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On Extremal Bases for the h-range Problem, I

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1 Introduction

A set of k positive integers $A_k = \{a_1 = 1 < a_2 < a_3 < \cdots < a_k\} \subset \mathrm{N}$ is called a *basis*. Let now $h \in \mathrm{N}$, $M \in \mathrm{N}_0 = \mathrm{N} \cup \{0\}$ be integers. We say M is *h-representable by A_k*, if there exist non-negative integer coefficients $x_1, x_2, \ldots, x_k \in \mathrm{N}_0$ such that

$$\sum_{i=1}^{k} x_i a_i = M \text{ and } \sum_{i=1}^{k} x_i \leq h.$$

The set of all integers which are h-representable by A_k is denoted by hA_k. We consider now the least positive integer N without such an h-representation by A_k, and call $N-1$ the *h-range* $n_h(A_k)$ of A_k:

$$n_h(A_k) = \min\{n \in \mathrm{N} \mid n \notin hA_k\} - 1.$$

Often we call a basis an *h-range basis*.

With h_0 we denote the least number of addends that is sufficient for the h-range to reach the largest basis element a_k:

$$h_0 = h_0(A_k) = \min\{h \in \mathrm{N} \mid n_h(A_k) \geq a_k\}.$$

It is very easy to show that for $h \geq h_0 - 1$ we have

$$n_{h+1}(A_k) \geq a_k + n_h(A_k). \tag{1}$$

For sufficiently large h even equality holds:

$$n_{h+1}(A_k) = a_k + n_h(A_k), \text{ for } h \geq h_1 \geq h_0 - 1. \tag{2}$$

For a proof see Selmer [17] or Meures [10]. By h_1 we denote the least bound for (2) to hold. Selmer [17] could show that if

$$n_h(A_k) \geq (h+1)a_{k-1} - a_k, \tag{3}$$

then $h_1 \leq h$. If (2) holds for all $h \geq h_0$, we say that the basis A_k is h_0-*stable*.

An h-representation of $M \in \mathbf{N}$ by $A_k = \{1, a_2, a_3, \dots, a_k\}$:

$$\sum_{i=1}^{k} x_i a_i = M, \quad \sum_{i=1}^{k} x_i \leq h,$$

is called the *regular h-representation by* A_k if a_k, the largest element of the basis, is used as often as possible, and then a_{k-1} is used as often as possible to represent the rest $M - x_k a_k$, and so on. A representation of $M \in \mathbf{N}$ is called *minimal* if the number of addends used in this representation is minimal among all possible representations. The regular representation need not always be minimal.

On the other hand, the minimal representation can always be achieved by starting with the regular one and performing several "transfers" of basis element. In order to describe this process we need the so called *normal form* of the basis A_k, introduced by Hofmeister [5]. Let A_k be a basis, then there exist uniquely determined integers $\gamma_i \geq 2$ for $i = 1, 2, \dots k-1$ and $\beta_j^{(i)} \geq 0$ for $i = 2, 3, \dots, k$ and $j = 1, 2, \dots, i-2$ such that

$$\gamma_{i-1} = \left\lceil \frac{a_i}{a_{i-1}} \right\rceil, \quad a_i = \gamma_{i-1} a_{i-1} - \sum_{j=1}^{i-2} \beta_j^{(i)} a_j \text{ for } i = 2, 3, \dots, k, \tag{4}$$

where $\sum_{j=1}^{i-2} \beta_j^{(i)} a_j$ is the regular representation of $\gamma_{i-1} a_{i-1} - a_i$. Here $\lceil x \rceil$ denotes the least integer $\geq x \in \mathbf{R}$.

Let now

$$M = \sum_{i=1}^{k} e_i a_i$$

be the regular representation of $M \in \mathbf{N}$, and $s_i \in \mathbf{Z}$ for $i = 2, 3, \ldots k$. Then

$$\begin{aligned} M &= \sum_{i=1}^{k} e_i a_i + \sum_{i=2}^{k} s_i \left(\gamma_{i-1} a_{i-1} - a_i - \sum_{j=1}^{i-2} \beta_j^{(i)} a_j \right) \\ &= \sum_{j=1}^{k} \left(e_j - s_j + s_{j+1}\gamma_j - \sum_{i=j+2}^{k} s_i \beta_j^{(i)} \right) a_j, \end{aligned}$$

where we put $s_1 = s_{k+1} = \gamma_k = 0$. If we write

$$x_j = e_j - s_j + s_{j+1}\gamma_j - \sum_{i=j+2}^{k} s_i \beta_j^{(i)} \text{ for } j = 1, 2, \ldots, k, \tag{5}$$

we get

$$M = \sum_{j=1}^{k} x_j a_j. \tag{6}$$

Now Hofmeister [5] showed that for every representation – also for the minimal one – there exist *non-negative* integers s_i, $i = 2, 3, \ldots k$ such that the representation is given by (6) and (5) holds. So every representation of $M \in \mathbf{N}$ is defined uniquely by an integer vector $(s_2, s_3, \ldots, s_k) \in {\mathbf{N}_0}^{k-1}$. This integer vector is called a *transfer* or *substitution of basis elements.*

By the *gain* $G(s_2, s_3, \ldots, s_k)$ of a transfer $(s_2, s_3, \ldots, s_k) \in {\mathbf{N}_0}^{k-1}$, we mean the reduction in the number of addends caused by this transfer in comparison to the number of addends in the regular representation:

$$G(s_2, s_3, \ldots, s_k) = \sum_{j=1}^{k} (e_j - x_j) = \sum_{j=1}^{k} \left(s_j - s_{j+1}\gamma_j + \sum_{i=j+2}^{k} s_i \beta_j^{(i)} \right).$$

For the minimal representation of a number $M \in \mathbf{N}$ the gain is always ≥ 0.

2 Extremal Bases

2.1 Known Results

Now we fix the number of elements k in our h-range basis and ask for the basis $A_k^* = \{1, a_1^*, a_2^*, \ldots, a_k^*\}$, or possibly those bases, with largest h-range for a given integer h. These bases are called *extremal* or *optimal*. Our main interest is not the particular extremal basis but the sequence $A_k^*(h)$ of extremal bases, when h, the number of addends allowed in the representations, increases to infinity. Throughout this paper we regard k as a fixed number. Rohrbach [16] could show by a simple combinatorial argument that for all bases A_k there is a common bound for the h-range:

$$n_h(A_k) < \binom{h+k}{k},$$

and Rödseth [15] was able to sharpen this bound to

$$n_h(A_k)) \leq \frac{(k-1)^{k-1}}{(k-1)!} \left(\frac{h}{k}\right)^k + O(h^{k-1}). \tag{7}$$

Since these bounds are of course also valid for the extremal bases $A_k^*(h)$, we can find a real positive constant $C \in \mathbf{R}$ such that

$$n_h(A_k^*(h)) \leq C(h/k)^k.$$

Now it turns out that $(h/k)^k$ already is the right "size" of the extremal h-range, since Stöhr [19] could show that there exists a real positive constant $c \in \mathbf{R}$ such that

$$c(h/k)^k \leq n_h(A_k^*(h)) \leq C(h/k)^k. \tag{8}$$

In fact, by Stöhr's result [19] we have $c \geq 1$ and by Rödseth's bound (7) we have $C \leq (k-1)^{k-1}/(k-1)!$.

Hofmeister [5] could show that for all parameter bases $A_k(h)$ that satisfy (8) – and there are of course many more such bases than the extremal ones – the "size" of the basis elements $a_i(h)$ is given by a simple formula. He showed that there exist real positive constants $c_i, C_i \in \mathbf{R}$, $i = 1, 2, \ldots, k$ such that

$$c_i h^{i-1} \leq a_i(h) \leq C_i h^{i-1}, \text{ for } i = 1, 2, \ldots, k. \tag{9}$$

The constants c_i and C_i, $i = 1, 2, \dots, k$ are depending on the number k of basis elements but not on h.

For $k = 2$ we know the extremal bases by Stöhrs result [19]. He showed that

$$\begin{aligned} A_2^*(h) &= \{1, (h+3)/2\}, \text{ if } h \text{ is odd, and} && (10) \\ A_2^*(h) &= \{1, (h+3 \pm 1)/2\}, \text{ if } h \text{ is even.} && (11) \end{aligned}$$

The corresponding extremal h-range is given by one formula:

$$n_h(A_2^*(h)) = \left\lfloor \frac{h^2 + 6h + 1}{4} \right\rfloor .$$

Here $\lfloor x \rfloor$ denotes the largest integer $\leq x \in \mathbf{R}$.

In 1968 Hofmeister [4] found out how to determine the extremal h-ranges and the corresponding extremal bases in the case $k = 3$. Let

$$\beta(h) = \left\lfloor \frac{4h+4}{9} \right\rfloor + 2, \ \gamma(h) = \left\lfloor \frac{2h}{9} \right\rfloor + 2.$$

If $h \geq 23$, the extremal bases $A_3^*(h)$ are given by

$$a_2^*(h) = 2\beta(h) - \gamma(h) + 1, \ a_3^*(h) = \gamma(h)a_2^*(h) - \beta(h), \qquad (12)$$

with the corresponding h-range

$$n_h(A_3^*(h)) = (h + 4 - \beta(h) - \gamma(h))a_3^*(h) + (\gamma(h) - 2)a_2^*(h) + \beta(h) - 2.$$

Originally, Hofmeister's proof was only valid for sufficiently large h. A student of his, Hertsch [2], showed that it was enough to claim $h \geq 500$. In a new attempt, Hofmeister [6] could reduce this to $h \geq 200$, and Mossige [11] verified the theorem for $23 \leq h \leq 200$ on a computer. For $h < 23$ the extremal bases and their h-ranges can easily be determined by a computer. Table 1 below contains all these bases and their h-ranges. Note that for $h = 11$ and $h = 22$ there are two extremal bases.

Table 1.

h	a_2	a_3	$n_h(A_3^*)$	h	a_2	a_3	$n_h(A_3^*)$
1	2	3	3	12	11	37	212
2	3	4	8	13	13	34	259
3	4	5	15	14	12	52	302
4	5	8	26	15	12	52	354
5	6	7	35	16	15	54	418
6	7	12	52	17	14	61	476
7	8	13	69	18	15	80	548
8	9	14	89	19	18	65	633
9	9	20	112	20	17	91	714
10	10	26	146	21	17	91	805
11	9	30	172	22	19	102	902
11	10	26	172	22	20	92	902

Quite a lot of people, Hofmeister and Schell [5], Mossige [12], [13], Braunschädel [1], Selmer [18] and the author have spent great effort on the determination of the extremal h-ranges and the corresponding extremal bases in the case $k = 4$. Nevertheless the final answer, the value of the extremal h-range is not known and we are not able to give the answer in this report either.

According to Hofmeister's notation (4), we write the basis elements of A_4 in the *normal form*:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \gamma_1 \\ a_3 &= \gamma_2 a_2 - \beta_1^{(3)}, 0 \le \beta_1^{(3)} < a_2 \\ a_4 &= \gamma_3 a_3 - \beta_2^{(4)} a_2 - \beta_1^{(4)}, \\ & \quad 0 \le \beta_1^{(4)} < a_2 \text{ and } 0 \le \beta_2^{(4)} a_2 + \beta_1^{(4)} < a_3. \end{aligned} \tag{13}$$

Hofmeister and Schell presented in [5] a concrete parameter basis with a quite large h-range. They put $h = 12d$, $d \in \mathbf{N}$ and constructed their basis for each value of d. By Mrose [14] this fact does not cause any restriction to the general problem, if our interest is the "asymptotic size" of the h-range.

The normal form of this basis is:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 9d-6 \\ a_3 &= 3da_2-(5d-3) \\ a_4 &= 2da_3-(d-1)a_2-(6d-4). \end{aligned} \tag{14}$$

They could show that for this choice of $A_4(h)$ we have

$$n_h(A_4(h)) \geq (3d+6)a_4 = 2(h/4)^4 + O(h^3). \tag{15}$$

For a long time the coefficient 2 in front of the $(h/4)^4$ term was believed to be the largest possible for the h-range of $A_4(h)$, until Mossige [12] in 1986 by a slight alteration of the Hofmeister-Schell basis could achieve a coefficient 2.008 instead of 2 in front of the $(h/4)^4$ term in (15). The original Mossige basis has very complicated non-rational coefficients. A basis $A_4(h)$ with an h-range quite close to the one that is achieved by the Mossige basis, and where the coefficients are rational, is given thus: If we put $h = 2472d$, $d \in \mathrm{N}$ and

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1869d \\ a_3 &= (603d+3)a_2-1031d \\ a_4 &= (392d+1)a_3-(193d+1)a_2-1242d, \end{aligned} \tag{16}$$

Mossige showed that the h-range is given by

$$n_h(A_4(h)) = 663da_4 + 193da_2 + (1646d-2) = 2.0080397(h/4)^4 + O(h^3).$$

Recently Selmer [18] found another basis – in some sense a dual to the Mossige basis – with the same highest coefficient in the h-range. He deleted the constant terms, and gave the basis for $h = 2472d$, $d \in \mathrm{N}$ as

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1869d \\ a_3 &= 603da_2-1441d \\ a_4 &= 392da_3-193da_2-826d. \end{aligned} \tag{17}$$

If we apply the upper bound (7) to the case $k = 4$ we get

$$n_h(A_4(h)) \leq 4.5(h/4)^4 + O(h^3),$$

so we are left with quite a big gap between the best known upper and lower bounds for the extremal h-range. In section 2.3 we treat this problem, and there we will tighten a big part of the mentioned gap.

2.2 Properties of the Extremal Bases A_k^*

For parameter bases $A_k(h)$ satisfying (8) we have (9). Now put

$$\bar{h} = h_0(A_k(h)) + 2\lceil C_{k-1}/c_k \rceil,$$

then by (1)

$$\begin{aligned} n_{\bar{h}}(A_k(h)) &\geq 2\lceil C_{k-1}/c_k \rceil a_k(h) \geq 2\lceil C_{k-1}/c_k \rceil c_k h^{k-1} \\ &\geq 2h a_{k-1}(h) \geq 2h_0 a_{k-1}(h) \geq \bar{h} a_{k-1}(h) \end{aligned}$$

for sufficiently large h, since $2\lceil C_{k-1}/c_k \rceil$ is a constant and h_0 is increasing with h, because

$$h_0 = h_0(A_k(h)) \geq h_0(\{1, a_2(h)\}) = a_2(h) - 1 \geq c_2 h - 1.$$

This means that for sufficiently large h, we have by (3):

$$h_0 - 1 \leq h_1 = h_1(A_k(h)) \leq h_0 + 2\lceil C_{k-1}/c_k \rceil,$$

the first inequality stemming from the definition of h_1. Asymptotically spoken, this means that the bases $A_k(h)$ with (8) are h_0-*stable*, and

$$\lim_{h\to\infty} \frac{h_1}{h_0} = \lim_{h\to\infty} \frac{h_1(A_k(h))}{h_0(A_k(h))} = 1. \tag{18}$$

Similarly we have for bases with (8):

$$n_h(A_k(h)) \geq \frac{ch^k}{k^k} \geq \left(\frac{ch}{k^k C_{k-1}}\right) h a_{k-1}(h) \geq h a_{k-1}(h)$$

for sufficiently large h. Thus by (3) we have that

$$h_1(A_k(h)) \leq h \text{ for sufficiently large } h, \tag{19}$$

so equality (2) is valid for sufficiently large h. Note that this is not trivial, since also the basis changes if h increases.

The problem of finding the extremal bases A_k^* for given k has shown to be very difficult. Apart from the cases $k = 2$ and $k = 3$, where the whole truth is known, we know little about the general case. From Stöhr [19] we learned that there exist real positive constants $c, C \in \mathbf{R}$ such that

$$c \leq \frac{n_h(A_k^*(h))}{(h/k)^k} \leq C.$$

Whether the $\lim_{h\to\infty} n_h(A_k^*)/(h/k)^k$ exists or not is hard to say. Of course

$$T = \limsup_{h\to\infty} \frac{n_h(A_k^*(h))}{(h/k)^k}$$

exists and is positive, and we may choose a sequence $(h_i)_{i\in\mathrm{N}}$ with

$$\lim_{i\to\infty} \frac{n_{h_i}(A_k^*(h_i))}{(h_i/k)^k} = T.$$

Therefore we focus our attention on this sequence $(h_i)_{i\in\mathrm{N}}$. By (8) and (9) we know that

$$\frac{c}{C_k k^k} \leq \frac{n_{h_i}(A_k^*(h_i))}{h_i a_k^*(h_i)} \leq \frac{C}{c_k k^k},$$

and therefore we may choose a subsequence $\left(h_{i_j}\right)_{j\in\mathrm{N}}$ of $(h_i)_{i\in\mathrm{N}}$ such that also

$$t = \lim_{j\to\infty} \frac{n_{h_{i_j}}(A_k^*(h_{i_j}))}{h_{i_j} a_k^*(h_{i_j})}$$

exists. In order to reduce the number of indices we write $(h_j)_{j\in\mathrm{N}}$ for the latter sequence.

Now all numbers which have got an h_j-representation are contained in the interval $[0,\ h_j a_k^*(h_j)]$, and the magnitude t now measures the number of consecutive h_j-representable numbers starting from 0 to the h_j-range in comparison to the largest h_j-representable number $h_j a_k^*(h_j)$. So t may

be interpreted as the *covering percentage* of the h_j-range in the interval of h_j-representable numbers.

If we put

$$n_{h_j}(A_k^*(h_j)) = \epsilon_k(h_j)a_k^*(h_j) + \epsilon_{k-1}(h_j)a_{k-1}^*(h_j) + \cdots + \epsilon_1(h_j)$$

for the *regular* representation of the h_j-range of $A_k^*(h_j)$, we see at once that

$$t = \lim_{j\to\infty} \frac{\epsilon_k(h_j)}{h_j}.$$

Assume that

$$t < \frac{1}{k},$$

and put $H_j = \lceil (1+m)h_j \rceil$ for a real positive m. Then by (2)

$$n_{H_j}(A_k^*(h_j)) = n_{h_j}(A_k^*(h_j)) + mh_j a_k^*(h_j) + O(h_j^{k-1}). \tag{20}$$

On the other hand

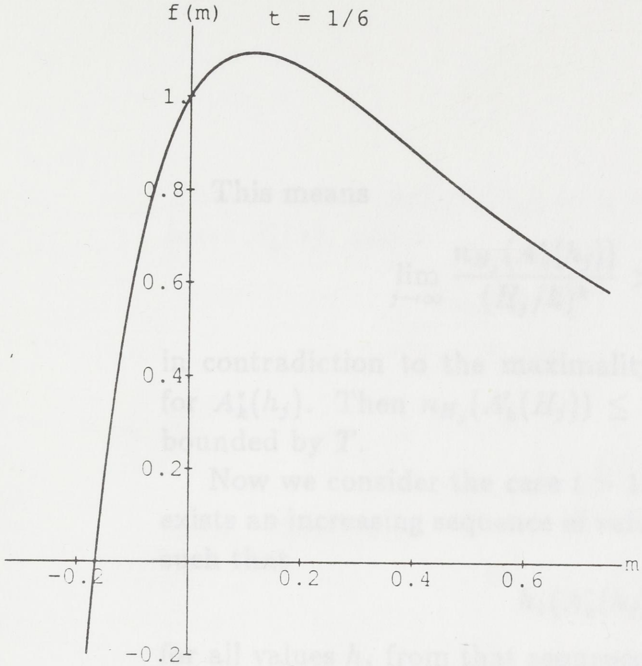
$$\left(\frac{(1+m)h_j}{k}\right)^k - \left(\frac{H_j}{k}\right)^k = O(h_j^{k-1}).$$

Therefore we can write

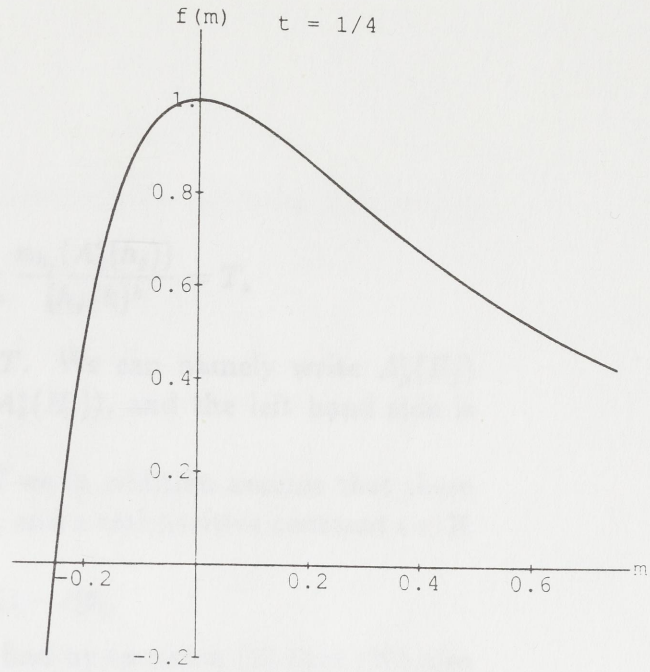
$$\begin{aligned}
\lim_{j\to\infty} \frac{n_{H_j}(A_k^*(h_j))}{(H_j/k)^k} &= \lim_{j\to\infty} \frac{n_{h_j}(A_k^*(h_j)) + mh_j a_k^*(h_j) + O(h_j^{k-1})}{(1+m)^k (h_j/k)^k + O(h_j^{k-1})} \\
&= \lim_{j\to\infty} \frac{\left(\frac{n_{h_j}(A_k^*(h_j))}{h_j a_k^*(h_j)} + m\right) h_j a_k^*(h_j)}{(1+m)^k (h_j/k)^k} \\
&= \lim_{j\to\infty} \frac{\frac{n_{h_j}(A_k^*(h_j))}{h_j a_k^*(h_j)} + m}{(1+m)^k \frac{n_{h_j}(A_k^*(h_j))}{h_j a_k^*(h_j)}} \lim_{j\to\infty} \frac{n_{h_j}(A_k^*(h_j))}{(h_j/k)^k} \\
&= \frac{t+m}{t(1+m)^k} T = f(m)T \text{ (say)}.
\end{aligned}$$

Now we choose m optimally namely $m = m_0 = \frac{1-kt}{k-1} > 0$, giving

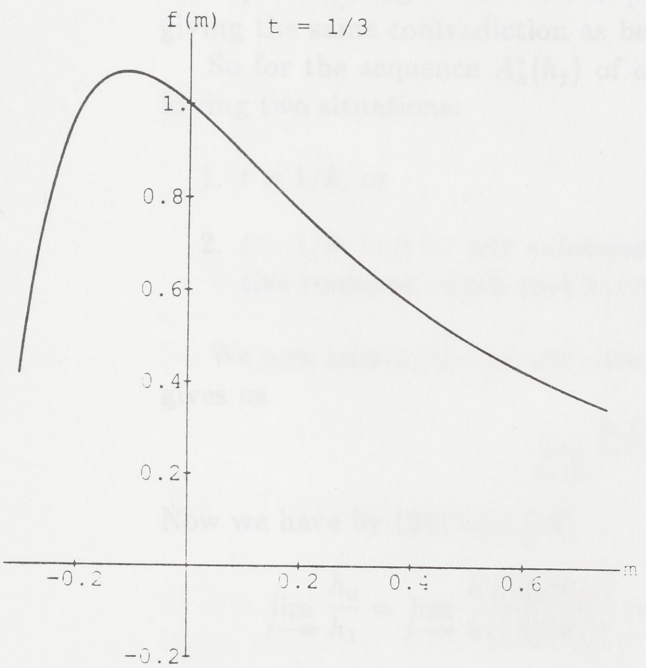
$$f(m_0) > f(0) = 1.$$



Picture 1.



Picture 2.



Picture 3.

Picture 1, 2 and 3 show the interesting part of the graph of the function

$$f(m) = \frac{t+m}{t(1+m)^k}$$

for $k = 4$ and different values of t.

This means

$$\lim_{j\to\infty} \frac{n_{H_j}(A_k^*(h_j))}{(H_j/k)^k} > \lim_{j\to\infty} \frac{n_{h_j}(A_k^*(h_j))}{(h_j/k)^k} = T,$$

in contradiction to the maximality of T. We can namely write $A'_k(H_j)$ for $A_k^*(h_j)$. Then $n_{H_j}(A'_k(H_j)) \leq n_{H_j}(A_k^*(H_j))$, and the left hand side is bounded by T.

Now we consider the case $t > 1/k$. If we in addition assume that there exists an increasing sequence of values h_j and a real positive constant $\epsilon \in \mathbf{R}$ such that

$$h_1(A_k^*(h_j)) \leq (1-\epsilon)h_j$$

for all values h_j from that sequence, we find by equation (2) that (20) also holds for negative $m \in [-\epsilon, 0]$, and the whole of the preceding proof goes through if also $m_0 = \frac{1-kt}{k-1} \geq -\epsilon$.

If $m_0 < -\epsilon$ we choose $m = m_1 = -\epsilon$. We know that the function $f(m)$ is strictly decreasing in the interval $[m_0, 0]$, and therefore $f(m_1) > f(0) = 1$, giving the same contradiction as before.

So for the sequence $A_k^*(h_j)$ of extremal bases we are left with the following two situations:

1. $t = 1/k$, or

2. $t > 1/k$, and for any subsequence of $(h_j)_{j\in\mathbf{N}}$ there exists no real positive constant ϵ such that $h_1(A_k^*(h_j)) \leq (1-\epsilon)h_j$ for this subsequence.

We now look at the second case. The last statement together with (19) gives us

$$\lim_{j\to\infty} \frac{h_1(A_k^*(h_j))}{h_j} = 1. \tag{21}$$

Now we have by (21) and (18)

$$\lim_{j\to\infty} \frac{h_0}{h_1} = \lim_{j\to\infty} \frac{h_0(A_k^*(h_j))}{h_1(A_k^*(h_j))} = \lim_{j\to\infty} \frac{h_0(A_k^*(h_j))}{h_j} = \lim_{j\to\infty} \frac{h_0}{h_j} = 1.$$

We collect all this information in one theorem.

Theorem 1 . *Let $A_k^*(h_j)$ be a subsequence of the sequence of extremal bases $A_k^*(h)$, where*

$$\lim_{j\to\infty} \frac{n_{h_j}(A_k^*(h_j))}{(h_j/k)^k} = \limsup_{h\to\infty} \frac{n_h(A_k^*(h))}{(h/k)^k} = T,$$

and where

$$t = \lim_{j\to\infty} \frac{n_{h_j}(A_k^*(h_j))}{h_j a_k(h_j)}$$

exists. Then either

$$t = \frac{1}{k}, \text{ or}$$

$$t > \frac{1}{k} \text{ and } \lim_{h_j\to\infty} \frac{h_1}{h_j} = \lim_{h_j\to\infty} \frac{h_0}{h_j} = 1.$$

Remark 1. For $k = 2$ the extremal bases (10) and (11) given by Stöhr satisfy $t = 1/2$ and $\lim_{h\to\infty} h_0/h = 1/2 \neq 1$. For $k = 3$ the extremal bases (12) found by Hofmeister satisfy $t = 1/3$ and $\lim_{h\to\infty} h_0/h = 8/9 \neq 1$, whilst the "good" bases (16) and (17) for $k = 4$ satisfy $t = 221/824 > 1/4$ and $\lim_{h\to\infty} h_0/h = 1$. The basis (14) satisfies both conditions $t = 1/4$ and $\lim_{h\to\infty} h_0/h = 1$. The best known bases for $k = 5$, found in Kolsdorf [8] and [9], have $t > 1/5$ and $\lim_{h\to\infty} h_0/h = 1$. So both alternatives of the theorem seem to be realistic.

Remark 2. Braunschädel [1] showed that for the extremal bases with $k = 4$ we have $\limsup_{h\to\infty} \epsilon_4(h)/h \geq 81/512$, while our theorem gives $\limsup_{h_j\to\infty} \epsilon_4(h_j)/h_j \geq 1/4$ for the sequence h_j defined above.

2.3 Bounds for the Extremal h-range $n_h(A_4^*)$

The method developed in this section is a extension of results obtained by Braunschädel [1] in 1988. In the introduction we already mentioned upper and lower bounds for the extremal h-range $n_h(A_4^*)$:

$$2.008(h/4)^4 + O(h^3) < n_h(A_4^*) \leq 4.5(h/4)^4 + O(h^3).$$

We now look at a parameter basis $A_4 = A_4(h)$. We often leave out the parameter h of the basis $A_4(h)$. If $n_h(A_4(h)) \leq 2(h/4)^4 + O(h^3)$, the basis cannot be an extremal one. Therefore we consider only bases $A_4(h)$ with

$$n_h(A_4(h)) > 2(h/4)^4 + O(h^3). \tag{22}$$

We use the normal form (13), and write

$$n_h(A_4) = \epsilon_4 a_4 + \epsilon_3 a_3 + \epsilon_2 a_2 + \epsilon_1 \tag{23}$$

for the regular representation of the h-range of a given basis A_4. Now Hofmeister [4] showed that if

$$\epsilon_4 + a\gamma_3 + b\gamma_2 + c\gamma_1 \leq h + \delta \tag{24}$$

for positive constants $a, b, c \in \mathbf{R}$, and $\delta \in \mathbf{R}$ then

$$\begin{aligned} n_h(A_4) &< (\epsilon_4 + 1)a_4 \leq \epsilon_4\gamma_3\gamma_2\gamma_1 + \gamma_3\gamma_2\gamma_1 \\ &\leq \frac{1}{abc}\left(\frac{h+\delta}{4}\right)^4 + O(h^3) = \frac{1}{abc}\left(\frac{h}{4}\right)^4 + O(h^3), \end{aligned} \tag{25}$$

a consequence of the fact that the geometric mean cannot exceed the arithmetic one. This means that if we can establish an inequality (24) where $\frac{1}{abc} \leq 2$, then the sequence of bases $A_4(h)$ cannot be extremal and can therefore be excluded from further consideration. Now given such an inequality (24), we may refine the result we can get from (25) by additional information. Look at the number

$$M = (\gamma_2 - 2)a_2 + (\gamma_1 - 1) < a_3 < n_h(A_4). \tag{26}$$

Here no a_4 or a_3 transfers are possible, since there are no such elements in the the regular representation (26), and this representation must therefore be minimal, giving

$$\gamma_2 + \gamma_1 \leq h + 3. \tag{27}$$

Now multiplying (27) by a positive weight x and adding it to (24) yields

$$\epsilon_4 + a\gamma_3 + (b + x)\gamma_2 + (c + x)\gamma_1 \leq (1 + x)h + \delta + 3x,$$

and in analogy with (25) this gives

$$n_h(A_4) \leq \frac{(1+x)^4}{a(b+x)(c+x)} \left(\frac{h}{4}\right)^4 + O(h^3). \tag{28}$$

Minimizing the coefficient by the optimal choice of

$$x = \max\left\{0, \sqrt{\left(\frac{3(b+c)-2}{4}\right)^2 + \frac{b+c-4bc}{2}} - \frac{3(b+c)-2}{4}\right\}$$

usually gives a better result than the one we could obtain from the original inequality (24).

By these means we can show at once that (22) implies

$$\begin{array}{lll}
\epsilon_4 + 8\gamma_3 & & > h+\delta \\
\epsilon_4 + 7\gamma_3 & +\frac{1}{6}\gamma_2 & > h+\delta \\
\epsilon_4 + 6\gamma_3 & +\frac{1}{3}\gamma_2 & > h+\delta \\
\epsilon_4 + 5\gamma_3 & +\frac{1}{2}\gamma_2 & > h+\delta \\
\epsilon_4 + 4\gamma_3 & +\frac{4}{5}\gamma_2 & > h+\delta \\
\epsilon_4 + 3\gamma_3 & +\frac{6}{5}\gamma_2 & > h+\delta \\
\epsilon_4 + 2\gamma_3 & +2\gamma_2 & > h+\delta \\
\epsilon_4 + \gamma_3 & +5\gamma_2 & > h+\delta.
\end{array}$$

Here we may put $t_7 = 0$, $t_6 = 1/6$, $t_5 = 1/3$, $t_4 = 1/2$, $t_3 = 4/5$, $t_2 = 6/5$, $t_1 = 2$, $t_0 = 5$ and write all these inequalities in one form

$$\epsilon_4 + (j+1)\gamma_3 + t_j\gamma_2 > h + \delta, \; j = 0, 1, \ldots, 7. \tag{29}$$

This is going to be used several times later on.

In the sequel we want to present a method that provides inequalities (24). We do this by looking at several "key numbers" and their representations by A_4. These representations will in general contain the variables h, ϵ_4, and $\gamma_3, \gamma_2, \gamma_1, \beta_1^{(3)}, \beta_1^{(4)}, \beta_2^{(4)}$ of (13). Combining several representations, we can get rid of $\beta_1^{(3)}$ and $\beta_1^{(4)}$. Since by (13) $0 \leq \beta_2^{(4)}/\gamma_2 < 1$, we divide the interval $[0,1]$ into the following 12 smaller intervals and choose $\beta_2^{(4)}/\gamma_2$ from one of them:

$$
\begin{array}{llllll}
I_1 & = \left[0, \frac{1}{6}\right), & I_2 & = \left[\frac{1}{6}, \frac{1}{5}\right), & I_3 & = \left[\frac{1}{5}, \frac{1}{4}\right), \\
I_4 & = \left[\frac{1}{4}, \frac{1}{3}\right), & I_5 & = \left[\frac{1}{3}, \frac{2}{5}\right), & I_6 & = \left[\frac{2}{5}, \frac{1}{2}\right), \\
I_7 & = \left[\frac{1}{2}, \frac{3}{5}\right), & I_8 & = \left[\frac{3}{5}, \frac{2}{3}\right), & I_9 & = \left[\frac{2}{3}, \frac{3}{4}\right), \\
I_{10} & = \left[\frac{3}{4}, \frac{4}{5}\right), & I_{11} & = \left[\frac{4}{5}, \frac{5}{6}\right), & I_{12} & = \left[\frac{5}{6}, 1\right).
\end{array}
$$

Then we know lower and upper bounds for $\beta_2^{(4)}$ in terms of γ_2, and we will get inequalities containing only $h, \epsilon_4, \gamma_3, \gamma_2, \gamma_1$ and of course constants not depending on h.

We look first at the number N_1 that has the largest regular coefficient sum of all numbers $< n_h(A_4)$ and its regular representation:

$$N_1 = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + (\gamma_1 - 1).$$

Now N_1 must have an h-representation by A_4 using the transfer $(s_2^{(1)}, s_3^{(1)}, s_4^{(1)})$, where the upper index denotes that the transfer belongs to N_1:

$$
\begin{aligned}
N_1 \;=\; & (\epsilon_4 - 1 - s_4^{(1)})a_4 + ((s_4^{(1)} + 1)\gamma_3 - s_3^{(1)} - 2)a_3 \\
& + ((s_3^{(1)} + 1)\gamma_2 - s_4^{(1)}\beta_2^{(4)} - s_2^{(1)} - 2)a_2 \\
& + ((s_2^{(1)} + 1)\gamma_1 - s_3^{(1)}\beta_1^{(3)} - s_4^{(1)}\beta_1^{(4)} - 1).
\end{aligned}
$$

Since the coefficient sum has to be $\leq h$, we have

$$
\begin{aligned}
& \epsilon_4 + (s_4^{(1)} + 1)\gamma_3 + (s_3^{(1)} + 1)\gamma_2 - s_4^{(1)}\beta_2^{(4)} \\
& + (s_2^{(1)} + 1)\gamma_1 - s_3^{(1)}\beta_1^{(3)} - s_4^{(1)}\beta_1^{(4)} \\
& \leq h + 6 + s_2^{(1)} + s_3^{(1)} + s_4^{(1)} \leq h + \delta,
\end{aligned}
$$

where δ is a constant not depending on h, since we know by Hofmeister [5] that for bases A_4 with (22), $s_2^{(1)}, s_3^{(1)}$ and $s_4^{(1)}$ are bounded independently of h.

For the reduction of the constant term, we now assume

$$\kappa_1 = s_3^{(1)}\beta_1^{(3)} + s_4^{(1)}\beta_1^{(4)} - s_2^{(1)}\gamma_1 > 0,$$

and consider the next "key number", with constant term $\kappa_1 - 1$:

$$N_2 = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + (s_3^{(1)}\beta_1^{(3)} + s_4^{(1)}\beta_1^{(4)} - s_2^{(1)}\gamma_1 - 1).$$

Clearly $N_2 < N_1 < n_h(A_4)$ must have an h-representation not using $(s_2^{(1)}, s_3^{(1)}, s_4^{(1)})$, since otherwise we would get a coefficient -1 in the last position. So N_2 uses $(s_2^{(2)}, s_3^{(2)}, s_4^{(2)}) \neq (s_2^{(1)}, s_3^{(1)}, s_4^{(1)})$, and we get

$$\begin{aligned} N_2 &= (\epsilon_4 - 1 - s_4^{(2)})a_4 + ((s_4^{(2)} + 1)\gamma_3 - s_3^{(2)} - 2)a_3 \\ &\quad + ((s_3^{(2)} + 1)\gamma_2 - s_4^{(2)}\beta_2^{(4)} - s_2^{(2)} - 2)a_2 \\ &\quad + (s_3^{(1)}\beta_1^{(3)} + s_4^{(1)}\beta_1^{(4)} - s_2^{(1)}\gamma_1 + (s_2^{(2)}\gamma_1 - s_3^{(2)}\beta_1^{(3)} - s_4^{(2)}\beta_1^{(4)}) - 1), \end{aligned}$$

giving

$$\begin{aligned} &\epsilon_4 + (s_4^{(2)} + 1)\gamma_3 + (s_3^{(2)} + 1)\gamma_2 - s_4^{(2)}\beta_2^{(4)} \\ &+ (s_3^{(1)}\beta_1^{(3)} + s_4^{(1)}\beta_1^{(4)} - s_2^{(1)}\gamma_1) + (s_2^{(2)}\gamma_1 - s_3^{(2)}\beta_1^{(3)} - s_4^{(2)}\beta_1^{(1)}) \\ &\leq h + 6 + s_2^{(2)} + s_3^{(2)} + s_4^{(2)} \leq h + \delta. \end{aligned}$$

Now we continue constructing N_i in the same way as before. For the reduction in the last position caused by $(s_2^{(i)}, s_3^{(i)}, s_4^{(i)})$, we write

$$\kappa_i = s_3^{(i)}\beta_1^{(3)} + s_4^{(i)}\beta_1^{(4)} - s_2^{(i)}\gamma_1$$

for $i = 1, 2, \ldots, l$ and $\kappa_0 = \gamma_1$ We stop the process for $i = l$, when for the first time

$$\kappa_l = s_3^{(l)}\beta_1^{(3)} + s_4^{(l)}\beta_1^{(4)} - s_2^{(l)}\gamma_1 \leq 0. \tag{30}$$

Since the reduction κ_{i+1} cannot exceed the constant term $\kappa_i - 1$, we have $\kappa_i > \kappa_{i+1}$ for $1 \leq i \leq l-1$, so each N_i needs a new transfer that has not been used earlier. Since there are only finitely many possible transfers – the numbers $s_2^{(i)}, s_3^{(i)}$ and $s_4^{(i)}$ are bounded independently of h – and since there is always a transfer satisfying (30), namely $(0,0,0)$, the described process has to terminate after a bounded number (independently of h) of steps.

We collect the inequalities for the coefficient sums of $N_1, N_2, \ldots, N_l$ in an array:

$$\begin{array}{rcl} \epsilon_4 + (1 + s_4^{(1)})\gamma_3 + (1 + s_3^{(1)})\gamma_2 - s_4^{(1)}\beta_2^{(4)} + \gamma_1 - \kappa_1 & \leq & h + \delta \\ \epsilon_4 + (1 + s_4^{(2)})\gamma_3 + (1 + s_3^{(2)})\gamma_2 - s_4^{(2)}\beta_2^{(4)} + \kappa_1 - \kappa_2 & \leq & h + \delta \\ \ldots\ldots\ldots\ldots\ldots & & \ldots \\ \epsilon_4 + (1 + s_4^{(l)})\gamma_3 + (1 + s_3^{(l)})\gamma_2 - s_4^{(l)}\beta_2^{(4)} + \kappa_{l-1} - \kappa_l & \leq & h + \delta. \end{array}$$

Averaging gives

$$\epsilon_4 + \left(1 + \frac{\sum_{i=1}^{l} s_4^{(i)}}{l}\right)\gamma_3 + \left(1 + \frac{\sum_{i=1}^{l} s_3^{(i)}}{l}\right)\gamma_2 - \frac{\sum_{i=1}^{l} s_4^{(i)}}{l}\beta_2^{(4)} + \frac{\gamma_1}{l} \leq h + \delta, \tag{31}$$

where we have used that $\gamma_1 - \kappa_l \geq \gamma_1$ by (30).

In the sequel we shall characterize the possible transfers that can be used for $N_1, N_2, \ldots, N_l$, and shall find bounds for l, $\sum_{i=1}^{l} s_4^{(i)}$, $\sum_{i=1}^{l} s_3^{(i)}$ and $\beta_2^{(4)}$. Thus we get inequalities (24), which we were looking for.

The number N_i in our list has got the regular representation

$$N_i = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + (\kappa_{i-1} - 1),$$

since $0 \leq \kappa_{i-1} - 1 < \gamma_1$ for $i = 1, 2, \ldots, l$. For this kind of numbers, only few transfers are possible if we claim (22). Assume N_i uses (s_2, s_3, s_4) in order to achieve the *minimal representation* – here we leave out the upper index for a moment. Then as usual

$$\begin{aligned} N_i &= (\epsilon_4 - 1 - s_4)a_4 + ((1 + s_4)\gamma_3 - s_3 - 2)a_3 \\ &\quad + ((1 + s_3)\gamma_2 - s_4\beta_2^{(4)} - s_2 - 2)a_2 + \kappa_{i-1} - \kappa_i - 1 \end{aligned} \tag{32}$$

and

$$\epsilon_4 + (1 + s_4)\gamma_3 + (1 + s_3)\gamma_2 - s_4\beta_2^{(4)} + \kappa_{i-1} - \kappa_i \leq h + \delta. \tag{33}$$

Here we find at once that

$$0 \leq s_4 \leq 6,$$

since $s_4 \geq 7$ together with (33) contradicts (29) for $j = 7$. For $0 \leq s_4 \leq 6$, (33) together with (29) for $j = s_4$ implies

$$0 < (1 + s_3)\gamma_2 - s_4\beta_2^{(4)} < t_{s_4}\gamma_2, \tag{34}$$

where the left inequality stems from the non-negativity of the coefficients in (32). Now for $3 \leq s_4 \leq 6$ we have $t_{s_4} < 1$, and therefore (34) implies

$$\frac{s_4\beta_2^{(4)}}{\gamma_2} - 1 < s_3 < \frac{s_4\beta_2^{(4)}}{\gamma_2} \text{ or } s_3 = \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor,$$

if there is a solution of (34) at all. Once $3 \leq s_4 \leq 6$ is chosen, s_3 is already determined uniquely, if (34) is soluble. Otherwise (s_2, s_3, s_4) cannot be used in (33). For $s_4 = 1, 2$ we have $t_1 = 2, t_2 = 1.2$, so (34) gives

$$s_3 = \left\lfloor \frac{s_4 \beta_2^{(4)}}{\gamma_2} \right\rfloor \quad \text{or} \quad s_3 = \left\lfloor \frac{s_4 \beta_2^{(4)}}{\gamma_2} \right\rfloor + 1.$$

The second alternative arises only when (34) has two solutions.

In the remaining case $s_4 = 0$, we have by (33)

$$\epsilon_4 + \gamma_3 + (1 + s_3)\gamma_2 \leq h + \delta.$$

By (29) for $j = 0$ it follows that $0 \leq s_3 \leq 3$ if $s_4 = 0$.

What about the choice of s_2? For the minimal representation (32), we always have

$$0 \leq \kappa_{i-1} - \kappa_i - 1 = s_2\gamma_1 - s_3\beta_1^{(3)} - s_4\beta_1^{(4)} + \kappa_{i-1} - 1 < \gamma_1.$$

This means that

$$\frac{s_3\beta_1^{(3)} + s_4\beta_1^{(4)} - \kappa_{i-1}}{\gamma_1} < s_2 \leq \frac{s_3\beta_1^{(3)} + s_4\beta_1^{(4)} - \kappa_{i-1}}{\gamma_1} + 1.$$

Thus s_2 is uniquely determined. Since $0 < \kappa_{i-1} \leq \gamma_1$ we have

$$s_2 = \left\lfloor \frac{s_3\beta_1^{(3)} + s_4\beta_1^{(4)}}{\gamma_1} \right\rfloor \quad \text{or} \quad s_2 = \left\lfloor \frac{s_3\beta_1^{(3)} + s_4\beta_1^{(4)}}{\gamma_1} \right\rfloor + 1.$$

We collect all the possible transfers in four sets:

$$\begin{aligned}
A &= \left\{ (s_2, s_3, s_4) \mid 1 \leq s_4 \leq 6, s_3 = \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor, s_2 = \left\lfloor \frac{s_3\beta_1^{(3)} + s_4\beta_1^{(4)}}{\gamma_1} \right\rfloor \right\} \\
B &= \left\{ (s_2, s_3, s_4) \mid 1 \leq s_4 \leq 2, s_3 = \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor + 1, s_2 = \left\lfloor \frac{s_3\beta_1^{(3)} + s_4\beta_1^{(4)}}{\gamma_1} \right\rfloor \right\} \\
C &= \left\{ (s_2, s_3, s_4) \mid 1 \leq s_4 \leq 6, s_3 = \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor, s_2 = \left\lfloor \frac{s_3\beta_1^{(3)} + s_4\beta_1^{(4)}}{\gamma_1} \right\rfloor + 1 \right\} \\
&\quad \cup \{(0,0,0)\} \\
D &= \left\{ (s_2, s_3, 0) \mid 1 \leq s_3 \leq 3, s_2 = \left\lfloor \frac{s_3\beta_1^{(3)}}{\gamma_1} \right\rfloor \right\}.
\end{aligned}$$

The possibilities

$$s_2 = \left\lfloor \frac{s_3\beta_1^{(3)} + s_4\beta_1^{(4)}}{\gamma_1} \right\rfloor + 1, \; s_3 = \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor + 1.$$

cannot be combined. In this case the gain of the transfer would be negative. Since then $s_2\gamma_1 > s_3\beta_1^{(3)} + s_4\beta_1^{(4)}$ and $s_3\gamma_2 > s_4\beta_2^{(4)}$, we have

$$\begin{aligned} G(s_2, s_3, s_4) &= s_4(\beta_1^{(4)} + \beta_2^{(4)} - \gamma_3 + 1) + s_3(\beta_1^{(3)} - \gamma_2 + 1) + s_2(1 - \gamma_1) \\ &< s_4 + s_3 + s_2 - s_4\gamma_3 < 0 \end{aligned}$$

for large h, because the s_i are bounded, and by (9) γ_3 increases when h increases.

By the same argument, we cannot have $s_2 = \lfloor (s_3\beta_1^{(3)} + s_4\beta_1^{(4)})/\gamma_1 \rfloor + 1$ for transfers where $s_4 = 0$. A transfer from C is assumed to stand at the end of our representation list for N_l, since only for those transfers $s_3\beta_1^{(3)} + s_4\beta_1^{(4)} - s_2\gamma_1 \leq 0$, whilst transfers from A, B, D have to be used earlier in the list.

Now we introduce 11 variables $r_1, r_2, \ldots, r_6, q_1, q_2, d_1, d_2, d_3$ taking values from $\{0, 1\}$, indicating whether the corresponding transfer is used for some N_i or not. Here r_j stands for the transfers $(s_2, s_3, j) \in A$, q_j for $(s_2, s_3, j) \in B$ and d_j for $(s_2, j, 0) \in D$. In addition we introduce $s \in \{0, 1, \ldots, 6\}$ for $(s_2, s_3, s) \in C$ used at the end of the list. We then choose values for r_j, q_j, d_j and s. All possible lists of representations for the N_i using these corresponding transfers under consideration will give rise to the same average inequality (31), so the ordering of the used transfers within the list does not play any role for us. In fact this is the crucial advantage of our method. Regarding all different orderings would make the problem very large and possibly unmanageable. Altogether we have to consider "only" $12 \cdot 7 \cdot 2^{11}$ cases, since $\beta_2^{(4)}/\gamma_2$ is chosen from one of the 12 intervals I_p, $p = 1, 2, \ldots, 12$.

Bounds for the values of the interesting magnitudes can now be computed:

$$l = \sum_{j=1}^{6} r_j + \sum_{j=1}^{2} q_j + \sum_{j=1}^{3} d_j + 1,$$

the last 1 standing for the ultimate line in the list. Further

$$\sum_{i=1}^{l} s_4^{(i)} = \sum_{j=1}^{6} j r_j + \sum_{j=1}^{2} j q_j + s, \text{ and}$$

$$\sum_{i=1}^{l} s_3^{(i)} = \sum_{j=1}^{6} \left\lfloor \frac{j\beta_2^{(4)}}{\gamma_2} \right\rfloor r_j + \sum_{j=1}^{2} \left(\left\lfloor \frac{j\beta_2^{(4)}}{\gamma_2} \right\rfloor + 1 \right) q_j + \sum_{j=1}^{3} j d_j + \left\lfloor \frac{s\beta_2^{(4)}}{\gamma_2} \right\rfloor .$$

Now the intervals $I_p, p = 1, 2, \ldots, 12$ are chosen in such a way that the values $\lfloor j\beta_2^{(4)}/\gamma_2 \rfloor$ for $j = 1, 2, \ldots, 6$ are constant over the whole of each I_p, and so $\sum_{i=1}^{l} s_3^{(i)}$ is constant over I_p. If $I_p = [w_p, z_p)$ and $\beta_2^{(4)}/\gamma_2 \in I_p$, we have $\lfloor j\beta_2^{(4)}/\gamma_2 \rfloor = \lfloor j w_p \rfloor$. Then

$$\sum_{i=1}^{l} s_3^{(i)} = \sum_{j=1}^{6} \lfloor j w_p \rfloor r_j + \sum_{j=1}^{2} (\lfloor j w_p \rfloor + 1) q_j + \sum_{j=1}^{3} j d_j + \lfloor s w_p \rfloor$$

and

$$\sum_{i=1}^{l} s_4^{(i)} \beta_2^{(4)} \leq \sum_{i=1}^{l} s_4^{(i)} z_p \gamma_2,$$

and we have computed bounds for all the values we are interested in. For each choice of the variables r_j, q_j, d_j and s, (31) now gives us an inequality (24). We combine this inequality with (27) as we did before, and get by (28) a bound for the h-range coefficient.

Now we try to reduce the number of cases to consider. Remember that we have chosen $\beta_2^{(4)}/\gamma_2 \in I_p = [w_p, z_p)$, and can find out whether (34) is soluble or not. If

$$1 + \lfloor s_4 w_p \rfloor - s_4 z_p \geq t_{s_4},$$

then

$$(1 + s_3)\gamma_2 - s_4 \beta_2^{(4)} \geq t_{s_4} \gamma_2$$

throughout I_p, and the corresponding transfer $(s_2, s_3, s_4) \in A$ or C cannot be used, a fact that reduces the amount of work considerably. In the same way $(s_2, s_3, s_4) \in B$ can be excluded if

$$2 + \lfloor s_4 w_p \rfloor - s_4 z_p \geq t_{s_4}.$$

The first runs on the computer showed that the cases where $s > 0$ and $r_s = 1$ were the most critical ones, giving large coefficient bounds for the

h-range. Therefore we refined our method in this case. Now $s > 0$ and $r_s = 1$ mean that the transfer (s_2, s_3, s) is used at the end of the list, and the transfer $(s_2 - 1, s_3, s)$ at another place, let us say in line f. We leave out line f from our list and average. Since the transfers of line f and line l only differ in the s_2 position, with one unit, we have $\kappa_f = \kappa_l + \gamma_1$, hence $\gamma_1 - \kappa_{f-1} + \kappa_f - \kappa_l \geq \gamma_1$. Now $\sum_{i=1,i\neq f}^{l} s_4^{(i)} = \sum_{i=1}^{l} s_4^{(i)} - s$ and $\sum_{i=1,i\neq f}^{l} s_3^{(i)} = \sum_{i=1}^{l} s_3^{(i)} - \lfloor sw_p \rfloor$. Thus averaging gives

$$\epsilon_4 + \left(1 + \frac{\sum_{i=1}^{l} s_4^{(i)} - s}{l-1}\right)\gamma_3 + \left(1 + \frac{\sum_{i=1}^{l} s_3^{(i)} - \lfloor sw_p \rfloor}{l-1}\right)\gamma_2$$
$$-\frac{\sum_{i=1}^{l} s_4^{(i)} - s}{l-1}\beta_2^{(4)} + \frac{\gamma_1}{l-1} \leq h + \delta. \tag{35}$$

This inequality usually gives much better results than (31). A computer run of the described program, taking care of the maximal coefficient bound occuring in each interval I_p, yields the following situation:

Table 2.

Interval I_p	Largest coefficient bound	Total number of cases	Cases with coefficient bound > 2.008
$I_1 = [0, \frac{1}{6})$	2.00	7168	0
$I_2 = [\frac{1}{6}, \frac{1}{5})$	2.16	3072	9
$I_3 = [\frac{1}{5}, \frac{1}{4})$	2.37	1280	29
$I_4 = [\frac{1}{4}, \frac{1}{3})$	2.59	1280	46
$I_5 = [\frac{1}{3}, \frac{2}{5})$	2.42	1280	31
$I_6 = [\frac{2}{5}, \frac{1}{2})$	2.78	6144	437
$I_7 = [\frac{1}{2}, \frac{3}{5})$	2.30	1280	39
$I_8 = [\frac{3}{5}, \frac{2}{3})$	2.60	3072	143
$I_9 = [\frac{2}{3}, \frac{3}{4})$	2.56	3072	134
$I_{10} = [\frac{3}{4}, \frac{4}{5})$	2.64	1280	94
$I_{11} = [\frac{4}{5}, \frac{5}{6})$	2.78	1280	98
$I_{12} = [\frac{5}{6}, 1)$	3.97	14336	2968

The computer result in I_1 is 1.97, but since we have used (22) on our way, we cannot conclude better than 2.0.

In order to get even better results, we consider now some other types of "key numbers". We look at

$$M(m) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (m\beta_2^{(4)} - \left\lfloor \frac{m\beta_2^{(4)}}{\gamma_2} \right\rfloor \gamma_2 - 1)a_2 + (\gamma_1 - 1)$$

for $m = 1$ in the intervals I_8, I_9, I_{10}, for $m = 2$ in the intervals I_6, I_{11}, I_{12} and for $m = 3$ in the interval I_4. In all these intervals, the above representation of $M(m)$ is the regular one, since there $0 \leq m\beta_2^{(4)} - \lfloor m\beta_2^{(4)}/\gamma_2 \rfloor \gamma_2 - 1 \leq \gamma_2 - 2$. Again we construct the corresponding list of the $M(m)_i$. Here the transfer $(s_2, \lfloor m\beta_2^{(4)}/\gamma_2 \rfloor, m) \in A$ or C is impossible throughout the list, since it would give us a negative coefficient for a_2. The set B has to be extended with the transfers

$$(s_2, \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor + 1, s_4),\ 3 \leq s_4 \leq 6$$

and $(s_2, 2, 1)$. In addition D has to be extended with $(s_2, 4, 0)$. We introduce the corresponding variables $q_3, q_4, q_5, q_6, q, d_4 \in \{0, 1\}$ and compute

$$l = \sum_{j=1}^{6} r_j + \sum_{j=1}^{6} q_j + q + \sum_{j=1}^{4} d_j + 1$$

$$\sum_{i=1}^{l} s_4^{(i)} = \sum_{j=1}^{6} j(q_j + r_j) + q + s$$

$$\begin{aligned} \sum_{i=1}^{l} s_3^{(i)} &= \sum_{j=1}^{6} \left\lfloor \frac{j\beta_2^{(4)}}{\gamma_2} \right\rfloor (r_j + q_j) + \sum_{j=1}^{6} q_j + \sum_{j=1}^{4} j d_j + 2q + \left\lfloor \frac{s\beta_2^{(4)}}{\gamma_2} \right\rfloor \\ &= \sum_{j=1}^{6} \lfloor j w_p \rfloor (r_j + q_j) + \sum_{j=1}^{6} q_j + \sum_{j=1}^{4} j d_j + 2q + \lfloor s w_p \rfloor. \end{aligned}$$

In the same way as before we may reduce the number of cases to consider, when we notice that the use of $(s_2, s_3, s_4) \in A$ or C is possible only if

$$0 \leq m\beta_2^{(4)} - \left\lfloor \frac{m\beta_2^{(4)}}{\gamma_2} \right\rfloor \gamma_2 + \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor \gamma_2 - s_4\beta_2^{(4)} - 1 < t_{s_4}\gamma_2.$$

If $m \geq s_4$ we must demand

$$(m - s_4)w_p - \lfloor mw_p \rfloor + \lfloor s_4 w_p \rfloor < t_{s_4},$$

and

$$0 < (m - s_4)z_p - \lfloor mw_p \rfloor + \lfloor s_4 w_p \rfloor.$$

If $m < s_4$ we must demand

$$(m - s_4)z_p - \lfloor mw_p \rfloor + \lfloor s_4 w_p \rfloor < t_{s_4},$$

and

$$0 < (m - s_4)w_p - \lfloor mw_p \rfloor + \lfloor s_4 w_p \rfloor.$$

Here we used again that $\lfloor j\beta_2^{(4)}/\gamma_2 \rfloor = \lfloor jw_p \rfloor$ in the interval I_p. For transfers from B we get similar conditions. As shown before, the cases with $s > 0$ and $r_s = 1$ can be treated seperately and do not cause that much headache.

A new computer run gives new bounds for our coefficient in the mentioned intervals:

Table 3.

Interval I_p	Largest coefficient bound	m	Total number of cases	Cases with coefficient bound > 2.008
$I_4 = [\frac{1}{4}, \frac{1}{3})$	2.78	3	1024	28
$I_6 = [\frac{2}{5}, \frac{1}{2})$	2.33	2	5120	44
$I_8 = [\frac{3}{5}, \frac{2}{3})$	2.38	1	768	21
$I_9 = [\frac{2}{3}, \frac{3}{4})$	2.35	1	1024	18
$I_{10} = [\frac{3}{4}, \frac{4}{5})$	2.30	1	1024	11
$I_{11} = [\frac{4}{5}, \frac{5}{6})$	2.30	2	1536	27
$I_{12} = [\frac{5}{6}, 1)$	2.31	2	10240	99

The largest coefficient bound 2.78, the only one exceeding 2.5, arises now for $\beta_2^{(4)}/\gamma_2 \in I_4 = [1/4,\ 1/3)$. Here we use a new method to reduce the coefficient bound. We consider all lists for N_i, where the coefficient bound by the averaging method became > 2.008. In each case we stored the

values of $l, \sum_{i=1}^{l} s_4^{(i)}$ and $\sum_{i=1}^{l} s_3^{(i)}$ corresponding to the average inequality (31). Then we considered all the lists for $M(3)$, where the coefficient bound became > 2.008. Here we stored the values say L, $\sum_{i=1}^{L} S_4^{(i)}$ and $\sum_{i=1}^{L} S_3^{(i)}$ corresponding to the inequality

$$\epsilon_4 + \left(1 + \frac{\sum_{i=1}^{L} S_4^{(i)}}{L}\right)\gamma_3 + \frac{\sum_{i=1}^{L} S_3^{(i)}}{L}\gamma_2 + \left(3 - \frac{\sum_{i=1}^{L} S_4^{(i)}}{L}\right)\beta_2^{(4)} + \frac{\gamma_1}{L} \leq h + \delta. \tag{36}$$

Note that $\lfloor 3\beta_2^{(4)}/\gamma_2 \rfloor = 0$ in I_4. It turned out that in all these 28 cases $3 - \sum_{i=1}^{L} S_4^{(i)}/L > 0$, so $\beta_2^{(4)}$ appears with positive factor in (36) and with negative factor in (31). Now we combine the two inequalities by multiplying (31) with the weight $w_1 = 3 - \sum_{i=1}^{L} S_4^{(i)}/L > 0$ and (36) with $w_2 = \sum_{i=1}^{l} s_4^{(i)}/l > 0$, add both and divide the result by the sum of the weights $w_1 + w_2$. In this way we get rid of $\beta_2^{(4)}$ and find

$$\epsilon_4 + \frac{w_1(1 + \sum_{i=1}^{l} s_4^{(i)}/l) + w_2(1 + \sum_{i=1}^{L} S_4^{(i)}/L)}{w_1 + w_2}\gamma_3$$
$$+\frac{w_1(1 + \sum_{i=1}^{l} s_3^{(i)}/l) + w_2 \sum_{i=1}^{L} S_3^{(i)}/L}{w_1 + w_2}\gamma_2 + \frac{w_1/l + w_2/L}{w_1 + w_2}\gamma_1 \leq h + \delta.$$

Optimal combination with (27), as in (28), yields that in each of the $28 \cdot 46 = 1288$ weighted averages, the coefficient bound becomes < 2.43. This is the largest value occuring. The best known upper coefficient bounds for the different intervals I_p are given in the following table 4.

Table 4.

Interval I_p	Largest coefficient bound	Interval I_p	Largest coefficient bound
$I_1 = [0, \frac{1}{6})$	2.00	$I_7 = [\frac{1}{2}, \frac{3}{5})$	2.30
$I_2 = [\frac{1}{6}, \frac{1}{5})$	2.16	$I_8 = [\frac{3}{5}, \frac{2}{3})$	2.38
$I_3 = [\frac{1}{5}, \frac{1}{4})$	2.37	$I_9 = [\frac{2}{3}, \frac{3}{4})$	2.35
$I_4 = [\frac{1}{4}, \frac{1}{3})$	2.43	$I_{10} = [\frac{3}{4}, \frac{4}{5})$	2.30
$I_5 = [\frac{1}{3}, \frac{2}{5})$	2.42	$I_{11} = [\frac{4}{5}, \frac{5}{6})$	2.30
$I_6 = [\frac{2}{5}, \frac{1}{2})$	2.33	$I_{12} = [\frac{5}{6}, 1)$	2.31

So we get our final result:

Theorem 2 . *Given a sequence of bases with four elements $A_4(h)$, then*

$$n_h(A_4(h)) \leq 2.43 \left(\frac{h}{4}\right)^4 + O(h^3).$$

Of course this bound is also valid for the extremal bases $A_4^*(h)$.

A computer program, written in "Pascal", that performs the computations which prove the above result can be found in Kirfel [7].

2.4 Bounds for the h-range $n_h(A_4)$, when special transfers are used

The method presented in the previous section may also be applied in order to obtain upper bounds for the h-range $n_h(A_4)$, when we know in advance which transfers we are going to use in the interval $[0,\ n_h(A_4)]$ in order to obtain minimal representations. Here we shall show that, using transfers $(s_2, s_3, 0)$, $s_3 \geq 0$ and $(0, 0, 1)$, we cannot get a coefficient larger than 2 in (22), and so we cannot get the extremal h-range.

Let us start with a simple example, in order to show how our method works. Assume that we only allow transfers of the kind $(s_2, s_3, 0)$, $s_3 \geq 0$. Look again at

$$\begin{aligned} N_1 &= (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + (\gamma_1 - 1) \\ &= (\epsilon_4 - 1)a_4 + (\gamma_3 - s_3^{(1)} - 2)a_3 + ((s_3^{(1)} + 1)\gamma_2 - s_2^{(1)} - 2)a_2 \\ &\quad + ((s_2^{(1)} + 1)\gamma_1 - s_3^{(1)}\beta_1^{(3)} - 1), \end{aligned}$$

and the corresponding list of N_i, as we did in section 2.3. Since $s_4^{(i)} = 0$ for $i = 1, 2, \ldots, l$, the averaging process (31) gives

$$\epsilon_4 + \gamma_3 + (1 + \frac{\sum_{i=1}^{l} s_3^{(i)}}{l})\gamma_2 + \frac{\gamma_1}{l} \leq h + \delta.$$

As we have seen before, only different tranfers are used throughout the list. But $(s_2, s_3, 0) \neq (\bar{s}_2, \bar{s}_3, 0)$ implies $s_3 \neq \bar{s}_3$ for transfers with positive

gain. Therefore

$$\sum_{i=1}^{l} s_3^{(i)} \geq 0 + 1 + 2 + \cdots + (l-1) = \frac{l(l-1)}{2},$$

giving

$$\epsilon_4 + \gamma_3 + \frac{l+1}{2}\gamma_2 + \frac{\gamma_1}{l} \leq h + \delta,$$

and by (25)

$$n_h(A_4) \leq \frac{2l}{l+1}\left(\frac{h}{4}\right)^4 + O(h^3).$$

For $l \leq 4$ this gives us a coefficient bound ≤ 1.6. If $l \geq 5$ there was an i, $1 \leq i \leq l$ such that $s_3^{(i)} \geq 4$. This would imply

$$\epsilon_4 + \gamma_3 + 5\gamma_2 \leq h + \delta$$

in line number i of our list. Inequality (28) then gives

$$n_h(A_4) \leq 1.78(h/4)^4 + O(h^3).$$

The last formula of course covers both cases. This gives us the following theorem:

Theorem 3 . *Let a sequence of bases $A_4(h)$ be given. Assume that only transfers of the type $(s_2, s_3, 0)$, $s_3 \geq 0$ are used in order to achieve minimal representations in the interval $[0,\ n_h(A_4(h))]$, then*

$$n_h(A_4) \leq 1.78(h/4)^4 + O(h^3).$$

In addition to the transfers $(s_2, s_3, 0)$ we now want to use $(0,0,1)$. The average inequality (31) for the N_i-list, where we now allow the use of $(0,0,1)$, then reads

$$\epsilon_4 + \frac{l+1}{l}\gamma_3 + \frac{l^2-l+2}{2l}\gamma_2 - \frac{\beta_2^{(4)}}{l} + \frac{\gamma_1}{l} \leq h + \delta. \tag{37}$$

Now assume $\beta_2^{(4)} > 0$ and look at

$$M(1) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\beta_2^{(4)} - 1)a_2 + (\gamma_1 - 1).$$

The corresponding list of $M(1)_i$, having length L, cannot contain the transfer $(0,0,1)$, since otherwise we would get a negative coefficient for a_2. So only transfers of the form $(s_2, s_3, 0)$, $s_3 \geq 0$ can be used here, and the corresponding average inequality reads

$$\epsilon_4 + \gamma_3 + \frac{L-1}{2}\gamma_2 + \beta_2^{(4)} + \frac{\gamma_1}{L} \leq h + \delta. \tag{38}$$

The weighted average of (37) and (38), using the weights l and 1 respectively, gives

$$\epsilon_4 + \frac{l+2}{l+1}\gamma_3 + \left(\frac{l^2-l+2}{2(l+1)} + \frac{L-1}{2(l+1)}\right)\gamma_2 + \frac{1+1/L}{l+1}\gamma_1 \leq h + \delta.$$

Running through all values $1 \leq l \leq 5$, $1 \leq L \leq 5$ and using (28), this gives coefficient bounds ≤ 1.94. If $l \geq 6$ or $L \geq 6$ we get, as before, a coefficient bound ≤ 1.78.

The case $\beta_2^{(4)} = 0$ can in fact be treated in general, not only in connection with the transfers $(0,0,1)$ and $(s_2, s_3, 0)$. The average inequality (31) for the list of the N_i now reads

$$\epsilon_4 + \left(1 + \frac{\sum_{i=1}^{l} s_4^{(i)}}{l}\right)\gamma_3 + \left(1 + \frac{\sum_{i=1}^{l} s_3^{(i)}}{l}\right)\gamma_2 + \frac{\gamma_1}{l} \leq h + \delta.$$

If $s_4^{(i)} > 0$ and $s_3^{(i)} > 0$ for some i, $1 \leq i \leq l$, this implies $\epsilon_4 + 2\gamma_3 + 2\gamma_2 \leq h + \delta$ in line i in our list. By (28) we then get a coefficient bound ≤ 2. So we may assume $s_4^{(i)} > 0 \Longrightarrow s_3^{(i)} = 0$ and $s_3^{(i)} > 0 \Longrightarrow s_4^{(i)} = 0$. Note that for $\beta_2^{(4)} = 0$ we have $(s_2, 0, s_4) \neq (\bar{s_2}, 0, \bar{s_4}) \Longrightarrow s_4 \neq \bar{s_4}$. Let now g denote the number of transfers $(s_2, 0, s_4)$, $s_4 > 0$ used in the list and v the number of transfers $(s_2, s_3, 0)$, $s_3 > 0$. Then

$$l = g + v + 1,$$

the last 1 standing for the regular representation at the end of the list. Since $\beta_2^{(4)} = 0$, no other transfer could occur in the final line. The averaging inequality (31) now becomes

$$\epsilon_4 + \left(1 + \frac{g(g+1)}{2(g+v+1)}\right)\gamma_3 + \left(1 + \frac{v(v+1)}{2(g+v+1)}\right)\gamma_2 + \frac{\gamma_1}{g+v+1} \leq h + \delta.$$

Running through the values $0 \leq g \leq 2$, $0 \leq v \leq 3$ and using (28), we get coefficient bounds ≤ 1.8. Here $v \geq 4$ would, as before, give a coefficient bound ≤ 1.78. On the other hand, $g \geq 3$ would imply the use of $(s_2, 0, s_4)$, $s_4 \geq 3$, and this again gives $\epsilon_4 + 4\gamma_3 + \gamma_2 \leq h + \delta$ in the corresponding line of the list, giving a coefficient bound ≤ 1.69 by (28). These results are collected in the following theorems:

Theorem 4 . *Let $A_4(h)$ be a sequence of bases, where $\beta_2^{(4)} = 0$ in (13), then*

$$n_h(A_4(h)) \leq 2(h/4)^4 + O(h^3).$$

In fact, this result could have been read off table 4. But the proof given here is not based on a computer result and is thus more transparent.

Theorem 5 . *Let $A_4(h)$ be a sequence of bases, where only transfers of the type $(s_2, s_3, 0)$, $s_3 \geq 0$ and $(0,0,1)$ are used in order to achieve minimal representations in the interval $[0, n_h(A_4(h))]$, then*

$$n_h(A_4(h)) \leq 2(h/4)^4 + O(h^3).$$

In fact this result can be sharpened in the following way:

Theorem 6 . *Let $A_4(h)$ be a sequence of bases, where only transfers of the type (s_2, s_3, s_4), $s_4 \leq 1$ are used in order to achieve minimal representations in the interval $[0, n_h(A_4(h))]$, then*

$$n_h(A_4(h)) \leq 2(h/4)^4 + O(h^3).$$

The details of the very technical proof are found in Kirfel [7].

Remark 3. It is enough to claim that no other transfers than the ones under consideration are used in order to achieve minimal representations in the interval

$$[(\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3,\ (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + \gamma_1 - 1],$$

since the lists for N_i and $M(1)_i$ only use information from this interval.

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