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# SUB-BASES OF PLEASANT h-BASES

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We assume knowledge of the "postage stamp problem", see for instance [4]. A comprehensive treatment of this problem is contained in the author's research monograph [5] (freely available on request).

A "stamp" basis (an h-basis)

$$A_k = \{1, a_2, \ldots, a_k\}, \quad 1 = a_1 < a_2 < \ldots < a_k,$$

is pleasant if and only if the regular representation $n = \Sigma_1^k e_i a_i$ has a minimal coefficient sum among all possible representations $n = \Sigma_1^k x_i a_i$, for all natural numbers $n$. Then the h-range $n_h(A_k)$ equals the regular h-range $g_h(A_k)$, which is easily determined.

Let $A_i = \{1, a_2, \ldots, a_i\}$, $2 \leq i \leq k$, be a "partial basis" of $A_k$. Then $A_2$ is always pleasant, and Djawadi [1] gave the following criterion for pleasantness in general: Let $\langle x \rangle$ denote the smallest integer $\geq x$, and put

$$(1) \quad a_i = \gamma_i a_{i-1} - \underbrace{\sum_{j=1}^{i-2} \beta_j^{(i)} a_j}_{\text{regular by } A_{i-2}}, \quad \gamma_i = \left\langle \frac{a_i}{a_{i-1}} \right\rangle .$$

Let further $A_{i-1}$ be pleasant. Then $A_i$ is pleasant if and only if

$$(2) \quad \gamma_i > \sum_{j=1}^{i-2} \beta_j^{(i)} .$$

Djawadi's proof has been simplified by the author [5, Ch. X].

If the condition (2) is satisfied for all $i = 3, 4, \ldots, k$, then all partial bases $A_i$ are pleasant, and we call $A_k$ completely pleasant.

Zöllner [6] showed that

$$(3) \quad k \geq 4, \ A_k \text{ pleasant} \Rightarrow \{1, a_2, a_i\} \text{ pleasant}, \quad 3 \leq i \leq k .$$

The condition was weakened to "$A_k$ weakly pleasant" by Kirfel [3].

In particular, a pleasant $A_k$ always has a pleasant partial basis $A_3$, and a pleasant $A_4$ is thus completely pleasant. For $k \geq 5$, there are pleasant $A_k$ which are not completely pleasant. For $k = 5$, all such bases were determined by Djawadi [2]:

(4) $$A_5 = \{1, 2, b, b+1, 2b\}, \quad b \geqq 4$$

(where $A_4$ is non-pleasant for $b \geqq 4$).

For $k = 6$, the similar bases were characterized by Zöllner [6]. On the average, probably "most" pleasant bases are completely pleasant.

Even if the complete set of conditions (2), for $i = 4, 5, \ldots, k$, is not always necessary for pleasantness of $A_k$, there are <u>some</u> cases of necessity. Djawadi writes (1) as

(5) $$a_i + \sum_{j=1}^{i-2} \beta_j^{(i)} a_j = \gamma_i a_{i-1},$$

where the left hand side is a <u>regular</u> representation by $A_i$. If then (2) fails, this representation has a <u>larger</u> coefficient sum than the non-regular representation $\gamma_i a_{i-1}$, and $A_i$ is then not pleasant by definition. In particular, the condition (2) for $i = k$ is thus always necessary for pleasantness of $A_k$ (whether $A_{k-1}$ is pleasant or not).

We have observed the following trivial but perhaps useful generalization: If $i < k$, and $\gamma_i a_{i-1} < a_{i+1}$, the left hand side of (5) is also a regular representation by the full basis $A_k$. Hence, <u>if</u>

(6) $$\left\langle \frac{a_i}{a_{i-1}} \right\rangle a_{i-1} < a_{i+1} \quad (i < k),$$

<u>the condition</u> (2) <u>is necessary for pleasantness of</u> $A_k$.

If $k > 3$, and we remove the basis elements $a_3, a_4, \ldots, a_{k-1}$, it follows from (3) with $i = k$ that the "<u>sub-basis</u>" $\{1, a_2, a_k\}$ is pleasant if $A_k$ is pleasant (or only weakly pleasant by [3]). We can prove the following generalization:

THEOREM. <u>If</u> $k \geqq 5$, $3 \leqq \kappa \leqq k-2$, <u>and the partial bases</u> $A_i$, $i = \kappa, \kappa+1, \ldots, k$, <u>are all pleasant</u>, <u>then</u>

$$A_k^{(\kappa)} = \{1, a_2, \ldots, a_\kappa, a_k\}$$

<u>is also pleasant. If in particular</u> $A_k$ <u>is completely pleasant, so is</u> $A_k^{(\kappa)}$ <u>for all</u> $\kappa$.

Before proving this, we make some comments:

(i) We must remove a "block" $a_{\kappa+1}, \ldots, a_{k-1}$ of elements in $A_k$ up to $a_{k-1}$. The simplest counter-example is given by the completely pleasant basis $A_5 = \{1, 2, 3, 5, 7\}$. Removing $a_3$, we get the non-pleasant basis $\{1, 2, 5, 7\}$.

(ii) The condition $A_i$ pleasant for <u>all</u> $i = \kappa, \kappa + 1, \ldots, k$ is not always necessary. For instance, the Djawadi basis (4) leads to $A_5^{(3)} = \{1, 2, b, 2b\}$, which is pleasant by (2).

(iii) As an example where the Theorem <u>fails</u> when $A_i$ is not pleasant for all $i = \kappa, \kappa + 1, \ldots, k$, consider the following extension of (4):

$$A_6 = \{1, 2, b, b + 1, 2b, a_6\}, \quad b \geqq 4,$$

which is pleasant if $a_6 > 2b$ is chosen such that (2) holds for $i = 6$. However, $A_6^{(4)} = \{1, 2, b, b + 1, a_6\}$ is not of the form (4), and is consequently not pleasant since the partial basis $A_4$ is not.

To prove the Theorem, it will clearly suffice to use repeated removal of the <u>next largest</u> element, hence to show that

$$(7) \qquad A_k^{(k-2)} = \{1, a_2, \ldots, a_{k-2}, a_k\}$$

is pleasant. For this purpose, we substitute $a_{k-1}$ from (1) with $i = k - 1$ into (1) with $i = k$, and get $a_k$ expressed by $A_{k-2}$ as

$$a_k = (\gamma_k \gamma_{k-1} - \beta_{k-2}^{(k)}) a_{k-2} - \sum_{j=1}^{k-3} (\gamma_k \beta_j^{(k-1)} + \beta_j^{(k)}) a_j$$

$$(8) \qquad = \tilde{\gamma} a_{k-2} - \sum_{j=1}^{k-3} \tilde{\beta}_j a_j \quad \text{(say)} .$$

Using (2) for $i = k - 1$ and $i = k$, this gives

$$\tilde{\gamma} - \sum_{j=1}^{k-3} \tilde{\beta}_j = \gamma_k (\gamma_{k-1} - \sum_{j=1}^{k-3} \beta_j^{(k-1)}) - \sum_{j=1}^{k-2} \beta_j^{(k)}$$

$$\geqq \gamma_k - \sum_{j=1}^{k-2} \beta_j^{(k)} > 0 ,$$

in analogy with (2). However, we do not know if (8) corresponds to the form (1) for the basis (7), where we now need

$$(9) \qquad a_k = \gamma a_{k-2} - \underbrace{\sum_{j=1}^{k-3} \beta_j a_j}_{\text{regular by } A_{k-3}}, \qquad \gamma = \left\langle \frac{a_k}{a_{k-2}} \right\rangle .$$

Equating the two expressions for $a_k$, we get

$$\tilde{\gamma} a_{k-2} + \sum_{j=1}^{k-3} \beta_j a_j = \gamma a_{k-2} + \sum_{j=1}^{k-3} \tilde{\beta}_j a_j .$$

The left hand side is a regular representation by the pleasant basis $A_{k-2}$, and thus has a minimal coefficient sum:

$$\tilde{\gamma} + \sum_{j=1}^{k-3} \beta_j \leqq \gamma + \sum_{j=1}^{k-3} \tilde{\beta}_j$$

$$\gamma - \sum_{j=1}^{k-3} \beta_j \geqq \tilde{\gamma} - \sum_{j=1}^{k-3} \tilde{\beta}_j > 0 .$$

This shows that (2) is satisfied for the form (9). Since $A_{k-2}$ is pleasant, so is also the basis (7), and the Theorem is proved.

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