## Department of APPLIED MATHEMATICS

Unconditionally stable Methods for Hamilton-Jacobi Equations

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### UNCONDITIONALLY STABLE METHODS FOR HAMILTON-JACOBI EQUATIONS

#### KENNETH HVISTENDAHL KARLSEN AND NILS HENRIK RISEBRO

ABSTRACT. We present new numerical methods for constructing approximate solutions to the Cauchy problem for Hamilton-Jacobi equations of the form  $u_t + H(D_x u) = 0$ . The methods are based on dimensional splitting and front tracking for solving the associated (non-strictly hyperbolic) system of conservation laws  $p_t + D_x H(p) = 0$ , where  $p = D_x u$ . In particular, our methods depends heavily on a front tracking method for one-dimensional scalar conservation laws with discontinuous coefficients. The proposed methods are unconditionally stable in the sense that the time step is not limited by the space discretization and they can be viewed as "large time step" Godunov type (or front tracking) methods. We present several numerical examples illustrating the main features of the proposed methods. We also compare our methods with several methods from the literature.

#### 1. INTRODUCTION

In this paper we present unconditionally stable numerical methods for the Cauchy problem for multi-dimensional Hamilton-Jacobi equations

(1) 
$$\begin{cases} u_t + H(D_x u) = 0, & \text{in } \mathbb{R}^d \times \{t > 0\}, \\ u = u_0, & \text{on } \mathbb{R}^d \times \{t = 0\}. \end{cases}$$

In (1), u = u(x,t) is the scalar unknown function that is sought,  $u_0 = u_0(x)$  is a Lipschitz continuous initial function, H is a Lipschitz continuous Hamiltonian, and  $D_x$  denotes the gradient with respect to  $x = (x_1, \ldots, x_d)$  defined by  $D_x u = (u_{x_1}, \ldots, u_{x_d})$ . Hamilton-Jacobi equations arise in a variety of applications, ranging from image processing, via mathematical finance, to the description of evolving interfaces (front propagation problems).

It is well known that solutions of (1) generically develop discontinuous derivatives in finite time even with a smooth initial condition. Moreover, generalized solutions (i.e., locally Lipschitz continuous functions satisfying the equation almost everywhere) are not uniquely determined by their initial data and an additional selection principle — a so-called entropy condition — is needed to single out physically relevant generalized solution. The most commonly used entropy condition is the vanishing viscosity condition which requires that the (correct) solution of (1) should be the vanishing viscosity limit of smooth solutions of corresponding viscous problems.

The vanishing viscosity entropy condition gives raise to the notion of viscosity solutions introduced by Crandall and Lions [7]. In particular, these authors established the existence, uniqueness and stability of a viscosity solution of (1). Since then the theory of viscosity solutions has been intensively studied and even extended to large class of fully nonlinear second order partial differential equations. We refer to Crandall, Ishii, and Lions [6] for an up-to-date overview of the viscosity solution theory. In passing, we mention that Kružkov has developed an alternative (equivalent) theory for Hamilton-Jacobi equations with a convex Hamiltonian, see, e.g., [28].

It is known that the Hamilton-Jacobi equations are closely related to (scalar) conservation laws

(2) 
$$\begin{cases} v_t + \sum_{i=1}^d f_i(v)_{x_i} = 0, & \text{in } \mathbb{R}^d \times \{t > 0\}, \\ v(x, 0) = v_0(x) & \text{on } \mathbb{R}^d \times \{t = 0\}. \end{cases}$$

Date: May 3, 2000.

Key words and phrases. Hamilton-Jacobi equation, conservation law, discontinuous coefficient, numerical method, front tracking, operator splitting, numerical example.

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#### L'ANTRODUCTION I

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Here v = v(x, t) is the scalar unknown,  $v_0 = v_0(x)$  is a bounded initial function, and  $f_1, \ldots, f_d$  are Lipschitz continuous flux functions. In contrast to the Hamilton-Jacobi equations, which possess at least continuous solutions, solutions of (2) develop discontinuities (shock waves) in finite time and therefore one has to consider solutions of (2) in the sense of distributions. However, distributional solutions are not uniquely determined by their initial data and one needs also here the vanishing viscosity entropy condition to pick out the correct solution. In the context of scalar conservation laws (2), the vanishing viscosity condition gives raise to the notion of entropy solutions in the sense of Kružkov. Kružkov [30] proved that the well-posedness of (2) is ensured within the framework of entropy solutions.

In the one-dimensional case (d = 1), it is well known that the existence of viscosity solutions of (15) is equivalent to the existence of entropy solutions of (16), see [28, 34, 5, 20, 22]. More precisely, if u = u(x,t) is the unique viscosity solution of (1), then  $v = D_x u$  is the unique entropy solution of (2). Conversely, if v = v(x,t) is the unique entropy solution of (2), then u defined via  $u(x,t) = \int_{-\infty}^{x} p(\xi,t) d\xi$  is the unique viscosity solution (1). In the multi-dimensional case (d > 1), this one-to-one correspondence no longer exists. Instead the gradient  $p = (p_1, \ldots, p_d) = D_x u$ satisfies (at least formally) a  $d \times d$  system of conservation laws [28, 34, 20]

(3) 
$$\begin{cases} (p_1)_t + H(p_1, \dots, p_d)_{x_1} = 0 \\ & \ddots \\ (p_d)_t + H(p_d, \dots, p_d)_{x_d} = 0. \end{cases}$$

If p is known, one may recover u from p by integrating the ordinary differential equation

(4)  $u_t + H(p_1, \dots, p_d) = 0.$ 

One should notice that (3) is non-strictly hyperbolic in the sense that the Jacobian does not possess a complete set of eigenvectors. Nevertheless, in [28, 34, 20] it is proved that the vanishing viscosity limit solutions of (1) and (3) (when such exist of both problems!) are equivalent. Roughly speaking, one may therefore in the multi-dimensional case also think of viscosity solutions of (1) as primitives of (entropy) solutions of (3).

Equipped with this view, it becomes natural to exploit some of the numerical concepts developed for hyperbolic conservation laws when developing numerical methods for Hamilton-Jacobi equations. Indeed, many well known shock-capturing methods for conservation laws have been extended to Hamilton-Jacobi equations, see [8, 35] for finite difference schemes of upwind type (see also [29]); [1, 27] for finite volume schemes; [38, 39, 19] for (W)ENO schemes; [33, 31] for central schemes; [2, 17] for finite element methods; and [20] for relaxation schemes.

In contrast to shock-capturing schemes just cited, we will in this paper be concerned with extending to Hamilton-Jacobi equations (1) a so-called front tracking method for conservation laws. The front tracking method was introduced by Dafermos [9] as a (mathematical) tool for constructing entropy solutions to one-dimensional scalar conservation laws. Holden, Holden, and Høegh-Krohn [13, 14] later proved that Dafermos' construction procedure was well-defined and developed it into an  $L^1$  linearly(!) convergent numerical method. Front tracking was later extended to systems of equations by Bressan [3] and Risebro [41], who used the method to give an alternative proof of Glimm's famous existence result for hyperbolic systems. Very recently a modification of the front tracking method was used by Bressan, Liu, and Yang [4] to prove stability and uniqueness of weak solutions of strictly hyperbolic systems of conservation laws. The front tracking method was used by Risebro and Tveito [42, 43] to numerically solve the Euler equations of gas dynamics and a non-strictly hyperbolic system modeling polymer flow.

Holden and Risebro [16] extended the scalar front tracking method to multi-dimensional scalar conservation laws by means of dimensional splitting. These authors also proved that the method converges to the unique entropy solution of the governing problem. An  $L^1$  error estimate of order 1/2 was proved in [21]. Although the convergence rate drops from 1 in the one-dimensional case to 1/2 in the multi-dimensional case, it should be noted that no CFL condition is associated with the multi-dimensional numerical method, which implies that the method is fast compared with conventional difference methods. Computations using CFL numbers as high as 10 - 20 (with

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Here v = v(x, t) is the social neighbors,  $v_0 = v_0(x)$  is a constant mixed mixed variable of the constant of the formula the formula is constant to the formula - levels equations, which proves at least continuous solutions, solutions (1) develop devolutions which wavel is make there and the formula - level of the f

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satisfactory results) have been reported, see Lie *et al.* [32]. Computational results for multidimensional hyperbolic systems can be found in Holden *et al.* [15, 12] and Haugse *et al.* [11].

The purpose of this paper is to device front tracking methods for Hamilton-Jacobi equations. In the one-dimensional case (see [22]), we simply rely on the equivalence between (1) and (2) and define a numerical method for (1) by "integrating" the front tracking method [13, 14]. The resulting numerical method for (1) is well-defined and  $L^{\infty}$  linearly convergent towards the unique viscosity solution of the governing problem. The linear convergence rate follows from the results in [13, 14] or [36], see also [22].

The multi-dimensional case is much more difficult and is the main focus of this paper. The basis for our numerical methods is the (formal) relation between (1) and the non-strictly hyperbolic system (3). The methods that we present can all be written as explicit marching schemes of the type

(5) 
$$u_J^{n+1} = u_J^n - \Delta t \mathcal{H}_J,$$

where  $J = (j_1, \ldots, j_d) \in \mathbb{Z}^d$  is a multi-index and  $\mathcal{H}_J$  is the numerical Hamiltonian that has to be determined. Typically,  $\mathcal{H}_J$  is a convex combination of one-dimensional numerical Hamiltonians  $\mathcal{H}_J^1, \ldots, \mathcal{H}_J^d$ .

To construct the numerical Hamiltonians  $\mathcal{H}_J^1, \ldots, \mathcal{H}_J^d$ , we first apply a sort of dimensional splitting to reduce the  $d \times d$  system of conservation laws (3) to a sequence of (decoupled) onedimensional scalar conservation laws of the form

(6) 
$$(p_i)_t + H(p_1, \dots, p_i, \dots, p_d)_{x_i} = 0, \qquad i = 1, \dots, d_y$$

where  $p_j = p_j(x)$ ,  $j \neq i$ , are fixed, possibly discontinuous coefficients. These equations can all be viewed as one-dimensional scalar conservation laws of the type

$$v_t + f(a, v)_x = 0, \qquad x \in \mathbb{R}, \ t > 0,$$

where f is some flux function and a = a(x) is a given, possibly discontinuous coefficient. The fact that a(x) can be discontinuous makes analysis of numerical methods for such conservation laws rather difficult. Front tracking for conservation laws with a flux function that depends discontinuously on the space variable is analyzed in Gimse and Risebro [10], Klingenberg and Risebro [25, 26], and Klausen and Risebro [24]. Recently some difference schemes for such conservation laws were proved to be convergent by Towers [45]. Roughly speaking, we shall in this paper build our numerical Hamiltonians  $\mathcal{H}_{J}^{1}, \ldots, \mathcal{H}_{J}^{d}$  (to be used in (5)) by applying the front tracking method to the scalar conservation laws in (6).

The rest of this paper is organized as follows: In the next section, we describe the front tracking algorithm for one-dimensional scalar conservation laws with discontinuous coefficients. Section 3 first describes a front tracking method for Hamilton-Jacobi equations in one dimension, then we detail the various numerical methods for multi-dimensional Hamilton-Jacobi equations which can be build from the front tracking method. These schemes are then tested on several problems in section 4. Finally, we draw some conclusions in Section 5.

#### 2. FRONT TRACKING IN ONE DIMENSION

In this section we describe the front tracking algorithm for one-dimensional conservation laws in some detail. Therefore we consider the one-dimensional scalar conservation law

(7) 
$$v_t + f(a(x), v)_x = 0, \quad v(x, 0) = v_0(x).$$

Here the unknown v = v(x, t) is a scalar and the "coefficient" a(x) is assumed to be a bounded, piecewise differentiable function, but not necessarily continuous. We shall always assume that f is a Lipschitz continuous function.

Front tracking is a method to compute approximate weak solutions to (7). Let first  $\delta$  be a parameter indicating the accuracy of the approximation, and let  $v_0^{\delta}$  and  $a^{\delta}$  be piecewise constant approximations to  $v_0$  and a respectively such that

$$v_0^{\delta} \to v_0$$
 and  $a^{\delta} \to a$  in  $L_{\text{loc}}^1$  as  $\delta \to 0$ .

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astudactory results) have been reported, an Lie et al. [33]. Computational romain for ratiodimensional hyperbolic systems and he found in Holden et al. [16, 12] and Haugar al of [11].

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where  $J = (k_1, \dots, k_n) \in \mathbb{Z}^n$  is a multi-order, and  $N_0$  is the numerical framilionian black distribution determined. Typically,  $K_0$  is a convert combination of one-dimensional maintained (framilional  $M_0$ 

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where  $p_1 = p_1(x)$ ,  $f \neq i$ , are littled. possibly districtions as confictionized little equations can all  $d\sigma$ , viewed as one dimensional scalar bornerviction invested with type

where f is some flux function and a so of all its grown, passible descontinuous technictors. The hast that q(z) are be discontinuous infines visibles of mittained methods for such construction have esther difficult. From tracking for conservation is a web with a distribution from depende difficution construction of the space variable is many act in Games and Rischer [11]. Komouneer and the in-[25, 26], and Riampa and Rimeiro [24]. Franching some infine is subsection with a mittaine have were proved to be communication for foreign (15]. Reacting and and a subsection with an analysis of a our manufaction framitorization for foreign (15]. Therefore a subsection with an analysis for the our manufaction framitorization for foreign (15].

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The Riemann problem for (7) is the initial value problem where  $v_0$  and a take the form

(8) 
$$v_0(x) = \begin{cases} v_l, & \text{for } x \le 0, \\ v_r, & \text{for } x > 0, \end{cases} \quad a(x) = \begin{cases} a_l, & \text{for } x \le 0, \\ a_r, & \text{for } x > 0, \end{cases}$$

Hence,  $v_0^{\delta}$  and  $a^{\delta}$  defines a series of Riemann problems located at their discontinuities. If  $\partial f/\partial v$  is bounded, then (7) has finite speed of propagation, and the solutions of neighboring Riemann problems will not interact for small t. Therefore, if we can compute the entropy solutions to the initial Riemann problems, and thereby the solution of (7) if  $v_0 = v_0^{\delta}$  for sufficiently small t. However, being able to compute the solution of Riemann problems does not help us to compute the solution past the time where waves from different Riemann problems interact. Generally, the solution of the Riemann problem (8) is a function of x/t, and is not always piecewise constant.

Front tracking is a strategy to remedy this. We choose a piecewise constant (in x/t) approximation  $v^{\delta}(x,t)$  to the solution of the Riemann problem such that

$$v^{o}(\cdot, t) \to v(\cdot, t) \quad \text{in } L^{1} \text{ as } \delta \to 0.$$

If we approximate all the initial Riemann problems defined by  $v_0^{\delta}$  and  $a^{\delta}$  in this manner, the resulting function will be piecewise constant in x, with discontinuities emanating in fans from each initial discontinuity. Collisions between these discontinuities will define new Riemann problems (since  $v^{\delta}$  is piecewise constant). We can approximately solve these Riemann problems in the same way (giving new discontinuities that move in straight lines) and thereby continuing the approximation beyond the interaction time. We call the function defined in this way  $v^{\delta}$  and the discontinuities in  $v^{\delta}$  fronts. The approximation process we call front tracking.

Note that it is not clear whether we are able to continue the front tracking approximation up to any prescribed time t (this depends on how we construct the approximate Riemann solution). Moreover, we must be able to construct an approximate solution of any Riemann problem arising from collisions. For the equations considered in this paper, front tracking is well-defined and converges to the entropy solution of the conservation law.

2.1. Convex f. If f(a, v) is uniformly convex in u and monotone in a, front tracking is welldefined. More precisely, from [24] we have the following theorem:

**Theorem 2.1.** Assume that a is in  $L^1 \cap BV$  and is piecewise  $C^1$  with a finite number of discontinuities. Assume also that  $v_0(x)$  is such that  $f(a, v_0)$  is of bounded variation. Then there exists a unique weak solution u to (7) such that  $v^{\varepsilon} \to u$  in  $L^1$ , where  $v^{\varepsilon}$  solves the "regularized" problem

(9) 
$$\begin{cases} v_t^{\varepsilon} + f \left(a^{\varepsilon}, v^{\varepsilon}\right)_x = 0, & \text{in } \mathbb{R} \times \{t > 0\}, \\ v^{\varepsilon} = v_0 * \omega_{\varepsilon}, & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

where  $a^{\varepsilon} = a * \omega_{\varepsilon}$  and  $\omega_{\varepsilon}$  being the usual mollifying kernel with radius  $\varepsilon$ . Furthermore, u satisfies the wave entropy condition

(10) 
$$\operatorname{sign}(f_{vv}) \partial_x (f_v(a, v)) \ge K \left(\frac{1}{t} + |a'|\right)$$

in each interval where a' exists. The constant K depends on f,  $||a||_{\infty}$ , and  $v_0$ , but not on a'. Furthermore if  $v^{\delta}$  denotes the front tracking approximation to v, then

$$\lim_{\delta \to 0} v^{\delta} = v \text{ in } L^1_{loc}$$

Also, there are only a finite number of collisions between fronts in  $v^{\delta}$  for all  $t \in [0, \infty)$ .

The proof of this theorem can be found in [24]. Here we detail the approximate solution of the Riemann problem. The assumptions on f imply that for each a there is a unique  $v_T$  such that

$$f_v\left(a, v_T\right) = 0.$$

For simplicity we set  $v_T = 0$ . Let z(v, a) and b(a) be defined as

(11) 
$$z = z(a, v) = \text{sign} (v - v_T) (f(a, v) - f(a, v_T))$$
$$b(a) = f(a, v_T).$$

The Rieman problem for (7) is the tritical value problem where we and a take the form -

$$(6)$$

Hence, v<sub>0</sub> and a<sup>2</sup> defines a varies of Neumann problems repleted at their decompanities. If 3//30 is bounded, then (7) has finite speet of procession, and the solutions of negativities H 3//30 problems will not incorrect for proof 1. Teaching, if we can compare the contropy solutions to the initial Riemann problems, and therefore the solution of (7) if to = q he collisions and to illowever, being able to compare the solution of Hermitic problems totae on tells of an to the solution past the solutions (3) to a found-or of solutions problems totae problems to solution of the Hiemann problems. (5) to a found-or of solutions of a solution of the solutions of solution of the Hiemann problems. (5) to a found-or of solutions of a solution of the solution of the Hiemann constant.

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2.1. Convert f. H f(a, a) is unlikently convert in a suid monotone intra, then that any is welldefined. More precisely, from [24] we have whe following theorem:

Theorem 2.1. Assume that a 15 m L of SV and 5 presentes C with a finite construct of discontinuities. Assume also that od(b) is such that I (2, 11) is 25 formulat construct. Then more anothe a unique mark solution to 10 (1) such that w -2 to 21 C, tobers of solver the "consistent" ambieur

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$$\left(\Gamma_{0}\right) = \frac{1}{2} \left(\Lambda_{1} \leq \left(\Lambda_{1}, \eta_{1}, \Lambda_{1}\right) \otimes \left(\Lambda_{1}, \eta_{2}, \eta_{3}\right) \otimes \left(\Lambda_{1}, \eta_{3}\right) \otimes \left(\Lambda_{$$

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Also, there are only a finite number of collisions leaves a fronts intell formula E (0.00).

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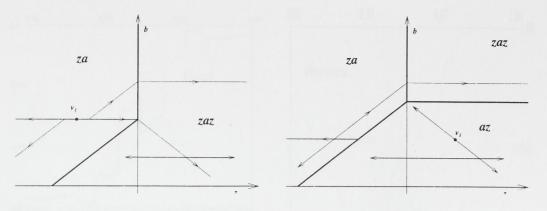


FIGURE 1. The solution of the Riemann problem.

Note that since  $f_a \neq 0$ , the mapping b(a) is one-to-one. Hence, the mapping

 $(a, v) \mapsto w = (b, z)$ 

is injective and regular everywhere except for z = 0. Thus the Riemann problem is determined by two states  $w_l$  and  $w_r$ .

In the following we use the notation  $f_l$  for  $f(a_l, v_l)$ , and similarly for other functions of the left and right states. We say that two states  $w_l$  and  $w_r$  are connected by an *a*-wave if sign  $(z_l) =$ sign  $(z_r)$  and  $f_l = f_r$ , similarly we say that they are connected by a z wave if  $a_l = a_r$ . The solution of the Riemann problem in the (z, b) plane is depicted in Figure 1. To find a particular solution, pick a right state  $v_r$  and follow the arrows from  $v_l$  to  $v_r$ . This traces out a series of waves, e.g., zaz, and the solution is then a found by connecting the  $v_l$  to the state to the right of the first z wave and so on. This diagram is entirely similar to the corresponding diagrams in [26] or [44]. The actual waves occurring in a z wave is found by solving the scalar Riemann problem with constant a (either  $a_l$  or  $a_r$ ) and  $v_l$  and  $v_r$  given by the endpoints of the curve. If the solution is determined by a zaz sequence, the first z wave will have non-positive speed, a waves will always have zero speed (they are discontinuities in a(x)) and the second z wave will have nonnegative speed. Note in particular that in the (z, b) plane, all waves trace curves which are either horizontal lines (z waves) or straight lines at an angle of 45° slope (a waves). Hence if we fix a grid  $(z,b) = (i\delta, j\delta)$ (for  $i, j \in \mathbb{Z}$  and some small number  $\delta > 0$ ) in the (z, b) plane and if the initial states  $w_l$  and  $w_r$ are points on the grid, then all intermediate states will also be points on the grid. Furthermore, if we interpolate f(a, v) linearly between grid points, the solution of the scalar Riemann problems determined by the z waves will consist of piecewise constant functions in x/t, see, e.g., [14]. Let this interpolation of f be denoted by  $f^{\delta}$ . Then the above construction yields an entropy weak solution to the initial value problem

(12) 
$$v_t^{\delta} + f^{\delta} (a(x), v^{\delta})_x = 0, \quad v^{\delta}(x, 0) = \begin{cases} w^{-1}(i\delta, b(a_l)), & x \le 0, \\ w^{-1}(j\delta, b(a_r)), & x > 0, \end{cases}$$

for any integers *i* and *j*. This solution will be piecewise constant in x/t where  $w(v^{\delta}(x,t), a(x))$  will be on the grid for all *x* and *t*.

We can also construct the approximation of initial function  $v_0^{\delta}$  and  $a^{\delta}$  such that

 $w\left(v_0^{\delta}(x), a^{\delta}(x)\right)$ 

is on the grid in the (z, b) plane. For a fixed  $\delta$ , we can then solve the initial value problem

(13) 
$$v_t^{\delta} + f^{\delta} \left( a^{\delta}, v^{\delta} \right)_{\tau} = 0, \quad v^{\delta}(x, 0) = v_0^{\delta}(x)$$

exactly using front tracking, see, e.g., [26]. Furthermore, for each  $\delta$ , there will only be a finite number of collisions between fronts in  $v^{\delta}$ .



FIGURE I. The solution of the Alendan metalin.

Note that since  $f_{s} \neq 0$  , the mapping  $\delta(s)$  as substituting the respect to  $\delta(s)$  and  $\delta(s)$ 

u injective and regular everys have extend for z = 0. Time the Namuni problem is ristermund by

In the following we use the notation h, for free, m, and similarly for other functions of the left and right states. We say that two make it multicless are connected by a restriction h and h = h, similarly we say that two anticommeted by a restriction h and h = h. Similarly we say that two anticommeted by a restriction h and h = h. Similarly we say that two h are depended by a restriction h and h = h. Similarly we say that two h are depended by a restriction h and h = h. Similarly h = h. Similarly we say that two h are depended by a restriction h and h = h. Similarly h = h are dependent in the two h are two h and h = h. The solution h are the following the solution h then a torus the restrict h = h are the following the solution h then a torus h = h and h = h are the restriction h = h. This traces can a sector h = h and h = h are the restrict h = h and h = h. The solution h = h are the restriction h = h are the restrict h = h and h = h. The solution h = h are the restrict h = h. The solution h = h are the restriction h = h are the restriction h = h. The solution h = h are the restriction h = h are the restriction h = h are the restriction h = h and h = h. The restriction h = h are the restriction h = h. There is the restriction h = h are the restriction h = h. The restriction h = h are the restriction h = h arestrestrestres h = h and h = h are the restriction h = h ar

$$0 \ge a = \left\{ \left( a(a), a^{*} \right)^{*} = \left( a, a^{*} \right)$$

for any integers it and it. This vehicles will be pleased to constant in mit relieve we(o'(x,x), eich) will be en the said for all r and t.

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is on the mid to the (2.4) phone (for a fixed 5, we can this solve the initial value problem.

$$f_{2}(x) = (0, x)^{-1} = 0 = 0$$

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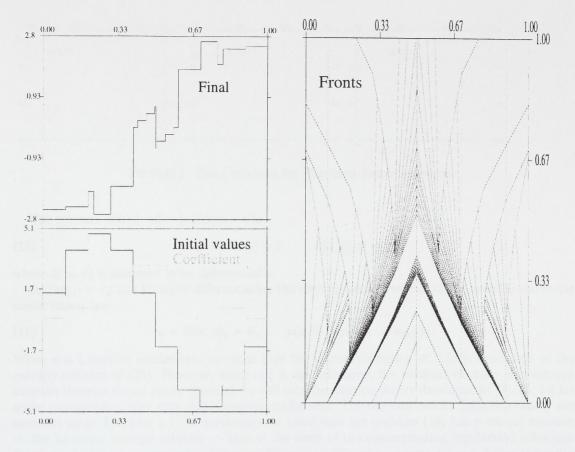


FIGURE 2. An example of front tracking.

In Figure 2 we show the fronts and the initial and final state for the initial value problem (13) with

(14) 
$$f(a,v) = \sqrt{1+a^2+v^2}, \quad v_0(x) = \frac{\pi^2}{2}\sin(2\pi x), \quad a(x) = \pi \left(1-\cos(2\pi x)\right),$$

and periodic boundary data. In this example we used  $\delta = 0.25$ . Figure 2 shows  $v_0^{\delta}$  and a in the lower left corner, and  $v^{\delta}(x, 1)$  in the upper left corner. To the right we see the fronts in the (x, t) plane. The z waves are shown as solid lines and the a waves as broken lines.

2.2. Other flux functions. Front tracking has been extended to other flux functions. In [26] it was shown to be well defined and convergent for Lipschitz continuous functions f(a, v) satisfying the requirement that there are values  $\alpha$ ,  $\beta$  and  $\gamma$  such that

$$f(a,\alpha) = f(a,\beta) = f(a,\gamma) = 0,$$

for all a. Furthermore,  $f_a < 0$ ,  $f_{vv} > 0$  in  $(\alpha, \beta)$  and  $f_a > 0$ ,  $f_{vv} < 0$  in  $(\beta, \gamma)$ . The prototype of such functions is  $f(a, v) = -a \sin(v)$ .

Another extension that we shall use later is to periodic f. Let g be a bounded Lipschitz continuous function of one real variable, and set f(a, v) = g(a + v). For such f front tracking for (13) is well-defined, see [23].

#### 3. NUMERICAL ALGORITHMS

3.1. **One-dimensional algorithm.** In this section, we recast the front tracking method from the previous section as a method for solving one-dimensional Hamilton-Jacobi equations with a discontinuous coefficient. This method will be used as an important building block in the multi-dimensional algorithm presented in the next section.



FIGURE BUSIES OF STREETS OF WORK SERVICE

In Figure 2 we show the fronts and the initial and final state for the induct value problem (13). with

$$(14) \quad f(a,b) = e^{(1+a^2+b^2)} \quad order = \frac{1}{2} \quad su(2\pi\pi), \quad s(a) = e^{(1+ab)^2} \\ (2a^2) = e^{(1+a^2+b^2)} \quad s(a) = \frac{1}{2} \quad s(a) = \frac{1}$$

and periodic boundary data. An this submule we used it is in the copper 5 shows as and a in the lower last corner, and e<sup>2</sup>(x, 1) in the upper left cornety. To the regit we see the fields in the (x, t) plane. The x waves are shown as solid lines and the a waves as headen lines.

2.2. Other flux functions. Featuressing has been estended to other dur-functions. In 200 it was shown to be web defined and convergent for deposition monthmous functions. Just a staffing the the requirement that there are values or of and reach while.

for all a. Furthermore,  $f_a < 0$ ,  $f_{ac} > 0$  in (a, b) and  $f_a > 0$ ,  $f_{ac} < 0$  in (b, c). The putturnes of each functions in  $f(a, v) = -a \sin(v)$ .

Another extension that we shall use later is to periodic f. Let g be a bounded Lipschitz continuous function of one real variable, and set f(g,u) = g(g eau). For such f from functing for (13) is well-defront, and [23].

#### RESIDENCIAL INCOMPANY R.

3.1. One-dimensional algorithms. In this section, we recast this irons against analog material iron the previous section as a mathod for advise one-dimensional function-found arous with a discontinuous coefficient. This method will be used as an industriate the biologies the mathod will be used as an industriate with a discontinuous coefficient. This method will be used as an industriate the biologies the mathod will be used as an industriate the biologies the section.

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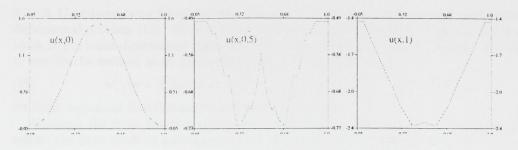


FIGURE 3. Front tracking for Hamilton-Jacobi equations.

The relevant initial value problem reads

(15) 
$$u_t + H(a, u_x) = 0, \qquad u(x, 0) = u_0(x),$$

where H(a, p) is assumed to be differentiable.

Setting  $p = u_x$  and formally differentiating the above problem, we find that p satisfies the scalar conservation law

(16) 
$$p_t + H(a, p)_x = 0, \qquad p(x, 0) = p_0(x) := u_{0x}(x).$$

When a is Lipschitz continuous, we recall that the viscosity solution of (15) is equivalent to the entropy solution of (16). However, when  $a(\cdot)$  is discontinuous, the classical viscosity and entropy solution theories do not apply. Instead, we will rely the solution theory developed in [26, 25, 24] for scalar conservation laws with discontinuous coefficients. The relevant results from this theory are summed up in Theorem 2.1. In particular, we know that the problem (16) has a unique solution — the so-called entropy solution — that is the limit of the corresponding regularized solutions. Furthermore by a recent results of Ostrov [40], if  $p \mapsto H(a, p)$  is convex, we can define "viscosity solutions" of (15) even when a has a finite number of discontinuities but elsewhere  $C^1$ . These are defined as the (unique!) limit of viscosity solutions of the "smoothed" equations

$$u^{\varepsilon} + H(a^{\varepsilon}, u^{\varepsilon}_{\tau}) = 0, \qquad a^{\varepsilon} = a * \omega_{\varepsilon}.$$

Let  $p^{\delta}$  be the front tracking approximation of (16). This algorithm is viable also as an algorithm for (15) almost without alterations. To define front tracking for (15), we need to keep track of the value of the approximate solution  $u^{\delta}$  along each front in  $p^{\delta}$ . Since  $p^{\delta}$  is piecewise constant,  $u^{\delta}$  will be piecewise linear between fronts. All fronts in  $p^{\delta}$  will move with constant speed between collision points, so the position of a front is given by

$$x(t) = x_0 + s(t - t_0)$$

where  $(x_0, t_0)$  is the starting point of the front. Let  $(p_l, a_l)$  and  $(p_r, a_r)$  denote the left and right states of the front. Then

(17)  
$$u^{\delta}(x(t),t) = u^{\delta}(x_{0},t_{0}) + (t-t_{0})(sp_{l} - H(a_{l},p_{l}))$$
$$= u^{\delta}(x_{0},t_{0}) + (t-t_{0})(sp_{r} - H(a_{r},p_{r})),$$

because of the Rankine-Hugoniot condition

$$s(p_l - p_r) = H(a_l, p_l) - H(a_r, p_r).$$

Figure 3 shows the front tracking approximation to the "Hamilton-Jacobi version" of (14), with H(a, p) = f(a, p) and

$$u_0(x) = \frac{\pi}{4} \left( 1 - \cos(2\pi x) \right).$$

To the left we see the initial approximation, in the middle  $u^{\delta}(x, 0.5)$ , and to the right  $u^{\delta}(x, 1)$ .

Fromes 3. From tracking for Romalion-laced and a second

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where H (a.p) is assumed to be differentiable.

Setting  $\mathbf{p} = \mathbf{u}_{e}$  and firmally differentiating the above problem, we find that provides the scalar conservation law

$$(16) = 0, \quad p(x, 0) = 0, \quad p(x, 0) = p_0(x) =$$

Les p<sup>2</sup> he the trest mocking approximation of (10). This algorithm is voluble algorithm in the first of a lagorithm in the trest in the second of a lagorithm the second of the second

where (20,40) is the statting points of the brack. Les (2, 2) and (2, 20.) denote the felt have spin states of the front. Then

$$(1, 2, 3, 3) = (2, 3) + (2, 3, 3) + (2, -2, 1) (2, 3, -2, 3)$$

because of the P.aniana-Hugoniot condition

Figure 3 shows the front trading approximation to the "Mamilton-Jacoid varian" of (14), with

$$((\pi)\pi)((\alpha) - 1) = (\pi)(\alpha)$$

Its the left we are the initial contraction along in the neidine  $w^{0}(x,0.6)$ , and to the right of (x,t)

3.2. A Godunov type formulation. The method just described is a good method for the onedimensional problem (15). We now present a formulation which makes it easy to use as a building block in the multi-dimensional algorithms described in the next section. Namely, we would like to to rewrite the method as an explicit marching scheme of the form

(18) 
$$u(\cdot, (n+1)\Delta t) = u(\cdot, n\Delta t) - \Delta t \mathcal{H}(a, u(\cdot, n\Delta t)), \quad n = 0, 1, 2, \dots,$$

for some numerical Hamiltonian  $\mathcal{H}$  and time step  $\Delta t > 0$ .

Although the front tracking approximation did not use any predefined time step  $\Delta t$ , we can restart the front tracking algorithm at  $t_n = n\Delta t$ , n = 1, 2, 3, ... To this end, let  $\Delta x$  be given and set

$$p_j^n = \frac{1}{\Delta x} \left( u^n (x_{j+1/2}) - u^n (x_{j-1/2}) \right),$$
  

$$a_j = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} a(x) \, dx,$$
  

$$a_{\Delta x}(x) = a_j \quad \text{and} \quad p_{\Delta x}^n(x) = p_j^n \quad \text{for } x \in [x_{j-1/2}, x_{j+1/2}),$$

where  $x_j = j\Delta x$ . Let  $\delta$  be some small parameter and define a grid in the (z, b) plane by combining a regular grid of size  $\delta$  with the grid determined by the points  $w(p_j^0, a_j)$ . This grid is the one we use to interpolate H, giving a function we label  $H^{\delta}$ . For  $t_n \leq t < t_{n+1}$ , let  $u^n$  be the front tracking solution of

(19) 
$$u_t^n + H^{\delta}(a_{\Delta x}, u_x^n) = 0, \qquad u^n(x, n\Delta t) = u^n(x_{1/2}) + \int_{x_{1/2}}^x p_{\Delta x}^n(\sigma) \, d\sigma.$$

Finally set

$$p_j^{n+1} = \frac{1}{\Delta x} \left( u^n \left( x_{j+1/2}, t_{n+1} - \right) - u^n \left( x_{j-1/2}, t_{n+1} - \right) \right).$$

We start this process by setting  $u^0(x) = u_0(x)$ . Note that  $u^n(x, t_{n+1})$  is a piecewise linear function in x, with breakpoints located at  $\{x_{j+1/2}\}$ . This method can also be recast in a simpler notation by noting that

$$u_{j+1/2}^{n+1} = u_{j+1/2}^n - \int_{t_n}^{t_{n+1}} H^{\delta} \left( a_{j+1/2}, u^n \left( x_{j+1/2}, t \right) \right) \, dt.$$

The integral term here defines our numerical Hamiltonian  $\mathcal{H}$ . The integral does not have to be computed explicitly, as this is already done in the front tracking process. By (17), we directly read off the value of  $u_{j+1/2}^{n+1}$  from the front located at  $x_{j+1/2}$ . Since this is a point of discontinuity for  $a_{\Delta x}$  there will be a front present at this location. If by chance  $a_{\Delta x}$  is continuous here, we can easily add an extra front in the front tracking process.

**Remark.** Although we have (re)formulated the front tracking method as an explicit marching scheme (18) and thereby introduced a time step into the method, one should note that there is no CFL condition associated with (18), i.e., large time steps are allowed.

3.3. Multi-dimensional algorithms. Now we use the Godunov-type method (18) to formulate "large time step" methods for the multidimensional problem (1). For ease of presentation, we shall restrict ourselves to two space dimensions, but the generalization to three or more dimensions is obvious. Therefore we study problems of the form

(20) 
$$\begin{cases} u_t + H(u_x, u_y) = 0, & \text{in } \mathbb{R}^2 \times \{t > 0\}, \\ u = u_0, & \text{on } \mathbb{R}^2 \times \{t = 0\}, \end{cases}$$

where the Hamiltonian H is of the types discussed in the previous sections. We can write (20) as a  $2 \times 2$  system conservation laws formally obtained by differentiating (20):

(21) 
$$p_t + H(p,q)_x = 0, q_t + H(p,q)_y = 0,$$

3.2. A Godunov type formulation. The method just described is a good method for the one dimensional problem (16). We now present a formulation which makes it easy to use as a building block in the main-dimensional algorithms described in the rest section. Namely, we would like to to rewrite the method as an explicit matching otherse of the form.

$$(18) = u(\cdot(n+1)\Delta n) = u(\cdot(n\Delta n) + \Delta n(t(n,0), n+0,1), n = 0, 1, 2, \dots, n)$$

for some numerical Ramiltonian F and time step 21 > 6.

Although the front fracking approximation (iii) not not any predshinot time step also we can restart the front presiding signation at  $t_0 = d\Omega t$ , n = 1, 2, 3. To this cod, let the to press and set

$$m = \frac{1}{2\pi} \left( \frac{m^2}{m} \left( \frac{m^2}{m} \right) - \frac{m^2}{m} \left( \frac{m^2}{m} \right) - \frac{m^2}{m} \left( \frac{m^2}{m} \right) \right)$$

$$m = \frac{1}{2\pi} \left( \frac{m^2}{m} \left( \frac{m^2}{m} \right) + \frac{m^2}{m} \left( \frac{m^2}{m$$

where  $z_i = j\Delta x$ . Let  $\delta$  be some small parameter and define a prick in the (a, b) passed by combining a regular grid of aize  $\delta$  with the sure dimension by the points  $z_i[x_i]_{i=1}^{k}$ . This grid is the one we use to interpolate H, giving a familion we label  $H^{k}$ . For  $k_i \leq i \leq k_{max}$ , let  $w^{k}$  be front tracking solution of

(39) 
$$a_{i}^{2} + B^{*}(a_{2}, a_{2}^{2}) = 0, \quad a^{*}(a_{2}, a_{2}^{2}) = a^{*}(a_{1}, a_{2}^{2}) + \int_{a_{1}a_{2}}^{a} B_{2}(a_{1}, a_{2}^{2}) + \int_{a_{1}a$$

Finally set

$$\left(\left(-n+ih_{2}(n_{1})^{2}\right)^{2}\right)^{2}=\left(-n+ih_{1}(n_{1})^{2}\right)^{2}\left(n\right)^{2}=\frac{1}{2\Delta}=1$$

We start this process by setting  $\sigma(z) = \sigma_0(z)$ . Note that  $\sigma'(z_1, z_2, z_{22})$  is a piecewise linear function in  $z_1$  with log-stochast located as  $\{\sigma_1 + z_2\}$ . This method can also be reast in a surplear broatlear by noting that

$$u_{j+1,n}^{(n)} = u_{j+1,n}^{(n)} \cdots \int H^{n} \left( u_{j+1,n} , \mu^{n} \left( x_{j+1,n} , \mu^{n} \right) \right) dt$$

The integral term here defines our adjusted Haminosine K. The integral does not involve to be computed explicitly, as this is already done in the front tracking process. By (17), we directly read of the value of  $a_{p,1/2}^{(1)}$  from the front icented at  $a_{p+1/2}$ . Since this is a point of discontinuity for  $a_{p,2}$  three will be a heat present at this location. If by drance  $a_{p,2}$  is continuous here, we can easily add on vertex from to the front tracking process.

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$$(u_{n} + B(u_{n}, u_{n}) = 0, \quad \text{in } E^{*} \times \{t > 0\}, \\ u_{n} = u_{0}, \quad \text{in } E^{*} \times \{b = 0\},$$

where the Hamiltonian H is of the types discussed in the provious sectors. We can write (20) at a 2 × 2 sectors contervation have formally obtained by differentiable (20): z

2

where  $(p,q) = (u_x, u_y)$  and

$$(p,q)(x,y,0) = ((u_0)_x, (u_0)_y)(x,y)$$

One should notice (20) is weakly hyperbolic in the sense that there is no complete set of eigenvectors. As already mentioned, in [34, 20] it is shown that the vanishing viscosity limit solution of (20) is equivalent to the vanishing viscosity limit solution of (21).

In order to define our scheme, we let  $\delta > 0$  be some small number. All our computed quantities will depend on this number, but for simplicity our notation does not always indicate this dependency. We use a computational grid  $x_j = j\Delta x$ ,  $y_k = k\Delta y$ , and  $t_n = n\Delta t$ , for small numbers  $\Delta x, \Delta y, \Delta t$  and integers  $j, k \in \mathbb{Z}$ ,  $n = 0, \ldots, N$ , where  $N\Delta t = T$ . To integrate (21) numerically, we can use dimensional splitting or a direct approach.

3.3.1. Dimensional splitting. Dimensional splitting for (20) is based on the sequential solution of the two conservation laws in (21) for a time step  $\Delta t$ , using the result for one equation as coefficients in the other. Concretely, this gives the following scheme: First set

(22) 
$$U_{j+1/2,k}^{0} = u_0 \left( x_{j+1/2}, y_k \right),$$

(23) 
$$V_{j,k+1/2}^{0} = u_0 \left( x_j, y_{k+1/2} \right),$$

(24) 
$$p_{j,k}^{0} = \frac{1}{\Delta x} \left( U_{j+1/2,k}^{0} - U_{j-1/2,k}^{0} \right) ,$$

(25) 
$$q_{j,k}^0 = \frac{1}{\Delta y} \left( V_{j,k+1/2}^0 - V_{j,k-1/2}^0 \right).$$

For t in the interval  $[t_n, t_{n+1})$  and for each k, we let  $U_k^n(t)$  be the front tracking solution to

(26) 
$$(U_k^n)_t + H^{\delta} \left( (U_k^n)_x, q_k^n \right) = 0, \qquad U_k^n \left( x, t_{n-1} \right) = U^n(x_{1/2}) + \int_{x_{1/2}}^x p_k^n(\sigma) \, d\sigma,$$

where the functions  $p_k^n$  and  $q_k^n$  are defined as

$$\begin{array}{l} q_k^n(x) = q_{j,k}^n \\ p_k^n(x) = p_{j,k}^n \end{array} \right\} \quad \text{for } x \in \big[ x_{j-1/2}, x_{j+1/2} \big\rangle.$$

Similarly to (19) and (17), this gives us an update based on a numerical Hamiltonian  $\mathcal{H}_{j+1/2,k}^{p,n}$  (which we never have to compute)

(27) 
$$U_{j+1/2,k}^{n+1} = U_{j+1/2,k}^n - \Delta t \mathcal{H}_{j+1/2,k}^{p,n}$$

Then we set

(28)

$$p_{j,k}^{n+1} = \frac{1}{\Delta x} \left( U_{j+1/2,k}^{n+1} - U_{j-1/2,k}^{n+1} \right)$$

This finishes the first part of the splitting step. As  $U^n$  was the solution of the first equation in (21), we let  $V^n$  denote the solution of the second. Precisely, for t in the interval  $[t_n, t_{n+1})$  and for each j, define  $V_j^n$  as the front tracking solution of

(29) 
$$\left(V_{j}^{n}\right)_{t} + H^{\delta}\left(p_{j}^{n+1}, \left(V_{j}^{n}\right)_{y}\right) = 0, \qquad V_{j}^{n}\left(t_{n}\right) = V_{j}^{n}\left(y_{1/2}\right) + \int_{y_{1/2}}^{y} q_{j}^{n}(\sigma) \, d\sigma,$$

where

$$p_j^{n+1}(y) = p_{j,k}^n \\ q_j^n(y) = q_{j,k}^n$$
 for  $y \in [y_{k-1/2}, y_{k+1/2}).$ 

Similarly to (27), we now can define  $V_{j,k}^{n+1}$  by a numerical Hamiltonian  $\mathcal{H}_{j,k+1/2}^{q,n}$  as follows

(30) 
$$V_{j,k+1/2}^{n+1} = V_{j,k+1/2}^n - \Delta t \mathcal{H}_{j,k+1/2}^{q,n}.$$

where (p.q) == (p.q) and v

One should notice (20) is weatly hyperiodic in the street that there is no compare net of againstetors. As already meanimed, m (34, 30) it is shown that the vanishing viscosity limit solution of (20) is emittalent to the variabing viscosity with solution of (21).

In order to define our ediants, we let  $\delta > 0$  be seen much mapper. All our computed quarttities will depend on this marcher, but its daughents our accession need not devers indense this dependency. We use a computational gold  $x_2 = 0$  for  $y_1 = 0$  by and  $t_2 = 0$  for scall minimum  $\Delta x$ ,  $\Delta y_1$ ,  $\Delta t$  and integers  $y_1 \in \mathbb{Z}$ , n = 0, ..., k, wowe  $K\Delta t = 1$ . To integrate (24) monochrolic, we can use dimensional splitting or a direct approach.

3.3.1: Dimensional splitting. Dimensional splitting for (29) is based on the sequential solution of the two conservation have in (21) for a time step Mr. using the result for one equiption as unlikelying in the other. Concretely, this gaves the following concrete First w:

$$\alpha_{i,k} = \sum_{i=1}^{n} \left( \left[ \alpha_{i,k} \right]_{i=1}^{n} \left( \left[ \alpha_{i,k} \right]_{i=1}^{n} \left( \alpha_{i,k} \right]_{i=1}^{n} \left( \alpha_{i,k} \right)_{i=1}^{n} \left( \alpha_{i,k} \right)_{i=1}^{n}$$

For t is the interval  $\{t_n, t_{n-1}\}$  and for each k, we let  $b_1^{(n)}(n)$  be the laser exclusive colution as

$$(26) \qquad (U_{k}^{2})_{k} + M^{2} \left( (U_{k}^{2})_{k}, u_{k}^{2} \right) = 0, \qquad U_{k}^{2} \left( u_{k}, u_{k}, u \right) = U^{2} \left( u_{k}, u_{k} \right) + \int_{-\infty}^{\infty} U_{k}^{2} \left( u_{k}^{2}, u_{k}^{2} \right) + \int_{-\infty}^{\infty} U$$

where the functions of and of ace defined as

Similarly to (19) and (17), this rives us an update bread on a numerical Harningman  $H_{2,1/2,k}^{(1)}$ 

$$U_{1,1,2,3}^{(n)} = U_{1,1,2,3,4}^{(n)} = U_{1,1,2,3,4}^{(n)} = \Delta U_{2,2,3,3,4}^{(n)}$$

Then sur north

$$\frac{1}{2}\left(122\right) = \frac{1}{22}\left(122\right) = \frac{1}{22}\left(1$$

$$= b(xy)_{1} + b(x_{1})_{2} + b(x_{2})_{3} = 0, \quad yy = b(x_{1}) + b(x_{2})_{3} +$$

where

Similarly to (27), we now can deline 1775 by a numerical Elemitication 30 and

To start the next time step, we define

(31) 
$$q_{j,k}^{n+1} = \frac{1}{\Delta y} \left( V_{j,k+1/2}^{n+1} - V_{j,k-1/2}^{n+1} \right)$$

This process is then continued for n = 0, 1, 2, ..., N - 1, where  $T = N\Delta t$ . Now we have two approximations to the solution of (20), namely  $U_{j+1/2,k}^n$  and  $V_{j,k+1/2}^n$ . Note that these are defined on spatial grids which are staggered with respect to each other. We can define the final approximation by linear interpolation between these two grids to the grid defined by the points  $(x_j, y_k)$ . This corresponds to using the update formula

(32)  
$$u_{j,k}^{n+1} = u_{j,k}^{n} - \frac{\Delta t}{4} \left( \mathcal{H}_{j-1/2,k}^{p,n} + \mathcal{H}_{j+1/2,k}^{p,n} + \mathcal{H}_{j,k-1/2}^{q,n} + \mathcal{H}_{j,k+1/2}^{q,n} \right)$$
$$= \frac{1}{4} \left( U_{j+1/2,k}^{n+1} + U_{j-1/2,k}^{n+1} + V_{j,k+1/2}^{n+1} + V_{j,k-1/2}^{n+1} \right).$$

This update formula does not have to be used, we can merely interpolate at the end of the splitting process where n + 1 = N.

A variant of this method is to update  $u^{n+1}$  before setting  $p^{n+1}$  and  $q^{n+1}$  to be used in the next time step. These are then defined by

(33)  
$$p_{j,k}^{n+1} = \frac{1}{\Delta x} \left( u_{j+1/2,k}^{n+1} - u_{j-1/2,k}^{n+1} \right),$$
$$q_{j,k}^{n+1} = \frac{1}{\Delta y} \left( u_{j,k+1/2}^{n+1} - u_{j,k-1/2}^{n+1} \right),$$

where  $u_{j+1/2,k}^{n+1}$  is the interpolated value of the piecewise linear function defined by (32) at the point  $((j + 1/2)\Delta x, k\Delta y)$ . Similarly for  $u_{j,k+1/2}^{n+1}$ . We call this method dimensional splitting with restarting.

3.3.2. A direct method. Rather than solve the p equation and the q equation sequentially, we can solve both for  $p_k^n$  and  $q_j^n$  using the values from the previous time step as coefficients. This we call a direct method. The initial values are defined as before, (24), (25). For  $t \in [t_n, t_{n+1})$ , we define  $U_k^n$  and  $V_j^n$  to be the front tracking solutions of

(34) 
$$(U_k^n)_t + H^{\delta} \left( (U_k^n)_x, q_k^n \right) = 0, \quad U_k^n \left( x, t_n \right) = U_{1/2,k}^n + \int_{x_{1/2}}^x p_k^n(\sigma) \, d\sigma,$$

(35) 
$$\left(V_{j}^{n}\right)_{t} + H^{\delta}\left(p_{j}^{n}, \left(V_{j}^{n}\right)_{y}\right) = 0, \qquad V_{j}^{n}\left(y, t_{n}\right) = V_{j,1/2}^{n} + \int_{y_{1/2}}^{y} q_{j}^{n-1}(\sigma) \, d\sigma.$$

We can use either  $U^n$  or  $V^n$  as an approximation to u, or use the interpolation defined by (32). We can define  $p^{n+1}$  and  $q^{n+1}$  by (33). This method is then called a direct method with restarting.

**Remark.** The reader should be cautioned that, in order to keep the presentation simple, our notation is somewhat misleading. The functions denoted " $H^{\delta}$ " in, e.g., (35) and (34) are not the same function! But rather two different piecewise linear approximations of H. Remember that when doing front tracking for, e.g., (35) we use a piecewise linear (in q) and piecewise constant (in x) approximation to H(p(x), q). This approximation depends on  $\delta$ , so that the distance between the interpolation points tends to zero as  $\delta \to 0$ , as well as on the initial values q(x, 0) and the coefficients p(x). Since only  $\delta$  is the same for (35) and (34),  $H^{\delta}$  in (35) and (34) are not the same, nor are they the same for different j and k. The same also applies to the dimensional splitting equations (26) and (29).

Note that none of the methods we present are monotone. This makes a convergence analysis complicated, and we have not been able to prove that the methods produce a sequence of approximate solutions that converges to the unique viscosity solution. However, the numerical experiments indicate that the approximations all converge to the viscosity solution.

#### COMPANY AND REPARTO

To start the next time step, we define

$$(\alpha) = \frac{1}{2} \left( (\alpha) + \frac{1}{2} \right) \frac{1}{2} = \frac{1}{2} \left( (\alpha) + \frac{1}{2} \right)$$

$$(32) = (12)^{-1} (12)^{-$$

This update formula does not have to be used, we can metery fine points at the end of the splitting process where n + 1 = N.

A variant of this method is to unders a "" serve atting p"", and v"" to be used in the first time step. These are then defined in

where  $u_{1,1,2,1}^{\pm}$  is the interpolated with of the piecewise factor faction defined by (32) at the point  $(j \pm 1/2)\Delta z$ ;  $\delta \Delta y$ ). Similarly for  $u_{1,1,2,2}^{\pm}$ . We call this are find dimensional splitting with restarting

3.3.2. A drivent method. Rather than tokwe the proposition and the q imposition any subsectivity, we can solve both for  $p_1^*$  and  $q_2^*$  using the values from the propose time rich as coefficients. This we call a direct method. The initial values are defined as brintly, (24), (25). For  $t \in \{t_0, t_{0+1}\}$ , we define  $W^*$  to be the from tracking contribution of

$$(24) \qquad (07)_{1} \rightarrow 17^{2} ((07)_{1}, q_{1}^{2}) = 0, \quad 07 (q_{1}, q_{1}^{2}) = 0, \quad (77)_{1} q_{1}^{2} (q_{1}^{2}) = 0, \quad (77)_{1} q_{1}^{2} (q_{$$

$$(35) \qquad (17) + 17 (17) = 0 \qquad (77) = 0 \qquad (77$$

We can use efficie  $U^{\infty}$  or  $V^{\infty}$  as an appendimention to  $u_i$  or use the interpolation defined by (32). We can define  $u^{n+1}$  and  $e^{n+1}$  by (23). This method is deep called a direct method with remaining

Remark. The reactor stantic to the maximum denoted that, is earlier to interp the presentation anopy, and notation is somewhat massimations. The fluctuous denoted "B" in e.g. (.16) and (.20) are out the same function. But reacher and different piecewise listent approximations of H. Remarker that when doing into tracking for, e.g., (.35) we can a parcestar have (in g) and piecewise construct. (is a) approximation to H(p(x), g). This approximation downids on h so that the distance between the interpolation points tends to zero as  $b \rightarrow 0$ , as seek as on the initial values q(x, 0) and the coefficients p(x). Since only h is the same for (.35) and (N), H' in (.35) and (24) are not the nor are they the same for different j and h. The same also applies to the dimensional epidemic equations (.26) and (.29).

Note that none of the methods we present are monstorie. Fine makes a commutate analysis complicated, and we have not been able to prove that the methods produce a sequence of approximate solutions that converges to the adique vistority solution. However, the numerical experiments indicate that the approximations all convergento the viscosity aduation.

#### 4. NUMERICAL EXAMPLES

To test the above methods, we have compared them with several other methods: the Lax-Friedrichs method, the Engquist-Osher scheme, and the relaxation method by Jin and Xin [20], more precisely with the method called III in [20]. This method is based on replacing (20) by the following system

(36)  

$$p_{t} + w_{x} = 0,$$

$$q_{t} + w_{y} = 0,$$

$$w_{t} + a(p_{x} + q_{y}) = -\frac{1}{\varepsilon} (w - H(p, q)),$$

$$u_{t} + w = 0,$$

$$w(x, 0) = H (p_{0}, q_{0}),$$

where  $\varepsilon$  is a (very) small parameter. For the implementation of scheme III, we have followed the recipe in [20]. The Lax-Friedrichs scheme we used is given by

$$(37) \quad u_{j,k}^{n+1} = \frac{1}{4} \left( u_{j-1,k}^n + u_{j+1,k}^n + u_{j,k-1}^n + u_{j,k+1}^n \right) - \Delta t H \left( \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2\Delta x}, \frac{u_{j,k+1}^n - u_{k,j-1}^n}{2\Delta y} \right)$$

Finally the Engquist-Osher scheme reads

$$p_{1} = \frac{1}{\Delta x} \left( u_{j,k}^{n} - u_{j-1,k}^{n} \right), \quad p_{2} = \frac{1}{\Delta x} \left( u_{j+1,k}^{n} - u_{j,k}^{n} \right),$$

$$q_{1} = \frac{1}{\Delta y} \left( u_{j,k}^{n} - u_{j,k-1}^{n} \right), \quad q_{2} = \frac{1}{\Delta y} \left( u_{j,k+1}^{n} - u_{j,k}^{n} \right),$$

$$^{1} = u_{i,j}^{n} - \Delta t \left( H \left( p_{1}, q_{1} \right) + \int_{-}^{p_{2}} \min \left( \frac{\partial H}{\partial n} \left( p, q_{1} \right), 0 \right) dp + \int_{-}^{q_{2}} \min \left( \frac{\partial H}{\partial q} \left( p_{1}, q \right), 0 \right)$$

 $q_1$ 

Details on the implementation of front tracking for one-dimensional Hamilton-Jacobi equations can be found in [22], and for details of implementation of front tracking and dimensional splitting, see [42], [16], [15]. In the numerical examples we use all our methods: (32) and (27)-(30) as well as the method with restarting (33) and (34)-(35). Furthermore, when applicable, we used Strang splitting, i.e., we start and finish with (27) using a time step  $\Delta t/2$ .

In our first two examples we use the convex Hamiltonian

(39) 
$$H(p,q) = \sqrt{1+p^2+q^2}.$$

**Example 1.** Our first example is taken from [20]. The initial data is given by

 $\tilde{p}_1$ 

(40) 
$$u = \frac{1}{4} \left( \cos(2\pi x) - 1 \right) \left( \cos(2\pi y) - 1 \right) + 1,$$

for x and y in the unit square  $0 \le x \le 1$ ,  $0 \le y \le 1$ , and we use periodic boundary data. The exact solution is unknown, and as a reference solution we used an approximation computed by scheme III with  $\Delta x = \Delta y = 1/511$ . We calculated<sup>1</sup> the approximate solutions until t = 0.6, at this time the surface has moved down and a sharp peak has formed. In Figure 4 we show solutions computed on a  $50 \times 50$  grid by scheme III and by dimensional splitting using CFL=5.

When doing dimensional splitting, both the quality of the solution as well as the CPU time depends on the parameter  $\delta$ . In order to avoid too many parameters, we set  $\delta = \sqrt{\min(\Delta x, \Delta y)}$ .

In Table 1 we show the supremum errors and the CPU time (in seconds) for dimensional splitting as well as for scheme III, the Engquist-Osher scheme and the Lax-Friedrichs scheme on several grid sizes (indicated by N in the table). The most salient feature of this table is that the error and the CPU time for dimensional splitting seem to be independent of the CFL number. All schemes seem to have a numerical convergence rate of about 1/2, and a dependence "CPU time  $\sim \text{error}^{-5/2}$ ". The Lax-Friedrichs scheme was very fast, but produced much larger errors than the other schemes.

(38)

 $u_{i,k}^{n+1}$ 

<sup>&</sup>lt;sup>1</sup>All computations were done on a Power Macintosh G3, 267MHz, and the CPU times reported include only the computations, not the time used for initializations and memory allocations. All algorithms were coded in the "C" programming language.

#### A DESERVER AL STRACT STRACT

To test the above methods, we have compared them with several other methods: the Law Friedrichs method, the Engquase-Orber scheres, and the minimation method by Jin and Xin (20), more precisely with the method called III in (20). This motiond is taked on replacing (20) by she following system

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where s is a (very) small parameter. For the implementation of tokenic fill, we have relieved the recipe in [20]. The Lox-Frindrich's scheme we used is given by

$$(37) \quad u_{22}^{-1} = \frac{1}{2} \left( u_{2-1}^{-1} + u_{2+1}^{-1} + u_{2-1}^{-1} + u_{2-1}^{-1} \right) = \frac{1}{242} \left( \frac{u_{22}^{-1} + u_{2-1}^{-1} +$$

r mally the Engquist-Okber whene routs

$$(as) = \frac{1}{2} (a_{2} - a_{2} - a_{3}) + a_{2} - a_{2} - a_{3} - a_{3} + a_{$$

Details on the implementation of front tracking for one-dimensional Statiston-facult equations can be found in [22], and for details of implementation of front receiping and dynamonal splitting as [42], [16], [16]. In the numerical converses to any all our methods. (53) and (27)-(50) as well as the method with restarting (53) and (54)-(56). Forthermore, steer applicable, we need Strang splitting, i.e., we atter and fairst with (37) using a time step (29/2

In our first two examples we pay the convex Hard tonian

Example 1. [Our field stampletis table from Diff. The fullief data is print by

for x and y in the unit square  $0 \le x \le 1$ ,  $0 \le y \le 1$ , and we use periodic boundary data. The exact solution is inductive site as a reference solution we need as approximation computed by acherical field y = 1/50. We colorished the approximate solutions and t = 0.5, at this time the surface has moved down and a situated the site format. In Figure 4 we show solutions to compute down the site of x and y is a site of x and y and y are the surface of x and y = 1/50. We colorished the surface of x and y = 1/50. We colorished the surface of x and y = 1/50. The colorished the surface of x and y are the site of x and y are the surface of x

When doing dimensional splitting, beam the quality of the solution as well as size CPU time depends on the parameter  $\delta$ . In order to avoid too many parameters, we set  $\delta = \sqrt{\sinh Ch}$ , but

as well as for advance III, the Engraph Crear advance and the Laz-Freedorichs advance on several grad nizes (indicated by .N in the table). The most selicor feature of this table is that the error and the CPU time for dimensional splitting seems to be independent of the OFU, connect: All when a view to have a numerical convergence rate of about 1/2; and a depindence. CPU time - most <sup>-170</sup>. The Laz-Friedrichs convergence rate of about 1/2; and a depindence. CPU time - most <sup>-170</sup>.

All computations were done on a Fourier Macriferian GC, 2074 first and the GFU three-reported indicate brill, the computations, not the time used for initializations and musicary allocatifies. All algorithms very confed in the 20% group interact first structure were confed in the 20% and an and the 20% a

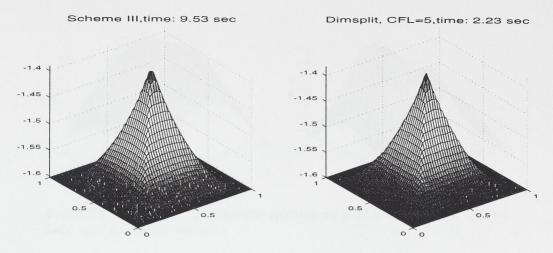


FIGURE 4. Example 1. Scheme III (left) and dimensional splitting (right).

	Dimensional splitting-Front tracking						Scheme	e III	Lax-Friedrichs		Engquist-Osher	
	CFL=	CFL=5 CFL=10 CFL=15										
N	$l^{\infty}$ -error	time	$l^{\infty}$ -error	time	$l^{\infty}$ -error	time	$l^{\infty}$ -error	time	$l^{\infty}$ -error	time	$l^{\infty}$ -error	time
16	0.0074	0.2	0.0217	0.1	0.1532	0.1	0.0210	0.6	0.0773	0.2	0.0142	0.2
32	0.0023	0.9	0.0047	0.6	0.0079	0.5	0.0203	2.8	0.0357	0.7	0.0087	0.4
64	0.0013	5.6	0.0047	4.9	0.0028	4.5	0.0061	19.0	0.0126	1.9	0.0051	1.9
128	0.0012	51.3	0.0009	50.5	0.0010	48.5	0.0035	143.6	0.0114	5.2	0.0031	13.3

TABLE 1. Supremum errors and CPU time for Example 1.

Figure 5 shows the supremum errors for the various methods as a function of the grid size. We see that on a given grid, the front tracking/dimensional splitting approach compares favorably with the three other schemes, and we again stress that the error is not very sensitive to the choice of the CFL number.

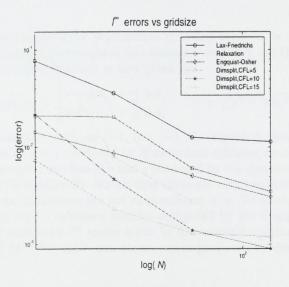


FIGURE 5. A log-log plot of the supremum errors versus the grid size for Example 1.





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TABLE 1. Supremum errors and CPU ther for Example 1.

Figure 5 shows the supremum errors for the various methods as a function of tan grid size. We can that on a given grid, the treat tradicity/dimensional splitting approach conjustes from filly with the three other schemes, and we again spread that the ampril is not rear sensitive to the choice of the CFL number.



Flouise 5. A log-log plot of the missenning strong vents and and state proteined from the parameter for

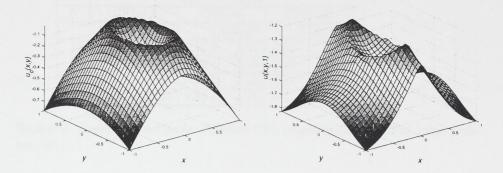


FIGURE 6. Example 2. Dimensional splitting on a  $32 \times 32$  grid with CFL=2. Left:  $u_0(x, y)$ . Right: u(x, y, 1).

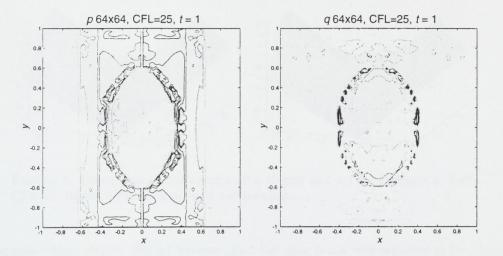


FIGURE 7. Dimensional splitting on a  $64 \times 64$  with CFL=25. Left: p(x, y, 1). Right: q(x, y, 1).

**Example 2.** The errors produced by dimensional splitting seem to be quite insensitive to the choice of  $\Delta t$ . Our next example investigates this feature closer. We use the same Hamiltonian as before (see (39)), but the initial function is now given by

(41) 
$$u_0(x,y) = \begin{cases} r - 0.4, & r \le 0.4, \\ 0.4 - r, & r \ge 0.4, \end{cases} \qquad r = \sqrt{x^2 + 0.4y^2},$$

and we use periodic boundary data on  $[-1,1] \times [-1,1]$ . Figure 6 shows the initial data and the approximate solution at t = 1 produced by dimensional splitting on a  $32 \times 32$  grid with CFL number 2. Figure 7 shows contour plots of p and q produced by the dimensional splitting algorithm on a  $64 \times 64$  grid with CFL number 25. Notice that these are very oscillatory in the vicinity of the shocks. Fortunately, these oscillations are not as prominent in u. To check the errors produced by dimensional splitting, we used a reference solution computed by the Engquist-Osher scheme on a  $512 \times 512$  grid. Table 2 shows the  $l^{\infty}$  errors made by dimensional splitting on various grids with CFL numbers 2, 5, and 25. This table also shows errors produced by dimensional splitting with restarting, see (33).

We remark that the errors produced by dimensional splitting in this example were of roughly the same order as those produced by the Lax-Friedrichs scheme and larger than those produced by



FIGURE 6. Lixenske 2. Dimensional quintus; on a 22 x 32 god well CFL-a. Left.  $u_0(x,y)$ . Eight u(x,y,1).



FIGURE 7. Dimensional splitting on a ST x 64 with CEL=25. Loit: 2(2, 3, 1). Right: o(2, 3; 1).

Example 2. The errors produced by dimensional splitting seem to be quite anomalies an the choice of At. Our next example investigates this feature does. We use the same Hamiltonian of before (see (391), but the initial function is now given by

$$\frac{1}{2} \left\{ \frac{1}{2} \left$$

and we use periodic boundary data on  $[-1,1] \times [-1,1]$ . Figure 6 shows the initial data and the approximate existion at  $t \approx 1$  produced by dimensional splitning on a 32 x 35 grid with CFL attorber 2. Figure 7 shows contain plots of p and g produced by the dimensional splitning algorithm on a 64 x 65 grid with CFL mumber 25. Notice that these are very oscillatory in the vicinity of the shocks. For these oscillations are not as prominent in a To chick that errors produced by dimensional splitting, we used a reference solution computed by the Engquint-Daim Mantee on 5 512 x 512 grid. Table 3 shows the  $\Gamma^{\circ}$  errors made by dimensional relating on various grids with CFL mumbers 2, 6, and 25. This table also shows or out produced by dimensional splitting the various grids restarting, see (33).

We remark that the errors produced by dimensional spiriting in this ascrupte vary of nonghey plot accordingly plot.

		$100 \times l^{\infty}$ error									
N	CI	FL=2	CI	FL=5	CFL=25						
		restart		restart		restart					
16	7.72	9.82	5.25	5.21	5.67	5.67					
32	4.56	6.60	3.22	3.56	6.22	6.22					
64	4.65	3.97	3.98	2.28	4.45	4.36					
128	2.88	2.25	2.38	1.23	3.49	4.57					

TABLE 2. Supremum errors for different CFL numbers for the initial value problem (41).

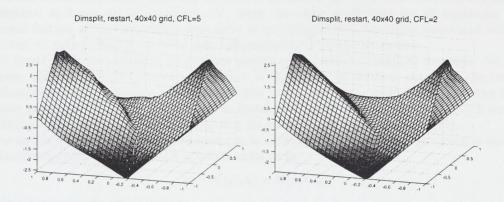


FIGURE 8. Dimensional splitting on a  $40 \times 40$  grid at t = 1 for Example 3. Left: CFL=5. Right: CFL=2. hva med ref losning....

the Engquist-Osher scheme. We see however, that the error is not very sensitive to the choice of CFL number. With CFL number 25, dimensional splitting still produces acceptable results. If we were interested in the derivatives, we would perhaps not find the accuracy in p and q quite satisfactory for this large CFL number, see Figure 7. We also remark that restarting seems to do little to improve the accuracy.

**Example 3.** To test dimensional splitting on a nonconvex case, we chose an example taken from Osher and Shu [39]. The relevant Hamiltonian reads

(42) 
$$H(p,q) = \sin(p+q).$$

The initial function is given by

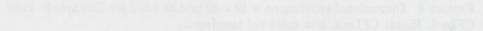
(43) 
$$u_0(x,y) = \pi(|y| - |x|).$$

We compute approximations in the square  $[-1,1] \times [-1,1]$  and impose no boundary conditions, i.e., fronts are allowed to pass the boundary undisturbed. This example was tested on the Lax-Friedrichs scheme and a number of ENO type schemes in [39].

To test out one of the methods described in the present paper, we used the direct method with restarting as described in Section 3.3.2. For this example, perhaps due to the nonconvex Hamiltonian, we found that the errors were more sensitive to the choice CFL number, see Figure 8 for an illustration of this. Here we show the computed solutions on a  $40 \times 40$  grid using CFL=5 and CFL=2. We clearly see the erroneous oscillations in the vicinity of the shocks for the solution with CFL=5.

LABLE 2. Supremum terrols for different CEL manipers for the tritual raise proh-





the Engquist-Other acheme. We see iterative, that the ender is not very sensitive to the choice of OFL mumber. With CFL manker, 26, dismustrational appreciate with procharge setuption results. If we were interested in the Serivatives, we would period more field the accuracy to p and g quite astisfactory for this impe CFL sumber, see Figure 1. We tiss remark that restarting storms to de little to improve the accuracy

Example 3. To test dimensional collities on a marchives case, we chose an arrangle taken from Other and Sha 15%. The relevant Estimitation reads

 $(42) \qquad (42)$ 

The initial fasterion is given by

$$(43) = \pi([n] - [n]) = \pi([n] - [n]).$$

We compute approximations in the square  $[-1, 1] \times [-1, 1]$  and tensors to be induce the allowed to pass the boundary undisturbed. This excerpts was teaced at the lass. Friedrichs scheme and a number of ENO type schemes is [33].

To test out one of the methods described in the present parent, we used the sense method with restarcing as described in Soction 3.3.2. For this cumple, generate me to the unnover-Hamiltonian, we found that the errors were more sensitive to the during OFE muchor, see Figure 8 for an Hustration of this. Here we show the encrysted coherens on a 50 x-50 grid using OFLe-5 and OFL-2. We dearly see the errors of desilicitums in the valuation of the model for one solution with CFL-5.

#### 5. CONCLUSIONS

We have devised and implemented a family of numerical methods for solving the initial value problem the Hamilton-Jacobi equation

$$u_t + H\left(D_x u\right) = 0.$$

The methods are all based on solving the *d* conservation laws (with discontinuous coefficients)

$$(p_1)_t + H(p_1, p_2, \dots, p_d)_{r_1} = 0$$

$$(p_d)_t + H(p_1, p_2, \dots, p_d)_r = 0$$

by a front tracking method. This can be done sequentially, in which case we label the method dimensional splitting, or "in parallel", i.e., using the same coefficients for all equations. The pertinent feature of our methods is that there is no intrinsic CFL condition associated with the time step, so we can choose our time step independently of other parameters.

We found that these method all produce results comparable to standard methods. We have not been able to show theoretical convergence of these type of method, except in the (trivial) one-dimensional case (see [22]), but our examples indicate that the methods all converge to the viscosity solutions. Moreover, the errors were found to be largely independent of the CFL number, something also found for dimensional splitting for scalar conservation laws, see [32].

The numerical methods developed herein can be easily extended to yield large time step methods for Hamilton-Jacobi equations with a zeroth order term

$$u_t + H(D_x u) = G(x, t, u)$$

by solving sequentially the equations

$$u_t + H(D_x u) = 0$$
 and  $u_t = G(x, t, u)$ 

sequentially, using the methods presented here for the first equation. In Jakobsen, Karlsen, and Risebro [18], it was shown that temporal error associated with the above "source term" splitting is linear in the splitting (time) step, and as such the splitting can be used in conjunction with the methods proposed herein without loss of accuracy.

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#### S. Conductions

We have devised and implemented a femily of normatical mechads for miving the initial value problem the Hamilton-Jerobi evolution

The methods are all based on whythe the allocation have (with disciplinations methods are

$$(p_d)_1 + h (p_1, p_2, \dots, p_d)_{d_d} + 0$$

by a hom tracking mathod. This can be done sequentially, in which pade an label the motion dimensional splitting, or "in paralle", i.e., using the state coefficients for all equations. The perturent feature of our methods is that them is no intrinsic (FL condition associated with the time step, so we can choose our time step independently of other directeristics.

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The nonserical methods developed needo can as cequir burended towned large time else methods for Hamilton-Jacobi equations with a strong order term

by solving secuencially the equations

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sequentially, using the methods presented have for vio that squaston. In Jakohara, Katakao, and Risebro [18], it was shown that temporal artor esseciated with the shows "source tech" splitting is linear in the splitting (time) step, and as such the splitting can be used in tradjunction with the methods proposed herein without loss of accuracy.

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