Approximating cube roots of integers, after Heron's *Metrica III.20*

Trond Steihaug [a] **and D. G. Rogers** [b]

[a]Institutt for Informatikk
Universitetet i Bergen
PB7803, N5020
Bergen
trond.steihaug@ii.uib.no
[b] dgrbgsu@gmail.com

For Christian Marinus Taisbak,
Institut for Græsk og Latin, Københavns Universitet, 1964–1994,
On his eightieth birthday, 17 February, 2014

Heron did not need any other corroboration than the fact that the method
works, and that the separate results are easily confirmed by multiplica-
tion.

C. M. Taisbak [28, §2]

1 Taisbak's conjecture

How often, in the happy Chinese idiom, do we search high and low for our shoulder pole, only at last to notice it again on our shoulder where we left it? For all that the learned commentator might reassure us that some mathematician of the past could not help but make some pertinent observation, just as surely we know, from our own experience, that such acuity might escape us for half a lifetime, before, all at once, perhaps of a Summer's night, the øre drops. This is, indeed, the story behind Christian Marinus Taisbak's conjecture in [28], as divulged in a recent letter [29]. So, we too were set thinking. We report here on some of our findings.

Heron, in *Metrica III.20–22*, is concerned with the the division of solid figures — pyramids, cones and *frustra* of cones — to which end there is a need to extract cube roots [15, II, pp. 340–342] (see also [16, p. 430]). A case in point is the cube root of 100, for which Heron obligingly outlines a method of approximation in *Metrica III.20* as follows (adapted from [15, 2, 28], noting that the addition in [28, p. 103, fn. 1] appears earlier in [2, p. 69]; cf. [20, p. 191, fn. 124]):

Take the cube numbers nearest 100 both above and below, namely 125 and 64.
Then, $125 - 100 = 25$ *and* $100 - 64 = 36$.
Multiply 25 by 4 and 36 by 5 to get 100 and 180; and then add to get 280.
Divide 180 by 280, giving 9/14. Add this to the side of the smaller cube; this
gives $4\frac{9}{14}$ *as the cube root of 100 as nearly as possible.*

It seems short, unobjectionable work to turn this descriptive algorithm into a general formula for approximating the cube root of some given integer N. We first locate N among the cubes of the integers:

$$m^3 < N < (m+1)^3.$$

Writing $d_1 = N - m^3$ and $d_2 = (m+1)^3 - N$, Heron would then have us approximate the cube root of N by

(1) $$m + \frac{(m+1)d_1}{(m+1)d_1 + md_2}.$$

The text of *Metrica* as we have it today only came to light in the mid-1890s, with a scholarly edition [24] published in 1903. How little was known for sure about *Metrica* in the years immediately prior to this is suggested by [12]. Fragments were known by quotation in other sources and Eutocius, in a commentary on the works of Archimedes, reports that Heron used the same methods for square and cube roots as Archimedes. But clearly this does not have the same *cachet* as a text — and we still lack anything by Archimedes on finding cube roots. Gustave Wertheim (1843–1902) proposed (1) in 1899 in [33], to be followed a few years latter by Gustaf Hjalmar Eneström (1852–1923) in [9] with an exact (if tautological) expression, given below in §5.2 as (22), for the cube root of N from which (1) follows on discarding cubes of positive terms less than unity. (Besides work in mathematics and statistics, Eneström had interests in the history of mathematics, as seen, for example, in his note [8] on rules of convergence in the 1700s: he is perhaps best remembered today for introducing the *Eneström index* to help identify the writings of Leonhard Euler (1707–1783); but, while there seems to be little written about him in English, the very first volume of *Nordisk Matematisk Tidskrift* carried a centenary profile [13].)

To be sure, other formulae might fit Heron's numerical instance in *Metrica III.20*: a nod is made to one in [18, pp. 137–138]:

(2) $$m + \frac{d_1\sqrt{d_2}}{N + d_1\sqrt{d_2}}.$$

At first sight, this gesture might seem *pro forma*, as it is conceded straightaway that (2), when compared with (1), is both less easy to justify and not so accurate for other values of N. But the record has not always been so clear-cut and it is (2), not (1), that we find on looking back to [30, pp. 62–63], where reference is made to an article [5] by Ernst Ludwig Wilhelm Maximilian Curtze (1837–1903) of 1897, along with [33, 9]. Both Curtze's tentative contribution (2) and another, similar

formula,

$$(3) \qquad m + \frac{(m+1)d_1}{N + (m+1)d_1},$$

had, in fact, been compared adversely for accuracy with (1) in 1920 by Josiah Gilbart Smyly (1867–1948) in [26]; Smyly attributes to George Randolph Webb (1877–1929; Fellow, Trinity College, Dublin) an estimate that the error in (1) is of the order of $1/m^2$ (see further §5.2, especially (33)). For the record, we might note here that Smyly alludes to the work of Curtze, but not that of Wertheim or Eneström; on the other hand, Heath [15] cites them, but not Curtze or Smyly (truely the vagaries of citation are not easily explicable: in [25, p. 256, fn. 2], we find Smyly footnoted as *correcting* Curtze, only for (3), rather than (1), to be printed).

It is also worth observing that the effect of emendations is to move our understanding of the received text in favour of the most accurate candidate, namely (1). As it happens, in *Metrica III.22*, Heron needs the estimate of another cube root, that of 97050 according to [24], but in fact of $97804\frac{4}{5}$, as pointed out in [3, pp. 338–340]. The approximation taken is 46, which cubes to 97336, so is not too far off either way, suggesting that Heron did not allow himself to be blinded by science.

If the consensus on (1) is by now reasonably settled, there remains the question of how Heron might have come upon (1), as well as the somewhat different question of how (1) might be justified. A formal derivation of (1) might well fail to satisfy those who want some heuristic insight into the approximation; and Eneström may have lost sight of the simplicity of his identity (26) in the manner he derives it (see further §5.2). Taisbak strikes out on his own account in [28] from the constancy of the third difference of the sequence of cubes of integers and builds up to the observation that the gradient of the chord between $m-1$ and m is to the gradient of the chord between m and $m+1$ approximately as $m-1 : m+1$. In effect, Taisbak sums up his thinking with a question [28, §3]: "*Did the Ancients know and use sequences of differences?*"

As far as Taisbak's mathematics goes, a rather similar argument was advanced some thirty years ago by Henry Graham Flegg (1924–) in a book [11, p. 137] (pleasingly enough it was reissued in 2013). Others have been here, too: Oskar Becker (1889–1964) in [2, pp. 69–71] in 1957; Evert Marie Bruins (1909-1990) in [3, p. 336] in 1964; Wilbur Richard Knorr (1945–1997) in [20, pp. 191–194] in 1986. It has also been noted how (1) can be adapted for iterative use, although the accuracy of (1), as remarked on by Smyly, coupled with the opportunity for rescaling it provides, might make iteration otiose (cf. §5.4). But, in fairness, it might be remarked that the main difference between these writers and Eneström is that their approximative sleight of hand takes good care to wipe away small terms as they go, rather than in one fell swoop at the end (we return to these comparisons in §5.2).

Our concerns are rather different. For a start, might there be more to discern in the numerical instance Heron presents in *Metrica*? This prompts two further questions. Why is no comparison made with the more straightforward cube root bounds (as in (6) and (7)) analogous to those (as in (14) and (15)) seemingly in common use by Archimedes, Heron and others for square roots? And, why do we not hear anything

like (1) in regard to square roots? Then, again, might there not be more to say about (1) itself?

Our concerns in these regards are mathematical, not historical. Perforce, we respect Taisbak's stricture, as endorsed by Unguru [31], that we adopt as our epigraph. Truly, the proof of the pudding is in the eating; and if, perhaps like Eutocius commenting on the works of Archimedes, you have nothing more imaginative to offer, arithmetical confirmation remains a safe recourse, if not always a sure one (cf. [21, pp. 522, 540]). But we suspect that, if anything, others before us may have been too abashed to descend our level of *naiveté*. Our excuse, if one is needed, is that, even at this level, there is still much with which to be usefully engaged.

2   Heron's example

The difference between successive cubes is

$$(4) \qquad\qquad (m + 1)^3 - m^3 = 3m^2 + 3m + 1.$$

More generally, we may picture the difference between cubes by cutting up the larger cube into smaller cube with various other slabs and blocks, a three-dimensional analogue of the pictures we might draw for the difference of two squares, perhaps as an *aide mémoire* to our reading of Euclid's *Elements II* (one traditional mode of visualising the cube of a binomial expression is shown in Fig. 1; an alternative dissection appears in Fig. 3 in conjunction with (16)).

Thus, as $d_1 = N - m^3$ and $d_2 = (m+1)^3 - N$ sum to this difference, we can ensure some cancellation in working with (1) if we arrange to take d_1 to be $k(m+1)+1$ for some k with $0 \le k \le 3m$. Perhaps Heron had something of this in mind in taking an example in which $d_1 = (2m-1)(m+1) + 1 = m(2m+1)$ and $d_2 = (m+1)^2$ for $m = 4$. At all events, generalising Heron's example in this way, we obtain from (1) a bound on the cube root of $N = m^3 + m(2m + 1) = m(m+1)^3 - (m+1)^2$:
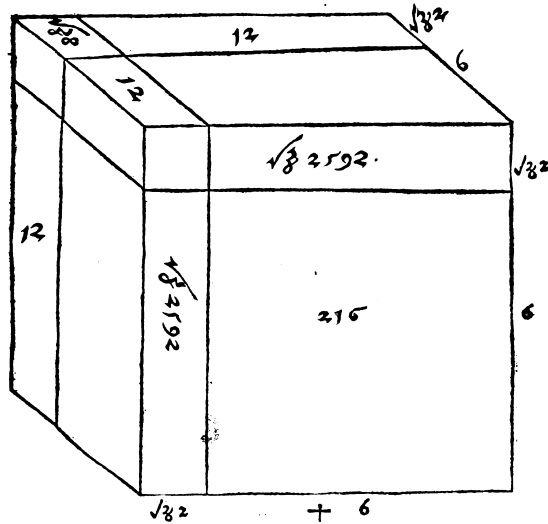
$$(5) \qquad\qquad m + \frac{2m + 1}{3m + 2} = m + 1 - \frac{m + 1}{3m + 2}.$$

It is a simple matter of verification to check that this is an *upper* bound.

But not only is this pleasing in itself, the form of these expressions suggests — invites? — a comparison with the upper bounds obtained more straightforwardly from binomial expressions analogous to those familiar for square roots (as in (14) and (15)), of which Gerolamo Cardono (1501–1576) made celebrated use in *Practica Arithmetice* (1539) [22, §2.4] (but cf. also (20)). Thus, for $N = m^3 + d_1$, the cube root is bounded above by

$$(6) \qquad\qquad m + \frac{d_1}{3m^2},$$

Figur 1: Picture of a cubed binomial from *Arithmetica Integra* (1544)

while for $N = (m + 1)^3 - d_2$, the cube root is bounded above by

$$(7) \qquad m + 1 - \frac{d_2}{3(m + 1)^2}.$$

So, in generalising Heron's example, we have hit on a case where the upper bounds in (6) and (7) also come out rather neatly:

$$m + \frac{2m + 1}{3m}; \quad m + 1 - \frac{1}{3}.$$

Of course, the former is not so good as the latter, reflecting the closer proximity of *this N* to $(m + 1)^3$ than to m^3. Rather more strikingly neither of these bounds is as good as that in (5) obtained from (1); indeed,

$$\frac{2m + 1}{3m + 2} < \frac{2}{3} < \frac{2m + 1}{3m}.$$

It is possible to squeeze (6) further by increasing the denominator in the fraction, and some writers in Arabic in the early 1000s worked with $3m^2 + 1$ in place of $3m^2$ (cf. [22, §3.2]. But this still does not give an improvement over (5).

Whether or not Heron may have indulged himself in such exercises, a few numerical instances like this would surely convey to any impressionable mind that (1) cannot be completely without merit. Trouble might spring more from the opposite corner, not to run away with too favourable an endorsement based *only* on evidence of this sort. However, as we show in §5.3, an approximate construction of two mean proportionals examined by Pappus early in *Synagogue III* allows us to improve on (5), indicating that it is by no means the best the Greeks could have done, had they put their minds to it.

3   Square roots

3.1   Elementary theory of proportions

When we look at the formulation of (1), it would seem that it is a recipe we could write down for other functions besides cubes and cube roots; and, if for cubes and cube roots, why not before that for squares and square roots? In fact, we might recognize (1) in the setting of the elementary theory of proportions that was well-articulated by the Greeks. For, given $a/b > c/d > 0$, an early result in that theory gives

$$\frac{c}{d} < \frac{a + c}{b + d} < \frac{a}{b},$$

and, more generally, for weights w_1 and w_2,

$$(8) \qquad \frac{c}{d} < \frac{aw_1 + cw_2}{bw_1 + dw_2} < \frac{a}{b}.$$

In particular (cf. (11), (13), (22) and (26))

$$(9) \qquad m = \frac{m^2}{m} < \frac{(m+1)^2 w_1 + m^2 w_1}{(m+1)w_1 + mw_2} < \frac{(m+1)^2}{m+1} = m+1,$$

where the central expression can then be rewritten as (cf. (1))

$$\frac{(m+1)^2 w_1 + m^2 w_1}{(m+1)w_1 + mw_2} = m + \frac{(m+1)w_1}{(m+1)w_1 + mw_2}.$$

This is pudding that anyone can eat, but it might not always satisfy Winston Churchill's demand that pudding have a *theme*. For, how to explain the choice of weights for different functions?
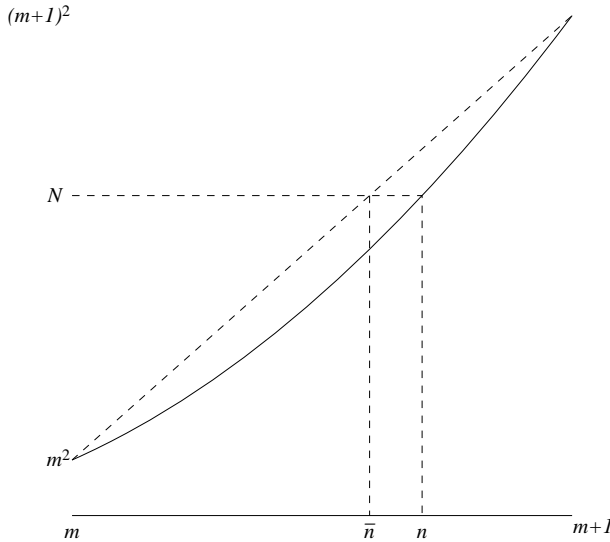
3.2 Curves and chords

Figur 2: Approximating square roots from below

For any increasingly increasing function, such as squaring or cubing, chords lie above the curve, so a particular height N will be encountered on the chord before it is encountered on the curve, giving a simple means of finding a lower bound on the ordinate for which N is attained, after the manner of solution traditionally known as "*double false position*" (a brief introduction to the history of which is recently to hand in [17]). Let us illustrate the thinking here rather naively in the case of squares. So, suppose now that we are given N, with

$$m^2 < N < (m+1)^2,$$

and we are interested in the square root $n = \sqrt{N}$. Then we expect that the gradient of the chord between m and n, that is, $d_1/(n-m)$, to be less than the gradient of

the chord going on from n to $m+1$, that is, $d_2/(n+1-m)$, where for our present purposes in this section we write $d_1 = n^2 - m^2$ and $d_2 = (m+1)^2 - n^2$ in analogy with the notation for (1). But, if

$$(10) \qquad \frac{d_1}{n-m} < \frac{d_2}{m+1-n},$$

then it follows that, for $0 \le d_1 \le 2m+1$,

$$(11) \qquad n > \frac{(m+1)d_1 + md_2}{d_1 + d_2} = m + \frac{d_1}{2m+1}.$$

Equality would hold here if the two gradients were equal, in which case the common value would be the gradient of the chord from m to $m+1$, confirming that this lower bound on n is the ordinate $\bar{n}$ at which N is attained on this chord (as in Fig. 2).

Of course, in this case, d_1 and d_2 are just differences of squares,

$$d_1 = n^2 - m^2 = (n-m)(n+m); \quad d_2 = (m+1)^2 - n^2 = (m+1-n)(m+1+n),$$

so

$$\frac{d_1}{n-m} = n+m; \quad \frac{d_2}{m+1-n} = m+1+n.$$

Hence, (10) holds trivially:

$$n + m < m + 1 + n.$$

But, looking at this last inequality, we see that it is readily reversed by judicious counterpoised weighting, mutiplying the left-hand side by $m+1$ and the right hand side by m:

$$(m+1)(n+m) > m(m+1+n).$$

So, in addition to (10), we also have

$$(12) \qquad \frac{(m+1)d_1}{n-m} > \frac{md_2}{m+1-n}$$

from which we deduce in turn the upper bound

$$(13) \qquad n < \frac{(m+1)^2 d_1 + m^2 d_2}{(m+1)d_1 + md_2} = m + \frac{(m+1)d_1}{(m+1)d_1 + md_2},$$

thereby providing easy confirmation that the analogue of (1) for square roots. But the algebra here is such that conversely, if a upper bound of the form (13) holds, then the weighted gradients stand as in (12), a point to bear in mind when considering (1).

3.3 Square root bounds

However, the sad fact of the matter is that (13) is not much help because we already do better with one or other of the standard upper bounds for square roots obtained from binomial expressions that complement the lower bound (11); the implicit use of all the bounds (11), (14) and (15) in antiquity is examined *in extenso* in [14, pp. lxxvii–xcix] (cf. [12, pp. 53–57]). We recall that, for $N = m^2 + d_1$,

$$(14) \qquad n = \sqrt{N} < m + \frac{d_1}{2m}$$

while, for $N = (m + 1)^2 - d_2$,

$$(15) \qquad n = \sqrt{N} < m + 1 - \frac{d_2}{2(m + 1)}.$$

We work with (14) for $0 < d_1 \leq m$, switching to (15) for $0 < d_2 \leq m + 1$.

Notice that (14) and (15) also follow from the iterative scheme that Heron sketches by example for $N = 720$ in *Metrica I.8*:

$$m_1 = \frac{1}{2} \left(\frac{N}{m_0} + m_0 \right),$$

with $m_0 = m$ for (14) and $m_0 = m + 1$ for (15). Whether Heron recognised (15) explicitly depends in large part on what inference can be drawn from the way fractions are recorded (cf. [15, II, p. 326]). There are other puzzles in relation to Heronian iteration. For instance, samplings in [14, p. lxxxii] and [7, p. 6] of estimates used by Heron for square roots includes that for $\sqrt{75}$ as $8\frac{11}{16}$ (cf. (14)), rather than $8\frac{2}{3}$ (cf. (15); and see further [3, pp. 10–11]), which is simpler, as well as more accurate; and a further example is raised in §5.4.

Now, in these ranges for d_1 and d_2 for (14) and (15),

$$(m + 1)d_1 + md_2 \leq 2m(m + 1),$$

with equality if and only if $d_1 = m$ and $d_2 = m + 1$. Hence (13) is only as good as (14) or (15) in the case where $d_1 = m$ and $d_2 = m + 1$, when all three bounds come out the same, namely $m + \frac{1}{2}$ (but see §5.4 for a reprieve of sorts for (9)). This points up the altered situation for cube roots, where the evidence of the previous section shows that (1) does better than (4) and (5), at least in a family of instances generalizing Heron's example in *Metrica III.20*. Clearly, we need to examine how the arguments leading to (11) and (3) for square roots go over to cube roots, especially as it is the innocent use of counterpoised weighting in shifting from (10) to (12) that lies at the heart of Taisbak's musings in [28].

3.4 Mellema's formula for quadratics

But before leaving this discussion of square roots it may be instructive in comparison with the derivation of Eneström's identity (26) to take a brief look at a formula

developed by Elcie Edouard Leon Mellema (1544–1622) as a *baroque* example of
the method of false position (cf. [17]). Suppose that a function $f(x)$ has a root at
n with $a < n < b$, then, trivially,

$$(f(n) - f(a))f(b) = (f(n) - f(b))f(a).$$

However, in the case of a quadratic function where the square has been completed,
that is, where
$$f(x) = (x + p)^2 - q,$$

rearranging this equation to make $(n + p)^2$ the subject yields Mellema's formula:

$$(n + p)^2 = \frac{(a + p)^2 f(b) - (b + p)^2 f(a)}{f(b) - f(a)}.$$

In contrast with (26), from which (1) follows as an approximation, the best that
can be said of Mellema's formula is that it is a trick on him, if not also on any who
might be taken in by it, as it just recomputes q, which we might suppose would be
known more swiftly on completing the square in the quadratic.

4   Cube roots

So, let us now return to cube roots and our initial supposition that we are given
N, with
$$m^3 < N < (m + 1)^3,$$

and write
$$d_1 = N - m^3; \quad d_2 = (m + 1)^3 - N.$$

If n is the cube root of N, so $n^3 = N$, then, possibly calling to mind Heron's account
of *frustra* of pyramids and cones in *Metrica II.6, 9* (cf. [15, II, pp. 332–334]; that
the formulae Heron provides were not always used with sufficient care is suggested
in [27, pp. 107–108]),

(16) $$d_1 = n^3 - m^3 = (n - m)(n^2 + nm + m^2),$$

so that

(17) $$\frac{d_1}{n - m} = n^2 + nm + m^2.$$

Similarly

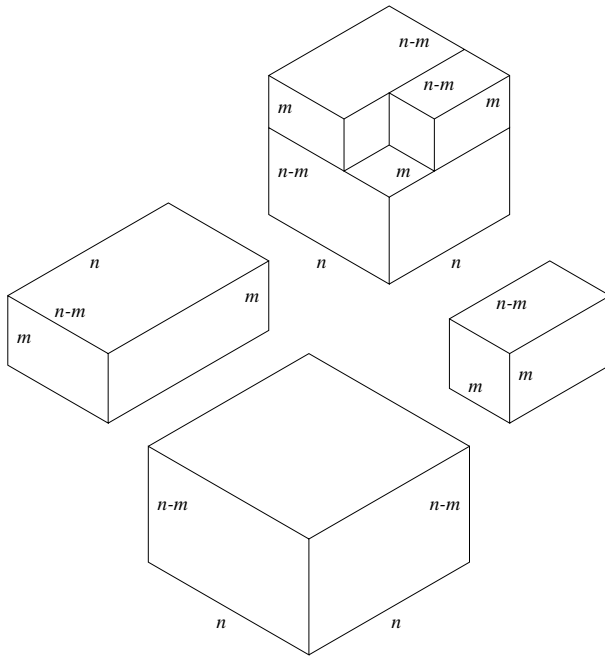(18) $$\frac{d_2}{m + 1 - n} = (m + 1)^2 + (m + 1)n + n^2.$$

Figur 3: Difference of cubes dissected according to (16)

It follows that, on the lines of (10), we have

$$(19) \qquad \frac{d_2}{m+1-n} - \frac{d_1}{n-m} = 2m+n+1 > 0,$$

from which we deduce, in perfect analogy with (11), the lower bound

$$(20) \qquad m_l = \frac{(m+1)d_1 + md_2}{d_1 + d_2} = m + \frac{d_1}{3m(m+1)+1},$$

and then, iterating the argument, the further refined lower bound

$$m_l + \frac{N - m_l^3}{3m_l(m+1) + (m+1-m_l)^2}.$$

By way of illustration, in Heron's example with $N = 100$, neither the the lower bound $m_l = 4\frac{36}{61}$ obtained from (20) nor the refined one, which involves much heavier computation, are as close to the cube root of 100 as Heron's upper bound $4\frac{9}{14}$. Yet, as a matter of historical record, Leonardo Pisano (Fibonacci; 1170?–1250?), in *Liber Abaci* (1202) [22, §2.3] and again in *De Practica Geometrie* (1223) [19, pp. 260–262], approximates cube roots by means of (20), sometimes in sequence with its improvement, knowing to ignore the term $(m+1-m_l)^2$ in the denominator of the fraction in the latter and even the analogous 1 in the denominator of the last

fraction in (20) if it suits the calculation (the textual problem raised in [22, p. 92, fn. 7] as to the use of the improved bound is resolved on cross-reference with [19, p. 262]). A version of (20) appears again in use in the 1500s (cf. [25, p. 255, fn. 4]; [17]).

So far, so good, although this is entirely as we might expect. But what about applying Taisbak's hunch on counterpoised weightings to (17) and (18) that, as we have seen in the previous section, does lead in the case of square roots to the analogue (13) of (1)?

Thus, in place of (19), we shall need to consider:

$$(21) \qquad \frac{(m+1)d_1}{n-m} - \frac{md_2}{m+1-n} = n^2 - m(m+1).$$

Now, with (21), we see the contingent nature of the expression in (1) as a bound on the cube root of N. For, if $N^2 > m^3(m+1)^3$, as is certainly the case when $N > (m+\frac{1}{2})^3$, then the right-hand side of (21) is positive, and, as, in the previous section, it follows that (1) gives an upper bound. On the other hand, if $N^3 < m^3(m+1)^3$, (1) will give another lower bound along with (20), although one that improves on (20), as it is a matter of easy algebra to check that the expression in (1) is always larger than its counterpart in (20):

$$(a^2p + b^2q)(p+q) \geq (ap+bq)^2.$$

In this latter case, let us take by way of illustration $N = 85$, so $d_1 = 21$ and $d_2 = 40$; the two lower bounds then come out as $4\frac{21}{61}$, for (20), and $4\frac{21}{53}$, for (1).

Of course, we can always up the ante by further loading the weights. Moving up from (21), we find that

$$\frac{(m+1)^2d_1}{n-m} - \frac{m^2d_2}{m+1-n} = (2m+1)n^2 + m(m+1)n > 0,$$

so at least we have the upper bound

$$(22) \qquad n < \frac{(m+1)^3d_1 + m^3d_2}{(m+1)^2d_1 + m^2d_2},$$

throughout the range $m^3 < N < (m+1)^3$, for what it is worth. But, in the test case $N = m^3 + m(2m+1)$ considered in §2, (22) gives the upper bound

$$m + \frac{2m+1}{3m+1}.$$

Thus, (22) loses the advantage we found (1) has over (7) for such N (even if it remains better than (6)).

5   Comparisons

All comparisons, it is has often been said, are odious, but, as an anonymous reviewer wryly rejoined in the *Edinburgh Review* [1, p. 400] for September, 1818:

> *No man, when he learns that the three angles of every triangle are equal to two right angles, ever thought of saying, that the series of comparisons by which that truth is demonstrated was invidious; neither has the fate of those interesting portions of space ever been deemed particularly hard, for having been subjected to such an investigation.*

The Greeks did debate the propriety of geometrical procedures — we turn to one example in §5.3. But their practical arithmetical competence was more pragmatic it seems. Approximations tend to be stated blankly, without supporting argument, but also without comparison with other methods, as though truly, as Taisbak has it with (1), the Greeks *did not need any other corroboration than the fact that the method works*.

In contrast, for us today proposal of an approximative method is incomplete unless accompanied by examination of how well it performs against both rivals and the target. So, in this section, we first look at an instance where Heron provides, not only a demonstration, but compares the resulting bound with an older rule of thumb; we then make a more thorough investigation of Eneström's identity; and we go on to show how a geometric scheme considered by Pappus can be adapted to improve on (1) for the family of numerical cases in §2. We conclude by observing how the improving accuracy of (1), as revealed by (33), allows us to make good effect of rescaling (returns to scale). The Newton-Raphson and Halley methods of approximating cube roots in (29) and (31), in contrast, do not guarantee such improving accuracy, even if some juggling may be possible (a rather more obvious distinction is that (1) is exact when N is the cube of an integer).

5.1   *Metrica I.27–32*: Area of a circular segment

Heron, in *Metrica I.27–32*, is concerned with formulae for the area of a circular segment (see [15, II, pp. 330–331]). Let AB be the arc of a circle subtending a segment less than a semicircle and let C be the midpoint of the arc. Then Heron asserts that the area subtended by AB is greater than four thirds the area of the triangle $\triangle ABC$; that is, if the arc AB has sagitta h and subtended chord b, the subtended segment between arc and chord has area at least

$$(23) \qquad\qquad \frac{4}{3}\left(\frac{hb}{2}\right).$$

But, rather out of character for him, Heron goes further, proving (23) in a manner reminiscent of Archimedes' *De quadratura parabolae, Prop. 24*. However, despite being game to take on this task, Heron does not seem entirely sure of himself: he sets up his diagram as if intending to argue in one way, but then heads off in another; and underlying this dithering is a certain uneasiness in handling inequalities (at issue, in a sense, are returns to scale resulting from the circle's convexity, cf. §5.4).

So, it may be some surprise to find that, in *Metrica I.30, 31*, Heron volunteers comparison of (23) with a more traditional approximation, namely

$$(24) \qquad\qquad \frac{h(b+h)}{2},$$

even stating, but without further comment, when one is to be preferred to the other.

This is all rather remarkable, and not unnaturally *Metrica I.27–32* has caught the attention of commentators. Wilbur Knorr, in particular, has made much of the passage, returning to tease it out several times, as for example, in his books [20, pp. 168–169] and [21, pp. 498–501], as well as in earlier papers on which the books build. Knorr adjudicates the comparison of (23) and (24) in a footnote [20, p. 168, fn. 63] (in a further footnote [21, p. 501, fn. 34], he reports how advantage was not always taken of the improved bound):

> *[Hero] adds that one should use this rule when b is less than three times h,*
> *but the former rule when b is greater. He does not explain this criterion, but*
> *one can see how it results from considering where the two rules yield the same*
> *result, namely, $2bh/3 = h(b+h)/2$, whence $b = 3h$. ...*
> *The [former] rule, by virtue of its association with that for the parabolic seg-*
> *ment, suggests an Archimedean origin. One suspects that the rather sophisti-*
> *cated effort reported by Hero to assess the relative utility of these two rules*
> *for the circular segments is also due to an Archimedean insight.*

Now, there is no doubt that inequalities are more tricky to handle than equalities for pupils today, no less than in the past; and we all resort to simple means of reassurance that we have them right. But, if Knorr's comments here arrest our attention, it is because of the incongruity between the supposed Archimedean origin of the comparison and the method advanced for seeing that it holds. Perhaps Knorr is empathising too much with the difficulty Heron might have encountered in understanding some abstruse Archimedean proto-text. Comparison of (23) and (24) would surely present little challenge to those, such as Archimedes, if not also Heron, for whom thinking in terms of areas was stock-in-trade.

In terms of areas, (23) tells us that the area of the subtended segment is a third more than the area of the triangle $\triangle ABC$, in keeping with the way the proof presented by Heron runs. So, in place of (23), we might write the bound as

$$(25) \qquad\qquad \frac{hb}{2} + \frac{1}{3}\left(\frac{hb}{2}\right) = \frac{h(b+b/3)}{2}.$$

Our areal intuition then suggests seeing in (24) and (25) triangles with common height h and bases

$$b+h; \quad b+\frac{b}{3},$$

respectively. Which triangle has the larger area is simply a matter of which base is longer, leading to the conclusion that (25) is a better lower bound when the latter base is the larger, that is, when $b/3$ is bigger than h, as Heron claimed.

But, with Taisbak's stricture as our epigraph, the point to remember here — and the point of this excursus — is that this is only *our* intution, not necessarily that of Heron or Archimedes, however plausible we fancy it to be. On the other hand, they were clearly not in want of competence of their own.

5.2   Eneström's identity

It would be wrong to give the impression that the papers of Curtze [5] and Wertheim [33] are confined to the elaboration of Heron's text as discussed in the opening section. For example, Curtze includes a list of quadratic approximations. Wertheim anticipates the spirit of Taisbak in [28], providing a foundation on which Eneström builds in [9]. Indeed, as Taisbak [29] playfully observes of any purported "*new insight*," on comparing Wertheim's contribution with his own,

> *If someone else said the same, it must be true. If not, it is high time to have said it.*

Now, if we write

$$\Delta_1 = d_1 - (n - m)^3; \quad \Delta_2 = d_2 - (m + 1 - n)^3,$$

then Eneström, in [9], goes through a series of algebraic manipulations that brings n out in this notation as

$$(26) \qquad n = m + \frac{(m + 1)\Delta_1}{(m + 1)\Delta_1 + m\Delta_2}.$$

Clearly, if we ignore terms that are cubes of positive numbers less than unity, the right-hand side of (26) is just (1). But (26) must hold as an identity, so going through a routine of solving for n, as Eneström does, might seem somewhat artificial. Why not proceed more simply by direct computation with Δ_1 and Δ_2? We have

$$(27) \qquad \Delta_1 = 3mn(n - m); \quad \Delta_2 = 3(m + 1)n(m + 1 - n),$$

expressions already familiar from [28] as approximations for d_1 and d_2. So, it readily follows that

$$(m + 1)^i \Delta_1 + m^i \Delta_2 = 3m(m + 1)n^i, \quad i = 1, 2.$$

Hence (cf. (9), (11), (13) and (22)),

$$(28) \qquad n = \frac{(m + 1)^2 \Delta_1 + m^2 \Delta_2}{(m + 1)\Delta_1 + m\Delta_2} = m + \frac{(m + 1)\Delta_1}{(m + 1)\Delta_1 + m\Delta_2},$$

as desired.

Looked at in this way, we see both that there is less mystery about Eneström's exact expression (26), but also less difference between him and later writers whose strategy is to get in early with the approximations for d_1 and d_2 given by (27),

rather than waiting to the end. Either way, while it is apparent that (1) *is* an approximation for the cube root of N, because we are modifying both numerator and denominator in the fraction we form in (28), we are left uncertain how *good* an approximation it is, or even whether we obtain an upper bound or a lower bound. As Taisbak draws inspiration from the gradient of chords between successive integers and their cubes, his approach inherently sets up the expectation of an upper bound.

Naturally, a version of (28), and so of (1), can be developed for general intervals, as in [20, p. 192] and [6, p. 29, (1)] (that thoroughness is needed here can be seen from [22, §2.1]). But Knorr's description in [20, p. 192] of a prospective iterative application of such an extension of (1) also appears to be written in the expectation that the result gives an upper bound. If, for some a and b not necessarily integers we have $a^3 < N < (a+b)^3$ and we obtain the approximation $a+b'$ after the manner of (1), as Knorr has us imagine, then certainly, at the next round of the iteration, we substitute for $a + b'$ for $a + b$, but only *if this approximation is an upper bound*. In view of (21), we shall need to check this. If, in the event, it turns out that $a + b'$ is a *lower* bound, we shall have to substitute it for a, *not $a + b$*, at the next round.

Knorr rightly goes on to question the authenticity of wiping away of small quantities, whenever in the scheme of things it happens, noting that we can reach the approximations in (27) in greater conformity with the Greek style by replacing the three terms on the left-hand side of (17) and (18) by three times their respective middle terms, rather than being tied to versions of the binomial expansion (4) (see [20, p. 193]). So far as this approach goes, it is on a par with a Newton-Raphson approximation for the cube root of N, such as

$$(29) \qquad \frac{N + 2m^3}{3m^2}$$

obtained by similarly replacing the same three terms by three times the last term, as Knorr also remarks.

For that matter, we could take this line of discussion further, by replacing the same three terms by three times the first term to obtain an approximation for the *square* of the cube root of N,

$$(30) \qquad \frac{2N + m^3}{3m},$$

and then cap this cleverness, by observing that an improved approximation for the cube root of N proposed by Edmund Halley is given as the *ratio* of the expressions in (29) and (30):

$$(31) \qquad m\left(\frac{2N + m^3}{N + 2m^3}\right).$$

Halley's approximation in (31) does at least serve to remind us that in (1) we are also involved with a ratio, a ratio moreover, as (28) makes clear, of two blends of the approximations in (27). Strangely enough, Knorr seems distracted from the significance of these differences between (1) and, say, (29), even while digressing at length on discoveries in approximation theory.
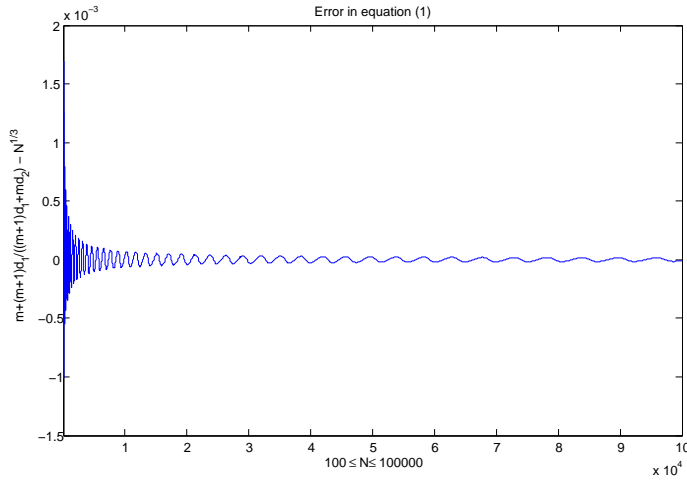
Figur 4: Damped oscillation exhibited by error in (1), as given by (32)

It may also be worth remembering that the statement of a result for illustrative purposes by way of a succinct algorithmic description, such as suits Heron's purpose in *Metrica III.20* might not be the formulation used were the result recast as a more formal proposition. It is natural that historians of mathematics should wish to adhere to the text as they understand it, that is, to (1) as encapsulating the numerical instance in *Metrica III.20*; and that is what we find, with proposed proofs in which the manipulations of ratios closely follows the form of (1). But, considering (13), (22), and now (28), in the general setting provided (8) and (9), we might suspect that it is these more symmetric equivalents of (1) that lend themselves more readily both to proof and to further examination.
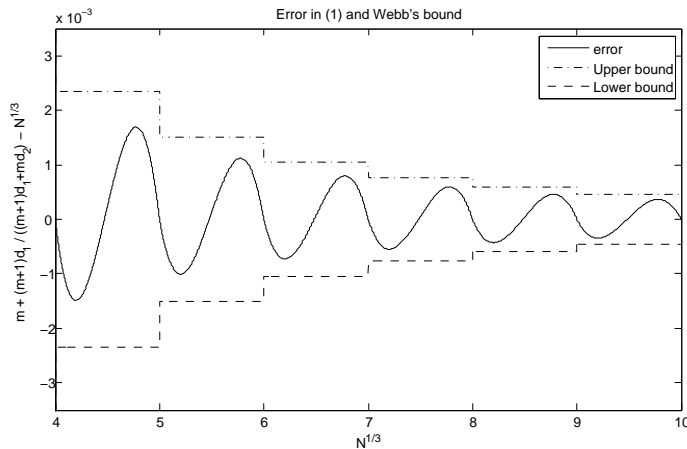

Figur 5: Heron's Wave: error in (1) with Ward's bound superimposed

Thus, starting from (21), we find that

(32) $$\frac{(m+1)^2 d_1 + m^2 d_2}{(m+1)d_1 + m d_2} - n = \frac{(n^2 - m(m+1))(n-m)(m+1-n)}{(m+1)d_1 + m d_2}.$$

To bound the absolute value of the left-hand side of (32) without going into too much fine detail, we note, first of all, that

$$|n^2 - m(m+1)| \le m + 1;$$

secondly, by the inequality between geometric and arithmetic means (cf. *Elements VI.27*)

$$(n-m)(m+1-n) \le \frac{1}{4},$$

with equality if and only if $n = m + 1/2$; and thirdly

$$(m+1)d_1 + m d_2 > m(d_1 + d_2) \ge 3m^2(m+1).$$

Hence, putting these ingredients together, we conclude that

(33) $$\left| \frac{(m+1)^2 d_1 + m^2 d_2}{(m+1)d_1 + m d_2} - n \right| < \frac{1}{12m^2},$$

of comparable order of magnitude to the bound $3/(80m^2)$ that Smyly tells us in [26] had been obtained by Webb. Another elementary bound is proved in [6, Theorem 3], but on the interval $(m, m+1)$ is is weaker than (33).

5.3 *Synagogue III*: Two mean proportionals

Pappus musters in *Synagogue III* a collection of constructions of two mean proportionals between two line segments by non-planar means. Perhaps by way of cautionary prologue, he also describes a geometrical solution, purportedly by plane considerations only, from some unnamed source, specifically with a view to showing that it fails. The flaws in the construction are fairly transparent, and Pappus' demolition of them is not especially edifying. However, for all the imperfections Pappus would have us see in it, the construction is not without other merits. Knorr offers a sensitive geometrical re-appraisal at some length in [21, pp. 64–70]; more recently, Serafina Cuomo has returned to the construction in a study [4, §4.1] of Pappus' mathematics in the setting of Late Antiquity. Earlier attempts at rehabilitating the construction tended to recast it as an iterative scheme of approximation to the mean proportionals, using an algebraic notation alien to the spirit of Pappus' *Synagogue*. Nevertheless, what we might notice about this algebra for our present purposes is how well it meshes with the family of numerical examples in §2 generalising Heron's case, $N = 100$, in *Metrica III.20*.

In this regard, the pioneering effort was made by Richard Pendlebury (1847–1902; Senior Wrangler, 1870) in a note [23] published in 1873, as reported in [15, I, pp. 268–270] (see further [21, p. 64, fn. 8]; [4, p. 130]). Suppose that $N = m^3 - lm^2$,

for some l and m, then Pendlebury shows that iteration of the construction faulted by Pappus in *Synagogue III* can be generalised as a recursive computation,

$$(34) \qquad n_{i+1} = m - \frac{(m - n_i)lm^2}{m^3 - n_i^3},$$

for some given n_0, with the n_i successively better approximations to the cube root of N, giving upper bounds when n_0 is bigger than this cube root, and lower bounds when it is smaller.

Now, the family of N in §2 generalising Heron's example is given by taking $l = 1$. If we start with our Heronian upper bound (5),

$$n_0 = m - \frac{m}{3m - 1} = m(1 - \frac{1}{3m - 1}),$$

then (34) gives the improved upper bound

$$(35) \qquad n_1 = m - \frac{(3m - 1)^2}{3(3m - 1)(3m - 2) + 1}.$$

In particular, for Heron's example, $N = 100$ is the case $m = 5$, when (35) yields

$$(36) \qquad n_1 = 5 - \frac{196}{547} = 4\frac{351}{547},$$

an improvement on Heron's upper bound $4\frac{9}{14}$ for the cube root of 100.

In this exercise, we may be scrabbling after crumbs, waiting for a spark from heaven to fall. This particular construction never seems to have attracted much attention until analysed by Pendlebury, although Leonardo Pisano and Gerolamo Cardano retained geometrical accounts of second mean proportionals in their discussions of cube root extraction. But, over the course of countless Greek lives, there was presumably time for many other failed constructions and, in amongst them, some near-misses, possibly the occasional success — after all, we still have Archimedes' *On the Measurement of a Circle*.

5.4 Rescaling

None of the ingredients we use in producing (33) could reasonably be said to be beyond the competence of the ancient Greek mathematicians, and yet we would naturally hesitate when it comes to an error bound like (33) itself. Nevertheless, if we do have a sense that the going gets better, however we might come by it, we can always try rescaling. Thus, to estimate the cube root in Heron's example, $N = 100$, we might divide the estimate from (1) for the cube roots, say, of 800 or 2700 by 2 or 3 respectively to get

$$4\frac{322}{502}; \quad 4\frac{7328}{11421};$$

the first of these estimates is a lower bound not as close to the cube root of 100 as the upper bound in (36) while the second is an upper bound improving on that in (36).

Of course, (1) is most in error for some small values of N. About the worst offender proportionately is $N = 5$, when the estimate from (1) is $1\frac{8}{11}$, with a cube greater than 5.153. It is here that we can use rescaling to good advantage. Amusingly enough, if we divide the estimates from (1) for 40 or 135 by 2 or 3 respectively, we come out with the same lower bound for the square root of 5, namely $1\frac{22}{31}$, with a cube greater than 4.997. Going further and dividing the estimate from (1) for 320 by 4 gives the upper bound $1\frac{615}{866}$, with a cube now less than 5.002.

Maybe there is some redemption to be found here, too, for the comparatively weak upper bound for square roots in (13), because, if we continue with the algebra there, we find that the diminution in the error is on the order of $1/m$. For example, Heron, in *Metrica I.9*, wants to compute $\sqrt{1575}$ and notes he can get at this as $\frac{10}{2}\sqrt{63}$, offering the upper bound $7\frac{15}{16}$ for $\sqrt{63}$, either by Heronian iteration as in *Metrica I.8* or possibly as an application of (15) (cf. *Stereometrica I.33*). Of course, if we stick with the same method and use it to approximate $\sqrt{1575}$ directly we come out with the same estimate either way. However, as it so happens, Heron also alludes to $\sqrt{1575}$ in passing as "*the square root of the fourth part of 6300*" (cf. [3, p. 203]). But, if we divide the estimate of $\sqrt{6300}$ from (13) by 10, we obtain a (slightly) improved upper bound: $7\frac{1183}{1262}$. Similarly, when Heron wants an approximation for $\sqrt{720}$ in *Metrica I.8*, his first estimate is the upper bound $26\frac{5}{6}$, whereas working (13) with $72,000$ improves this to $26\frac{30002}{36023}$.

Then, again, in any practical example, the convenience of working with an estimate may outweigh its accuracy, so such gains are largely a matter of theory. Moreover, elsewhere, in *Geometrica 53, 54* (cf. [15, II, p. 321], when dealing with the 4-6-8 triangle, Heron seems to show some awareness that gains can be made from delay in the taking of square roots, initially proposing a_1, an upper bound with

$$N = 4\sqrt{8\frac{7}{16}} < 11\frac{2}{3} = a_1,$$

but then, on rewriting N by multiplying into the square root, observing that we can do better using a_2, with

$$N = \sqrt{135} < 11\frac{13}{21} = a_2.$$

Typically, nothing is said about the derivation of these bounds. Interestingly enough though, Heronian iteration, as in (15), applied to N gives $11\frac{5}{8}$, which falls in between the two bounds,

$$a_1 = 11\frac{2}{3} > 11\frac{5}{8} > 11\frac{13}{21} = a_2;$$
(37)

a_1 results on applying Heronian iteration, or (15), to $\sqrt{136} = 4\sqrt{8\frac{1}{2}}$; and a_2 improves on a_1 precisely by Heronian iteration,

$$(38) \qquad a_2 = \frac{1}{2}\left(\frac{135}{a_1} + a_1\right).$$

A possible alternative derivation of a_1, in line with Heron's handing of $\sqrt{75}$ noted in §3.3, might be to stick with Heronian iteration in the form (14) for N, giving a less good upper bound $11\frac{14}{22}$, which, however, encourages nudging up to the simpler fraction a_1. But all of this is speculative, and those who enjoy numerical coincidences will be amused to see the early Fibonacci numbers showing up in (37), still more perhaps to learn that these bounds are the 4th, 6th and 8th convergents of the continued fraction for $\sqrt{135}$. Notice, however, that Heronian iteration with the middle bound in (37) yields $11\frac{307}{496}$, which does improve on a_2, if only just.

Thus, it is uncertain whether the improvement Heron notes here derives from *his* rescaling *per se* or from a change in the method of approximation. Indeed, (38) may run slightly counter to the view in [15, II, p. 326] on Heron's own use of Heronian iteration, while leaving it a mystery as to how he obtained bounds that improve on a first instance of the method. Something similar might be at work in the handling of $\sqrt{28}$ as discussed in [3, p. 309]. In this case, we might expect the bound $5\frac{3}{10}$ (cf. (14)), but the weaker bound $5\frac{1}{3}$ (cf. (15)) lends itself more easily to improvement by Heronian iteration, giving $5\frac{7}{24}$. However, what might require us to rethink, or at least re-express, the matter is the observation that rescaling combined with (15) does allow us to give the supposedly *improved* bounds in both cases more directly:

$$(39) \qquad \sqrt{28} = \frac{1}{3}\sqrt{252} < \frac{1}{3}\left(16 - \frac{4}{32}\right) = \frac{1}{3}\left(15\frac{7}{8}\right) = 5\frac{7}{24};$$

$$(40) \qquad \sqrt{135} = \frac{1}{3}\sqrt{1215} < \frac{1}{3}\left(35 - \frac{10}{70}\right) = \frac{1}{3}\left(34\frac{6}{7}\right) = 11\frac{13}{21}.$$

Fortunately, under Taisbak's dispensation, we are not so pressed to account for the rather *weak* estimates Heron also uses on occasion, as, for example, $43\frac{1}{3}$ for $\sqrt{1875}$ or $14\frac{1}{3}$ for $\sqrt{207}$, the former squaring to more than 1877, the latter to less than 206 (see [15, II, pp. 326, 328]).

Smyly [26, p. 67], in extolling the virtues of (1) for N of the order of 10^6 in comparison with tables of seven-figure logarithms, and Knorr [20, p. 192], in dilating on iterative use of (1), possibly overlook this simple trick of rescaling to obtain improved estimates for smaller N. Scaling, in the elementary sense of the law of indices, is one thing; the notion of returns to scale another, rather more subtle. Some accounts of Greek approximations for $\sqrt{2}$ and $\sqrt{3}$ would have us believe that the Greeks were great self-improvers, working their way to better estimates through solutions of Pell equations or the convergents of continued fractions which might be seen as implicitly involving a form of rescaling (indeed, not unlike (39) and (40)). Taisbak [28, §3] asks in regard to his conjecture whether the Ancients knew and used sequences of differences. With an eye to (39) and (40), we follow suit: did the Ancients know and use rescaling?

6   A last reckoning

Numerical corroboration, of course, might not be to everyone's taste. Bartel Le-
endert van der Waerden (1903–1996), for one, in the original Dutch edition of
Ontwakende Wetenschap (*Science Awakening*) [32, p. 306], in 1950, places Heron
in heavily weighted scales.

> *Laten we blij zijn, dat we de meesterwerken van Archimedes en Apollonios
> hebben, en niet treuren om het verlies van talloze rekenboekjes à la Heron.*
>
> *[Let us rejoice in the masterworks of Archimedes and of Apollonius and not
> mourn the loss of numberless little accounting books after the manner of Her-
> on.]*

The translation in English in 1954 is less pointed, but, recalling Heron's own mat-
hematical outlook as expressed in the preface to *Metrica*, it is likely that he could
at least hold his own (cf. [10]).

Litteratur

[1] Anonymous Reviewer, Shipwrecks of the Alceste and the Medusa, *The Edin-
burgh Review, Or Critical Journal*, **30** (1818), 388–408.

[2] O. Becker, *Das matematische Denken der Antike* (Vanderhoeck & Ruprecht,
Göttigen, 1957; 2nd ed., 1966).

[3] E. M. Bruins, *Codex Constantinopolitanus Palatii Veteris No. 1, Pt. 3: Trans-
lation and Commentary* (Brill, Leiden, 1964).

[4] S. Cuomo, *Pappus of Alexandria and the Mathematics of Late Antiquity* (Cam-
bridge University Press, Cambridge, 2007).

[5] E. L. W. M. Curtze, Quadrat- und Kubikwurzeln bei din Griechen nach Her-
ons neu aufgefundenen *Metrica*, *Z. für Math. und Physik*. **42** (1897) Hist.-lit.
Abth., 113–120.

[6] G. Deslauriers and S. Dubuc, Le calcul de la racine cubique selon Héron,
Elemente der Mathematik, **51** (1996), 28–34.

[7] W. C. Eells, Greek methods of solving quadratic equations, *Amer. Math. Mont-
hly*, **18** (1911), 3–14.

[8] G. H. Eneström, Note upon the history of the rules of convergence in the
eighteenth century, *Bull. New York Math. Soc.*, **3** (1894), 186–187.

[9] G. H. Eneström, Kleine Bemerkungen zur letzten Auflage von Cantors " *Vor-
lesungen über Geschichte der Mathematik*, I^3; 374. *Bibliotheca Math., 3rd.
ser.*, **8** (1907–1908), 412–413.

[10] J. Feke, Meta-mathematical rhetoric: Hero and Ptolemy against the philosop-
hers, *Historia Math.*, **41** (2014), 261–276.

[11] H. G. Flegg, *Numbers: Their History and Meaning* (Schocken Books, New York, NY, 1983; reissued Dover, Mineola, NY, 2002; 2013).

[12] J. Gow, *A Short History of Greek Mathematics* (Cambridge University Press, Cambridge, 1884).

[13] K.-G. Hagström, Gustaf Eneström, *Nordisk Matematisk Tidskrift*, **1** (1953), 145–155.

[14] T. L. Heath, *The Works of Archimedes, edited in modern notation with introductory chapters. With a supplement, The Method of Archimedes, recently discovered by Heiberg* (Cambridge University Press, Cambridge, 1897; 2nd ed., with supplement, 1912; reprt., Dover Pub., New York, NY, 1953)

[15] T. L. Heath, *A History of Greek Mathematics* (Clarendon Press, Oxford, 1921).

[16] T. L. Heath, *A Manual of Greek Mathematics* (Oxford University Press, Oxford, 1931).

[17] A. Heeffer, Was Uncle Tom right that quadratic problems can't be solved with the rule of false position? *Mathematical Intelligencer*, **36** (2014), No. 3, 65–69.

[18] A. Holme, *Geometry: Our Cultural Heritage* (Springer, New York, NY, 2010).

[19] B. Hughes, ed., *Fibonacci's* De Practica Geometrie (Springer, New York, NY, 2008).

[20] W. R. Knorr, *The Ancient Tradition of Geometric Problems* (Birkhäuser, Boston, MA, 1986).

[21] W. R. Knorr, *Textual Studies in Ancient and Medieval Geometry* (Birkhäuser, Boston, MA, 1989).

[22] H. L'Huillier, Concerning the method employed by Nicolas Chuquet for the extraction of cube roots, *in* C. Hay, ed., Mathematics from Manuscript to Print (Clarendon Press, Oxford, 1988), pp. 89–95.

[23] R. Pendlebury, On a method of finding two mean proportionals, *Messenger of Mathematics, ser., 2*, **2** (1873), 166-169.

[24] H. Schöne, ed., *Heronis Alexandrini Opera Quae Supersunt Omnia, III* (B. G. Teubner, Leipzig, 1903).

[25] D. E. Smith, *History of Mathematics, Vol. 2* (Ginn, Boston, MA, 1925; rev. ed. reprt., Dover Pub., New York, NY, 1958).

[26] J. G. Smyly, Heron's formula for cube root, *Hermathena*, **19** (1920), No. 42, 64–67.

[27] J. G. Smyly, Some examples of Greek arithmetics, *Hermathena*, **19** (1920), No. 42, 105–114.

[28] C. M. Taisbak, Cube roots of integers. A conjecture about Heron's method in *Metrika III.20*, *Historia Math.*, **41** (2014), 103–104.

[29] C. M. Taisbak, personal communication, 19 May, 2014.

[30] I. Thomas, *Selections illustrating the History of Greek Mathematics, I: From Thales to Euclid. Loeb Classical Library, 335* (Wm. Heinemann, London; Harvard University Press, Cambridge, MA, 1939).

[31] S. Unguru, C. M. Taisbak (Marinus), historian of Greek mathematics and friend: on his eightieth birthday, in *AIGIS Supplementum III 2014*, available at ⟨http://aigis.igl.ku.dk/CMT80/Forside.html⟩.

[32] B. L. van der Waerden, *Ontwakende Wetenschap. Egyptische, Babylonische en Griekse Wiskunde* (P. Noordhoff, Groningen, 1950); translated as, *Science Awakening* (P. Noordhoff, Groningen, 1954).

[33] G. Wertheim, Herons Ausziehung der irrationalen Kubikwurzeln, *Z. für Math. und Physik*, **44** (1899), hist.-lit. Abth., 1–3.