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Theoretical Investigation of Heat and Streaming generated by high Intensity Ultrasound

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THEORETICAL INVESTIGATION OF HEAT AND STREAMING GENERATED BY HIGH INTENSITY ULTRASOUND

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Abstract

Some nonlinear effects associated with intense sound fields in fluids, are considered theoretically. The analysis is based on the fundamental equations of motion for a homogeneous, thermoviscous fluid, for which thermal equations of state exist. Model equations are derived and used to analyze nonlinear sources for generation of heat and streaming, and other changes in the ambient state of the fluid.

Introduction

The rate at which heat is generated per unit volume by absorption of high intensity sound in a thermoviscous fluid is commonly predicted from the equation[1]

$$<q> = -\nabla \cdot <\mathbf{I}> , \tag{1}$$

where $\mathbf{I} = (p - p_0)\mathbf{v}$ is the intensity, and $<>$ denotes the mean value (temporal average). This leads to

$$<q> = \frac{\alpha P_0^2}{\rho_0 c_0} \tag{2}$$

to dominant order of magnitude, for the case of a time-harmonic, plane travelling wave. Here α is the absorption coefficient, P_0 the pressure amplitude, and ρ_0, c_0 the ambient values of density and sound speed, respectively. This result is also valid for a beam in the paraxial approximation, since it follows from Eq. (1) by substitution of the linear impedance relation, $p - p_0 = \rho_0 c_0 v$, which is consistent with the paraxial approximation.[2,3] Equation (2) can also under special conditions be used for the case of a standing sound field. Thus, Nyborg[1] considered two identical but oppositely directed traveling plane, continuous waves, forming approximately a standing wave field. He then obtained Eq. (2) under the assumption of zero shear viscosity, and that absorption was due to relaxation alone (or bulk viscosity). However, Eq. (1) is not generally true when q is interpreted as the source term in the heat-exchange (entropy) equation.

In the present paper we present some more general results for the intensity and heat generation of a nonlinear sound field and examine the validity of Eq. (1). Model equations are obtained for the case of a travelling sound beam in the paraxial approximation. We also consider the force through which streaming is generated in a sound beam. For the case of a time-harmonic field, this force is commonly obtained from the formula[3,4]

$$\mathbf{f} = \frac{2\alpha}{c_0}\mathbf{I} \tag{3}$$

to dominant order of magnitude, and generalized to[5]

$$f = -\frac{1}{c_0}\frac{dI}{dx} \tag{4}$$

for one-dimensional propagation of high intensity sound. Here, $I = |\mathbf{I}|$. "Taperfunctions," or "effective" absorption coefficients are introduced ad hoc, to account for higher order nonlinearity.[6] The results presented in the present paper have wider range of validity, and can be used for pulsed signals as well.

I. Nonlinear propagation; energy equation

The nonlinear theory of sound propagation in a homogeneous thermoviscous fluid is based on the Navier-Stokes equation, the heat-exchange (entropy) equation, the continuity equation, and the equation of state. The basic equations are:

$$\rho\frac{d\mathbf{v}}{dt} + \nabla p = \mathbf{F}\,, \tag{5}$$

$$\rho\theta\frac{ds}{dt} = \nabla\cdot(\rho c_v K\nabla\theta) + \Delta\,, \tag{6}$$

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0\,, \tag{7}$$

$$p = p(\rho,\theta)\quad,\qquad s = s(\rho,\theta)\,, \tag{8}$$

where $\mathbf{v}$ is the particle velocity, $p\,,\rho\,,\theta\,,s$ the pressure, density, temperature and specific entropy, respectively, t the time, and $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$. The viscous force is $\mathbf{F}$ and the viscous dissipation function Δ:

$$\begin{aligned}\mathbf{F} &= (\mu_B + \frac{\mu}{3})\nabla\nabla\cdot\mathbf{v} + \mu\nabla^2\mathbf{v}\\ &= (\mu_B + \frac{4}{3}\mu)\nabla\nabla\cdot\mathbf{v} - \mu\nabla\times\nabla\times\mathbf{v}\,,\end{aligned} \tag{9}$$

$$\Delta = \frac{\mu}{2}\sum_{k,j}[\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} - \frac{2}{3}\delta_{j,k}\nabla\cdot\mathbf{v}]^2 + \mu_B(\nabla\cdot\mathbf{v})^2\,, \tag{10}$$

where μ_B and μ are the coefficient of bulk and shear viscosity, K the coefficient of thermal conductivity, and c_v is the specific heat at constant volume.

From Eq. (8) we obtain

$$\rho\theta\frac{ds}{dt} = \rho c_p\frac{d\theta}{dt} - \eta\theta\frac{dp}{dt}\,, \tag{11}$$

and

$$\rho\theta\frac{ds}{dt} = \rho c_v\frac{d\theta}{dt} - \frac{\eta\theta c^2}{\gamma}\frac{d\rho}{dt}\,, \tag{12}$$

where c_p is the specific heat at constant pressure, $\gamma = c_p/c_v$, c the isentropic sound speed, and $\eta = -\rho^{-1}(\partial\rho/\partial\theta)_p$ is the coefficient of thermal expansion at constant pressure.

Model equations are derived by introducing various approximations. A common approximation is to neglect cubic and higher order terms in the acoustic variables, and to account for dissipative effects through the linear terms only. This corresponds to an expansion up to order $O(\epsilon^2, \epsilon S)$ in powers of the acoustic Mach number, $\epsilon = V/c_0$, and the Stokes number, $S = \delta/c_0^2T$, where T and V are characteristic time and velocity amplitude, respectively, and δ is the sound diffusivity defined below. Equation (5) is then reduced to

$$\rho_0\frac{\partial \mathbf{v}}{\partial t} + \nabla(p+\mathcal{L}) = (\mu_B + \frac{4}{3}\mu)_0\nabla\nabla\cdot\mathbf{v} - \mu_0\nabla\times\Omega + \rho_0\mathbf{v}\times\Omega\,, \qquad (13)$$

where $\Omega = \nabla\times\mathbf{v}$ and $\mathcal{L}$ is the Lagrangian density function defined by

$$\mathcal{L} = \frac{\rho_0 v^2}{2} - \frac{(p-p_0)^2}{2\rho_0 c_0^2}\,. \qquad (14)$$

Here $v = |\mathbf{v}|$ and the subscript zero denotes the ambient values in the unperturbed, homogeneous fluid at rest ($\mathbf{v}_0 = 0$). The velocity and the excess values $p - p_0\,, \rho - \rho_0\,, \theta - \theta_0$, *etc.*, are referred to as the acoustic variables.

Combining Eqs. (6), (7), (11) and (12), we obtain to the same order of approximation

$$\frac{\partial}{\partial t}(p+\mathcal{L}) + \rho_0 c_0^2\nabla\cdot\mathbf{v} = (\frac{\gamma-1}{\gamma}K)_0\nabla^2 p + \frac{\beta}{\rho_0 c_0^2}\frac{\partial}{\partial t}(p-p_0)^2 + 2\frac{\partial\mathcal{L}}{\partial t}\,, \qquad (15)$$

where $\beta = 1 + B/2A$ denotes the parameter of nonlinearity, and

$$\frac{B}{A} = \left(\frac{\rho}{c^2}(\frac{\partial^2 p}{\partial\rho^2})_{s,\rho}\right)_0.$$

The last term in Eq. (13) vanishes when we assume $\Omega = 0$ to $O(\epsilon)$, which is permitted at range from the transducer large compared to the viscous boundary layer of thickness $\delta_{ac} = (2\nu/\omega)^{1/2}$, where ω is a characteristic angular frequency. Furthermore, Eq. (13) yields

$$\rho_0\frac{\partial\Omega}{\partial t} - \mu_0\nabla^2\Omega = \rho_0\nabla\times(\mathbf{v}\times\Omega)\,, \qquad (16)$$

which shows that vortical motion can only, in this approximation, be generated nonlinearly within the boundary layer where $\Omega \neq 0$ to $O(\epsilon)$. This approximation was discussed in Secs. 4 and 6 of Ref. 3, and an energy equation of the form

$$\frac{\partial\mathcal{E}}{\partial t} + \nabla\cdot\mathbf{I} = -D \qquad (17)$$

was obtained. It is a direct consequence of Eqs. (13) and (15). Various forms of $\mathcal{E}$, $\mathbf{I}$, and D are possible. With

$$\mathbf{I} = (p - p_0 + E)\mathbf{v} - \mu_0\mathbf{v}\times(\nabla\times\mathbf{v})\,, \qquad (18)$$

we have

$$D = \frac{1}{c_0^2}(\mu_B + \frac{4}{3}\mu)_0 |\frac{\partial \mathbf{v}}{\partial t}|^2 + (\frac{\gamma - 1}{\gamma}K)_0 \frac{1}{\rho_0 c_0^4}(\frac{\partial p}{\partial t})^2 + \mu_0 |\nabla \times \mathbf{v}|^2 \tag{19}$$

and

$$\mathcal{E} = \frac{\rho_0 v^2}{2} + \frac{(p - p_0 + \mathcal{L})^2}{2\rho_0 c_0^2} + (1 - \frac{2}{3}\beta)\frac{(p - p_0)^3}{\rho_0^2 c_0^4} - \frac{\partial}{\partial t}\left[(\mu_B + \frac{4}{3}\mu)_0 \frac{v^2}{2c_0^2} + (\frac{\gamma - 1}{\gamma}K)_0 \frac{(p - p_0)^2}{2\rho_0 c_0^4}\right], \tag{20}$$

where

$$E = \frac{\rho_0 v^2}{2} + \frac{(p - p_0)^2}{2\rho_0 c_0^2}. \tag{21}$$

Here, D is a sum of positive definite quadratic forms that represent rates of energy dissipation per unit volume due to viscosity and heat conduction. Similar terms could be added in order to account for other possible diffusive effects (relaxation). Equations (17) and (18) contain terms of cubic order in ϵ that are not found in the similar energy equations derived from the linearized equations.

Alternative versions of Eq. (19), in our approximation, are:

$$D = \frac{\delta_0}{\rho_0 c_0^4}(\frac{\partial p}{\partial t})^2 + (\mu_B + \frac{4}{3}\mu)_0 \frac{1}{\rho_0 c_0^2}\left[|\nabla p|^2 - \frac{1}{c_0^2}\left(\frac{\partial p}{\partial t}\right)^2\right] + \mu_0 |\nabla \times \mathbf{v}|^2, \tag{22}$$

$$D = \frac{\delta_0}{\rho_0 c_0^2}|\nabla p|^2 - (\frac{\gamma - 1}{\gamma}K + R)_0 \frac{1}{\rho_0 c_0^2}\left[|\nabla p|^2 - \frac{1}{c_0^2}(\frac{\partial p}{\partial t})^2\right] + \mu_0 |\nabla \times \mathbf{v}|^2, \tag{23}$$

where

$$\delta = (\mu_B + \frac{4}{3}\mu)\frac{1}{\rho} + \frac{\gamma - 1}{\gamma}K + R. \tag{24}$$

We have here included a coefficient R in order to account for relaxation.

Considerable simplifications can be obtained in the special case of a travelling, narrow sound beam. If the nonlinear terms are accounted for only in the paraxial (parabolic) approximation (See, for example, Refs. 2,3 for a discussion of this approximation), we obtain:

$$D = \frac{\delta_0}{\rho_0 c_0^4}(\frac{\partial p}{\partial t})^2 \tag{25}$$

and

$$\nabla \cdot [(p - p_0)\mathbf{v}] = -\frac{\delta_0}{\rho_0 c_0^4}(\frac{\partial p}{\partial t})^2 - (1 - \frac{\delta_0}{2c_0^2}\frac{\partial}{\partial t})\frac{1}{\rho_0 c_0^2}\frac{\partial}{\partial t}(p - p_0)^2 + \frac{2\beta}{3\rho_0^2 c_0^4}\frac{\partial}{\partial t}(p - p_0)^3. \tag{26}$$

This can, for time periodic but otherwise arbitrary boundary (on-source) conditions be computed following the procedure with Fourier series expansions described in Refs. 7–9 (Bergen Code).

II. Mean values

The mean value, $<a>$, of a field variable, a, or any function of such variables, is obtained by integrating over the fast time scale, $t = t_0$ (which corresponds to the oscillations with typical period T) at position $\mathbf{x}$, keeping the slow time variable $t_1 = \epsilon t$ constant:

$$<a(\mathbf{x}, t_0, t_1)>= \frac{1}{T} \int_{t_0-T}^{T} a(\mathbf{x}, s, t_1)\, ds\,. \tag{27}$$

Here, ϵ is the acoustic Mach number, which is used as expansion parameter

$$a = \epsilon a_1 + \epsilon^2 a_2 + O(\epsilon^3)\,. \tag{28}$$

The time derivative of a second order term is then defined by

$$\frac{\partial a_2}{\partial t} = \frac{\partial a_2}{\partial t_0} + \frac{\partial a_1}{\partial t_1}\,. \tag{29}$$

[See Ref. 10 for a discussion of this method.]

When the sound source is periodic, we obtain from Eq. (17)

$$\nabla\cdot <\mathbf{I}>= - <D>\,, \tag{30}$$

since $<\partial \mathcal{E}/\partial t>= 0$ in this case.

From Eq. (26) we obtain in the paraxial approximation

$$\nabla\cdot <(p - p_0)\mathbf{v}>= -\frac{\delta_0}{\rho_0 c_0^4} <(\frac{\partial p}{\partial t})^2>\,, \tag{31}$$

which can readily be computed, for example, by expanding the finite amplitude pressure field in a Fourier series

$$p - p_0 = \sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty}(g_n \sin n\tau + h_n \cos n\tau)\,. \tag{32}$$

For each harmonic component we then obtain from Eq. (26)

$$\nabla\cdot <\mathbf{I}_n>= -2n^2\alpha|I_n| + \frac{\beta}{\rho_0^2 c_0^4} <p_n \frac{\partial}{\partial t}(p - p_0)^2>\,, \tag{33}$$

in agreement with the result derived in Ref. 7. Here, $|I_n| =<p_n^2> /\rho_0 c_0$, and $\alpha = \delta\omega^2/2c^3$ is the absorption coefficient for the fundamental component. The last term in Eq. (33) represent finite amplitude attenuation.

III. Heat generation

From the basic equations we derive model equations for heat exchange by expanding to $O(\epsilon^2 S)$, and time averaging ($S = O(\epsilon)$, but no term of $O(\epsilon^3)$ occurs, since the fluctuations in entropy are of $O(\epsilon S)$ in the linear approximation). We obtain

$$\rho_0\theta_0\frac{D <s>}{Dt} - \kappa_0\nabla^2 <\theta>=<q> , \tag{34}$$

where

$$<q>=<\Delta> +\frac{1}{\rho_0 c_0^4}(\frac{\gamma-1}{\gamma}K)_0 <(\frac{\partial p}{\partial t})^2> -\nabla\cdot <\rho_0\theta_0(s-s_0)\mathbf{v}> +\nabla\cdot <(\kappa-\kappa_0)\nabla\theta> , \tag{35}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + <\mathbf{v}> \cdot\nabla . \tag{36}$$

Here, $\kappa = \rho(c_v)K$, and the subscript nought refers to the ambient values.

Further,

$$\begin{aligned}\nabla\cdot <\rho_0\theta_0(s-s_0)> &= (\frac{c_p}{\eta c^2})_0\frac{\kappa_0}{\theta_0}\nabla\cdot <(\theta-\theta_0)\nabla\theta> \\ &= (\frac{c_p}{\eta c^2})_0(\frac{\gamma-1}{\gamma}K)_0\frac{1}{\rho_0 c_0^2} <|\nabla p|^2 - \frac{1}{c_0^2}(\frac{\partial p}{\partial t})^2>\end{aligned} \tag{37}$$

to our degree of approximation.

When $\kappa = \kappa(\rho,\theta)$, we have to $O(\epsilon)$

$$\kappa - \kappa_0 = (\frac{\partial\kappa}{\partial\theta} + \frac{\rho\eta}{\gamma-1}\frac{\partial\kappa}{\partial\rho})_0(\theta-\theta_0) . \tag{38}$$

When substituted in the last term of Eq. (35), this leads to a term of the same type as Eq. (37). The two last terms of Eq. (35) are zero in the case of a travelling wave computed in the paraxial approximation. Since also Δ is simplified in this approximation, we obtain

$$<q>= \frac{\delta_0}{\rho_0 c_0^4} <(\frac{\partial p}{\partial t})^2> . \tag{39}$$

This has the same form as Eq. (31), and can be computed accordingly. Equations (31) and (39) also show that

$$<q>= -\nabla\cdot <(p-p_0)\mathbf{v}> \tag{40}$$

is valid in this case. However, the simple relationship shown in Eq. (40) between the intensity and heat source in a finite amplitude sound field may not always be true, as the following example shows:

Consider a periodic sound field, and $\nabla\times\mathbf{v}=0$ to $O(\epsilon)$ (which is a good approximation away from the boundaries). When shear and bulk viscosity represent the main contribution to sound diffusivity, we have

$$<q>=<\Delta>\,, \tag{41}$$

and from Eq. (30):

$$\nabla\cdot <(p-p_0+E)\mathbf{v}>= -\frac{\delta_0}{\rho_0 c_0^2} <|\nabla p|^2>\,, \tag{42}$$

where now $\delta=(\mu_B+\frac{4}{3}\mu)/\rho$.

In the case of a plane wave, this leads to

$$<q>= \frac{\delta_0}{\rho_0 c_0^4} <(\frac{\partial p}{\partial t})^2>\,, \tag{43}$$

which is not equal to $-\nabla\cdot <\mathbf{I}>$ for a standing wave field, as seen from Eq. (42).

Equation (43) is a more general result than Eq. (2), as it accounts for the effects of finite amplitude attenuation in a sound field, and is valid for a pulsed signal as well. It contains the results of Eq. (2) for a time harmonic field when computed to the lowest order of approximation only, and provided $<q>$ is interpreted as source term in the heat exchange equation.

For the special case $\mu=0$ considered in Ref. 1, we have, when the thermal terms in Eq. (35) are neglected,

$$<q>= (\frac{\mu_B}{\rho^2 c^4})_0 <(\frac{\partial p}{\partial t})^2> \tag{44}$$

even for a nonplanar wave, since $\Delta=\mu_B(\nabla\cdot\mathbf{v})^2$ in this case.

To $O(\epsilon^2 S,\epsilon^2)$ we derive the following equation of state in the mean value:

$$\rho_0\theta_0\frac{D}{Dt} <s> = \rho_0(c_p)_0\frac{D}{Dt} <\theta> -\eta_0\theta_0\frac{D}{Dt} <p>\,, \tag{45}$$

$$\rho_0\theta_0\frac{D}{Dt} <s> = \rho_0(c_v)_0\frac{D}{Dt} <\theta> -\frac{\eta_0\theta_0 c_0^2}{\gamma_0}\frac{D}{Dt} <\rho>\,. \tag{46}$$

These equations are especially simple to this order of approximation, as they have the same form as the exact equations of state. Fluctuations in the coefficients caused by the sound have been fully accounted for to our order of approximation (Expansions to $O(\epsilon^3)$, however, could here add new terms).

When we substitute these equations in Eq. (34), we obtain model equations that can be used to compute the temperature elevation produced by the dissipation of sound under conditions of either constant pressure ($D <p> /Dt = 0$), or constant density ($D <\rho> /Dt = 0$). The nonlinear term $<\mathbf{v}> \cdot\nabla <\theta>$ in $D <\theta> /Dt$ is of order of magnitude $R_s P_r/\gamma$, where R_s is the streaming Reynolds number (cf. Sec. IV) and P_r the Prandtl number, $P_r = \gamma\mu/\rho K$. For a typical ultrasound beam in water, $R_s P_r/\gamma$ is of order one. Therefore we cannot in general neglect the nonlinear term in $D <\theta> /Dt$ when computing the mean temperature in a fluid as water.

IV. Acoustic streaming

In order to analyze the source of streaming we start with the vorticity equation:

$$\frac{\partial \Omega}{\partial t} - \nu_0 \nabla^2 \Omega - \nabla \times (\mathbf{v} \times \Omega) = \frac{1}{\rho^2} \mathcal{F} \times \nabla \rho - \nu_0 \frac{\rho - \rho_0}{\rho} \nabla^2 \Omega - \frac{\mu}{\rho^2} \nabla \rho \times (\nabla \times \Omega) . \quad (47)$$

where $\Omega = \nabla \times \mathbf{v}$, and

$$\mathcal{F} = (\mu_B + \frac{4}{3}\mu) \nabla \nabla \cdot \mathbf{v} - (\frac{\partial p}{\partial s})_\rho \nabla s . \quad (48)$$

Equation (47) is a consequence og Eq. (5) when we assume constant μ. Effects from fluctuations in μ caused by the sound can be significant under special conditions.[4] Such effects can be accounted for by assuming $\mu = \mu(\rho, \theta)$ and expanding to $O(\epsilon)$ as for κ in Eq. (38). They are discussed in Sec. V, but are left out here for reason of simplicity. The last two terms in Eq. (47) are zero to $O(\epsilon^2 S)$ in the region outside the boundary layer of thickness δ_{ac}, since $\Omega = 0$ to order $O(\epsilon)$ in this region. Neglecting these terms, we obtain to $O(\epsilon^2 S)$:

$$\frac{\partial \Omega}{\partial t} - \nu_0 \nabla^2 \Omega - \nabla \times (\mathbf{v} \times \Omega) = \frac{\delta_0}{\rho_0^2 c_0^4} \nabla \times [(p - p_0) \nabla \frac{\partial p}{\partial t}] , \quad (49)$$

where the term on the right-hand side of the equation is the nonlinear source through which vortical motion is generated in the sound field.

The corresponding momentum equation can be written

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} - \mu_0 \nabla^2 \mathbf{v} - \rho_0 \mathbf{v} \times \Omega + \nabla \pi = \mathbf{f} , \quad (50)$$

where

$$\mathbf{f} = \frac{\delta_0}{\rho_0 c_0^4} (p - p_0) \nabla \frac{\partial p}{\partial t} , \quad (51)$$

and π is a generalized pressure defined by

$$\pi = p + \mathcal{L} + (\frac{\gamma - 1}{\gamma} K)_0 \frac{1}{2 \rho_0 c_0^4} \frac{\partial}{\partial t} (p - p_0)^2 - (\mu_B + \frac{\mu}{3})_0 \nabla \cdot \mathbf{v} . \quad (52)$$

Vortical motion is thus generated only when $\delta \neq 0$ and $\nabla \times \mathbf{f} \neq 0$.

For the mean value we obtain

$$\rho_0 \frac{\partial}{\partial t} <\mathbf{v}> - \mu_0 \nabla^2 <\mathbf{v}> - \rho_0 <\mathbf{v} \times \Omega> + \nabla <p + \mathcal{L}> = <\mathbf{f}> , \quad (53)$$

where the nonlinear source term <f> has a special simple form when computed for a sound beam in the paraxial approximation. Then <f> is directed along the beam axis and of magnitude

$$<f>= \frac{\delta_0}{\rho_0 c_0^5} <(\frac{\partial p}{\partial t})^2> . \tag{54}$$

Comparing this with Eqs. (31) and (39), we obtain

$$<f>= \frac{1}{c_0} <q>= -\frac{1}{c_0}\nabla\cdot <(p-p_0)\mathbf{v}> , \tag{55}$$

which can be computed as described in Sec. I. For a time harmonic wave, we obtain from Eq. (54):

$$<f>= \frac{2\alpha I}{c_0} \tag{56}$$

in accordance with Eq. (3).

The nonlinear term $-\rho_0 <\mathbf{v}\times\Omega>$ in Eq. (53) is for the case of a sound beam (outside the boundary layer) of order of magnitude $R_s = U l_c^2/\nu_0 L_c$ compared to the viscous term $\mu_0\nabla^2 <\mathbf{v}>$. Here, U is a characteristic value of <**v**>, and L_c, l_c are characteristic lengths for changes in <**v**> in the directions along the beam axis and transverse to this axis, respectively. R_s is typically of order one in ultrasound beams, which indicates that nonlinearity in the flow velocity field may be important in this case.

The flow velocity may also have a non vortical component in the Eulerian description. To the same order of expansion as above, $O(\epsilon^2 S)$, we obtain

$$\nabla\cdot <\mathbf{v}>= -\frac{1}{\rho_0}\frac{D<\rho>}{Dt} + \frac{\delta_0}{\rho_0^2 c_0^4} <|\nabla p|^2> , \tag{57}$$

where again $\Omega = 0$ has been assumed.

Thus

$$\nabla\cdot <\mathbf{v}>= \frac{\delta_0}{\rho_0^2 c_0^4} <|\nabla p|^2> \tag{58}$$

if we assume $D<\rho>/Dt = 0$, as commonly is the case in the theory of acoustic streaming. If, on the other hand, $D<p>/Dt = 0$, as assumed in the theory of heating, we obtain, using Eqs. (45) and (46)

$$\nabla\cdot <\mathbf{v}>= \frac{\delta_0}{\rho_0^2 c_0^4} <|\nabla p|^2> +\eta_0\frac{D<\theta>}{Dt} , \tag{59}$$

which leads to

$$\nabla\cdot <\mathbf{V}>= (1+\frac{\eta c^2}{c_p})_0\frac{\delta_0}{\rho_0^2 c_0^6} <(\frac{\partial p}{\partial t})^2> , \tag{60}$$

when the square order terms are computed in the paraxial approximation. Here, $\mathbf{V}$ is defined by

$$\mathbf{V} = \mathbf{v} - \frac{\eta_0 K_0}{\gamma_0} \nabla \theta \,. \tag{61}$$

We can now determine the flow field $<\mathbf{V}>$ instead of $<\mathbf{v}>$, since $\nabla \times \mathbf{V} = \Omega = \nabla \times \mathbf{v}$ and the vorticity equation remains the same.

For an ideal gas $(\eta c^2/c_p)_0 = \gamma_0 - 1$, and there is a significant difference between $\mathbf{V}$ and $\mathbf{v}$. For liquids the effect of heat conduction is small, and $(1 + \eta c^2/c_p)_0 \simeq 1$ (For water $(\eta c^2/c_p)_0 \simeq 1/20$). Then we can put $\mathbf{V} \simeq \mathbf{v}$ with good approximation. However, the vorticity component of the mean flow velocity will dominate in a typical ultrasound field.

In experiments on acoustic streaming the motion of the fluid is commonly studied by introducing some kind of particles or tracing points. The particles will oscillate and drift away in the second order flow field with a mean Lagrangian, or particle velocity,

$$<\mathbf{v}_p>=<\mathbf{v}> + <\int^t \mathbf{v}\, dt \cdot \nabla \mathbf{v}> \,. \tag{62}$$

The mean mass flow velocity is defined by

$$<\mathbf{U}>= \frac{1}{\rho_0} <\rho \mathbf{v}> \tag{63}$$

in Eulerian description. The following transformation formulas can then be found[4]:

$$<\mathbf{v}_p> = <\mathbf{U}> - \frac{1}{2} \nabla \times <\int^t \mathbf{v}\, dt \times \mathbf{v}> \tag{64}$$

$$<\mathbf{U}> = <\mathbf{v}> + \frac{<\mathbf{I}>}{\rho_0 c_0^2} - \left(\frac{\gamma - 1}{\gamma} K\right)_0 \frac{1}{\rho_0^2 c_0^4} <|\nabla p|^2 - \frac{1}{c_0^2} \left(\frac{\partial p}{\partial t}\right)^2> \,. \tag{65}$$

Here, the last term is zero for the case of a travelling plane wave or for a bounded beam when computed in the paraxial approximation. Both $<\mathbf{v}_p>$ and $<\mathbf{U}>$ are solenoidal fields (incompressible flow) when we neglect the effect from changes in mean density.

V. Effects from fluctuations in the coefficients of viscosity

In order to estimate the effect of $\nabla\mu$ as a source for streaming generation in a sound beam we start with the momentum (Navier-Stokes) equation written as

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla\left[p - (\mu_B + \frac{1}{3}\mu)_0\nabla\cdot\mathbf{v} - \mathbf{v}\cdot\nabla\mu\right] + \mu\nabla^2\mathbf{v} + \nabla\times(\mathbf{v}\times\nabla\mu) - \mathbf{v}\nabla^2\mu\,. \tag{66}$$

From this we derive the vorticity equation

$$\frac{\partial\Omega}{\partial t} - \nu_0\nabla^2\Omega - \nabla\times(\mathbf{v}\times\Omega) = \frac{d\mathbf{v}}{dt}\times\frac{\nabla\rho}{\rho} - \nu_0\frac{\rho-\rho_0}{\rho}\nabla^2\Omega + \mathbf{A}\,, \tag{67}$$

where

$$\mathbf{A} = \frac{1}{\rho}\left[(\mu-\mu_0)\nabla^2\Omega + \nabla\mu\times\nabla^2\mathbf{v} + \nabla\times\nabla\times(\mathbf{v}\times\nabla\mu) + \mathbf{v}\times\nabla^2\nabla\mu - \Omega\nabla^2\mu\right] \tag{68}$$

represents the new source terms, when compared with Eq. (47). Fluctuations in μ_B do not lead to new source terms in the vorticity equation. Let $\mu = \mu(\rho,\theta)$. Then $d\mu = \zeta d\rho$ to order $O(\epsilon)$, where

$$\zeta = (\frac{\partial\mu}{\partial\rho})_\theta + \frac{\gamma-1}{\rho\eta}(\frac{\partial\mu}{\partial\theta})_\rho\,. \tag{69}$$

If we now assume $\Omega = 0$ to $O(\epsilon)$, i.e., a sound field away from the boundary layers, we find to $O(\epsilon^2 S)$:

$$\begin{aligned}\mathbf{A} = \frac{1}{\rho_0}\nabla\times\nabla\times(\mathbf{v}\times\nabla\mu) &= -\frac{1}{\rho_0}\nabla^2(\mathbf{v}\times\nabla\mu)\\ &= \frac{1}{\rho_0}\nabla^2\nabla\times[(\mu-\mu_0)\mathbf{v}]\\ &= \frac{\zeta_0}{\rho_0}\nabla^2\nabla\times[(\rho-\rho_0)\mathbf{v}]\\ &\simeq \frac{\zeta_0}{\rho_0 c_0^2}\nabla^2\nabla\times\mathbf{I}\,,\end{aligned} \tag{70}$$

as the other terms in $\mathbf{A}$ are zero due to $\Omega = 0$, or cancel within our approximation. This yields an additional source term $(\zeta/\rho c^2)_0\nabla^2\mathbf{I}$ in the corresponding momentum equation, Eq. (50).

Typically, ζ_0/ν_0 is of magnitude unity or smaller, which means that the new source terms due to $\nabla\mu$ will generate a vortical flow field comparable to $\mathbf{I}/\rho_0 c_0^2$, the second term of Eq. (65).

We can also use $<\mathbf{U}>$ as flow variable. A governing equation in $<\mathbf{U}>$ is obtained by substituting Eq. (65) in the equation for $\mathbf{v}$. Within the same approximation as in Eq. (70), we obtain

$$\rho_0 \frac{\partial}{\partial t} <\mathbf{U}> + \rho_0 <\mathbf{U}> \cdot \nabla <\mathbf{U}> - \mu_0 \nabla^2 <\mathbf{U}> + \nabla <p> = \frac{\delta_0}{\rho_0 c_0^5} <(\frac{\partial p}{\partial t})^2> + (\frac{\zeta - \nu}{\rho c^2})_0 \nabla^2 <\mathbf{I}> . \tag{71}$$

Further, $\nabla \cdot <\mathbf{U}> = 0$ when we assume $<\partial \rho / \partial t> = 0$.

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