla\hat{\psi}_1\mid^2\right\}.$$

By multiplying eq.(A.12) by $x_0^2 + \dot{x}_0^2 = x_0^2 Q_0^2$ we obtain

$$\left(\frac{d\psi_2}{d\theta}\right)^2 + A\frac{d\psi_2}{d\theta} + x_0^2 Q_0^2 B = 0\,. \tag{A.13}$$

After some algebra we can rewrite the expression for A as

$$A = \frac{ix_0}{2}Q_0^2\frac{\partial}{\partial x_0}\left\{\nabla\hat{\psi}_1^* \times \nabla\hat{\psi}_1 \cdot \mathbf{e}_z + 2(\hat{\chi}_2 - \hat{\chi}_2^*)\right\}. \tag{A.14}$$

Finally this can be cast into the form given by eq.(1.16). Here we have used the relation

$$\frac{1}{N} = \frac{1}{hR_0} = \frac{1}{ha}\frac{a}{R_0} = \frac{\varepsilon}{x_{av}} \tag{A.15}$$

where a is the average plasma radius defined so that $x_{av} = ha$ and

$$x_{av} = \sqrt{\frac{A}{\pi}} \tag{A.16}$$

where A is the cross-sectional area enclosed by the curve $x = x_0(\theta)$.

A.4 Helical Transform

The equations of the magnetic field lines are given as

$$\frac{dr}{B_r} = \frac{rd\theta}{B_\theta} = \frac{dz}{B_z} \quad \text{or} \quad \frac{dx}{B_r} = \frac{xd\theta}{B_\theta} = \frac{ds}{B_z} \tag{A.17}$$

Thus

$$\frac{d\theta}{ds} = \frac{B_\theta}{xB_z} \tag{A.18}$$

By averaging this equation over z or $s = hz$, over one helical period and keeping terms to second order we obtain after some algebra

$$\delta\theta_h = \int_0^{2\pi R/N} \frac{d\theta}{dz} dz = \frac{2\pi}{4x_0} \frac{\partial F(x_0, \theta)}{\partial x_0} = \frac{2\pi}{4x_0} F_{x_0} \tag{A.19}$$

Here $\delta\theta_h$ is the transform over one helical period. Assuming this to be small we can write

$$\frac{d\theta}{dz} \approx \frac{\delta\theta_h}{2\pi R/N}. \tag{A.20}$$

Let the length a field line must travel to obtain a transform 2π be L. Then by integrating along a field line we obtain

$$L = \int_0^L dz = \int_0^{2\pi} \frac{dz}{d\theta} d\theta = \int_0^{2\pi} \frac{d\theta}{d\theta/dz}. \tag{A.21}$$

The transform for the vacuum field is now given by

$$\iota_H = \frac{2\pi R}{L} \cdot 2\pi \text{ or } \hat{\iota}_H = \frac{\iota_H}{2\pi} = \frac{2\pi R}{L}. \tag{A.22}$$

Thus we have

$$\hat{\iota}_H = \frac{2\pi R}{\displaystyle\int_0^{2\pi} \frac{(R/N) d\theta}{\frac{1}{4x_0} F_{x_0}}} = \frac{2\pi N}{\displaystyle\int_0^{2\pi} \frac{d\theta}{\frac{1}{4x_0} F_{x_0}}}. \tag{A.23}$$

Introducing the arclength variable v, we have

$$\frac{d\theta}{dv} = \frac{C}{x_0 Q_0} \text{ and } C = \int_0^{2\pi} x_0 Q_0 d\theta, \tag{A.24}$$

where C is the circumference of the plasma- vacuum boundary $x = x_0(\theta)$ (to leading order) and $v \epsilon [0, 1]$. Since $\frac{dF}{d\theta} = 0$ and $\nabla F = \mathbf{e}_r \frac{\partial F}{\partial x_0} + \mathbf{e}_\theta \frac{1}{x_0} \frac{\partial F}{\partial \theta}$ we obtain $|\nabla F| \, \mathrm{sgn} F_{x_0} = Q_0 F_{x_0}$, and we may write

$$\hat{\iota}_H = \frac{2\pi \; x_{av}}{C} \frac{1}{\int\limits_0^1 \frac{dv}{\hat{\iota}_h}} , \tag{A.25}$$

where

$$\hat{\iota}_h = \frac{1}{4\varepsilon} Q_0 F_{x_0} = \frac{1}{4\varepsilon} \mid \nabla F \mid \mathrm{sgn}\,(F_{x_0}) \; . \tag{A.26}$$

We notice that the scale factor $\frac{1}{C} 2\pi x_{av}$ takes the value one if $x = x_0(\theta)$ is a circle, in which case $x_0 = \text{const}$, and $Q_0 = 1$.

A.5 Total Transform

The total transform includes the effect of the current flowing in the plasma-vacuum interface. This can be evaluated by computing the transform just outside the interface which means that the poloidal field produced by the plasma current must be included. The result is formula (A.25) with $\hat{\iota}_h$ replaced by $\hat{\iota}_h + b$ where b is given by eq.(1.16).

A.6 Magnetic Well

We take the magnetic well quantity to be U and given by

$$U = \frac{B_0}{2\pi R} \int \frac{d\ell}{B} = \frac{B_0}{2\pi R} \int\limits_0^{2\pi} \frac{\rho d\varphi}{B_z} \; . \tag{A.27}$$

Computing the contribution to the magnetic well over one helical period we obtain

$$\delta U_h = \frac{1}{N} \left[\frac{1}{2} \mid \nabla \hat{\psi}_1 \mid^2 + \frac{2x_0}{N} \cos\theta \right] . \tag{A.28}$$

As we integrate along a field line, the corresponding change in angle is $\delta\theta_h$ given by eq.(A.19). Again assuming these quantities δU_h and $\delta\theta_h$ to be small we replace them by the continuous function

$$\frac{dU}{d\theta} \approx \frac{\delta U_h}{\delta\theta_h}. \tag{A.29}$$

With this approximation we have

$$\begin{aligned}\frac{dU}{d\theta} &= \frac{1}{N}\frac{\left[\frac{1}{2} \mid \nabla\hat{\psi}_1 \mid^2 + \frac{2x_0}{N}\cos\theta\right]}{\frac{2\pi}{4x_0}F_{x0}} \\ &= \frac{\varepsilon}{2\pi x_{av}}\frac{\left[\frac{1}{2\varepsilon} \mid \nabla\hat{\psi}_1 \mid^2 + \frac{2x_0}{x_{av}}\cos\theta\right]}{\frac{1}{4\varepsilon x_0}F_{x0}}.\end{aligned}$$

Thus

$$U = \frac{\varepsilon C}{2\pi x_{av}}\int_0^1 \frac{w dv}{\hat{\iota}_h}, \tag{A.30}$$

and w is given by eq.(1.20) and $\hat{\iota}_h$ by eq.(1.19).

COMMENTS

By the relations presented here the quantities ι_h, and w are given a physical interpretation in terms of helical transform ι_H, current-transform, total transform and magnetic well. It should, however, be pointed out that in the stability analysis it is only the local quantities of the transform and well that appear explicitly. That is local with respect to the variable θ.

Appendix B

Surface Contribution to δW

$$
\begin{aligned}
\delta W_s &= \frac{1}{2}\int_S |\,\mathbf{n}\cdot\boldsymbol{\xi}\,|^2 \;\; \mathbf{n}\cdot\left[\!\left[\nabla(p+\frac{1}{2}B^2)\right]\!\right] ds \\
&= -\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi R_0} |\,\mathbf{n}\cdot\boldsymbol{\xi}\,|^2 \left\{ \left[\!\left[B_\theta^2\right]\!\right]\left(1+2\frac{r_\theta^2}{r^2}-\frac{r_{\theta\theta}}{r}\right)\right. \\
&+ \; 2[\![B_\theta B_z]\!]\left(\frac{r_\theta r_z}{rR}-\frac{1}{R}\,r_{\theta z}+\frac{r_z}{R^2}\frac{\partial}{\partial\theta}\,(r\cos\theta)\right)R_0 \\
&+ \; \left[\!\left[B_z^2\right]\!\right]\left[-\left(\frac{R_0}{R}\right)^2 rr_{zz}+\frac{r}{R}\cos\theta\left(1+2\left(\frac{R_0}{R}r_z\right)^2\right)\right. \\
&\qquad\qquad\qquad \left.\left. +\frac{1}{R}r_\theta\sin\theta\right]\right\}\frac{R}{R_0}d\theta dz\,. \qquad\qquad (\mathrm{B}.1)
\end{aligned}
$$

$$
\begin{aligned}
\left[\!\left[B_\theta^2\right]\!\right] &= B_0^2\,\left\{\beta\;b_{\theta 1}^2+2b_{\theta 1}\hat{b}_{\theta 3}+2\hat{b}_{\theta 2}(b_{\theta 1}+b_{\theta 2}+\frac{1}{2}\hat{b}_{\theta 2})\right\}+O(\delta^5)\,, \\
[\![B_\theta B_z]\!] &= B_0^2\left\{\beta b_{\theta 1}+\hat{b}_{\theta 3}+\hat{b}_{\theta 2}(1+b_{z1})+b_{\theta 1}\hat{b}_{z2}\right\}+O(\delta^4)\,, \\
\left[\!\left[B_z^2\right]\!\right] &= B_0^2\left\{(1+2b_{z1})+2\hat{b}_{z3}\right\}+O(\delta^4)\,.
\end{aligned}
$$

We may now write

$$
\delta W_s = -\pi R_0 \int_0^1 (W_0+W_1)\frac{d\theta}{dv}dv\,, \qquad (\mathrm{B}.2)
$$

where we have performed the z-integration, and where θ has been replaced by the arclength variable v, $(d\theta/dv = C/(x_0Q_0))$. Here W_0 are the terms that do not contain β explicitly and W_1 are the terms proportional to β. We first consider the β-dependent terms

B.1 β-dependent Terms

Collecting all the β-dependent terms we find

$$
\begin{aligned}
W_1 &= \mid \mathbf{n}\cdot\boldsymbol{\xi}\mid^2 B_0^2\Bigg[\Big\{\beta b_{\theta 1}^2 + 2b_{\theta 1}\hat{b}_{\theta 3}^{(\beta)}\Big\}\Big\{1+2\Big(\frac{r_\theta}{r}\Big)^2 - \frac{r_{\theta\theta}}{r}\Big\} \\
&+ 2\Big\{\beta b_{\theta 1} + \hat{b}_{\theta 3}^{(\beta)}\Big\}\Big\{\frac{r_\theta r_z}{rR} - \frac{r_{\theta z}}{R} + \frac{r_z}{R^2}\frac{\partial}{\partial\theta}(r\cos\theta)\Big\}R_0 \\
&+ \Big\{\beta(1+2b_{z1}) + 2\hat{b}_{z3}^{(\beta)}\Big\}\Big\{-\Big(\frac{R_0}{R}\Big)^2 rr_{zz} \\
&\qquad + \frac{r}{R}\cos\theta\Big(1+2\Big(\frac{R_0}{R}r_z\Big)^2\Big) + \frac{r_\theta}{R}\sin\theta\Big\}\Bigg]. \qquad \text{(B.3)}
\end{aligned}
$$

Here we have written

$$
b_{\theta 3} \equiv \hat{b}_{\theta 3}^{(0)} + \hat{b}_{\theta 3}^{(\beta)}\ ,\ \hat{b}_{z3} = \hat{b}_{z3}^{(0)} + \hat{b}_{z3}^{(\beta)}\ , \qquad \text{(B.4)}
$$

where the last term in each expression is proportional to β, the first term does not include β explicitly. We proceed to derive expressions for these terms. Starting from the equation for $b_{\theta 3}$ we have

$$
\hat{b}_{\theta 3} = \frac{i}{2}\Big\{\hat{x}_1\frac{\partial}{\partial x_0}\Big(\frac{1}{x_0}\frac{\partial\psi_2}{\partial\theta}\Big) - \frac{1}{x_0}\frac{\partial\hat{\psi}_3}{\partial\theta}\Big\}e^{is} + c.c. \qquad \text{(B.5)}
$$

From the third order $\mathbf{n}\cdot\mathbf{B} = 0$, and pressure balance conditions we have

$$
\begin{aligned}
\frac{1}{2i}\frac{d}{dn}\hat{\psi}_3 e^{is} + c.c. &= \frac{1}{x_0}\frac{d}{d\theta}\Big\{\frac{x_1}{x_0}\frac{\partial\psi_2}{\partial\theta}\Big\}, \qquad \text{(B.6)} \\
\hat{\psi}_3 &= -i\beta\hat{\psi}_1 - \frac{1}{x_0^2}\frac{d\hat{\psi}_1}{d\theta}\frac{\partial\psi_2}{\partial\theta}\ , \qquad \text{(B.7)}
\end{aligned}
$$

respectively.
Notice that eqs.(B.4) and (B.5) are valid at the plasma surface only and that in order to evaluate $b_{\theta 3}$ given by eq.(B.3). We need to know $\frac{\partial\psi_3}{\partial\theta}$ at the surface. This can be obtained from the formula

$$\frac{\partial\psi_3}{\partial\theta} = \frac{1}{Q_0^2}\Big(\frac{d}{d\theta}\psi_3 - x_0\frac{d}{dn}\psi_3\Big). \tag{B.8}$$

Notice that $d/d\theta$ is a derivative along the surface and $d\psi_3/d\theta$ can be obtained by taking the derivative of eq.(B.7). After some algebra where we also use the fact that [1]

$$\frac{d}{d\theta}\frac{d\psi_2}{dn} = 0\,, \tag{B.9}$$

we obtain

$$\begin{aligned} \hat{b}_{\theta 3}^{(0)} &= \frac{i}{2x_0^3}\frac{\partial\hat{\psi}_1}{\partial\theta}\frac{d}{d\theta}\frac{\partial\psi_2}{\partial\theta} + \frac{i}{2x_0Q_0^2}\Big\{\frac{d}{d\theta}\Big(\frac{1}{x_0^2}\frac{d\hat{\psi}_1}{d\theta}\Big) \\ &+ \frac{\dot{x}_0}{x_0}\frac{d}{d\theta}\Big(\frac{1}{x_0}\frac{d\hat{\psi}_1}{dn}\Big) + \frac{g(x_0)}{x_0}\frac{d\hat{\psi}_1}{dn}\Big\}\frac{\partial\psi_2}{\partial\theta} + c.c.\,, \end{aligned} \tag{B.10}$$

$$\hat{b}_{\theta 3}^{(\beta)} = \frac{-\beta}{2x_0Q_0^3}\frac{d\hat{\psi}_1}{d\theta} + c.c.\,. \tag{B.11}$$

Turning to $\hat{b}_{z3}$ we have

$$\hat{b}_{z3} = \frac{1}{2}\hat{\psi}_3 e^{is} + c.c.\,, \tag{B.12}$$

and by using eq.(B.7)

$$\hat{b}_{z3}^{(0)} = \frac{1}{2x_0^2}\frac{d\hat{\psi}_1}{d\theta}\frac{\partial\psi_2}{\partial\theta}\,, \tag{B.13}$$

$$\hat{b}_{z3}^{(\beta)} = -\frac{i}{2}\beta\hat{\psi}_1\,. \tag{B.14}$$

[1] $d\psi_2/dn = 0$ along the surface, therefore its derivative along the surface is also zero.

Now we write

$$T_1 \equiv \ll \mid \mathbf{n}\cdot\boldsymbol{\xi} \mid^2 \; B_0^2 \left\{ \beta b_{\theta 1}^2 + 2 b_{\theta 1} \hat{b}_{\theta 3}^{(\beta)} \right\} \left\{ 1 + 2\left(\frac{r_\theta}{r}\right)^2 - \frac{r_{\theta\theta}}{r} \right\} \gg$$

$$= \frac{\mid \xi_0 \mid^2}{Q_0^2} B_0^2 \beta \left\{ \frac{1}{2x_0^2}\left(1 - \frac{2}{Q_0^2}\right) \mid \frac{\partial \hat{\psi}_1}{\partial \theta} \mid^2 - \frac{\dot{x}_0}{2x_0^2 Q_0^2} \frac{\partial \hat{\psi}_1}{\partial \theta} \frac{\partial \hat{\psi}_1^*}{\partial x_0} + c.c. \right\} g(x_0) \quad \text{(B.15)}$$

where $\ll Q \gg$ means averaging Q over z.

$$\begin{aligned}
T_2 &\equiv \ll 2 \mid \mathbf{n}\cdot\boldsymbol{\xi} \mid^2 \; B_0^2 \left\{ \beta b_{\theta 1} + \hat{b}_{\theta 3}^{(\beta)} \right\} \left\{ -x_0 \frac{d}{d\theta}\left(\frac{1}{x_0} \frac{\partial x_1}{\partial s} \right) \right\} \gg \\
&= -\frac{\mid \xi_0 \mid^2}{2Q_0^2} B_0^2 \beta \left\{ \frac{\partial \hat{\psi}_1}{\partial \theta}\left(1 - \frac{1}{Q_0^2}\right) - \frac{\dot{x}_0}{Q_0^2} \frac{\partial \hat{\psi}_1}{\partial x_0} \right\} \frac{d}{d\theta}\left(\frac{1}{x_0} \frac{d\hat{\psi}_1^*}{dn} \right), \qquad \text{(B.16)} \\
T_3 &\equiv \ll \mid \mathbf{n}\cdot\boldsymbol{\xi} \mid^2 \; B_0^2 \beta \left\{ 1 + 2b_{z1} + 2b_{z3}^{(\beta)} \right\} \left\{ \frac{1}{N} \frac{d}{d\theta}(x_0 \sin\theta) \right. \\
&\qquad\qquad \left. - (x_0 + x_1) \frac{\partial^2 x_1}{\partial s^2} \right\} \gg \\
&= \frac{\mid \xi_0 \mid^2}{Q_0^2} B_0^2 \beta \left\{ \frac{1}{2} \mid \frac{d\hat{\psi}_1}{dn} \mid^2 + \frac{1}{N} \frac{d}{d\theta}(x_0 \sin\theta) \right\} - \frac{\beta B_0^2}{4 x_0 Q_0^2} \frac{\partial \hat{\psi}_1}{\partial \theta} \frac{d\hat{\psi}_1}{dn} \frac{d}{d\theta} \mid \xi_0 \mid^2 \\
&- \frac{\mid \xi_0 \mid^2}{2Q_0^2} B_0^2 \beta \left\{ \frac{d\hat{\psi}_1^*}{dn} \frac{d}{d\theta}\left(\frac{1}{x_0} \frac{\partial \hat{\psi}_1}{\partial \theta} \right) + \frac{\dot{x}_0}{Q_0^2} \frac{d\hat{\psi}_1^*}{dn} \frac{d}{d\theta}\left(\frac{1}{x_0} \frac{d\hat{\psi}_1}{dn} \right) + c.c. \right. \\
&\qquad\qquad \left. + 2 \mid \frac{d\hat{\psi}_1}{dn} \mid^2 - x_0 \hat{\psi}_1 \frac{d\hat{\psi}_1^*}{dn} + c.c. \right\}. \qquad \text{(B.17)}
\end{aligned}$$

Collecting these terms we obtain

$$\begin{aligned}
W_1 &= T_1 + T_2 + T_3 \\
&= \frac{\mid \xi_0 \mid^2}{2Q_0^2} B_0^2 \beta \left[\left\{ \frac{1}{x_0^2}\left(1 - \frac{2}{Q_0^2}\right) \mid \frac{\partial \hat{\psi}_1}{\partial \theta} \mid^2 - \frac{\dot{x}_0}{x_0^2 Q_0^2} \frac{\partial \hat{\psi}_1}{\partial \theta} \frac{\partial \hat{\psi}_1^*}{\partial x_0} + c.c. \right\} g(x_0) \right. \\
&+ \left\{ \frac{\partial \hat{\psi}_1}{\partial \theta}\left(1 - \frac{1}{Q_0^2}\right) - \frac{\dot{x}_0}{Q_0^2} \frac{\partial \hat{\psi}_1}{\partial x_0} \right\} \frac{d}{d\theta}\left(\frac{1}{x_0} \frac{d\hat{\psi}_1^*}{dn} \right) + c.c.
\end{aligned}$$

$$
\begin{aligned}
&+ \quad | \frac{d\hat{\psi}_1}{dn} |^2 + \frac{2}{N}\frac{d}{d\theta}(x_0 \sin\theta) - \frac{d\psi_1^*}{dn}\frac{d}{d\theta}\Big(\frac{1}{x_0}\frac{\partial\hat{\psi}_1}{\partial\theta}\Big) \\
&- \quad \frac{\dot{x}_0}{Q_0^2}\frac{d\hat{\psi}_1}{dn}\frac{d}{d\theta}\Big(\frac{1}{x_0}\frac{d\hat{\psi}_1}{dn}\Big) \\
&- \quad 2 | \frac{d\hat{\psi}_1}{dn} |^2 + x_0\hat{\psi}_1\frac{d\hat{\psi}_1^*}{dn} + \frac{1}{2}Q_0^2\frac{d}{d\theta}\Big(\frac{1}{Q_0^2}\frac{\partial\hat{\psi}_1}{\partial\theta}\frac{1}{x_0}\frac{d\hat{\psi}_1^*}{dn}\Big) + c.c.\Big] \\
&+ \quad \text{term integrating to zero}\,.
\end{aligned}
$$

After some algebra this expression can be written in the compact form

$$
W_1 = \frac{| \xi_0 |^2}{2Q_0^2} B_0^2 \beta \; x_0 \frac{d}{dn}\Big\{\frac{1}{2} | \nabla\hat{\psi}_1 |^2 + \frac{2}{N} x_0 \cos\theta\Big\}\,. \qquad \text{(B.18)}
$$

B.2 β-independent Terms

We first consider the terms T_4 containing $b_{\theta 3}^{(0)}$ and $b_{z3}^{(0)}$ which we write as

$$
T_4 = \alpha^2 (t_1 + t_2 + t_3) \qquad \text{(B.19)}
$$

$$
\begin{aligned}
t_1 &= 2 b_{\theta 1}\hat{b}_{\theta 3}^{(0)}\Big\{1 + 2\Big(\frac{r_\theta}{r}\Big)^2 - \frac{r_{\theta\theta}}{r}\Big\}\,, \\
t_2 &= 2\hat{b}_{\theta 3}^{(0)}\Big\{\frac{r_\theta r_z}{rR} - \frac{r_\theta z}{R} + \frac{r_z}{R^2}\frac{\partial}{\partial\theta}(r\cos\theta)\Big\} R_0\,, \\
t_3 &= 2\hat{b}_{z3}^{(0)}\Big\{-\Big(\frac{R_0}{R}\Big)^2 r r_{zz} + \frac{r}{R}\cos\theta\Big[1 + 2\Big(\frac{R}{R_0} r_z\Big)^2\Big] + \frac{r_\theta}{R}\sin\theta\Big\}\,, \\
\alpha^2 &= \frac{| \xi_0 |^2 B_0^2}{2Q_0^2 x_0^2}\,.
\end{aligned}
$$

After some algebra we find

$$
\begin{aligned}
t_1 &= i\frac{g(x_0)}{Q_0^2}\frac{\partial\hat{\psi}_1^*}{\partial\theta}\left\{\frac{d}{d\theta}\left(\frac{1}{x_0}\frac{d\hat{\psi}_1}{d\theta}\right)+\frac{\dot{x}_0}{x_0}\frac{d}{d\theta}\left(\frac{1}{x_0}\frac{d\hat{\psi}_1}{dn}\right)\right. \\
&\qquad\qquad \left.+\frac{g(x_0)}{x_0}\frac{d\hat{\psi}_1}{dn}\right\}\frac{\partial\psi_2}{\partial\theta}+c.c.\,, \\
t_2 &= -i\left\{\frac{x_0^2}{Q_0^2}\frac{d}{d\theta}\left(\frac{1}{x_0^2}\frac{d\hat{\psi}_1}{d\theta}\right)+\frac{x_0 g(x_0)}{Q_0^2}\frac{d\hat{\psi}_1}{dn}\right\}\frac{d}{d\theta}\left(\frac{1}{x_0}\frac{d\hat{\psi}_1^*}{dn}\right)\frac{\partial\psi_2}{\partial\theta} \\
&\qquad\qquad -i\frac{\partial\hat{\psi}_1}{\partial\theta}\frac{d}{d\theta}\left(\frac{1}{x_0}\frac{d\hat{\psi}_1^*}{dn}\right)\frac{d}{d\theta}\frac{\partial\psi_2}{\partial\theta}+c.c.\,, \\
t_3 &= i\,x_0^2\,Q_0^2\frac{1}{x_0}\frac{\partial\hat{\psi}_1}{\partial x_0}\frac{\partial\hat{\psi}_1^*}{\partial\theta}\frac{\partial\psi_2}{\partial\theta}+c.c.\,.
\end{aligned}
$$

The remaining terms we call T_5 and they are given by

$$
\begin{aligned}
T_5 &\equiv B_0^2 \ll 2\hat{b}_{\theta 2}(b_{\theta 1}+b_{\theta 2}+\frac{1}{2}\hat{b}_{\theta 2})(1+2(\frac{r_\theta}{r})^2-\frac{r_{\theta\theta}}{r})\mid \xi_n\mid^2 \\
&+\ 2\hat{b}_{\theta 2}(1+b_{z1})\frac{R_0}{R}(\frac{r_\theta r_z}{r}-r_{\theta z})\mid \xi_n\mid^2 \gg \\
&= R_1+R_2+\ldots+R_7\,,
\end{aligned}
$$

where

$$
\mid \xi_n\mid^2=\mid \xi_0\mid^2+\mid \xi_n\mid_1^2\,, \tag{B.20}
$$

and $\mid \xi_n\mid_1^2$ is obtained form eq.(C.3) and (C.6). For convenience we write

$$
\begin{aligned}
R_1 &= 2B_0^2\,\hat{b}_{\theta 2} \ll\mid \xi_n\mid_1^2\left\{g(x_0)b_{\theta 1}-x_0\frac{d}{d\theta}\left(\frac{1}{x_0}\frac{\partial x_1}{\partial s}\right)\right\}\gg, \\
R_0 &= 4x_0^2\alpha^2\,\hat{b}_{\theta 2}\ll b_{\theta 2}\gg g(x_0)\,, \\
R_3 &= -4x_0^2\,\alpha^2\hat{b}_{\theta z}\hat{b}_{z1}x_0 \ll \frac{d}{d\theta}\left(\frac{1}{x_0}\frac{\partial x_1}{\partial s}\right)\gg,
\end{aligned}
$$

$$R_4 = 4x_0^2\,\alpha^2\,\hat{b}_{\theta 2} \ll b_{\theta 1} x_1 \gg \left(\frac{\ddot{x}_0}{x_0^2} - 4\frac{\dot{x}_0^2}{x_0^3}\right),$$

$$R_5 = 4x_0^2\,\alpha^2\,\hat{b}_{\theta 2}\Big\{ \ll b_{\theta 1}\dot{x}_1 \gg \frac{4\dot{x}_0}{x_0^2} - \ll b_{\theta 1}\ddot{x}_1 \gg \frac{1}{x_0}\Big\},$$

$$R_6 = 4x_0^2\,\alpha^2\,\hat{b}_{\theta 2} \ll \left(\frac{r_\theta r_z}{r} - r_{\theta z}\right)_2 \gg,$$

$$R_7 = 2\alpha^2\,\hat{b}_{\theta 2}^2\,g(x_0)\,.$$

Substituting for $\mid \xi_n \mid$, by eqs.(C.3) and (C.6) and for x_1 by eqs.(1.9) and (1.14), we find after some algebra

$$R_1 = -\frac{iB_0^2}{2x_0^2 Q_0^2}\frac{\partial\psi_2}{\partial\theta}\frac{\partial\hat{\psi}_1}{\partial\theta}\frac{d}{d\theta}\left(\frac{1}{x_0}\frac{d\hat{\psi}_1^*}{dn}\right)\frac{d}{d\theta}\mid \xi_0 \mid^2$$

$$+ \; i\alpha^2\Big[\Big\{2x_0^2\frac{d}{dn}\frac{\partial\hat{\psi}_1}{\partial x_0} + \frac{2\dot{x}_0}{x_0}\Big(2 - \frac{g(x_0)}{Q_0^2}\frac{\partial\hat{\psi}_1}{\partial\theta}\Big\}\frac{d}{d\theta}\left(\frac{1}{x_0}\frac{d\hat{\psi}_1^*}{dn}\right)$$

$$- \; 2g(x_0)\frac{\partial\hat{\psi}_1}{\partial\theta}\frac{d}{dn}\frac{\partial\hat{\psi}_1}{\partial x_0}\Big]\frac{\partial\psi_2}{\partial\theta} + c.c.\,,$$

$$R_2 = i\alpha^2\Big\{\frac{1}{x_0}\frac{\partial\hat{\psi}_1}{\partial x_0}\frac{\partial\hat{\psi}_1^*}{\partial\theta} + \frac{\partial^2\psi_1}{\partial\theta\partial x_0}\frac{d\hat{\psi}_1^*}{dn}\Big\}\frac{\partial\hat{\psi}_2}{\partial\theta}g(x_0) + c.c.$$

$$+ \; 4\alpha^2\frac{\partial\psi_2}{\partial\theta}\frac{\partial\chi_2}{\partial\theta}g(x_0)\,,$$

$$R_3 = -i\alpha^2 x_0^2\,\hat{\psi}_1\frac{d}{d\theta}\left(\frac{1}{x_0}\frac{d\hat{\psi}_1^*}{dn}\right)\frac{\partial\psi_2}{\partial\theta} + c.c.\,,$$

$$R_4 = i\alpha^2\left(g(x_0) - 1 + \frac{2\dot{x}_0^2}{x_0^2}\right)\frac{1}{x_0}\frac{\partial\hat{\psi}_1}{\partial x_0}\frac{\partial\hat{\psi}_1^*}{\partial\theta}\frac{\partial\psi_2}{\partial\theta} + c.c.\,,$$

$$R_5 = i\alpha^2\Big\{\frac{4\dot{x}_0^2}{x_0^3}\frac{d\hat{\psi}_1^*}{dn} + \frac{4\dot{x}_0}{x_0}\frac{d}{d\theta}\left(\frac{1}{x_0}\frac{d\hat{\psi}_1^*}{dn}\right)$$

$$- \; \frac{1}{x_0}\frac{d^2}{d\theta^2}\left(\frac{d\hat{\psi}_1^*}{dn}\right)\Big\}\frac{\partial\hat{\psi}_1}{\partial\theta}\frac{\partial\psi_2}{\partial\theta} + c.c.\,,$$

$$R_6 = i\alpha^2 x_0 \frac{d\hat{\psi}_1}{dn}\frac{d}{d\theta}\Big(\frac{1}{x_0}\frac{d\hat{\psi}_1^*}{dn}\Big)\frac{\partial\psi_2}{\partial\theta} + c.c.\,,$$

$$R_7 = 2\alpha^2\Big(\frac{\partial\psi_2}{\partial\theta}\Big)^2 g(x_0)\,.$$

After some more algebra these terms can be collected to give

$$\begin{aligned}
W_0 &= T_4 + T_5 = T_4 + R_1 + \ldots + R_7 \\
&= i\alpha^2\Big[\Big\{\frac{\dot{x}_0}{x_0}\Big(2 + \frac{g(x_0)}{Q_0^2}\Big)\frac{\partial\hat{\psi}_1}{\partial\theta} + 2x_0^2\frac{d}{dn}\frac{\partial\hat{\psi}_1}{\partial x_0} - x_0^2\hat{\psi}_1\Big\}\frac{d}{dn}\Big(\frac{1}{x_0}\frac{d\hat{\psi}_1^*}{dn}\Big) \\
&+ \Big\{\frac{g^2(x_0)}{Q_0^2} + g(x_0) - \frac{2\dot{x}_0^2}{x_0^2 Q_0^2}g(x_0) + Q_0^2 x_0^2\Big\}\frac{1}{x_0}\frac{\partial\hat{\psi}_1}{\partial x_0}\frac{\partial\psi_1^*}{\partial\theta} \\
&+ g(x_0)\Big\{\frac{1}{x_0^2 Q_0^2}\frac{\partial\hat{\psi}_1^*}{\partial\theta}\frac{d^2\hat{\psi}_1}{d\theta^2} + 2\frac{\partial\psi_1}{\partial\theta}\frac{d}{dn}\frac{\partial\hat{\psi}_1^*}{\partial x_0} \\
&\qquad + \frac{\partial^2\hat{\psi}_1}{\partial\theta\partial x_0}\frac{d\hat{\psi}_1^*}{dn}\Big\}\Big]\frac{\partial\psi_2}{\partial\theta} + c.c. \\
&+ 2\alpha^2 g(x_0)\Big\{2\frac{\partial\chi_2}{\partial\theta}\frac{\partial\psi_2}{\partial\theta} + \Big(\frac{\partial\psi_2}{\partial\theta}\Big)^2\Big\} \\
&= i\alpha^2\, Q_0^2 x_0^2\Big\{\frac{1}{x_0}\frac{\partial\hat{\psi}_1}{\partial x_0}\frac{\partial\hat{\psi}_1^*}{\partial\theta} + \frac{\partial}{\partial x_0}\Big(\frac{1}{x_0}\frac{\partial\hat{\psi}_1^*}{\partial\theta}\Big)\Big(2\frac{\partial^2\hat{\psi}_1}{\partial x_0^2} - \hat{\psi}_1\Big)\Big\}\frac{\partial\psi_2}{\partial\theta} + c.c. \\
&+ 2\alpha^2 g(x_0)\Big\{\Big(\frac{\partial\psi_2}{\partial\theta}\Big)^2 + \frac{x_0}{2}\frac{\partial F}{\partial x_0}\frac{\partial\psi_2}{\partial\theta}\Big\},
\end{aligned}$$

where the last step also takes a fair amount of algebra. In order to arrive at our final form we need one more step. The following identity can be proven to be true

$$\begin{aligned}
\nabla^2 F &= i\nabla^2\Big\{\frac{1}{x_0}\frac{\partial\hat{\psi}_1^*}{\partial x_0}\frac{\partial\hat{\psi}_1}{\partial\theta} + 2i\hat{\chi}_2 + c.c.\Big\} \\
&= i\nabla^2\Big\{\frac{1}{x_0}\frac{\partial\hat{\psi}_1^*}{\partial x_0}\frac{\partial\hat{\psi}_1}{\partial\theta}\Big\} + c.c.
\end{aligned}$$

$$= \quad 2i\left\{\frac{1}{x_0}\frac{\partial\hat{\psi}_1^*}{\partial x_0}\frac{\partial\psi_1}{\partial\theta} + \frac{\partial}{\partial x_0}\left(\frac{1}{x_0}\frac{\partial\psi_1}{\partial\theta}\right)\left(2\frac{\partial^2\hat{\psi}^*}{\partial x_0^2} - \hat{\psi}^*\right)\right\} + c.c.\,,$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_0^2} + \frac{1}{x_0}\frac{\partial}{\partial x_0} + \frac{1}{x_0^2}\frac{\partial^2}{\partial\theta^2}. \tag{B.21}$$

From this identity it follows that

$$i\left\{\frac{1}{x_0}\frac{\partial\hat{\psi}_1}{\partial x_0}\frac{\partial\hat{\psi}_1^*}{\partial\theta} + \frac{\partial}{\partial x_0}\left(\frac{1}{x_0}\frac{\partial\hat{\psi}_1^*}{\partial\theta}\right)\left(2\frac{\partial^2\hat{\psi}}{\partial x_0^2} - \hat{\psi}_1\right)\right\} + c.c.$$
$$= \quad -\frac{1}{2}\nabla^2 F\,.$$

Using this result we obtain the final form of W_0.

$$\begin{aligned} W_0 \quad &= \quad -\frac{1}{2}\alpha^2 Q_0^2 x_0^2 \nabla^2 F \frac{\partial\psi_2}{\partial\theta} + 2\alpha^2 g(x_0)\left\{\left(\frac{\partial\psi_2}{\partial\theta}\right)^2 + \frac{x_0}{2}F_{x_0}\frac{\partial\psi_2}{\partial\theta}\right\} \\ &= \quad -2\alpha^2\varepsilon^2 Q_0 x_0^3\left\{b\nabla\cdot\hat{\boldsymbol{\imath}}_h - \frac{g(x_0)}{x_0 Q_0^3}(b^2 + 2\hat{\imath}_h b)\right\} \end{aligned} \tag{B.22}$$

Notice that

$$b = \frac{1}{\varepsilon x_0 Q_0}\frac{d\psi_2}{d\theta}, \quad Q_0 = \sqrt{1 + \frac{\dot{x}_0^2}{x_0^2}}$$

$$\hat{\boldsymbol{\imath}}_h = \frac{1}{4\varepsilon}\nabla F\,,$$

$$\hat{\imath}_h = |\hat{\boldsymbol{\imath}}_h|\,\mathrm{sgn}(F_{x_0}) = \frac{1}{4\varepsilon}Q_0 F_{x_0} \qquad \text{(helical transform)},$$

$$\frac{\partial}{\partial\theta}\psi_2 = \frac{1}{Q_0^2}\frac{d}{d\theta}\psi_2$$

B.3 The Surface Contribution to δW

We add the two terms from eqs.(B.18) and (B.22) to obtain

$$\frac{\delta W_s}{2\pi R_0} = -\frac{1}{2}\int_0^1 (W_0 + W_1)\frac{d\theta}{dv}dv$$

$$= \frac{\varepsilon^2 B_0^2 C}{2}\int_0^1 \mid \xi_0 \mid^2 \left\{ b\nabla \cdot \hat{\imath} - \frac{g(x_0)}{x_0 Q_0^3}\left(b^2 + 2\hat{\imath}_h b\right) - \frac{\beta}{2\varepsilon}\frac{dw}{d\hat{n}} \right\} dv\,, \qquad \text{(B.23)}$$

where

$$\frac{d}{d\hat{n}} = \frac{1}{Q_0}\frac{d}{dn} \quad \text{and} \quad w = \frac{1}{2\varepsilon} \mid \nabla\hat{\psi}_1 \mid^2 + \frac{2x_0}{x_{av}}\cos\theta\,. \qquad \text{(B.24)}$$

Appendix C

Plasma and Vacuum δW

The boundary-condition $(\mathbf{n}\cdot\mathbf{B}\,|_{r=r_p}=0)$ to first order can be written

$$\mathbf{n}\cdot\nabla V\,|_{r=r_p}=\frac{1}{r_pRQ}\left\{\frac{d}{d\theta}(RQB_\theta\xi)+R_0\frac{d}{dz}(r_pQB_z\xi)\right\}. \qquad (C.1)$$

Since the contribution to δW from both plasma and vacuum are positive definite, and since the destabilizing surface terms are small of order δ^4, this requires that we carefully taylor our perturbation in the plasma- and vacuum-region such that δW_v and δW_p also are small of order δ^4. From the boundary-condition eq.(C.1) we see that $\mathbf{n}\cdot\nabla V\,|_{r=r_p}$ is at most of order δ. Consistent with this fact we assume that V is also at most of order δ. From eqs.(1.35) we see that this makes δW_p small of order δ^2, but since the largest contribution from δW_p for any unstable mode can at most be of order δ^4, this requires that we choose the perturbation such that the R.H. side of eq.(C.1) becomes small of order δ^2. It then follows that the first order part of this expression must be zero i.e.

$$\left[\frac{d}{d\theta}(RQ\xi B_\theta)+R_0\frac{d}{dz}(r_pQ\xi B_z)\right]_1=0. \qquad (C.2)$$

Let

$$Q\xi=\left[\xi_0+\xi_+e^{is}+\xi_-e^{-is}+\xi_2\right]e^{iks} \qquad (C.3)$$

$$Q^2 = 1 + \frac{\dot{x}^2}{x^2}\,, \quad Q_0^2 = 1 + \frac{\dot{x}_0^2}{x_0^2}\,, \tag{C.4}$$

and

$$Q^2 = Q_0^2 + 2\frac{\dot{x}_0}{x_0}\frac{d}{d\theta}\Big(\frac{x_1}{x_0}\Big) + 0(\delta^2)\,. \tag{C.5}$$

After some algebra this determines ξ_+ and ξ_- as

$$\xi_+ = \frac{i}{2}G(\xi_0, \hat{\psi}_1)\,, \quad \xi_- = -\frac{i}{2}G(\xi_0, \hat{\psi}_1^*)\,, \tag{C.6}$$

where

$$G(\xi_0, \hat{\psi}_1) = \frac{1}{x_0}\Big\{\frac{d}{d\theta}\Big(\frac{\xi_0}{x_0}\frac{\partial\hat{\psi}_1}{\partial\theta}\Big) + x_0\xi_0\Big(\frac{d\hat{\psi}_1}{dn} - \hat{\psi}_1\Big)\Big\}\,. \tag{C.7}$$

By expanding V and $\mathbf{n}\cdot\nabla V$ in Fourier series in the variable s, one can prove that the different Fourier modes decompose. And by choosing ξ_2 properly δW_p can be minimized by making all contributions coming from terms having the z-dependency vanish. By writing

$$(\mathbf{n}\cdot\nabla V ds)_2 = B_0 R_0\Big\{a_0(\theta) + \sum_{\substack{n=-2\\(n\neq 0)}}^{2} a_n(\theta)\; e^{ins}\Big\}e^{iks}\; d\theta dz\,, \tag{C.8}$$

and

$$V = \Big\{V_0(\theta) + \sum_{\substack{n=-2\\(n\neq 0)}}^{2} V_n(\theta)\; e^{ins}\Big\}e^{iks}\,, \tag{C.9}$$

we obtain

$$\frac{\delta W_p}{2\pi R_0} = \frac{B_0}{2}\int_0^{2\pi} V_0^*(\theta) a_0(\theta) d\theta\,. \tag{C.10}$$

After some algebra $a_0(\theta)$ can be determined from eq.(C.1)

$$a_0(\theta) = \varepsilon \frac{d}{d\theta}\left(\frac{\xi_0}{Q_0}\hat{\imath}_h\right) + ikx_0\xi_0 \,. \tag{C.11}$$

Then taking $\xi = \frac{\xi_0}{Q_0}$, $V_0(\theta) = B_0\varepsilon \; V(v)$, $k' = \frac{kc}{\varepsilon}$ and using the arclength variable v we obtain

$$\frac{\delta W_s}{2\pi R_0} = \frac{\varepsilon^2 B_0^2}{2} \int_0^1 V_p^*(v) \; a(v) dv \,, \tag{C.12}$$

where

$$a(v) = \frac{d}{dv}(\xi \hat{\imath}_h) + ik'\xi \,, \tag{C.13}$$

and we have omitted the subscript 0 on $a(v)$.

The contribution from vacuum can be derived in the same way, the only difference now is that the "transform term" is modified by the current flowing in the surface $(\hat{\imath}_h \;\rightarrow\; \hat{\imath}_h + b)$

$$\hat{a}(v) \;=\; \frac{d}{dv}\Big\{\xi(\hat{\imath}_h + b)\Big\} + ik'\xi \tag{C.14}$$

$$\frac{\delta W_v}{2\pi R_0} \;=\; -\frac{\varepsilon^2 B_0^2}{2} \int_0^1 \hat{V}^*(v)\hat{a}(v) dv \,. \tag{C.15}$$

Appendix D

Matrix Representation

D.1 Perturbation in Plasma and Vacuum

In order to determine the perturbation in the vacuum-magnetic field we need to know $\hat{V}$, eqs.(1.35) - (1.37). In order to determine $\hat{V}$ at the surface we shall use a Greens-function technique similar to that used in [4]. From Greens formulae we have

$$\int_V \left\{ \hat{V}\nabla^2 U - U\nabla^2 \hat{V} \right\} d\tau = \int_S \left\{ \hat{V}\ \mathbf{n}\cdot\nabla U - U\mathbf{n}\cdot\nabla\hat{V} \right\} ds\,. \tag{D.1}$$

Here V is the volume outside (inside for the plasma region) the surface S. $\hat{V}$ is the potential such that $\mathbf{B}_1 = \nabla\hat{V}$ and U is the Green's function. We assume $\hat{V}$ goes to zero sufficiently rapidly so the volume integral exists. Which means that we do not have any boundary in the vacuum- region. This makes our perturbation slightly pessimistic, since a conducting boundary would limit the motion somewhat. (However, on a long time scale the real effect of a conducting boundary would be limited to the resistive time-scale of that boundary). We then have

$$\nabla^2 U = \delta(\mathbf{r} - \mathbf{r}')\ ,\ \nabla^2\hat{V} = 0\,. \tag{D.2}$$

Since we do not take into account an outer boundary and since the integration the long way around the torus can be done analytically to the significant order, we need only to consider the two dimensional problem with

$$\begin{aligned} U &= U(r, r', \theta, \theta') \\ \nabla^2 U &= \delta(r - r', \theta - \theta') \text{ (deltafunction)}, \end{aligned}$$

and

$$\hat{V}(r,\theta) = \int_C \left\{ \hat{V}'\hat{\mathbf{n}}' \cdot \nabla' U - U\hat{\mathbf{n}}' \cdot \nabla'\hat{V}' \right\} x_0'\, Q_0'\, d\theta', \tag{D.3}$$

where a prime($'$) refers to the coordinates r', θ' and $V' = V(r', \theta')$ etc. Notice that $\mathbf{n}'$ is the unit normal vector pointing outward from the surface. We change to arclength variable and obtain

$$\hat{V}(x,\theta) = -C \int_0^1 \left\{ \hat{V}'\hat{\mathbf{n}}' \cdot \nabla' U - U\hat{\mathbf{n}}' \cdot \nabla'\hat{V}' \right\} dv'. \tag{D.4}$$

When the observation point (x, θ) moves on to the surface we have $x = x(\theta)$ or in arclength variable $x = x(v)$. The integral from the deltafunction is then reduced by a factor 1/2. Therefore when we evaluate $\hat{V}$ at the surface we obtain

$$\hat{V}(x(v), v) = -2C \int_0^1 \left\{ \hat{V}'\mathbf{n}' \cdot \nabla' U - U\hat{\mathbf{n}}' \cdot \nabla'\hat{V}' \right\} dv'. \tag{D.5}$$

The two dimensional Green's function for the Laplaces equation is given by

$$U = \frac{1}{2\pi} \ln r, \tag{D.6}$$

where

$$\begin{aligned} r &= \sqrt{(x - x')^2 + (y - y')^2}, \\ x' &= r' \cos\theta', \quad y' = r' \sin\theta' \\ x &= r\cos\theta, \quad y = r\sin\theta. \end{aligned}$$

We introduce

$$hr = \sqrt{x_0'^2 + x_0^2 - 2x_0x_0'\cos(\theta - \theta')}\,, \tag{D.7}$$

and obtain

$$\hat{V}(x(v), v) = -\frac{C}{\pi}\int_0^1 \left\{\hat{V}'\hat{\mathbf{n}}' \cdot \nabla \ln(hr) - \ln(hr)\hat{\mathbf{n}}' \cdot \nabla'\hat{V}'\right\} dv\,. \tag{D.8}$$

It is convenient to introduce the function

$$G = G(v, v') = -\frac{C}{\pi} \ln\left\{x_0' + x_0^2 - 2x_0x_0'\cos(\theta - \theta')\right\}^{\frac{1}{2}}\,, \tag{D.9}$$

and we can write

$$\hat{V}(v) = \int_0^1 \left\{\hat{V}(v')\hat{\mathbf{n}}' \cdot \nabla G - G\hat{\mathbf{n}} \cdot \nabla\hat{V}(v')\right\} dv'\,. \tag{D.10}$$

We then evaluate $\hat{\mathbf{n}}' \cdot \nabla' G$ at the surface and obtain

$$\mathbf{n}' \cdot \nabla' G = -\frac{C}{\pi Q_0'} \frac{x_0' - x_0\cos(\theta - \theta') + (\dot{x}_0'/x_0')x_0\ \sin(\theta - \theta')}{x_0'^2 + x_0^2 - 2x_0x_0'\cos(\theta - \theta')}\,. \tag{D.11}$$

In the limit $v \to v'$ we find

$$\lim_{v \to v'} \mathbf{n}' \cdot \nabla G = -\frac{C}{2\pi} \frac{g(x_0)}{x_0 Q_0^3}\,. \tag{D.12}$$

It is also convenient to resolve the singularity of G at $v = v'$. After some algebra we find the dominant term in this limit, which can be written as

$$\frac{C}{\pi} \ln\left[\frac{C}{\pi} \sin\left\{\pi(v - v')\right\}\right]\,. \tag{D.13}$$

For the numerical evaluation it is convenient to write

$$G = \hat{G}(v, v') + G_s(v, v') \,, \tag{D.14}$$

where

$$\hat{G}(v, v') = -\frac{C}{\pi} \ln \left\{ \frac{x_0'^2 + x_0^2 - 2x_0 x_0' \cos(\theta - \theta')}{\frac{C}{\pi} \mid \sin \pi(v' - v) \mid} \right\} , \tag{D.15}$$

$$G_s(v, v') = -\frac{C}{\pi} \ln \left\{ \frac{C}{\pi} \mid \sin \ \pi(v' - v) \mid \right\} . \tag{D.16}$$

Summarizing these result we have:

$$\hat{V}(v) = - \int_0^1 \Big\{ G(v, v') \hat{\mathbf{n}}' \cdot \nabla \hat{V}(v') - \hat{V}(v') \hat{\mathbf{n}}' \cdot \nabla' G(v, v') \Big\} dv' , \tag{D.17}$$

$G = \hat{G} + G_s$ given by eqs.(D.14) - (D.16), $\lim\limits_{v' \to v} \hat{G} = 0$, G_s is singular at $v = v'$, $\hat{\mathbf{n}} \cdot \nabla G$ and $\lim\limits_{v \to v'} \hat{\mathbf{n}} \cdot \nabla G$ are given by eqs.(D.11) and (D.12), $\hat{\mathbf{n}}' \cdot \nabla' \hat{V}'$ is given by eq.(C.1) or in elaborated form by eq.(C.14). We can therefore regard eq.(D.17) as an integral equation for determining $\hat{V}(v)$ at the surface. We shall solve this equation by Fourier expansion in v. Let

$$\xi = \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \xi_n \, e^{2\pi i n v} , \tag{D.18}$$

notice that the summation omits the $n = 0$ terms. This follows from the fact that we have already by analytic minimization determined the displacement vector $\boldsymbol{\xi}$ to have $\nabla \cdot \boldsymbol{\xi} = 0$ to leading order. Since the variation in z (or s) is zero to leading order this "incompressibility" condition implies that in the fourier series for $\xi(v)$, $\xi_0 = 0$. Notice that a $\xi_0 \neq 0$ would correspond to a uniform contraction or expansion of the surface, inconsistent with the leading order perturbation being incompressible. We now write

$$\hat{V}(v) = i\varepsilon B_0 e^{iks} \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \hat{V}_n e^{2\pi i n v}, \tag{D.19}$$

$$\hat{\mathbf{n}} \cdot \nabla' \hat{V}(v) = i\varepsilon B_0 e^{iks} \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \hat{a}_n \, e^{2\pi i n v}, \tag{D.20}$$

$$b = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} b_n \, e^{2\pi i n v}, \tag{D.21}$$

$$\hat{\imath}_h = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} C_n \, e^{2\pi i n v}, \tag{D.22}$$

$$\hat{G}(v, v') = \sum_{\substack{m,m'=-\infty \\ (m,m'\neq 0)}}^{\infty} \hat{g}_{mm'} \, e^{2\pi i m v} \, e^{-2\pi i m' v'}, \tag{D.23}$$

$$G_s(v, v') = \sum_{\substack{m,m'=-\infty \\ (m,m'\neq 0)}}^{\infty} g^s_{mm'} \, e^{2\pi i m v} \, e^{-2\pi i m' v'}, \tag{D.24}$$

$$\hat{\mathbf{n}}' \cdot \nabla' G = \sum_{\substack{m,m'=-\infty \\ (m,m'\neq 0)}}^{\infty} \gamma_{mm'} \, e^{2\pi i m v} \, e^{-2\pi i m' v'}. \tag{D.25}$$

Notice that the Fourier transform of G_s , $g^s_{m,m'}$ can be calculated analytically, (which was the purpose of extracting the singular part of G in that special form). After some calculation we obtain

$$\begin{aligned} g^s_{m,m'} &= \int_0^1 \int_0^1 G_s(v, v') e^{-2\pi i m v} \, e^{2\pi i m' v'} \, dv dv' \\ &= -\frac{C}{\pi} \int_0^1 \int_0^1 \Big\{ \cos 2\pi(mv - m'v') + i \sin 2\pi(mv - m'v') \Big\} \end{aligned}$$

$$\times \ln \left\{ \frac{C}{\pi} \mid \sin \pi(v - v') \mid \right\} dv dv'$$

$$= \begin{cases} \frac{C}{2\pi} \frac{1}{|m|} \delta_{mm'} & m \neq 0 \,, \\ -\frac{C}{\pi} \ln \frac{C}{2\pi} \delta_{mm'} & m = 0 \,. \end{cases} \tag{D.26}$$

Continuing we have

$$\hat{g}_{mm'} = \int_0^1 \int_0^1 \hat{G}(v, v') \left\{ \cos\ 2\pi(mv - m'v') + i\ \sin\ 2\pi(mv - m'v') \right\} dv dv' \,, \tag{D.27}$$

$\hat{g}_{mm'}$ has to be evaluated numerically and since $\hat{G}(v, v')$ is regular in the limit $v \to v'$, this is a straight forward matter. By utilizing the symmetry properties of $\hat{G}(v, v')$ we can show that

$$\hat{g}_{mm'} = \int_0^1 \int_0^1 \hat{G}(v, v') \cos\ 2\pi(mv - mv') dv dv' \,. \tag{D.28}$$

In a similar way we find

$$\gamma_{mm'} = \int_0^1 \int_0^1 \hat{\mathbf{n}}' \cdot \nabla' G\ \cos\ 2\pi(mv - m'v') dv dv' \,. \tag{D.29}$$

When evaluating $\hat{g}_{mm'}$ and $\bar{g}_{mm'}$ it is convenient to expand the cosine term

$$\cos 2\pi(mv - m'v') = \cos 2\pi mv \cos 2\pi m'v' + \sin 2\pi mv\ \sin 2\pi m'v' \,,$$

then $\hat{g}_{mm'}$ and $\bar{g}_{mm'}$ can readily be evaluated by using fast Fourier transform routines in real space. The Fourier transform of $G(v, v')$ is now given as

$$g_{mm'} = \hat{g}_{mm'} + \frac{C}{2\pi} \frac{1}{\mid m \mid} \delta_{mm'} \,. \tag{D.30}$$

Notice that $\hat{g}_{00}$ will not appear in the problem because of the incompressibility condition discussed after eq.(D.18).

We now substitute by eqs.(D.11) - (D.16), (D.18) - (D.29) in eq.(D.17) to obtain

$$\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} V_n \, e^{2\pi inv} = - \sum_{\substack{m,m'=-\infty \\ (m,m'\neq 0)}}^{\infty} \{g_{mm'} \, e^{2\pi imv} \, \hat{a}_\ell \, \delta_{\ell m'} - \gamma_{mm'} \, e^{2\pi imv} \, V_\ell \, \delta_{\ell m'}\} \, , \tag{D.31}$$

eq.(D.31) can be rearranged and written as a matrix equation

$$\{\mathbf{I} - \mathbf{\Gamma}\} \cdot \hat{\mathbf{V}} = -\mathbf{G} \cdot \hat{\mathbf{a}} \, , \tag{D.32}$$

where $\mathbf{I}$ is the unit matrix and the matrices $\mathbf{\Gamma}$ and $\mathbf{G}$ have the elements $\gamma_{m\ell}$ and $g_{m\ell}$ respectively. $\hat{\mathbf{V}}$ is a vector with components $\hat{V}_n$ and $\hat{\mathbf{a}}$ is a vector with components $\hat{a}_n$. By solving eq.(D.32) we obtain

$$\hat{\mathbf{V}} = -(\mathbf{I} - \mathbf{\Gamma})^{-1} \cdot \mathbf{G} \cdot \hat{\mathbf{a}} \, . \tag{D.33}$$

The next step is to find an expression for $\hat{\mathbf{a}}$. From eq.(C.13) we have

$$\hat{a}(v) = \frac{d}{dv} \Big\{ \xi(\hat{\imath}_h + b) \Big\} + ik'\xi \, . \tag{D.34}$$

Substituting the Fourier expansion given by eqs.(D.18), (D.21) and (D.22) we obtain

$$\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \hat{a}_n \, e^{2\pi inv} =$$

$$\frac{d}{dv} \Big\{ \sum_{\substack{m=-\infty \\ (m\neq 0)}}^{\infty} \xi_m \, e^{2\pi imv} \sum_{\substack{\ell=-\infty \\ (\ell\neq 0)}}^{\infty} (C_\ell + b_\ell) e^{2\pi i\ell v} \Big\} + i \sum_{\substack{m=-\infty \\ (m\neq 0)}}^{\infty} k'\xi_m \, e^{2\pi imv}$$

$$= i \sum_{\substack{m=-\infty \\ (m\neq 0)}}^{\infty} \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \Big\{ 2\pi n \Big(C_{n-m} + b_{n-m} \Big) \xi_m + k'\delta_{nm} \, \xi_m \Big\} e^{2\pi inv} \, .$$

In matrix notation we then have

$$\hat{\mathbf{a}} = i\hat{\mathbf{A}} \cdot \boldsymbol{\xi}\,, \tag{D.35}$$

where

$$\hat{A}_{mn} = 2\pi n(C_{n-m} + b_{n-m}) + k'\delta_{mn}\,, \tag{D.36}$$

and $\boldsymbol{\xi}$ is now a vector in Fourier space having components given by eq.(D.18). From eq.(D.23) we obtain

$$\hat{\mathbf{V}} = -(\mathbf{I} - \boldsymbol{\Gamma})^{-1} \cdot \mathbf{G} \cdot \hat{\mathbf{A}} \cdot \boldsymbol{\xi}\,. \tag{D.37}$$

D.2 δW Matrix

We now return to the expression for δW_v, eq.(C.15). By substituting the Fourier expansion into this equation we obtain

$$\begin{aligned} \frac{\delta W_v}{2\pi R_0} &= -\frac{\varepsilon^2 B_0^2}{2} \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \hat{a}_n\, \hat{V}_n^* \\ &= -\frac{\varepsilon^2 B_0^2}{2} \hat{\mathbf{a}} \cdot \hat{\mathbf{V}}^* = -\frac{\varepsilon^2 B_0^2}{2} \hat{\mathbf{a}}^* \cdot \hat{\mathbf{V}}\,, \end{aligned}$$

and in matrix notation

$$\frac{\delta W_v}{2\pi R_0} = \frac{\varepsilon^2 B_0^2}{2}\, \boldsymbol{\xi}^* \cdot \hat{\mathbf{A}}^{T*} \cdot (\mathbf{I} - \boldsymbol{\Gamma})^{-1} \cdot \mathbf{G} \cdot \hat{\mathbf{A}} \cdot \boldsymbol{\xi}\,. \tag{D.38}$$

The plasma contribution can now be determined in exactly the same way. The only difference is a change in sign on $\boldsymbol{\Gamma}$, (the other sign-changes cancel) and a different $\mathbf{a}$,

$$\mathbf{a} = i\mathbf{A} \cdot \boldsymbol{\xi}\,, \quad A_{mn} = 2\pi n\, C_{n-m} + k'\delta_{mn}\,, \tag{D.39}$$

$$\frac{\delta W_p}{2\pi R_0} = \frac{\varepsilon^2 B_0^2}{2}\, \boldsymbol{\xi}^* \cdot \mathbf{A}^{T*} \cdot (\mathbf{I} + \boldsymbol{\Gamma})^{-1} \cdot \mathbf{G} \cdot \mathbf{A} \cdot \boldsymbol{\xi}\,. \tag{D.40}$$

The surface term matrix can easily be obtained from eq.(B.23). Let

$$b\nabla \cdot \boldsymbol{\imath}_h - \frac{g(x_0)}{x_0 Q_0^3}\left(b^2 + 2\hat{\imath}_h b\right) - \frac{\beta}{2\varepsilon}\frac{d}{d\hat{n}} w = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} s_n \; e^{2\pi i n v}\,, \tag{D.41}$$

it then follows

$$\frac{\delta W_s}{2\pi R_0} = \frac{\varepsilon^2 B_0^2}{2}\boldsymbol{\xi}^* \cdot \mathbf{S} \cdot \boldsymbol{\xi}\,, \tag{D.42}$$

where the elements of the $\mathbf{S}$ matrix are given by

$$S_{mn} = C \; s_{n-m}\,. \tag{D.43}$$

Finally we obtain the complete W-matrix formulation as

$$\frac{\delta W}{2\pi R_0} = \frac{\varepsilon^2 B_0^2}{2}\boldsymbol{\xi}^* \cdot \mathbf{W} \cdot \boldsymbol{\xi}\,, \tag{D.44}$$

where

$$\mathbf{W} = \hat{\mathbf{A}}^{T*} \cdot (\mathbf{I} - \boldsymbol{\Gamma})^{-1} \cdot \mathbf{G} \cdot \hat{\mathbf{A}} + \mathbf{A}^{T*} \cdot (\mathbf{I} + \boldsymbol{\Gamma})^{-1} \cdot \mathbf{G} \cdot \mathbf{A} + \mathbf{S}\,. \tag{D.45}$$