Report (Universitetet i Bergen. Matematisk institutt)

Department of PURE MATHEMATICS

On Extremal Bases for the h-range Problem, II

Christoph Kirfel

February 2, 1990

Dedicated to Professor Ernst S. Selmer on the occasion of his 70th birthday, February 11, 1990.

Report No. 55 ISSN - 0332 - 5407

UNIVERSITY OF BERGEN
Bergen, Norway

Department of Mathematics

University of Bergen

5014 Bergen - U

NORWAY

On Extremal Bases for the h-range Problem, II

Christoph Kirfel

February 2, 1990

Dedicated to Professor Ernst S. Selmer on the occasion of his 70th birthday, February 11, 1990.

Report No. 55 ISSN - 0332 - 5407

On Extremal Bases for the h-range Problem, II

Christoph Kirfel

February 2, 1990

This report is a direct continuation of [20]. All references to the formulas (1) – (38) and the text sources [1] – [19] go back to this report. The promised proof of Theorem 6 in section 2.4 is our first aim in the present paper.

2.5 The proof of Theorem 6

First we recall the theorem we want to prove.

Theorem 6 . *Let $A_4(h)$ be a sequence of bases, where only transfers of the type (s_2, s_3, s_4), $s_4 \leq 1$ are used in order to achieve minimal representations in the interval $[0,\, n_h(A_4(h)]$, then*

$$n_h(A_4(h)) \leq 2(h/4)^4 + O(h^3).$$

In addition to the transfers $(s_2, s_3, 0)$, we want to consider those of the form $(s_2, s_3, 1)$. As before we mainly study numbers from the interval

$$[(\epsilon_4-1)a_4+(\gamma_3-2)a_3+(\beta_2^{(4)}-1)a_2, (\epsilon_4-1)a_4+(\gamma_3-2)a_3+(\gamma_2-2)a_2+(\gamma_1-1)],$$

so the use of $(s_2, s_3, 1)$ with $s_3 \geq 2$ would imply

$$\epsilon_4 + 2\gamma_3 + 2\gamma_2 \leq h + \delta,$$

and by (28) the coefficient bound is ≤ 2. So we only have to to study the transfers $(0,0,1)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$ in addition to those of the form $(s_2, s_3, 0)$. It is easy to show that $s_2 \leq s_3 + s_4$. Then $s_2 = 2$ implies $s_3 = s_4 = 1$, but the corresponding transfer $(2,1,1)$ has no positive gain, so we need not look at transfers with $s_2 \geq 2$.

Now 16 different cases arise for the N_i-list, depending on which transfers are used in addition to those of the form $(s_2, s_3, 0)$. The following table 5 gives an overview. The symbols + and − show whether a particular transfer is used or not.

Table 5.

Case number	Transfers used in the N_i-list in addition to $(s_2, s_3, 0)$			
	$(0,0,1)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
1	−	−	−	−
2	+	−	−	−
3	−	+	−	−
4	+	+	−	−
5	−	−	+	−
6	+	−	+	−
7	−	+	+	−
8	+	+	+	−
9	−	−	−	+
10	+	−	−	+
11	−	+	−	+
12	+	+	−	+
13	−	−	+	+
14	+	−	+	+
15	−	+	+	+
16	+	+	+	+

In the same way as in section 2.3, we can show that if $(0,0,1)$ and $(1,0,1)$ are used in the N_i-list — in the list for the $M(1)_i$ neither of the two transfers are admissible — then we may leave out the line where $(0,0,1)$ occurs. The situation then obtained equals a case where $(1,0,1)$ is used and not $(0,0,1)$. So we may assume that the situations where $(0,0,1)$ and $(1,0,1)$ are in use, are covered by those where $(1,0,1)$ occurs and not $(0,0,1)$. We did not exploit this aspect in section 2.3, since we were interested in the number of cases where the coefficient bound was > 2.008. Here we only want to show that no coefficient bound is > 2.

We can also show that neither of the transfers $(0,1,1)$ nor $(1,1,1)$ can terminate neither the N_i-list nor the $M(1)_i$-list. Remember that the condition for the transfer in the last line is given by (30). If $\beta_1^{(3)} + \beta_1^{(4)} \leq 0$ then $\beta_1^{(3)} = \beta_1^{(4)} = 0$ and

$$\begin{aligned} G(0,1,1) &= (\beta_1^{(4)} + \beta_2^{(4)} - \gamma_3 + 1) + (\beta_1^{(3)} - \gamma_2 + 1) \\ &= \beta_2^{(4)} - \gamma_2 - \gamma_3 + 2 < 0. \end{aligned}$$

If $\beta_1^{(3)} + \beta_1^{(4)} - \gamma_1 \leq 0$ then

$$\begin{aligned} G(1,1,1) &= (\beta_1^{(4)} + \beta_2^{(4)} - \gamma_3 + 1) + (\beta_1^{(3)} - \gamma_2 + 1) + (1 - \gamma_1) \\ &\leq \beta_2^{(4)} - \gamma_2 - \gamma_3 + 3 \leq 0. \end{aligned}$$

Now case 1 and 2 are covered by Theorems 3 and 5, respectively.

In case 3 we get for the average inequality (31):

$$\epsilon_4 + \frac{l+1}{l}\gamma_3 + \frac{l+1}{2}\gamma_2 - \frac{\beta_2^{(4)}}{l} + \frac{\gamma_1}{l} \leq h + \delta,$$

since $(1,0,1) \in C$ has to stand in the final line instead of $(0,0,0)$. For $l \geq 2$ this implies

$$\epsilon_4 + \frac{l+1}{l}\gamma_3 + \frac{l}{2}\gamma_2 + \frac{\gamma_1}{l} \leq h + \delta,$$

giving a coefficient bound < 2 by (25). For $l = 1$, going back to section 2.3 we get

$$N_1 = (\epsilon_4 - 2)a_4 + (2\gamma_3 - 2)a_3 + (\gamma_2 - \beta_2^{(4)} - 3)a_2 + (2\gamma_1 - \beta_1^{(4)} - 1)$$

for the minimal representation of N_1 with a coefficient $2\gamma_1 - \beta_1^{(4)} - 1 \geq \gamma_1$ in the last position, a contradiction.

Case 4 now reduces to case 3 as described above.

Case 5 and 9. Here $(0,1,1)$ or $(1,1,1)$, say $(s_2,1,1)$, is used in the N_i-list, giving the following average inequality (31):

$$\epsilon_4 + \frac{l+1}{l}\gamma_3 + \frac{l^2-l+2}{2l}\gamma_2 + \frac{\gamma_2 - \beta_2^{(4)}}{l} + \frac{\gamma_1}{l} \le h + \delta.$$

Since $\gamma_2 - \beta_2^{(4)} > 0$, the coefficient bound of (25) does not exceed

$$\frac{2l^3}{(l+1)(l^2-l+2)} < 2.$$

Case 6. The transfers $(0,0,1)$ and $(0,1,1)$ are now going to be used in the N_i-list. Assume that $(0,0,1)$ occurs in line f of the list. Now if

$$n = x_4a_4 + x_3a_3 + x_2a_2 + x_1$$

is a minimal representation of $n \in \mathbf{N}$, and $z = x_4a_4 + x_3a_3 + y_2a_2 + x_1$, where $0 \le y_2 \le x_2$, then also the representation of z is minimal. This means that the transfers $(s_2^{(i)}, s_3^{(i)}, s_4^{(i)})$, $1 \le i \le f-1$ from the N_i-list coincide with the first $f-1$ ones from the $M(1)_i$-list. Remember that all transfers from the N_i-list except $(0,0,1)$ are also allowed in the $M(1)_i$-list. Further, only transfers with different reductions κ_i are used in the N_i-list, and therefore the reduction in the last position of $(0,1,1)$ is larger than for $(0,0,1)$, thus the transfer $(0,1,1)$ is used earlier in the list than $(0,0,1)$, and $f \ge 2$. This implies that $(0,1,1)$ is also used in the $M(1)_i$-list. Now the first $f-1$ lines in the $M(1)_i$-list can be exchanged by those of the N_i-list without destroying the cancellation effect of the averaging process. Since these corresponding lines differ by $\gamma_2 - \beta_2^{(4)}$, we therefore get an additional contribution $(f-1)(\gamma_2 - \beta_2^{(4)})/L$ to the average inequality of the $M(1)_i$-list, which now reads

$$\epsilon_4 + \frac{L+1}{L}\gamma_3 + \frac{L^2-3L+4}{2L}\gamma_2 + (L-1)\frac{\beta_2^{(4)}}{L} + (f-1)\frac{\gamma_2 - \beta_2^{(4)}}{L} + \frac{\gamma_1}{L} \le h + \delta \quad (39).$$

For the N_i-list we get an average inequality (31):

$$\epsilon_4 + \frac{l+2}{l}\gamma_3 + \frac{l^2-3l+8}{2l}\gamma_2 - \frac{2\beta_2^{(4)}}{l} + \frac{\gamma_1}{l} \le h + \delta.$$

If $f \geq 3$, weighting and combining these inequalities and running through the actual values $3 \leq l \leq 6$ and $3 \leq L \leq 6$ give coefficient bounds ≤ 1.98.

For $f = 2$ we get a coefficient bound > 2 only for $L = 4$ and $L = 5$. Now since $f = 2$, we see that $(0,1,1)$ stands at the top of the $M(1)_i$-list and $(0,0,0)$ at the end, and the two or three transfers, $(0,1,0)$, $(s_2,2,0)$ or $(0,1,0)$, $(s_2,2,0)$ and $(s_2',3,0)$ in between. If "higher" transfers occurred, we would get an additional contribution γ_2/L in the list, and this would be enough to get coefficient bounds < 2. Now since $L = 4$ or 5, we must have two transfers of the type $(s_2,s_3,0)$, $(s_2',s_3+1,0)$ in two consecutive lines, say line i and $i+1$. This situation is going to be studied closer.

Assume $(s_2,s_3,0)$ stands first. Then line i and $i+1$ read:

$$\begin{aligned}
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + s_3\gamma_2 + \kappa_{i-1} - s_3\beta_1^{(3)} + s_2\gamma_1 &\leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + (s_3+1)\gamma_2 + (s_2' - s_2)\gamma_1 - \beta_1^{(3)} &\leq h + \delta.
\end{aligned}$$

Since $(s_2' - s_2)\gamma_1 - \beta_1^{(3)} - 1$ is the constant term of N_{i+1} and so has to be between 0 and $\gamma_1 - 1$, we have $s_2' = s_2 + 1$. Now compare the occurring constant terms. If

$$\kappa_{i-1} - s_3\beta_1^{(3)} + s_2\gamma_1 > (s_2' - s_2)\gamma_1 - \beta_1^{(3)} = \gamma_1 - \beta_1^{(3)},$$

then $s_2 = 0$ is impossible, so

$$\kappa_{i-1} - (s_3 - 1)\beta_1^{(3)} + (s_2 - 1)\gamma_1 > 0$$

implies that the transfer $(s_2 - 1, s_3 - 1, 0)$ would have been possible in line i, giving a better gain than $(s_2,s_3,0)$, a contradiction. So

$$\kappa_{i-1} - s_3\beta_1^{(3)} + s_2\gamma_1 \leq \gamma_1 - \beta_1^{(3)},$$

and therefore

$$\epsilon_4 + \gamma_3 + \beta_2^{(4)} + (s_3+1)\gamma_2 + \kappa_{i-1} - s_3\beta_1^{(3)} + s_2\gamma_1 \leq h + \delta.$$

Now replacing line i, where we have $s_3\gamma_2$ in the second position, by this new one, where we have $(s_3+1)\gamma_2$ in the second position, we get an additional γ_2/L in the average inequality of our list.

If $(s_2', s_3+1, 0)$ is used first, the corresponding lines look like:

$$\begin{aligned} \epsilon_4 + \gamma_3 + \beta_2^{(4)} + (s_3+1)\gamma_2 + \kappa_{i-1} - (s_3+1)\beta_1^{(3)} + s_2'\gamma_1 &\leq h+\delta \\ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + s_3\gamma_2 + (s_2 - s_2')\gamma_1 + \beta_1^{(3)} &\leq h+\delta, \end{aligned}$$

so $s_2 = s_2'$. Now $(0,0,0)$ is used in the last line and there $\kappa_{l-1} < \beta_1^{(3)}$, otherwise this representation would not be minimal, since an additional transfer $(0,1,0)$ could be performed. The gain of $(0,1,0)$ is always positive if there is a transfer $(s_2, s_3, 0)$ with positive gain, a fact we have assumed in our situation. But then the information in line $i+1$ implies

$$\epsilon_4 + \gamma_3 + \beta_2^{(4)} + \gamma_2 + \kappa_{l-1} \leq h + \delta.$$

Again replacing the last line by this new one, we get an additional contribution γ_2/L in our average inequality. In any case we now get such a contribution and the average inequaliy (39) holds for $f = 3$, and we are through.

Case 7 and 11. Here $(1,0,1)$ again terminates the list. Inequality (31) now reads:

$$\epsilon_4 + \frac{l+2}{l}\gamma_3 + \frac{l-1}{2}\gamma_2 + \frac{2(\gamma_2 - \beta_2^{(4)})}{l} + \frac{\gamma_1}{l} \leq h + \delta.$$

Since $\gamma_2 > \beta_2^{(4)}$, this gives a coefficient bound

$$\frac{2l^3}{l(l+2)(l-1)} = \frac{2l^2}{l^2+l-2} \leq 2,$$

since of course $l \geq 2$.

Case 8 and 12. These two cases reduce to case 7 and 11, respectively.

Case 10. Now $(0,0,1)$ and $(1,1,1)$ are used in the N_i-list. Assume that $(1,1,1)$ occurs in line $l-f+1$ and assume also that $(1,1,1)$ is used in the $M(1)_i$-list. Then the transfers in the lines $l-f+2, l-f+3, \ldots, l$ from the N_i-list coincide with those in the lines $L-f+2, L-f+3, \ldots, L$ from the $M(1)_i$-list by the same argument as in case 6. Exchanging the corresponding lines, we get an additional contribution $(f-1)(\gamma_2 - \beta_2^{(4)})/L$ in the average inequality exactly like in (39). Since $(0,0,0)$ terminates both lists, we have $f \geq 2$. Now also (31) coincides with the average inequality from case 6 and we are finished if $f \geq 3$.

If $f = 2$ we get coefficient bounds > 2 only if $L = 3$ or $L = 4$. Since $(1,1,1)$ and $(0,0,0)$ then form the last two lines in the $M(1)_i$-list, we know that there must be two "consecutive" transfers in two consecutive lines, and by the same argument as in case 6 we are through.

Assume now that $(1,1,1)$ does not occur in the $M(1)_i$-list at all. Then only transfers of the type $(s_2, s_3, 0)$, $s_3 \geq 0$ occur in the $M(1)_i$- list, and the average inequality is given by (38). Now combining (31) given in case 6 with (38), and running through the actual values $3 \leq l \leq 6$ and $1 \leq L \leq 6$, give coefficient bounds ≥ 2 only in two cases where $L = 3$. But then $(0,0,0)$, $(0,1,0)$ and $(s_2, 2, 0)$ are used in the $M(1)_i$- list, and the same argument as in case 6 applies, giving an additional contribution $\gamma_2/L = \gamma_2/3$ in the average inequality (38). Again running through the actual values $L = 3$ and $3 \leq l \leq 6$ gives only coefficient bounds < 2.

Case 13 — 16. It is easy to see that $(0,1,1)$ and $(1,1,1)$ cannot both be used, since the use of $(0,1,1)$ implies $\beta_1^{(3)} + \beta_1^{(4)} < \gamma_1$, and then

$$\begin{aligned} G(1,1,1) &= (\beta_1^{(4)} + \beta_2^{(4)} + 1 - \gamma_3) + (\beta_1^{(3)} + 1 - \gamma_2) + (1 - \gamma_1) \\ &= (\beta_1^{(3)} + \beta_1^{(4)} - \gamma_1) + (\beta_2^{(4)} + 1 - \gamma_2) + 2 - \gamma_3 < 0. \end{aligned}$$

So none of the situations 13 – 16 arise for us, completing the proof of the theorem.

We conclude this section with another result on the coefficient bound for the h-range, when special transfers are used, a generalization of Theorem 5.

Theorem 7 . *Let $A_4(h)$ be a sequence of bases, where only transfers of the type $(s_2, s_3, 0)$, $s_3 \geq 0$ and one transfer of the type (s_2, s_3, s_4), $s_4 \geq 1$ are used in order to achieve minimal representations in the interval $[0, n_h(A_4(h)]$, then*

$$n_h(A_4(h)) \leq 2(h/4)^4 + O(h^3).$$

Proof. By Theorem 6 we may assume that (s_2, s_3, s_4) with $s_4 \geq 2$ is used in the N_i-list. Now the average inequality (31) reads

$$\epsilon_4 + \frac{l + s_4}{l}\gamma_3 + \frac{l^2 - l + 2 + 2s_3}{2l}\gamma_2 - \frac{s_4\beta_2^{(4)}}{l} + \frac{\gamma_1}{l} \leq h + \delta. \qquad (40)$$

Now $(s_3+1)\gamma_2 - s_4\beta_2^{(4)} > 0$, since $s_2 - s_3\gamma_2 + s_4\beta_2^{(4)}$ is the reduction in the second position and has to be $< \gamma_2$. This together with (40) implies

$$\epsilon_4 + \frac{l+2}{l}\gamma_3 + \frac{l^2-l}{2l}\gamma_2 + \frac{\gamma_1}{l} \leq h + \delta.$$

By (25) this gives a coefficient bound

$$\frac{2l^3}{(l+2)(l^2-l)} = \frac{2l^2}{l^2+l-2} \leq 2$$

for $l \geq 2$. If $l = 1$ this means that (s_2, s_3, s_4) occurs in the "last" line implying $\kappa_1 = s_4\beta_1^{(4)} + s_3\beta_1^{(3)} - s_2\gamma_1 \leq 0$, and therefore by the representation of N_1 we have

$$\epsilon_4 + 3\gamma_3 + \gamma_1 \leq h + \delta.$$

Since $l = 1$ there is no $N_2, N_3, \ldots$ and no transfer with $s_4 = 0$ can have a positive gain. This means that the regular representation of the number

$$(\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + (\gamma_1 - 1)$$

has to be minimal, giving

$$\gamma_3 + \gamma_2 + \gamma_1 \leq h + 5.$$

Adding the last two inequalities gives

$$\epsilon_4 + 4\gamma_3 + \gamma_2 + 2\gamma_1 \leq 2h + \delta + 5,$$

and by (25) the coefficient bound is again ≤ 2.

2.6 A New Bound for the Extremal h-range $n_h(A_4^*(h))$

In the computations performed in order to get the results contained in the tables 2, 3 and 4, we had to consider very many different cases. In order to reduce this huge number, we look at pairs of transfers that cannot occur together under certain circumstances. The reduction obtained by these means is so essential that the number of cases we are left with is rather small.

Eleven such "pairs" are considered. We use the notation of section 2.3 and put $m = 0$, when we consider the N_i-list.

1. $(m = 0)$ and $((r_5 = 1)$ or $(r_6 = 1)$ or $(s > 4))$ and $((d_1 = 1)$ or $(d_2 = 1)$ or $(d_3 = 1))$ is impossible for the extremal bases A_4^*, since

$$(r_5 = 1) \text{ or } (r_6 = 1) \text{ or } (s > 4) \implies \epsilon_4 + 6\gamma_3 \leq h + \delta$$

in the corresponding line of the list, and

$$(m = 0) \text{ and } ((d_1 = 1) \text{ or } (d_2 = 1) \text{ or } (d_3 = 1)) \implies \epsilon_4 + \gamma_3 + 2\gamma_2 \leq h + \delta$$

in the corresponding line of the N_i-list. Adding these two inequalities and applying (28) give a coefficient bound < 2.

2. $(m = 2)$ and $(p = 12)$ and $((r_5 = 1)$ or $(r_6 = 1)$ or $(s > 4))$ and $((d_1 = 1)$ or $(d_2 = 1)$ or $(d_3 = 1))$ is impossible for the extremal bases A_4^*, since

$$((r_5 = 1) \text{ or } (s = 5)) \text{ and } (m = 2) \text{ and } (p = 12) \implies \epsilon_4 + 6\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} \leq h + \delta$$

in the corresponding line of the $M(2)_i$-list, while

$$((r_6 = 1) \text{ or } (s = 6)) \text{ and } (m = 2) \text{ and } (p = 12) \implies \epsilon_4 + 7\gamma_3 + 4\gamma_2 - 4\beta_2^{(4)} \leq h + \delta$$

in the corresponding line of the $M(2)_i$-list, and

$$(m = 2) \text{ and } ((d_1 = 1) \text{ or } (d_2 = 1) \text{ or } (d_3 = 1)) \Rightarrow \epsilon_4 + \gamma_3 + 2\beta_2^{(4)} \leq h + \delta$$

in the corresponding line of the $M(2)_i$-list. Adding either of the two first inequalities to the last one and applying (28) again give a coefficient bound < 2.

3. $(m = 1)$ and $(p = 10)$ and $((r_6 = 1)$ or $(s = 6))$ and $((d_1 = 1)$ or $(d_2 = 1)$ or $(d_3 = 1))$ is impossible for the extremal bases A_4^*, since

$$(p = 10) \text{ and } (m = 1) \text{ and } ((r_6 = 1) \text{ or } (s = 6)) \implies \epsilon_4 + 7\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} \leq h + \delta$$

in the corresponding line of the $M(1)_i$-list, and

$$(m = 1) \text{ and } ((d_1 = 1) \text{ or } (d_2 = 1) \text{ or } (d_3 = 1)) \implies \epsilon_4 + \gamma_3 + \gamma_2 + \beta_2^{(4)} \leq h + \delta$$

in the corresponding line of the $M(1)_i$-list. Adding these two inequalities, applying (28), and using the bound for $\beta_2^{(4)}$ in the interval I_{10}, give a coefficient bound < 2.

4. $(m = 1)$ and $(p = 8)$ and $((r_4 = 1)$ or $(s = 4))$ and $((d_2 = 1)$ or $(d_3 = 1))$ is impossible for the extremal bases A_4^*, since

$$(p = 8) \text{ and } (m = 1) \text{ and } ((r_4 = 1) \text{ or } (s = 4)) \implies \epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} \leq h + \delta$$

in the corresponding line of the $M(1)_i$-list, and

$$(m = 1) \text{ and } ((d_2 = 1) \text{ or } (d_3 = 1)) \implies \epsilon_4 + \gamma_3 + 2\gamma_2 + \beta_2^{(4)} \leq h + \delta$$

in the corresponding line of the $M(1)_i$-list. Adding these two inequalities, applying (28), and using the bound for $\beta_2^{(4)}$ in I_8, give a coefficient bound < 2.

5. $(m = 0)$ and $(p = 8)$ and $((r_4 = 1)$ or $(s = 4))$ and $((d_2 = 1)$ or $(d_3 = 1))$ is impossible for the extremal bases A_4^*, since

$$(p = 8) \text{ and } (m = 0) \text{ and } ((r_4 = 1) \text{ or } (s = 4)) \implies \epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} \leq h + \delta$$

in the corresponding line of the N_i-list, and

$$(m = 0) \text{ and } ((d_2 = 1) \text{ or } (d_3 = 1)) \implies \epsilon_4 + \gamma_3 + 3\gamma_2 \leq h + \delta$$

in the corresponding line of the N_i-list. Adding these two inequalities and applying (28) give a coefficient bound < 2.

6. $((m = 0)$ or $(m = 2))$ and $((r_6 = 1)$ or $(s = 6))$ and $(q_2 = 1)$ is impossible for the extremal bases A_4^*, since

$$(m = 2) \text{ and } ((r_6 = 1) \text{ or } (s = 6)) \implies \epsilon_4 + 7\gamma_3 \leq h + \delta$$

in the corresponding line of the $M(2)_i$-list, and

$$(m = 2) \text{ and } (q_2 = 1) \implies \epsilon_4 + 3\gamma_3 + \gamma_2 \leq h + \delta$$

in the corresponding line of the $M(2)_i$-list. Adding these two inequalities and applying (28) give a coefficient bound < 2. For $m = 0$ in both inequalities the left hand side increases, and we get the same result.

7. $(m = 1)$ and $(s > 0)$ and $((q_1 = 1)$ or $(d_1 = 1)$ or $(d_2 = 1)$ or $(d_3 = 1))$ is impossible for the extremal bases A_4^*, since one of the last four statements implies that the second position of the representation in the corresponding line of the $M(1)_i$-list is $\geq \gamma_2 - 1$. But then an additional (s_2, s_3, s) transfer would be possible, since such a transfer does not decrease the constant term. Thus we get a lower coefficient sum, a contradiction.

8. $(m = 2)$ and $(s > 0)$ and $((q_2 = 1)$ or $(d_1 = 1)$ or $(d_2 = 1)$ or $(d_3 = 1))$ is impossible by the same arguments as above used for the $M(2)_i$- list.

9. $(m = 0)$ and $(s > 1)$ and $((q_1 = 1)$ or $(q_2 = 1)$ or $(d_1 = 1)$ or $(d_2 = 1)$ or $(d_3 = 1))$ is impossible by the same arguments as above used for the N_i-list.

10. $(m = 2)$ and $(s > 1)$ and $(q_1 = 1)$ and $(p = 12)$ is impossible for the extremal bases A_4^*, since the second position of the representation in the line of the $M(2)_i$-list where $q_1 = 1$ is $\beta_2^{(4)} - 1$. An additional (s_2, s_3, s) transfer would be possible, because

$$\beta_2^{(4)} - 1 - s\beta_2^{(4)} + (s-1)\gamma_2 = (s-1)(\gamma_2 - \beta_2^{(4)}) - 1 \geq 0,$$

since $s > 1$. Thus we get a lower coefficient sum, a contradiction.

11. Theorem 6 implies that a transfer (s_2, s_3, s_4) with $s_4 \geq 2$ has to occur in the N_i-list or the $M(1)_i$-list. Either of these statements gives

$$\epsilon_4 + 3\gamma_3 \leq h + \delta$$

in the corresponding line. Now $((d_3 = 1)$ and $(m = 0))$ or $((d_4 = 1)$ and $(m > 0))$ implies

$$\epsilon_4 + \gamma_3 + 4\gamma_2 \leq h + \delta.$$

Adding these two inequalities and applying (28) give a coefficient bound ≤ 2.

When incorporating these conditions into the computer program, the amount of work is greatly reduced. Table 6 shows the effect of this incorporation for the N_i-list. (Compare with Table 2 in section 2.3.)

Table 6.

Interval I_p	Largest coefficient bound	Total number of cases	Cases with coefficient bound > 2.008
$I_1 = [0, \frac{1}{6})$	2.00	608	0
$I_2 = [\frac{1}{6}, \frac{1}{5})$	2.16	320	8
$I_3 = [\frac{1}{5}, \frac{1}{4})$	2.37	192	29
$I_4 = [\frac{1}{4}, \frac{1}{3})$	2.59	144	44
$I_5 = [\frac{1}{3}, \frac{2}{5})$	2.42	144	28
$I_6 = [\frac{2}{5}, \frac{1}{2})$	2.78	448	186
$I_7 = [\frac{1}{2}, \frac{3}{5})$	2.30	144	36
$I_8 = [\frac{3}{5}, \frac{2}{3})$	2.60	288	99
$I_9 = [\frac{2}{3}, \frac{3}{4})$	2.56	320	95
$I_{10} = [\frac{3}{4}, \frac{4}{5})$	2.64	144	62
$I_{11} = [\frac{4}{5}, \frac{5}{6})$	2.78	144	57
$I_{12} = [\frac{5}{6}, 1)$	3.97	768	604

Also for the $M(m)_i$-lists we get large reductions when we incorporate the mentioned conditions in our computer program. But the number of cases where the coefficient bound exceeds 2.008 is not reduced essentially. Table 7 shows the new situation and should be compared with Table 3.

Table 7.

Interval I_p	Largest coefficient bound	m	Total number of cases	Cases with coefficient bound > 2.008
$I_2 = [\frac{1}{6}, \frac{1}{5})$	2.37	5	1280	40
$I_3 = [\frac{1}{5}, \frac{1}{4})$	2.60	4	512	34
$I_4 = [\frac{1}{4}, \frac{1}{3})$	2.78	3	512	28
$I_5 = [\frac{1}{3}, \frac{2}{5})$	2.67	2	304	52
$I_6 = [\frac{2}{5}, \frac{1}{2})$	2.33	2	512	43
$I_8 = [\frac{3}{5}, \frac{2}{3})$	2.38	1	96	19
$I_9 = [\frac{2}{3}, \frac{3}{4})$	2.35	1	152	18
$I_{10} = [\frac{3}{4}, \frac{4}{5})$	2.30	1	96	10
$I_{11} = [\frac{4}{5}, \frac{5}{6})$	2.30	2	288	27
$I_{12} = [\frac{5}{6}, 1)$	2.31	2	448	65

In section 2.3 we presented a method of generating two types of average inequalities, one with positive and one with negative prefactors for $\beta_2^{(4)}$, and used that method in I_4. We did the same in the other intervals, and were lucky in I_2, I_3 and I_5. In fact all occuring maximal coefficient bounds in these intervals now became < 2.31. In I_6 we performed the same computations for $m = 2$. Here we got two cases with negative prefactor, but the corresponding coefficient bounds did not exceed 2.2. The remaining cases were combined with those from Table 6 for $m = 0$, and the largest coefficient bound occurring was 2.28. The intervals I_4 and I_8 were treated separately.

In I_8 we performed the computations for $m = 1$ and found three cases with coefficient bound > 2.3. Two of them had negative prefactor. Combing them with all situations for $m = 3$, where only positive prefactors occur, gave coefficient bounds < 2.31. The third case had a positive prefactor and was combined with the situations for $m = 0$, where of course all prefactors are negative. Here 2.35 was the maximal value occurring.

We remember that we got the maximal coefficient bound 2.43 in I_4, when we combined the situations for $m = 0$ and $m = 3$. The largest coefficient bound arose for the situation $r_1 = 1$, $r_2 = 1$, $s = 0$ and $l = 3$ for the $M(3)_i$-list and $r_1 = 1$, $r_2 = 1$, $r_3 = 1$, $s = 0$ and $l = 4$ for the N_i-list. In all other situations the coefficient bound did not exceed 2.33.

We now study this special situation closer. Assume first that $(0,0,2)$, $(0,0,1)$ and $(0,0,0)$ are used in the $M(3)_i$-list in this ordering. The list then reads:

$$\begin{array}{rcl} \epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + & \gamma_1 - 2\beta_1^{(4)} & \leq h + \delta \\ \epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} + & \beta_1^{(4)} & \leq h + \delta \\ \epsilon_4 + \gamma_3 + 3\beta_2^{(4)} + & \beta_1^{(4)} & \leq h + \delta. \end{array}$$

If now $r_3 = 1$ corresponds to $(0,0,3)$ in the N_i-list, this list reads

$$\begin{array}{rcl} \epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + & \gamma_1 - 3\beta_1^{(4)} & \leq h + \delta \\ \epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + & \beta_1^{(4)} & \leq h + \delta \\ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + & \beta_1^{(4)} & \leq h + \delta \\ \epsilon_4 + \gamma_3 + \gamma_2 + & \beta_1^{(4)} & \leq h + \delta. \end{array}$$

Now replacing the last two lines of the first list by the last two of the second one gives an average inequality

$$\epsilon_4 + 2\gamma_3 + 2\gamma_2/3 + \gamma_1/3 \leq h + \delta,$$

giving a coefficient bound < 2.3 by (28).

Now all other orderings of the actual transfers in both lists either lead to contradictions or to situations very similar to the one above, where two lines have to be replaced and the coefficient bound does not exceed 2.3.

Collecting all this information we get the follwing Table 8.

Table 8.

Interval I_p	Largest coefficient bound	Interval I_p	Largest coefficient bound
$I_1 = [0, \frac{1}{6})$	2.00	$I_7 = [\frac{1}{2}, \frac{3}{5})$	2.30
$I_2 = [\frac{1}{6}, \frac{1}{5})$	2.14	$I_8 = [\frac{3}{5}, \frac{2}{3})$	2.35
$I_3 = [\frac{1}{5}, \frac{1}{4})$	2.31	$I_9 = [\frac{2}{3}, \frac{3}{4})$	2.35
$I_4 = [\frac{1}{4}, \frac{1}{3})$	2.33	$I_{10} = [\frac{3}{4}, \frac{4}{5})$	2.30
$I_5 = [\frac{1}{3}, \frac{2}{5})$	2.31	$I_{11} = [\frac{4}{5}, \frac{5}{6})$	2.30
$I_6 = [\frac{2}{5}, \frac{1}{2})$	2.28	$I_{12} = [\frac{5}{6}, 1)$	2.31

Thus we get our final result sharpening Theorem 2:

Theorem 8 . *Given a sequence of bases with four elements $A_4(h)$, then*

$$n_h(A_4(h)) \leq 2.35 \left(\frac{h}{4}\right)^4 + O(h^3).$$

2.7 Asymptotic h-ranges

Now we go back to the general problem of the extremal h-range for an extremal k element basis $A_k^*(h)$, where k, as before, is a fixed number and h tends to infinity. In section 2.2 we mentioned that we did not know whether the limit

$$\lim_{h\to\infty} \frac{n_h(A_k^*(h))}{(h/k)^k}$$

does exist or not. Meanwhile we have been able to show that the answer to this question is yes. In this section we want to present the proof. Things shown in section 2.2 then get much easier and we need not so many subsequences and subsubsequences in the formulation of Theorem 1. Our main result is the following

Theorem 9 . *Let $A_k^*(h)$ denote a sequence of extremal bases. Then*

$$\lim_{h\to\infty} \frac{n_h(A_k^*(h))}{(h/k)^k}$$

exists.

Proof. Recalling (8) we already know that there exist positive constants $c, C \in \mathbf{R}$ such that

$$c \leq \frac{n_h(A_k^*(h))}{(h/k)^k} \leq C \qquad \text{for all } h.$$

Hofmeister [5] could show that for all parameter bases $A_k(h)$ with $0 < d(h/k)^k \leq n_h(A_k(h))$, the number of possible transfers giving a positive gain is bounded independently of h.

Let now ρ_j be the reduction of the j-th component in a regular representation caused by the transfer $(s_2, s_3, \ldots, s_k)$. From (5), we get

$$\rho_j = e_j - x_j = s_j - s_{j+1}\gamma_j + \sum_{b=j+2}^{k} s_b \beta_j^{(b)}.$$

Note that $\rho_j \leq e_j \leq \gamma_j - 1$ for $j \leq k-1$, and that some of these "reductions" may be negative. The reductions ρ_j and their sum $G(s_2, s_3, \ldots, s_k)$, the gain of the transfer, are linear functions in the variables γ_j and $\beta_j^{(b)}$ with integer coefficients.

We now look at the set of possible transfers $\tau^{(i)} = (s_2^{(i)}, s_3^{(i)}, \ldots, s_k^{(i)})$, $i = 1, 2, \ldots, F$ for the sequence $A_k^*(h)$. This set has to be finite as mentioned above. To each $\tau^{(i)}$ we can find the corresponding vector $\rho_j^{(i)}$ of reductions. Consider now the set of all possible orderings of the corresponding gains and the reductions:

$$\begin{aligned} G(\tau^{(i_1)}) &\geq G(\tau^{(i_2)}) \geq \cdots \geq 0 \geq \cdots \geq G(\tau^{(i_F)}) \\ \rho_j(\tau^{(l_1^{(j)})}) &\geq \rho_j(\tau^{(l_2^{(j)})}) \geq \cdots \geq 0 \geq \cdots \geq \rho_j(\tau^{(l_F^{(j)})}) \text{ for } j = 1, 2, \ldots, k. \end{aligned} \tag{41}$$

This set of orderings must of course also be finite. Each such ordering is what Braunschädel [1] called a *structure*, and we get finitely many structures $S_1, S_2, \ldots, S_N$.

We now choose a sequence h_m such that

$$\lim_{m\to\infty} \frac{n_{h_m}(A_k^*(h_m))}{(h_m/k)^k} = \limsup_{h\to\infty} \frac{n_h(A_k^*(h))}{(h/k)^k} = T.$$

For each h_m the corresponding basis $A_k^*(h_m)$ belongs to one of the structures $S_1, S_2, \ldots, S_N$. So there must be at least one structure to which infinitely many bases $A_k^*(h_m)$ belong. We call this structure S_L, and choose a subsequence $(h_{m_l})_{l\in\mathbb{N}}$ of $(h_m)_{m\in\mathbb{N}}$, where all $A_k^*(h_{m_l})$ belong to S_L. In order to reduce the number of indices, we denote also this subsequence by h_m. Now we write

$$n_{h_m}(A_k^*(h_m)) = \epsilon_k(h_m)a_k^*(h_m) + \epsilon_{k-1}(h_m)a_{k-1}^*(h_m) + \cdots + \epsilon_1(h_m)$$

for the regular representation of the h_m-range of $A_k^*(h_m)$, and introduce a new vector $\rho_j^{(0)}$, $j = 1, 2, \ldots, k$, that does not correspond to any transfer, by the following definition (we write ϵ_k for $\epsilon_k(h_m)$):

$$\begin{aligned}
\rho_1^{(0)} &= \gamma_1, \\
\rho_j^{(0)} &= \gamma_j - 1, \text{ for } j = 2, 3, \ldots, k-1 \\
\rho_k^{(0)} &= \epsilon_k - 1.
\end{aligned}$$

We now build "key numbers" by running through all positive reductions $\rho_j^{(i)}$ and combining them to regular representations by $A_k^*(h_m)$ in the following way:

$$\sum_{j=1}^{k} \max\{\rho_j^{(l_j)} - 1,\ 0\} a_j^* \leq \epsilon_k a_k^*.$$

Since all these numbers are h_m-representable, we can find a transfer τ for each of them, such that the coefficient sum for the minimal representation is $\leq h_m$, giving

$$\sum_{j=1}^{k} \rho_j^{(l_j)} - G(\tau) \leq h_m + \delta.$$

Here $\rho_j^{(l_j)}$ and $G(\tau)$ are linear functions in our variables, and the magnitude δ that corresponds to the constant terms in the inequality, is bounded independently of h_m, since at most k units from the key numbers and possibly a number of $s_j^{(i)}$ are involved. For each key number we thus get an inequality

$$\sum_{j=1}^{k-1} p_j \gamma_j + p_k \epsilon_k + \sum_{j=1}^{k} \sum_{b=j+2}^{k} p_j^{(b)} \beta_j^{(b)} \leq h_m + \delta.$$

The system of these inequalities together with (41) forms what we call the *inequality system associated with S_L*. Remember that (41) can be written as a number of inequalities of the form

$$\sum_{j=1}^{k-1} q_j \gamma_j + \sum_{j=1}^{k} \sum_{b=j+2}^{k} q_j^{(b)} \beta_j^{(b)} \leq \delta,$$

where again δ is a constant that is bounded independently of h_m.

We now introduce new variables

$$\begin{aligned}
x_j &= \gamma_j / h_m, \text{ for } j = 1, 2, \ldots, k-1, \\
x_k &= \epsilon_k / h_m, \qquad\qquad (42) \\
x_l &= \beta_j^{(b)} / h_m, \text{ for } l > k, \text{ suitable.}
\end{aligned}$$

Let R denote the total number of variables. We now form the *reduced inequality system associated with* S_L, by dividing all the earlier inequalities by h_m and leaving out the constant term divided by h_m. Renumbering the coefficients p_j, $p_j^{(b)}$, q_j and $q_j^{(b)}$ in a suitable manner, this gives

$$\sum_{i=1}^{R} p_i x_i \leq 1, \qquad \text{and} \qquad \sum_{i=1}^{R} q_i x_i \leq 0.$$

Since we possibly strengthened the conditions in the inequality system by disregarding δ/h_m, it is not evident that there is a solution of the reduced inequality system associated with S_L. Since for all $j = 1, 2, \ldots, k-1$ the number $(\gamma_j - 1)a_j^*(h_m)$ is h_m-representable, and no transfer applies, we must have $\gamma_j - 1 \leq h_m$, thus $\gamma_j \leq 2h_m$. The same argument can be used to show that $\epsilon_k \leq h_m$. In addition we know that $0 \leq \beta_j^{(b)} \leq \gamma_j - 1 \leq h_m$, and thus $0 \leq x_i \leq 2$ for all of our variables x_i, $i = 1, 2, \ldots, R$. Therefore we can find a subsequence $(h_{m_l})_{l \in \mathbf{N}}$ of $(h_m)_{m \in \mathbf{N}}$ such that for all $1 \leq i \leq R$

$$\lim_{l \to \infty} x_i(h_{m_l}) = \bar{x}_i$$

exists. But then $\sum_{i=1}^{R} p_i x_i(h_{m_l}) \leq 1 + \delta/h_{m_l}$ and $\sum_{i=1}^{R} q_i x_i(h_{m_l}) \leq \delta/h_{m_l}$ imply

$$\sum_{i=1}^{R} p_i \bar{x}_i \leq 1, \qquad \text{and} \qquad \sum_{i=1}^{R} q_i \bar{x}_i \leq 0,$$

so there are (not necessarily inner) points in the simplex corresponding to the reduced linear inequality system. We even have

$$\bar{x}_1 \bar{x}_2 \cdots \bar{x}_k = \lim_{l \to \infty} \gamma_1 \gamma_2 \cdots \gamma_{k-1} \epsilon_k / h_{m_l}^k.$$

We now look at the object function

$$E(x_1, x_2, \ldots, x_R) = \prod_{j=1}^{k} x_j$$

defined on the simplex corresponding to the reduced inequality system. Since this simplex is contained in the "cube" where $0 \leq x_i \leq 2$, $1 \leq i \leq R$, and the constraints only include "$\leq$" symbols, the definition set for E is compact. Since E of course is continous, we can find a maximal value M for E in a point $\vec{x}^* = (x_1^*, x_2^*, \ldots, x_R^*)$ in the simplex. The point $\vec{x}^*$ need not be unique.

Since

$$\epsilon_k a_k^*(h_{m_l}) \leq n_{h_{m_l}}(A_k^*(h_{m_l})) < (\epsilon_k + 1)a_k^*(h_{m_l}),$$

we find by (9) that

$$\begin{aligned}
&\lim_{l\to\infty} n_{h_{m_l}}(A_k^*(h_{m_l}))/(h_{m_l})^k = \lim_{l\to\infty} \epsilon_k a_k^*(h_{m_l})/(h_{m_l})^k \\
&= \lim_{l\to\infty} \epsilon_k \gamma_{k-1} a_{k-1}^*(h_{m_l})/(h_{m_l})^k = \cdots \\
&= \lim_{l\to\infty} \epsilon_k \gamma_{k-1}\gamma_{k-2}\cdots\gamma_1/(h_{m_l})^k.
\end{aligned}$$

Because of the existence of every single $\lim_{l\to\infty} x_j(h_{m_l}) = \lim_{l\to\infty} \gamma_j/h_{m_l}$ for all $j = 1, 2, \ldots, k-1$ and $\lim_{l\to\infty} x_k(h_{m_l}) = \lim_{l\to\infty} \epsilon_k/h_{m_l}$, we have

$$\begin{aligned}
M &= E(x_1^*, x_2^*, \ldots, x_R^*) \geq \bar{x}_1\bar{x}_2\cdots\bar{x}_k = \lim_{l\to\infty} \gamma_1\gamma_2\cdots\gamma_{k-1}\epsilon_k/(h_{m_l})^k \\
&= \lim_{l\to\infty} n_{h_{m_l}}(A_k^*(h_{m_l}))/(h_{m_l})^k = T/k^k. \qquad (43)
\end{aligned}$$

Now we try to find a *rational* point in the simplex not too far away from $\vec{x}^*$. It is not quite evident how to do this, since the intuitive way – reducing all variables x_i^* to a "near" rational – may violate the inequality constraints since there may occur negative coefficients.

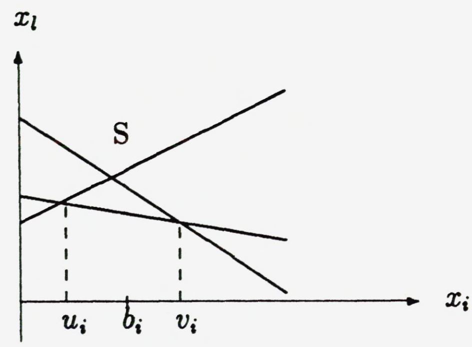
Let now $\epsilon > 0$. We shall show that for $h = Kt + \delta$, for fixed δ, $K \in \mathbf{N}$ and t running through the positive integers, we can find a sequence of bases $A_k(t)$, such that the prefactor in front of $(h/k)^k$ for the h-range is $\geq T - 2\epsilon$. Choose $\delta_j > 0$, $j = 1, 2, \ldots, k$ such that

$$\begin{aligned}
x_j^* - \delta_j \;&\in\; \mathbf{Q} \qquad \text{for } j = 1, 2, \ldots, k \\
\prod_{j=1}^{k}(x_j^* - \delta_j) \;&>\; M - \epsilon.
\end{aligned}$$

We now introduce additional linear constraints for our variables:

$$x_j \geq x_j^* - \delta_j \qquad \text{for } j = 1, 2, \ldots, k,$$

and get a new non-empty simplex S contained in the first one. If S only consists of one point, this point has rational coordinates since it is the intersection of a number of linear equalities with integer or rational coefficients. If there are two points in the simplex, their connecting line segment will also be contained in the simplex because of its convexity, and we can find a variable x_i such that the projection of the simplex onto the x_i- axis will contain an interval $[u_i,\ v_i]$, and we can choose a rational number b_i from this interval, $u_i \leq b_i \leq v_i$. See also picture 4.



Picture 4.

Look now at the intersection of the simplex S and the hyperplane $x_i = b_i$ and continue inductively. Then we find a rational point in S, where $x_i = b_i \in \mathbf{Q}$, for all indices $i = 1, 2, \ldots, R$, and

$$\prod_{j=1}^{k} x_j = \prod_{j=1}^{k} b_j > M - \epsilon.$$

Let K be the common denominator for $b_1, b_2, \ldots, b_R$ and look at the basis $A_k(t)$ given by

$$\gamma_j = b_j Kt + 1,\ \beta_j^{(b)} = b_l Kt \qquad \text{corresponding to the definition (42).}$$

Now we go back from the reduced inequality system to an unreduced one. In the expression for the gain of a transfer, the constant terms now cancel, and we get the same ordering for the actual gains $G(\tau)$ as for the "reduced" ones $\tilde{G}(\tau)$, since

$$\begin{aligned}
G(\tau) &= \sum_{j=1}^{k}\left(s_j - s_{j+1}\gamma_j + \sum_{b=j+2}^{k} s_b\beta_j^{(b)}\right)\\
&= \sum_{j=1}^{k}\left(s_j - s_{j+1}(b_jKt+1) + \sum_{b=j+2}^{k} s_b\beta_j^{(b)}\right)\\
&= \sum_{j=1}^{k}\left(-s_{j+1}b_jKt + \sum_{b=j+2}^{k} s_b\beta_j^{(b)}\right).
\end{aligned}$$

Assume first that we never have equality for the "reduced" versions $\tilde{\rho}_i$ expressed by the b_i-values. For large t, we then have the same ordering for the reductions for $A_k(t)$ as for the earlier reduced ones. This ordering then coincides with that in the second line of the original inequality system (41).

Consider now the positive integer $n \leq b_kKta_k(t)$ with regular representation

$$n = e_ka_k(t) + e_{k-1}a_{k-1}(t) + \cdots + e_1.$$

We then find indices l_j such that

$$\rho_j^{(l_j)} > e_j \geq \rho_j^{(l_j+1)}.$$

There are no other ρ_j values between the upper and the lower bound, meaning that the reduction in the j-th component caused by the transfer τ which produces the minimal representation of $\sum_{j=1}^{k}(\rho_j^{(l_j)}-1)a_j(t)$, is $\leq \rho_j^{(l_j+1)}$, and the minimal representation of n is obtained by exactly the same transfer as for $\sum_{j=1}^{k}(\rho_j^{(l_j)}-1)a_j(t)$. Now

$$\sum_{j=1}^{k}\tilde{\rho}_j - \tilde{G}(\tau) \leq 1$$

from the reduced system implies that for $A_k(t)$

$$\sum_{j=1}^{k}\rho_j - G(\tau) \leq Kt + \delta,$$

since the values of γ_j are increased by a unit. Here again δ is bounded independently of t. Thus we cover all integers $\leq b_kKta_k$ with $Kt+\delta$

addends, since $\rho_1^{(0)} - 1 = \gamma_1 - 1$, and $\rho_j^{(0)} = \gamma_j - 1$, $j = 2, 3, \ldots, k-1$ are the maximal coefficients in the regular representations.

If some of the $\tilde{\rho}_j$ are equal, say $\tilde{\rho}_j^{(l_m^{(j)})} = \tilde{\rho}_j^{(l_m^{(j)}+1)}$, then the actual reductions ρ_j may occur in reverse order in comparison to (41), so $\rho_j^{(l_m^{(j)})} < \rho_j^{(l_m^{(j)}+1)}$ since the additional constant terms may be different. This difference is then bounded independently of t. If e_j was chosen between two such values, we then use the representation corresponding to the lower one and increase δ of $Kt + \delta$ to cover the extra addends.

By the definition of $A_k(t)$ we get

$$\begin{aligned} n_h(A_k(t)) &\geq b_k Kta_k(t) \geq b_k Ktb_{k-1}Kta_{k-1}(t) \geq \cdots \\ &\geq b_k b_{k-1} \cdots b_1 (Kt)^k. \end{aligned}$$

If we put $h = Kt + \delta$, and choose t so large that $(M-\epsilon)(Kt/(Kt+\delta))^k \geq M - 2\epsilon$, we obtain

$$\frac{n_h(A_k(t))}{(h/k)^k} \geq (M-\epsilon)(Kt/(Kt+\delta))^k k^k \geq (M-2\epsilon)k^k.$$

Mrose [14] showed that if we can construct a sequence of bases $A_k(t)$, for $h = Kt + \delta$, where $\delta, K \in \mathbf{N}$ are fixed positive integers and t runs through the positive integers, such that the the asyptotic h-range $\geq (M - 2\epsilon)(h/k)^k$, then $\liminf_{h\to\infty} \frac{n_h(A_k^*(h))}{(h/k)^k} \geq M - 2\epsilon$. So here we get by (43)

$$T - 2\epsilon k^k \leq (M - 2\epsilon)k^k \leq \liminf_{h\to\infty} \frac{n_h(A_k^*(h))}{(h/k)^k},$$

and we are through, since the difference between $\limsup_{h\to\infty} n_h(A_k^*(h))/(h/k)^k$ and $\liminf_{h\to\infty} n_h(A_k^*(h))/(h/k)^k$ can be made as small as wanted.

Acknowledgement. I would like to thank Prof. E. S. Selmer for his thorough reading and reviewing several versions of the present paper. He supplied helpful comments to my rather brief first version, which both I and the reader should appreciate.

References

[20] C. Kirfel, *On extremal bases for the h-range problem, I*, Inst. Rep. No **53**, Math. Inst., Univ. Bergen, 1989.

Appendix. The computer program

Here we present the promised computer program written in "Pascal". We tried to use the same letters for the variables in the program as in the text, so the interested reader can check the program himself by the results from the theory. The results from section 2.6 are not incorporated.

```
program sepkon;
var
qe,re:array[1..6] of integer;
t:array[1..6] of real;
w,z:array[1..12] of real;
a,b,c,opt,eps,beta2min,beta2max,tot,min,optold,bound:real;
l,ss4,ss3,j,p,de4,eq,m,g,out,counter1,counter2:integer;
r1,r2,r3,r4,r5,r6,q1,q2,q3,q4,q5,q6,d1,d2,d3,d4,q,s:integer;

procedure koef(a,b,c:real;
                var opt:real);
var
v,x:real;
begin    (* procedure *)
v:=(3*(b+c)-2)/4; x:=sqrt(v*v+(b+c-4*b*c)/2)-v; x:=(x+abs(x))/2;
opt:=(1+x)*(1+x)*(1+x)*(1+x)/(a*(b+x)*(c+x));
end;     (* procedure *)

begin (* Main program *)
writeln('Choose m to determine M(m). m=0 corresponds to the Ni-list. ');
readln(m);
writeln('Choose an interval for the magnitude beta2 / gamma2 ');
writeln('|----|----|----|----|----|----|----|----|----|----|----|----|');
writeln('0   1/6  1/5  1/4  1/3  2/5  1/2  3/5  2/3  3/4  4/5  5/6   1');
writeln('Nr.1    2    3    4    5    6    7    8    9    10   11   12');
readln(p);
writeln('Do you want an output for each actual case? Yes = 1, no = 0.');
readln(out);
if out>0.5 then
   begin          (* Output option *)
   writeln(' Coefficientbound for the output. Choose a bound! ');
   readln(bound);
   end;           (* Output option *)
counter1:=0; counter2:=0; eps:=0.00001; tot:=0;
w[1]:=0; w[2]:=1/6; w[3]:=1/5; w[4]:=1/4; w[5]:=1/3; w[6]:=2/5;
w[7]:=1/2; w[8]:=3/5; w[9]:=2/3; w[10]:=3/4; w[11]:=4/5; w[12]:=5/6;
z[1]:=1/6; z[2]:=1/5; z[3]:=1/4; z[4]:=1/3; z[5]:=2/5; z[6]:=1/2;
z[7]:=3/5; z[8]:=2/3; z[9]:=3/4; z[10]:=4/5; z[11]:=5/6; z[12]:=1;
t[1]:=2; t[2]:=6/5; t[3]:=4/5; t[4]:=1/2; t[5]:=1/3; t[6]:=1/6;
```

```
if m=0 then
      begin      (* m=0 *)
      g:=-1;         de4:=0;      eq:=0;     end        (* m=0 *)
else
      begin      (* m>0 *)
      g:=trunc(m*w[p]+eps);       de4:=1;
      if (m-1)*w[p]-g+2>=t[1]  then  eq:=0
                               else  eq:=1;
      end;       (* m>0 *)
for j:=1 to 6 do
  begin             (* Computation of the ends of the loops for rj and qj *)
  if m>=j then
     begin          (* m>=j *)
     if ((m-j)*w[p]-g+trunc(j*w[p]+eps) >= t[j]) or
        ((m-j)*z[p]-g+trunc(j*w[p]+eps) <= eps) then  re[j]:=0
                                                else  re[j]:=1;
     if ((m-j)*w[p]-g+trunc(j*w[p]+eps)+1 >= t[j]) or
        ((m-j)*z[p]-g+trunc(j*w[p]+eps)+1 <= eps) then  qe[j]:=0
                                                  else  qe[j]:=1;
     end            (* m>=j *)
  else
     begin          (* m<j *)
     if ((m-j)*z[p]-g+trunc(j*w[p]+eps) >= t[j]) or
        ((m-j)*w[p]-g+trunc(j*w[p]+eps) <= eps) then  re[j]:=0
                                                else  re[j]:=1;



      if ((m-j)*z[p]-g+trunc(j*w[p]+eps)+1 >= t[j]) or
         ((m-j)*w[p]-g+trunc(j*w[p]+eps)+1 <= eps) then  qe[j]:=0
                                                   else  qe[j]:=1;
     end;           (* m<j *)
  end;              (* Computation of the ends of the loops for rj and qj *)
writeln('The ends of the loops:');
for j:=1 to 6 do
begin                    (* Writing the ends of the loops for rj *)
   write('re[',j:1,'] = ',re[j]:2,'      ');
   if m>=j then
   write('x2min  = ',((m-j)*w[p]-g+trunc(j*w[p]+eps)):4:3,'    ')
   else
   write('x2min  = ',((m-j)*z[p]-g+trunc(j*w[p]+eps)):4:3,'    ');
   if m>=j then
   writeln('x2max = ',((m-j)*z[p]-g+trunc(j*w[p]+eps)):4:3)
   else
   writeln('x2max = ',((m-j)*w[p]-g+trunc(j*w[p]+eps)):4:3);
   end;                  (* Writing the ends of the loops for rj*)
writeln;
for j:=1 to 6 do
write('qe[',j:1,'] = ',qe[j]:2,'  '); write('  eq =',eq:2,'  de4 = ',de4:2);
writeln('     m = ',m:2,'      g = ',g:2); writeln;
for r6:=0 to re[6] do
  for r5:=0 to re[5] do
    for r4:=0 to re[4] do
      for r3:=0 to re[3] do
        for r2:=0 to re[2] do
          for r1:=0 to re[1] do
            for q6:=0 to qe[6] do
              for q5:=0 to qe[5] do
                for q4:=0 to qe[4] do
                  for q3:=0 to qe[3] do
                    for q2:=0 to qe[2] do
                      for q1:=0 to qe[1] do
                        for d4:=0 to de4 do
                          for d3:=0 to 1 do
                            for d2:=0 to 1 do
                              for d1:=0 to 1 do
                                for q:=0 to eq do
                                  for s:=0 to 6 do
```

```
begin   (* Inner loop *)
if  ((s=0) or ((s>0) and (re[s]=1)))  then
  begin (* Last transfere allowed *)
  counter1:=counter1+1;
  l:=r1+r2+r3+r4+r5+r6+q1+q2+q3+q4+q5+q6+q+d1+d2+d3+d4+1;
  ss4:=q+q1+r1+2*(q2+r2)+3*(q3+r3)+4*(q4+r4)+5*(q5+r5)+6*(q6+r6)+s;
  a:=1+ss4/l;
  ss3:=d1+2*d2+3*d3+4*d4+2*q+q1+q2+q3+q4+q5+q6;
  ss3:=ss3+(q2+r2)*trunc(2*w[p]+eps)+(q3+r3)*trunc(3*w[p]+eps);
  ss3:=ss3+(q4+r4)*trunc(4*w[p]+eps)+(q5+r5)*trunc(5*w[p]+eps);
  ss3:=ss3+(q6+r6)*trunc(6*w[p]+eps)+trunc(s*w[p]+eps);
  if m>ss4/l then b:=(m-ss4/l)*w[p]+ss3/l-g
             else b:=(m-ss4/l)*z[p]+ss3/l-g;
  c:=1/l;
  koef(a,b,c,opt);
  optold:=opt;
if (opt >2.008) and
(((s=1) and (r1=1)) or ((s=2) and (r2=1)) or ((s=3) and (r3=1))
or
((s=4) and (r4=1)) or ((s=5) and (r5=1)) or ((s=6) and (r6=1))) then
      begin      (* Truncated list *)
      l:=l-1;
      ss4:=ss4-s;
      ss3:=ss3-trunc(s*w[p]+eps);



        a:=1+ss4/l;
        if m>ss4/l then b:=(m-ss4/l)*w[p]+ss3/l-g
                   else b:=(m-ss4/l)*z[p]+ss3/l-g;
        c:=1/l;
        koef(a,b,c,opt);
        end       (* Truncated list *)
      else  opt:=10;
     if opt>optold then min:=optold
                   else min:=opt;
     if (min>2.008) then  counter2:=counter2+1;
     if (out>0.5) and (min>bound) then
       begin                (* Output *)
       writeln('opt ',opt:6:4,' optold ',optold:6:4);
       writeln(' a ',a:6:4,' b ',b:6:4,' c ',c:6:4);
       writeln(' r1 ',r1:2,' r2 ',r2:2,' r3 ',r3:2);
       writeln(' r4 ',r4:2,' r5 ',r5:2,' r6 ',r6:2);
       writeln(' q1 ',q1:2,' q2 ',q2:2,' q3 ',q3:2);
       writeln(' q4 ',q4:2,' q5 ',q5:2,' q6 ',q6:2);
       writeln(' q  ',q:2,' d1 ',d1:2,' d2 ',d2:2);
       writeln(' d3 ',d3:2,' d4 ',d4:2);
       writeln;
       end;                 (* Output *)
     if (min>tot) then tot:=min;
     end;  (* Last transfere allowed *)
   end;    (* Inner loop *)
writeln('The largest coefficientbound occuring is :',tot:6:4);
writeln('Total number of cases ',counter1:6,'.');
writeln('Coefficientbound >2.008 ',counter2:6,'times.');
end.  (* Main program *)
```