## Department of APPLIED MATHEMATICS

The propagation of discontinuities for linear hyperbolic partial differential equations

by

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### Abstract.

The propagation of discontinuities for solutions of linear hyperbolic systems of the first order is studied. The transport equations for systems with characteristics of nonuniform multiplicity are found in general. These transport equations are studied in detail in the nonsingular cases, and it is shown how discontinuous initialvalue problems can be solved. The propagation of discontinuities

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Abstract

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### INTRODUCTION.

Discontinuous solutions of hyperbolic partial differential equations have been extensively studied in the literature. The earlier treatments dealt mainly with second order equations. In [2] R.Courant and P.D.Lax extended the theory to first order linear hyperbolic systems with distinct characteristics. In [3] R.M.Lewis extended the theory further to symmetric hyperbolic systems with characteristics of constant multiplicity. D.Ludwig and B.Granoff [5], and J.V.Ralston [6] have considered some problems for hyperbolic systems with characteristics of nonuniform multiplicity.

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In this paper we shall try to resolve the problem in general for hyperbolic systems of the first order with characteristics of nonuniform multiplicity. We are forced to restrict the class of problems somewhat, but we hope that the theory covers most of the interesting cases.

The study of propagation of discontinuities is of course important in itself since it gives us information about the solutions. However, since the asymptotic behaviour of solutions of hyperbolic equations is closely related to the propagation of discontinuities, a study of this is more important than one may realize at first glance (for further details on this see Courant-Hilbert [1] and D.Ludwig [4]). The author will consider some of the problems in this connection elsewhere, especially we shall study how the problem of stability is related to the propagation of discontinuities.

Biscontinucus solutions of hoperbolic partial differential equations have been extendicely studied in the literature. The earlier treatments dealth m**ai**nly with second order equalitans. (a [2] A. Sverant and P. D. Lax estanded the theory to first order instr hyperbolic systems with distinct churscheristics. In [3] avaters with characteristics of constant martiplicity. D.Lanwig and S. Grenorr (5), and J.V. Raiston (6) have considered some problems for hyperbolic systems with characteristics of nonmulform Inticipiter.

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## 1. ASSUMPTIONS AND FORMULATION OF THE PROBLEM.

We shall study hyperbolic systems of the following form

$$Lu = u_t + \sum_{i=1}^{n} A^i u_{xi} + Bu = 0$$
 (1.1)

where  $u = \{u^1, \ldots, u^k\}$ , while B, A<sup>1</sup>, i = 1, ..., n are k × k matrices which may depend on the independent variables t and  $x = \{x^1, \ldots, x^n\}$ . The independent variable t (time) is separated from x mostly for practical reasons, but also because this separation is needed in later applications. It is well known that any linear hyperbolic system of the first order can be transformed to a system of the type (1.1) at least locally. Our study will be local in x,t-space. We shall only briefly indicate how the local results can be glued together in hopefully wide classes of problems.

As far as this author knows, a general theory of hyperbolic equations and systems is still not well established. The meaning of the notion hyperbolic above is therefore not clear. In this work we shall by the notion hyperbolic mean that the assumptions later in this section are satisfied, and furthermore that the Cauchy problem for (1.1) is well-posed in "suitable" metric spaces. We shall not give a precise definition of what we mean by a "suitable" metric space, but it will suffice if for instance the solution of (1.1) is in  $C^N$  (the space of N-times continuously differentiable functions) when the Cauchy data is in  $C^{N_O}$  for some  $N_O \ge N$ . - Symmetric hyperbolic systems, which we are particularA CONTRACTOR AND A CONTRACTOR OF AND A CONTRACTOR AND A C

we should study hyperbolits systems of the fullowing form

$$(1.1) \qquad 0 = u \delta + \frac{1}{2^2} A^2 u \delta + \frac{1}{2^2} \delta u = 0$$

If the second is atter more, a general theory of hyperbluc optations and fortend is still not well established. The headle of the mobion hyperbolic above is therefore not dient. In this we call to the totich hyperbolic mean that the assumptions have in this the totich hyperbolic mean that the assumptions (datably problem or (...) is well-could in "autsable" metric spates, we shall not give a preside of mission of which we mean by a "pute able" metric space, but it will suffrie if the instance the colloent of the the space, but it will suffrie a if the instance the colloentities in the space, but it will suffrie a if the instance the colloentities in the the space of Mistars is the  $M_{\rm coll}$  for some evolution instance hyperbole the space of Mistars is in  $M_{\rm coll}$  for some  $M_{\rm colloced}$  is the the could be space is the weather be the evolution instance hyperbole to systems which we are particular ily interested in, and which are covered by a well established theory in the literature, are easily seen to satisfy all our requirements (see [1]).

The characteristic equation associated with (1.1) is

det 
$$\left\{ -\lambda I + \sum_{i=1}^{n} \xi^{i} A^{i} \right\} = 0$$
 (1.2)

where I denotes the  $k \times k$  unit matrix and  $\xi^1, \ldots, \xi^n$  are real numbers which are not all zero simultaneously. Let the roots in the equation (1.2) be given by

$$\lambda = \Omega^{\alpha}(\mathbf{x}, \mathbf{t}, \boldsymbol{\xi}^{1}, \ldots, \boldsymbol{\xi}^{n}), \alpha = 1, \ldots, \gamma.$$
 (1.3)

We shall assume that the functions  $\Omega^{\alpha}$  depend on the variables  $x, t, \xi^1, \ldots, \xi^n$  sufficiently smooth. The phrase "sufficiently smooth" is chosen here and elsewhere in this paper to mean sufficiently smooth for our later arguments to be valid. The functions  $\Omega^{\alpha}$  are obviously homogeneous of degree one with respect to the variables  $\xi = \{\xi^1, \ldots, \xi^n\}$ . Except in special cases (weakly coupled hyperbolic systems), some or all of the functions  $\Omega^{\alpha}$  will have a branchpoint for  $\xi = 0$  and possibly also for other vectors  $\xi$ . Since  $\xi = 0$  is already excluded above from the smoothness requirements etc., only branchpoints for  $\xi \neq 0$  can cause trouble. If branchpoints exist for  $\xi \neq 0$ , they have to be treated separately; we shall give some comments on such cases in section 5.

Let the eigenvectors associated with the eigenvalues (1.3) be given by

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$$r^{\alpha\beta}(x,t,\xi)$$

$$\alpha = 1, ..., \gamma$$
  
 $\beta = 1, ..., q^{\alpha}$  (1.4)

 $l^{\alpha\beta}(x,t,\xi)$ 

where  $r^{\alpha\beta}$  denote the right- and  $1^{\alpha\beta}$  the left-eigenvectors. We shall assume that the eigenvectors (1.4) depend sufficiently smooth on x,t, $\xi$ . In general, this smoothness requirement will be only partially valid, because we have to allow discontinuities in the set of eigenvectors  $r^{\alpha\beta}$ ,  $1^{\alpha\beta}$ . The discontinuities are connected with changes in the multiplicity of a characteristic manifold, and the points x,t, $\xi$  where  $r^{\alpha\beta}$ ,  $1^{\alpha\beta}$  are discontinuous have to be treated separately. We shall in section 5 give some comments on how the problem can be handled.

We may without loss of generality assume that  $r^{\alpha\beta}$  and  $l^{\alpha\beta}$  are normalized by the relations

$$l^{\alpha\beta} \cdot r^{ab} = s^{\alpha a} s^{\beta b} \tag{1.5}$$

$$r^{\alpha\beta} \cdot r^{\alphab} = \delta^{\beta b} \tag{1.6}$$

(1.7)

We assume furthermore that  $r^{\alpha\beta}$  and  $l^{\alpha\beta}$  form complete sets, i.e. that  $\sum_{\alpha=1}^{\gamma} q^{\alpha} = k$ . The relations (1.6) will not be used anywhere in this paper, but will be needed in later applications. By definition we have the following identities

$$\left\{ -\Omega^{\alpha} \mathbf{I} + \sum_{i=1}^{n} \xi^{i} \mathbf{A}^{i} \right\} \cdot r^{\alpha\beta} \equiv 0$$

$$\alpha = 1, \dots, \gamma$$

$$\beta = 1, \dots, q^{\alpha}$$

$$\mathbf{I}^{\alpha\beta} \cdot \left\{ -\Omega^{\alpha} \mathbf{I} + \sum_{i=1}^{n} \xi^{i} \mathbf{A}^{i} \right\} \equiv 0$$

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where "" isonce the right and "" the lett-sign number is that assume that the elementors (...) repart antifictions arcount on root . In general, this anothere redirection will be only periodily willing promote a love to dilot discontinuities in the the het of elementors is "in" to the the discontinuities are concepted will be explicit and the mittailies of a discontinuities are manifold, and the polists root a same pol 100 are discontinuities are also to be traced apprending. The mall in section 5 give

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We assume that all  $\Omega^{\alpha}$ ,  $r^{\alpha\beta}$ ,  $l^{\alpha\beta}$  are real for any real  $\xi$ , and that

$$\Omega^{\alpha} \neq \Omega^{a}$$
 when  $\alpha \neq a$  (1.8)

If we for every choice of x,t, \$ have that

$$\Omega^{\alpha} \neq \Omega^{a}$$
 when  $\alpha \neq a$  (1.9)

then the hyperbolic system (1.1) is said to have characteristics of <u>constant multiplicity</u>. If in addition  $\gamma = k$  (or equivalently  $q^{\alpha} = 1$  for  $\alpha = 1, \ldots, \gamma$ ), (1.1) is said to have <u>distinct</u> <u>characteristics</u> and the system (1.1) is called <u>totally hyper-</u> <u>bolic</u>. In general (1.9) will not be satisfied even locally in x,t, but the multiplicities of the characteristics will be dependent on  $\xi$  at every point x,t. There seems to be little known for such systems in the literature; they are, however, not excluded from the discussion in this paper (see also [5] and [6]).

In this paper we want to study propagation of discontinuities for solutions of the hyperbolic system (1.1). Since classical solutions in the strict sense cannot have any discontinuities at all, we have to define what we shall mean by a solution. We shall work within the class of so-called "weak solutions". To define this we introduce the space S of all smooth k-dimensional vectortestfunctions  $\eta(x,t)$  with compact support in the region under consideration. We define the adjoint operator M to the operator L in (1.1) by

$$Mv = -v_t - \sum_{i=1}^{n} (A^{i*}v)_{x^i} + B^*v$$

(1.10)

We assume that all of raf, 19 are real for any real f a

 $\Omega^{\alpha} \neq \Omega^{\beta}$  when  $\alpha \neq \alpha$  (1.6)

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 $(K_{n}(f)) \qquad \qquad p \quad \text{ then } f \in Q_{n}$ 

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where \* denotes transposing of the matrix. A measurable function u is defined as being a weak solution of the equation (1.1) if

$$\int u M\eta \, dx dt = 0 \quad \forall \eta \in S \tag{1.11}$$

By partial integration in (1.11) it is easily seen that a differentiable weak solution of (1.1) is a solution in the strict sense, and that a solution in the strict sense is a weak solution.

The problem of propagation of discontinuities in the whole class of weak solutions is too involved to be studied in detail (some results can be found in [6]). We shall therefore restrict our study to weak solutions which locally are piece-wise smooth. Here, a piece-wise smooth function is defined as being a function for which there exists a finite set of smooth hypersurfaces dividing the domain of definition into a finite set of subdomains in which the function is smooth, and furthermore that the limit of the function and its derivatives exist in every subdomain when we move out to the boundaries. Thus we assume that the discontinuities of the function and its derivatives are everywhere finite, and that locally they are located on a finite set of hypersurfaces.

We are now able to formulate the problem we are going to study in the rest of the paper: Suppose that a solution of (1.1)has a discontinuity at the point  $x_0, t_0$ , what then are the equations for the hypersurfaces in a neighbourhood of  $x_0, t_0$  where the solution is discontinuous? How are the magnitudes of the discontinuities related on these hypersurfaces? where \* desired as being of the marries of the same solution of the equaliton (1,1).

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We are now only to remain the property of are gring to study in the rest of the new Support that a solution of (1.1) of editor in the rest of the new Support of the property of the other of the hypersurfaces in a styleoder of the solution of solution is discontinuous? How are the mentioned of the continuities related on these hypercent sets. 2. THE TRANSPORT EQUATIONS.

We shall in this section consider the special case where the discontinuous solution we are considering, u, is smooth everywhere in a neighbourhood of  $x_0, t_0$  except on a smooth hypersurface C given by the equation

$$p^{O}(t,x) = 0 \tag{2.1}$$

where  $\varphi^{\circ}$  is nonsingular at  $x_{\circ}, t_{\circ}$ . By our assumptions, u and its derivatives have finite jump discontinuities across C, and the jump discontinuities are smooth functions defined on the manifold C in a neighbourhood of  $x_{\circ}, t_{\circ}$  (which by assumption lies on C).

In a neighbourhood of  $x_0, t_0$  we introduce a regular coordinate transformation

$$y^{j} = \phi^{j}(t, x); \quad j = 0, 1, ..., n$$
 (2.2)

which utilizes C as a coordinate surface. The equation for C becomes  $y^{\circ} = 0$ . In the new coordinates we have

$$Lu = \sum_{j=0}^{n} H^{j} u_{j} + Bu \qquad (2.3)$$

Here we have introduced the matrices H<sup>j</sup> defined by

$$H^{j} = \varphi_{t}^{j} I + \sum_{\nu=1}^{n} \varphi_{x\nu}^{j} A^{\nu}, j = 0, 1, ..., n \qquad (2.4)$$

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Let  $D_1$  and  $D_2$  denote the regions on either side of C and let  $u = u^{I} + u^{II}$  where  $u^{I} \equiv 0$  in  $D_2$  and  $u^{II} \equiv 0$  in  $D_1$ . By Gauss' theorem it easily follows that

$$\int_{D_{1}\cup D_{2}} u^{I} M\eta \, dxdt = \int_{D_{1}} \eta L u^{I} \, dxdt - \int_{C} \left\{ \eta u^{I} \varphi_{t}^{\circ} + \sum_{\nu=1}^{n} \eta A^{\nu} u^{I} \varphi_{x}^{\circ} \right\} \theta dS$$

$$(2.5)$$

$$\int u^{II}M\eta \, dxdt = \int \eta L u^{II}dxdt + \int \left\{ \eta u^{II}\phi_t^{\circ} + \sum_{\nu=1}^n \eta A^{\nu}u^{II}\phi_x^{\circ} \right\} \theta ds$$

$$D_1 U D_2 \qquad D_2 \qquad C \qquad C \qquad V = 1$$

where  $\theta = \theta(x,t)$  is a scalarfunction such that  $\theta \left\{ \varphi_t^0, \varphi_{x_1}^0, \dots, \varphi_{x_n}^0 \right\}$  is a unitvector pointing out of the region  $D_1$ . Since Lu = 0 everywhere except on C, we are only left with the following when we add the equations (2.5) and introduce the notion  $[u] = u^{II} - u^{I}$ 

$$\int \Theta \eta \left\{ \varphi_t^{o}[u] + \sum_{\nu=1}^{n} \varphi_{x\nu}^{o} A^{\nu}[u] \right\} dS = 0$$
(2.6)

Since  $\theta \neq 0$  everywhere and the components of  $\eta$  are arbitrary, it follows that

$$H^{O}[u] = 0$$
 on C (2.7)

Here [u] is simply the jump of u across C, thus  $[u] \neq 0$ wherever u is discontinuous on C. From equation (2.7) we see that in these points the matrix H<sup>O</sup> must be singular, i.e. Let  $D_{i}$  and  $D_{i}$  denote the regions of slither ship of 0 and let  $u = u^{T} + u^{T}$  where  $u^{T} = 0$  in  $D_{i}$  and  $u^{T} = 0$  in  $D_{i}$ . By Cashir theorem if eactly follows that

$$(0.5) \qquad 0 = 2b \left( [0]^{*} A \ (0)^{*} A \right) = \left( \frac{1}{2} + (0)^{*} A \right) = 0$$

Styde d d o everynnere and the despense of a are arbitrary. It tellere the

$$(u) = 0 = (u)^{C}$$

shift ful is clithiy the jund of a social C, thus (a) = 0 white both () is disconstructed on 0. Even equation (2.7) we see that is broke Bothes the sate is and he street and the

$$\det\left(\varphi_{t}^{O} I + \sum_{\nu=1}^{H} \varphi_{x}^{O} A^{\nu}\right) = 0 \qquad (2.8)$$

This is the characteristic partial differential equation for the hyperbolic system (1.1). By definition the hypersurface C, given by  $\phi^{\circ} = 0$  and satisfying (2.8), is a characteristic manifold for the operator L. Thus we can conclude that if u is discontinuous across a hypersurface C, then C must be characteristic.

The function  $\varphi^{\circ}$  which determines the hypersurface C in equation (2.1), need not satisfy the characteristic differential equation (2.8) identically; we only know that  $\varphi^{\circ}$  satisfies (2.8) on C, i.e. for  $\varphi^{\circ} = 0$ . At every point x,t on this characteristic manifold C, there is at least one  $\alpha$  such that

$$\phi_t^{\circ} + \Omega^{\alpha}(x, t, \phi_1^{\circ}, \dots, \phi_n^{\circ}) = 0$$
 (2.9)

On the other hand, if a hypersurface  $\varphi^{\circ} = 0$  satisfies (2.9) for some choice of  $\alpha$  at every point, then the hypersurface must be a characteristic manifold. In this sense the family of equations (2.9) is equivalent to the characteristic equation (2.8), we therefore call (2.9), with  $\alpha = 1, \ldots, \gamma$ , the family of characteristic partial differential equations associated with (1.1).

In general a characteristic manifold may, at some or all points, satisfy more than one of the equations in the family of characteristic partial differential equations (2.9). Furthermore there need not be a single index  $\alpha$  such that (2.9) is satisfied

- 9 -

$$(8.8) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The motion  $\Phi^{0}$  which decomplies the hypersumbers ( ), where motion ( ), need not satisfy the characteristic sinfemential constant ( ), identically: we only know that  $\phi^{0}$  satisfies ( )) on ( ), i.e. for  $\phi^{0} = 0$  is every point x,t on this decomplete ( ), include the close of the constant of

(2.2) which is a northerapy of a first of addist and the second state of a construction of the second construction of a mailfold. In this some frails of quations (2.2) is equivelent to the characteristic frails of all of a first out (3.9) which a = 1, ..., p, she find (5.9) which constructions associated with (1.1).

To gereral a controlenteric scale (14 may, at each of ell solution as clary more anon one of the production derates with persion de construction and tone '

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at every point on a characteristic manifold. However, if the hyperbolic system (1.1) has characteristics of constant multiplicity, then a characteristic manifold satisfies (2.9) for one choice of  $\alpha$  only, and this  $\alpha$  is the same all over the manifold.

We shall now study the special case where the hypersurface C in a neighbourhood of  $x_0, t_c$  satisfies (2.9) for r different choices of  $\alpha$ , say  $\alpha_1, \ldots, \alpha_r$ , and that nowhere in this neighbourhood C satisfies (2.9) for any other choice of  $\alpha$ than  $\alpha_1, \ldots, \alpha_r$ ; we shall later see that the general result can be deduced from this special case. Again the function  $\varphi^0$ is only known to satisfy (2.9) for  $\alpha_1, \ldots, \alpha_r$  on the hypersurface C given by (2.1). However, we may here without loss of generality assume that the function  $\varphi^0$  in a neighbourhood of  $x_0, t_0$  satisfies (2.9) identically for at least <u>one</u> of the indices  $\alpha_1, \ldots, \alpha_r$ . In general it will not be possible to get (2.9) satisfied identically for more than one of the indices  $\alpha_1, \ldots, \alpha_r$ , but  $\varphi^0$  can be chosen such that (2.9) is satisfied identically for any choice of one of these indices (compare with Courant-Hilbert [1]).

If the hypersurface C satisfies (2.9) for only one choice of  $\alpha$  at the point  $x_0, t_0$ , then, by continuity, there exists a neighbourhood of  $x_0, t_0$  where C satisfies (2.9) for this choice of  $\alpha$  only. Thus this is a special case of the situation we are considering, namely the case where r = 1; this is the only case that arises for hyperbolic systems with characteristics of constant multiplicity. There seems to be little known for the cases r > 1in the literature, some results are obtained in [5] and [6].

From (2.7) we see that [u] is in the right nullspace of  $H^{\circ}$ . Since we assume that the equation for C satisfies (2.9) for

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Definition of the solution of the second state of the solution of the second state of n lo seterior with an of (2.3) to an other cheirs of a them appended the entit ister for the gommal result  $^{\circ}$  whereas from the spectal case. As in most success  $\phi^{\circ}$ the only known to swallong (2.9) nor ever (2.9) the inverter

In the momentation of satisfies (2.9) for only one choice of a stind data  $x_0$  is then, by continuity, there exists matrixed theod of  $x_0$  is a second case of the standic data of a only, then this is a second case of the standic on an considering, manely the second data case of the standic on an that stress for upperhance to been with conservation of constant stress into 11000 when a constitution of the standard of constant in the lifest case the be it due to the case > 1is the lifest of the second the second in (5) and (6).

Since we assume that the emetion form ( matifies (2.9) for

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 $\alpha_1, \ldots, \alpha_r$ , and these only, [u] can be expanded in the following way

$$[u] = \sum_{i=1}^{r} \sum_{\beta=1}^{q^{i}} \sigma_{\beta}^{i} r^{\alpha_{i}\beta}$$
(2.10)

where  $\sigma_{\beta}^{i}$  are scalarfunctions to be determined, and  $\varphi_{x^{\nu}}^{\circ}$  are substituted for  $\xi^{\nu}$ ,  $\nu = 1, ..., n$ , in the expressions for  $r^{\alpha_{i}\beta}$ . Since Lu = 0 on both sides of C, (2.3) gives on C

$$\sum_{j=0}^{n} H^{j}[u_{y^{j}}] + B[u] = 0$$
 (2.11)

For  $j \neq 0$ , u, is a tangential derivative to C, so that  $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} u \end{bmatrix}_{j}$ . Thus (2.11) may be written as

$$H^{0}[u_{y^{0}}] + \sum_{j=1}^{n} H^{j}[u]_{y^{j}} + B[u] = 0 \qquad (2.12)$$

We multiply (2.12) on the left by  $1^{\alpha_{\nu}\mu}$ ,  $\nu = 1, ..., r$  &  $\mu = 1, ..., q^{\alpha_{\nu}}$ . Since by hypothesis  $1^{\alpha_{\nu}\mu}$  H<sup>O</sup> = 0, we get

$$\sum_{i=1}^{n} 1^{\alpha_{\nu}\mu} H^{j}[u]_{j} + 1^{\alpha_{\nu}\mu} B[u] = 0$$
 (2.13)

In view of (2.10) we see that (2.13) is a system of  $k_c = \sum_{i=1}^{r \alpha_i} \frac{1}{i=1}$ partial differential equations with respect to the  $k_c$  unknown functions  $\sigma_{\beta}^{i}$ . (2.13) is a system of equations on the manifold C, and [u] is only defined there. From this point of view  $\alpha_1,\ldots,\alpha_n$  and there calls, [11] can be separated in the

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i$$

where  $c_{i}^{\dagger}$  are sectarfunctions to be decomposition  $c_{i}^{\dagger}$ , the fiber  $c_{i}^{\dagger}$ , the sector  $c_{i}^{\dagger}$ ,  $c_$ 

Tor  $3 \neq 0$ , n, the changer tick derivative to 0,  $2 \neq 0$ ,  $n \neq 0$ , n

$$\frac{u}{2} = \frac{u}{2} = \frac{u^2(u)}{2} = \frac{u^2(u)}{2} = 0$$

We rectally (2.12) on the tells by  $1^{\alpha}$  ,  $2^{\alpha}$  ,  $2^{\alpha}$  ,  $2^{\alpha}$  ,  $2^{\alpha}$  ,  $2^{\alpha}$ 

The view of (k, k) we can him (k, j) the analysis of (k, j) is a second of (k, j) is a second of (k, j) we can him (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) where (k, j) is a second of the other (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) of (k, j) is a second of the (k, j) is a secon

it is meaningless to treat [u] as a function depending on  $y_0$ . However, if we in (2.13) let [u] depend on  $y_0$  as a parameter, it does not affect our results as long as we remember that (2.13) has relevance to our problem only for  $y_0 = 0$ . Thus we shall treat [u] as a function also dependent on  $y_0$ , because this will simplify our study. Since  $1^{\alpha_V \mu} H^0 = 0$  on C by hypothesis, we may add the following term to (2.13)

$$\mathbf{1}^{\alpha_{\nu}\mu} \mathbf{H}^{\mathsf{o}}[\mathbf{u}]_{\mathbf{y}_{\mathsf{o}}}$$
(2.14)

Thus the following system of equations will be equivalent to (2.13) on C

$$\sum_{j=0}^{n} 1^{\alpha_{v}\mu} H^{j}[u]_{y^{j}} + 1^{\alpha_{v}\mu} B[u] = 0 \qquad (2.15)$$

$$v = 1, \dots, r \& \mu = 1, \dots, q^{\alpha_{v}}$$

If we introduce x,t as independent variables instead of  $y^{\circ}$ , ...  $y^{n}$ , (2.15) becomes

$$l^{\alpha_{\nu}\mu}[u]_{t} + \sum_{j=1}^{n} l^{\alpha_{\nu}\mu} A^{j}[u]_{x^{j}} + l^{\alpha_{\nu}\mu} B[u] = 0 \qquad (2.16)$$

$$v = 1, \dots, r \& \mu = 1, \dots, q^{\alpha_{\nu}}$$

to it meaning to the set of as a could a specific of the set of o

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We substitute (2.10) into (2.16) and get

We shall call (2.17) the system of transport equations for the hyperbolic system (1.1), it tells us how the discontinuities of u propagate along C. The system of transport equations (2.17) is a hyperbolic system of a very special type. To see this we differentiate (1.7) with respect to  $\xi^{\mu}$  and get

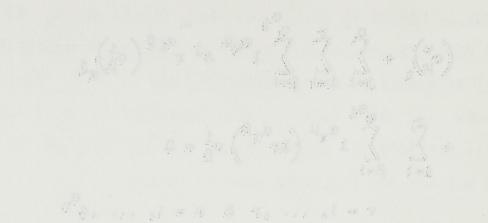
$$\left\{-\frac{\partial\Omega^{\alpha}}{\partial\xi^{\mu}}\mathbf{I} + A^{\mu}\right\} \cdot \mathbf{r}^{\alpha\beta} + \left\{-\Omega^{\alpha}\mathbf{I} + \sum_{i=1}^{n} \xi^{i}A^{i}\right\} \cdot \frac{\partial\mathbf{r}^{\alpha\beta}}{\partial\xi^{\mu}} = 0 \qquad (2.18)$$

Multiplication on the left by 1<sup>ab</sup> and using (1.5) and (1.7), gives us

$$1^{ab} A^{\mu} r^{\alpha\beta} = \frac{\partial \Omega^{\alpha}}{\partial \xi^{\mu}} \sigma^{\alpha\beta} \delta^{\betab} + (\Omega^{\alpha} - \Omega^{a}) 1^{ab} \frac{\partial r^{\alpha\beta}}{\partial \xi^{\mu}}$$
(2.19)

In particular, if  $\Omega^{\alpha} = \Omega^{\alpha}$  which is the case either if  $\alpha = a$  or if we consider a multiple characteristic, (2.19) gives us

and the contract (or so and an an a



We small coll (2.17) with respect to 6<sup>1</sup> and 48t hyperbolic system (1.1), it tolls us her the discentions (2.17) is propagate along 0 , the system of thenspert equations (2.17) is a hyperbolic system (1.1), it tolls us her her the discentions (2.17) differentiate (1.7) with respect to 6<sup>1</sup> and 48t individual (2.17)

Extended to the fact by  $\mathbb{R}^{n}$  and each (1.5) and (1.7), where  $\mathfrak{C}$ 

$$\frac{22\alpha_{0}}{12\alpha_{0}} \frac{d\alpha_{1}}{d\alpha_{1}} \left(\frac{\alpha_{0}}{\alpha_{0}} - \frac{\alpha_{1}}{\alpha_{1}}\right) = \frac{\alpha_{0}}{\alpha_{0}} \frac{d\alpha_{0}}{\alpha_{0}} = \frac{\alpha_{1}}{\alpha_{0}} \frac{d\alpha_{1}}{\alpha_{0}} \frac{d\alpha_{2}}{\alpha_{0}}$$

in particular, is at a - or marker to the second singles if a - or .

$$1^{ab} A^{\mu} r^{\alpha\beta} = \frac{\partial \Omega^{\alpha}}{\partial \xi^{\mu}} \delta^{a\alpha} \delta^{b\beta}$$
(2.20)

The expressions (2.20) can now be substituted for the coefficients to  $(\sigma_{\beta}^{i})_{x^{j}}$  in (2.17), which gives us on C

- 14 -

$$\begin{pmatrix} \sigma_{\mu}^{\nu} \end{pmatrix}_{t} + \sum_{j=1}^{n} \frac{\partial \alpha^{\nu}}{\partial \phi_{xj}^{0}} \begin{pmatrix} \sigma_{\mu}^{\nu} \end{pmatrix}_{xj} + \sum_{i=1}^{r} \sum_{\beta=1}^{q^{\alpha_{i}}} 1^{\alpha_{\nu}\mu} \begin{pmatrix} \Gamma^{\alpha_{i}\beta} \end{pmatrix} \sigma_{\beta}^{i} = 0$$
(2.21)  
$$\nu = 1, \dots, r \quad \& \quad \mu = 1, \dots, q^{\alpha_{\nu}}$$

We see that the system of transport equations (2.21) is only coupled through the nondifferentiated terms, i.e. (2.21) is a weakly coupled system. Hence the system of transport equations is trivially seen to be a symmetric hyperbolic system; we shall discuss it further in the next section.

In an analogous way we can derive the corresponding equations for the discontinuities in the derivatives of u. From (2.3) we have for \*e = 1, 2, ...

$$\frac{\partial^{e}}{\partial y^{o^{e}}} \left( Lu \right) = \sum_{j=0}^{n} H^{j} u_{y^{o^{e}}y^{j}} + Bu_{y^{o^{e}}}$$
(2.22)

+  $\sum_{j=0}^{n} (H^{j})_{j} u_{j} + N(ze^{-j}) u$ 

The expressions (2, 7) or (3, 7) and (2, 7) which (2, 7) which (2, 7)

$$(15.5) \quad \mathcal{P} = \mathcal{P} \left( \begin{array}{c} \mathcal{P} \\ \mathcal{P} \\$$

We are that the system of that will appressions to differ to differ coupled through the development the second to the second to the first the workly coupled system. Hend's the evolute of the bolls of attained to trivially adds to be a symboly of a system of the bolls of attained.

$$\frac{1}{2} = \frac{1}{2} = \frac{1}$$

Here the expression  $N^{(x-1)}u$  involves derivatives of u of order at most x-1. Since  $Lu \equiv 0$  on both sides of C

$$H^{O}\left[u_{y\circ (xe+i)}\right] + \sum_{j=1}^{n} H^{j}\left[u_{y\circ xe}\right]_{yj}$$

$$+ \left\{B + \varepsilon (H^{O})_{y\circ}\right\} \left[u_{y\circ xe}\right] \qquad (2.23)$$

$$+ \varepsilon \sum_{j=1}^{n} (H^{j})_{y\circ}\left[u_{y\circ (xe-i)}\right]_{yj} + \left[N^{(xe-i)}u\right] = 0$$

If all the derivatives of u of order less than \*e+1 are continuous across C, (2.23) shows that  $H^{O}\left[u_{g^{O}C^{\infty}+1}\right] = 0$ . If some  $(\infty+1)^{st}$  derivative has a non-zero jump, then  $\left[u_{g^{O}C^{\infty}+1}\right] \neq 0$ and hence  $H^{O}$  must be singular. Thus we may assert that if u or any of its derivatives has a jump discontinuity across C, then C is a characteristic manifold.

Suppose now that  $\begin{bmatrix} u \\ y^{\circ 5} \end{bmatrix}$  is known for  $s = 0, 1, \dots, \infty - 1$ . Then the jumps in all the derivatives of u of order  $\infty - 1$  or

less are known. By substituting  $\approx$  for  $\approx +1$ , we can rewrite (2.23) in the form

$$H^{O}\left[u_{y^{o}}\right] + \sum_{i=1}^{n} H^{j}\left[u_{y^{o}}\right]_{y^{i}} + \left[P^{(x^{e}-i)}u\right] = 0 \qquad (2.24)$$

Here  $[P^{(se-1)}u]$  involves derivatives of u of order at most x-1, so it is known. We now expand  $[u_{y^{o,se}}]$  in terms of <u>all</u> the right eigenvectors of  $H^{O}$ 

 $(x,-1)_{u}$  involves derivatives of u of  $(x,-1)_{u}$  involves derivatives of u of

 $(-+1)^{22}$  do available a non-serve jump, is an  $\left[ a_{22}, \dots \right] \neq 0$ and horne if the chargeler. Thus we say creat the is of the serve section has a jump direct index server is each a sector conficult.

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$$\left[a_{ij}a_{j}\right] = \sum_{k=1}^{N} \left[a_{ij}a_{k} - a_{k}\right] = \left[a_{ij}a_{k}\right] = \left[a_{ij}a_{k}\right] = 0 \quad (2.24)$$

to the track of the second of a 97 centre of the second of

- 15 -

Substituting (2.25) into (2.24) gives

$$H^{0}\left[u_{y^{\alpha}x^{\alpha}}\right] = -\sum_{\alpha=1}^{\gamma} \sum_{\beta=1}^{q^{\alpha}} \omega_{\beta}^{\alpha} c_{\beta} r^{\alpha\beta}$$
$$= -\sum_{j=1}^{n} H^{j}\left[u_{y^{\alpha}}c_{x^{\alpha-j}}\right]_{y^{j}} - \left[P^{(x^{\alpha-j})}u\right] \qquad (2.26)$$

When we make the same assumptions on C as before, i.e. that C satisfies (2.9) for  $\alpha_1, \ldots, \alpha_r$  and these only, then we see that

$$\omega_{\beta}^{\alpha} = \Omega^{\alpha}^{j} - \Omega^{\alpha}, \quad \gamma = 1, \dots, \gamma \& \quad \beta = 1, \dots, q^{\alpha} \quad (2.27)$$

In (2.27) j can be chosen to be 1,...,r-1 or r, and  $\varphi_{x'x}^{\circ}$  are substituted for the  $\xi^{\mathbb{Z}'_{x}}$  in the  $\Omega$ 's. In particular we have that

$$\omega_{\beta}^{\alpha} \equiv 0 \quad \text{for } \alpha = \alpha_{i}, \quad i = 1, \dots, r$$

$$\omega_{\beta}^{\alpha} \neq 0 \quad \text{everywhere for all other } \alpha's \quad \& \quad \beta's \qquad (2.28)$$

Multiplication on the left in (2.26) by 1<sup>ab</sup> gives

$$\sigma_{b}^{a} = \frac{1}{\omega_{b}^{a}} \left\{ \sum_{j=1}^{H} 1^{ab} H^{j} \left[ u_{\gamma \circ (\infty - i)} \right]_{y^{j}} + 1^{ab} \left[ P^{(\infty - i)} u \right] \right\}$$

$$a \neq \alpha_{1}, \quad i = 1, \dots, r \quad \& \quad b = 1, \dots, q^{a}$$

$$(2.29)$$

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when we sake the ISIM 25803ptions on C as before, 1.0. that C satisfies (2.9) for C<sub>1</sub>, ..., C<sub>2</sub> and these only, then we tee that

$$(75.8)$$
  $p_{1.1.1} = 0 + s_{1.1.1} = s_{1.1} + \frac{n}{2} = \frac{1}{3}\omega$ 

ia (2.27) 3 can be chosen to be intervent or 2. and whe are substituted for the §<sup>2</sup>'s in the fet. In particular we have that

$$\frac{\partial x}{\partial y} = 0 \quad for \quad x = \alpha_{1}, \quad y = 1, \dots, p \quad (2.26)$$

$$\frac{\partial x}{\partial y} \neq 0 \quad for \quad x = \alpha_{1}, \quad y = 1, \dots, p \quad (2.26)$$

Mittinication on the left in (2.26) by 1<sup>30</sup> gives

To obtain the other coefficients in the expansion (2.25), we multiply (2.23) on the left by  $l^{a_ib}$ 

$$\sum_{j=1}^{n} 1^{\alpha_{i}b} H^{j} \left[ u_{y^{\circ}} \right]_{y^{j}} + 1^{\alpha_{i}b} \left\{ B + \lambda e (H^{\circ})_{y^{\circ}} \right\} \left[ u_{y^{\circ}} \right]_{y^{\circ}} = 1^{\alpha_{i}b} \left[ T^{(\infty - 1)} u \right] \quad (2.30)$$

 $i = 1, ..., r \& b = 1, ..., q^{\alpha_i}$ 

Here  $\begin{bmatrix} T^{(x-1)} & u \end{bmatrix}$  involves only functions we have assumed to be known. The equations (2.30) are valid on the manifold C, and  $\begin{bmatrix} u_{y^{\bullet}x^{\bullet}} \end{bmatrix}$  is only defined there. However, we may look at  $\begin{bmatrix} u_{y^{\bullet}x^{\bullet}} \end{bmatrix}$  as a function also dependent on  $y_0$ , and apply the same arguments as we used in going from (2.13) to (2.16). Thus in the independent variables x,t (2.30) becomes

$$l^{\alpha,b} \left[ u_{y^{\circ}x^{c}} \right]_{t}^{t} + \sum_{j=1}^{n} l^{\alpha,b} A^{j} \left[ u_{y^{\circ}x^{c}} \right]_{x^{j}}$$

$$+ l^{\alpha,b} \left\{ B + \infty (H^{0})_{y^{\circ}} \right\} \left[ u_{y^{\circ}x^{c}} \right] = l^{\alpha,b} \left[ T^{(x^{c}-t)} u \right]$$

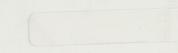
$$(2.31)$$

If we substitute (2.25) into (2 31) and use the relations (2.20), the system (2.31) can be written on the following form

.To ofterm the other contribution and the Che Repaired of (42.25), We wanted the Chester (5.25), we

Here  $\begin{bmatrix} 1 & (1-1) & 0 \end{bmatrix}$  annalizes only functions to have meaned to be interval (the equations ((2.30)) are with to a the methoded 4 , and  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$  its cally distinged there. Moreous, as may have the two is functions along thereadent on  $Y_0$  , and really the follow  $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$  in a distribution of the componism on  $Y_0$  , and really the follow enguments are used in point adaption ((2.30)) to ((2.30)). There is

Tr ve substituete (2.25) inite (2 31) and use this relations (2.20). the system (2.31) and he writichet as the following form



$$\left( \left( e_{\sigma} \frac{\alpha_{i}}{b} \right)_{t} + \sum_{\eta=1}^{n} \frac{\partial \Omega^{\alpha_{i}}}{\partial \phi_{\chi^{\chi}}^{\circ}} \left( \left( e_{\sigma} \frac{\alpha_{i}}{b} \right)_{\chi^{\chi}} + \sum_{m=1}^{r} \sum_{\beta=1}^{q} e_{\tau} \frac{\partial \sigma_{\alpha_{m}}}{\partial \beta} = g_{\chi^{\chi}}^{ib}$$
(2.32)

$$i = 1, ..., r \& b = 1, ..., q^{\alpha}$$

Here we have introduced the following functions

$$\tau_{m\beta}^{b} = 1^{\alpha_{i}b} \left( Lr^{\alpha_{m}\beta} \right) + \alpha 1^{\alpha_{i}b} \left( H^{\circ} \right)_{y} r^{\alpha_{m}\beta}$$
(2.33)

$$g_{\star}^{ib} = 1^{\alpha_i b} \left[ T^{(\infty - i)} u \right]$$

$$- \sum_{\substack{\alpha = 1 \\ \alpha \neq \alpha_{\psi} \\ \nu = 1, \dots, r}}^{\gamma} \sum_{\beta = 1}^{q_{\star}^{\alpha} b} \left\{ L \left( \varepsilon_{\sigma_{\beta}}^{\alpha} r^{\alpha_{\beta}} \right) - \varepsilon (H^{\circ})_{\psi} \circ \varepsilon_{\sigma_{\beta}}^{\alpha} r^{\alpha_{\beta}} \right\} \right]$$

$$(2.34)$$

When we have found the equation for C, the expressions (2.33) and (2.34) are known from (2.29) and our assumptions.

The system of equations (2.32) constitutes <u>the transport</u> <u>equations of higher order</u> for the hyperbolic system (1.1). We see that they only differ from the transport equations (2.21) in the nondifferentiated terms.



Hore we have introduced and following function

$$\frac{4}{12} = 1^{2} \frac{1}{2} \left(12^{2} e^{\frac{2}{3}}\right) + 2^{2} e^{\frac{2}{3}} \frac{1}{2} \left(12^{2}\right) \frac{1}{3} e^{\frac{2}{3}} \frac{1}{2} e^{\frac{2}{3}} \frac{1}{2} e^{\frac{2}{3}} \frac{1}{3} e^{\frac{2}{3$$

Man ve des read and builded or 2, and explored (2.3.2) and (2.3.4) are near from (2.30) and ere assumptions. We system of equiviliant (2.30) constitutes <u>ene transmits</u> <u>hereiting</u> of hipher and y for the hyperbolic system (1.1). We say that they are allow from the transport equations (2.01) and the mondification to the transport equations (2.01).

## 3. PROPERTIES OF THE TRANSPORT EQUATIONS.

We shall in this section study more closely the transport equations which we obtained in the previous section. We shall restrict ourselves to study only the transport equations of lowest order (2.21), but since the difference between these and the higher order transport equations (2.32) is only in the nondifferentiated terms, similar results can be obtained for the higher order transport equations.

The transport equations (2.21) tell us how the discontinuities in u propagate along C. Even though the transport equations (2.21) may be defined in the whole x,t-space, their only relevance to our problem is on the hypersurface C. The hypersurface C, given by the equation (2.1), was in the construction assumed to satisfy (2.9) for  $\alpha = \alpha_1, \ldots, \alpha = \alpha_r$ , and these choices of  $\alpha$  only. Each of the equations (2.9) is a first order partial differential equation with respect to the scalarfunction  $\phi^{\circ}$ , and can therefore be solved by the well-known method of characteristics. For  $\alpha$  given, the characteristic equations associated with (2.9) are

$$\frac{dt}{ds} = 1 , \quad \frac{dx^{i}}{ds} = \frac{\partial \Omega^{\alpha}}{\partial \varphi^{\circ}} \qquad i = 1, \dots, n \qquad (3.1)$$

$$\frac{d\phi_{\chi^{\mu}}}{ds} = -\frac{\partial\Omega^{\alpha}}{\partial\chi^{\mu}} \qquad \mu = 1, \dots, n \qquad (3.2)$$

This closed system of ordinary differential equations is called the bicharacteristic system of equations associated with the hyper-

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PROTEMENTS OF THE TRANSPORT DOUATIONS

We shall in this section and/y more closely the transport equations which us obtained in the previous section. We shall restrict ourselves to study only the transport equations of lowest order (2.21), hut since the informate between these and the higher order transport equations (2.32) is only in the ADDdifferentiated terms, sincilar results can be obtained for the biotector creations equations

The transport solutions (2.2), (eff at her the discontinutties in a propagate where G from the theory the transport edeations (2.2) may be defined in the whole of access their only relevance to our problem is on the hypersurface G. The hypercontace G , given by the equation (2.1), the in the construction assumed to sector, (2.9) for  $\alpha = \alpha_1$ . . .  $\alpha = \alpha_m$ , and there choices of  $\alpha$  only isoble the equations (2.9) is a first order partial aliferential equations with respect to the scalartraction of, and can therefore by solved by the collection condet of characteristics. For  $\alpha$  given, in discontraction action of a solution of the equations (2.9) is a first condet partial aliferential equation with respect to the scalar-

This closed system of ordinary differential equations is called the bicharmeteristic system of ecostions susceized with the hyper-

bolic system (1.1), and the solutions of (3.1 & 3.2) are called the bicharacteristic strips for (1.1). The t,x-components of the bicharacteristic strips are usually called the bicharacteristic curves or simply the bicharacteristics for the hyperbolic system (1.1). There are  $\gamma$  different bicharacteristic systems associated with (1.1), namely one for each  $\alpha = 1, \ldots, \gamma$ , and thus there are  $\gamma$  different families of bicharacteristics.

From (1.8) and the fact that  $\Omega^{\alpha}$  is homogeneous of degree 1 with respect to  $\xi$ , it is clear that for any pair  $\alpha, \alpha = 1, \ldots$  $\ldots, \gamma$ , with  $\alpha \neq \alpha$ , there is at least one  $\mu = 1, \ldots$ , n such that

$$\frac{\partial \Omega^{\alpha}}{\partial \xi^{\mu}} \neq \frac{\partial \Omega^{a}}{\partial \xi^{\mu}}$$
(3.3)

This means that no two of the  $\gamma$  families of bicharacteristics are identical. However, in general it may happen that the n equations

$$\frac{\partial \Omega^{\alpha}}{\partial \xi^{\mu}} = \frac{\partial \Omega^{a}}{\partial \xi^{\mu}} \quad \mu = 1, \dots, n$$
 (3.4)

are all satisfied simultaneously at certain points x,t, $\xi$  for  $\alpha \neq a$ . At such points the directions of bicharacteristics from two different families are the same. If the equations (3.4) are satisfied at all points on a bicharacteristic strip of the family with index  $\alpha$  say, then the families of bicharacteristics with indices  $\alpha$  and a must have at least one bicharacteristic in common. If the directions of the bicharacteristics from different families are different at all points, i.e. if the n equations

and the twelver (a) and the subblems of (a) & a.2, and celled the bicks activity of the 'or (b). The the concentrate along the state of the bicks of the bickstoile derives of and a the bickstoile for the hyperbolic system (a) There are a sufferent bickstochtetic systems associated with (c.) makely one for each of the bickstoile. And thus there are for the bickstochtetic state of the bickstoile.

Steam (13) and the rach that S<sup>2</sup> is homogeneous of degree 1 with respect to \$ it is clear that for any pair one + '.'... ... ?. with a set a there is at least one + - '.... n such that

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(3.4) for no points x,t, $\xi$  are simultaneously satisfied, we say that the hyperbolic system (1.1) has <u>bicharacteristics of con-</u> <u>stant multiplicity</u>. If in addition  $\gamma = k$ , (1.1) is said to have distinct bicharacteristics.

It is readily seen that a hyperbolic system with characteristics of constant multiplicity also has bicharacteristics of constant multiplicity. The opposite is, however, not true as is easily seen for instance for weakly coupled hyperbolic systems. Thus it is less restrictive to consider the case with bicharacteristics of constant multiplicity than the case with characteristics of constant multiplicity.

We shall now study the transport equations (2.21) in view of the above considerations on bicharacteristics. Let us first consider the special case where on C we have

$$\frac{\partial \Omega^{\alpha_i}}{\partial \varphi^{\circ}_{\chi^{\eta}}} \equiv \frac{\partial \Omega^{\alpha_j}}{\partial \varphi^{\circ}_{\chi^{\eta}}}$$
(3.5)

for every i, j = 1,...,r and  $\eta = 1,...,n$ . The equations (3.5) obviously contain no restrictions if r = 1. Thus the special case we are considering includes all cases where the hyperbolic system (1.1) has characteristics of constant multiplicity. When r > 1, the equations (3.5) means that the bicharacteristics of the r different families with indices  $\alpha_1, \ldots, \alpha_r$ , are identical on the hypersurface C.

In the transport equations (2.21) we see that the functions  $\sigma^{\nu}_{\mu}$  are differentiated along the bicharacteristics of the family with index  $\alpha_{\nu}$ . With the above assumption (3.5), all the functions  $\sigma^{\nu}_{\mu}$  are in (2.21) differentiated in the same direction, we may

) for no points which we simultaneously rather on we say the life type about  $x = x_1$  (i.e.  $x_2 = x_3$ ) has bioharachemistics of one start multiplicates is the set on  $x = k_3$  (i.e.  $x_2 = x_3$ ) distinct blaims to item.

It is readily seen that a hyperbolic system of constant addition by also has a determent which addited a start for each determined a start multiplies. The opposite is, we have the set of the set of

We shall now support on the second fine (2.2) to view for above second and the second field of a second field of the second field of a second field of the second field of the second field of the second field of the second of t

$$(3.8)$$
  $\frac{3}{2.66} = \frac{3}{2.66}$ 

The equation (1, ..., N) = 1 is the equation (1, ..., N)is the contained of the original field (1, ..., N) is the end of (1, ..., N) has character the end of expected in the second the (1, ..., N) has character the end of expected in the second the (1, ..., N) has character the end of expected in the second the second in the second the second

We consider the rate of the set one (2, 1) we see that the frequence  $Q_{\rm L}^{\rm V}$  are different along the bicharacteristic. If we can be up with in the time the above assumption (3, 2), all when turn of  $Q_{\rm L}^{\rm V}$  are in the line to and the constant of the set of the second term of the second term.

therefore interpret the system of transport equations (2.21) as ordinary differential equations along the bicharacteristics

$$\frac{d\sigma_{\mu}^{\nu}}{ds} = -1^{\alpha_{\nu}\mu} \left\{ \sum_{i=1}^{r} \sum_{\beta=1}^{q^{\alpha_{i}}} \left( Lr^{\alpha_{i\beta}} \right) \sigma_{\beta}^{i} \right\}$$
(3.6)  
$$\nu = 1, \dots, r \quad \& \quad \mu = 1, \dots, q^{\alpha_{\nu}}$$

We shall now study the expressions on the right hand side in (3.6) a little closer. In general we have for  $a = 1, ..., \gamma$  and  $\beta = 1, ..., q^{a}$ 

$$Lr^{a\beta} = \frac{\partial r^{a\beta}}{\partial t} + \sum_{i=1}^{n} A^{i} \frac{\partial r^{a\beta}}{\partial x^{i}} + Br^{a\beta}$$

$$+ \sum_{\eta=1}^{n} \left\{ \varphi^{\circ}_{x^{\eta}t} + \sum_{i=1}^{n} \varphi^{\circ}_{x^{i}x^{\eta}} A^{i} \right\} \frac{\partial r^{a\beta}}{\partial \varphi^{\circ}_{x^{\eta}}}$$
(3.7)

We assume that  $\phi^{O}$  satisfies (2.9) identically, by differentiation with respect to  $x^{2}$  we get

$$\varphi_{x^{\tau}t}^{\circ} + \sum_{i=1}^{n} \frac{\partial \Omega^{\alpha}}{\partial \varphi_{x^{i}}^{\circ}} \varphi_{x^{i}x^{\tau}}^{\circ} + \frac{\partial \Omega^{\alpha}}{\partial x^{\tau}} = 0 \qquad (3.8)$$

From (3.8) we see that (3.7) can be written

$$Lr^{a\beta} = \frac{\partial r^{a\beta}}{\partial t} + \sum_{i=1}^{n} A^{i} \frac{\partial r^{a\beta}}{\partial x^{i}} + Br^{a\beta}$$

$$- \sum_{\eta=1}^{n} \frac{\partial \Omega^{\alpha}}{\partial x^{2}} \frac{\partial r^{a\beta}}{\partial \varphi_{x^{2}}^{\circ}} + \sum_{\eta=1}^{n} \sum_{i=1}^{n} \varphi_{x^{i}x^{2}}^{\circ} \left\{ -\frac{\partial \Omega^{\alpha}}{\partial \varphi_{x^{i}}^{\circ}} I + A^{i} \right\} \frac{\partial r^{a\beta}}{\partial \varphi_{x^{2}}^{\circ}}$$

$$(3.9)$$

therefore interpret the system of transport diantiches (2.21) as

$$\frac{d\omega}{d\xi} = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{$$

The shall now sindy the starts since on the right Mark Side the (3.6) a listed closer. In general we fixed to the  $100^{\circ}$  a = 1....) and  $20^{\circ}$ 

$$F = t$$

We assume that of satisfies (5.9) Livitically, by differentiation

From (3.8) we see that (3.7) can be will then

$$\frac{1}{2} = \frac{1}{2} = \frac{1}$$

2.8)

$$\frac{\partial s_{n6}}{\sqrt{266}} \left\{ \frac{1}{4} + \frac{1}{266} + \frac{1}{26$$

If we differentiate (2.18) with respect to  $\xi^{\nu}$  we get

From

$$-\frac{\partial^{2}\Omega^{\alpha}}{\partial\xi^{\mu}\partial\xi^{\nu}}r^{\alpha\beta} + \left\{-\Omega^{\alpha}I + \sum_{i=1}^{n}\xi^{i}A^{i}\right\} \cdot \frac{\partial^{2}r^{\alpha\beta}}{\partial\xi^{\mu}\partial\xi^{\nu}}$$

$$+ \left\{-\frac{\partial\Omega^{\alpha}}{\partial\xi^{\mu}}I + A^{\mu}\right\} \frac{\partial r^{\alpha\beta}}{\partial\xi^{\nu}} + \left\{-\frac{\partial\Omega^{\alpha}}{\partial\xi^{\nu}}I + A^{\nu}\right\} \frac{\partial r^{\alpha\beta}}{\partial\xi^{\mu}} \equiv 0$$
(3.10)
(3.5), (3.9) and (3.10) we get on C that

$$l^{\alpha,\mu}(Lr^{\alpha;\beta}) = l^{\alpha,\mu} \left\{ \frac{\partial r}{\partial t}^{\alpha;\beta} + \sum_{j=1}^{n} A^{j} \frac{\partial r^{\alpha;\beta}}{\partial x^{j}} + Br^{\alpha;\beta} \right\}$$
$$- \sum_{j=1}^{n} \frac{\partial \Omega^{\alpha}}{\partial x^{j}} l^{\alpha,\mu} \frac{\partial r^{\alpha;\beta}}{\partial \phi_{x^{j}}^{\alpha}}$$
$$+ \frac{1}{2} \sum_{j=1}^{n} \sum_{\eta=1}^{n} \phi_{x^{j}x^{\eta}}^{\alpha} \frac{\partial^{2} \Omega^{\alpha;}}{\partial \phi_{x^{j}}^{\alpha,j}} \delta_{\mu}^{\gamma} \delta_{\beta}^{\mu}$$
$$(3.11)$$

Here  $\alpha$  is fixed, and equal to one of the indices  $\alpha_1, \dots, \alpha_r$ . Thus we see that in general the right hand side of (3.6) depends on the second derivatives of  $\varphi^0$ . However, in (3.6) we only need the second derivatives along the bicharacteristics, there they must satisfy the following equations

$$\frac{d\varphi_{x^{i}x^{j}}^{\circ}}{ds} = -\sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \frac{\partial^{2} \Omega^{\alpha}}{\partial \varphi_{x^{\mu}}^{\circ} \partial \varphi_{x^{\nu}}^{\circ}} \varphi_{x^{\nu}x^{i}}^{\circ} \varphi_{x^{\mu}x^{i}}^{\circ} - \frac{\partial^{2} \Omega^{\alpha}}{\partial z^{i} \partial z^{j}}$$
(3.12)  
$$-\sum_{\mu=1}^{n} \left\{ \frac{\partial^{2} \Omega^{\alpha}}{\partial \varphi_{x^{\mu}}^{\circ} \partial z^{i}} \varphi_{x^{\mu}x^{i}}^{\circ} + \frac{\partial^{2} \Omega^{\alpha}}{\partial \varphi_{x^{\mu}}^{\circ} \partial z^{i}} \varphi_{x^{\mu}x^{i}}^{\circ} \right\}$$

if we differentiate (2.18) with respect to the we get

$$\frac{2\alpha_{2}^{2}}{\sqrt{36}} \left\{ \frac{1}{\sqrt{2}} + \frac{1}{$$

(21. -)

$$0 = \frac{4x_{26}}{4x_{36}} \left\{ {}^{\vee}A + \pm \frac{2006}{\sqrt{36}} + \frac{4x_{26}}{\sqrt{36}} \left\{ \frac{4x_{26}}{\sqrt{36}} + \pm \frac{2126}{\sqrt{36}} + \frac{2}{\sqrt{36}} \right\} + \frac{4x_{26}}{\sqrt{36}} \left\{ \frac{4x_{26}}{\sqrt{36}} + \pm \frac{2x_{26}}{\sqrt{36}} + \frac{2}{\sqrt{36}} \right\} + \frac{4x_{26}}{\sqrt{36}} \left\{ \frac{4x_{26}}{\sqrt{36}} + \pm \frac{2x_{26}}{\sqrt{36}} + \frac{2}{\sqrt{36}} + \frac{2}$$

From (3.5), (3.9) and (3.10) we get on 0 that

$$\frac{(a_{-}\mu_{-}(\mathbf{r}^{\alpha}, \mathbf{r}))}{(\mathbf{r}^{\alpha}, \mathbf{r})} = \frac{(a_{+}\mu_{-})}{(a_{+}\mu_{-})} + \sum_{j=1}^{m} A_{-} \frac{a_{+}\mu_{-}}{a_{j}} + \frac{a_{+}\mu_{$$

Hare a 10 itsed and equal to one of the indices approved frame we see that in general the right hand side of (3.6) depende on the second derivatives of 9<sup>2</sup>. However, to (3.6) we only work the second derivatives along the bicharacteristics, there they ensit satisfy the following equations

In the special case we are considering, we have now obtained a <u>closed</u> system of <u>ordinary</u> differential equations which must be satisfied by the discontinuity functions and the discontinuity surface. In fact, if we denote  $\varphi_{z}^{\circ}$ ; by  $\xi^{i}$  and  $\varphi_{z'z'}^{\circ}$ ; by  $\xi_{j}^{i}$ , we see that the following closed system of equations must be satisfied:

$$\begin{aligned} \frac{dt}{ds} &= 1 \quad , \quad \frac{dx^{\perp}}{ds} = \frac{\partial\Omega^{\alpha}}{\partial\xi^{\perp}} \\ i &= 1, \dots, n \end{aligned}$$

$$\begin{aligned} \frac{dt}{ds} &= -\frac{\partial\Omega^{\alpha}}{\partialx^{\perp}} \\ \frac{dt}{ds} &= -\frac{\partial\Omega^{\alpha}}{\partialx^{\perp}} \\ \frac{dt}{ds} &= -\sum_{\nu=1}^{n} \sum_{\mu=1}^{n} \frac{\partial^{2}\Omega^{\alpha}}{\partial\xi^{\nu} \partial\xi^{\mu}} \xi_{1}^{\nu} \xi_{1}^{\mu} - \frac{\partial^{2}\Omega^{\alpha}}{\partialx^{\perp} \partialx^{\perp}} \\ &- \sum_{\mu=1}^{n} \left\{ \frac{\partial^{2}\Omega^{\alpha}}{\partial\xi^{\mu} \partialx^{\perp}} \xi_{1}^{\mu} + \frac{\partial^{2}\Omega^{\alpha}}{\partial\xi^{\mu} \partialx^{\perp}} \xi_{1}^{\mu} \right\} \\ \xi_{1}^{\perp} &= \xi_{1}^{\perp} , \quad i, j = 1, \dots, n \end{aligned}$$

$$\begin{aligned} \frac{d\sigma_{\mu}^{\nu}}{ds} &= -\sum_{\perp=1}^{r} \sum_{\beta=1}^{\alpha} \frac{1}{2^{\alpha_{\mu}\mu}} \left\{ \frac{\partial\alpha^{\alpha}}{\partialt} + \sum_{j=1}^{n} A^{j} \frac{\partial\alpha^{\alpha}}{\partialx^{j}} + Br^{\alpha_{\perp}\beta} \right\} \\ &+ \sum_{\perp=1}^{r} \sum_{\beta=1}^{q} \sum_{j=1}^{n} \frac{\partial\Omega^{\alpha}}{\partialx^{j}} 1^{\alpha_{\nu}\mu} \left\{ \frac{\partial r^{\alpha_{\nu}\beta}}{\partial\xi^{j} - \delta\xi^{j}} \sigma_{\beta}^{i} \\ &- \frac{1}{2} \sum_{j=1}^{n} \sum_{\eta=1}^{n} \xi_{\eta}^{\perp} \frac{\partial^{2}\Omega^{\alpha}}{\partial\xi^{j} - \delta\xi^{\eta}} \sigma_{\mu}^{\nu} \\ &\nu = 1, \dots, r \quad \& \quad \mu = 1, \dots, q^{\alpha_{\nu}} \end{aligned}$$

(3.13)

 $\sigma_{\beta}^{i}$ 

The second second second second

In (3.13) the index  $\alpha$  is fixed, and equal to one of the indices  $\alpha_1, \ldots, \alpha_r$ . From the construction of the equations (3.13) and the general theory of characteristics, we know that the initial-valueproblem for (3.13) with relevant initialvalues, is equivalent to the initialvalueproblem for (2.9) and (2.21). We shall therefore also call (3.13) the system of transport equations. We conclude that the discontinuities propagate along the bicharacteristics when (3.5) is satisfied on C.

When (3.5) is not valid on C, the above conclusions will no longer be true. The discontinuities will no longer propagate along the bicharacteristics, but will spread out on C governed by the transport equations (2.21). Since (2.21) is a symmetric hyperbolic system, there is a well established theory for existence, uniqueness and other properties of solutions, see for instance Courant-Hilbert [1]. Since (2.21) is a weakly coupled system, there is also a more direct approach available. In fact, essentially the same method as that used in [1] for hyperbolic systems with two independent variables can be applied.

In our study which led to the system of transport equations (2.21), we assumed that the jumpdiscontinuity for u across C was a smooth function on C. When u is a piece-wise smooth function, the jumpdiscontinuities of u on the finite number of smooth hypersurfaces will also be piece-wise smooth functions on these surfaces. It is therefore of interest to study how the discontinuities of the jumpdiscontinuities of u propagate. Thus we want to study how discontinuities in the solutions of the system of transport equations (2.21) propagate on C. This problem is of course a special case of the problem we started out with,

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In (3.13) the index a is 10.5248, and equal 99 or 90 the indices of ..., a. . From the construction of the severition (3.13) and the general theory of electrolifies, we know that the figures valueproblem for (3.13) with relevant inibial values, is souther tent to the initial velue robits for (2.9) and (3.91). Is show therefore also cold (3.13) the values of branch spectice conclude that the disconstructure percepts along the blain of therefore also cold (3.13) the value of the souther.

Strand State (15.5) Jonis (15.2) and tamper incaratest eda to hyperbolic sectors a well established theory for every sence and queners and prime proverties of solutions and the was subded four tion on 0 . When a frid piece which subout we can therefore apply the results we have found so far. Since the system of transport equations (2.21) is weakly coupled and contains one independent variable less than the original problem (because of the restriction to C), the problem we now want to study is considerably simpler than the problem we started out with. The assumptions in section 1 are trivially seen to be satisfied, and the functions corresponding to  $\Omega^{\alpha}$  and the eigenvectors corresponding to  $r^{\alpha\beta}$  and  $1^{\alpha\beta}$  are easily found. In fact, if we let  $y^1, \ldots, y^{n-1}$ ,  $y^n = t$  be the coordinates on C (since the hyperplanes t = constant are spacelike, there is no loss of generality to take  $y^n = t$  as one of the independent variables on C), the system (2.21) can be written on the following form on C

$$(\sigma_{\mu}^{\nu})_{t} + \sum_{i=1}^{n-1} d_{i}^{\nu} (\sigma_{\mu}^{\nu})_{y^{i}} + \sum_{i=1}^{r} \sum_{j=1}^{q^{\alpha_{i}}} e_{j\mu}^{i\nu} \sigma_{j}^{i} = 0 \qquad (3.14)$$

$$\nu = 1, \dots, r \& \mu = 1, \dots, q^{\alpha_{\nu}}$$

where the coefficients  $d_i^{\nu}$ ,  $e_{j\mu}^{i\nu}$  are functions of t,  $y^1, \ldots, y^{n-1}$ . The functions corresponding to  $\Omega^{\alpha}$  are

$$\Omega^{*\nu} = \sum_{i=1}^{n-1} \xi^{i} d_{i}^{\nu}, \nu = 1, \dots, r \qquad (3.15)$$

and the eigenvectors corresponding to  $r^{\alpha\beta}$  and  $l^{\alpha\beta}$  are simply the unitvectors

we can therefore apply the results we have found to far. All the system of transport souchions (2.2) is really coupled and contains one interendent variable for that the original problem (because of the restriction to 0), the problem we started out study is considerably timpler that the problem we started out the The assumptions in section is an trivially seen to be satisfied, and the functions corresponding to  $\Omega^{ch}$  and the elecriset, if we let  $y^{1} \dots y^{n-1} \dots y^{n} = t$  be the coordinates on the less of generality to take  $y^{n} = t$  as one of the independenne less of generality to take  $y^{n} = t$  as one of the independention of the system (2.21) can be written on the following form on C

$$[o_{1}^{v})_{t} + \sum_{k=1}^{n-1} a_{k}^{v} (o_{k}^{v})_{t} + \sum_{k=1}^{n} \sum_{j=1}^{n} a_{jk}^{t} o_{j}^{t} = 0$$

$$v = 1, \dots, r \in k = 1, \dots, c^{n_{k}}$$

where the coefficients  $d_1^v$ ,  $e_{3\mu}^{1v}$  are functions of  $b_1^v$ ,  $e_{3\mu}^{1v}$ ,  $e_{3\mu}^v$ ,  $e_{3\mu}^{n-1}$ . The functions corresponding to  $R^0$  are

$$(ar.z) = \sum_{i=1}^{N} a_i^x a_i^x, \quad r = r, \quad ..., r = v \quad (3.15)$$

and the eigenvectors corresponding to  $e^{\alpha\beta}$  and  $i^{\alpha\beta}$  are simply the unitvectors

$$r^{*\nu\mu} = 1^{*\nu\mu} = \left\{ \delta_{\mu 1}^{\nu 1}, \dots, \delta_{\mu q}^{\nu r} \right\}$$
(3.16)  
$$v = 1, \dots, r \quad \& \quad \mu = 1, \dots, q^{\alpha_{\nu}}$$

In general some of the functions (3.15) may be identical; for instance will all of them be identical when (3.5) is satisfied. If this is the case, one would have to renumber the functions  $\Omega^{*\nu}$  and the eigenvectors  $r^{*\nu\mu} = 1^{*\nu\mu}$  in order to get the assumption corresponding to (1.8) satisfied. Obviously this would complicate the notations, we shall therefore for simplicity restrict our study to the case where the hyperbolic system (1.1) has bicharacteristics of constant multiplicity, since such problems cannot arise in that case. At the end of this section we shall make a few comments on what the differences may be in the general case.

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Now, if we study the cases where the hyperbolic system (1.1) has bicharacteristics of constant multiplicity, we know that all the assumptions in section 1 are satisfied for the system of equations (3.14). We can therefore apply the same procedure to (3.14) as we did in section 2 to (1.1) when we wanted to study the propagation of discontinuities. Again, we get that the discontinuities propagate along the characteristic hypersurfaces. If we pull any characteristic surface for (3.14) back to the t,x-space, we get an n-1 dimensional submanifold of C which is generated by a n-2 parameter family of bicharacteristic curves. If we restrict ourselves to characteristic hypersurfaces for (3.14) which satisfy conditions analogous to those imposed on C on page 10, the transport equations will be of the same type

(酒和, 图)

 $w^{\mu} V^{\mu} = \mathbf{1}^{*} V^{\mu} = \left\{ b_{\mu 1}^{*}, \dots, b_{\mu q}^{*} \right\},$ 

in general same of the functions (2.15) may be identical; for instance will all of them be identical them (3.5) is solicited if this is the case, one would note to require the functions of and the eigenvectors and the to require the functions addingetime corrected and (1.8) activited. Overlaphy talk whether on redy to the case boars the typerbolic events (1.1) activity and in that case to are not all therefore for simplicity council and the case boars of a the typerbolic events addingetime to be case boarse the typerbolic events and the section and the case boars of a the section and the council and the total of the the case boarse the typerbolic events and the section and the case boarse the typerbolic events and the section and the case the section and the section and the case of the company of the the the difference of the the section and the section and the case of the case boarse the typerbolic events and the section and the case of the difference of the the section and the case of

as (2.21) but now the number of independent variables are reduced to n-1 essentially. The number of equations in this system of transport equations will depend on the multiplicity of the characteristic hypersurface considered for (3.14). When n > 2 the system (3.14) has characteristics of nonuniform multiplicity since it is weakly coupled, the number of equations in the system of transport equations for (3.14) may therefore be difficult to tell a priori.

However, if we apply the same procedure over and over again, i.e. find the transport equations for the transport equations for the transport equations etc. for (1.1), it is clear that sooner or later (i.e. after at most n steps) we will arrive at a stage where these transport equations are of the type (2.21) with r = 1.<sup>\*)</sup> As we saw in the beginning of this section, these transport equations will therefore be equivalent to a system of ordinary differential equations of the form (3.13) with r = 1. As a result of this we can say that when the hyperbolic system (1.1) has bicharacteristics of constant multiplicity, then the discontinuities of sufficiently high order (i.e. the discontinuities of the discontinuities etc., sufficiently many times) will always propagate along the bicharacteristics and be governed by (3.13) with r = 1. In general we do not know a priori the lowest order of the discontinuities that propagate along the bicharacteristics

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<sup>\*)</sup> At each step we have to restrict ourselves to characteristic hypersurfaces which satisfy conditions analogous to those imposed on C on page 10. This will normally require that we restrict ourselves to a sufficiently small neighbourhood of  $x_0, t_0$ . To obtain the global behaviour of the discontinuity functions at each step, we have to apply the construction described in section 4.

The  $(2, 2^{+})$ : that now the number of Smargentieur variables are solved; to an-i ensemblishly. The number of solutions is these evelope of creasure the quattions will depend on the solution is these evelope densities the propersurface considered for (2, 14). then  $n > 2^{+}$  desystem (2, 14) has constituted for (2, 14). then  $n > 2^{+}$  desystem (2, 14) has constituted or require from (2, 14). the  $n > 2^{+}$  determine the matrice constituted of action of require the nthat is weakly complete, the matter of action in the solution ntransport operation (5, 14), may there are n = 0 defined on  $2^{+}$  dematrix (2, 14) is a constitute of action of action  $n > 2^{+}$  determine (2, 14) has constituted at the matter (2, 14) of (2, 14).

However, if we apply the sets provider outer 5% even 25% i.e. i.e. find the transport equations etc. for the transport equations etc. for (1.1), it is clear that scontrained the transport equations etc. for (1.1), it is clear that scontrain for or later (1.2, after at most in stops) we will arrive at a stop or later (1.2, after at most in stops) we will arrive at a stop  $e^{-1}$ . As we set in the beginning of this section, these even equations etc.  $e^{-1}$  has we set in the beginning of this section, these even equations will arrive at a stop aport equations will therefore to contrain (2.2) with end of the equations of the tops (2.2) with end of the equations of the tops (2.2) with end of the equations of the tops (2.2) with end of the equations of the tops (2.2) with end of the tops of the discontinuities of constant multiplicity, then the discontinuities of extincted the end of the enverted by (2.1) the discontinuities of the bicherocorrected stops and by (3.1) with end of the discontinuities of the bicherocorrected stops (2.1) with end of the discontinuities of the bicherocorrected stops (3.2) with end of the discontinuities of the discontinuities are another (1.2) and (2.2) with end of the discontinuities of the discontinuities of the discontinuities are another (1.2) are and by (3.2) with end of the discontinuities of the discontinuities are do not cause a priori the isometeristics of the discontinuities of the discontinuities are do not cause a priori the isometeristice of the discontinuities of the discontinuities of the discontinuities are do not cause a priori the isometeristice of the discontinuities of the discontinuities that propagate along the bicherocorrecteristics are to be discontinuities of the discontinuities of the discontinuities are do not cause a priori the isometeristice of the discontinuities of the discontinuities that propagate along the bicherocorrecteristice are to be bicherocorrecteristice are to be bicherocorrecteristice are to be bicherocorecteristice are to be bicherocorecteri

At each step we have to restrict ourselves to therefore the hypersarisces which satisfy conditions analogous to those imposed on 0 on eage 10. This will normally require that we restrict ourselves to a sufficiently small notphourhood of the to the obtain the global behaviour of the discontinuity functions at grad stop, we have to apoly the construction described in section .

. (1.3 ....

except in the case where the hyperbolic system (1.1) has characteristics of constant multiplicity, in this case the discontinuities of all orders propagate along the bicharacteristics. Finally we note that in general the lowest order of the discontinuities that propagate along the bicharacteristics depends on the normals of the characteristic and subcharacteristic manifolds, and may also vary from point to point in x,t-space.

In the general case where the hyperbolic system (1.1) does not have bicharacteristics of constant multiplicity, the situation may be much more complicated than above. However, in the non-pathological cases one can also here apply the technique which we are going to describe in the next section, to glue the results together. In short, we can describe the situation as follows: The discontinuities of sufficiently high order will propagate along the bicharacteristics, and locally the transport equations will be of the form (3.13). However, in general we will not have r = 1 in (3.13), but we will have that r may vary from point to point on C.

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In the control the where the specified of (1.1) and not have bicknesseriates of constant which this is is. ation may be also accordance is and there is the issue non-pathological case on an allohant which the technique to are going of control to be and the second of the technic to discontation of an electron as the second of the technicus for discontation of an electron high order will propagate atoms the bicknesseriation and locally of brunssort equation will be a second of the second of the second of tave to point of the second of the order, is general we will not have to point on the second of the second of the second of the second to point on the second of the second of the second of the second to point on the second of the second

## 4. DISCONTINUOUS INITIALVALUE PROBLEMS.

We shall now consider initialvalue problems for the hyperbolic system (1.1). The initialvalues considered

$$u\Big|_{t=t_0} = u_0(x) \tag{4.1}$$

are assumed to be piece-wise smooth functions. Thus  $u_0$  is assumed to be smooth everywhere except on a finite number of smooth hypersurfaces; the jumpdiscontinuities of  $u_0$  and the derivatives of  $u_0$  are assumed to be piece-wise smooth functions on these hypersurfaces.

Since we assume that the initialvalue problem is well-posed when the initial values are in  $C^{N_O}$  for some N , it suffices to study how the initial discontinuities of u and its derivatives up to the order No propagate. In fact, if this is known the discontinuous initialvalue problem (1.1) & (4.1) can be solved by another initialvalue problem with CNO initialvalues (for the details on this, see [3]). Furthermore, it suffices to consider the case where u is smooth everywhere except on one n-1 dimensional smooth manifold F. In fact, if u for instance is discontinuous along two crossing manifolds, the discontinuity of u is smooth everywhere on these n-1 dimensional manifolds except on one n-2 dimensional submanifold. Thus the initialvalue problem for the discontinuity of the discontinuity function of u is of the above type in view of the considerations in the preceding section. If this problem is solved first, the initial value problem for the discontinuities of u and its derivatives

## L. DESCURPTINENTS INTUALVALUE. PROBLEMS.

We shall now consider initialvance problems for the appened bolic system (1.1). The initial values considered

 $(x, y) = u_0(x)$ 

are assumed to be plece-wise enough functions (the unbit of anoth essumed to be anoth everyoners except on a finite outfor of anoth hypersurfaces; the jumpdiceendinalities of u and the dorivatives of u are assumed to be plece-wise anoth functions on theory baresarfaces.

Since we assume that the initial value problem is well-booked when the initial man are to  $C^{2}$  for even N<sub>0</sub>, it suffices to ether how the initial discontinuities of a and its derivatives on to the order N<sub>0</sub> properate. In each, if this is known the ediscontinuous initial value problem (0.1) is (0.1) can be solved by another initial value problem with  $C^{20}$  initial values (for the derivative the case where V<sub>0</sub> is smooth overphere except on QMO 0.1 the case where V<sub>0</sub> is smooth overphere except on QMO 0.1 the case where V<sub>0</sub> is smooth overphere except on QMO 0.1 the case where V<sub>0</sub> is smooth overphere except on QMO 0.1 the case where V<sub>0</sub> is smooth overphere except on QMO 0.1 the case where V<sub>0</sub> is smooth overphere except on QMO 0.1 the case where V<sub>0</sub> is smooth overphere except on QMO 0.1 the case where V<sub>0</sub> is smooth exception. The discussion of V<sub>0</sub> is mooth everphere on these 1.1 v<sub>0</sub> for instance is discontinuous along two crossing manifolds. The discussion of V<sub>0</sub> is mooth everphere on these 1.1 v<sub>0</sub> for instance walve problem for the discontinuity of the considerations in the value problem for the discontinuity of the discontinuity function of V<sub>0</sub> is of the above type in view of the considerations in the presenting excitent. If this problem is actived first, the initial can by the same construction as referred to above be transferred to problems where the discontinuities of  $u_0$  and its derivatives are smooth on a n-1 dimensional manifold. This corresponds to cases where  $u_0$  is smooth everywhere except on one n-1 dimensional manifold. - If several discontinuity manifolds have a submanifold in common we would have to start the construction by considering the discontinuity of sufficiently high order and then successively solve the problems for the lower order discontinuities. Since the discontinuities of sufficiently high order always propagate along the bicharacteristics, the construction will consist of a finite number of steps.

We have thus reduced the problem to the problem of finding out how the initial discontinuities along  $\Gamma$  of u and the derivatives of u propagate. In the following we shall restrict ourselves to the study of how the discontinuities of u itself propagate. The discussion of the propagation of the discontinuities of the derivaties of u is completely analogous and is therefore omitted (see [3] for the construction in the case of symmetric hyperbolic systems with characteristics of constant multiplicity).

In section 2 we found that discontinuities can only propagate along the characteristics. Since the discontinuities are initially located on  $\Gamma$ , we can therefore conclude that the discontinuities must be located on the characteristics going through  $\Gamma$ . If the hyperbolic system (1.1) has characteristics of constant multiplicity, there are exactly  $\gamma$  different characteristic manifolds going through  $\Gamma$ , namely one for each of the characteristic partial differential equations (2.9). In the general case, however,

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ear by the same continuent of a continuent as of up and to contain a to problem there the discontinuent as of up and the derivatives are shooth on a set discontinuent area of up and the set of the control discontinuent of a continuent area of up and the set of the control discontinuent of a continuent area of up and the set of the stated where up is around a control of a set of the control discontinuent stated where the discontinuent of and the lower of the optic and the discontrol discontinuent of the problem for the lower of a set of the states whe discontinue to the lower of a set of the lower of the discontinuent discontrol discontinuent of a set of the control discontinuent states and the problem for the lower of a set of the lower of the discontrol discontinue to a set of the control discontinuent states along the biohermologication of the control discontinuent states and the problem for the control discontinuent states along the biohermologication of the control discontinuent states and the problem of the control discontinuent states and the problem of the control discontinuent states along the biohermologication of the construction will state state of a finite curves of the control discontinue to a state of the construction will state state of a finite curves of the control discontinue to a state of the construction will state state of a finite curves of the construction of the construction of the construction of the state of the construction of the curves of th

We have this reduced the problem to the problem of finding out how the initial discontinuities along of a shall testify derretive of a propagate. In the following we shall testify her actives to the study of some the discontinuities of a steep propagate. The discontinuities propagation of the discontinuities of the derivaties of a te completely analogous and to the description diffice (dec [3] for the construction in the second symmetries

In section 2 we found that discontinuities cap only propagate along the characteristics. Since the discontinuities are initially recated on 1, we can therefore constate that the discertinitian area be located on the characteristics point through F. If the hyperbolic system (1.1) has characteristics of constant multiplic dive there are exactly of different characteristics of constants (0.1) has for any the for and the distribute point (0.1) has the for and the exactly of different characteristics of constants (0.1) has the for and the point characteristics of constants (1.1) has characteristics of constants and the different characteristics of the constants of (0.1) has the for and the point of the characteristic point (1.1) has the formation (2.1). In the seneral cose, by ever that differential equations (2.1). In the seneral cose, by ever the point (1.1) has the formation (2.1). there may be a lot more characteristic manifold going through  $\Gamma$ . In fact, a characteristic manifold may in this case satisfy different characteristic equations (2.9) in different regions. Globally there may therefore be an infinite set of characteristic manifolds going through  $\Gamma$ . In the following we shall study what happens locally and briefly indicate the global aspects.

Let  $C^{(\alpha)}$  be the characteristic manifold satisfying (2.9) for the index  $\alpha$  and going through  $\Gamma$ ,  $\alpha = 1, \ldots, \gamma$  (some of the characteristics  $C^{(\alpha)}$  may partially or completely be equal). We let  $[]^{\alpha}$ ,  $[]^{\Gamma}$  denote jumps across  $C^{(\alpha)}$  and  $\Gamma$ respectively. Since the eigenvectors  $r^{\alpha\beta}$  form a complete set, the jumps in the initial values of u across  $\Gamma$  have a unique decomposition

$$[u_{o}]^{\Gamma} = \sum_{a=1}^{\gamma} \sum_{\beta=1}^{q^{a}} \sigma_{o\beta}^{a} r^{a\beta}$$

In the same way we may set

$$[u]^{\alpha} = \sum_{a=1}^{\gamma} \sum_{\beta=1}^{q^{a}} \sigma_{\alpha\beta}^{a} r^{a\beta}$$

(4.3)

(4.2)

The initial conditions are

$$\sum_{x=1}^{\gamma} [u]^{\alpha} = [u_0]^{\Gamma} \text{ on } \Gamma$$
 (4.4)

Multiplication by  $l^{\mu\nu}$  gives

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 $(a,b) = \frac{1}{2} \left[ \left( a^{0} \right) \right]_{T} = \left[ \left( a^{0}$ 

$$\sum_{\alpha=1}^{\gamma} \sigma_{\alpha\nu}^{\mu} = \sigma_{\sigma\nu}^{\mu} \quad \text{on} \quad \Gamma$$
 (4.5)

From the discussion in section 2 we know that in every point on  $C^{(\alpha)}$  where  $C^{(\alpha)}$  only satisfies one of the equations (2.9),  $\sigma_{\alpha\beta}^{a} = 0$  if  $a \neq \alpha$ . We may without loss of generality assume that on  $\Gamma$  this is true everywhere, since we shall see that we are then led to a well defined construction of how the discontinuities propagate.

Now let  $x_0, t_0$  be an arbitrary point on  $\Gamma$ . We want to study how the discontinuities in the neighbourhood of  $x_0, t_0$ propagate along one of the characteristic manifolds  $C^{(\alpha)}$ , say  $C^{(\alpha_1)}$ , near  $x_0, t_0$ . By definition  $C^{(\alpha_1)}$  satisfies (2.9) for  $\alpha = \alpha_1$ . If  $C^{(\alpha_1)}$  satisfies (2.9) at  $x_0, t_0$  for  $\alpha = \alpha_1$  only, then as we saw in section 2, the propagation of the discontinuities in a neighbourhood of  $x_0, t_0$  on  $C^{(\alpha_1)}$  is governed by the transport equations found in section 2. So in this case everything is nice, the discontinuities are propagated along the bicharacteristics and are described by the system of ordinary differential equations (3.13) with r=1 as we saw in section 3. The initial conditions for the equations (3.13) are found from the initial conditions given above.

The case above is the "normal" case in the sense that this is the case most frequently met in applications. In general, however,  $c^{(\alpha_1)}$  may satisfy (2.9) at  $x_0, t_0$  for one or several a's different from  $\alpha_1$ . Assume therefore that  $c^{(\alpha_1)}$  satisfies

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Trees the discussion in section 2 we know that in every point on q(3) where q(4) one of the every point on q(3) (a) where q(4) one of the exection (2.9)  $q_{13}$  and  $q_{23}$  and  $q_{33}$  and  $q_{34}$  are exected on q(3, 2, 3) and  $q_{34}$  are exected on  $q_{34}$  and  $q_{34}$  are exected (2.9)  $q_{34}$  and  $q_{34}$  are exected on  $q_{34}$  are exected are exected on  $q_{34}$  are exected on  $q_{34}$ .

How let  $x_{0}, x_{0}$  be an ifficienty point on  $1^{\circ}$ , ho want to study how the discenticulies is sin ack manned of  $x_{0}$ , and  $f^{(1)}$ , acar  $x_{0}, x_{0}$ . By definition  $f^{(0)}$ , setting  $f^{(0)}$ , and  $f^{(1)}$ , acar  $x_{0}, x_{0}$ . By definition  $f^{(0)}$ , setting  $f^{(0)}$ , and  $f^{(1)}$ , acar  $x_{0}, x_{0}$ . By definition  $f^{(0)}$ , setting  $f^{(0)}$ , and  $f^{(1)}$ , acar  $x_{0}, x_{0}$ . By definition  $f^{(0)}$ , setting  $f^{(0)}$ , and  $f^{(1)}$ , at a set  $f^{(1)}$  satisfies  $f^{(1)}$ , setting  $f^{(1)}$ , and  $f^{(1)}$ . These we are the control of  $x_{0}$  to  $f^{(1)}$ , and  $f^{(2)}$ , is these we are the control of  $x_{0}$  on  $f^{(1)}$ . The case events the in a interval in control of  $x_{0}$  on  $f^{(2)}$ . The case events the field in a interval of  $f^{(2)}$ ,  $f^{(2)}$ , thing is interval the difference of  $f^{(2)}$ ,  $f^{(2)}$ ,  $f^{(2)}$ ,  $f^{(2)}$ ,  $f^{(2)}$ , there events is the discontrol is a set properiod of  $f^{(2)}$ , there events is the discontrol is a set properiod of  $f^{(2)}$ , there even is the discontrol is a set of  $f^{(2)}$ ,  $f^{(2)}$  with  $f^{(2)}$  is the order of  $f^{(2)}$ , there events is the distribution  $f^{(2)}$ ,  $f^{(2)}$ , there events is the distribution  $f^{(2)}$ ,  $f^{(2)}$ ,  $f^{(2)}$ ,  $f^{(2)}$ ,  $f^{(2)}$ ,  $f^{(2)}$ ,  $f^{(2)}$ , there events is the distribution  $f^{(2)}$ ,  $f^{(2)}$ 

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(2.9) at  $x_0, t_0$  for  $\alpha_1, \ldots, \alpha_r$ . By continuity there is then a neighbourhood of  $x_0, t_0$  where  $C^{(\alpha_1)}$  does not satisfy (2.9) for any <u>other</u> choice of  $\alpha$  than  $\alpha_1, \ldots, \alpha_r$ . We may without loss of generality assume that the equation for  $C^{(\alpha_1)}$  near  $x_0, t_0$  is given by

$$\varphi(x,t) \equiv \psi(x) - t = 0$$
 (4.6)

Consider now the function

$$\Omega(\mathbf{x}) \stackrel{\text{def}}{=} \Omega^{\alpha_{1}}(\mathbf{x}, \psi(\mathbf{x}), \psi_{\mathbf{x}'}(\mathbf{x}), \dots, \psi_{\mathbf{x}''}(\mathbf{x})) - \Omega^{\alpha_{2}}(\mathbf{x}, \psi(\mathbf{x}), \psi_{\mathbf{x}'}(\mathbf{x}), \dots, \psi_{\mathbf{x}''}(\mathbf{x}))$$
(4.7)

From the above assumptions we see that  $\Omega(x_0) = 0$ , furthermore we see that

$$\Omega(\mathbf{x}) = 0 \tag{4.8}$$

is the equation for the points on  $C^{(\alpha_1)}$  where  $C^{(\alpha_1)}$  satisfies (2.9) for  $\alpha_1$  and  $\alpha_2$  in the neighbourhood of  $x_0, t_0$ . If  $\frac{\partial}{\partial x}\Omega(x_0) \neq 0$ , which is the normal case, the solution of (4.8) is an n-1 dimensional manifold going through  $x_0$ , which defines an n-1 dimensional manifold S on  $C^{(\alpha_1)}$ . In a neighbourhood of  $x_0, t_0, C^{(\alpha_1)}$  satisfies in this case (2.9) for  $\alpha = \alpha_2$ only on this manifold S going through  $x_0, t_0$ . If  $\frac{\partial}{\partial x}\Omega(x_0) = 0$ the situation is much more complicated. In this case the function  $\Omega(x)$  may at  $x = x_0$  either ne dhe contra da contra da

$$(0, -1) = (x, x) + (x, -1)$$

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$$\gamma(\mathbf{x}) = 0 \tag{4.8}$$

- (a) have an extremum,
- (b) have a saddlepoint,
- (c) = 0 in a neighbourhood of  $x_0$ , or
- (d) be "pathological" in a neighbourhood of  $x_0$ .

In (a) we shall by an extremum mean that there is a neighbourhood of x, where  $\Omega(x) \neq 0$  everywhere except at  $x = x_0$ . In this case there is a neighbourhood of  $x_0, t_0$  on  $C^{(\alpha_1)}$ , where  $c^{(\alpha_1)}$  satisfies (2.9) for  $\alpha = \alpha_2$  only at  $x_0, t_0$ . In (b) we shall by a saddlepoint mean that there is a finite number of manifolds, each of dimension at most n-1 and containing  $x_0$ , such that (4.8) is satisfied everywhere on these manifolds in a neighbourhood of  $x_0$ , and furthermore that in each of the open subregions (we assume that the number of such regions is finite) which these submanifolds divide the neighbourhood of x into, either  $\Omega \equiv 0$  or  $\Omega \neq 0$  everywhere. In this case  $C^{(\alpha_1)}$  satisfies (2.9) for  $\alpha = \alpha_2$  only on a finite set of submanifolds of  $C^{(\alpha_1)}$  of dimension at most n-1 and going through  $x_0, t_0$ , and  $(\alpha_1)^{(\alpha_1)}$ in a finite (possibly empty) set of sectors on  $c^{(\alpha_1)}$ going out from  $x_0, t_0$ . By a sector going out from  $x_0, t_0$  we here mean a region bounded by a finite set of n-1 dimensional manifolds all containing  $x_0, t_0$ . In case (c)  $C^{(\alpha_1)}$  satisfies (2.9) for  $\alpha = \alpha_2$  in addition to  $\alpha = \alpha_1$  everywhere in a neighbourhood of  $x_0, t_0$ . Case (d) is by definition all cases which are not contained in (a), (b) or (c). In this case we see that x may for instance be an accumulation point for at least one sequence of points, all satisfying (4.8), and such that this sequence of points does not belong to a finite number of connected manifolds where (4.8) is everywhere satisfied. A simple example of the case (d) is given by the function

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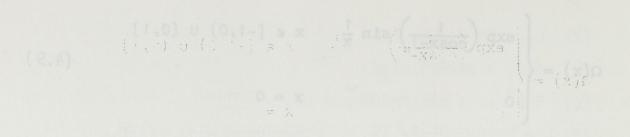
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$$\Omega(x) = \begin{cases} \exp\left(\frac{1}{\cos x - 1}\right) \sin \frac{1}{x} & x \in [-1, 0) \cup (0, 1] \\ 0 & x = 0 \end{cases}$$
(4.9)

 $x_0 = 0$  is here an accumulation point of the type described above. We are not able to treat case (d) in full generality, and we have not been able to find simple criteria on the coefficients in (1.1) to avoid these cases when (1.1) does not have characteristics of constant multiplicity.

We find the points where  $C^{(\alpha_1)}$  in a neighbourhood of  $x_0, t_0$ satisfies (2.9) for  $\alpha = \alpha_3, \ldots, \alpha_r$  by comparing  $\Omega^{\alpha_i}$ ,  $i = 3, \ldots, r$  with  $\Omega^{\alpha_i}$  in the same way as we did above with  $\Omega^{\alpha_2}$ . Thus we may conclude that there is a neighbourhood of  $x_0, t_0$ on  $C^{(\alpha_1)}$  where  $C^{(\alpha_1)}$  is a multiple characteristic on a point set which is a finite union of sets of the above types. If we exclude the pathological possibilities from our discussion, we can therefore summarize the above in the following way: If  $C^{(\alpha_1)}$ satisfies (2.9) at  $x_0, t_0$  for  $\alpha = \alpha_1, \ldots, \alpha_r$  then there is a neighbourhood of  $x_0, t_0$  on  $C^{(\alpha_1)}$  which can be divided into a finite number of subregions with the property that all interior points are of the type we considered in section 2, when we were able to find the transport equations.

From the above discussion we see that locally we know the transport equations for the propagation of discontinuities everywhere on  $c^{(\alpha_1)}$ , except possibly on a finite set of submanifolds of  $c^{(\alpha_1)}$  of dimension at most n-1. It should be clear that those of these exceptional submanifolds which have the property



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int officering randport of the contract of the int as a mboy, discussion is a shall radius we have a set of a contract officer of the contract and a contract of the contract officer of the contractor and it as a contract of the contract of the contract is a shall be contract of the contract of the interval dimension of the contract of the contract is a shall be contract of the contract of the interval dimension of the contract of the contract is a shall be contract of the contract of the interval of the contract of the contract of the interval of the contract of the contract of the contract of the interval of the contract of the contract of the contract of the interval of the contract of the contract of the contract of the interval of the contract of the contract of the contract of the interval of the contract of the contract of the contract of the interval of the contract of the contract of the contract of the interval of the contract of the contract of the contract of the interval of the contract of the contract of the contract of the interval of the contract of the contract of the contract of the contract of the interval of the contract of the contract of the contract of the contract of the interval of the contract of the contract of the contract of the contract of the interval of the contract of the interval of the contract of that the multiplicity of  $C^{(\alpha_1)}$  is the same \*) everywhere in a neighbourhood of the submanifold on  $C^{(\alpha_1)}$  (except on the submanifold itself), cannot affect the propagation of the discontinuities on  $C^{(\alpha_1)}$  because of the continuity properties the discontinuities necessarily must have. In particular the exceptional manifolds of dimension at most n-2 are all of this type. In exactly the same way we see that those pathological cases which involve a countable set of submanifolds of the above type, such that this set of submanifolds has a finite number of submanifolds as accumulation points, cannot affect the propagation of the discontinuities.

We have thus seen that except in the "most" pathological cases, the exceptional submanifolds which <u>can</u> affect the propagation of the discontinuities, are those n-1 dimensional exceptional submanifolds which are such that  $C^{(\alpha_1)}$  has a different multiplicity on either side of the submanifolds. That  $C^{(\alpha_1)}$ has a different multiplicity on either side of an n-1 dimensional submanifold, means that there is an  $\alpha$  such that  $C^{(\alpha_1)}$ satisfies (2.9) for this  $\alpha$  on one side of the submanifold and not on the other. We shall call this type of n-1 dimensional exceptional submanifolds on  $C^{(\alpha_1)}$  <u>multiplicitychange-manifolds</u>, and we shall now see that such manifolds really affect the propagation of discontinuities in general.

Consider now an isolated multiplicitychange-manifold on  $C^{(\alpha_1)}$ . We shall restrict our study to the case where the change in multi-

<sup>\*)</sup> We say that the multiplicity of a characteristic manifold is the same at two different points, if the characteristic manifold satisfies (2.9) for exactly the same indices  $\alpha$  at the two points.

Spect (see matrix) to the maximization (1, 1) is the ender (1, 2) where to an ender the maximization (1, 2) is the ender (1, 2) where the maximization (1, 2) is the ender (1, 2) is the

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<sup>(</sup>i) the any track the multiplication of a second second second contract of a concentration second contract, contract when the second second contract of a second second second contract of a memory of the second second contract of a second second second contract of a memory of the second contract of a memory of the second second

plicity of  $C^{(\alpha_1)}$  is exactly one, i.e. we shall suppose that there is one and only one  $\alpha$  such that  $C^{(\alpha_1)}$  satisfies (2.9) for this  $\alpha$  on one side of the multiplicitychange-manifold and not on the other. The general case is then an easy extension of this special case, and will be left to the reader. It is natural to divide the multiplicitychange-manifold into three disjoint sets, namely the sets where  $C^{(\alpha_1)}$  1) loses multiplicity, 2) gains multiplicity, 3) neither loses nor gains multiplicity. We define these concepts in the following way: consider a point on the multiplicitychange-manifold, and consider the bicharacteristic direction for increasing t associated with the equation (2.9) for the exceptional  $\alpha$  at that point. If this bicharacteristic direction is tangent to the multiplicitychangemanifold, we say that  $c^{(\alpha_1)}$  neither loses nor gains multiplicity at that point. If the bicharacteristic direction is pointing into the region of  $c^{(\alpha_1)}$  where  $c^{(\alpha_1)}$  satisfies (2.9) for the exceptional  $\alpha$ , we say that  $C^{(\alpha_1)}$  gains multiplicity at that point. Finally, if the bicharacteristic direction is pointing out of the region of  $c^{(\alpha_1)}$  where  $c^{(\alpha_1)}$  satisfies (2.9) for the exceptional  $\alpha$ , we say that  $c^{(\alpha_1)}$  loses multiplicity at that point.

To be able to get a finite process in the following construction, we are also here forced to exclude some "pathological" cases. Namely, we shall assume that locally each of the three types of sets defined above on the multiplicitychange-manifold consists of a finite number of connected sets. Then we may consider each of the three types of pointsets separately, and afterwards glue the results together.

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Let now  $E^{(\alpha)}$  denote the characteristic manifold which is going through the multiplicitychange-manifold, and which satisfies (2.9) for the exceptional  $\alpha$ . From the assumptions we have made above, we see that  $E^{(\alpha)}$  is identical with  $C^{(\alpha_1)}$  on one side of the multiplicitychange-manifold, while  $E^{(\alpha)}$  on the other side of the multiplicitychange-manifold satisfies (2.9) only for the exceptional  $\alpha$  (at least locally) if we assume that  $C^{(\alpha_1)}$  satisfies (2.9) for no more indices at the multiplicitychange-manifold than in a neighbourhood of this manifold.

From the discussion in section 2 we see that the characteristic manifold  $E^{(\alpha)}$  is a possible carrier of discontinuities. In fact, if  $c^{(\alpha_1)}$  loses multiplicity everywhere along the multiplicitychange-manifold, there will be a "branching" of the propagation of the discontinuities there; that part of the discontinuity which is associated with the eigenvectors  $r^{\alpha\beta}$ , for the exceptional  $\alpha$ , will follow the characteristic manifold  $E^{(\alpha)}$ , while the rest will continue to follow  $C^{(\alpha_1)}$ . We know the transport equations everywhere on these manifolds except on the multiplicitychange-manifold, but there the continuity properties of the discontinuities solve the problem. On the other hand, if  $c^{(\alpha_1)}$  gains multiplicity everywhere along the multiplicitychangemanifold, the opposite can happen. Namely, if both  $c^{(\alpha_1)}$  and  $E^{(\alpha)}$  carry discontinuities before they run together in the multiplicitychange-manifold, the discontinuities will propagate along a single manifold (namely  $c^{(\alpha_1)}$  and  $E^{(\alpha)}$  which are identical) on the other side of the multiplicitychange-manifold. Here also the transport equations are known everywhere except on the multiplicitychange-manifold, but there again the continuity solves the problem.

 $B^{(0)}$  is obe contacted to be which the balance of balance

Finally, we consider what happens at the points on the multiplicitychange-manifold where  $C^{(\alpha_1)}$  neither loses nor gains multiplicity. If these pointsets locally are contained in a finite set of n-2 dimensional submanifolds of  $C^{(\alpha_1)}$ , the continuity properties of the discontinuities solve the problem. If on the other hand the multiplicitychange-manifold everywhere is such that  $C^{(\alpha_1)}$  neither loses nor gains multiplicity, then the situation is entirely different. In fact,  $E^{(\alpha)}$  is then not uniquely determined by  $C^{(\alpha_1)}$ , but in view of the results found in section 3 we know that there is no coupling between the discontinuities carried by  $E^{(\alpha)}$  on either side of the multiplicitychange-manifold, and no "information" is carried over this manifold on  $E^{(\alpha)}$ .

Using the above local results and the continuity properties of the discontinuities, it will in principle be possible to glue the results together, and find out how the discontinuities propagate up to the nearest caustic. In general caustics will exist due to the focussing effects (blow up of  $\varphi_{ziz}^{o}$ , j), so the discontinuous solution will in general not exist globally as a piece-wise smooth function. Exceptions to this are the weakly coupled hyperbolic systems, since focussing phenomena do not occur for such systems. The focussing effect will be discussed in a later work on stability for hyperbolic systems. -Handler and the hard hard and the points on the multi-(difficult constants and there  $C^{(2+)}$  methods into the parity and the set of the point and issolly are constants for a continue set of the constant inters of  $C^{(2+)}$ , the continue set of the constant inters of  $C^{(2+)}$ , the continue set of the constant inters of  $C^{(2+)}$ , the are also also also the restriction of the constant inters of the the set of the restriction of the constant inters of the point is the set of the restriction of the set of the constant inters intertion of the set of the restriction of the restriction of the restriction of the set of the restriction of the restriction of the restriction is the set of the restriction of the restriction of the restriction of the section of the restriction of the restriction of the restriction found in section of the restriction of the re

(attached and the above (see all the and the book four ty proportion of the discontinuities, it will in privation to discontinuities to give the real to be been and first out for the discontinuities will eccl and to the focusing affect blow to of one i custor will eccl de to the focusing affect blow to of one i, to the discontinues solution will in several and of cole (solution will eccl been-wise should be been affect blow to of the discontinuity as a continue solution will in several and the cole (solution the cole been-wise should be been affect blow to of the discontinuity as a couched by probable because allow to be the cole (solution the cole for suc the cole of the beached blow to be the cole (solution to solution of the cole of the beached blow to be the cole for suc the cole of the beached blow to be the cole of the work on reaction for the beached blow to be the beached in work on reaction for the beached blow to be the beached in the beached blow to be the beached blow to be the beached in work on reaction for the beached blow to be the beached in the beached blow to be the beached blow to be the beached in the beached blow to be the beached blow to be the beached in the beached blow to be blow t 5. SOME REMARKS.

In the previous sections we have assumed that  $\Omega^{\alpha}$ ,  $r^{\alpha\beta}$ ,  $1^{\alpha\beta}$ are smooth for  $\xi \neq 0$ . As we mentioned in section 1, this assumption will not be satisfied in general. On the one hand the functions  $\Omega^{\alpha}$  may have branchpoints for vectors  $\xi$  other than  $\xi = 0$ , and on the other hand the eigenvectors  $r^{\alpha\beta}$ ,  $1^{\alpha\beta}$  may be discontinuous at points x,t, $\xi$  where the multiplicity of a characteristic root changes, since the dimensions of the eigensubspaces change there. The system of equations in ideal magnetohydrodynamics is an example where such problems arise at the points on a characteristic surface where the magnetic field is either tangent or orthogonal to the surface.

The functions  $\Omega^{\alpha}$  are solutions of the algebraic equation (2.8) with smooth coefficients. Since we assume that the system of equations (1.1) is hyperbolic, the equation (2.8) belongs to a special class of algebraic equations. This author does not know whether there exist any theory for this class of equations with regard to solvability by root extraction, but in any case it seems likely that the smoothness-requirements etc. we have to impose on the functions  $\Omega^{\alpha}$ , essentially will limit our theory to the cases where it is possible to find the functions  $\Omega^{\alpha}$  by extraction of roots. Since the coefficients in the equation (2.8) as well as the roots  $\Omega^{\alpha}$  are all real, it seems likely that the expressions  $\Omega^{\alpha}$  involve square-roots only.

In view of the above, we limit ourselves to cases where the only singularities of the functions  $\Omega^{\alpha}$  are branchpoints for square-roots. Since the coefficients in (2.8) are smooth, we see that at least two different  $\Omega^{\alpha}$ 's become equal at each such branch-

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An when of the shore, we limit surseive to cases where to only at suit these so the inetions of and paints for equations of the resolutions in (e.F) are smooth, we see that at inet two states of a case of a case of the back have brack that at inet two states of a case of a case of the back have brack that at inet two states of a case of the case of the back have brack that at inet the back of the back of the case of the back of the back of the that at inet the back of the back of the case of the back of the back of the that at the back of the back of the case of the back of the back of the the back of the the back of the the back of the the back of the the back of the the back of the the back of the the back of the the back of the the back of the the back of the the back of the back o point. Thus a branchpoint for  $\xi \neq 0$  will always involve a change in multiplicity for the characteristics, hence such branchpoints cannot exist for hyperbolic systems with characteristics of constant multiplicity.

The subset of x,t, $\xi$ -space where either  $1^{\alpha\beta}$ ,  $r^{\alpha\beta}$  are discontinuous or  $\Omega^{\alpha}$  have branchpoints, will in the following be referred to as critical points (note that  $\xi = 0$  is always excluded). We shall for simplicity assume that the multiplicity for every  $\Omega^{\alpha}$  is the same in every connected set of critical points in x,t, $\xi$ -space, and thus that  $1^{\alpha\beta}$ ,  $r^{\alpha\beta}$  are continuous on such sets. In general there may probably exist double critical points, i.e. points in the set of critical points where the multiplicity of  $\Omega^{\alpha}$  changes. The following discussion will essentially also cover such cases, because the critical points of different types are always separated on a characteristic surface by multiplicitychange-manifolds.

The continuity properties of the discontinuities are easily seen to imply that critical points cannot affect the propagation of discontinuities along a characteristic surface C, unless there is a domain on C where every point is critical. By the same arguments as those we used in the preceding section, we see that this domain may be divided by multiplicitychange-manifolds into a finite number of subdomains where the assumptions imposed on C on page 10 are satisfied. We shall now study what happens in each of these subdomains, the discussion of what happens at the multiplicitychange-manifolds will then be completely analoguous to the discussion in the preceding section and will therefore be left to the reader. Thus we consider a critical point  $x_0, t_0$  on C and

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assume that the hypersurface C satisfies (2.9) for  $\alpha = \alpha_1, \dots, \alpha_r$  in a neighbourhood of  $x_0, t_0$ , and that nowhere in this neighbourhood C satisfies (2.9) for any other choice of  $\alpha$ . We see that in the case we now are discussing, we must have  $r \ge 2$ . It is easily checked that the arguments which in section 2 led to the system of transport equations (2.17), also apply in the case we are considering here, while the arguments which led from (2.17) to (2.21) in general do not apply.

In domains of critical points therefore, the system of transport equations (2.17) describe the propagation of discontinuities, but this system will in general <u>not</u> be weakly coupled any longer. In fact, in [5] it is shown that the system of transport equations is a strongly coupled hyperbolic system in special cases. At least when (1.1) is symmetric hyperbolic it is clear that the system of transport equations is hyperbolic, thus the initialvalue problem can be solved for it.

From the above considerations we can now conclude that apart from the fact that the transport equations may become a strongly coupled hyperbolic system, the critical points do not change the picture we have given in the earlier sections of the propagation of discontinuities, in any essential way. The qualitative properties for the propagation of discontinuities is described in section 4, while the quantitative properties are given by the system of transport equations (2.17) (which simplifies to (2.21) at noncritical points). Obviously the construction of a global picture may be very tedious in concrete problems, we shall not go into further details of this here. where a their bite inperduction 0 are determined (2.9) for x = 0, ...,  $a_{i_1}$  is a neighborhood of  $x_0$ ,  $b_{i_2}$ , and the rewriter in units neighbourhood T sectration (2.1) to any differentiation of q. We see the first the case we are discussible, at must leave a = 2. There exists the case we are an ediscussible, at must leave a = 2. There exists the case we are argumenter which in section 2.1 if is a sector of the case  $a_{i_1}$  is a sector  $a_{i_2}$  is a sector  $a_{i_1}$  is the case  $a_{i_2}$  is a sector  $a_{i_1}$  is a sector  $a_{i_2}$  is a sector  $a_{i_2}$  is a sector  $a_{i_1}$  is a sector  $a_{i_2}$  and  $a_{i_2}$  is a sector  $a_{i_1}$  is a sector  $a_{i_1}$  is a sector  $a_{i_2}$  is a sector  $a_{i_1}$  is a sector  $a_{i$ 

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irrentic above conditinations we can now conclude that apart from the Back the tradition equations may becore a through oughted hyperbolic system. The oritical points do not the proparities at discontinuities, in any densitial way. The condition proprotion by while the propagation of discontinuities its ideactibed in section by while the quantifacture proparties are given by the SVtem. I: repropri equations (SIP) (which simplifies to (2.2)) at bem. I: repropri equations (SIP) (which simplifies to (2.2)) at binomary at a postival point of a proparties are given by the SVbinomary at a state of the construction of a global picter. Intriner is a state of the state of the state of the binomary at a state of the state of the state of the state binomary at a state of the state of the state of the state binomary at a state of the state of the state of the state binomary at a state of the state of the state of the state binomary at a state of the state of the state of the state binomary at a state of the state of the state of the state binomary at a state of the state of the state of the state binomary at a state of the state of the state of the state of the binomary at the state of the state of the state of the state of the state binomary at the state of th Finally, we would like to remark that the discussion in this paper seems to be fairly easy to modify to mixed boundary-initialvalue problems for the hyperbolic system (1.1), in the same way as carried out in the cases treated in [3]. Furthermore higher order hyperbolic systems and semilinear hyperbolic systems seem to be fairly easy to study by the same methods which we have used in this paper. We would also like to remark that the discussion in section 3 makes it possible to generalize the WKB method to cover certain hyperbolic systems with characteristics of varying multiplicity. To a certain extent this will be studied in a later work on stability for hyperbolic systems.

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