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DIVIDED DIFFERENCES AND IDEALS GENERATED BY SYMMETRIC POLYNOMIALS

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INTRODUCTION

This note arose from a comparison of [F] and [P1]. In [F], the author proved the following result. Let $\mathcal{F} \subset \mathbb{Z}[A,B]$ be the ideal in the ring of polynomials in the variables $A=(a_1,\dots,a_n)$ and $B=(b_1,\dots,b_n)$, which consists of all polynomials $F(A,B)$ such that for all ring homomorphisms $f\colon \mathbb{Z}[A,B] \longrightarrow K$ (a field) the following holds :

$$\{f(a_1),\dots,f(a_n)\} = \{f(b_1),\dots,f(b_n)\} \quad \text{implies} \quad f(F(A,B)) = 0.$$

Then $\mathcal{F}$ is generated by

$$\sum (a_{i_1}\dots a_{i_k} - b_{i_1}\dots b_{i_k}) ,$$

where the sum is over all sequences $1\le i_1<\dots<i_k\le n$, $k=1,\dots,n$; in other words $\mathcal{F}$ is generated by differences of elementary symmetric polynomials in A and B. In the present note we generalize this result by describing the following more general ideals. Let $A=(a_1,\dots,a_n)$, $B=(b_1,\dots,b_m)$ be two sequences of independent variables.
Fix $r\ge 0$ and let $\mathcal{F}_r \subseteq \mathbb{Z}[A,B]$ be the ideal of all polynomials $F(A,B)$ such that for every ring homomorphism $f\colon \mathbb{Z}[A,B] \longrightarrow K$ (a field) :
$$\operatorname{card}(\{f(a_1),\dots,f(a_n)\} \cap \{f(b_1),\dots,f(b_n)\}) \ge r+1 \quad \text{implies} \quad f(F(A,B))=0.$$

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We give an explicit description of the ideal $\mathcal{F}_r$, with the help of Schur S-polynomials, in Theorem 2.2. Note that if we replace $\mathbb{Z}[A,B]$ by the ring of polynomials symmetric in A and B, then the analogous ideal was described in [P1]. The key trick used in this note is a reduction of a description of $\mathcal{F}_r$ to the latter case with the help of a scalar product on $\mathbb{Z}[A]$ which was defined in [L-S 1] using divided differences. This method allows us to obtain a certain criterion when an G-invariant ideal is actually generated by G-invariants, G being a product of symmetric groups.

1. DIVIDED DIFFERENCES AND A SCALAR PRODUCT ON A POLYNOMIAL RING.

Let $A=(a_1,\ldots,a_n)$ be a sequence of independent variables. We will use actions of different operators on the polynomial ring $\mathbb{Z}[A]$. Preserving the convention used in [L-S 1,2] we assume that these operators act from the right hand side.

Firstly, elements of the symmetric group $\mathfrak{S}_n$ act on $\mathbb{Z}[A]$ by permuting the variables; if $\mu\in\mathfrak{S}_n, F\in\mathbb{Z}[A]$ then the formula $F\mu(a_1,\ldots,a_n) = F(a_{\mu(1)},\ldots,a_{\mu(n)})$ defines a structure of a (right) $\mathfrak{S}_n$-module on $\mathbb{Z}[A]$.

Secondly we have operators $\partial_i=\partial_i^A : \mathbb{Z}[A] \longrightarrow \mathbb{Z}[A]$, $i=1,\ldots,n-1$ defined by

$$F\,\partial_i = \frac{F - F\tau_i}{a_i - a_{i+1}} ,$$

where $\tau_i = (1,\ldots,i-1,i+1,i,i+2,\ldots,n)$, $i=1,\ldots,n-1$, denotes the i-th simple transposition. It turns out (see [B-G-G],[D]) that for a given permutation μ we can define an operator $\partial_\mu=\partial_\mu^A$ as $\partial_{i_1}\circ\ldots\circ\partial_{i_k}$ independently of the reduced decomposition $\mu = \tau_{i_1}\circ\ldots\circ\tau_{i_k}$.

Denote by ω the (longest) permutation $(n,n-1,\ldots,1)$. It is easy to check that:

(1.1) For every $i=1,\ldots,n-1$, $\omega\partial_i\omega = -\partial_{n-i}$; which implies that $\partial_{\omega\mu\omega} = (\operatorname{sgn}\mu)\,\omega\partial_\mu\omega$ for $\mu\in\mathfrak{S}_n$.

$\mathbb{Z}[A]$ is a free rank $n!$ - module over the ring $\mathcal{S}ym(A)$ of symmetric polynomials in A. The following form:

$$< \, , \, > \; : \; \mathbb{Z}[A] \times \mathbb{Z}[A] \longrightarrow \mathcal{S}ym(A)$$

is useful in a description of the module structure. For $F,G \in \mathbb{Z}[A]$ we define following [L-S 1],[L-S 2], $<F,G> = (F\cdot G)\,\partial_{\omega}$. This gives us a bilinear form over $\mathcal{S}ym(A)$ which has the property

(1.2) For every $i=1,\dots,n-1$; $F,G \in \mathbb{Z}[A]$ $<F\partial_i,G> = <F,G\partial_i>$. This implies that for every $\mu\in\mathfrak{S}_n$, $<F\partial_{\mu},G> = <F,G\partial_{\mu^{-1}}>$.

<u>Convention.</u> Given a sequence $I=(i_1,\dots,i_n)$ of nonnegative integers we write a^I for $a_1^{i_1}\dots a_n^{i_n}$. Moreover for two such sequences I,J , we write $I\subset J$ iff $i_1\le j_1,\dots,i_n\le j_n$ and $I+J$ (resp. $I-J$) for the sequence $(i_1+j_1,\dots,i_n+j_n)$ (resp. $(i_1-j_1,\dots,i_n-j_n)$). The sequence $(n-1,n-2,\dots$ $\dots,1,0)$ will be denoted by E_n .

The monomials $\{a^I\}$ where $I \subset E_{n-1}$ form a basis of $\mathbb{Z}[A]$ over $\mathcal{S}ym(A)$. Another such a basis is given by Schubert polynomials indexed by permutations in $\mathfrak{S}_n = \mathrm{Aut}(A)$. Recall that for a given permutation $\mu \in \mathfrak{S}_n$ one defines, following [L-S 1], the Schubert polynomial $X_{\mu} = X_{\mu}(A)$, by

$$X_{\mu} = a^{E}\,\partial_{\omega\mu}.$$

where, here and in the sequel, $E=E_n$. The action of the ∂_{ν}'s on Schubert polynomials is described by

$$(1.3) \qquad X_{\mu}\,\partial_{\nu} = \begin{cases} X_{\mu\nu} & \text{if } \ell(\mu\nu) = \ell(\mu)-\ell(\nu) \\ 0 & \text{otherwise} \end{cases}$$

The scalar product $< \, , \, >$ is nondegenerate. The following proposition describes, for instance, the dual bases of the bases mentioned above.

Denote by $\Lambda_r(A)$ the r-th elementary symmetric polynomial in A.

We have

Proposition 1.4

(i) Let $e_I = a^I$, $I \subset E_{n-1}$ and $f_J = (-1)^{|K|} \prod \Lambda_{k_p}(A \backslash A_p)$, where for $J \subset E_{n-1}$ we put $K = E_{n-1} - J$ and the product is over $p=1,\dots,n-1$. Then

$$<e_I, f_J> = \delta_{I,J}$$

(ii) Let $e_\mu = X_\mu(A)$, $\mu \in \mathfrak{S}_n$ and $f_\nu = X_{\nu\omega}(-A)\omega$, $\nu \in \mathfrak{S}_n$. Then

$$<e_\mu, f_\nu> = \delta_{\mu,\nu}$$

(i) stems from [L-S1] and (ii) stems from [L-S2]. We give here a sketch of the proof of (ii). We will show that

$$<X_\mu \omega, X_{\nu\omega}> = (\mathrm{sgn}\ \mu)\ \delta_{\mu,\nu}$$

for every $\mu, \nu \in \mathfrak{S}_n$. We have $(E=E_n)$

$$
\begin{aligned}
<X_\mu \omega, X_{\nu\omega}> &= <X_\mu \omega, a^E \partial_{\omega\nu\omega}> \\
&= <(X_\mu \omega)\partial_{\omega\nu^{-1}\omega}, a^E> && \text{(by 1.2)} \\
&= (\mathrm{sgn}\ \nu)\ <(X_\mu\ \partial_{\nu^{-1}})\omega, a^E> && \text{(by 1.1)} \\
&= \begin{cases} (\mathrm{sgn}\ \nu)\ <(X_{\mu\nu^{-1}})\omega, a^E> & \text{if } \ell(\mu)-\ell(\nu^{-1}) = \ell(\mu\nu^{-1}) \\ 0 & \text{otherwise}. \end{cases} && \text{(by 1.3)}
\end{aligned}
$$

Write $X_{\mu\nu^{-1}} = \sum \alpha_I\ a^I$ $(\alpha_I \in \mathbb{Z})$, the sum over $I \subset E$. Then $(X_{\mu\nu^{-1}})\omega \cdot a^E = \sum \beta_J\ a^J$ $(\beta_J \in \mathbb{Z})$, the sum over J where $J = I\omega + E \subset$ $\subset (n-1,\dots,n-1)$ (n-times). Finally, invoking that $a^J \partial_\omega = 0$, unless all the components of J are distinct, one sees that the only possibility for a nonzero scalar product is $\mu=\nu$. In this case, by the above calculations, $<X_\mu \omega, X_{\mu w}> = (\mathrm{sgn}\ \mu)<1, a^E> = \mathrm{sgn}\ \mu$. □

2. SOME IDEALS IN THE POLYNOMIAL RING GENERALIZING RESULTANT.

Let $A = (a_1,\dots,a_n)$, $B = (b_1,\dots,b_m)$ be two sequences independent variables. By $\mathcal{S}ym(A)$ we denote the ring of symmetric polynomials in A. Moreover we write $\mathcal{S}ym(A,B) = \mathcal{S}ym(A) \otimes \mathcal{S}ym(B)$. For the purposes of this note we need the following families of polynomials.

Schur S-polynomials

Define $S_i(A-B) \in \mathcal{S}ym(A,B)$ by

$$\prod_{i=1}^{n} (1-ta_i)^{-1} \prod_{j=1}^{m} (1-tb_j) = \sum_{k=0}^{\infty} S_i(A-B)\ t^i,$$

and if $I = (i_1,\dots,i_k)$ is a partition (i.e., $i_1 \geq \dots \geq i_k \geq 0$) , we put

$$S_I(A-B) := \mathrm{Det}\left[S_{i_p-p+q}(A-B) \right] \qquad 1 \leq p,q \leq k$$

Schur Q-polynomials

Define $Q_i(A) \in \mathcal{S}ym(A)$ by

$$\prod_{i=1}^{n} (1+a_it)(1-a_it)^{-1} = \sum_{i=1}^{\infty} Q_i(A)\ t^i$$

Then for nonnegative integers i,j we put

$$Q_{i,j}(A) = Q_i(A)\ Q_j(A) + 2\sum_{p=1}^{j} (-1)^p\ Q_{i+p}(A)\ Q_{j-p}(A)$$

It is easy to see that for $i > 0$, $Q_{(i,0)}(A) = Q_i(A)$ and for $i+j>0$, $Q_{i,j}(A) = -Q_{j,i}(A)$.

Finally, if $I = (i_1,\dots,i_k)$ is a partition and k is even, we put

$$Q_I(A) := \mathrm{Pfaffian}\left[Q_{i_s,i_t}(A) \right] \qquad 1 \leq s,t \leq k$$

and for k-odd, $Q_I(A) := Q_{(i_1,\dots,i_k,0)}(A)$. Since $Q_i(A) = 2\sum_p S_{(p,1^{i-p})}(A)$, we infer that for every partition I, $Q_I(A) = 2^{\ell(I)} P_I(A)$ for some $P_I(A) \in \mathbb{Z}[A]$ uniquely defined by this equation ($\ell(I)$ is the number of nonzero parts of I).

Let $\square_r$ denote the partition $(m-r,\dots,m-r)$ ($(n-r)$-times).

Let $\mathcal{I}_r \subset \mathcal{S}ym(A,B)$ be the ideal generated by $S_{\square_r+I}(A-B)$ where $I \subset (r,\dots,r)$ ($(n-r)$-times).

Let $\mathcal{I}'_r \subset \mathcal{S}ym(A)$ be the ideal generated by $P_{E_{n-r}+I}(A)$ where $I \subset (r,\dots,r)$ ($(n-r)$-times), and finally, let $\mathcal{I}''_r \subset \mathcal{S}ym(A)$ be the ideal generated by $P_{E_{n-r-1}+I}(A)$ where $I \subset (r,\dots,r)$ ($(n-r)$-times), r-even.

Let $\mathcal{T}_r \subset \mathcal{S}ym(A,B)$ be the ideal of all polynomials $T(A,B) \in \mathcal{S}ym(A,B)$ such that for every ring homomorphism $f:\mathcal{S}ym(A,B) \longrightarrow K$ (a field), if $\mathrm{card}(\{f(a_1),\dots,f(a_n)\} \cap \{f(b_1),\dots,f(b_m)\}) \geq r+1$, then $f(T(A,B))=0$. Similarly, let $\mathcal{T}'_r \subset \mathcal{S}ym(A)$ (resp. $\mathcal{T}''_r \subset \mathcal{S}ym(A)$ r-even) be the ideal of all polynomials $T(A)$ such that for every ring homomorphism $f:\mathcal{S}ym(A) \longrightarrow K$ (a field of characteristic $\neq 2$), if

$$\mathrm{card}\,(\{f(a_1),\dots,f(a_n)\} \cap \{f(-a_1),\dots,f(-a_n)\}) \geq r+1 ,$$

(resp. $\mathrm{card}\,(\{f(a_1),\dots,f(a_n)\} \cap \{f(-a_1),\dots,f(-a_n)\} \cap K^*) \geq r+1$), then $f(T(A)) = 0$.

The following result stems from [P1] and [P2, Theorem 5.3].

<u>Theorem 2.1</u>

(i) In $\mathcal{S}ym(A,B)$, $\mathcal{T}_r = \mathcal{I}_r$.

(ii) In $\mathcal{S}ym(A)$, $\mathcal{T}'_r = \mathcal{I}'_r$.

(iii) In $\mathcal{S}ym(A)$, for even r, $\mathcal{T}''_r = \mathcal{I}''_r$.

Define now the ideals $\mathcal{F}_r \subset \mathbb{Z}[A,B]$, $\mathcal{F}'_r \subset \mathbb{Z}[A]$ and $\mathcal{F}''_r \subset \mathbb{Z}[A]$ (r-even) by replacing in the above definitions $\mathcal{S}ym(A,B)$ by $\mathbb{Z}[A,B]$ and $\mathcal{S}ym(A)$ by $\mathbb{Z}[A]$ respectively.

We now state the main result of this note.

<u>Theorem 2.2</u>

(i) In $\mathbb{Z}[A,B]$, $\mathcal{F}_r = \mathcal{I}_r\mathbb{Z}[A,B]$.

(ii) In $\mathbb{Z}[A]$, $\mathcal{F}'_r = \mathcal{I}'_r\mathbb{Z}[A]$.

(iii) In $\mathbb{Z}[A]$, for even r, $\mathcal{F}''_r = \mathcal{I}''_r\mathbb{Z}[A]$.

We will prove (i), for instance. Let $\{e_\alpha\}_{\alpha\in\Lambda}$ be a basis of $\mathbb{Z}[A]$ over $\mathcal{S}ym(A)$ and let $\{f_\alpha\}_{\alpha\in\Lambda}$ be its dual basis. Then for any $F=F(A)$ in $\mathbb{Z}[A]$ we have

$$F = \sum \langle f_\alpha, F\rangle \cdot e_\alpha = \sum (F\cdot f_\alpha)\partial_\omega \cdot e_\alpha \quad .$$

Denoting by $\{e_\alpha'\}_{\alpha\in\Lambda'}$, $\{f_\alpha'\}_{\alpha\in\Lambda'}$ a similar pair of bases of $\mathbb{Z}[B]$ over $\mathcal{S}ym(B)$ (card $\Lambda'=m!$), we have for $F=F(A,B) \in \mathbb{Z}[A,B]$

$$(*) \qquad F = \sum (F\cdot f_\alpha)\partial_\omega \cdot (F\cdot f_\beta')\partial_{\omega'} \cdot e_\alpha \cdot e_\beta' ,$$

where the sum over $\alpha\in\Lambda$, $\beta\in\Lambda'$, and ω' is the longest permutation in $\mathfrak{S}_m = \mathrm{Aut}(B)$. Now, if $F \in \mathcal{F}_r$ then both $F\cdot f_\alpha$ and $F\cdot f_\beta'$ belong to $\mathcal{F}_r$. Moreover for every $G\in\mathbb{Z}[A,B]$, if $G\in\mathcal{F}_r$ then $G\partial_i^A \in \mathcal{J}_r$, $i=1,\dots,n-1$ and $G\partial_j^B \in \mathcal{J}_r$, $j=1,\dots,m-1$. Finally $(*)$ shows that for $F\in\mathcal{F}_r$

$$F = \sum d_{\alpha,\beta}\cdot e_\alpha \cdot e_\beta' \quad ,$$

where $d_{\alpha,\beta}\in \mathcal{J}_r$. This gives the assertion. □

<u>Remark 2.3</u> If $m=n$ $r=n-1$, then Theorem 2.2(i) gives the main result of [F]. Indeed, it is proved in [P2, Proposition 5.8] that $\mathcal{J}_{n-1}$ is generated by $\Lambda_k(A-B) = \sum_{p=0}^{k} (-1)^{k-p}\Lambda_p(A)\ S_{k-p}(B)$ $k=1,\dots,n$. Then the relation $\Lambda_k(A) = \sum_{p=0}^{k} \Lambda_p(A-B)\ \Lambda_{k-p}(B)$ implies that $\mathcal{J}_{n-1}$ is generated by the differences of the elementary symmetric polynomials in A and B.

<u>Corollary 2.4</u> Let $e_1,\dots,e_{n!}$ be a $\mathcal{S}ym(A)$-basis of $\mathbb{Z}[A]$, and let $f_1,\dots,f_{m!}$ be a $\mathcal{S}ym(B)$-basis of $\mathbb{Z}[B]$. (For example, one can take $\{e_i\} = \{a^I ; I \subset E_{n-1}\}$ or $\{e_i\} = \{X_\mu(A) ; \mu\in\mathfrak{S}_n\}$.) Then a $\mathbb{Z}$-basis of the d-th component of $\mathcal{F}_r$ is given by

$$S_{I_k}(A-B)\ S_{J_k}(B)\ e_p\ f_q$$

where, for some $k=0,1,\dots,r$, I_k contains $(m-k)^{n-k}$ but does not contain $(m-k+1)^{n-k+1}$ and $\ell(J_k) \le k$; $p=1,\dots,n!$, $q=1,\dots,m!$; $|I_k| + |J_k| + \deg e_p + \deg f_q = d$. This follows from Theorem 2.2 by invoking a description of a $\mathbb{Z}$-basis of $\mathcal{I}_r$ given in [P2, Proposition 5.9] (see also the references there).

3. WHEN AN INVARIANT IDEAL IS GENERATED BY SYMMETRIC POLYNOMIALS ?

The argument used in the proof of Theorem 2.2 can be summarized in the following way. Let $A^{(1)},\dots,A^{(k)}$ be sequences of independent variables, $A^{(i)} = (a_1^{(i)},\dots,a_{n_i}^{(i)})$. Then the product of symmetric groups $G = \mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_k}$ acts on $\mathbb{Z}[A^{(1)},\dots,A^{(k)}]=\mathbb{Z}[A^{(\cdot)}]$ by permuting the variables. Let $I \subset \mathbb{Z}[A^{(\cdot)}]$ be an ideal and let $\mathcal{S}ym(A^{(\cdot)})$ denote the ring $\mathcal{S}ym(A^{(1)}) \otimes \dots \otimes \mathcal{S}ym(A^{(k)})$ of polynomials symmetric in $A^{(1)},\dots,A^{(k)}$ separately.

<u>Proposition 3.1</u> Let $I \subset \mathbb{Z}[A^{(\cdot)}]$ be an ideal satisfying:

1) I is G-invariant.

2) For some set of generators $F_1,\dots,F_t$ of I, $F_p\, \partial_j^{A^{(i)}}$ belongs to I for $i=1,\dots,k$; $j=1,\dots,n_i-1$; $p=1,\dots,t$.

Then $I = J\,\mathbb{Z}[A^{(\cdot)}]$, where $J = I \cap \mathcal{S}ym(A^{(\cdot)})$, i.e. I is generated by G-invariants.

By arguing as in the proof of Theorem 2.2 we see that if for every $F \in I$, $F\, \partial_j^{A^{(i)}} \in I$, $i=1,\dots,k$, $j=1,\dots,n_i-1$, then our assertion is true. For every $G \in \mathbb{Z}[A^{(\cdot)}]$ we have

$$(G \cdot F_p)\, \partial_j^{A^{(i)}} = G \cdot (F_p\, \partial_j^{A^{(i)}}) + (G\, \partial_j^{A^{(i)}}) \cdot (F_p\, \tau_j^{(i)}).$$

where $\tau_j^{(i)}$ denotes the simple transposition which exchanges $a_j^{(i)}$ and $a_{j+1}^{(i)}$. The first summand belongs to I by 2), the second - by 1) .Since every element from I is a $\mathbb{Z}[A^{(\cdot)}]$-combination of the F_p's, the desired claim now follows. □

Sometimes, it is more convenient to rewrite the above fact as follows. Assume that a subscheme $V \subset \operatorname{Spec} \mathbb{Z}[A^{(\cdot)}]$ is given. For every field K, denote by $\sigma_j^{(i)} : K^{n_1} \times \ldots \times K^{n_k} \longrightarrow K^{n_1} \times \ldots \times K^{n_k}$, $i=1,\ldots$ $\ldots,k$; $j=1,\ldots,n_i-1$, the map which exchanges the j-th with the (j+1)-th component in the i-th factor of the above product. Let $I \subset \mathbb{Z}[A^{(\cdot)}]$ be the ideal of all polynomials which vanish on V_K (:= V after a specialization in the field K) for every such a specialization in some field.

<u>Proposition 3.2</u> Assume that for every field K, V_K has the following properties:

1) If $a \in V_K$ then $\sigma_j^{(i)}(a) \in V_K$ for every $i=1,\ldots,k$; $j=1,\ldots,n_i-1$.
2) $V_K \not\subset \text{Zeros}\ (a_j^{(i)} - a_{j+1}^{(i)}) \subset K^{n_1} \times \ldots \times K^{n_k}$ for every $i=1,\ldots,k$; $j=1,\ldots,n_i-1$.

Then $I = J\, \mathbb{Z}[A^{(\cdot)}]$, where $J = I \cap \mathcal{S}ym(A^{(\cdot)})$.

Indeed, the above assumptions guarantee that for $F \in I$ and $G \in \mathbb{Z}[A^{(\cdot)}]$, $(F \cdot G)\, \partial_j^{A^{(i)}}$ belongs to I, $i=1,\ldots,k$; $j=1,\ldots,n_i-1$; and the assertion follows. □

For example, the situation considered in Theorem 2.2(i) was: $k=2$, $A=A^{(1)}$, $B=A^{(2)}$, $n=n_1$, $m=n_2$, $V = \bigcup V_{I,J}$ the sum over all pairs of sequences $I = (1 \le i_1 < \ldots < i_{r+1} \le n)$, $J = (1 \le j_1 < \ldots < j_{r+1} \le m)$ and $V_{I,J} =$ $= \text{Zeros}\ (a_{i_1} - b_{j_1}, \ldots, a_{i_{r+1}} - b_{j_{r+1}})$.

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