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On multivariate Vernic recursions

by

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Abstract

In the present paper we extend a recursive algorithm developed by Vernic (1999) for compound distributions with bivariate counting distribution and univariate severity distributions to more general multivariate counting distributions.

1 Introduction

1A. Panjer (1981) described a procedure for recursive evaluation of a compound distribution when the counting distribution belongs to a certain class. Vernic (1999) developed a bivariate version of this recursion, assuming that the counting distribution is bivariate and the severity distributions univariate. In the present paper we discuss a generalisation of the result of Vernic to a situation with m-variate counting distribution and univariate severity distributions. As special cases we obtain the recursion of Panjer in the univariate case ($m = 1$) and the recursion of Vernic in the bivariate case ($m = 2$).

The recursions of Panjer and Vernic are briefly recapitulated in Sections 2 and 3 respectively, and the multivariate extension is introduced in Section 4. In Section 5 we look at some examples, and, finally, in Section 6 we briefly indicate some possible extensions of the theory.

1B. In the recursions that we study in the present paper, the distributions are expressed through their probability functions. For simplicity we shall therefore normally mean the probability function when referring to a distribution.

We make the convention that a summation over an empty set is equal to zero and multiplication over an empty set is equal to one.

2 The recursion of Panjer

In the univariate case, a compound distribution is the distribution of the sum of independent and identically distributed random variables where the number of terms is itself a random variable assumed to be independent of the terms. We shall assume that the terms are distributed on the positive integers. Let p be the distribution of the number of terms (the counting distribution), f the distribution of the terms (the severity distribution), and g the compound distribution. Then $g = \sum_{n=0}^{\infty} p(n) f^{n*}$. As f is a confined to the positive integers, we must have $f^{n*}(x) = 0$ for all integers $n > x$, and thus

$$g(x) = \sum_{n=0}^{x} p(n) f^{n*}(x); \qquad (x = 0, 1, 2, \dots)$$

in particular we have $g(0) = p(0)$.

If p satisfies the recursion

$$p(n) = \left(a + \frac{b}{n}\right) p(n-1), \qquad (n = 1, 2, \dots)$$

then

$$g(x) = \sum_{y=1}^{x} \left(a + b\frac{y}{x}\right) f(y) g(x-y). \qquad (x = 1, 2, \dots)$$

This recursion was described by Panjer (1981).

3 The recursion of Vernic

When extending the concept of compound distributions to the multivariate case, one can go in two directions:

1. Let the severities by independent and identically distributed random vectors.

2. Let the counting distribution be multivariate and the severities one-dimensional; we consider the distribution of, say, m random variables with compound distributions whose counting variables are dependent whereas the severities are mutually independent and independent of the counting variables.

The two approaches can be combined by letting the severities in Case 2 be random vectors.

For Case 1 recursions have been studied by Sundt (1999); for Case 2 by Hesselager (1996) and Vernic (1999) in the bivariate case.

In Case 2 the compound distribution is given by

$$g = \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} p(n_1, \ldots, n_m) \prod_{i=1}^{m} f_i^{n_i *}. \tag{1}$$

When assuming that the severity distributions are restricted to the positive integers, like in the univariate case we obtain that the infinite summations become finite when we insert an argument in g:

$$g(x_1, \ldots, x_m) = \sum_{n_1=0}^{x_1} \cdots \sum_{n_m=0}^{x_m} p(n_1, \ldots, n_m) \prod_{i=1}^{m} f_i^{n_i *}(x_i); \qquad (x_1, \ldots, x_m = 0, 1, 2 \ldots)$$

in particular we have $g(0, \ldots, 0) = p(0, \ldots, 0)$.

Let us turn to the bivariate case. Vernic (1999) assumed that

$$p(n_1, n_2) = \psi_{12}(n_1, n_2) p(n_1 - 1, n_2 - 1) + \psi_1(n_1, n_2) p(n_1 - 1, n_2) + \psi_2(n_1, n_2) p(n_1, n_2 - 1) \tag{2}$$

when at least one of n_1 and n_2 are positive, with

$$\psi_{12}(n_1, n_2) = \begin{cases} a_0 + \frac{a_1}{n_1} + \frac{a_2}{n_2} + \frac{a_{12}}{n_1 n_2} & (n_1, n_2 = 1, 2, \ldots) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\psi_1(n_1, n_2) = \begin{cases} b_0 + \frac{b_1}{n_1} & (n_1, n_2 = 1, 2, \ldots) \\ d_0 + \frac{d_1}{n_1} & (n_1 = 1, 2, \ldots; \; n_2 = 0) \\ 0 & (n_1 = 0; \; n_2 = 1, 2, \ldots) \end{cases}$$

$$\psi_2(n_1, n_2) = \begin{cases} c_0 + \frac{c_2}{n_2} & (n_1, n_2 = 1, 2, \ldots) \\ e_0 + \frac{e_2}{n_2} & (n_1 = 0; \; n_2 = 1, 2, \ldots) \\ 0, & (n_1 = 1, 2, \ldots; \; n_2 = 0) \end{cases}$$

and showed that then

$$g(x_1,x_2) = \sum_{y_1=1}^{x_1}\sum_{y_2=1}^{x_2}\varphi_{12}(y_1,y_2;x_1,x_2)\,f_1(y_1)\,f_2(y_2)\,g(x_1-y_1,x_2-y_2)+$$
$$\sum_{y_1=1}^{x_1}\varphi_1(y_1;x_1,x_2)\,f_1(y_1)\,g(x_1-y_1,x_2)+$$
$$\sum_{y_2=1}^{x_2}\varphi_2(y_2;x_1,x_2)\,f_2(y_2)\,g(x_1,x_2-y_2) \qquad (3)$$

when at least one of x_1 and x_2 are positive, with

$$\varphi_{12}(y_1,y_2;x_1,x_2) = \begin{cases} a_0 + a_1\dfrac{y_1}{x_1} + a_2\dfrac{y_2}{x_2} + a_{12}\dfrac{y_1y_2}{x_1x_2} & (y_i = 1,\ldots,x_i;\ x_i = 1,2,\ldots;\ i=1,2) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\varphi_1(y_1;x_1,x_2) = \begin{cases} b_0 + b_1\dfrac{y_1}{x_1} & (y_1 = 1,\ldots,x_i;\ x_1,x_2 = 1,2,\ldots) \\ d_0 + d_1\dfrac{y_1}{x_1} & (y_1 = 1,\ldots,x_1;\ x_1 = 1,2,\ldots;\ x_2 = 0) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\varphi_2(y_2;x_1,x_2) = \begin{cases} c_0 + c_2\dfrac{y_2}{x_2} & (y_i = 1,\ldots,x_i;\ x_i = 1,2,\ldots;\ i=1,2) \\ e_0 + e_2\dfrac{y_2}{x_2} & (y_1 = 1,\ldots,x_1;\ x_1 = 0;\ x_2 = 1,2,\ldots) \\ 0. & (\text{otherwise}) \end{cases}$$

Some special cases are studied by Hesselager (1996).

We see that already in the bivariate case the formulae and notation start getting rather messy, and unfortunately it will get even worse when extending the theory to a more general multivariate case. We shall therefore abstain from writing out a general theory in full and rather give a rough outline of what can be done.

4 General results

4A. When considering extension of the Vernic recursions from the bivariate case to the m-variate case, it will be convenient to use some vector notation. We shall denote an $m \times 1$ column vector by a bold-face letter and its elements by the corresponding italic with the number of the element as subscript; subscript $\cdot$ denotes the sum of the elements, e.g. $\mathbf{x} = (x_1,\ldots,x_m)'$ and $x_{\cdot} = \sum_{i=1}^{m} x_i$. By $\mathbf{y} \leq \mathbf{x}$ we shall mean that $y_i \leq x_i$ for $i = 1,\ldots,m$, and

by $\mathbf{y} < \mathbf{x}$ that $\mathbf{y} \leq \mathbf{x}$ with $\mathbf{y} \neq \mathbf{x}$. By $\mathbf{e}_{i_1 \dots i_h}$ we shall mean the vector whose i_jth element is equal to one for $j = 1, \dots, h$ and all other elements are equal to zero. We also introduce the vector $\mathbf{0}$ where all elements are equal to zero.

It is tacitly assumed that all vectors introduced have integer-valued elements.

4B. Let $\mathbf{N}$ be an $m \times 1$ vector of non-negative integer-valued random variables. We introduce positive, integer-valued random variables Y_{ij} ($i = 1, \dots, m$; $j = 1, 2, \dots$), assumed to be independent of $\mathbf{N}$ and mutually independent, and for fixed i identically distributed with common distribution f_i. Let p denote the distribution of $\mathbf{N}$. We introduce the random vector $\mathbf{X} = (X_1, \dots, X_m)'$ with $X_i = \sum_{j=1}^{N_i} Y_{ij}$ for $i = 1, \dots, m$. Then the distribution of $\mathbf{X}$ is the compound distribution g given by (1).

4C. When trying to extend (2) and (3) to an m-variate situation, it is natural to look for pairs of functions $\left(\psi_{i_1 \dots i_h}, \varphi_{i_1 \dots i_h}\right)$ such that

$$p(\mathbf{n}) = \sum_{h=1}^{m} \sum_{1 \leq i_1 < \dots < i_h \leq m} \psi_{i_1 \dots i_h}(\mathbf{n})\, p\left(\mathbf{n} - \mathbf{e}_{i_1 \dots i_h}\right) \quad (\mathbf{n} > \mathbf{0}) \tag{4}$$

$$g(\mathbf{x}) = \sum_{h=1}^{m} \sum_{1 \leq i_1 < \dots < i_h \leq m} \sum_{s=1}^{h} \sum_{y_s=1}^{x_{i_s}} \varphi_{i_1 \dots i_h}(y_1, \dots, y_h; \mathbf{x})\, g\left(\mathbf{x} - \textstyle\sum_{j=1}^{h} y_j \mathbf{e}_{i_j}\right) \displaystyle\prod_{j=1}^{h} f_{i_j}(y_j). \quad (\mathbf{x} > \mathbf{0}) \tag{5}$$

Like in the Vernic recursion, we would normally have that for $i \in \{1, \dots, m\} \sim \{i_1 \dots i_h\}$ $\psi_{i_1 \dots i_h}(\mathbf{n})$ and $\varphi_{i_1 \dots i_h}(y_1, \dots, y_h; \mathbf{x})$ depend on n_i and x_i respectively only to the extent of whether they are equal to zero or not.

The following lemma describes the relation we need between a ψ and the corresponding φ.

Lemma 1. *If for different integers* $i_1, \dots, i_h \in \{1, \dots, m\}$

$$\mathrm{E}\left[\varphi\left(Y_{i_1 1}, \dots, Y_{i_h 1}; \mathbf{x}\right) \middle| \bigcap_{j=1}^{h} \left(\sum_{r=1}^{n_{i_j}} Y_{i_j r} = x_{i_j}\right)\right] = \psi(\mathbf{n}) \tag{6}$$

for all $\mathbf{x}, \mathbf{n} > \mathbf{0}$ *such that* $\prod_{i=1}^{m} f_i^{n_i *}(x_i) > 0$, *then*

$$\sum_{\mathbf{n} > \mathbf{0}} \psi(\mathbf{n})\, p\left(\mathbf{n} - \mathbf{e}_{i_1 \dots i_h}\right) \prod_{i=1}^{m} f_i^{n_i *}(x_i) =$$

$$\sum_{s=1}^{h} \sum_{y_s=1}^{x_{i_s}} \varphi(y_1, \dots, y_h; \mathbf{x})\, g\left(\mathbf{x} - \textstyle\sum_{j=1}^{h} y_j \mathbf{e}_{i_j}\right) \displaystyle\prod_{j=1}^{h} f_{i_j}(y_j). \quad (\mathbf{x} > \mathbf{0})$$

Proof. We extend the set $\{i_1, \ldots, i_h\}$ to a permutation $\{i_1, \ldots, i_m\}$ of $\{1, \ldots, h\}$. For all $\mathbf{x} > \mathbf{0}$ we have

$$\sum_{\mathbf{n}>\mathbf{0}} \psi(\mathbf{n})\, p(\mathbf{n} - \mathbf{e}_{i_1 \ldots i_h}) \prod_{i=1}^{m} f_i^{n_i *}(x_i) =$$

$$\sum_{\mathbf{n}>\mathbf{0}} p(\mathbf{n} - \mathbf{e}_{i_1 \ldots i_h})\, \mathrm{E}\left[\varphi(Y_{i_1 1}, \ldots, Y_{i_h 1}; \mathbf{x}) \middle| \bigcap_{j=1}^{h} \left(\sum_{r=1}^{n_{i_j}} Y_{i_j r} = x_{i_j}\right)\right] \prod_{i=1}^{m} f_i^{n_i *}(x_i) =$$

$$\sum_{\mathbf{n}>\mathbf{0}} p(\mathbf{n} - \mathbf{e}_{i_1 \ldots i_h}) \sum_{s=1}^{h} \sum_{y_s=1}^{x_{i_s}} \varphi(y_1, \ldots, y_h; \mathbf{x}) \times$$

$$\left(\prod_{j=1}^{h} f_{i_j}(y_j)\, f_{i_j}^{(n_{i_j}-1)*}(x_{i_j} - y_j)\right) \prod_{j=h+1}^{m} f^{n_{i_j}*}(x_{i_j}) =$$

$$\sum_{s=1}^{h} \sum_{y_s=1}^{x_{i_s}} \varphi(y_1, \ldots, y_h; \mathbf{x}) \left(\prod_{j=1}^{h} f_{i_j}(y_j)\right) \times$$

$$\sum_{\mathbf{n}>\mathbf{0}} p(\mathbf{n} - \mathbf{e}_{i_1 \ldots i_h}) \left(\prod_{j=1}^{h} f_{i_j}^{(n_{i_j}-1)*}(x_{i_j} - y_j)\right) \prod_{j=h+1}^{m} f^{n_{i_j}*}(x_{i_j}) =$$

$$\sum_{s=1}^{h} \sum_{y_s=1}^{x_{i_s}} \varphi(y_1, \ldots, y_h; \mathbf{x})\, g\left(\mathbf{x} - \textstyle\sum_{j=1}^{h} y_j \mathbf{e}_{i_j}\right) \prod_{j=1}^{h} f_{i_j}(y_j)\,. \qquad \text{Q.E.D.}$$

In the univariate case Lemma 1 is closely related to Theorem 2 in Sundt & Jewell (1981).

It is clear that if the pairs $(\varphi_1, \psi_1), \ldots, (\varphi_w, \psi_w)$ satisfy the conditions of Lemma 1, then $(\sum_{v=1}^{w} c_v \varphi_v, \sum_{v=1}^{w} c_v \psi_v)$ also satisfies the conditions of Lemma 1 for all constants $c_1, \ldots, c_w$.

As the severities are positive, $X_i = 0$ if and only if $N_i = 0$. This implies that if the pairs (φ_1, ψ_1) and (φ_2, ψ_2) satisfy the conditions of Lemma 1, then these conditions are also satisfied by the pair (φ, ψ) given by

$$\varphi(y_1, \ldots, y_h; \mathbf{x}) = \begin{cases} \varphi_1(y_1, \ldots, y_h; \mathbf{x}) & (x_i = 1, 2, \ldots) \\ \varphi_2(y_1, \ldots, y_h; \mathbf{x}) & (x_i = 0) \end{cases}$$

$$\psi(\mathbf{n}) = \begin{cases} \psi_1(\mathbf{n}) & (n_i = 1, 2, \ldots) \\ \psi_2(\mathbf{n})\,. & (n_i = 0) \end{cases}$$

We have already seen one application of such a construction in the Vernic recursion, where the coefficients were allowed to depend on whether some of the variables were equal to zero.

We are now ready to prove our main theorem.

Theorem 1. *If there exist pairs of functions* $\left(\psi_{i_1\ldots i_h}, \varphi_{i_1\ldots i_h}\right)$ *that* (4) *holds and each pair satisfies* (6) *for all* $\mathbf{x}, \mathbf{n} > \mathbf{0}$ *such that* $\prod_{i=1}^{m} f_i^{n_i *}(x_i) > 0$, *then* (5) *holds.*

Proof. From Lemma 1 we obtain that for all $\mathbf{x} > \mathbf{0}$

$$g(\mathbf{x}) = \sum_{\mathbf{n}>\mathbf{0}} p(\mathbf{n}) \prod_{i=1}^{m} f_i^{n_i *}(x_i) =$$

$$\sum_{\mathbf{n}>\mathbf{0}} \sum_{h=1}^{m} \sum_{1 \le i_1 < \cdots < i_h \le m} \psi_{i_1\ldots i_h}(\mathbf{n})\, p(\mathbf{n} - \mathbf{e}_{i_1\ldots i_h}) \prod_{i=1}^{m} f_i^{n_i *}(x_i) =$$

$$\sum_{h=1}^{m} \sum_{1 \le i_1 < \cdots < i_h \le m} \sum_{\mathbf{n}>\mathbf{0}} \psi_{i_1\ldots i_h}(\mathbf{n})\, p(\mathbf{n} - \mathbf{e}_{i_1\ldots i_h}) \prod_{i=1}^{m} f_i^{n_i *}(x_i) =$$

$$\sum_{h=1}^{m} \sum_{1 \le i_1 < \cdots < i_h \le m} \sum_{s=1}^{h} \sum_{y_s=1}^{x_{i_s}} \varphi_{i_1\ldots i_h}(y_1, \ldots, y_h; \mathbf{x})\, g\left(\mathbf{x} - \sum_{j=1}^{h} y_j \mathbf{e}_{i_j}\right) \prod_{j=1}^{h} f_{i_j}(y_j).$$

Q.E.D.

Our next theorem shows a way to construct additional recursions for g if there are more than one set of recursions that satisfy the conditions of Theorem 1.

Theorem 2. *If for* $v = 1, \ldots, w$ (5) *is satisfied with*

$$\varphi_{i_1\ldots i_h} = \varphi_{i_1\ldots i_h}^{(v)}, \qquad (1 \le i_1 < \cdots < i_h \le m;\ h = 1, \ldots, m)$$

then (5) *is satisfied with*

$$\varphi_{i_1\ldots i_h}(y_1, \ldots, y_h; \mathbf{x}) = \sum_{v=1}^{w} c_v(\mathbf{x})\, \varphi_{i_1\ldots i_h}^{(v)}(y_1, \ldots, y_h; \mathbf{x}),$$
$$\left(y_j = 1, \ldots, x_{i_j};\ j = 1, \ldots, h;\ 1 \le i_1 < \cdots < i_h \le m;\ h = 1, \ldots, m;\ \mathbf{x} > \mathbf{0}\right)$$

where the weight functions c_v *are chosen such that* $\sum_{v=1}^{w} c_v(\mathbf{x}) = 1$ *for all* $\mathbf{x} > \mathbf{0}$.

Proof. By assumption we have

$$g(\mathbf{x}) =$$
$$\sum_{h=1}^{m} \sum_{1 \le i_1 < \cdots < i_h \le m} \sum_{s=1}^{h} \sum_{y_s=1}^{x_{i_s}} \varphi_{i_1\ldots i_h}^{(v)}(y_1, \ldots, y_h; \mathbf{x})\, g\left(\mathbf{x} - \sum_{j=1}^{h} y_j \mathbf{e}_{i_j}\right) \prod_{j=1}^{h} f_{i_j}(y_j),$$
$$(\mathbf{x} > \mathbf{0};\ v = 1, \ldots, w)$$

and the theorem follows by multiplication by $c_v(\mathbf{x})$ and summation over v. Q.E.D.

In Section 5 we shall consider an application of Theorem 2.

The condition (6) in Lemma 1 is satisfied by the pairs

$$\varphi(y_1, \ldots, y_h; \mathbf{x}) = \prod_{j=1}^{q} \frac{y_j}{x_{i_j}}; \qquad \psi(\mathbf{n}) = \frac{1}{\prod_{j=1}^{q} n_{i_j}} \qquad (q = 0, 1, \ldots, h)$$

and consequently by

$$\varphi(y_1, \ldots, y_h; \mathbf{x}) = a + \sum_{q=1}^{h} \sum_{1 \leq s_1 < \cdots < s_q \leq h} b_{i_{s_1} \ldots i_{s_q}} \prod_{j=1}^{q} \frac{y_{s_j}}{x_{i_{s_j}}} \tag{7}$$

$$\psi(\mathbf{n}) = a + \sum_{q=1}^{h} \sum_{1 \leq s_1 < \cdots < s_q \leq h} \frac{b_{i_{s_1} \ldots i_{s_q}}}{\prod_{j=1}^{q} n_{i_{s_j}}}. \tag{8}$$

Like in the Vernic recursion the coefficients could depend on whether $x_i = n_i = 0$ for some i's; in particular this should be done to avoid division by zero. To give a general expression for (5) based on these functions would be notationally rather messy, and we shall therefore abstain from that and rather suggest that one develops the formulae in special cases.

In the univariate case, (7) and (8) reduce to

$$\varphi(y; x) = a + b\frac{y}{x}; \qquad \psi(n) = a + \frac{b}{n}.$$

From Theorem 3 in Sundt & Jewell (1981) follows that these are the only (ψ, φ)'s for which (6) is satisfied for any possible choice of severity distribution. The present author believes that also in the multivariate case (7) and (8) give the only (ψ, φ)'s that satisfy the condition (6) of Lemma 1 for any possible choice of severity distributions.

5 Examples

5A. The following model is discussed by Hesselager (1996) in the bivariate case. We assume that the distribution $p_.$ of $N_.$ satisfies the Panjer recursion

$$p_.(n_.) = \left(a + \frac{b}{n_.}\right) p_.(n_. - 1), \qquad (n_. = 1, 2, \ldots)$$

and that the conditional distribution of $\mathbf{N}$ given that $N. = n.$ is the multinomial distribution

$$q(\mathbf{n}) = n.! \prod_{i=1}^{m} \frac{w_i^{n_i}}{n_i!}.$$

We have $q = q_1^{n.*}$ with

$$q_1(\mathbf{y}) = \begin{cases} w_i & (\mathbf{y} = \mathbf{e}_i;\ i = 1, 2, \dots) \\ 0. & (\text{otherwise}) \end{cases} \tag{9}$$

Hence p is the compound distribution $p = \sum_{n.=0}^{\infty} p.(n.)\, q_1^{n.*}$ with univariate counting distribution $p.$ and multivariate severity distribution q_1. Such compound distributions are discussed by Sundt (1999). From his Theorem 1 follows that for $h = 1, \dots, m$ and $\mathbf{n} > \mathbf{0}$ we have the recursion

$$n_h p(\mathbf{n}) = \sum_{\mathbf{0} < \mathbf{u} \leq \mathbf{n}} (a n_h + b u_h)\, q_1(\mathbf{u})\, p(\mathbf{n} - \mathbf{u}),$$

and insertion of (9) gives

$$n_h p(\mathbf{n}) = b w_h p(\mathbf{n} - \mathbf{e}_h) + a n_h \sum_{i=1}^{m} w_i p(\mathbf{n} - \mathbf{e}_i).$$

When $\mathbf{n} > \mathbf{e}_h$, we can divide by n_h, and we then obtain

$$p(\mathbf{n}) = b \frac{w_h}{n_h} p(\mathbf{n} - \mathbf{e}_h) + a \sum_{i=1}^{m} w_i p(\mathbf{n} - \mathbf{e}_i) =$$
$$\left(a + \frac{b}{n_h}\right) w_h p(\mathbf{n} - \mathbf{e}_h) + a \sum_{i \neq h} w_i p(\mathbf{n} - \mathbf{e}_i).$$

Hence

$$g(\mathbf{x}) = b \frac{w_h}{x_h} \sum_{y_h=1}^{x_h} y_h f_h(y_h)\, g(\mathbf{x} - y_h \mathbf{e}_h) + a \sum_{i=1}^{m} w_i \sum_{y_i=1}^{x_i} f_i(y_i)\, g(\mathbf{x} - y_i \mathbf{e}_i) =$$
$$w_h \sum_{y_h=1}^{x_h} \left(a + b \frac{y_h}{x_h}\right) f_h(y_h)\, g(\mathbf{x} - y_h \mathbf{e}_h) + a \sum_{i \neq h} w_i \sum_{y_i=1}^{x_i} f_i(y_i)\, g(\mathbf{x} - y_i \mathbf{e}_i). \tag{10}$$

Formula (10) gives m recursions for g. We shall now combine these recursions by using Theorem 2. Multiplying (10) by $x_h / x.$ and summing over

those values of h where $x_h > 0$, gives

$$g(\mathbf{x}) = \sum_{h=1}^{m} w_h \sum_{y_h=1}^{x_h} \left(a + b\frac{y.}{x.} \right) f_h(y_h)\, g(\mathbf{x} - y_h \mathbf{e}_h). \qquad (\mathbf{x} > \mathbf{0}) \qquad (11)$$

Compared to (10), this recursion has the advantage that it holds for all $\mathbf{x} > \mathbf{0}$. On the other hand, as it involves more algebraic operations, it would presumably be more time-consuming.

As a special case of (11) we obtain

$$p(\mathbf{n}) = \left(a + \frac{b}{n.} \right) \sum_{h=1}^{m} w_h p(\mathbf{n} - \mathbf{e}_h). \qquad (\mathbf{n} > \mathbf{0})$$

This recursion was also given by Sundt (1999).

5B. Teicher (1954) discusses a class of multivariate Poisson distributions that satisfy the recursion

$$p(\mathbf{n}) = \frac{1}{n_m} \left(c p(\mathbf{n} - \mathbf{e}_m) + \sum_{h=1}^{m-1} \sum_{1 \leq i_1 < \ldots i_h \leq m-1} d_{i_1 \ldots i_h} p(\mathbf{n} - \mathbf{e}_{i_1 \ldots i_h m}) \right). \quad (\mathbf{n} > \mathbf{e}_m)$$

as well as analogous recursions where we divide by n_k instead of n_m; $k = 1, \ldots, m-1$. In the bivariate case the corresponding compound distributions are discussed by Hesselager (1996) and Vernic (1999).

6 Extensions

6A. In the univariate case Sundt (1992) gave the following extension to Panjer's (1981) recursion.

Theorem 3. *If p satisfies the recursion*

$$p(n) = \sum_{i=1}^{k} \left(a_i + \frac{b_i}{n} \right) p(n-i), \qquad (n = 1, 2, \ldots)$$

then

$$g(x) = \sum_{y=1}^{x} g(x-y) \sum_{i=1}^{k} \left(a_i + \frac{b_i}{i} \frac{y}{x} \right) f^{i*}(y). \quad (x = 1, 2, \ldots)$$

An analogous extension of the theory in Section 4 would mean to allow the recursion for p to go k steps back. In that connection we would need the following extension of Lemma 1.

Lemma 2. *If for different integers $i_1, \ldots, i_h \in \{1, \ldots, m\}$ and positive integers $k_1, \ldots, k_h$*

$$\mathrm{E}\left[\varphi\left(\sum_{j=1}^{k_1} Y_{i_1 j}, \ldots, \sum_{j=1}^{k_h} Y_{i_h j}; \mathbf{x}\right) \middle| \bigcap_{j=1}^{h}\left(\sum_{r=1}^{n_{i_j}} Y_{i_j r} = x_{i_j}\right)\right] = \psi(\mathbf{n}) \qquad (12)$$

for all $\mathbf{x}, \mathbf{n} > \mathbf{0}$ such that $\prod_{i=1}^{m} f_i^{n_i *}(x_i) > 0$, then

$$\sum_{\mathbf{n}>\mathbf{0}} \psi(\mathbf{n})\, p\left(\mathbf{n} - \sum_{j=1}^{h} k_j \mathbf{e}_{i_j}\right) \prod_{i=1}^{m} f_i^{n_i *}(x_i) =$$

$$\sum_{s=1}^{h} \sum_{y_s=1}^{x_{i_s}} \varphi(y_1, \ldots, y_h; \mathbf{x})\, g\left(\mathbf{x} - \sum_{j=1}^{h} y_j \mathbf{e}_{i_j}\right) \prod_{j=1}^{h} f_{i_j}^{k_j *}(y_j). \qquad (\mathbf{x} > \mathbf{0})$$

Theorem 1 can be extended analogously.

The condition (12) in Lemma 2 is in particular satisfied in the two cases

$$\varphi(y_1, \ldots, y_h; \mathbf{x}) = \prod_{j=1}^{q} \frac{y_j}{k_j x_{i_j}}; \qquad \psi(\mathbf{n}) = \frac{1}{\prod_{j=1}^{q} n_{i_j}}. \qquad (q = 0, 1, \ldots, h)$$

6B. Analogous to Sundt's (1999) extension of Panjer's recursion to Case 1 of Section 3, we could extend the results of the present paper to the case when the severity distributions are multivariate.

6C. In the present paper we have concentrated on recursions for multivariate distributions. In practice one will often approximate distributions by functions that are not necessarily distributions themselves, and thus it can be of interest to have recursions for more general functions. In the univariate case some recursions originally developed for distributions have been extended to more general functions by Dhaene & Sundt (1998) and Sundt, Dhaene, & De Pril (1998); Dhaene, Willmot, & Sundt (1999) discuss recursions for some classes of functions related to distributions, in particular cumulative distribution functions. Some multivariate extensions have been given in Sundt (1998). Analogously, the recursions of the present paper could be extended to more general functions. However, as the deductions in the present paper depend heavily on the conditional expectation in (6), which could obviously not be applied if we leave the realm of distributions, we would then need other proofs.

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