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Recursions for Convolutions of discrete uniform Distributions revisited

by

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Abstract

In the present paper we apply the De Pril transform to obtain recursions for convolutions of discrete uniform distributions. Some of the recursions represent improvements and generalisations of recursions presented by Sundt (1988) and seem to be very suitable for application in spread-sheet programs.

1 Introduction

1A. From De Pril's (1985) recursion for convolutions of an arithmetic distribution, Sundt (1988) deduced a recursion for the special case when that distribution is a discrete uniform distribution. In the present paper we deduce some related recursions by using the De Pril transform. More generally, this paper can be considered as an illustration on how the De Pril transform can be applied to derive convenient recursive algorithms.

Before considering discrete uniform distributions in Section 3 we recapitulate some of the properties of the De Pril transform in Section 2.

1B. In this paper we shall consider functions on the non-negative integers. Unless explicitly stated, it will be silently assumed that the value of these functions is zero for negative arguments.

We apply the convention that $\sum_{x=a}^{b} = 0$ when $b < a$.

2 De Pril transforms

2A. Let $\mathcal{F}_0$ denote the class of functions on the non-negative integers with a positive mass at zero. We define the *De Pril transform* φ_f of a function

$f \in \mathcal{F}_0$ by the recursion

$$\varphi_f(x) = \frac{1}{f(0)} \left(x f(x) - \sum_{y=1}^{x-1} \varphi_f(y) f(x-y) \right). \qquad (x = 1, 2, \ldots) \tag{1}$$

When φ_f and $f(0)$ are known, we can evaluate f recursively by

$$f(x) = \frac{1}{x} \sum_{y=1}^{x} \varphi_f(y) f(x-y), \qquad (x = 1, 2, \ldots) \tag{2}$$

which is found by solving (1) for $f(x)$.

The De Pril transform was defined for probability functions in $\mathcal{F}_0$ by Sundt (1995), and the definition was extended to general functions in $\mathcal{F}_0$ by Dhaene & Sundt (1996).

2B. The convolution $f * g$ of two functions $f, g \in \mathcal{F}_0$ is defined by

$$(f * g)(x) = \sum_{y=0}^{x} f(y) g(x-y). \qquad (x = 0, 1, 2, \ldots) \tag{3}$$

Dhaene & Sundt (1996) showed that if $f_1, f_2, \ldots, f_m \in \mathcal{F}_0$, then

$$\varphi_{*_{i=1}^m f_i} = \sum_{i=1}^{m} \varphi_{f_i}. \tag{4}$$

2C. We define the *cumulation operator* Γ of a function $f \in \mathcal{F}_0$ by

$$\Gamma f(x) = \sum_{y=0}^{x} f(y). \qquad (x = 0, 1, 2, \ldots)$$

The inverse operator Γ^{-1} is defined by

$$\Gamma^{-1} f(x) = \begin{cases} f(0) & (x = 0) \\ f(x) - f(x-1). & (x = 1, 2, \ldots) \end{cases} \tag{5}$$

We see that $\Gamma f, \Gamma^{-1} f \in \mathcal{F}_0$, and that $\Gamma(\Gamma^{-1} f) = \Gamma^{-1}(\Gamma f) = f$.

If f is a probability function, then Γf is the corresponding cumulative distribution function.

Dhaene, Willmot, & Sundt (1996) showed that

$$\varphi_{\Gamma^t f}(x) = \varphi_f(x) + t \qquad (x = 1, 2, \ldots) \tag{6}$$

for non-negative integers t. This result trivially extends to negative integers.

3 Uniform distributions

3A. Let $f \in \mathcal{F}_0$ be the probability function of a uniform distribution on the integers $0, 1, \ldots, k$, that is,

$$f(x) = \begin{cases} \dfrac{1}{k+1} & (x = 0, 1, \ldots, k) \\ 0. & \text{otherwise} \end{cases} \tag{7}$$

By insertion of (7) in (5) we obtain

$$\Gamma^{-1} f(x) = \begin{cases} \dfrac{1}{k+1} & (x = 0) \\ -\dfrac{1}{k+1} & (x = k+1) \\ 0. & \text{otherwise} \end{cases}$$

Insertion in (1) gives

$$\varphi_{\Gamma^{-1} f}(x) = \begin{cases} 0 & (x = 1, 2, \ldots, k) \\ -(k+1) & (x = k+1) \\ \varphi_{\Gamma^{-1} f}(x - k - 1), & (x = k+2, k+3, \ldots) \end{cases}$$

from which we obtain

$$\varphi_{\Gamma^{-1} f}(x) = -(k+1)\,\delta_k(x) \qquad (x = 1, 2, \ldots)$$

with $\delta_k(x) = 1$ for $x = k+1, 2(k+1), \ldots$ and zero otherwise. Application of (6) gives

$$\varphi_f(x) = 1 - (k+1)\,\delta_k(x). \qquad (x = 1, 2, \ldots) \tag{8}$$

By application of (8), (4), and (6) we obtain

$$\varphi_{\Gamma^t f^{m*}}(x) = m\varphi_{\Gamma^t f^{m*}}(x) + t = (m+t) - m(k+1)\,\delta_k(x), \qquad (x = 1, 2, \ldots)$$

and insertion in (2) gives

$$\Gamma^t f^{m*}(x) = \tag{9}$$
$$\frac{1}{x}\left((m+t)\,\Gamma^{t+1} f^{m*}(x-1) - m(k+1) \sum_{1 \le z \le \frac{x}{k+1}} \Gamma^t f^{m*}(x - z(k+1))\right).$$

Together with the initial value

$$\Gamma^t f^{m*}(0) = (k+1)^{-m}, \tag{10}$$

(9) can be applied for recursive evaluation of $\Gamma^t f^{m*}$.

3B. The recursion (9) is easily extended to convolutions of different unifiorm distributions. Let $g = *_{i=1}^{m} f_i$, where f_i denotes the probability function of a uniform distribution on the integers $0, 1, \ldots, k_i$ for $i = 1, 2, \ldots, m$. Then

$$\Gamma^t g(x) = \frac{1}{x}\left((m+t)\Gamma^{t+1} g(x-1) - \sum_{i=1}^{m} (k_i+1) \sum_{1 \le z \le \frac{x}{k_i+1}} \Gamma^t g(x - z(k_i+1))\right)$$

with initial value

$$\Gamma^t g(0) = \prod_{i=1}^{m} \frac{1}{k_i+1}.$$

3C. In subsections 3C-D we shall deduce some simplifications of the recursion (9).

We first want to get rid of the summation sign. By multiplying (9) by x we obtain

$$x\Gamma^t f^{m*}(x) = (m+t)\Gamma^{t+1} f^{m*}(x-1) - m(k+1) \sum_{1 \le z \le \frac{x}{k+1}} \Gamma^t f^{m*}(x - z(k+1)). \tag{11}$$

Replacing x with $x-k-1$ and changing the summation variable gives

$$\begin{aligned}&(x-k-1)\Gamma^t f^{m*}(x-k-1) = (m+t)\Gamma^{t+1} f^{m*}(x-k-2) - \\ &m(k+1) \sum_{2 \le z \le \frac{x}{k+1}} \Gamma^t f^{m*}(x - z(k+1)).\end{aligned} \tag{12}$$

We subtract (12) from (11) and obtain

$$\begin{aligned}&x\Gamma^t f^{m*}(x) - (x-k-1)\Gamma^t f^{m*}(x-k-1) = \\ &(m+t)\left(\Gamma^{t+1} f^{m*}(x-1) - \Gamma^{t+1} f^{m*}(x-k-2)\right) - \\ &m(k+1)\Gamma^t f^{m*}(x-k-1),\end{aligned}$$

which after some rearranging gives

$$\begin{aligned}&\Gamma^t f^{m*}(x) = \frac{m+t}{x}\left(\Gamma^{t+1} f^{m*}(x-1) - \Gamma^{t+1} f^{m*}(x-k-2)\right) + \\ &\left(1 - \frac{(k+1)(m+1)}{x}\right)\Gamma^t f^{m*}(x-k-1),\end{aligned} \tag{13}$$

which can be applied together with the initial values given by (10) and

$$\Gamma^t f^{m*}(x) = 0. \qquad (x = -k-1, -k, \ldots, -1) \tag{14}$$

For $t = 0$, (13) reduces to

$$f^{m*}(x) = \frac{m}{x}\left(\Gamma f^{m*}(x-1) - \Gamma f^{m*}(x-k-2)\right) +$$
$$\left(1 - \frac{(k+1)(m+1)}{x}\right) f^{m*}(x-k-1)\,. \tag{15}$$

This recursion was given by Sundt (1988).

3D. It is displeasing that the recursion (13) for $\Gamma^t f^{m*}$ depends on $\Gamma^{t+1} f^{m*}$, although the values of this function can be easily found by summing the values of $\Gamma^t f^{m*}$. Sundt (1988) converted the recursion (15) for f^{m*} to a recursion for Γf^{m*} by replacing f^{m*} with differences of Γf^{m*} and solving for $\Gamma f^{m*}(x)$. Let us now do the same with the more general recursion (13). We obtain

$$\Gamma^{t+1} f^{m*}(x) - \Gamma^{t+1} f^{m*}(x-1) =$$
$$\frac{m+t}{x}\left(\Gamma^{t+1} f^{m*}(x-1) - \Gamma^{t+1} f^{m*}(x-k-2)\right) +$$
$$\left(1 - \frac{(k+1)(m+1)}{x}\right)\left(\Gamma^{t+1} f^{m*}(x-k-1) - \Gamma^{t+1} f^{m*}(x-k-2)\right),$$

and some rearranging gives

$$\Gamma^{t+1} f^{m*}(x) = \left(1 + \frac{m+t}{x}\right)\Gamma^{t+1} f^{m*}(x-1) +$$
$$\left(1 - \frac{(k+1)(m+1)}{x}\right)\Gamma^{t+1} f^{m*}(x-k-1) -$$
$$\left(1 - \frac{k(m+1)-t+1}{x}\right)\Gamma^{t+1} f^{m*}(x-k-2)\,.$$

By replacing t with $t-1$ we obtain

$$\Gamma^{t} f^{m*}(x) = \left(1 + \frac{m+t-1}{x}\right)\Gamma^{t} f^{m*}(x-1) +$$
$$\left(1 - \frac{(k+1)(m+1)}{x}\right)\Gamma^{t} f^{m*}(x-k-1) - \tag{16}$$
$$\left(1 - \frac{k(m+1)-t+2}{x}\right)\Gamma^{t} f^{m*}(x-k-2)\,.$$

This recursion can be applied together with the initial values given by (10) and (14).

In the special cases $t = 1$ and $t = 0$ we obtain

$$\begin{aligned}
&\Gamma f^{m*}(x) = \left(1 + \frac{m}{x}\right)\Gamma f^{m*}(x-1) + \\
&\left(1 - \frac{(k+1)(m+1)}{x}\right)\Gamma f^{m*}(x-k-1) - \\
&\left(1 - \frac{k(m+1)+1}{x}\right)\Gamma^t f^{m*}(x-k-2)
\end{aligned}$$

$$\begin{aligned}
&f^{m*}(x) = \left(1 + \frac{m-1}{x}\right) f^{m*}(x-1) + \\
&\left(1 - \frac{(k+1)(m+1)}{x}\right) f^{m*}(x-k-1) - \\
&\left(1 - \frac{k(m+1)+2}{x}\right) f^{m*}(x-k-2).
\end{aligned} \tag{17}$$

The former recursion was given by Sundt (1988). The latter recursion seems to be more convenient than (15); in particular we do not need to evaluate Γf^{m*}.

In the special case $m = 1$ (17) reduces to

$$f(x) = f(x-1) + \left(1 - \frac{2(k+1)}{x}\right)(f(x-k-1) - f(x-k-2)).$$

Application of Corollary 1 in Sundt (1992) to this recursion brings us back to (17).

The recursion (16) is very suitable for application in spread-sheet programs. The author has tried it out in Excel.

3E. We define the *stop loss transform* $\overline{g}$ of a distribution with probability function $g \in \mathcal{F}_0$ by

$$\overline{g}(x) = \sum_{y=x+1}^{\infty} (y-x)\, g(y). \qquad (x = \ldots, -1, 0, 1, \ldots)$$

We easily get

$$\Gamma^2 g(x) = \overline{g}(x+1) + x + 1 - \mu_g, \tag{18}$$

where μ_g denotes the mean of the distribution. When x is less than or equal to zero, we have

$$\overline{g}(x) = \mu_g - x. \tag{19}$$

We shall now apply (16) and (18) to obtain a recursion for $\overline{f^{m*}}$. We easily obtain that the mean of the distribution is given by

$$\mu_{f^{m*}} = m\mu_f = \frac{mk}{2}.$$

By letting $t = 2$ in (16) we obtain

$$\Gamma^2 f^{m*}(x) = \left(1 + \frac{m+1}{x}\right)\Gamma^2 f^{m*}(x-1) + \left(1 - \frac{(k+1)(m+1)}{x}\right)\Gamma^2 f^{m*}(x-k-1) - \left(1 - \frac{k(m+1)}{x}\right)\Gamma^2 f^{m*}(x-k-2).$$

Insertion of (18) and some manipulation give

$$\overline{f^{m*}}(x+1) = \left(1 + \frac{m+1}{x}\right)\overline{f^{m*}}(x) + \left(1 - \frac{(k+1)(m+1)}{x}\right)\overline{f^{m*}}(x-k) - \left(1 - \frac{k(m+1)}{x}\right)\overline{f^{m*}}(x-k-1),$$

and by replacing x with $x-1$ we obtain

$$\overline{f^{m*}}(x) = \left(1 + \frac{m+1}{x-1}\right)\overline{f^{m*}}(x-1) + \left(1 - \frac{(k+1)(m+1)}{x-1}\right)\overline{f^{m*}}(x-k-1) - \left(1 - \frac{k(m+1)}{x-1}\right)\overline{f^{m*}}(x-k-2).$$

By application of (19) we obtain the initial values

$$\overline{f^{m*}}(x) = \frac{mk}{2} - x \qquad (x = -k, -k+1, \ldots, 0)$$

$$\overline{f^{m*}}(1) = \mu_{f^{m*}} - 1 + f^{m*}(0) = \frac{mk}{2} - 1 + (k+1)^{-m}.$$

3F. Let us finally consider the compound probability function

$$g = \sum_{n=0}^{\infty} f^{m*}(n)\, h^{n*},$$

where h is a probability function on the positive integers.

From (17) we see that f^{m*} satisfies a recursion in the form

$$f^{m*}(n) = \left(a_1 + \frac{b_1}{n}\right) f^{m*}(n-1) + \left(a_2 + \frac{b_2}{n}\right) f^{m*}(n-k-1) +$$
$$\left(a_3 + \frac{b_3}{n}\right) f^{m*}(n-k-2). \qquad (n = 1, 2, \ldots)$$

Thus Theorem 9 in Sundt (1992) gives that g satisfies the recursion

$$g(x) = \frac{1}{w} \sum_{y=1}^{x} v(x,y)\, g(x-y) \qquad (x = 1, 2, \ldots)$$

with

$$v(x,y) = \left(a_1 + b_1 \frac{y}{x}\right) h(y) + \left(a_2 + b_2 \frac{y}{x}\right) h^{(k+1)*}(y) + \left(a_3 + b_3 \frac{y}{x}\right) h^{(k+2)*}(y)$$

$$w = 1 - a_1 h(0) - a_2 h(0)^{k+1} - a_2 h(0)^{k+2}$$

and initial value

$$g(0) = f^{m*}(0).$$

We obtain

$$v(x,y) = \left(1 + (m-1) \frac{y}{x}\right) h(y) + \left(1 - (k+1)(m+1) \frac{y}{x}\right) h^{(k+1)*}(y) -$$
$$\left(1 - (k(m+1)+2) \frac{y}{x}\right) h^{(k+2)*}(y) \qquad (20)$$

$$w = 1 - h(0) - h(0)^{k+1} + h(0)^{k+2} = (1 - h(0)) \left(1 - h(0)^{k+1}\right)$$
$$g(0) = (k+1)^{-m}.$$

The convolution $h^{(k+1)*}$ can be evaluated recursively by the algorithm of Theorem 3 in De Pril (1985). Then we can evaluate $h^{(k+2)*}$ by (3) as the convolution of $h^{(k+1)*}$ and h.

For a compound uniform distribution, that is, $m = 1$, (20) reduces to

$$v(x,y) = h(y) + \left(1 - 2(k+1) \frac{y}{x}\right) \left(h^{(k+1)*}(y) - h^{(k+2)*}(y)\right).$$

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