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POLYNOMIALS HOMOLOGICALLY SUPPORTED ON DETERMINANTAL LOCI

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Piotr Pragacz[1] & **Jan Ratajski**

INTRODUCTION

The aim of this paper, which should be considered as a supplement to [P], is to extend the main theorem of [P] to other homology theories. Let $H(\ )$ be a homology theory with properties specified in Section 1. Fix integers $m>0$, $n>0$ and $r\geq 0$. Assume that

$$(c.,c.') = (c_1,\dots,c_n,c_1',\dots,c_m')$$

is a sequence of $m+n$ variables with $\deg c_i = \deg c_i' = i$.

We say, following [P], that $P \in \mathbb{Z}[c.,c.']$ is *universally supported on r-th degeneracy locus* if for every scheme X, every morphism $\varphi\colon \mathcal{F} \longrightarrow \mathcal{E}$ of vector bundles on X, $\operatorname{rank} \mathcal{E} = n$, $\operatorname{rank} \mathcal{F} = m$ and every $\alpha \in H(X)$

$$P(c_1(\mathcal{E}),\dots,c_n(\mathcal{E});c_1(\mathcal{F}),\dots,c_m(\mathcal{F})) \cap \alpha \in \operatorname{Im} i_* .$$

Here, for

$$D_r(\varphi) := \{ x \in X \mid \operatorname{rank} \varphi(x) \leq r \},$$

the map $i\colon D_r(\varphi) \longrightarrow X$ is the inclusion, and $i_*\colon H(D_r(\varphi)) \longrightarrow H(X)$ is the induced morphism on homology.

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1 Research carried out during the author's stay at the University of Bergen (Norway). This stay was supported by the N.A.V.F.. Thanks are due to Stein Arild Stromme for creating this opportunity and to Department of Mathematics in Bergen - for hospitality.

Define $\mathcal{P}_r$ to be the *set of all polynomials universally supported on r-th degeneracy locus* . It follows from the projection formula for i that $\mathcal{P}_r \subset \mathbb{Z}[c.,c.']$ is an ideal.

In [P] the author gave a description of $\mathcal{P}_r$ in the case of the Chow groups. In this work we show that the same result holds true for other homology theories.

The homology we consider here are endowed with a "cl-map" $A_k(\ ) \longrightarrow H_{2k}(\ )$, where A_k are Chow homology, or, they are singular homology. The proof in [P] does not go through (at least verbatim) for these homology. An obstruction is provided by the fact that even for such a nice homology theory as the Borel-Moore homology, the schemes used in the proof in [P] have nontrivial odd homology (see Remark 2.3). Similar arguments show that complex affine determinantal varieties D_r can have nontrivial odd Borel-Moore homology. Therefore, the problem of computation of $H_*^{BM}(D_r)$ is more difficult than computation of $A(D_r)$ (see [P]) and $IH^*(D_r)$ (see [Z]) .

In order to overcome this obstruction we modify the construction from [P] by using a certain compactification of it. This allows us to proceed with schemes for which the cl-map is an isomorphism (in particular odd-homology vanish) and preserving the needed genericity properties at the same time. Then, it is possible to follow the lines of the proof given in [P]. This gives us a proof which is valid both for Chow homology and other homology theory simultaneously.

We treat also the case of morphisms with symmetries. This case is somehow more difficult to tackle than the "generic" one. In order to overcome additional difficulties we prove a certain fact about surjectivity of morphisms of Chow groups of stratified schemes (see Proposition 3.5). This fact appears to be quite useful, and thus it seems to be of independent interest.

The setup of the present paper is borrowed from an useful work [R-X]. In addition to the homology theory treated there we prove the theorem in the singular homology case. Note that this last version of the theorem simplifies significantly calculations from Section 1 in [P-P].

We thank A. Białynicki-Birula, W. Fulton and W. Żelazko for alerting us to think about this problem. Thanks are due to L. Kaup and Z. Marciniak for useful informations about different homology theories (especially concerning Poincaré duality) transmitted to us during the Algebraic Geometry School-Rajgrod 1990. We are also grateful to A. Parusiński for pointing out some corections and simplifications in response to a preliminary draft of this paper.

Notation

1. Homology groups.

Let X be a scheme.

$A_k(X)$ denotes the Chow group of k-dimensional cycles modulo rational equivalence; $A(X) := \oplus_k A_k(X)$ (also for singular X).

If the ground field is $\mathbb{C}$, $H_k(X,\mathbb{Z})$ denotes the k-th singular homology group (in the notation of [B] this corresponds to $H^c_k(X,\mathbb{Z})$); and $H^k(X,\mathbb{Z})$ denotes the k-th singular cohomology group (in the notation of [B] - $H^k_{cld}(X,\mathbb{Z})$). Moreover, $H^{BM}_k(X)$ denotes the k-th Borel-Moore homology (with closed supports) or "homology with locally finite supports" (in the notation of [B] - $H^{cld}_k(X,\mathbb{Z})$ or $H_k(X,\mathbb{Z})$).

2. Partitions

By a *partition* we mean a sequence of integers $I = (i_1,\dots,i_k)$ where $i_1 \ge i_2 \ge \dots \ge i_k \ge 0$.

Instead of $(i,\dots,i)$ (k-times) we will write $(i)^k$.

For partitions $I = (i_1,\dots,i_k)$, $J = (j_1,\dots,j_k)$, $I+J$ will denote the sequence $(i_1+j_1,\dots,i_k+j_k)$, and $I \subset J$ will mean that $i_h \le j_h$ for every h .

1. HOMOLOGY THEORIES USED IN THIS ARTICLE

Let **k** be an algebraically closed field. By a "scheme" we shall understand an algebraic **k**-scheme of finite type which can be embedded as a closed subscheme of a smooth **k**-scheme of finite type. The restriction on **k** comes from the fact that in our arguments we use an homology theory satisfying properties (a)-(e) below. In the characteristic 0 case it is the homology with locally finite supports, or the Borel-Moore homology ([B-M], [B,Ch.5], [F,Ch.19],[I,Ch.9]), and if **k** has positive characteristic p, then the homology theory is defined as some suitable ℓ-adic cohomology, ℓ-a prime number different from p ([L,Sect.6]).

Recall, for instance, that Borel-Moore homology of a complex variety X, denoted $H_i^{BM}(X)$, are defined as the singular homology of X if X is proper, and as the relative singular homology of $\bar{X}$ modulo $\bar{X}\setminus X$ if X is not proper and $\bar{X}$ is a compactification of X. In [B-M], [B,Ch.5] a sheaf-theoretic construction of $H_i^{BM}(X)$ is given (in the notation of [B] this is $H_i^{\varphi}(X,\mathcal{F})$ where $\mathcal{F}=\mathbb{Z}$ and φ=cld).

By H_i we will denote a "cl-homology" theory that is, a functor from schemes to abelian groups that is covariant for proper maps and contravariant for open embeddings. Moreover we assume that the following conditions are satisfied

(a) Let X be a scheme, Y a closed subscheme and $U = X \setminus Y$. Then there exists a long exact sequence

$$\dots \longrightarrow H_{i+1}(U) \longrightarrow H_i(Y) \longrightarrow H_i(X) \longrightarrow H_i(U) \longrightarrow \dots$$

(b) For any finite disjoint union of schemes $\dot{\cup} X_j$ and for all i

$$H_i(\dot{\bigcup_j} X_j) = \oplus_j H_i(X_j)$$

(c) For all schemes and all integers i there exists a map

$$cl_X : A_i(X) \longrightarrow H_{2i}(X)$$

that commutes with pushforward by proper morphism and with restriction to open sets. $A_i(X)$ is here and in the sequel the Chow group of i-dimensional cycles modulo rational equivalence (see [F] for a precise definition and properties).

In characteristic 0 we shall say that "cl_X is an isomorphism" if cl_X is an isomorphism and $H_{2i+1}(X) = 0$ for all i.

In characteristic $p > 0$ we shall say that "cl_X is an isomorphism" if for prime $\ell \neq p$

$$cl_X \otimes 1_{\mathbb{Z}_\ell} : A_i(X) \otimes \mathbb{Z}_\ell \longrightarrow H_{2i}(X)$$

is an isomorphism for all i, and $H_{2i+1}(X) = 0$ for all i.

(d) If X is a scheme such that cl_X is an isomorphism then for every vector bundle $\mathcal{E}$ on X the map $cl_{P(\mathcal{E})}$ is an isomorphism, where $P(\mathcal{E})$ is the Projective bundle associated with $\mathcal{E}$.

(e) (Chern classes) Given a vector bundle $\mathcal{E}$ on a scheme X there exist uniquely defined Chern classes $c_i(\mathcal{E}) \cap$ - operators on $H(X)$.

They satisfy the conditions specified e.g. in Theorem 3.2 in [F]. Note that [F,Theorem 3.2 (d) - the pullback property] requires $f^*: H(X) \longrightarrow H(X')$ associated with a flat morphism f. In the case of the Borel-Moore homology, such a f^* exists by [V, Sect.3.2].

In the case of cl-homology in char p, f^* exists for flat f by [L, Sect.5]. For a definition of Chern classes operators in this case see [L,Sect.7].

Note also that for every polynomial P in the Chern classes of a vector bundle $\mathcal{E}$ and every cycle α on X,

$$cl_X(P(c.(\mathcal{E})) \cap \alpha) = P(c.(\mathcal{E})) \cap cl_X(\alpha) .$$

Pushforward formulas for Grassmannian bundles, like [P,Proposition 2.2], are valid for these homology theories and singular homology $H(-,\mathbb{Z})$, when appriopriately formulated.

Finally, recall that for the Grassmannian bundle $\pi\colon G_r(\mathcal{E}) \longrightarrow X$, parametrizing rank r-(sub)bundles of $\mathcal{E}$, the map

$$\pi_*\colon A_i(G_r(\mathcal{E})) \longrightarrow A_i(X)$$

is surjective for every i. This follows, for instance, from [P,Proposition 2.2]; or can be obtained by Noetherian induction on X (cf.the second step in the proof of [P, Lemma 3.7]).

2. GENERIC MORPHISMS

Assume that a sequence of m+n variables

$$(c.,c.') = (c_1,\dots,c_n,c_1',\dots,c_m')$$

is given. Define s_i inductively as follows

$$s_i = s_{i-1}c_1 - s_{i-2}c_2 + \dots + (-1)^{i-1}c_i .$$

Then define $s_i(c.,c.')$ by the formula

$$s_i(c.,c.') = \sum_k (-1)^{i-k}\, s_k\, c'_{i-k} .$$

Finally, for a given partition $I = (i_1,\dots,i_k)$ we put

$$s_I(c.,c.') = \mathrm{Det}\left[s_{i_p-p+q}(c.,c.') \right]_{1 \le p,q \le k} .$$

Let $\square_r$ denote the partition $(m-r)^{n-r}$. Let us denote by $\mathcal{I}_r$ the ideal in $\mathbb{Z}[c.,c.']$ generated by $s_I(c.,c.')$ where $I \supset \square_r$. It is known [P, Proposition 6.1] that $\mathcal{I}_r$ is generated by a finite set $\{ s_{\square_r+I}(c.,c.') \mid I \subset (r)^{n-r} \}$[2].

The ideal $\mathcal{P}_r$ of all polynomials universally supported on r-th degeneracy locus (see Introduction) admits the following description.

Theorem 2.1 For any homology theory specified in Section 1, we have $\mathcal{P}_r = \mathcal{I}_r$.

The proof of the inclusion $\mathcal{I}_r \subset \mathcal{P}_r$ is verbatim after [P, Ch.3].[3] The essential problem is to prove an opposite inclusion. Let us introduce first some notation.

Let W, V be vector spaces over **k** of dimension $w = \dim W$, $v = \dim V$. Let $G^m = G^m(W)$ be a Grassmannian parametrizing m-quotients of W and let $G_n = G_n(V)$ be a Grassmannian parametrizing n-subspaces of V. Denote by $\mathcal{Q}$ the tautological rank m-quotient bundle on G^m and by $\mathcal{R}$ the tautological rank n (sub)bundle on G_n. Moreover let $Fl^{m,r} = Fl^{m,r}(W)$ be the flag variety parametrizing the flags of quotients of W of dimension m and r, and $Fl_{r,n} = Fl_{r,n}(V)$ be the flag variety parametrizing the flags of subspaces of V of dimension r, n . Let $\mathcal{R}^{(r)} \subset \mathcal{R}^{(n)}$ be the tautological flag on $Fl_{r,n}$.

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[2] It is an open problem, whether this set gives a minimal set of generators of the ideal for $m \ge n$. We thank S.A.Stromme for helping us to check with "MACAULAY" that this holds true for a large number of cases.

[3] We correct an inaccuracy in quotation in the mentioned proof: [P] p.427_{10} - replace [F, Proposition 1.7] by [F, Theorem 6.2(a)].

A forthcoming Remark 2.3 will show that the proof of $\mathcal{P}_r \subset \mathcal{I}_r$ from [P] does not work for the Borel-Moore homology. We begin with the following useful fact.

Lemma 2.2 Let X be a complex space and $Y \subset X$ be a closed subset. Assume that $X \setminus Y$ is a $2\dim X$ - homology manifold. Then there is an exact sequence

$$\ldots \longrightarrow H_i^{BM}(Y) \longrightarrow H_i^{BM}(X) \longrightarrow H^{2\dim X-i}(X\setminus Y,\mathbb{Z}) \longrightarrow H_{i-1}^{BM}(Y) \longrightarrow \ldots, \qquad (\#)$$

where $H^i(-,\mathbb{Z})$ denotes the singular cohomology.

Proof. The assertion follows from the long exact sequence (a) for the Borel-Moore homology and the isomorphism

$$H_i^{BM}(X) \cong H^{2\dim X-i}(X,\mathbb{Z}),$$

valid for $2\dim X$ - homology manifold X. The latter isomorphism follows from [B-M, Theorem 7.9 with ϕ=cld and $\mathcal{I}=\mathbb{Z}$] (see also [B, Ch.9]). For a particularly transparent treatment of such a Poincaré-type duality - see [K]. The isomorphism in question follows from [K, Theorem 2.1 with $A=\emptyset$, $\mathcal{F}=\mathbb{Z}$ and φ=cld] and [K, Theorem 4.2 with $\mathcal{F}=\mathbb{Z}$ and φ=cld] in the notation from loc.cit.. □

Remark 2.3 (A raison d'être of this article)

We prove that for D_1 from construction (13) in [P] we have $H_3^{BM}(D_1) \neq 0$. This construction will be recalled in Step 1 of the proof of Theorem 2.1, where a morphism φ' is defined. Here, we take $\mathbf{k}=\mathbb{C}$, $m,n\geq 2$ and write D_i for $D_i(\varphi')$. Note that obviously $D_i \setminus D_{i-1}$ is a $2\dim D_i$-homology manifold, so we can apply Lemma 2.2.

We have a locally trivial fibration

$$D_1 \setminus D_0 \longrightarrow Fl^{m,1} \times Fl_{1,n} = FF$$

with the fiber $Gl(1)$. We use the spectral sequence of fibration

$$E_2^{p,q} = H^p(FF,\ H^q(Gl(1),\mathbb{Z})) \implies H^{p+q}(D_1\setminus D_0,\mathbb{Z}).$$

Invoking $H^0(Gl(1),\mathbb{Z}) = H^1(Gl(1),\mathbb{Z}) = \mathbb{Z}$, $H^i(Gl(1),\mathbb{Z}) = 0$ for $i \geq 2$, we get $E_2^{p,q} = 0$ for $q \geq 2$ and all p. Moreover, denoting $d = \dim D_1$

we get in $E_2^{\cdot,\cdot}$

$$\begin{array}{ccc}
E_2^{2d-4,1}=H^{2d-4}(FF,H^1(Gl(1),\mathbb{Z})) & & \\
\quad =H^{2\dim FF-2}(FF,\mathbb{Z})=\mathbb{Z}^4 & \xrightarrow{\quad d_2 \quad} & \\
& E_2^{2d-3,0}= & E_2^{2d-2,0}= \\
& =H^{2\dim FF-1}(FF,\mathbb{Z})=0 & =H^{2\dim FF}(FF,\mathbb{Z})=\mathbb{Z} \\
(2d-4) & (2d-3) & (2d-2)
\end{array}$$

Thus $\operatorname{rk} H^{2d-3}(D_1\backslash D_0,\mathbb{Z})\geq 3$. The following segment of the exact sequence (#)

$$H_3^{BM}(D_1) \longrightarrow H^{2d-3}(D_1\setminus D_0,\mathbb{Z}) \longrightarrow H_2^{BM}(D_0),$$

where $H_2^{BM}(D_0) = H_2(G^m xG_n,\mathbb{Z}) = \mathbb{Z}^2$, shows $H_3^{BM}(D_1) \neq 0$.

In particular, if we take a standard desingularization

$$\eta\colon\ Z = \underline{\mathrm{Hom}}\left[W_{G^m xFl_{1,n}} , \mathcal{R}^{(1)}_{G^m xFl_{1,n}} \right] \longrightarrow D_1$$

we see that $\eta_*\colon H_*^{BM}(Z) \longrightarrow H_*^{BM}(D_1)$ is not surjective because the even Borel- Moore homology groups of Z are zero. This obstructs to extend the first proof of $\mathcal{P}_r \subset \mathcal{I}_r$ from [P, Ch.3] to the Borel-Moore homology case. The second proof (see [P,Ch.7]), not using a desingularization, does not go through as well because the remark shows that the restriction map $H_{2i}^{BM}(D_r) \longrightarrow H_{2i}^{BM}(D_r\setminus D_{r-1})$ is not surjective.

<u>Remark 2.4</u> Similar arguments show that for affine determinantal variety D_1 (over $\mathbf{k}=\mathbb{C}$) we have $H_3^{BM}(D_1) \neq 0$ (here, we use the notation of [P, Ch.4], and assume $m,n \geq 2$). We have a locally trivial fibration

$$D_1\setminus D_0 \longrightarrow G^1 \times G_1$$

with fiber $Gl(1)$, which gives the spectral sequence

$$E_2^{p,q} = H^p(\ G^1 xG_1,\ H^q(Gl(1),\mathbb{Z})\ ) \Longrightarrow H^{p+q}(D_1\setminus D_0,\mathbb{Z}) \quad .$$

We have $E_2^{p,q} = 0$ for $q \geq 2$ and all p. Moreover, in $E_2^{\cdot,\cdot}$, for $d=\dim D_1$,

$$\begin{tikzcd}[column sep=large]
E_2^{2d-4,1} = \mathbb{Z}^2 \arrow[rrd, "d_2"] & & \\
 & E_2^{2d-3,0} = 0 & E_2^{2d-2,0} = \mathbb{Z} \\
(2d-4) & (2d-3) & (2d-2)
\end{tikzcd}$$

Thus shows $\operatorname{rk} H^{2d-3}(D_1 \setminus D_0, \mathbb{Z}) \geq 1$. Again the exact sequence (#)

$$H_3^{BM}(D_1) \longrightarrow H^{2d-3}(D_1 \setminus D_0; \mathbb{Z}) \longrightarrow H_2^{BM}(D_0)$$

where $H_2^{BM}(D_0) = H_2(pt, \mathbb{Z}) = 0$, shows $H_3^{BM}(D_1) \neq 0$.

This remark shows that the problem of computation of $H_*^{BM}(D_r)$ (and probably also a similar question about singular homology) is more subtle than computation of $A(D_r)$ (see [P]) and $IH^*(D_r)$ (see [Z]) .

We give now a proof of the inclusion $\mathcal{P}_r \subset \mathcal{I}_r$, which is valid for homology theories from Section 1.

<u>Notation</u> Given two vector bundles $\mathcal{E}$ and $\mathcal{F}$, the polynomial $s_I(c.,c.')$ specialized with $c_i = c_i(\mathcal{E})$ and $c_j' = c_j(\mathcal{F})$ will be denoted $s_I(\mathcal{E}-\mathcal{F})$.

<u>Step 1</u> (A construction from [P])

Define

$$X' := \underline{\mathrm{Hom}}(Q_{GG}, \mathcal{R}_{GG}) \longrightarrow GG := G^m \times G_n$$

$$\mathcal{F}' := Q_{X'} \ , \quad \mathcal{E}' = \mathcal{R}_{X'}$$

On X' there exists a tautological morphism $\varphi' : \mathcal{F}' \longrightarrow \mathcal{E}'$. Note two features of this construction:

1) The Chern classes of $\mathcal{E}'$, $\mathcal{F}'$ are algebraically independent (over $\mathbb{Z}$) if $w, v \longrightarrow \infty$.
2) The matrix of φ' is given locally by $m \times n$ matrix of indeterminates.

Step 2 (A compactification of X')

The following construction is inspired by [K-L, p. 161]. Let

$$X := G_m(\mathcal{Q}_{GG} \oplus \mathcal{R}_{GG}) \longrightarrow GG .$$

X is a relative Grassmannian over GG and is endowed with the tautological rank m (sub)bundle $\mathcal{S} \subset (\mathcal{Q} \oplus \mathcal{R})_X$. We define a morphism (of fibrations over GG) from X' to X. Fix a point $(M,N) \in GG$. We assign to $f \in \mathrm{Hom}(M,N)$ (in $X'_{(M,N)}$) the point given by

$$\text{(The graph of } f) \hookrightarrow M \oplus N \qquad (\text{in } X_{(M,N)}).$$

This assignment defines an open immersion $X' \hookrightarrow X$. We have $\mathcal{S}_{X'} = \mathcal{F}'$ and the value of the restriction of $\mathcal{S} \hookrightarrow (\mathcal{Q} \oplus \mathcal{R})_X$ to X', in the point $(M,N,f:M \longrightarrow N) \in X'$, is given by

$$M \longrightarrow M\oplus N \quad \text{such that} \quad m \longmapsto (m, f(m)) \ , \ m\in M .$$

Therefore, if we define $\mathcal{F} := \mathcal{S}$, $\mathcal{E} := \mathcal{R}_X$ and φ as the composit:

$$\mathcal{F} = \mathcal{S} \hookrightarrow (\mathcal{Q} \oplus \mathcal{R})_X \xrightarrow{pr_{\mathcal{E}}} \mathcal{E} = \mathcal{R}_X ,$$

we have $\varphi|_{X'} = \varphi'$. Finally, we put $D_k := D_k(\varphi)$.

Lemma 2.5 (1) The map $D_r \subset X \longrightarrow GG$ is a locally trivial fibration; its fiber over a point $(M,N) \in GG$ is the r-th determinantal Schubert variety in $G=G_m(M\oplus N)$ given by the inequality

$$\mathrm{rk}\left(\mathcal{S}_G \hookrightarrow (M\oplus N)_G \xrightarrow{pr_{N_G}} N_G \right) \le r .$$

(2) If $w,v \longrightarrow \infty$, the Chern classes of $\mathcal{E}$ and $\mathcal{F}$ become algebraically independent (over $\mathbb{Z}$) in A(X).

Proof. (1) The required trivialization is given by $\{U^\beta \times U_\alpha\}$ where $\{U_\alpha\}$ is the standard covering of G_n trivializing the bundle $\mathcal{R}$ and $\{U^\beta\}$ is the standard covering of G^m trivializing the bundle Q.

(2) We have $\mathcal{E}|_{X'} = \mathcal{E}'$, $\mathcal{F}|_{X'} = \mathcal{F}'$. Then an eventual relation

$$\sum \alpha_{I,J}\, s_I(\mathcal{E})\, s_J(\mathcal{F}) = 0 \quad \text{in} \quad A(X) \quad (\alpha_{I,J} \in \mathbb{Z})$$

gives rise to the relation

$$\sum \alpha_{I,J}\, s_I(\mathcal{E}')\, s_J(\mathcal{F}') = 0 \quad \text{in} \quad A(X'),$$

which is not possible. □

Step 3 (A standard desingularization of D_r)

Consider the diagram of schemes

$$\begin{tikzcd} \mathbf{Z} = \mathrm{Zeros}\left(\mathcal{F}_{\mathbf{G}} \xrightarrow{\varphi_{\mathbf{G}}} \mathcal{E}_{\mathbf{G}} \longrightarrow \mathbf{Q} \right) \arrow[r, hook, "j"] \arrow[d, "\eta"] & \mathbf{G} = G_r(\mathcal{E}) \arrow[d, "\pi"] \\ D_r \arrow[r, hook, "i"] & X \end{tikzcd}$$

where $\mathbf{Q}$ is the tautological quotient bundle on $\mathbf{G}$.

Lemma 2.6 The inclusion $j\colon \mathbf{Z} \longrightarrow \mathbf{G}$ can be identified with the following inclusion of Grassmannian bundles on $GF = G^m \times Fl_{r,n}$:

$$\begin{tikzcd} j : G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(r)}_{GF}) \arrow[rr, "j"] \arrow[dr] & & G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(n)}_{GF}) \arrow[dl] \\ & GF & \end{tikzcd}$$

Proof. A point of $\mathbf{G}$ is represented by (M,N,K,L) where $W \twoheadrightarrow M$ and $\dim M = m$; $N \subset V$ and $\dim N = n$; $K \subset M \oplus N$ and $\dim K = m$; and finally $L \subset N$ and $\dim L = r$.

A point of $G_m(\,\mathcal{Q}_{GF} \oplus \mathcal{R}^{(n)}_{GF}\,)$ is represented by $(M, L\subset N, K)$ where $W \twoheadrightarrow M$ and $\dim M = m$; $N \subset V$ and $\dim N = n$, $\dim L = r$; finally $K \subset M \oplus N$ and $\dim K = m$.

This allows us to identify $\mathbf{G}$ and $G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(n)}_{GF})$. A point (M,N,K,L) belongs to $\mathbf{Z}$ iff the composit

$$K \hookrightarrow M \oplus N \xrightarrow{pr_N} N \twoheadrightarrow N / L$$

is zero. This means that $K \subset M \oplus L$ and thus $\mathbf{Z}$ is identified with $G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(r)}_{GF})$. □

Corollary 2.7 $j^*: A(\mathbf{G}) \longrightarrow A(\mathbf{Z})$ is surjective.

Proof. Let $\mathcal{S}$ be the tautological rank m (sub)bundle on $G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(n)}_{GF})$. Then the tautological rank m (sub)bundle on $G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(r)}_{GF})$ is $\mathcal{S}|_{\mathbf{Z}}$. The assertion now follows from a well-known description of $A\left[G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(n)}_{GF})\right]$ and $A\left[G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(r)}_{GF})\right]$ as free $A(GF)$-modules with bases given respectively by Schur polynomials $s_I(\mathcal{S})$, $I \subset (n)^m$ and $s_I(\mathcal{S}|_{\mathbf{Z}})$, $I \subset (r)^m$ (see e.g. [F, Chap.14]), and, from the equality $j^*(s_I(\mathcal{S})) = s_I(\mathcal{S}|_{\mathbf{Z}})$.
□

Define $\mathbf{Z}^k = \eta^{-1}(D_k)$, $k=0,1,\dots,r$.

Lemma 2.8[4] Under the above identification $\mathbf{Z}^k$ is given in $\mathbf{Z} = G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(r)}_{GF})$ by the inequality

$$\mathrm{rk}\left(\mathcal{S} \longrightarrow \mathcal{Q} \oplus \mathcal{R}^{(r)} \longrightarrow \mathcal{R}^{(r)} \right)_{\mathbf{Z}} \le k .$$

In other words $\mathbf{Z}^k$ is the k-th determinantal Schubert subvariety in $G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(r)}_{GF}) \longrightarrow GF$.

Proof. Let $x \in D_k$. Then x can be represented by (M,N,K) where $W \twoheadrightarrow M$ and $\dim M = m$, $N \hookrightarrow V$ and $\dim N = n$, $K \subset M \oplus N$ and $\dim K = m$. Moreover $\mathrm{rk}(K \hookrightarrow M \oplus N \longrightarrow N) \le k$. The point $\eta^{-1}(x)$ is then represented by (M,N,K,L) where $\dim L = r$, $L \subset N$ and $K \subset M \oplus L$. Since then

$$\mathrm{rk} (K \hookrightarrow M \oplus L \longrightarrow L) = \mathrm{rk} (K \hookrightarrow M \oplus N \longrightarrow N) \le k ,$$

[4] A similar analysis was done earlier in [Kl-La].

the assertion follows. $\square$

Step 4 (A theorem of Rosselló - Xambó)

We say, following [F, Ex.1.9.1] , that a scheme X has a cellular decomposition if there exists a filtration

$$X = X_n \supset X_{n-1} \supset \dots \supset X_0 \supset X_{-1} = \emptyset$$

such that X_i are closed, and each $X_i \setminus X_{i-1}$ is a disjoint union of locally closed subschemes C_{ij} isomorphic to affine spaces $\mathbb{A}^{m_{ij}}$.
The C_{ij} will be referred to as cells of cellular decomposition. It is well known (see e.g. [R-X, Corollary]) that if X admits a cellular decomposition then $A_i(X)$ is a finitely generated free abelian group for which the classes of closures of the i-dimensional cells form a basis.

We record the following result [R-X, Theorem 2].

Theorem 2.9 Let X be a scheme which admits a cellular decomposition and let $f\colon X' \longrightarrow X$ be a morphism such that for all cells C_{ij} of the decomposition $f^{-1}(C_{ij}) = C_{ij} \times F$ where F is a fixed scheme. Then

(i) For all i there exists an epimorphism

$$\bigoplus_{r+s=i} A_r(X) \otimes A_s(F) \longrightarrow A_i(X') \qquad (\#\#)$$

(ii) If cl_F is an isomorphism and $A_i(F)$ is free for all i, then (##) is an isomorphism for all i, and $cl_{X'}$ is an isomorphism.

We apply this result to D_k , Z^k.

Let,for a sequence $I\colon 1 \le i_1 < \dots < i_m \le w$, $\mathring{\Omega}(I)$ denote the (open) Schubert cell in $G^m(W)$ (taken with respect to a fixed flag in W) with generic point given by a matrix : ("*" means a place occupied by a free parameter, empty places are occupied by zeros).

$$\begin{array}{c}
\begin{bmatrix}
* & \cdots & * & 1 & & & & & & & & & \\
* & \cdots & * & 0 & * & \cdots & * & 1 & & & & & \\
* & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & 1 & \\
 & & & & & \vdots & & & & & & & \\
\end{bmatrix} \\
\begin{array}{ccccc}
\uparrow & \qquad\qquad & \uparrow & \qquad\qquad & \uparrow \\
i_1 & & i_2 & & i_3 \quad \ldots
\end{array}
\end{array}$$

The Plücker coordinate $p(I)$ given by the minor taken on columns $i_1,\ldots,i_m$ is not zero. Thus $\mathring{\Omega}(I) \subset G^m(W) \setminus \mathrm{Zeros}(p(I))$ which is a set over which the tautological bundles are trivial. If we repeat the same consideration with Schubert cells $\mathring{\Omega}(J)$ in $G_n(V)$ (here J: $1 \le j_1 < \ldots < j_n \le v$), then we see that the fibrations $D_k \longrightarrow GG$ and $Z^k \longrightarrow GF \longrightarrow GG$ are trivial over $\mathring{\Omega}(I) \times \mathring{\Omega}(J)$. Moreover, the fiber of $D_k \longrightarrow GG$ is a Schubert variety, and, the fiber of $Z^k \longrightarrow GG$ is a product of a Schubert variety and a Grassmannian. Thus these fibers have cellular decompositions, and we infer from Theorem 2.9 the following result.

<u>Corollary 2.10</u> For any "cl-homology" theory from Section 1, cl_{D_k} and cl_{Z^k} are isomorphisms. In particular, we have $H_{odd}(D_k) = H_{odd}(Z^k) = 0$.

<u>Step 5</u> (Final calculations)

From Step 4 , we get for every i a commutative diagram with exact rows

$$\begin{array}{ccccccc}
H_{2i}(Z^{k-1}) & \longrightarrow & H_{2i}(Z^k) & \longrightarrow & H_{2i}(Z^k \setminus Z^{k-1}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_{2i}(D_{k-1}) & \longrightarrow & H_{2i}(D_k) & \longrightarrow & H_{2i}(D_k \setminus D_{k-1}) & \longrightarrow & 0
\end{array} \qquad (\#\#\#)$$

Since cl_{D_k} , cl_{Z^k} are isomorphisms we have for $U = D_k \setminus D_{k-1}$ and $U = Z^k \setminus Z^{k-1}$, $H_{2i}(U) = A_i(U)$ if char $\mathbf{k} = 0$, and, $H_{2i}(U) =$

$A_i(U) \otimes \mathbb{Z}_\ell$ if char $\mathbf{k} = p$. Therefore, since $\mathbf{Z}^k \setminus \mathbf{Z}^{k-1} \longrightarrow D_k \setminus D_{k-1}$ is a Grassmannian bundle, the induced map

$$H_{2i}(\mathbf{Z}^k \setminus \mathbf{Z}^{k-1}) \longrightarrow H_{2i}(D_k \setminus D_{k-1})$$

is surjective (see Section 1). Thus by induction on k and a diagram chase in (###) we get

Proposition 2.11 $\eta_* : H_{2i}(\mathbf{Z}) \longrightarrow H_{2i}(D_r)$ is surjective for every i.

The Proposition implies $\mathrm{Im}\ i_* = \pi_*(\mathrm{Im}\ j_*)$. To compute the latter group we can use the Chow groups because of Corollary 2.10. Now, we will mimick the arguments from [P, p.431][5] and prove

$$\mathrm{Im}\ i_* = \left(s_{\square_r + I}(\mathcal{E}-\mathcal{F}) \mid I \subset (r)^{n-r} \right) \qquad (\#\#\#\#)$$

At first, $\mathrm{Im}\ j_*$ is a principal ideal in $A(\mathbf{G})$ generated by $[\mathbf{Z}] = c_{top}(\mathcal{F}_{\mathbf{G}}^{\vee} \otimes \mathbf{Q}) = s_{(m)^{n-r}}(\mathbf{Q}-\mathcal{F}_{\mathbf{G}})$. Indeed, by Corollary 2.7 for $z \in A(\mathbf{Z})$ there exists $g \in A(\mathbf{G})$ such that $z = j^*(g)$. Then, by the projection formula,

$$j_*(z) = j_*(j^*g) = [\mathbf{Z}]\cdot g\ .$$

Secondly, we know that every element $g \in A(\mathbf{G})$ has a presentation $g = \sum \alpha_I s_I(\mathbf{Q})$ where $\alpha_I \in A(X)$ and $I \subset (r)^{n-r}$ (see e.g. [F] Ch.14). Thus

$$\pi_*([\mathbf{Z}]\cdot g) = \pi_*\left[s_{(m)^{n-r}}(\mathbf{Q}-\mathcal{F}_{\mathbf{G}}) \cdot \sum_I \alpha_I s_I(\mathbf{Q}) \right]$$

$$= \pi_*\left[\sum_I \alpha_I\ s_{(m)^{n-r}+I}(\mathbf{Q}-\mathcal{F}_{\mathbf{G}}) \right]$$

---

5 We correct misprints in [P]: 431_1- read: $s_{(m-r)^{n-r}+I}(E-F)$ ...; $431_{5,10}$- read: ... Lemma 3.6 ... .

$$= \sum_I \alpha_I \, s_{\square_r + I}(\mathcal{E}-\mathcal{F})$$

by using succesively the factorization formula [P, Lemma 1.1] and the push forward formula [P, Proposition 2.2].

This proves Theorem 2.1 for "cl-homology" theory, because if $w, v \longrightarrow \infty$ the Chern classes of $\mathcal{E}$ and $\mathcal{F}$ are algebraically independent, so (####) is sufficient to get the assertion.

The same proof works for singular homology because D_r, $\mathbf{G}$ and $\mathbf{Z}$ are proper and thus their singular homology coincide with the Borel-Moore homology.

$\square\ \square\ \square$

Remark 2.12 The "singular homology" version of Theorem 2.1 allows us to perform the key calculations in [P-P, Proposition 1.6] without any use of the Chow groups; this gives a significant simplification.

3. MORPHISMS WITH SYMMETRIES

In this Section we will deal with symmetric and antisymmetric vector bundle morphisms. We assume here char $\mathbf{k} \neq 2$. We will treat first the symmetric case; necessary modifications needed for the antisymmetric case will be specified in Remark 3.10.

Assume that a sequence $(c.) = (c_1, \dots, c_n)$ of variables is given ($\deg c_i = i$). We say, following [P], that $P \in \mathbb{Z}[c.]$ is *universally supported on r-th symmetric degeneracy locus* if for every scheme X, every symmetric morphism $\varphi\colon \mathcal{E}^\vee \longrightarrow \mathcal{E}$ of vector bundles on X, $\operatorname{rank}\mathcal{E} = n$, and every $\alpha \in H(X)$

$$P(c_1(\mathcal{E}), \dots, c_n(\mathcal{E})) \cap \alpha \ \in \ \operatorname{Im} i_*$$

where $i_*\colon H(D_r(\varphi)) \longrightarrow H(X)$ is the induced homology-morphism associated to the inclusion $i\colon D_r(\varphi) \longrightarrow X$. Define $\mathcal{P}_r$ to be the *ideal of all polynomials universally supported on r-th symmetric degeneracy locus* .

In this Section the following polynomials $Q_I(c.)$ indexed by strict partitions I[6] will play a crucial role. First define s_i inductively as follows

$$s_i = s_{i-1}c_1 - s_{i-2}c_2 + \ldots + (-1)^{i-1}c_i \ .$$

Then define

$$Q_i(c.) := \sum_k s_k c_{i-k} \ ,$$

$$Q_{i,j}(c.) := Q_i(c.)Q_j(c.) + 2\sum_p (-1)^p Q_{i+p,j-p}(c.).$$

Finally, for a given strict partition $I = (i_1,\ldots,i_k)$ we put

$$Q_I(c.) = \text{Pfaffian}\left[Q_{i_p,i_q}(c.) \right]_{1 \le p,q \le k}$$

(we can assume k even by putting $i_k=0$ if necessary).

Let Δ_r denote the partition $(n-r,n-r-1,\ldots,2,1)$. Let us denote by $\mathcal{I}_r$ the ideal in $\mathbb{Z}[c.]$ generated by $Q_I(c.)$ where $I \supset \Delta_r$. It is known [P, Proposition 7.17] that $\mathcal{I}_r$ is generated by a finite set $\{ Q_{\Delta_r+I}(c.) \mid I \subset (r)^{n-r} \}$.

Theorem 3.1 For any homology theory specified in Section 1, we have $\mathcal{P}_r = \mathcal{I}_r$.

The proof of the inclusion $\mathcal{I}_r \subset \mathcal{P}_r$ is verbatim after [P Ch.7]. In the proof of the opposite inclusion we will follow the notation from Section 2. Moreover, given a vector bundle $\mathcal{E}$, the polynomial $Q_I(c.)$ specialized with $c_i = c_i(\mathcal{E})$ will be denoted by $Q_I(\mathcal{E})$.

Step 1 (A construction from [P])

Define

$$X' := \mathbf{S}^2\mathcal{R} \longrightarrow G_n \qquad \text{and} \qquad \mathcal{E}' = \mathcal{R}_{X'},$$

On X' there exists a tautological morphism $\varphi' : \mathcal{E}'^{\vee} \longrightarrow \mathcal{E}'$. Note two

[6] Recall that $I=(i_1,\ldots,i_k)$ is strict if $i_1>\ldots>i_k$.

features of this construction:

1) The Chern classes of $\mathcal{E}'$ are algebraically independent (over $\mathbb{Z}$) if $v \longrightarrow \infty$.

2) The matrix of φ' is given locally by nxn symmetric matrix of indeterminates.

<u>Step 2</u> (A compactification of X')

Let Φ be a symplectic form on $\mathcal{R}^{\vee} \oplus \mathcal{R}$ given by the matrix

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

where here, and in the sequel, I denotes the nxn identity matrix.

Denote by

$$X := G_n^{\Phi}(\mathcal{R}^{\vee} \oplus \mathcal{R}) \longrightarrow G_n$$

the relative Grassmannian parametrizing rank n subbundles of $\mathcal{R}^{\vee} \oplus \mathcal{R}$ that are isotropic with respect to Φ. X is endowed with the tautological rank n (sub)bundle $\mathcal{S} \subset (\mathcal{R}^{\vee} \oplus \mathcal{R})_X$. We define a morphism (of fibrations over G_n) from X' to X. Fix a point $N \in G_n$. We assign to a symmetric $f \in \mathrm{Hom}(N^{\vee}, N)$ (in X'_N) the point given by

$$(\text{The graph of } f) \hookrightarrow N^{\vee} \oplus N \qquad\qquad (\text{in } X_N).$$

We need

<u>Lemma 3.2</u> If f is symmetric then the graph of f is an isotropic subspace of $N^{\vee} \oplus N$ (with respect to Φ).

<u>Proof.</u> If A is a matrix of f then the graph of f is spanned by the columns of

$$\begin{bmatrix} I \\ A \end{bmatrix}.$$

Then the assertion follows from the equality

$$[I, A^t] \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} = [I, A^t] \begin{bmatrix} A \\ I \end{bmatrix} = A - A^t = 0,$$

where A is symmetric. □

The above assignment defines an open immersion $X' \hookrightarrow X$. Put

$\mathcal{E} := \mathcal{S}^\vee$, and define the following symmetric morphism on X,

$$\varphi\colon \mathcal{S} \hookrightarrow (\mathcal{R}^\vee \oplus \mathcal{R})_X \xrightarrow{\Psi} (\mathcal{R} \oplus \mathcal{R}^\vee)_X \twoheadrightarrow \mathcal{S}^\vee .$$

where Ψ is given by

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} .$$

<u>Lemma 3.3</u> We have $\varphi|_{X'} = 2\varphi'$.

<u>Proof.</u> The assertion follows from the equality

$$[I, A^t] \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} = [I, A^t] \begin{bmatrix} A \\ I \end{bmatrix} = A + A^t = 2A,$$

where A is symmetric. □

<u>Lemma 3.4</u> (1) The map $D_r(\varphi) \subset X \longrightarrow G_n$ is a locally trivial fibration; its fiber over a point $N \in G_n$ is "the r-th determinantal Schubert variety" in $G^\Phi = G_n^\Phi(N^\vee \oplus N)$ given by the inequality

$$\operatorname{rank}\,(\mathcal{S} \hookrightarrow (N^\vee \oplus N)_{G^\Phi} \xrightarrow{\Psi} (N \oplus N^\vee)_{G^\Phi} \twoheadrightarrow \mathcal{S}^\vee) \le r$$

(2) If $v \longrightarrow \infty$, the Chern classes of $\mathcal{E}$ and become algebraically independent (over $\mathbb{Z}$) in A(X).

<u>Proof.</u> The proof of (2) is analogous to the proof of Lemma 2.5 (2). As for (1), we invoke here the following fact from [L-S, page 36_6]. It follows from loc.cit. that there exists an irreducible Schubert subvariety in G^Φ such that its restriction to open subset S^2N is the r-th determinantal variety in S^2N. The above inequality defines also an irreducible subvariety in G^Φ as a calculation in local coordinates shows. Moreover, by Lemma 3.3 , the restriction of this subvariety to S^2N is the r-th determinantal variety. Our assertion follows. □

<u>Step 3</u> (A standard desingularization of $D_r(\varphi)$)

Consider the diagram of schemes (φ is symmetric)

$$\begin{array}{ccc}
\mathbf{Z} = \mathrm{Zeros}\left(\mathcal{E}_{\mathbf{G}}^{\vee} \xrightarrow{\varphi_{\mathbf{G}}} \mathcal{E}_{\mathbf{G}} \longrightarrow \mathbf{Q}\right) & \xhookrightarrow{\ j\ } & \mathbf{G} = G_r(\mathcal{E}) \\
= \mathrm{Zeros}\left(\mathcal{O}_{\mathbf{G}} \longrightarrow \mathcal{K}er(\mathcal{E}\otimes\mathbf{Q} \longrightarrow \Lambda^2\mathbf{Q})\right) & & \\
\Big\downarrow \eta & & \Big\downarrow \pi \\
D_r(\varphi) & \xhookrightarrow{\ i\ } & X
\end{array}$$

where $\mathbf{Q}$ is the tautological bundle on $\mathbf{G}$.

Now, in order to mimick the proof from Section 2 we will use the following fact.[7]

<u>Proposition 3.5</u> Let $D = D_r \supset D_{r-1} \supset \ldots \supset D_0 \supset D_{-1} = \emptyset$ be a sequence of closed schemes. Put $S_k = D_k \setminus D_{k-1}$. Let $\pi : G \longrightarrow D$ be a morphism and $j : Z \longrightarrow G$ a regular embedding. Assume that both

$$\pi: G_{S_k} \longrightarrow S_k \qquad \text{and} \qquad \pi_{|Z_{S_k}} : Z_{S_k} \longrightarrow S_k$$

are locally trivial fibrations; and there exists an open affine covering $\{U_\alpha^{(k)}\}$ of S_k trivializing them simultaneously. Under this trivialization the map $j: Z_{U_\alpha^{(k)}} \longrightarrow G_{U_\alpha^{(k)}}$ is equal to

$$U_\alpha^{(k)} \times F^{(k)} \xhookrightarrow{1\times h} U_\alpha^{(k)} \times G^{(k)},$$

where $h: F^{(k)} \hookrightarrow G^{(k)}$ is a regular embedding. Assume that h^*: $A(G^{(k)}) \longrightarrow A(F^{(k)})$ is surjective ($k=1,\ldots,r$).

Then $j^*: A(G) \longrightarrow A(Z)$ is surjective.

<u>Proof.</u> We claim that it suffices to show the surjectivity of $j_k^!$, where $j_k = j_{S_k} : Z_{S_k} \longrightarrow G_{S_k}$. We have a commutative diagram

---

[7] Note that Proposition 3.5 and Lemma 3.6 give an alternative proof of Corollary 2.7 and 2.10.

$$\begin{tikzcd}
A(G_{D_{k-1}}) \arrow[r] \arrow[d] & A(G_{D_k}) \arrow[r] \arrow[d] & A(G_{S_k}) \arrow[r] \arrow[d, "j_k^!"] & 0 \\
A(Z_{D_{k-1}}) \arrow[r] & A(Z_{D_k}) \arrow[r] & A(Z_{S_k}) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. To be more precise, the vertical maps are "refined Gysin homomorphisms" constructed as in [F, Ch. 6.2] from fibre squares

$$\begin{tikzcd}
Z_{D_{k-1}} \arrow[r] \arrow[d] & G_{D_{k-1}} \arrow[d] \\
Z \arrow[r] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_{D_k} \arrow[r] \arrow[d] & G_{D_k} \arrow[d] \\
Z \arrow[r] & G
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Z_{S_k} \arrow[r] \arrow[d] & G_{S_k} \arrow[d] \\
Z \arrow[r] & G
\end{tikzcd} .$$

We denote the Gysin morphism associated to the latter fibre square by $j_k^!$ to emphasis its dependence on k. The commutativity of the left hand side diagram follows from the fibre square

$$\begin{tikzcd}
Z_{D_{k-1}} \arrow[r] \arrow[d] & G_{D_{k-1}} \arrow[d] \\
Z_{D_k} \arrow[r] \arrow[d] & G_{D_k} \arrow[d] \\
Z \arrow[r] & G
\end{tikzcd}$$

and [F, Theorem 6.2(a)]. The commutativity of the right hand side diagram follows from [F, Theorem 6.2(b)]. Assuming by induction the surjectivity of the left vertical map (for k=1, it becomes $j_0^!$) and of $j_k^!$, we get the final assertion by a diagram chase.

In turn, the surjectivity of $j_k^!$ can be proved by Noetherian induction. Take $U \subset S_k$ an affine open subset trivializing simultaneously Z_{S_k} and G_{S_k}. We have a diagram with exact rows

$$\begin{array}{ccccccc}
A(G_{S_k \setminus U}) & \longrightarrow & A(G_{S_k}) & \longrightarrow & A(G_U) & \longrightarrow & 0 \\
\downarrow & & \downarrow j_k^! & & \downarrow & & \\
A(Z_{S_k \setminus U}) & \longrightarrow & A(Z_{S_k}) & \longrightarrow & A(Z_U) & \longrightarrow & 0
\end{array}$$

Again, the diagram is commutative by [F, Theorem 6.2 (a) and (b)]. Since $\dim(S_k \setminus U) < \dim S_k$, we get the surjectivity of the left vertical map by Noetherian induction. We have a commutative diagram [F, Theorem 6.2 (b)]

$$\begin{array}{ccc}
A(U\times G^{(k)}) & \xrightarrow{(1\times h)^*} & A(U\times F^{(k)}) \\
\uparrow p_2^* & & \uparrow p_2^* \\
A(G^{(k)}) & \xrightarrow{h^*} & A(F^{(k)})
\end{array} .$$

Since U is open affine, the p_2^*'s are epimorphisms. Finally, h^*-surjective implies the surjectivity of $A(G_U) \longrightarrow A(Z_U)$. This concludes the proof of the proposition.

□

We record also the following fact which combines Theorems 1 and 2 from [R-X].

<u>Lemma 3.6</u> Let $D = D_r \supset D_{r-1} \supset \ldots \supset D_0 \supset D_{-1} = \emptyset$ be a sequence of closed schemes. Put $S_k = D_k \setminus D_{k-1}$ and asssume that S_k has a cellular decomposition. Let $\pi : Z \longrightarrow D$ be a morphism such that the restriction of π: $Z_{S_k} \longrightarrow S_k$ is a locally trivial fibration. Assume that its fiber $F^{(k)}$ satisfies: $cl_{F^{(k)}}$ is an isomorphism and $A(F^{(k)})$ is free $(k=1,\ldots,r)$.

Then, for every k, $cl_{Z_{D_k}}$ is an isomorphism.

<u>Proof.</u> It follows from Theorem 2.9 and our assumptions that $cl_{Z_{S_k}}$ are isomorphisms. To end we proceed by induction on k. In char 0 case, it follows from the commutative diagram

$$\begin{array}{ccccccc}
A_i(Z_{D_{k-1}}) & \longrightarrow & A_i(Z_{D_k}) & \longrightarrow & A_i(Z_{S_k}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_{2i}(Z_{D_{k-1}}) & \longrightarrow & H_{2i}(Z_{D_k}) & \longrightarrow & H_{2i}(Z_{S_k}) & \longrightarrow & 0
\end{array}$$

that $A_i(Z_{D_k}) \cong H_{2i}(Z_{D_k})$. In char p case we tensorize all Chow groups by $\mathbb{Z}_\ell$ and repeat the arguments. Moreover,

$$0=H_{2i+1}(Z_{D_{k-1}}) \longrightarrow H_{2i+1}(Z_{D_k}) \longrightarrow H_{2i+1}(Z_{S_k})=0$$

implies $H_{2i+1}(Z_{D_k})=0$. □

In the notation before Proposition 3.5 we put $D_k: = D_k(\varphi)$ and $\mathbf{Z}^k: = \eta^{-1}(D_k)$ $(= \mathbf{Z}_{D_k})$.

<u>Corollary 3.7</u> In the notation before Proposition 3.5 , the map $j^*: A(\mathbf{G}) \longrightarrow A(\mathbf{Z})$ is surjective.

<u>Proof.</u> We use Proposition 3.5 and its notation. In our situation, it is sufficient to find an open covering of X, trivializing the bundle $\mathcal{S}$. Take first an open covering $\{U\}$ trivializing $\mathcal{R}$. Then denoting by p the projection $X= G_n^{\Phi}(\mathcal{R}^\vee \oplus \mathcal{R}) \longrightarrow G_n$, we have $p^{-1}(U) = U \times G_n^{\Phi}(N^\vee \oplus N)$ where dimN=n; so if we take an open covering $\{U'\}$ of $G_n^{\Phi}(N^\vee \oplus N)$ trivializing the tautological vector bundle on it, we obtain an open covering $\{U \times U'\}$ trivializing $\mathcal{S}$.

Since $D_k = D_k(\varphi)$ we have $G^{(k)}=G_r(A)$, dimA=n; $F^{(k)}=G_{r-k}(B), B\subset A$, dimB=n-k; and the embedding $h: F^{(k)} \hookrightarrow G^{(k)}$ is given as follows. Let $A=B \oplus C$, then $L \in G_{r-k}(B)$ is sent via h into $L\oplus C \in G_r(A)$. Clearly under this embedding the tautological quotient bundle on $G^{(k)}$ restricts to the tautological quotient bundle on $F^{(k)}$. This implies the surjectivity of j^* because of the well known description of the Chow ring of a Grassmannian in terms of Schur polynomials of the tautological quotient bundle (see e.g. [F,Ch.14]). □

Corollary 3.8 For any "cl-homology" theory from Section 1, cl_{D_k} and cl_{Z^k} are isomorphisms. In particular, we have $H_{odd}(D_k) = H_{odd}(Z^k)=0$.

Proof. Since the fiber of $D_k \longrightarrow G$ is a Schubert variety (in an isotropic Grassmannian), the assertion for D_k follows from Theorem 2.9. Since $D_k \setminus D_{k-1}$ as a difference of two Schubert varieties has a cellular decomposition, the assertion for Z^k is a consequence of Lemma 3.6. □

Step 4 (Final calculations)

From Step 3, we get as in Section 2:

Proposition 3.9 $\eta_* : H_{2i}(Z) \longrightarrow H_{2i}(D_r)$ is surjective for every i.

The Proposition implies $\operatorname{Im} i_* = \pi_*(\operatorname{Im} j_*)$. To compute the latter group we can use the Chow groups because of Corollary 3.8. Now, we will mimick the arguments from [P, Ch.7] and prove

$$\operatorname{Im} i_* = \left(Q_{\Delta_r + I}(\mathcal{E}) \mid I \subset (r)^{n-r} \right) \qquad (\#\#\#\#\#)$$

At first, $\operatorname{Im} j_*$ is a principal ideal in $A(G)$ generated by $[Z] = c_{top}(\mathcal{K}er\,(\mathcal{E}_G \otimes Q \longrightarrow \Lambda^2 Q)) = c_{top}(R\otimes Q + S^2Q) = c_{top}(R\otimes Q)\, Q_{\Delta_r}(Q)$, where R is the tautological subbundle on G. Indeed, by Corollary 3.7, for $z \in A(Z)$ there exists $g \in A(G)$ such that $z = j^*(g)$. Then

$$j_*(z) = j_*(j^*g) = [Z]\cdot g \ .$$

Secondly, we know that every element $g \in A(G)$ has a presentation $g = \sum \alpha_I s_I(Q)$ where $\alpha_I \in A(X)$ and $I \subset (r)^{n-r}$ (see e.g. [F] Ch.14). Thus

$$\pi_*([Z]\cdot g) = \pi_* \left[c_{top}(R\otimes Q)\, Q_{\Delta_r}(Q) \cdot \sum_I \alpha_I s_I(Q) \right]$$

$$= \pi_* \left[\sum_I \alpha_I c_{top}(R\otimes Q)\, Q_{\Delta_r + I}(Q) \right]$$

$$= \sum_I \alpha_I \, Q_{\Delta_r + I}(\mathcal{E})$$

by using succesively the factorization formula [P, Lemma 1.13] and the push forward formula [P, Proposition 2.8].

This proves Theorem 3.1 for "cl-homology" theory, because if $v \longrightarrow \infty$ the Chern classes of $\mathcal{E}$ are algebraically independent, so (#####) is sufficient to get the assertion.

The same proof works for singular homology because D_r, **G** and **Z** are proper and thus their singular homology coincide with the Borel-Moore homology.

□ □ □

Remark 3.10 One can prove similarly an analogous assertion for antisymmetric morphisms. In the proof of Theorem 3.1 one makes the following modifications: take r-even and in all stratifications use even k; replace Φ by Ψ and vice versa in all definitions and calculations; replace polynomials $Q_I(c.)$ and $Q_I(\mathcal{E})$ by P-polynomials $2^{-l(I)}Q_I(-)$ (see [P] for details); and finally, change Δ_r to the partition $\Delta'_r :=$ $(n-r-1, n-r-2, \dots, 2, 1)$. The "antisymmetric version" of Theorem 3.1 is :

$$" \mathcal{P}_r = \left(P_{\Delta'_r + I}(c.) \;\middle|\; I \subset (r)^{n-r} \right) ".$$

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