Fully dispersive water wave equations

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Abstract

The PhD project concerns the surface water wave theory. Liquid is presented as a three or two dimensional layer bounded from below by a rigid horizontal bottom. Above it can have either a free surface or an elastic layer such as an ice cover, for example. The fluid flow is considered to be inviscid, incompressible and irrotational. The flow is described by the Euler equations with some appropriate boundary conditions. Following Lagrange we assume that at the bottom liquid has zero vertical velocity component and at the free surface there is a constant atmospheric pressure. The first condition is very natural meaning that water cannot penetrate through the bottom and is always attached to it making no cavities. The second condition is based on the fact that the air fluctuations are negligible and the pressure is defined by the weight of the atmosphere. Note that it also assumed that the pressure is changing continuously in the air-water media. The latter can be modified in different ways. For example, assuming a strong capillarity effect one can assume that the liquid pressure at the top is proportional to the surface curvature. However, it turns out that this effect is more important in water tanks than in the ocean. Another more relevant modification for the real world is the modelling of ice cover. In such situation the surface tension is caused by deformation forces in the ice. The latter is assumed to be elastic.

Solving the Euler equations with the mentioned boundary conditions provides with the complete description of the flow dynamics. However, in many situations it is more important to know only the surface time evolution, whereas solving the full problem may be too demanding. Different nonlinear dispersive wave equations, such as the Korteweg-de Vries equation, allow to approximate the free surface dynamics without providing with the complete description of the fluid motion below the surface. There are several ways of testing the validity of these models. The two most important of them are tests on well-posedness and travelling wave existence. A relevant model should at least reflect adequately these two properties.

In this work we are mainly concerned with the fully dispersive bi-directional extensions of the mentioned KdV equation. Still being simple toy models they are believed to be better approximations to the full Euler equations. Moreover, they allow two directional water motion for the two dimensional liquid layer, whereas the scalar KdV equation describes the waves traveling in one direction. A recent interest in such models was caused by discovery of certain phenomena in the Whitham equation that is a direct fully dispersive one-directional extension of the KdV equation. Among them are solitary waves, the existence of a wave of greatest height predicted by Stokes, the existence of shocks and modulational instability of steady periodic waves. A natural step forward is to try to find an adequate two-directional extension of the scalar Whitham equation displaying the same phenomena or some other water wave properties.

In this project we have an overview of several such models recently put forward.

We pay a special attention to one particular system introduced by the author. We analyse situations demanding the use of such models, testing them with different surface boundary conditions, as for example, putting ice cover atop. Consideration is given to the initial value problem and solitary wave existence. We simulate periodic travelling waves for these models and also for one exact model the so-called Babenko equation, that is also an example of a fully dispersive model, in particular. The thesis is based on ten papers submitted during the work on the project, with eight of them being included in the text.

Contents

Chapter 1

Water waves

1.1 Introduction

We start with a short reminder of the very basic concepts of Hydrodynamics [4, 56]. Fluid flow in some domain Ω is governed by the fundamental conservation laws and boundary conditions. In our framework Ω represents a layer bounded from below by the flat bottom $z = -h$. The upper boundary is called a free surface. It is described by $z = \eta(x, y, t)$. The positive finite constant h is called an undisturbed depth. This notion needs some clarification, since the depth at each point $(x, y) \in \mathbb{R}^2$ at time moment $t \in \mathbb{R}$ equals $h + \eta(x, y, t)$, and so it is not definite. It will be convenient for us below to work with L^2-based Sobolev spaces $H^s(M)$ where $M = \mathbb{R}^d$ or $\mathbb{T}^d$ with $d = 1, 2$. For this reason we define the undisturbed depth h in a way that $\eta \to 0$ at infinity for the non-periodic domain Ω and $\int \eta dxdy = 0$ over a period otherwise. Such restriction gives us a reference length that will be convenient for approximations below. We use standard notations for the density ρ, pressure p, velocity $\mathbf{v} = (v_1, v_2, v_3)$ and the gravity acceleration $\mathbf{g} = (0, 0, -g)$.

The first fundamental principal is the mass conservation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$

that in the case of incompressible fluid takes the form

$$\nabla \cdot \mathbf{v} = 0. \tag{1.1}$$

The second principal is the momentum conservation (Newton's law)

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{g} - \frac{\nabla p}{\rho} \tag{1.2}$$

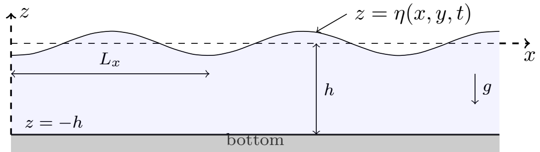

Figure 1.1: The domain of the fluid flow: longitudinal cross-section of a rectangular channel, η is the free surface, L_x is the wavelength, h is the undisturbed depth, g is the gravity.

where it is assumed that the liquid is inviscid. These two equations were put forward by Euler in 1755. Then Lagrange complemented them in 1781 by the boundary conditions

$$v_3 = 0 \quad \text{at } z = -h, \tag{1.3}$$

$$\partial_t \eta = v_3 - v_1 \partial_x \eta - v_2 \partial_y \eta, \quad p = p_0 \quad \text{at } z = \eta(x,y,t). \tag{1.4}$$

Here $p_0 = \text{const}$. Condition (1.3) reflects the fact that the bottom is impenetrable. The second surface boundary condition (1.4) means that the pressure at the top of the liquid coincides with atmospheric pressure assumed to be constant. The first equation of (1.4) is in fact the velocity definition of surface particles. Indeed, the material derivative in Euler coordinates is $\partial_t + (\mathbf{v}\cdot\nabla)$. To find the velocity $\mathbf{v}$ of liquid particles at the surface one has to apply it to the vector field $(x,y,\eta(x,y,t))$ describing the surface location. In particular, $v_3 = \partial_t \eta + (\mathbf{v}\cdot\nabla)\eta$ that is the first equation of (1.4).

Equations (1.1)-(1.4) represent the full gravity water wave problem. In some situations it is desirable to take into account surface tension effect [56]. Then the surface condition $p = \text{const}$ is substituted by the more general

$$p = p_0 = \text{const} - \rho g h^2 \varkappa \nabla \cdot \frac{\nabla \eta}{\sqrt{1+|\nabla\eta|^2}} \quad \text{at } z = \eta(x,y,t), \tag{1.5}$$

where it is assumed that the capillary pressure is proportional to the free surface curvature. The full gravity-capillary water wave problem consists of Equations (1.1)-(1.5). It can be significantly simplified admitting existence a velocity potential ϕ such that

$$\mathbf{v} = \nabla\phi$$

implying the flow to be irrotational. The complete problem reduces to the Laplace's equation

$$\Delta\phi = 0 \quad \text{in } \Omega, \tag{1.6}$$

the Neumann boundary condition at the flat bottom

$$\partial_z \phi = 0 \quad \text{at } z = -h, \tag{1.7}$$

the kinematic condition at the free surface

$$\partial_t \eta = \partial_z \phi - \partial_x \phi \partial_x \eta - \partial_y \phi \partial_y \eta \quad \text{at } z = \eta(x,y,t), \tag{1.8}$$

and the Bernoulli equation combined with the surface tension

$$\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + g\eta - g h^2 \varkappa \nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}} \right) = 0 \quad \text{at } z = \eta(x,y,t). \tag{1.9}$$

The latter is a consequence of the momentum conservation (1.2) restricted to the free surface. Note that we do not regard the Bernoulli equation inside of the fluid domain Ω, since it will only give us the pressure distribution provided the rest of the problem is solved.

System (1.6)-(1.9) poses a conserved quantity having the meaning of the total energy

$$\mathcal{H} = \int_{\mathbb{R}^2}\int_0^\eta gz\,dzdxdy + \frac{1}{2}\int_{\mathbb{R}^2}\int_{-h}^\eta |\nabla\phi|^2\,dzdxdy + \frac{\varkappa g h^2}{2}\int_{\mathbb{R}^2}\frac{|\nabla\eta|^2}{1+\sqrt{1+|\nabla\eta|^2}}dxdy \tag{1.10}$$

that is the sum of the potential energy due to gravity, the kinetic energy and the surface tension energy. In the periodic case the domain of integration $\mathbb{R}^2$ should be substituted by $\mathbb{T}^2$.

Our main concern is formation and propagation of surface waves. For this reason Equations (1.6)-(1.9) are restricted to the surface as follows. Firstly, we introduce the trace of the potential at the free surface as $\Phi(x,y,t) = \phi(x,y,\eta(x,y,t),t)$. With the elliptic problem (1.6), (1.7) and $\phi = \Phi$ on the surface we associate the Dirichlet-Neumann operator $G(\eta)$ by the formula

$$G(\eta)\Phi = \sqrt{1+|\nabla\eta|^2}\partial_n\phi \tag{1.11}$$

where $\partial_n\phi$ is the projection of the surface fluid velocity on the outer normal. For the more detailed definition of $G(\eta)$ taking into account the appropriate periodic or asymptotic conditions on ϕ we refer to [2, 44]. Note that the right hand side of (1.11) coincides with the right hand side of (1.8). The full system (1.6)-(1.9) is reduced to

$$\partial_t\eta = G(\eta)\Phi, \tag{1.12}$$

$$\partial_t\Phi = -g\eta + gh^2\varkappa\nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right) - \frac{1}{2}|\nabla\Phi|^2 + \frac{(\nabla\eta\cdot\nabla\Phi + G(\eta)\Phi)^2}{2(1+|\nabla\eta|^2)} \tag{1.13}$$

with a slight abuse of notation $\nabla = (\partial_x, \partial_y)$. A pair (η,Φ) solving System (1.12)-(1.13) describes the surface waves completely. An obvious drawback of this formulation is that the dependence of the Dirichlet-Neumann operator on the surface elevation η is implicit. As was shown by Zakharov [59], System (1.12)-(1.13) enjoys the Hamiltonian structure

$$\partial_t\eta = \frac{\delta\mathcal{H}}{\delta\Phi}, \quad \partial_t\Phi = -\frac{\delta\mathcal{H}}{\delta\eta} \tag{1.14}$$

with the total energy (1.10) serving as the Hamiltonian

$$\mathcal{H}(\eta,\Phi) = \frac{1}{2}\int_{\mathbb{R}^2}\left[g\eta^2 + \Phi G(\eta)\Phi + \frac{\varkappa g h^2|\nabla\eta|^2}{1+\sqrt{1+|\nabla\eta|^2}}\right]dxdy. \tag{1.15}$$

One can simplify the water wave problem further approximating the Dirichlet-Neumann operator by different explicit expressions.

1.2 Linear wave theory

It is a matter of common knowledge that ocean waves are small and long with respect to the water depth h. So the simplest possible approximation of (1.12)-(1.13) is the linearisation

$$\partial_t\eta = G_0\Phi, \tag{1.16}$$

$$\partial_t \Phi = -g\eta + gh^2 \varkappa \Delta \eta \tag{1.17}$$

where $G_0 = G(0)$ is the Dirichlet-Neumann operator corresponding to the undisturbed surface $\eta = 0$. We will provide with an exact expression for G_0.

Firstly, we remind the notion of Fourier multipliers. Let $\mathcal{S}'(M)$ with $M = \mathbb{R}^d$ or $\mathbb{T}^d$ and $d = 1, 2$ be the space of tempered distributions. The Fourier transform is defined by the formula

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_M f(x) e^{-i\xi \cdot x} dx$$

on Schwartz functions. By the Fourier multiplier operator $\varphi(D)$ with symbol $\varphi \in C^\infty(M)$ we mean the line $\mathcal{F}(\varphi(D)f) = \varphi(\xi)\widehat{f}(\xi)$ for any $f \in \mathcal{S}'(M)$. In particular, $D_1 = -i\partial_x$ is the Fourier multiplier associated with the symbol $\varphi(\xi) = \xi_1$.

For any given potential trace Φ the elliptic problem (1.6) with $\Omega = M \times (-h, 0)$, (1.7) and $\phi = \Phi$ at $z = 0$ can easily be solved in the frequency space. Indeed, defining $\widehat{\phi}(\xi, z)$ to be the Fourier transform of ϕ with respect to the horizontal variables one obtains

$$\begin{cases} \partial_z^2 \widehat{\phi}(\xi, z) = |\xi|^2 \widehat{\phi}(\xi, z) \quad \text{for } -h < z < 0, \\ \partial_z \widehat{\phi}(\xi, -h) = 0, \\ \widehat{\phi}(\xi, 0) = \widehat{\Phi}(\xi) \end{cases}$$

that is a second-order ODE in z with boundary conditions at $z = -h$ and $z = 0$. It has a unique solution given by

$$\widehat{\phi}(\xi, z) = \frac{\cosh((h+z)|\xi|)}{\cosh(h|\xi|)} \widehat{\Phi}(\xi)$$

where ξ lies in the frequency space and $z \in [-h, 0]$. Therefore by the definition of the Dirichlet-Neumann operator (1.11) we have $G_0\Phi = \partial_z \phi$ at $z = 0$ and so

$$\mathcal{F}(G_0\Phi)(\xi) = \partial_z \widehat{\phi}(\xi, 0) = |\xi| \tanh(h|\xi|) \widehat{\Phi}(\xi).$$

In other words, G_0 is a Fourier multiplier operator given by the formula

$$G_0 = |D| \tanh(h|D|) \tag{1.18}$$

where we use the notation $D = -i\partial_x$ if $d = 1$ and $D = -i\nabla = -i(\partial_x, \partial_y)$ if $d = 2$.

With the explicit expression (1.18) one can easily solve the linear Cauchy problem (1.16)-(1.17) with $\eta(0) = \eta_0 \in \mathcal{S}'$, $\Phi(0) = \Phi_0 \in \mathcal{S}'$ as

$$\eta(t) = \cos \omega t \eta_0 + G_0 \frac{\sin \omega t}{\omega} \Phi_0, \tag{1.19}$$

$$\Phi(t) = -g(1 + \varkappa h^2 |D|^2) \frac{\sin \omega t}{\omega} \eta_0 + \cos \omega t \Phi_0, \tag{1.20}$$

where $\omega = \omega(D)$ is the Fourier multiplier operator defined by the symbol

$$\omega(\xi) = \sqrt{g(1 + \varkappa h^2 |\xi|^2)|\xi| \tanh(h|\xi|)} \tag{1.21}$$

that is called the water wave dispersion relation. Note that η satisfies a wave type equation

$$\partial_t^2 \eta + \omega^2 \eta = 0$$

with frequency variable speed. Introducing the wave celerity

$$c(|\xi|) = \frac{\omega(\xi)}{|\xi|} = \sqrt{g(1 + \varkappa h^2 |\xi|^2) \frac{\tanh h|\xi|}{|\xi|}} \tag{1.22}$$

one can notice that $c'(|\xi|) < 0$ after setting $\varkappa = 0$. This phenomena results in the fact that the linear solution always disperse. The phase speed increases with the wave length approaching maximum $c_0 = \sqrt{gh}$.

Dispersion of linear waves makes it impossible to explain the so called solitary waves (or localised waves of translation) discovered by Russell in 1834, staying only in the framework of the linear theory. Moreover, neglecting dispersion by assuming waves to be long and substituting $\omega(D)$ with the long wave limit $c_0^2 |D|^2$ one arrives at the wave equation

$$\partial_t^2 \eta - c_0^2 \Delta \eta = 0$$

where $\Delta = -|D|^2 = \partial_x^2 + \partial_y^2$. It comes across with the fact that the height of a solitary wave depends on its speed, whereas this cannot be the case with the linear wave equation.

1.3 Nonlinear long waves

Nonlinear long wave approximation of the water wave problem (1.12)-(1.13) can be done through expansion of the Dirichlet-Neumann operator (1.11) developed in [20, 22]. It was shown in [49] that this operator depends analytically on the unknown η and can therefore be expanded in the power series

$$G(\eta)\Phi = \sum_{j=0}^{\infty} G_j(\eta)\Phi,$$

where each operator $G_j(\eta)$ is homogeneous of degree j in η. The first term G_0 of zero order was computed above in (1.18). The next term has the form

$$G_1(\eta)\Phi = D \cdot (\eta D\Phi) - G_0(\eta G_0 \Phi) \tag{1.23}$$

and the rest terms in the power series can be computed using a recursion formula [19, 21].

In the long wave framework it is generally assumed that the surface displacement and velocity are comparable with respect to smallness, at least after nondimensinal transformation $\eta \mapsto \eta/h$, $\mathbf{v} \mapsto \mathbf{v}/c_0$, see [6, 45], for example. For this reason we introduce the surface velocity variable $\mathbf{u} = \nabla\Phi = (\phi_x + \eta_x \phi_z, \phi_y + \eta_y \phi_z)$. Note that by the definition the new velocity $\mathbf{u}$ is curl free. It will be also convenient to introduce the operator

$$K = \sqrt{\frac{\tanh(h|D|)}{h|D|}}. \tag{1.24}$$

Note that $G_0\Phi = -hK^2\nabla\cdot\mathbf{u}$. We neglect cubic and higher order terms in $(\eta,\mathbf{u})$ in order to approximate formally (1.12)-(1.13) by the system

$$\partial_t\eta = -hK^2\nabla\cdot\mathbf{u} - \nabla\cdot(\eta\mathbf{u}) + hG_0(\eta K^2\nabla\cdot\mathbf{u}), \tag{1.25}$$

$$\partial_t\mathbf{u} = -g\left(1+\varkappa h^2|D|^2\right)\nabla\eta - \frac{1}{2}\nabla|\mathbf{u}|^2 \tag{1.26}$$

which can be found in [45]. Since we neglected the third order terms one could anticipate a good agreement with the full problem (1.12)-(1.13), however, numerical simulations carried out in [27] suggest that this system is probably ill-posed and so is not a relevant asymptotic model.

In order to proceed we need to make some assumptions on relations between wave amplitudes and lengths. Regard a wave-field with a characteristic non-dimensional amplitude $\alpha = \mathfrak{a}/h$. We suppose that a characteristic wavelength L_x along the x-axis is of the same order as L_y along the y-axis. Define the small parameter $\mu = h/L_x$. The squared value μ^2 is often referred as the shallowness parameter, whereas $\mu\alpha$ is called steepness [44]. A particular scaling regime is defined by assuming a dependence of α on μ, while $\eta = h\mathcal{O}(\alpha)$, $\mathbf{u} = \sqrt{gh}\mathcal{O}(\alpha)$ and $hD = \mathcal{O}(\mu)$. The latter means that the Fourier transformations of $\eta,\mathbf{u}$ are localised close to the origin, so the absolute value of frequencies involved do not exceed μ. One of the mostly used scaling is the Boussinesq regime $\alpha = \mathcal{O}(\mu^2)$.

Using Taylor expansions of symbols corresponding to G_0 and K in the first equation (1.25) one obtains

$$\partial_t\eta = -h(1-\frac{1}{3}h^2|D|^2)\nabla\cdot\mathbf{u} - \nabla\cdot(\eta\mathbf{u}) + \mathcal{O}(\mu^7)$$

in the Boussinesq regime, and so discarding the error term $\mathcal{O}(\mu^7)$ we arrive at the following Boussinesq system

$$\partial_t\eta = -h\left(1+\frac{1}{3}h^2\Delta\right)\nabla\cdot\mathbf{u} - \nabla\cdot(\eta\mathbf{u}), \tag{1.27}$$

$$\partial_t\mathbf{u} = -g\left(1-\varkappa h^2\Delta\right)\nabla\eta - \frac{1}{2}\nabla|\mathbf{u}|^2 \tag{1.28}$$

appeared in [24]. In the purely gravity case $\varkappa = 0$ it is also known as the integrable Boussinesq system [55]. Note that from the definition of velocity $\mathbf{u}$ and Equation (1.8) follows

$$\mathbf{u} = (v_1,v_2) + (\partial_t\eta + (v_1,v_2)\cdot\nabla\eta)\nabla\eta = (v_1,v_2) + c_0\mathcal{O}(\mu^6),$$

and so in the Boussinesq regime we can identify $\mathbf{u}$ and the surface horizontal velocity (v_1,v_2).

System (1.27)-(1.28) is a particular case of the following four-parameter Boussinesq system

$$\partial_t\eta = -h\left(1+ah^2\Delta\right)\nabla\cdot\mathbf{u} + bh^2\Delta\partial_t\eta - \nabla\cdot(\eta\mathbf{u}), \tag{1.29}$$

$$\partial_t\mathbf{u} = -g\left(1+(c-\varkappa)h^2\Delta\right)\nabla\eta + dh^2\Delta\partial_t\mathbf{u} - \frac{1}{2}\nabla|\mathbf{u}|^2 \tag{1.30}$$

introduced in [6, 9] without the surface tension $\varkappa = 0$. Here $\mathbf{u} = \mathbf{u}(x,y,t)$ is the horizontal velocity taken at some height in the fluid domain Ω, and a,b,c,d are modelling parameters satisfying the constraint $a+b+c+d = 1/3$. Such three degrees of freedom arise from the choice of height at which the horizontal velocity is taken and from the double use of the BBM trick [5].

In order to justify rigorously Equations (1.29)-(1.30) one has to include the shallowness parameter μ^2 explicitly. It can be done by introducing non-dimensional variables via changing $x,y \mapsto L_x x, L_x y$, $t \mapsto (L_x/c_0)t$, $\eta \mapsto \alpha\eta$, $\mathbf{u} \mapsto (g\alpha/c_0)\mathbf{u}$. This leads to

$$\partial_t \eta = -\left(1 + a\mu^2\Delta\right)\nabla\cdot\mathbf{u} + b\mu^2\Delta\partial_t\eta - \mu^2\nabla\cdot(\eta\mathbf{u}), \tag{1.31}$$

$$\partial_t \mathbf{u} = -\left(1 + (c-\varkappa)\mu^2\Delta\right)\nabla\eta + d\mu^2\Delta\partial_t\mathbf{u} - \frac{1}{2}\mu^2\nabla|\mathbf{u}|^2 \tag{1.32}$$

where we have set $\alpha = \mu^2$. Note that we have a balance here between dispersion and non-linearity. Well-posedness of this system was investigated in [8, 53] establishing long time $T = \mathcal{O}(\mu^{-2})$ of existence for some particular restrictions on the modelling parameters a,b,c,d. Consistency with the full water wave problem (1.12)-(1.13) with the optimal error estimate $\mathcal{O}(\mu^4 t)$ was proved in [9]. Note that the full problem is well-posed [44] on the time scale $\mathcal{O}(\mu^{-2})$.

In the one dimensional case $d = 1$ assuming waves travelling in one direction the Boussinesq system (1.27)-(1.28) can approximately, staying in the same framework of accuracy, be reduced to the scalar KdV equation

$$\partial_t\eta + c_0\partial_x\eta + (1-3\varkappa)\,\frac{c_0 h^2}{6}\partial_x^3\eta + \frac{3c_0}{2h}\eta\partial_x\eta = 0 \tag{1.33}$$

firstly introduced by Boussinesq in 1871 and then later in 1895 by Korteweg and de Vries. It is known to be globally well-posed in H^s with $s \geqslant 1$ (see [41], for instance). This is a quasilinear equation, which in particular, results in the fact that for a proof of the well-posedness one might need to study a regularized problem first, and then investigate convergence of solutions of the regularized Cauchy problem [10]. In their proof Bona and Smith have used the BBM regularization

$$\partial_t\eta + c_0\partial_x\eta - (1-3\varkappa)\,\frac{h^2}{6}\partial_x^2\partial_t\eta + \frac{3c_0}{2h}\eta\partial_x\eta = 0 \tag{1.34}$$

firstly introduced in [5]. In particular, they showed that the Cauchy problems for the KdV and BBM equations give close solutions in the long wave limit. A nice exposition of the well-posedness for (1.34) in H^s with $s \geqslant 0$ can be found in [7].

In the physically relevant case $\varkappa < 1/3$ one may rescale x and t by $\sqrt{1-3\varkappa}$ in both equations (1.33), (1.34) to exclude the surface tension. Thus the capillary effect does not play any significant role at this level of accuracy. One of the most important features of Equations (1.33), (1.34) is that they admit explicit solitary wave solutions. Indeed, one can easily check that the wave

$$\eta(x,t) = \eta_0\,\mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{4(1-3\varkappa)h^3}}(x-Ut)\right)$$

with the velocity depending on the amplitude η_0 according to

$$U = c_0 \left(1 + \frac{\eta_0}{2h}\right)$$

is a solution of the KdV equation (1.33). Solitary waves for the BBM equation (1.34) look similar. At this point we would like to make a remark that the general (a,b,c,d)-family (1.29)-(1.30) also has explicit solitary waves [16, 24].

As one can notice the solitary wave solution of the KdV equation (1.33) is defined for all $\eta_0/h > 0$. However, solitary waves are found to peak at a maximum height $\eta_0/h \approx 0.7$ experimentally. To overcome this difficulty and precise the model Whitham proposed [57, 58] the following equation

$$\partial_t \eta + c_0 \sqrt{1 + \varkappa h^2 |D|^2} K \partial_x \eta + \frac{3c_0}{2h} \eta \partial_x \eta = 0 \tag{1.35}$$

where we have the KdV type nonlinearity and the full dispersion (1.22). For the Whitham equation with the surface tension we refer the reader to [28]. In the case of pure gravity waves $\varkappa = 0$, it was proved to be locally well-posed in $H^s(M)$ with $M = \mathbb{R}$ or $M = \mathbb{T}$ and $s > 3/2$ [33]. Moreover, several interesting phenomena predicted by Whitham were confirmed, as for example, a solitary wave regime close to KdV [34], the existence of a wave of greatest height [35], the existence of shocks [37], and modulational instability of steady periodic waves [38]. With the nontrivial surface tension $\varkappa > 0$ the dynamics of (1.35) appears to be completely different [42]. Finally, we point out that the Whitham equation (1.35) was proved to be a relevant water wave model [42]. However, it is justified only in the Boussinesq regime. In other words, it is proved to be at least as accurate as the KdV equation (1.33). The question if the Whitham equation (1.35) gives a better approximation remains open. Various numerical simulations suggest the positive answer [11, 15, 28, 48]. We also refer to [42, 52, 54] for other interesting numerical experiments.

Chapter 2

Main results

2.1 The Whitham equation for hydroelastic waves

We introduced a Whitham type model for describing thin elastic water cover. The validity is checked numerically on periodic travelling waves. We also observed asymmetric waves. Some of them bifurcate sub-harmonically from curves starting from trivial solutions. And some of them cannot be obtained from the Crandall–Rabinowitz asymptotics.

In fact Paper 2.1 is a continuation of our other work [28], where we have showed how the surface tension naturally arises in asymptotic models from the Hamiltonian formulation of the water wave problem. Moreover, the use of the Whitham equation with surface tension was justified by a formal derivation and numerical experiments.

We work in the same variables η, u as they were introduced in Section 1.3 in the one dimensional setting $d = 1$. They are non-dimensionalised in a way that $h = g = c_0 = 1$. The Hamiltonian formulation has the form

$$\eta_t = -\partial_x \frac{\delta \mathcal{H}}{\delta u}, \qquad u_t = -\partial_x \frac{\delta \mathcal{H}}{\delta \eta} \tag{2.1}$$

with the energy

$$\mathcal{H}(\eta, u) = \frac{1}{2} \int_{\mathbb{R}} \left[\eta^2 + u D^{-1} G(\eta) D^{-1} u \right] \, dx + \mathcal{H}_{\varkappa}(\eta) \tag{2.2}$$

where $\mathcal{H}_{\varkappa}$ stands for the surface deformation energy due to either capillarity or elasticity. Correspondingly, $\varkappa$ is either the capillarity or elasticity parameter. Simplifying only the Hamiltonian functional in the long wave framework we obtain the system

$$\eta_t = -\frac{\tanh D}{D} u_x - (\eta u)_x, \tag{2.3}$$

$$u_t = -\eta_x - \varkappa |D|^r \eta_x - u u_x \tag{2.4}$$

where $r = 2$ for the surface tension problem and $r = 4$ for the hydro-elastic problem. Note that this system is Hamiltonian, and moreover, its Hamiltonian comparable with the total energy (2.2) of the full problem. On the contrary, in such formulation one inevitably has to impose more smoothness on variable η than on u. As a result operator $\eta \mapsto \partial_x(\eta u)$ with fixed u is singular regardless of the choice of domain, and so one can hardly anticipate a satisfactory well-posedness result here. This does not seem in line with the fact that surface tension sometimes is used for regularisation of approximate models. Indeed, the following system

$$\eta_t = -\frac{\tanh D}{D}(1+\varkappa|D|^2)w_x - (\eta w)_x, \tag{2.5}$$

$$w_t = -\eta_x - ww_x \tag{2.6}$$

can be found in [15, 42]. The corresponding initial value problem was considered in [51] for $\varkappa = 0$ and in [40] for $\varkappa > 0$. The latter work provides with a more satisfactory formulation of the Cauchy problem, because of the regularization effect due to the surface tension. Note that the new velocity variable $w = (1+|D|^2)^{-1}u$, and so formally Systems (2.3)-(2.4) and (2.5)-(2.6) differ only in the nonlinear part, negligibly from the long wave point of view. However, System (2.5)-(2.6) has a different Hamiltonian structure, and in particular, its Hamiltonian is not comparable with the total energy (2.2) of the full problem.

Assuming waves travelling in one direction we reduce System (2.3)-(2.4) to the Whitham equation

$$\eta_t + K\sqrt{1+\varkappa|D|^r}\eta_x + \frac{3}{2}\eta\eta_x = 0$$

where K is defined by (1.24).

We simulate periodic travelling waves $\eta(x,t) = \varphi(x-ct)$ satisfying

$$-c\varphi + \frac{3}{4}\varphi^2 + K\sqrt{1+\varkappa|D|^r}\varphi = B \tag{2.7}$$

with some constant B. Exploiting asymptotic Crandall–Rabinowitz expressions [23] one can bifurcate from trivial solutions and obtain a wide range of travelling wave solutions. However, there are some solutions that cannot be obtained from the linear theory. Those were obtained by trial and error method. We would like to point out a very good agreement between the Whitham approximation and the full hydro-elastic travelling wave problem even for high waves.

2.2 Fully dispersive models for moving loads on ice sheets

The paper is on modelling of ice response of a floating elastic plate to the time-dependent motion of a moving load. The final model combines the full dispersion together with nonlinearity, forcing and damping.

The main motivation comes from the previous study 2.1. Firstly, deformations of an ice cover are normally very small, and so nonlinearity is very weak. Hence presumably the KdV type nonlinearity should be enough. Secondly, elastic effects affect only high frequencies of the dispersion relation, so it makes more sense of using fully dispersive models. Thirdly, as was mentioned at the end of the last section there are travelling waves that cannot be deduced from the limit argument, but can be obtained from the Whitham asymptotic model. As mentioned in the appendix of Paper 2.2, waves of half a meter height and of 50 meters length in a channel of 4 meters depth might be of purely nonlinear nature.

2.3 Modified Babenko's Equation For Periodic Gravity Waves On Water Of Finite Depth

Known Babenko equations, as for example introduced in [17, 18, 43], have a flaw in the following sense. The operators involved are parametrised by non-physical conformal parameters. We modified the equation from our other paper [43] in order to exclude the shortcoming. As a result we were able to carry out numerical bifurcation with a fixed depth. To our knowledge this is the easiest way to calculate asymmetric periodic travelling waves, in particular.

It is a continuation of our paper [43]. It turns out that 2D periodic travelling water waves of finite depth can be described by the single differential equation

$$\mu \mathcal{J}_r w = w + w\mathcal{J}_r w + \frac{1}{2}\mathcal{J}_r(w^2)\,, \quad \text{where } \mathcal{J}_r = \sum_{n=1}^{\infty} \lambda_n P_n\,. \tag{2.8}$$

Here P_n is the projector onto the subspace of $L^2(0,\pi)$ spanned by $\cos nt$ and

$$\lambda_n = n\frac{1+r^{2n}}{1-r^{2n}} \tag{2.9}$$

is the corresponding eigenvalue of $\mathcal{J}_r = \mathcal{B}_r \partial_t$, $n \in \mathbb{N}$. Parameters $\mu > 0$ and $r \in (0,1)$ are called the Froude number and conformal radius, respectively. The Hilbert transform $\mathcal{B}_r$ in $L^2(\mathbb{T})$ is defined by the line

$$\mathcal{B}_r\left(\sum_{n=0}^{\infty}(a_n \cos nt + b_n \sin nt)\right) = \sum_{n=1}^{\infty}\frac{1+r^{2n}}{1-r^{2n}}(a_n \sin nt - b_n \cos nt).$$

If $w \in H^1(0,\pi)$ solves (2.8) for some $\mu > 0$ and $r \in (0,1)$ then the corresponding wave profile $z = \eta(x)$ can be restored from the parametrisation $x = -t - \mathcal{B}_r w(t), z = w(t)$ with w being evenly extended on the whole interval $t \in (-\pi,\pi)$. The square root $\sqrt{\mu}$ gives the speed of the wave up to some dimensional constant. The non-dimensional fluid depth $h = -\log r - P_0 w$ with a slight abuse of notation where $P_0 w$ stands for the mean value of w.

As one can see the conformal radius r, the undisturbed water depth h and the mean value $P_0 w$ are bounded by one relation. This makes it difficult to solve the problem as it was formulated in the introduction 1.1 for the fixed depth h using the Babenko equation (2.8). We modified (2.8) in the way

$$\mu(1-P_0)w = \mathcal{L}_h w - \mathcal{L}_h(-w\mathcal{J}_h w) + \frac{1}{2}(1-P_0)(w^2) \tag{2.10}$$

where the nonlinear operators $\mathcal{L}_h$, $\mathcal{J}_h$ parametrized by the physical depth $h > 0$ are defined as follows. Firstly, we introduce the nonlinear functional

$$r_h(w) = \exp\{-h - P_0 w\}. \tag{2.11}$$

Secondly, changing r to this functional in formula (2.9) giving the sequence of eigenvalues of $\mathcal{J}_r$, we come to the following functionals

$$\lambda_n^{(h)}(w) = n\frac{1+[r_h(w)]^{2n}}{1-[r_h(w)]^{2n}}, \quad n = 1,2,\ldots, \tag{2.12}$$

all of which are well defined provided $P_0 w \neq -h$. Changing $\{\lambda_n\}_{n=1}^{\infty}$ to these functionals in the definition of $\mathcal{J}_r$, we introduce the following nonlinear operator

$$\mathcal{J}_h w = \sum_{n=1}^{\infty} \lambda_n^{(h)}(w) P_n w, \quad w \in H^1(0,\pi), \; P_0 w > -h.$$

The second operator is defined on the whole $L^2(0,\pi)$ by the formula

$$\mathcal{L}_h w = P_0 w + \sum_{n=1}^{\infty} \mu_n^{(h)}(w) P_n w \,, \text{ where } \mu_n^{(h)}(w) = \frac{1-[r_h(w)]^{2n}}{n\{1+[r_h(w)]^{2n}\}} \,.$$

Note that according to Proposition 2.1 of the paper there is a bijection between solutions of the Babenko equations parametrised by h and solutions of the Babenko equations parametrised by r. However, this bijection is not so much of use. Solving the equations parametrised by r does not give us solutions of the initial problem for particular given h. As a result one has to somehow iteratively adapt r to converge to the correct solution of the physical problem for the given h. This is not very efficient. The modified version of Babenko's equation allows us to avoid this difficulty.

As in the case of infinite depth [3], we observed numerically sub-harmonic bifurcations. We believe that this Babenko equation can be used for analytical proof of existence of these bifurcations, as it was done for the infinite depth in [14]. Note that the infinite depth $h = \infty$ corresponds to $r = 0$.

2.4 A comparative study of bi-directional Whitham systems

We show how one can naturally come from the full water wave problem (1.12)-(1.13) to different dispersive models of a Whitham–Boussinesq type. Numerical experiments demonstrate that all these models approximate the full problem similarly. Moreover, it turned out later (see [25, 30]) that one of the systems proposed in this paper is well posed under physically satisfactory conditions.

To our understanding models of such type were introduced mostly *ad hoc* [1, 15, 31, 39, 42]. Moreover, the non-physically conditional well-posedness proved in [51] for one of the systems was not enough. Unfortunately, we were not aware of the system introduced in [31], while writing our paper, because they introduced a fully dispersive system with satisfactory results on the initial value problem [31] and on existence of solitary waves [32]. Other than that, it seemed that some tidiness were needed to be add to the existing results.

We start from the formulation (1.14)-(1.15) and introduce the new variable

$$\mathbf{v} = K^2 \mathbf{u} = K^2 \nabla \Phi$$

where Φ is the surface trace of the potential ϕ and K is given in (1.24). For simplicity in the paper we consider the one dimensional problem $d = 1$ that corresponds to 2D water waves. Since in [26, 30] we regard the initial value problem for both cases $d = 1, 2$ we will present here the content of Paper 2.4 more generally.

Under the change of variables $\Phi \mapsto \mathbf{v}$ the Hamiltonian structure (1.14) transforms visually to

$$\partial_t (\eta, \mathbf{v})^T = \mathcal{J} \nabla \mathcal{H}(\eta, \mathbf{v})$$

with the skew-adjoint matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -K^2\partial_{x_1} & -K^2\partial_{x_2} \\ -K^2\partial_{x_1} & 0 & 0 \\ -K^2\partial_{x_2} & 0 & 0 \end{pmatrix}.$$

Applying the Hamiltonian long wave approximation as was explained in Section 1.3 (see also [19, 22]), and keeping untouched the part of the Hamiltonian responsible for the linear waves, one simplifies the energy functional to the form

$$\mathcal{H}(\eta, \mathbf{v}) = \frac{1}{2}\int_{\mathbb{R}^2} \left(g\eta^2 + \varkappa g h^2 |\nabla\eta|^2 + h\left|K^{-1}\mathbf{v}\right|^2 + \eta|\mathbf{v}|^2 \right) dx. \tag{2.13}$$

Note that (2.13) is well defined on $H^1 \times \left(H^{1/2}\right)^d$ for $\varkappa > 0$ and on $L^2 \times \left(H^{1/2}\right)^d$ for $\varkappa = 0$. Such Hamiltonian structure generates evolution described by the model

$$\begin{cases} \partial_t\eta + h\nabla\cdot\mathbf{v} = -K^2\nabla\cdot(\eta\mathbf{v}), \\ \partial_t\mathbf{v} + gK^2\nabla(1+\varkappa h^2|D|^2)\eta = -K^2\nabla\left(|\mathbf{v}|^2/2\right). \end{cases} \tag{2.14}$$

There are two conserved quantities for this system. The first one is Energy (2.13). The second one has the meaning of momentum and the form

$$\mathcal{I}(\eta, \mathbf{v}) = \int_{\mathbb{R}} \eta K^{-2}\mathbf{v}\, dx. \tag{2.15}$$

The latter conserves under the restriction that $\mathbf{v}$ is a curl free vector field ($\nabla\times\mathbf{v} = 0$), which holds true according to the definition of the velocity variable $\mathbf{v}$ given above.

In addition to the introduction of System (2.14), we compared numerically performance of different fully dispersive models. In most simulations results for (2.14) were slightly better than for other Whitham–Boussinesq type models.

2.5 On well-posedness of a dispersive system of the Whitham–Boussinesq type

Soon after submission of Paper 2.4 we obtained energy estimates for Model (2.14) for the one dimensional case $d = 1$ and the pure gravity $\varkappa = 0$, that were presented in this short note. A natural choice of the energy norm is $E(\eta, v) = \|\eta, v\|_{H^s\times H^{s+1/2}}$. With the energy estimates in hand we claimed the local well-posedness in $H^s(\mathbb{R}) \times H^{s+1/2}(\mathbb{R})$ with any $s \geqslant 1/2$. This is a standard method to use for quasilinear equations. The result was extensively extended later in my work with Selberg and Tesfahun [30], when we realised that System (2.14) is actually of the semilinear nature in the absence of the surface tension $\varkappa = 0$.

In addition some numerical computations of solitary waves were carried out, which allowed us to make a hypothesis about their existence for System (2.14). It was later confirmed in my work with Nilsson [29]. This is a crucial milestone for the justification of the model.

2.6 Well-posedness for a dispersive system of the Whitham–Boussinesq type

We have studied well-posedness for the Whitham–Boussinesq system (2.14) in the pure gravity case $\varkappa = 0$ at a very low level of regularity proposed in Paper 2.4. For this purpose we improved dispersive estimates of Strichartz type for water waves and implemented them together with the fixed point argument. In fact we derived the frequency localised estimates. Conservation of Hamiltonian allows to extend globally well-posedness at least for small initial data in the one dimensional case $d = 1$. This is a nice complement to the existing initial value problem results on other fully dispersive models [31, 51].

We work in the non-dimensional settings $h = g = 1$ with zero surface tension. We take the initial data

$$\eta(0) = \eta_0 \in H^s\left(\mathbb{R}^d\right), \qquad \mathbf{v}(0) = \mathbf{v}_0 \in \left[H^{s+1/2}\left(\mathbb{R}^d\right)\right]^d \tag{2.16}$$

where $d = 1, 2$. The corresponding Sobolev product space is notated shortly by X^s. In case $d = 2$ we also have to impose the natural curl free condition $\nabla \times \mathbf{v}_0 = 0$. System (2.14) can be rewritten in the Duhamel form

$$(\eta, \mathbf{v})(t) = \mathcal{S}(t)u_0 - \int_0^t \mathcal{S}(t-t')\begin{pmatrix} K^2\nabla\cdot(\eta\mathbf{v}) \\ K^2\nabla\left(|\mathbf{v}|^2/2\right)\end{pmatrix}(t')dt' \tag{2.17}$$

in $X_T^s = C([0,T]; X^s)$. Here $\mathcal{S}(t)$ is the fundamental continuous group. In other words, for any fixed $u_0 = (\eta_0, \mathbf{v}_0)^T \in X^s$ function $\mathcal{S}(t)u_0$ solves the linear initial-value problem associated with (2.14). With the help of the inequality $\|f_1 f_2\|_{H^s} \lesssim \|f_1\|_{H^s}\|f_2\|_{H^{s+1/2}}$ that can be found, for example in [36], the X^s-norm of the right hand side integrand can be estimated by $\|\eta, \mathbf{v}\|_{X^s}^2$. This means that Equations (2.14) are semilinear, in particular. One can proceed applying the fixed point argument to obtain solution $u = (\eta, \mathbf{v})^T \in X_T^s$ at least for small enough $T > 0$. As one can notice the main ingredient here are suitable bilinear estimates for the right hand side of (2.14). These estimates hold true provided $s > 0$ for $d = 1$ and $s > 1/2$ for $d = 2$.

The main focus of the work is on lowering the regularity threshold for the local well-posedness through the implementation of the dispersive nature of Equations (2.14). However, the dispersion is weak in the sense that the time-decaying $L^1 \to L^\infty$-boundedness of the group $\mathcal{S}(t)$ is not available. So instead, we obtain the decay estimate on each component of the dyadic Littlewood-Paley decomposition with a sharp dependence on the dyadic number. From this local-in-frequency decay we deduce bilinear estimates in the Bourgain spaces $X_\pm^{s,b}(T)$ associated with the water wave dispersion relation (1.21). The local well-posedness is deduced from the contraction mapping principal applied to the Duhamel formula (2.17) with the help of these bilinear estimates. The dispersive estimate is given in Lemma 9 of the paper, whereas Lemma 10 provides with the Strichartz estimates. By $S_{m_d}(\pm t)$ we denote elements of the matrix $\mathcal{S}(t)$ obtained after diagonalisation, more precisely, $\mathcal{S}(t) = \mathcal{K}\,\mathrm{diag}\{S_{m_d}(t), S_{m_d}(-t)\}\mathcal{K}^{-1}$. The final Bourgain bilinear estimates are given in Lemma 12 with the regularity restriction $s > -1/16$ for $d = 1$ and $s > 1/4$ for $d = 2$.

Global bound for $d=1$ in $X^0 = L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ follows from the Hamiltonian conservation, since $\mathcal{H}(\eta, v) \approx \|\eta, v\|_{X^0}^2$ provided $\|\eta, v\|_{X^0}$ is small. Hence the global well-posedness in X^s with $s>0$ follows from the local result and an a priori bound obtained from the persistence of regularity and the Brezis-Gallouet inequality [12, 13]. In the two dimensional case the gap between the energy space X^0 and the solution existence spaces X^s with $s>1/4$ is too big to claim the global-in-time well-posedness. It worth to notice that a similar situation takes place in the capillarity case [26].

2.7 Solitary wave solutions of a Whitham–Boussinesq system

We have showed existence of solitary waves for the Whitham–Boussinesq system (2.14) introduced in Paper 2.4. It was proved to be well-posed in the previous two papers [25, 30]. Solitary waves we obtain from the Concentration–Compactness principle [46] by Lions. We transformed the travelling wave system to one scalar equation. There are two difficulties here: the symbol of the Fourier multiplier standing in the linear part is of positive order and nonlinear part is also nonlocal.

Here we set $h=g=1$, $\varkappa=0$ and $d=1$ in (2.14). Solitary waves $\eta(x,t)=\eta(x-ct)$ and $v(x,t)=v(x-ct)$, with $c>1$ standing for the Froude number, satisfy the system

$$c\eta = v + K^2(\eta v), \tag{2.18}$$

$$cv = K^2\eta + K^2 v^2/2 \tag{2.19}$$

where K is a bounded self-adjoint operator in $L_2(\mathbb{R})$ defined by (1.24). Expressing η via v by (2.19) and substituting to (2.18) one obtains

$$v = \frac{1}{c^2}K^2 v + \frac{1}{2c}K^2 v^2 + \frac{1}{c}K^4(vK^{-2}v) - \frac{1}{2c^2}K^4 v^3$$

that can be represented as the Euler–Lagrange equation

$$\mathrm{d}\mathcal{E}(u) + \lambda \mathrm{d}\mathcal{Q}(u) = 0,$$

after introducing the new variable $u = -K^{-1}v/c$. The Lagrange multiplier $\lambda = -1/c^2$. The functionals involved are

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}} \left(K^{-1}u + \frac{1}{2}(Ku)^2\right)^2 dx,$$

$$\mathcal{Q}(u) = \frac{1}{2}\int_{\mathbb{R}} u^2 dx.$$

Hence, in order to find solutions of (2.18)-(2.19) we can instead consider the constrained minimization problem

$$\inf_{u\in U_q} \mathcal{E}(u) \quad \text{with} \quad U_q = \left\{u \in H^{1/2}(\mathbb{R}) : \mathcal{Q}(u) = q\right\}$$

where q parametrises size of a solitary wave in some sense. Implementation of the Lions principle in the spirit of [47] to a minimizing sequence provides us with solitary waves. In addition we analyse the long wave asymptotic of the obtained solutions following closely arguments of [32]. Thus we complement other results on solitary wave existence for the fully dispersive bidirectional models [32, 50].

2.8 Well-posedness for a Whitham–Boussinesq system with surface tension

We have showed global well posedness for the Whitham–Boussinesq system (2.14) introduced in Paper 2.4. In contrast to the case considered in [30], we have now an additional half loss of regularity due to presence of the surface tension. It makes the technique based on Strichartz estimates [30] inapplicable. Modified energy method is used instead.

As above we stay in the non-dimensional settings $h = g = 1$ with the surface tension $\varkappa > 0$. Formally the Duhamel formula looks the same as (2.17). However, now the group $\mathcal{S}(t)$ is continuous in the space

$$X^s = H^{s+1/2}\left(\mathbb{R}^d\right) \times \left[H^s\left(\mathbb{R}^d\right)\right]^d$$

where $d = 1, 2$, since $\mathcal{S}(t)u_0$ solves a different linear system. So the integrand in (2.17) lies in $\left[H^s\left(\mathbb{R}^d\right)\right]^{1+d}$ instead of X^s. It means that Equations (2.14) are quasilinear. A natural way to tackle the problem is to find an a priori estimate using the energy method with the norm $\|\eta, \mathbf{v}\|_{X^s}$. However, it turns out that the straightforward use of X^s-norm as the energy does not allow to close the estimates. The main problem is to find an appropriate coercive energy functional.

We define the following energy

$$E^s(\eta, v) = \frac{1}{2}\|\eta, \mathbf{v}\|_{X^s}^2 + \frac{1}{2}\int \eta \left|J^{s-1/2}\mathbf{v}\right|^2$$

that up to norm equivalence coincides with Hamiltonian (2.13) for $s = 1/2$. From this perspective the choice seems natural. In order to guarantee that E^s is coercive one has to impose an additional condition, namely, the non-cavitation of the flow. The latter can be controlled locally by the first equation of the System (2.14) or globally by the Hamiltonian conservation.

Finally, we would like to point out that the System (2.14) introduced in Paper 2.4 satisfies all the necessary conditions to be a relevant water wave model, such as well-posedness, solitary wave existence (at least in case $\varkappa = 0$). It incorporates the surface tension in the natural physical way as was explained in Section 2.1. It is globally well-posed for small initial data. We anticipate the wave breaking for big enough initial data, that can be questioned by the future research.

Chapter 3

Selected works

Paper I

3.1 The Whitham equation for hydroelastic waves

DINVAY, E., KALISCH, H., MOLDABAYEV, D., AND PĂRĂU, E. I.

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The Whitham equation for hydroelastic waves

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ABSTRACT

A weakly nonlinear fully dispersive model equation is derived which describes the propagation of waves in a thin elastic body overlying an incompressible inviscid fluid. The equation is nonlocal in the linear part, and is similar to the so-called Whitham equation which was proposed as a model for the description of wave motion at the free surface of an inviscid fluid.

Steady solutions of the fully nonlinear hydro-elastic Euler equations are approximated numerically, and compared to numerical approximations to steady solutions of the fully dispersive but weakly nonlinear model equation.

The bifurcation curves for these two different models are compared, and it is found that the weakly nonlinear model gives accurate predictions for waves of small to moderate amplitude. For larger amplitude waves, the two models still agree on key qualitative features such as the bifurcation points, secondary bifurcations, and the number of oscillations in a given fundamental wave period.

1. Introduction

The present contribution is devoted to hydroelastic waves propagating along a thin elastic body overlying an incompressible inviscid fluid. The prime example for this situation is wave propagation in an ice sheet over a body of water, a topic which has attracted increasing attention among researchers in recent years.

One of the motivating problems for studying this situation has been the motion of trucks and other vehicles on frozen lakes and rivers (see for example [1,2]). In many such cases, hydroelastic waves are highly dispersive but only weakly nonlinear. For example the measurements recorded in [2] feature a large spectrum of wavelengths while nonlinearity is relatively weak. As a result, many researchers have chosen to disregard nonlinear effects altogether [1,3–5]. More recently, nonlinear effects have come into focus as some studies of weakly and fully nonlinear hydroelastic waves (see for example [6–9]) have suggested that nonlinearity does have an appreciable effect on hydro-elastic waves.

The nonlinear model system derived in [8] couples the well known Saint-Venant (shallow water) system with hydroelastic dispersion, but neglects gravity dispersion. Going further, the works [10,11] take into account gravity dispersion in addition to both elasticity and nonlinearity. These studies depend heavily on the long-wave assumption and therefore lose part of the information of the linearized problem, as wavelengths are restricted to be very long when compared with the undisturbed depth of the fluid and the elastic length scale. In order to improve the modeling accuracy of such long-wave systems, in the present work we are aiming at the derivation of a fully dispersive but weakly nonlinear system. As mentioned above, the motivation and need for such a system is given by experiments such as those reported in [2] where a weakly nonlinear but highly dispersive response is recorded which may not be adequately modeled by traditional long-wave equations.

The idea of using fully dispersive weakly nonlinear equations goes back to Whitham [12], and was recently formalized both mathematically [13–15] and asymptotically [16,17]. Fully dispersive equations have been the subject of a number of studies recently, especially regarding the existence and stability of traveling waves [18–23]. In the current work, we first present a formal derivation of a Whitham-type fully dispersive and weakly nonlinear system of evolution equations, and then reduce the system to a single equation in the case when it can be assumed that the waves travel in a single direction (such as is the case for traveling waves). Using a recently developed open-source *Python* code, we then provide numerical approximations of traveling-wave solutions of this system. In order to check whether the derivation and numerical approximation are valid, we compare these solutions against

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numerical approximations of traveling-waves solutions of the fully nonlinear Euler equations with an elastic surface layer. As will come to light in Section 4, there is very good agreement between solutions of the full model and the weakly nonlinear model.

The fully dispersive weakly nonlinear equation we are aiming for can be written as

$$\eta_t + W\eta_x + \frac{3}{2}\eta\eta_x = 0. \tag{1}$$

Elasticity and gravity dispersion are combined in the convolution Whitham operator $W\eta_x = w(-i\partial_x)\eta_x = (\mathcal{F}^{-1}w)^*\eta_x$ which is defined by the dispersive function

$$w(\xi) = \sqrt{(1 + \varkappa\xi^4)\frac{\tanh(\xi)}{\xi}}. \tag{2}$$

Here we have non-dimensionalized the variables so that we may take the gravitational acceleration $g = 1$ and the undisturbed depth of the fluid as $h_0 = 1$, and the corresponding long-wave speed is $c_0 = \sqrt{gh_0} = 1$. This non-dimensionalization is explained in the next section. The floating ice sheet is included here by means of the non-zero elasticity parameter $\varkappa = \mathcal{D}/\rho$ where $\mathcal{D}$ is the coefficient of flexural rigidity for the ice sheet and $\rho = 1$ is the normalized density of the fluid. The convolution operator W represents a Fourier multiplier operator with the symbol (2). Note also that the Whitham equation (1) has the conserved integral

$$Q(\eta) = \int \eta W\eta \mathrm{dx} + \frac{1}{2}\int \eta^3 \mathrm{dx}. \tag{3}$$

Firstly, in case of free surface $\varkappa = 0$, Eq. (1) was proposed by Whitham [12] as a fully dispersive alternative to the well known Korteweg-de Vries (KdV) equation. With non-zero $\varkappa$ the last equation [10] is of the fifth order

$$\eta_t + \eta_x + \frac{3}{2}\eta\eta_x + \frac{1}{6}\eta_{xxx} + \frac{1}{360}(19 + 180\varkappa)\eta_{xxxxx} = 0. \tag{4}$$

For very long waves the KdV equation (4) is thought to be a good model for hydro-elastic waves. In order to be able to model shorter waves one needs to use an equation which also gives a good description of shorter waves, such as for example Eq. (1).

In the present article, we arrive at the Whitham equation from the Hamiltonian formulation of the water-wave problem given in [24–26] and modified for the hydroelastic problem in [10]. Extending the results of [10,16,17] we justify the Whitham equation (1) as a Hamiltonian system. We also obtain numerical results on steady solutions of Eq. (1) and compare them with solutions of the fully nonlinear hydro-elastic system based on the Euler equations. The comparisons indicate that the Whitham equation is able to provide a good description of hydro-elastic waves. Indeed, small-amplitude solutions of the full Euler equations are always closely approximated by the weakly nonlinear model, regardless of the wavelength. For large-amplitude solutions, there is good quantitative agreement in many cases, and good qualitative agreement in essentially all cases.

2. The hydro-elastic system

We consider a thin elastic plate supported by a fluid below. The elastic layer is modelled by making use of the special Cosserat theory of hyperelastic shells in Cartesian coordinates [27]. As already stated, the fluid base is assumed to be inviscid and incompressible, and the fluid flow is assumed to be two-dimensional and irrotational, so that potential theory can be used to describe the flow.

In order to normalize the problem the following standard non-dimensionalization is used. Letting the dimensional variables be primed, the non-dimensional variables are defined in terms of the fluid depth h_0, the gravitational acceleration g and the long-wave speed $c_0 = \sqrt{gh_0}$ as $x' = h_0x$, $z' = h_0x$ and $t' = \sqrt{h_0/g}\,t$. The unknowns are the vertical deflection of the cover and the velocity potential, and these are non-dimensionalized as $\eta' = h_0\eta$ and $\phi' = h_0c_0\phi$, respectively. This non-dimensionalization is equivalent to using the fluid depth h_0 as a unit of length, and $\sqrt{h_0/g}$ as a unit of time.

In the non-dimensional setting, the fluid domain is given by $\{(x, z) \in \mathbb{R}^2 | -1 < z < \eta(x, t)\}$ extending to infinity in the positive and negative horizontal x-direction. The complete hydro-elastic Euler system [10] consists of the Laplace's equation in this domain

$$\phi_{xx} + \phi_{zz} = 0 \quad \text{for } x \in \mathbb{R}, \quad -1 < z < \eta(x, t), \tag{5}$$

the Neumann boundary condition at the flat bottom

$$\phi_z = 0 \quad \text{at } z = -1, \tag{6}$$

the kinematic condition at the free surface

$$\eta_t + \phi_x\eta_x - \phi_z = 0 \quad \text{for } x \in \mathbb{R}, \quad z = \eta(x, t), \tag{7}$$

and the Bernoulli equation combined with elasticity

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + \eta + \varkappa\left(\kappa_{ss} + \frac{1}{2}\kappa^3\right) = 0 \quad \text{for} \quad x \in \mathbb{R}, \quad z = \eta(x, t), \tag{8}$$

where $\kappa = \eta_{xx}(1 + \eta_x^2)^{-3/2}$ is the curvature of the shell and s is the arclength along this cover and therefore

$$\kappa_{ss} + \frac{1}{2}\kappa^3 = \frac{1}{\sqrt{1 + \eta_x^2}}\partial_x\left(\frac{1}{\sqrt{1 + \eta_x^2}}\partial_x\left(\frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}}\right)\right) + \frac{1}{2}\left(\frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}}\right)^3.$$

The total energy of the system is the sum of the kinetic energy, the potential energy and the shell deformation energy [10], so the Hamiltonian function for this problem is expressed as

$$H = \int_{\mathbb{R}}\int_0^{\eta} z\,\mathrm{d}z\,\mathrm{d}x + \int_{\mathbb{R}}\int_{-1}^{\eta}\frac{1}{2}|\nabla\phi|^2\,\mathrm{d}z\,\mathrm{d}x + \frac{\varkappa}{2}\int_{\mathbb{R}}\frac{\eta_{xx}^2}{(1 + \eta_x^2)^{5/2}}\mathrm{d}x.$$

Introducing the trace of the potential at the free surface as $\Phi(x, t) = \phi(x, \eta(x, t), t)$, one may integrate in z in the first integral and use the divergence theorem on the second integral to obtain

$$H = \frac{1}{2}\int_{\mathbb{R}}\left[\eta^2 + \Phi G(\eta)\Phi + \varkappa\frac{\eta_{xx}^2}{(1 + \eta_x^2)^{5/2}}\right]\mathrm{d}x. \tag{9}$$

This integral represents the Hamiltonian functional of the water wave problem with a floating thin elastic cover as found in [10]. The Hamiltonian is written in terms of the Dirichlet–Neumann operator $G(\eta)$. It was shown in [28] that this operator depends analytically on the unknown η and can therefore be expanded in a power series

$$G(\eta)\Phi = \sum_{j=0}^{\infty} G_j(\eta)\Phi,$$

where each operator $G_j(\eta)$ is homogeneous of degree j in η. The terms in the power series can be computed using a recursion formula (see [25,26]), and the first two terms have the form

$$G_0(\eta) = D\tanh(D), \quad G_1(\eta) = D\eta D - D\tanh(D)\eta D\tanh(D)$$

where $D = -i\partial_x$ is a self-adjoint operator on $L^2(\mathbb{R})$.

Our first aim is to give the Hamiltonian formulation of the hydro-elastic problem in terms of the displacement η and the variable $u = \Phi_x = \phi_x + \eta_x\phi_y = \phi_\tau\sqrt{1 + \eta_x^2}$ proportional to the velocity of the fluid ϕ_τ tangential to the surface. Formally integrating by parts one may rewrite the Hamiltonian in terms of new variables as

$$H = \frac{1}{2}\int_{\mathbb{R}}\left[\eta^2 + uD^{-1}G(\eta)D^{-1}u + \varkappa\frac{\eta_{xx}^2}{(1 + \eta_x^2)^{5/2}}\right]\mathrm{d}x. \tag{10}$$

The integration by parts can be made mathematically precise by using the following well known lemma.

Lemma 1. *Let f, g be real-valued square integrable functions on the real line*

$\mathbb{R}$. *Regard $D = -i\partial_x$ as self-adjoint on $L^2(\mathbb{R}, \mathbb{C})$ and a real-valued function φ that is measurable and almost everywhere finite with respect to Lebesgue measure. If f, g lie in the domain of the operator $\varphi(D)$ then*

$$\int f\varphi(D)g = \int g\varphi(-D)f.$$

$$\int f\varphi(D)g = \int g\varphi(-D)f.$$

3. Derivation of Whitham type evolution equations

We now give a formal analysis of the long-wave approximation of the Hamiltonian. For the sake of completeness, we summarize some arguments presented in [16]. Regarding a wave-field having a characteristic non-dimensional wavelength λ and a characteristic non-dimensional amplitude α, we define the small parameter $\mu = \frac{1}{\lambda}$ and in order to bring out the difference in the horizontal and vertical scales in the problem, the scalings $\eta = \alpha\tilde{\eta}$ and $\tilde{D} = \lambda D = -\lambda i\partial_x$ are used. Then the natural scaling for the velocity unknown u is $u = \alpha\tilde{u}$, and the Hamiltonian is scaled as $H = \alpha^2\tilde{H}$. Omitting terms of cubic and higher order in α, the scaled Hamiltonian (10) is then written using only the terms G_1 and G_2 as

$$\tilde{H} = \frac{1}{2}\int_{\mathbb{R}} \tilde{\eta}^2\,dx + \frac{1}{2}\int_{\mathbb{R}} \tilde{u}\left[1 - \frac{1}{3}\mu^2\tilde{D}^2 + \cdots\right]\tilde{u}\,dx + \frac{\alpha}{2}\int_{\mathbb{R}} \tilde{\eta}\tilde{u}^2\,dx$$
$$- \frac{\alpha}{2}\int_{\mathbb{R}} \tilde{u}\left[\mu\tilde{D} - \frac{1}{3}\mu^3\tilde{D}^3 + \cdots\right]\tilde{\eta}\left[\mu\tilde{D} - \frac{1}{3}\mu^3\tilde{D}^3 + \cdots\right]\tilde{u}\,dx$$
$$+ \mu^4\frac{\varkappa}{2}\int_{\mathbb{R}} (\tilde{D}^2\tilde{\eta})^2\left\{1 - \alpha^2\mu^2\frac{5}{2}(\tilde{D}\tilde{\eta})^2 + \cdots\right\}dx.$$

In the case of flexural-gravity waves, the linear terms are dominant and terms of all order in μ should be kept. We may assume that $\alpha = o(\mu)$ such as for example the exponential scaling appearing in [17]. Then if terms of order $O(\alpha^2)$ and $O(\alpha\mu)$ are disregarded, the Hamiltonian (10) reduces to

$$H = \frac{1}{2}\int_{\mathbb{R}}\left[\eta^2 + u\frac{\tanh D}{D}u + \eta u^2 + \varkappa\eta_{xx}^2\right]dx. \quad (11)$$

Now the hydro-elastic problem can be reformulated as a Hamiltonian system using the variational derivatives of the approximate Hamiltonian (11). We point out [26,29] that the pair (η, Φ) represents the canonical variables for the Hamiltonian functional (9). Our purpose is to derive the equations of motion in terms of η and $u = \Phi_x$ which is slightly more convenient in the situation at hand. The transformation $(\eta, \Phi) \mapsto (\eta, u)$ is associated with the Jacobian

$$\frac{\partial(\eta, u)}{\partial(\eta, \Phi)} = \begin{pmatrix} 1 & 0 \\ 0 & \partial_x \end{pmatrix}.$$

So in terms of η and u the Hamiltonian equations have the form

$$\eta_t = -\partial_x\frac{\delta H}{\delta u}, \quad u_t = -\partial_x\frac{\delta H}{\delta\eta} \quad (12)$$

which is not canonical since the associated structure map $J_{\eta,u}$ is symmetric:

$$J_{\eta,u} = \left(\frac{\partial(\eta, u)}{\partial(\eta, \Phi)}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(\frac{\partial(\eta, u)}{\partial(\eta, \Phi)}\right)^* = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}.$$

The Gâteaux derivative $\delta H/\delta u$ of the Hamiltonian (11) is defined by means of an arbitrary real-valued square integrable function h from the variation

$$\delta_u H(h) = \frac{d}{d\tau}\bigg|_{\tau=0} H(u + \tau h, \eta) = \frac{1}{2}\int_{\mathbb{R}}\left[h\frac{\tanh D}{D}u + u\frac{\tanh D}{D}h + 2u\eta h\right]dx.$$

Integrating by parts according to Lemma 1 and taking into account arbitrariness of the given function h one obtains

$$\frac{\delta H}{\delta u} = \frac{\tanh D}{D}u + \eta u$$

and similarly

$$\frac{\delta H}{\delta\eta} = \eta + \frac{1}{2}u^2 + \varkappa\eta_{xxxx}.$$

Substituting these variational derivatives into Eq. (12) we arrive at the Hamiltonian system

$$\eta_t = -\frac{\tanh D}{D}u_x - (\eta u)_x, \quad (13)$$

$$u_t = -\eta_x - uu_x - \varkappa\eta_{xxxxx}. \quad (14)$$

This system is fully dispersive in the linear part, which is the distinctive feature of the original Whitham equation [12]. At the same time it allows bi-directional wave propagation, so that it may be of independent interest.By changing variables in the Whitham system (13) and (14) one may arrive at a new system, where the equations will be uncoupled in the linear part. In other words we can separate solutions corresponding to waves moving left and right. To justify this we follow the approach of [16,17], and consider the linearization

$$\eta_t + \frac{\tanh D}{D}u_x = 0, \quad (15)$$

$$u_t + (1 + \varkappa D^4)\eta_x = 0. \quad (16)$$

Looking for solutions of this linear system as waves $\eta(x, t) = Ae^{i\xi x - i\omega t}$, $u(x, t) = Be^{i\xi x - i\omega t}$ gives rise to the necessary condition $\omega^2 - (\xi + \varkappa\xi^5)\tanh\xi = 0$. Introducing the phase speed as $c = \omega(\xi)/\xi$ one arrives at the dispersion relation

$$c^2(\xi) = (1 + \varkappa\xi^4)\frac{\tanh\xi}{\xi}$$

which coincides, up to the sign of c, with the Whitham dispersion relation (2). Clearly, the choice $c > 0$ corresponds to right-going wave solutions of the linear system (15), (16), while the phase speed $c < 0$ gives left-going waves. In order to split up left- and right-going waves we use the following transformation of variables:

$$r = \frac{1}{2}(\eta + \mathrm{K}\,u), \quad s = \frac{1}{2}(\eta - \mathrm{K}\,u), \quad (17)$$

where one anticipates K to be an invertible operator, or more precisely, an invertible function of the differential operator D. The inverse transformation has the form

$$\eta = r + s, \quad u = K^{-1}(r - s). \quad (18)$$

It turns out that the operator K can be chosen in a way that r and s correspond to right- and left-going waves, respectively. Substituting the transformation (18) to the linear system (15), (16) obtain

$$r_t + \partial_x(\mathcal{A}(D, K)r + \mathcal{B}(D, K)s) = 0,$$
$$s_t - \partial_x(\mathcal{A}(D, K)s + \mathcal{B}(D, K)r) = 0,$$

where the operators $\mathcal{A}$ and $\mathcal{B}$ depend on D and K as follows:

$$\mathcal{A} = \frac{1}{2}\left((1 + \varkappa D^4)K + \frac{\tanh D}{D}K^{-1}\right), \quad \mathcal{B} = \frac{1}{2}\left((1 + \varkappa D^4)K - \frac{\tanh D}{D}K^{-1}\right).$$

So to achieve independence of the obtained two equations we need to choose the transformation K in the way $\mathcal{B}(D, K) = 0$, so that

$$K = \sqrt{\frac{1}{1 + \varkappa D^4}\cdot\frac{\tanh D}{D}} \quad (19)$$

which leads to $\mathcal{A}(D, K) = W = w(D)$, in terms of the dispersive Whitham operator W which was introduced at the beginning of the paper by means of the dispersive function (2). In this way, we get the two independent linear equations

$$r_t + \partial_x \mathrm{Wr} = 0, \tag{20}$$

$$s_t - \partial_x \mathrm{Ws} = 0. \tag{21}$$

If we again look at the special solutions $r(x, t) = \exp(i\xi x - i\omega_r t)$ and $s(x, t) = \exp(i\xi x - i\omega_s t)$ then we conclude that the first equation (20) describes waves moving to the right with the phase velocity $c_r = \omega_r/\xi = w(\xi)$ and the second equation (21) corresponds to the left-going waves with $c_s = \omega_s/\xi = -w(\xi)$.

Returning to the nonlinear theory we want to obtain a new Hamiltonian system with respect to unknown functions (17). In new variables r and s after integrating by parts due to Lemma 1 the Hamiltonian (11) takes the form

$$H(r, s) = \frac{1}{2}\int_{\mathbb{R}} [(r + s)^2 + (r - s)(1 + \varkappa D^4)(r - s) + (r + s)(K^{-1}(r - s))^2 + \varkappa(r + s)_{xx}^2]\mathrm{dx}. \tag{22}$$

As explained in [29], the change of variables (17) transforms the structure map to

$$J_{r,s} = \left(\frac{\partial(r, s)}{\partial(\eta, u)}\right) J_{\eta,u} \left(\frac{\partial(r, s)}{\partial(\eta, u)}\right)^* = \begin{pmatrix} -\frac{1}{2}\partial_x K & 0 \\ 0 & \frac{1}{2}\partial_x K \end{pmatrix}$$

that leads to the Hamiltonian system

$$r_t + \partial_x\left(\frac{K}{2}\frac{\delta H}{\delta r}\right) = 0, \qquad s_t - \partial_x\left(\frac{K}{2}\frac{\delta H}{\delta s}\right) = 0. \tag{23}$$

As explained above, one calculates variational derivatives of H given by (22) with respect to r and s at a real-valued square integrable function. Then after integrating by parts as in Lemma 1 and applying operator $K/2$ one obtains as a result

$$\frac{K}{2}\frac{\delta H}{\delta r} = \mathrm{Wr} + \frac{14}{K}(K^{-1}(r - s))^2 + \frac{1}{2}(r + s)K^{-1}(r - s), \tag{24}$$

$$\frac{K}{2}\frac{\delta H}{\delta s} = \mathrm{Ws} + \frac{14}{K}(K^{-1}(r - s))^2 - \frac{1}{2}(r + s)K^{-1}(r - s), \tag{25}$$

If these expressions are substituted into (23), a Whitham system describing in terms of a left-going component s and a right-going component r is appears. This system corresponds to the Hamiltonian (22) which is the same as for the system (13), (14), and no further approximation has been made. Recalling that the approximate Hamiltonian (22) was obtained by discarding terms of order $O(\mu\alpha)$ and $O(\alpha^2)$ in (10) we may modify terms in the Hamiltonian on the of order $O(\mu\alpha)$ and $O(\alpha^2)$ without changing the overall order of approximation.Thus in order to simplify the above system, we use the long-wave approximation $D = O(\mu)$ which leads to $K = 1 + O(\mu^2)$, $K^{-1} = 1 + O(\mu^2)$ and since $r = O(\alpha)$, $s = O(\alpha)$, the nonlinear part of (24), (25) can be written as

$$H_1 = \frac{1}{2}\int_{\mathbb{R}} (r + s)(K^{-1}(r - s))^2\mathrm{dx} = \frac{1}{2}\int_{\mathbb{R}} (r + s)(r - s)^2\mathrm{dx} + O(\mu^2\alpha).$$

Thus neglecting again terms of order $O(\mu\alpha)$, and assuming that the left-going waves are $s = o(\alpha)$ such as in [26], we can write the nonlinear part of the Hamiltonian as $H_1 = 1/2\int_{\mathbb{R}} r^3\mathrm{dx}$. So eventually we arrive at

$$\frac{K}{2}\frac{\delta H}{\delta r} = \mathrm{Wr} + \frac{34}{K}r^2, \tag{26}$$

$$\frac{K}{2}\frac{\delta H}{\delta s} = \mathrm{Ws}. \tag{27}$$

These approximations in the Hamiltonian together with the equations (23) yield the system

$$r_t + Wr_x + \frac{32}{K}(\mathrm{rr}_x) = 0, \tag{28}$$

$$s_t - Ws_x = 0. \tag{29}$$

This system is a fully dispersive system describing waves mainly moving in the right direction. Since we have modified only the energy of the hydro-elastic problem, the system (28), (29) is still Hamiltonian. Indeed that system corresponds to the Hamiltonian

$$H(r, s) = \frac{1}{2}\int_{\mathbb{R}} [(r + s)^2 + (r - s)(1 + \varkappa D^4)(r - s) + r^3 + \varkappa(r + s)_{xx}^2]\mathrm{dx}.$$

As one can see the first equation (28) can be considered independently from the second one (29) which is actually linear and homogeneous. Moreover, one can easily see that the Hamiltonian splits additively into two functionals $H(r, s) = H(r) + H(s)$, where the first one has the form

$$H(r) = \int_{\mathbb{R}}\left[r(1 + \varkappa D^4)r + \frac{1}{2}r^3\right]\mathrm{dx}.$$

With this energy, comparable with the initial total energy (9), the new Whitham equation (28) becomes Hamiltonian in the sense (23).

One may now use the approximation $K = 1 + O(\mu^2)$ in the first equation (28) to arrive at the Whitham equation (1). This is consistent with disregarding terms of order $O(\alpha\mu)$ in the derivation of the system (13),(14). Note, however, that this equation is then not Hamiltonian in the current context. Indeed, we have changed the structure map $J_{r,s}$, and the unknown η in Eq. (1) has the same meaning as r in Eq. (28), though the Hamiltonian structure is different. We have

$$\eta_t + \partial_x\left(\frac{1}{2}\frac{\delta Q}{\delta\eta}\right) = 0$$

where $Q(\eta)$ is given in (3). Note that energy Q is not comparable with the original Hamiltonian H given in (9), and it is therefore not clear whether it represents the energy. Indeed, as indicated in [30], the mechanical energy might take a completely different form in the context of Eq. (3).

4. Numerical analysis

For the numerical analysis of Eq. (1) we use the method thoroughly described in [31,32]. Here we will just give a quick overview. Travelling wave solutions are under investigation, so the ansatz

$$\eta(x, t) = \varphi(x - \mathrm{ct}) \tag{30}$$

is employed which together with Eq. (1) yields

$$-c\varphi + \frac{34^2}{\varphi} + W\varphi = B \tag{31}$$

with some undetermined constant B. We look for solutions of the last equation that are even and $2L$-periodic, so we compute solutions on the interval $[0, L]$ and restore it symmetrically on the whole interval $[-L, L]$. We regard solutions with different wavelengths L and elasticity $\varkappa$. We also normalize solutions by requiring its mean-value to be zero. This means that we require $\int_0^L \varphi = 0$. We use a cosine collocation method, so solutions of Eq. (31) are represented as linear combinations of functions $\cos\frac{m\pi x}{L}$, $m = 0, 1, \ldots$, that form a basis in $L^2(0, L)$. As explained in [20,31], for the discretization of the problem we look for solutions in a subspace $\mathcal{S}_N$ spanned by the first N of these cosine functions and defined at the collocation points $x_n = L\frac{2n-1}{2N}$ for $n = 1, \ldots, N$. Let W^N be the discrete form of the Whitham operator W and $\varphi_N \in \mathcal{S}_N$ be the discrete cosine representation of a solution φ. Then the values of φ_N at the collocation points satisfy the algebraic equations

$$-c\varphi_N(x_n) + \frac{34^2}{\varphi_N}(x_n) + W^N\varphi_N(x_n) = B \tag{32}$$

with $n = 1, \ldots, N$. As indicated above, to this system one also needs to add the mean-value normalization

$$\sum_{n=1}^{N} \varphi_N(x_n) = 0. \tag{33}$$

To compute profiles along the bifurcation curve, we introduce a parameter θ as a coordinate on the curve. We define the pseudo-waveheight A by the equation

$$\max_{n=1,\dots,N} \varphi_N(x_n) - \min_{n=1,\dots,N} \varphi_N(x_n) = A, \tag{34}$$

and treat the pseudo-waveheight $A = A(\theta)$ and the phase speed $c = c(\theta)$ as functions of the parameter θ. This procedure is described in more detail in [31].

Eqs. (32)–(34) form the final system to be solved by Newton's method. The unknowns here are $\varphi_N(x_1)$, …, $\varphi_N(x_N)$, B, θ. Solutions bifurcating from the linearization of Eq. (31) can be obtained from the initial guess, provided by the Stokes approximation [31].The numerical algorithm used to solve the fully nonlinear Euler equations with an elastic term (5)–(8) is based on the conformal mapping techniques [33,34] and generalizes previous works on gravity waves or capillary waves [35,36] to the hydroelastic case [10]. We will give just some essentials here. The Bernoulli equation (8) at the free surface in the moving frame $X = x - ct$ and $Y = z$ has the form

$$\frac{1}{2}(\phi_X^2 + \phi_Y^2) + \eta + \varkappa\left(\kappa_{ss} + \frac{1}{2}\kappa^3\right) = \mathcal{B}$$

where $\mathcal{B}$ is the Bernoulli constant. As above (30) the surface is represented by $\eta(x, t) = \varphi(X)$, so the outward unit normal vector is $\mathbf{n} = (-\varphi_X, 1)/\sqrt{1 + \varphi_X^2}$.The physical domain bounded by the free surface and the bottom is transformed with a conformal mapping from $X + iY$ to $\xi + i\zeta$ into a strip of thickness h (called conformal modulus), with $\zeta = 0$ corresponding to the free surface and $\zeta = -h$ corresponding to the bottom. Solving a Dirichlet boundary-value problem for $Y(\xi, \zeta)$ in the mapped domain, using Cauchy-Riemann relations between the partial derivatives of $X(\xi, \zeta)$ and $Y(\xi, \zeta)$ and denoting $\hat{X}(\xi)$: $=X(\xi + i0)$m $\hat{Y}(\xi)$: $=Y(\xi + i0)$, we obtain after some algebra and dropping the hats (see e.g. [35])

$$X_\xi = 1 - \mathcal{T}Y_\xi,$$

where $\mathcal{T}$ is the operator $\mathcal{T} = i\coth \mathrm{h\,D}$. In the new variables the Bernoulli equation can be re-written

$$\frac{1}{2}\frac{c^2}{X_\xi^2 + Y_\xi^2} + Y + \varkappa\left(\kappa_{ss} + \frac{1}{2}\kappa^3\right) = \mathcal{B}. \tag{35}$$

The elastic term $\left(\kappa_{ss} + \frac{1}{2}\kappa^3\right)$ can be expanded in an explicit form in terms of derivatives of X and Y, as explained in [11]. The velocity potential on the free surface is satisfying $\phi = c\xi$. We expand the variable Y as a Fourier series

$$Y(\xi) = \sum_{n=-\infty}^{\infty} a_n e^{in\xi},$$

and we are using equally-spaced collocation points $\xi_j = \frac{2\pi(j-1)}{2N+1}$, $j = 1$, …, $2N + 1$. The non-local operator acts (for $n \neq 0$) as

$$\mathcal{T}(e^{in\xi}) = i\coth(\mathrm{nh})e^{in\xi}.$$

After truncating the Fourier series at $n = \pm N$, we satisfy the Eq. (35) at the $2N + 1$ collocation points for the $2N + 2$ unknowns a_n, $n = -N$, …, N and the Bernoulli constant $\mathcal{B}$. The remaining equation is obtained by fixing $a_0 = h$ and the nonlinear system obtained is solved in MATLAB using a Newton method. In practice, for any given c we choose an initial guess $h = 1$ of the conformal modulus and then we enforce that the dimensionless height of the fluid in the physical space remain unchanged and equal to 1 for all computations, so we actually use a fixed-point procedure to guarantee

$$\frac{1}{2L}\int_{-L}^{L} YX_\xi d\xi = 1,$$

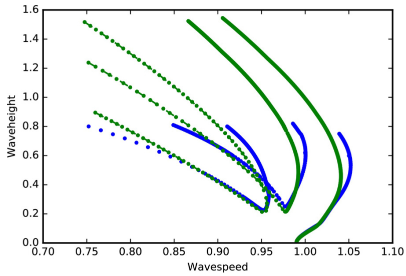

Fig. 1. Bifurcation diagrams for $L = 4\pi$ and $\varkappa = 0.1$. The blue curves represent the full hydroelastic system. The green curves represent the Whitham equation. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

until the correct value of h is found for each c. It was found that $N = 128$ is enough for most of the calculations to obtain a very good accuracy of solutions. When the curves are turning in the bifurcation diagrams, a version of this algorithm is used, where the height of the wave is enforced and c was found as part of the solution.

17 In Figs. 1–18, one can see how solutions change in the bifurcation diagrams. Solutions are plotted on the interval $[-L, L]$, where L ranges from $\pi/2$ to 4π, and several features are worth noting. First of all, in the bifurcation curves in Figs. 1, 8 and 11, for large to intermediate wavelengths, the bifurcation curves of the hydro-elastic Whitham equation are very close to the bifurcation curves for the Euler system as long as the waveheight, defined as $\max(Y) - \min(Y)$, is below about 0.4. A similar conclusion is reached by examining Figs. 2, 9, 12 and 16, where steady wave profiles are shown, and a close match between the two curves is evident. Whitham waves of larger amplitude still resemble the full hydro-elastic waves, qualitatively as shown in Figs. 3–7 for $L = 4\pi$ and $\varkappa = 1$, and similarly in Figs. 13, 14 and 18 for other parameter values. It is also worth noting that on the period $[-4\pi, 4\pi]$ solutions with large enough waveheight develop several secondary crests which actually start dominating the main crest at the center of the wave.

For small-amplitude solutions the error in the phase speed c is generally below 1%. For large-amplitude solutions, the error in the phase speed becomes larger. An extreme example is shown in Figs. 8 and 10. Concerning solutions high up on the branch in Fig. 8, if one

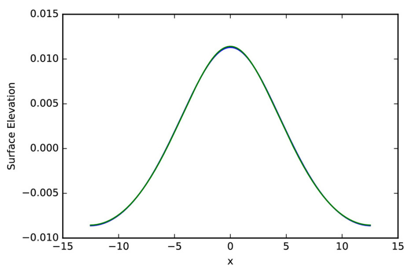

Fig. 2. Wave profiles for $L = 4\pi$, $\varkappa = 0.1$ near the bifurcation point of the rightmost curve in Fig. 1. The blue curve is an approximate solution of the full Euler system with $c = 0.9910$. The green curve an approximate solution of the Whitham equation with $c = 0.9911$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

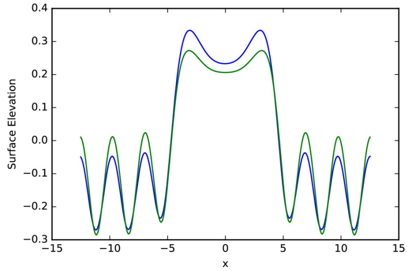

Fig. 3. Wave profiles for $L = 4\pi$, $\varkappa = 0.1$ higher up on the main branch in Fig. 1. The blue curve represents the Euler system with $c = 1.0506$. The green curve represents the Whitham equation with $c = 1.0391$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

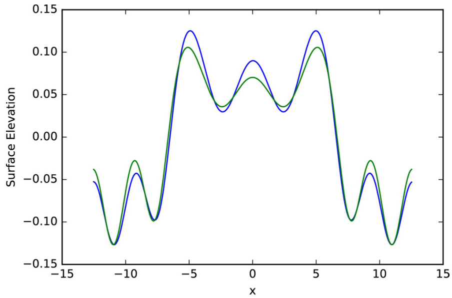

Fig. 4. Wave profiles for $L = 4\pi$, $\varkappa = 0.1$ on the second (middle) bifurcation point in Fig. 1. The blue curve represents the Euler system with $c = 0.9764$. The green curve represents the Whitham equation with $c = 0.9742$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

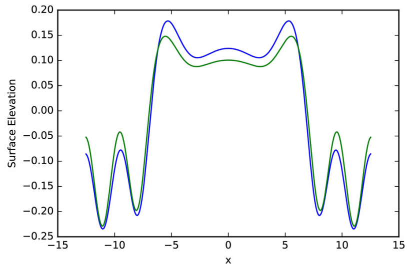

Fig. 5. Wave profiles for $L = 4\pi$, $\varkappa = 0.1$ on the left bifurcation branch emanating from the middle bifurcation point in Fig. 1. The blue curve represents the Euler system with $c = 0.9553$. The green curve represents the Whitham equation with $c = 0.9545$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

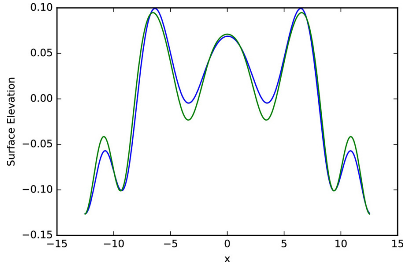

Fig. 6. Wave profiles for $L = 4\pi$, $\varkappa = 0.1$ on the third (leftmost) bifurcation point shown in Fig. 1. The blue curve represents the Euler system with $c = 0.9530$. The green curve represents the Whitham equation with $c = 0.9552$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

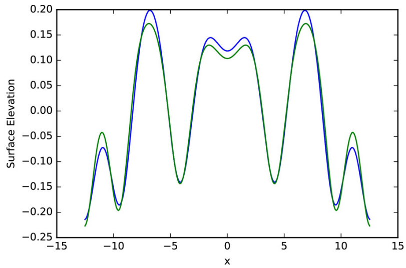

Fig. 7. Wave profiles for $L = 4\pi$, $\varkappa = 0.1$ on the right bifurcation branch emanating from the leftmost bifurcation point in Fig. 1. The blue curve represents the Euler system with $c = 0.9553$. The green curve represents the Whitham equation with $c = 0.9501$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

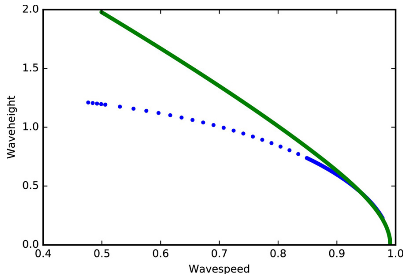

Fig. 8. Bifurcation diagrams for $L = 2\pi$ and $\varkappa = 1$. The blue curve represents the full hydroelastic system. The green curve represents the Whitham equation. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

matches the waveheight, then a near-exact match in the shape of the wave is obtained, such as in the blue and green profiles in Fig. 10. However, the phase velocities for these two solutions vary a great deal. On the other hand if one tries to get the best fit for the phase velocity, then the solution profiles have very different waveheights, though they still look qualitatively similar (cf. blue and red curve in Fig. 10). Fig. 10 also shows the interesting result (when compared with Figs. 13 and 14) that a larger elastic parameter $\varkappa$ actually leads to fewer oscillations in the steady wave profile.

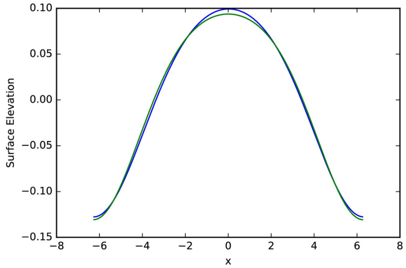

Fig. 9. Wave profiles for $L = 2\pi$, $\varkappa = 1$ for small-amplitude solutions on the bifurcation branches in Fig. 8. The blue curve represents the Euler system with $c = 0.9779$. The green curve represents the Whitham equation with $c = 0.9771$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

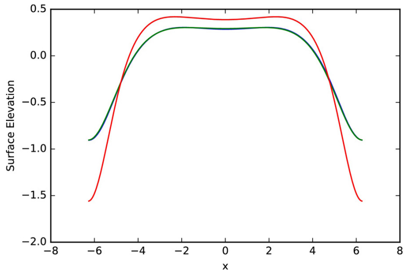

Fig. 10. Wave profiles for $L = 2\pi$, $\varkappa = 1$. The blue curve represents the Euler system with $c = 0.4767$. The green curve represents the Whitham equation with $c = 0.7431$. The red curve shows a different solutions of the Whitham equation with $c = 0.5002$ (that is 5% difference from $c = 0.4767$). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

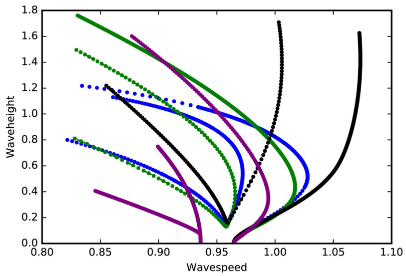

Fig. 11. Bifurcation diagrams for $L = 2\pi$ and $\varkappa = 0.1$. The blue curves represent the hydroelastic system. The green curves represent the Whitham equation (1). The black curves correspond to the fully nonlinear Whitham equation (28). The purple color is associated with the KdV-like equation (4). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

For large enough wavelengths and small enough elasticity, branches were found which do not connect to trivial solutions with vanishing waveheight (see Figs. 1 and 11). On the other hand, for short wavelengths there are secondary bifurcation from the main branch, such as shown in Fig. 15. In this figure, one can see how 2π-periodic solutions bifurcate from π-periodic ones. In all cases examined, the bifurcation picture of the Whitham model looks qualitatively similar to the fully nonlinear hydro-elastic system. Since the Whitham equation is much more easily approximated numerically, future work will include obtaining bifurcation curves for a larger parameter space of $\varkappa$ and L, such as for example presented for the similar Whitham equation with capillarity in [32].

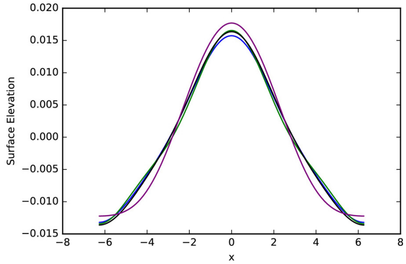

Fig. 12. Wave profiles for $L = 2\pi$, $\varkappa = 0.1$ for small-amplitude waves on the main branch in Fig. 11. The blue curve shows a solution of the full hydro-elastic system with $c = 0.9653$. The green curve shows a solution of the Whitham equation (1) with $c = 0.9657$. The black solution corresponds to the Hamiltonian Whitham equation (28) with $c = 0.9654$. The purple color is associated with the KdV equation (4) for $c = 0.9668$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

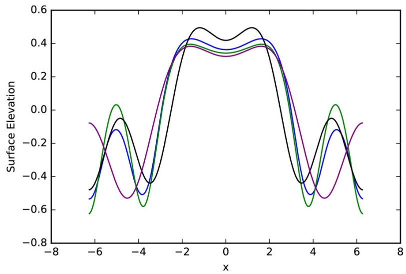

Fig. 13. Wave profiles for $L = 2\pi$, $\varkappa = 0.1$ for waves high up on the main branch in Fig. 11. The blue curve shows a solution of the full hydro-elastic system with $c = 0.9668$. The green curve shows a solution of the Whitham (1) equation with $c = 0.9685$. The black solution corresponds to the Hamiltonian Whitham equation (28) with $c = 1.0682$. The purple color is associated with the KdV equation (4) for $c = 0.9607$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Finally, we want to comment on two other models mentioned in the current paper, the 5th-order KdV equation (4) with a flexural term, and the fully nonlinear Whitham equation (28). Since the flexural effect corresponds to the term with the 5th derivative in (4), it appears that elasticity has a rather weak effect in the long-wave regime. Indeed, in the case of long waves, the Whitham equation (1) with a free surface (i.e. $\varkappa = 0$) gives similar small-amplitude solutions to solutions of other models depicted in Fig. 12. However, with increasing of amplitude elasticity causes the appearance of several crests on the fundamental wavelength $[-L, L]$, whereas the wave profiles of the free surface Whitham equation have exactly one crest and are strictly monotonic to both sides of it [20]. Some simulations of (4) and (28) are presented in

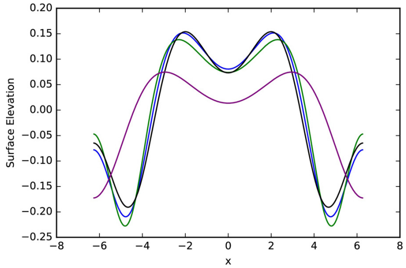

Fig. 14. Wave profiles for $L = 2\pi$, $\varkappa = 0.1$ on the leftmost branch in Fig. 11. The blue curve shows a solution of the full hydro-elastic system with $c = 0.9353$. The green curve shows a solution of the Whitham equation (1) with $c = 0.9261$. The black solution corresponds to the Hamiltonian Whitham equation (28) with $c = 0.9486$. The purple color is associated with the KdV equation (4) for $c = 0.8932$. It corresponds to the secondary bifurcation curve in Fig. 11. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

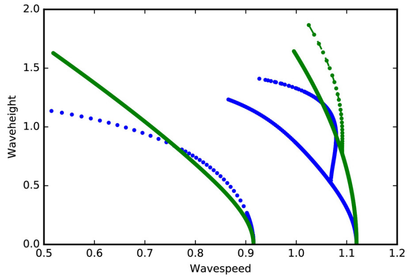

Fig. 15. Bifurcation diagrams for $L = \pi$ and $L = \pi/2$ with $\varkappa = 0.1$. The blue curves represent the full hydroelastic system. The green curves represent the Whitham equation. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

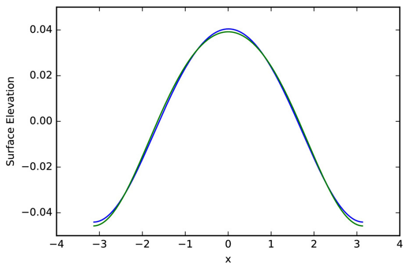

Fig. 16. Wave profiles for $L = \pi$, $\varkappa = 0.1$ for small-amplitudes solutions on the left branch in Fig. 15. The blue curve shows a solution of the full hydro-elastic system with $c = 0.9140$. The green curve shows a solution of the Whitham equation with $c = 0.9129$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Figs. 11–14. Note that both Whitham equations (1), (28) give travelling waves very similar to the fully nonlinear solutions. Whereas the elastic KdV (4) gives qualitatively different waves, i.e. with a different number

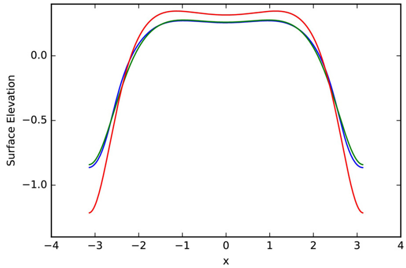

Fig. 17. Wave profiles for $L = \pi$, $\varkappa = 0.1$ on the left branch in Fig. 15. The blue curve shows a solution of the full hydro-elastic system with $c = 0.5148$. The green curve shows a solution of the Whitham equation with $c = 0.6790$. The red curve shows a solution of the Whitham equation with $c = 0.5402$ (that is 5% difference from $c = 0.5148$). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

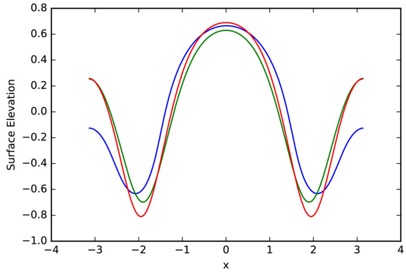

Fig. 18. Wave profiles for $L = \pi$, $\varkappa = 0.1$ on the secondary branch on the very right in Fig. 15. Note that these solutions have full wavelength 2π, the same as the solutions in the leftmost branch in Fig. 15. The blue curve shows a solution of the full hydro-elastic system with $c = 1.0173$. The green curve shows a solution of the Whitham equation with $c = 1.0786$. The red curve shows a solution of the Whitham equation with $c = 1.0658$ (that is 5% difference from $c = 1.0173$). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of wavecrests on a fundamental wavelength. Moreover, the KdV wave depicted in Fig. 14 is a solution of the secondary bifurcation type. It can be obtained by bifurcating from the $L = \pi$-branch as shown in Fig. 11. On the other hand, it does not seem possible to obtain corresponding solutions of the full system and equations (1), (28) in a similar way continuously from linear theory.

5. Conclusion

In order to understand properties of waves in a thin elastic sheet overlaying an inviscid fluid, a fully dispersive system of equations was derived using the Hamiltonian formulation of the hydroelastic surface wave system. The Hamiltonian was approximated using an asymptotic expansion of the Dirichlet–Neumann operator, and the derivation was based on fundamental ideas regarding Hamiltonian evolution systems [24–26,29]. It was also shown how the Hamiltonian equations can be restricted to model wave propagation in a single direction. Finally, a single fully dispersive weakly nonlinear equation of Whitham type (1) has been derived. This equation is the hydro-elastic version of the so-called Whitham equation put forward by Whitham [12] based on phenomenological considerations.

In order to further validate Eq. (1) as a model for hydro-elastic waves, a numerical study of traveling waves has been undertaken. Numerical approximations of steady solutions of (1) were compared to numerical approximations of steady solutions of the full hydro-elastic system introduced in Section 2. Bifurcation branches and wave profiles were compared, and it was shown that small-amplitude solutions of (1) resemble solutions of the full hydro-elastic system rather closely. Intermediate-amplitude solutions of the two approximations are still comparable, and even large-amplitude solutions of (1) show good qualitative agreement. In addition, it can be seen in Figs. 1, 8, 11 and 15 that bifurcation branches of (1) contain the same features as the bifurcation curves for the corresponding fully nonlinear hydro-elastic system. In particular, primary and secondary bifurcation points and the number of branches are closely matched.

Further work on these issues will include the study of whether the flow below the surface can be reconstructed from the surface profile $\eta(x, t)$. Such an analysis has recently been begun in the case of a free surface in [37]. In the present situation, this issue is more complicated because of the elastic layer. Nevertheless, analyzing for example the derivation used in [8] it seems possible that progress could be made.

Another important issue which deserves further study is the analysis of the nonlinear regime of the bifurcation curves, and the stability of the traveling waves. Such an analysis has been provided for the original Whitham equation [18,21,23], and the Whitham equation with surface tension in [22,32] but not yet for the equation derived in the current work.

The current model may also be extended by the inclusion of a variety of additional physical effects, such as a horizontal loading on the ice sheet, as described in [8], a vertical forcing [2], and the influence of nontrivial bathymetry [38]. These issues will be the focus of future work.

Acknowledgments

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Paper IV

3.4 A comparative study of bi-directional Whitham systems

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A comparative study of bi-directional Whitham systems

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A B S T R A C T

In 1967, Whitham proposed a simplified surface water-wave model which combined the full linear dispersion relation of the full Euler equations with a weakly linear approximation. The equation he postulated which is now called the Whitham equation has recently been extended to a system of equations allowing for bi-directional propagation of surface waves. A number of different two-way systems have been put forward, and even though they are similar from a modeling point of view, these systems have very different mathematical properties.

In the current work, we review some of the existing fully dispersive systems, such as found in [1,4,9,17,22,23]. We use state-of-the-art numerical tools to try to understand existence and stability of solutions to the initial-value problem associated to these systems. We also put forward a new system which is Hamiltonian and semi-linear. The new system is shown to perform well both with regard to approximating the full Euler system, and with regard to well posedness properties.

1. Introduction

Consideration is given to the two-dimensional water-wave problem for an inviscid incompressible fluid with a free surface over an even bottom. As this problem has not been completely resolved mathematically, there is still interest in developing new simplified models which yield an approximate description of the waves at the free surface in the case when the waves have distinctive properties, such as small amplitude or large wave period. In particular, there is the Boussinesq scaling regime which gives a good approximate description of long waves of small-amplitude. Recently, there has been interest in full-dispersion model which aims to give an exact description of “linear” waves while still being weakly nonlinear, and therefore accommodating some nonlinear processes such as wave steepening. The idea of representing the linear dynamics exactly goes back to the work of Whitham [27] who conceived the equation (now called Whitham equation)

$$\eta_t + g\mathcal{W}\eta_x + \frac{3}{2}\frac{c_0}{H}\eta\eta_x = 0, \tag{1.1}$$

where $\mathcal{W} = w(-i\partial_x) = \mathcal{F}^{-1} w \mathcal{F}$ is a Fourier multiplier operator defined by the dispersive function

$$w(\xi) = \sqrt{\frac{\tanh(H\xi)}{g\xi}}, \tag{1.2}$$

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and $c_0 = \sqrt{gH}$ is the limiting long-wave speed, defined in terms of the undisturbed fluid depth H and the gravitational acceleration g. The Fourier transform $\mathcal{F}$ and inverse transform $\mathcal{F}^{-1}$ are defined in the standard way, such as for example in [28]. It is clear that since the operator $\mathcal{W}$ reduces to the identity for very long waves ($\xi \to 0$), the Whitham equation reduces to the inviscid Burgers equation for very long waves. Properties of the Whitham equation have been investigated in [11,12,16,19,21,24].

Recently, Whitham's idea has been extended to the study of systems of evolution equation which allow for bi-directional wave propagation. In particular, in [1], Aceves-Sánchez, Minzoni and Panayotaros, found the Whitham system

$$\eta_t = -H\mathcal{K}u_x - (\eta u)_x, \tag{1.3}$$

$$u_t = -g\eta_x - uu_x, \tag{1.4}$$

and in [23], it was shown how this system arises as a Hamiltonian system from the Zakharov–Craig–Sulem formulation of the water-wave problem using an exponential long-wave scaling. The operator $\mathcal{K}$ is defined by the Fourier symbol $\frac{\tanh(H\xi)}{H\xi}$, so that we have the relation $H\mathcal{K} = g\mathcal{W}^2$. It can be seen that since the operator $\mathcal{K}$ reduces to the identity operator for very long waves ($\xi \to 0$), this Whitham system reduces to the classical shallow-water system for very long waves. In the remainder of this article, we will refer to the system (1.3), (1.4) as the ASMP system.

The system (1.3), (1.4) has been studied in a number of works. In particular, it was shown in [10] that it admits periodic traveling-wave solutions and features a highest cusped wave on the bifurcation branch. The modulational stability of its periodic traveling-wave solutions has been investigated numerically in [4], and the system has been studied numerically in the presence of an uneven bottom in [25]. Moreover, it was shown in [14] that the initial-value problem on the real line is well posed locally-in-time for data that are strictly positive and bounded away from zero.

On the other hand, the system

$$\eta_t = -Hv_x - (\eta v)_x, \tag{1.5}$$

$$v_t = -g\mathcal{K}\eta_x - vv_x \tag{1.6}$$

was put forward by Hur and Pandey in [17], and it was shown to behave somewhat more favorably than (1.3), (1.4) with regard to modulational instability and local well posedness (see also [4]). We will call this system the HP system.

In the current work, it is shown how the ASMP system (1.3), (1.4) and the HP system (1.5), (1.6) can be related by an asymptotic change of variables. Using the new variables, it is also possible to obtain a Hamiltonian system which is much less sensitive to instabilities than either the ASMP or HP system. We also show that the new system yields better approximations to the full water-wave problem than any of the other bi-directional Whitham system in use so far. We also present two other Hamiltonian systems, the right–left system, where dependent variables are chosen to represent wave propagating mainly to the left or to the right, and the essentially right-going system For the sake of completeness, we also include the Matsuno system in our study since it is easily obtained using the Hamiltonian theory.

2. The Hamiltonian formalism

A two-dimensional water-wave problem with the gravity g and the mean depth H is under consideration. The fluid is supposed to be inviscid and incompressible with irrotational flow. The unknowns are the surface elevation $\eta(x,t)$ and the velocity potential $\phi(x,z,t)$. The fluid domain is the set $\{(x,z) \in \mathbb{R}^2 | -H < z < \eta(x,t)\}$ extending to infinity in the positive and negative horizontal x-direction. Liquid motion is governed by the Euler system consisting of the Laplace's equation in this domain

$$\phi_{xx} + \phi_{zz} = 0 \quad \text{for} \quad x \in \mathbb{R}, \quad -H < z < \eta(x,t), \tag{2.1}$$

the Neumann boundary condition at the flat bottom

$$\phi_z = 0 \quad \text{at} \quad z = -H, \tag{2.2}$$

the kinematic condition at the free surface

$$\eta_t + \phi_x\eta_x - \phi_z = 0 \quad \text{for} \quad x \in \mathbb{R}, \quad z = \eta(x,t), \tag{2.3}$$

and the Bernoulli equation

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0 \quad \text{for} \quad x \in \mathbb{R}, \quad z = \eta(x,t). \tag{2.4}$$

The total energy of the fluid motion consists of potential and kinematic energy:

$$\mathcal{H} = \int_{\mathbb{R}} \int_0^{\eta} gz\,dzdx + \frac{1}{2}\int_{\mathbb{R}} \int_{-H}^{\eta} |\nabla\phi|^2\,dzdx. \tag{2.5}$$

It is known that the system (2.1)–(2.4) is equivalent to a certain Hamiltonian system. Indeed, with the trace $\Phi(x,t) = \phi(x,\eta(x,t),t)$ of the potential at the free surface and the Dirichlet–Neumann operator $G(\eta)$ the total energy (2.5) takes the form

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}} g\eta^2 dx + \frac{1}{2}\int_{\mathbb{R}} \Phi G(\eta)\Phi dx. \tag{2.6}$$

We regard $\mathcal{H}(\eta,\Phi)$ as a functional on a dense subspace of $L^2(\mathbb{R})\times L^2(\mathbb{R})$. We do not wish to specify smoothness of functions η, Φ and the exact domain of the functional $\mathcal{H}$ at this point, but we assume its variational derivatives lie in $L^2(\mathbb{R})$. The pair (η,Φ) represents the canonical variables for the Hamiltonian functional (2.6) with the structure map

$$J_{\eta,\Phi} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and so the Hamiltonian equations have the form

$$\eta_t = \frac{\delta\mathcal{H}}{\delta\Phi}, \qquad \Phi_t = -\frac{\delta\mathcal{H}}{\delta\eta}. \tag{2.7}$$

This evolutionary system in $L^2(\mathbb{R})$ is known to be equivalent to the Euler system (2.1)–(2.4). However, it does not simplify the problem since in general there is no explicit expression for the operator $G(\eta)$. More details on the Hamiltonian approach can be found in [5–7,30].

3. Weakly nonlinear approximations

In this section several approximations to Hamiltonian (2.6) will be presented. Each one will give rise to a system that can be considered as an approximate model to (2.7). The analysis is mainly heuristic consisting of arguments represented in [5,9], for example.

Regarding the self-adjoint operator $D = -i\partial_x$ in $L^2(\mathbb{R})$ we assume that the Dirichlet–Neumann operator appearing in (2.6) may be approximated by the sum $G(\eta) = G_0 + G_1(\eta)$ where

$$G_0(\eta) = D\tanh(HD), \qquad G_1(\eta) = D\eta D - G_0\eta G_0.$$

Such substitution should not change the Hamiltonian significantly since the remaining terms in the truncated operator $G(\eta)$ are of at least quadratic order in η and its derivatives. After integration by parts (Lemma 2.1 in [9]) it leads to

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}} \left(g\eta^2 + \Phi G_0\Phi - \eta(D\Phi)^2 - \eta(G_0\Phi)^2\right)dx. \tag{3.1}$$

One may notice the relative advantage of this approximation immediately. Instead of integrating the system (2.1)–(2.4), the much simpler system (2.7) with Hamiltonian (3.1) is to be solved.

In works on the surface water-wave problem, it has been common to use unknowns other than the potential Φ. Here, we use the variable $u = \Phi_x = \phi_x + \eta_x\phi_z = \varphi_\tau\sqrt{1+\eta_x^2}$, which is proportional to the velocity component of the fluid φ_τ which is tangent to the surface. This change of variables transforms the Hamiltonian (3.1) to

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}} \left(g\eta^2 + u\frac{\tanh HD}{D}u + \eta u^2 + \eta(\tanh HDu)^2\right)dx. \tag{3.2}$$

From now on, we will refer to the pair (η,u) as Boussinesq variables. Note that unlike (η,Φ) these new variables are not canonical. The corresponding structure map has the form

$$J_{\eta,u} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}$$

and the Hamiltonian system (2.7) transforms to

$$\eta_t = -\partial_x\frac{\delta\mathcal{H}}{\delta u}, \qquad u_t = -\partial_x\frac{\delta\mathcal{H}}{\delta\eta}. \tag{3.3}$$

It will become clear later that it is convenient to introduce yet another change of dependent variables. We define the new velocity variable $v = \mathcal{K}u$, where the transformation $\mathcal{K}$ is defined by the expression

$$\mathcal{K} = \frac{\tanh HD}{HD}, \tag{3.4}$$

which shows that it is an invertible and bounded Fourier multiplier operator. While the physical meaning of the new velocity variable $v = \mathcal{K}\partial_x\Phi = i\tanh(HD)\Phi/H$ is not clear, it will be shown later that it can be used to find a new system of equations which has desirable mathematical properties. In these new variables the Hamiltonian functional $\mathcal{H}(\eta, v)$ has the form

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}} \left(g\eta^2 + Hv\mathcal{K}^{-1}v + \eta(\mathcal{K}^{-1}v)^2 + \eta(HDv)^2\right)dx \tag{3.5}$$

with the structure map

$$J_{\eta,v} = \begin{pmatrix} 0 & -\mathcal{K}\partial_x \\ -\mathcal{K}\partial_x & 0 \end{pmatrix}$$

and the Hamiltonian system (2.7) transforming to

$$\eta_t = -\mathcal{K}\partial_x\frac{\delta\mathcal{H}}{\delta v}, \qquad v_t = -\mathcal{K}\partial_x\frac{\delta\mathcal{H}}{\delta\eta}. \tag{3.6}$$

In physical problems a question often arises if there is a way to split waves on right- and left-going components. One possible way of doing this splitting is to regard the linearization of the problem given in elevation-velocity variables and then change variables [23]. Namely, regard the following transformation

$$r = \frac{1}{2}(\eta + \mathcal{W}u), \qquad s = \frac{1}{2}(\eta - \mathcal{W}u) \tag{3.7}$$

where $\mathcal{W}$ is supposed to be an invertible function of the differential operator D. The inverse transformation has the form

$$\eta = r + s, \qquad u = \mathcal{W}^{-1}(r - s). \tag{3.8}$$

Omitting the details provided in [9] we notice that to split the linearized system into two independent equations one needs to take

$$\mathcal{W} = \sqrt{\frac{H}{g}\mathcal{K}} = \sqrt{\frac{\tanh HD}{gD}}. \tag{3.9}$$

The new variables r and s correspond to right- and left-going waves, respectively. Returning to the nonlinear theory we want to obtain a new Hamiltonian system with respect to unknown functions (3.7). Using the variables r and s and integrating by parts puts the Hamiltonian (3.2) into the form

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}} \left(2g(r^2 + s^2) + (r+s)(\mathcal{W}^{-1}(r-s))^2 + (r+s)(\sqrt{gG_0}(r-s))^2\right)dx, \tag{3.10}$$

with the structure map

$$J_{r,s} = \begin{pmatrix} -\mathcal{W}\partial_x/2 & 0 \\ 0 & \mathcal{W}\partial_x/2, \end{pmatrix}$$

and the Hamiltonian system (2.7) transforming to

$$r_t = -\frac{1}{2}\mathcal{W}\partial_x\frac{\delta\mathcal{H}}{\delta r}, \qquad s_t = \frac{1}{2}\mathcal{W}\partial_x\frac{\delta\mathcal{H}}{\delta s}. \tag{3.11}$$

In what follows we perform a Hamiltonian perturbation analysis based on the assumption of smallness of wave gradients. Regard a wave-field with a characteristic non-dimensional wavelength $\lambda = l/H$, amplitude $\alpha = a/H$ and velocity $\beta = b/\sqrt{gH}$ where l, a and b are typical dimensional parameters. Define the small parameter $\mu = 1/\lambda$. Usually α and β are identified and regarded as functions of wave-number μ. For justification of the models derived below there is no need for this identification or concretization of the dependence α, β on μ. The meaning of the scaling is of course that $\eta = H\mathcal{O}(\alpha)$, $u = \sqrt{gH}\mathcal{O}(\beta)$ and $HD = -iH\partial_x = \mathcal{O}(\mu)$. During our derivations, omission of higher-order terms is applied only to the Hamiltonian expressions (3.2), (3.5). The main idea is that high-order dispersive effects have little effect on the energy of the motion. Moreover, this approach guarantees that the obtained systems are Hamiltonian.

3.1. Matsuno model

The first useful system can be obtained if we take Hamiltonian (3.2) as it is and find the corresponding variational derivatives. Taking any real-valued square integrable smooth function h and using the definition

$$\int_{\mathbb{R}} \frac{\delta \mathcal{H}}{\delta u}(x)h(x)dx = \frac{d}{d\tau}\bigg|_{\tau=0} \mathcal{H}(u + \tau h, \eta) = \frac{1}{2}\int_{\mathbb{R}} (Hh\mathcal{K}u + Hu\mathcal{K}h + 2\eta uh + 2\eta(\tanh HDu)\tanh HDh)\, dx$$

one arrives after integration by parts to

$$\frac{\delta \mathcal{H}}{\delta u} = H\mathcal{K}u + \eta u - \tanh HD(\eta \tanh HDu)$$

and in the same way to

$$\frac{\delta \mathcal{H}}{\delta \eta} = g\eta + \frac{1}{2}u^2 + \frac{1}{2}(\tanh HDu)^2.$$

Thus System (3.3) transforms to

$$\eta_t = -H\mathcal{K}u_x - (\eta u)_x + \tanh HD(\eta \tanh HDu)_x, \tag{3.12}$$

$$u_t = -g\eta_x - uu_x - (\tanh HDu)\tanh HDu_x \tag{3.13}$$

which appeared in [18], and is similar to the systems found in [3] and [22]. It is not known so far if the system is well posed, but from a modeling point of view, it is sometimes regarded as the most exact model of all the so called bidirectional Whitham systems. Even though this system conserves the Hamiltonian (3.2), it turns out that this system is very sensitive to aliasing due to spatial discretization.

3.2. ASMP model

Simplifying the Hamiltonian through and appropriate scaling such as $\alpha = O(\mu^N)$ and thus discarding the last integrand in (3.2), one arrives at the system

$$\eta_t = -H\mathcal{K}u_x - (\eta u)_x,$$

$$u_t = -g\eta_x - uu_x.$$

This is the system (1.3), (1.4) mentioned in the introduction. The corresponding Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}} \left(g\eta^2 + Hu\mathcal{K}u + \eta u^2\right) dx. \tag{3.14}$$

This is also a Hamiltonian system with respect to the same Boussinesq variables η, u in the same sense as (3.3). This model started to attract attention after it appeared in [1] and [23]. The local well-posedness of the system (1.3)–(1.4) is proved [14] by imposing the additional condition $\inf \eta(x, 0) > 0$ on the initial surface elevation. It should be remarked that this condition may mean that the system is not useful from a physical point of view since all surface water wave models should have the property that the mean elevation be zero. However strictly positive solutions, like solitons for example, have always featured prominently in the analysis of such systems. In a recent paper by Claassen and Johnson [4] the well-posedness for more general initial data was questioned. In fact the authors showed numerically that the ASMP system is probably ill-posed in $L^2(\mathbb{T})$. However, our computations suggest to assume this is not the case in $L^2(\mathbb{R})$ and so that the system is probably well-posed on the real line. We also show that periodic discretization affects numerical computations significantly.

3.3. Hamiltonian version of the Hur–Pandey model

Regarding the Hamiltonian (3.5) given in the new variables defined above, one may discard the last integral in the expression and simplify the next one staying in the same framework of accuracy up to $\mathcal{O}(\mu^2\alpha\beta^2)$. This results in the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}} \left(g\eta^2 + Hv\mathcal{K}^{-1}v + \eta v^2\right) dx \tag{3.15}$$

with the Gâteaux derivatives

$$\frac{\delta\mathcal{H}}{\delta v} = H\mathcal{K}^{-1}v + \eta v,$$
$$\frac{\delta\mathcal{H}}{\delta \eta} = g\eta + \frac{1}{2}v^2.$$

Thus for the Hamiltonian (3.15), the system (3.6) has the form

$$\eta_t = -Hv_x - \mathcal{K}(\eta v)_x, \tag{3.16}$$
$$v_t = -g\mathcal{K}\eta_x - \mathcal{K}(vv_x). \tag{3.17}$$

To the best of our knowledge this system is completely new. One may notice that the nonlinear part of System (3.16)–(3.17) contains only the bounded operator $\mathcal{K}\partial_x$, which could mean that it is at least a locally well-posed system. Moreover we shall see later that among all bidirectional Whitham systems this is numerically the most stable one.

If one formally substitutes the operator $\mathcal{K}$ into the nonlinear part of (3.16)–(3.17) by unity according to the long wave approximation $\mathcal{K} = 1 + \mathcal{O}(\mu^2)$ then one arrives at the system

$$\eta_t = -Hv_x - (\eta v)_x,$$
$$v_t = -g\mathcal{K}\eta_x - vv_x,$$

i.e. system (1.5), (1.6) which was introduced by Hur & Pandey [17]. This system does well in the sense of numerical stability comparing with ASMP model but not as well as its Hamiltonian relative (3.16)–(3.17). Unlike the system (3.16)–(3.17) one cannot say for certain if the Hur–Pandey system is Hamiltonian with the same structure map as the original water-wave problem.

3.4. Right–left waves model

Again simplifying the Hamiltonian (3.10) up to $\mathcal{O}(\mu^2\alpha\beta^2)$ we obtain

$$\mathcal{H} = g\int_{\mathbb{R}} \left(r^2 + s^2 + \frac{1}{2H}(r+s)(r-s)^2 \right) dx \tag{3.18}$$

with the Gâteaux derivatives

$$\frac{\delta\mathcal{H}}{\delta r} = 2gr + \frac{g}{2H}(3r+s)(r-s),$$
$$\frac{\delta\mathcal{H}}{\delta s} = 2gs - \frac{g}{2H}(3s+r)(r-s).$$

Hence for the Hamiltonian functional (3.18) the bi-directional Whitham system has the form

$$r_t = -g\mathcal{W}r_x - \frac{g}{4H}\mathcal{W}\partial_x(3r+s)(r-s), \tag{3.19}$$
$$s_t = g\mathcal{W}s_x - \frac{g}{4H}\mathcal{W}\partial_x(3s+r)(r-s). \tag{3.20}$$

This system is also new even though it has implicitly appeared in a recently submitted paper [8], where it was not investigated further. Here we emphasize its usefulness and demonstrate that this system also outperforms the system (1.5)–(1.6) in the sense of numerical stability. Moreover, the variables r, s have a clear physical meaning and in particular initial data are easier to obtain. This means that sometimes the initial elevations $r(x, 0)$ and $s(x, 0)$ can be measured directly as opposed to velocity variables. We do not know if the system is well-posed. It deserves note that the symbol of the unbounded operator $\mathcal{W}\partial_x$ behaves like a square root at infinity. This fact might be enough to obtain well posedness. In any case, as shown later, the system has favorable numerical stability properties.

3.5. Uncoupled twin-unidirectional model

One may notice that in the system (3.19)–(3.20), the coupling between the dependent variables is due to the following part of Hamiltonian (3.18):

$$\mathcal{H}_{\text{coupling}} = -\frac{g}{2H}\int_{\mathbb{R}} rs(r+s)dx. \tag{3.21}$$

This part may sometimes be neglected. Then we arrive to the Hamiltonian

$$\mathcal{H} = g\int_{\mathbb{R}} \left(r^2 + s^2 + \frac{1}{2H}(r^3+s^3) \right) dx \tag{3.22}$$

and the corresponding Hamiltonian system consisting of the two independent equations

$$r_t = -g\mathcal{W}r_x - \frac{3g}{2H}\mathcal{W}rr_x, \tag{3.23}$$

$$s_t = g\mathcal{W}s_x + \frac{3g}{2H}\mathcal{W}ss_x. \tag{3.24}$$

The first equation is a modification of the equation proposed by Whitham [27,28]. The second one is its analogue for left-going waves. It is not known if they are well-posed even though for a large class of similar equations the answer is affirmative [13]. We shall see below that it is quite often the case that colliding waves almost do not affect each other and one may admit independence and regard basically just the equation (3.23). Up to small terms, the final result is obtained by linear superposition (3.8). Indeed in Fig. 2 the dependence on time of interaction energy (3.21) for the right–left system (3.19), (3.19) is represented. One can see that the interaction is going on for a short time and is of negligible order. This results in a small residual of solution after the interaction.

4. The numerical approach

All the models discussed in the project are solved by treating the linear part $\mathcal{L}$ and the nonlinear part $\mathcal{N}$ separately using a split-step scheme. In other words we solve a system of the form

$$z_t = \mathcal{L}(z) + \mathcal{N}(z) \tag{4.1}$$

which is treated by solving the systems $z_t = \mathcal{L}(z)$ and $z_t = \mathcal{N}(z)$. Denote by $\exp(t\mathcal{L})$ an integrator of the first one and $\exp(t\mathcal{N})$ an integrator of the second one. We make use of a symplectic integrator of 6th order introduced by Yoshida [29]. The main advantage of such an integrator is that the time step can be made relatively large which can accelerate calculations greatly. Yoshida developed his numerical scheme for separable finite Hamiltonian systems, however, it proved to be efficient also in water wave problems [2]. Below we describe the method in application to the models derived. Following Yoshida a one step integrator for the whole system (4.1) is approximated by the product

$$\exp[\delta t(\mathcal{L}+\mathcal{N}))] = \exp(c_1\delta t\mathcal{L})\exp(d_1\delta t\mathcal{N})\exp(c_2\delta t\mathcal{L})\cdot\ldots\cdot\exp(d_7\delta t\mathcal{N})\exp(c_8\delta t\mathcal{L})$$

where δt is the time step and c_i, d_i are constants given by

$$c_1 = c_8 = w_3/2, \quad c_2 = c_7 = (w_3 + w_2)/2, \quad c_3 = c_6 = (w_2 + w_1)/2, \quad c_4 = c_5 = (w_1 + w_0)/2$$

and

$$d_1 = d_7 = w_3, \quad d_2 = d_6 = w_2, \quad d_3 = d_5 = w_1, \quad d_4 = w_0.$$

Here we take the following set of weights

$$w_3 = 0.784513610477560, \quad w_2 = 0.235573213359357,$$
$$w_1 = -1.17767998417887, \quad w_0 = 1.315186320683906.$$

One can notice that the integrator is symmetric. The meaning of the product is that each time step is divided into substeps.

The systems $z_t = \mathcal{L}(z)$ and $z_t = \mathcal{N}(z)$ are solved using spectral methods. Moreover, the first one for each model can be solved exactly. For example, the linearization of the system (3.16)–(3.17) has the following solution

$$\eta(t) = \cos Ut\eta_0 - iHD\frac{\sin Ut}{U}v_0,$$
$$v(t) = -ig/H\tanh HD\frac{\sin Ut}{U}\eta_0 + \cos Utv_0,$$

with the initial data η_0, v_0. The operator U has the form

$$U = \sqrt{gG_0} = \sqrt{gD\tanh HD}. \tag{4.2}$$

These formulas represent the integrator $\exp(t\mathcal{L})$ for the systems (3.16)–(3.17) and (1.5)–(1.6) since the linear part $\mathcal{L}$ is the same for those two.

For the systems (1.3)–(1.4) and (3.12)–(3.13) the integrator $\exp(t\mathcal{L})$ has the form

$$\eta(t) = \cos Ut\eta_0 - i\tanh HD\frac{\sin Ut}{U}u_0,$$
$$u(t) = -igD\frac{\sin Ut}{U}\eta_0 + \cos Utu_0,$$

with the initial data η_0, u_0.

For the system (3.19)–(3.20), the operator $\exp(t\mathcal{L})$ is diagonal,

$$g\mathcal{W}\partial_x = ig\mathcal{W}D = i\sqrt{gG_0}\,\mathrm{sgn}\,D,$$

and the linearized problem has the solution

$$r(t) = \exp(-itU\,\mathrm{sgn}\,D)r_0,$$
$$s(t) = \exp(itU\,\mathrm{sgn}\,D)s_0,$$

where r_0, s_0 are initial right- and left-going waves, respectively, and U is defined by (4.2).

For all models discussed here, we use the standard Runge–Kutta scheme of 4th order as the nonlinear integrator $\exp(t\mathcal{N})$. It is explicit but not symplectic. One might argue that it makes the whole integrator $\exp(t\mathcal{L}+t\mathcal{N})$ not symplectic any more.

As an alternative we also ran all computations with a symplectic Euler scheme, such as described in [15]. This scheme turns out to be explicit for most of the models discussed here. Indeed, for example, for the ASMP model (1.3)–(1.4) one step of the semi-implicit Euler method has the form

$$\eta_{n+1} = \eta_n - \delta t\partial_x(H\mathcal{K}u_n + \eta_{n+1}u_n),$$
$$u_{n+1} = u_n - \delta t\partial_x(g\eta_{n+1} + \frac{1}{2}u_n^2)$$

that can be resolved with respect to η_{n+1} as follows. On the space l_2^N define operator $B_n f = -\delta t\partial_x(u_n f)$ that is bounded $\|B_n\| \leqslant \delta t N \max u_n$. Expecting uniform boundedness of solution u one can choose the time step $\delta t = O(1/N)$ so that $\|B_n\| \leqslant C < 1$. Thus

$$(1 - B_n)\eta_{n+1} = \eta_n - \delta t\partial_x H\mathcal{K}u_n$$

is resolved as

$$\eta_{n+1} = (1 + B_n + B_n^2 + \ldots)(\eta_n - \delta t\partial_x H\mathcal{K}u_n).$$

Hence η_{n+1}, u_{n+1} are resolved via η_n, u_n and the scheme is explicit and symplectic at the same time.

The numerical scheme of the free-surface problem for the Euler equations is based on a time-dependent conformal mapping of the fluid domain into a strip. A complete description of the method can be found in [20,26].

5. Numerical experiments

The model systems described above are now characterized with respect to numerical instability due to spatial discretization. For the numerical experiments we make the problem nondimensional by setting $H = 1$ and $g = 1$. The computational domain is $-L \le x \le L$, with $L = 70$. Initial conditions are imposed by means of

$$\eta_0(x; x_0, a, \lambda) = a \cdot \mathrm{sech}^2(f(x - x_0)) - C, \tag{5.1}$$

where

$$f(\lambda) = \frac{2}{\lambda}\log\left(1 + \sqrt{2}\right), \qquad C(\lambda) = \frac{a}{2fL}\left(\tanh f(L - x_0) + \tanh f(L + x_0)\right).$$

Here $C(\lambda)$ and $f(\lambda)$ are chosen so that $\int_{-L}^{L}\eta_0(x)dx = 0$, and the wave-length λ is the distance between the two points x_1 and x_2 at which $\eta_0(x_1) = \eta_0(x_2) = a/2$. Below we always take the wave-length $\lambda = \sqrt{5}$.

In all problems below we are interested in time evolution from $t_0 = 0$ to $t_{max} = 50$. In cases of collision of two waves we send them towards each other. So first of all we simulate problems that cannot be described by unidirectional models like KdV or Whitham equations. Secondly, one can see that all the models introduced are in line with the effect of quasi-elastic interaction of waves. So after collision waves behave as independent with slight tails. In all experiments below we provide initial data $\eta(x, 0)$ and $\Phi(x, 0)$ for the Euler system. Initial data for the approximate models can easily be obtained by applying transformations of variables $u(x, 0) = \partial_x\Phi(x, 0)$, (3.4) and (3.7). According to (3.8) one can make quasi-right moving waves taking the surface velocity $u(x, 0) = \mathcal{W}^{-1}\eta(x, 0)$.

As was already said the splitting method we are making use of allows us to take relatively large time steps. So we take $\delta t = 0.05$ when the number of Fourier harmonics is either $N = 512$ or $N = 1024$. This choice is dictated by the stiffness of the ASMP model (1.3)–(1.4) since the scheme becomes unstable for large N and might need filtering due to the probable ill-posedness of the model. In comparative experiments, on the other hand, we do not want to use any filtration.

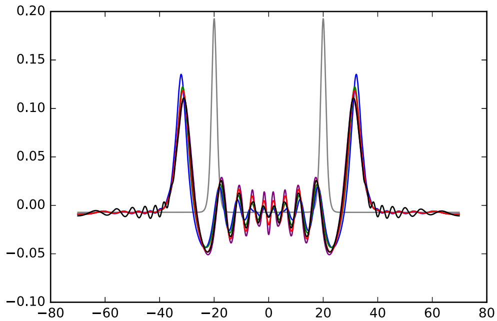

Fig. 1. Experiment (A). The thin grey curve represents the initial data. The black curve is the approximate solution of the full Euler system at $t = 50$. The color coding is as follows: purple – Hamiltonian HP system; red – right–left system; blue – ASMP system; green – HP system. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

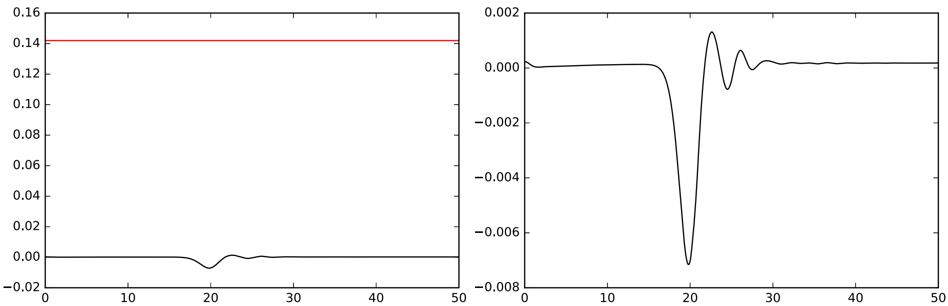

Fig. 2. Left panel: Development of the Hamiltonian for total initial energy $\mathcal{H} = 0.1420$, and the coupling term $\mathcal{H}_{\text{coupling}}$ for Experiment (A). Right panel: close-up of the graph of $\mathcal{H}_{\text{coupling}}$.

Experiment 5.1 *(A). Consider a collision of two approaching positive waves. Let* $a = 0.2$ *and* $x_0 = 20$. *Impose initial surface*

$$\eta(x, 0) = \eta_0(x; x_0) + \eta_0(x; -x_0)$$

and initial potential

$$\Phi(x, 0) = -\int_0^x \mathcal{W}^{-1}\eta_0(\xi; x_0)d\xi + \int_0^x \mathcal{W}^{-1}\eta_0(\xi; -x_0)d\xi.$$

All approximate systems in Experiment (A) are solved on the grid with $N = 1024$. (Fig. 1.)

Experiment 5.2 *(B). Consider a collision of a trough and a convex wave. Let* $a = 0.1$ *and* $x_0 = 20$. *Impose initial surface*

$$\eta(x, 0) = \eta_0(x; x_0) - \eta_0(x; -x_0)$$

and initial potential

$$\Phi(x, 0) = -\int_0^x \mathcal{W}^{-1}\eta_0(\xi; x_0)d\xi - \int_0^x \mathcal{W}^{-1}\eta_0(\xi; -x_0)d\xi.$$

All approximate systems in Experiment (B) are solved on the grid with $N = 512$. (Fig. 3.)

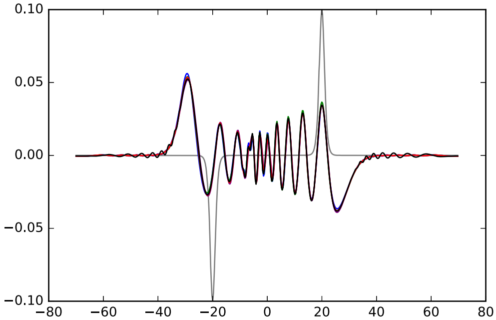

Fig. 3. Experiment (B). The thin grey curve represents the initial data. The black curve is the approximate solution of the full Euler system at $t = 50$. The color coding is as follows: purple – Hamiltonian HP system; red – right–left system; blue – ASMP system; green – HP system.

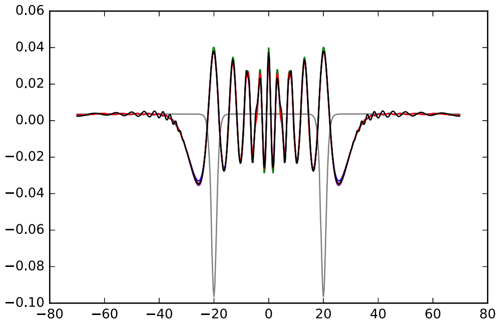

Fig. 4. Experiment (C). The thin grey curve represents the initial data. The black curve is the approximate solution of the full Euler system at $t = 50$. The color coding is as follows: purple – Hamiltonian HP system; red – right–left system; blue – ASMP system; green – HP system.

Experiment 5.3 *(C). Consider a collision of two troughs. Let $a = 0.1$ and $x_0 = 20$. Impose initial surface*

$$\eta(x, 0) = -\eta_0(x; x_0) - \eta_0(x; -x_0)$$

and initial potential

$$\Phi(x, 0) = \int_0^x \mathcal{W}^{-1}\eta_0(\xi; x_0)d\xi - \int_0^x \mathcal{W}^{-1}\eta_0(\xi; -x_0)d\xi.$$

All approximate systems in Experiment (C) are solved on the grid with $N = 512$. (Fig. 4.)

Experiment 5.4 *(E1–E3). Consider the evolution of waves with the initial surface elevation*

$$\eta(x, 0) = \eta_0(x; x_0 = 0)$$

where $a = 0.3$ and $x_0 = 0$. Impose firstly (E1) initial potential

$$\Phi(x, 0) = \int_0^x \mathcal{W}^{-1}\eta_0(\xi)d\xi,$$

than secondly (E2) initial potential

$$\Phi(x, 0) = \int_0^x \eta_0(\xi)d\xi,$$

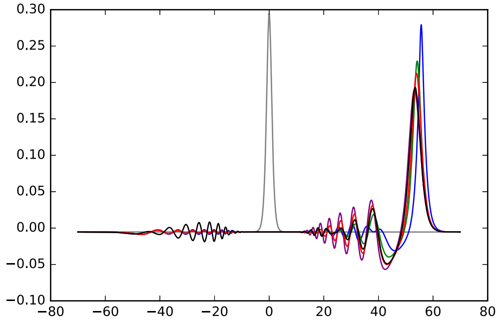

Fig. 5. Experiment (E1). The thin grey curve represents the initial data. The black curve is the approximate solution of the full Euler system at $t = 50$. The color coding is as follows: purple – Hamiltonian HP system; red – right–left system; blue – ASMP system; green – HP system.

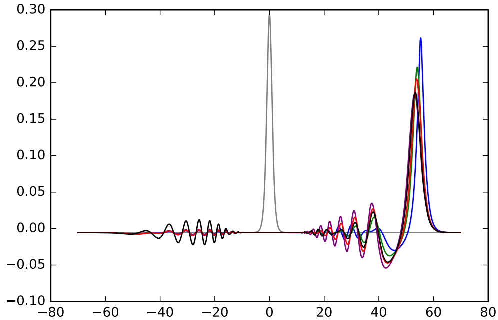

Fig. 6. Experiment (E2). The thin grey curve represents the initial data. The black curve is the approximate solution of the full Euler system at $t = 50$. The color coding is as follows: purple – Hamiltonian HP system; red – right–left system; blue – ASMP system; green – HP system.

and finely (E3) initial potential

$$\Phi(x, 0) = 0.$$

All approximate systems in Experiments (E1–E3) are solved on the grid with $N = 1024$. (Figs. 5–7.) Note that the initial potential of Experiment (E2) creates only approximately a right-going wave according to the linear long wave theory. Anyway neither the conditions of Experiment (E1) or of Experiment (E2) induce completely one way propagation as numerical results shows. Surprisingly, initial potentials of the type as in Experiment (E3) lead to better correspondence between approximate models and the Euler system then initial potentials of the type as in Experiment (E2). And moreover, of the type as in Experiment (E2) lead to the better correspondence then of the type as in Experiment (E1). We believe it is mainly a technical feature since the initial error of evaluation surface potential via $\mathcal{W}^{-1}$ and integration normally increases with the time.

In all presented figures initial elevation profiles are marked by grey lines. Solutions of the Euler system (2.1)–(2.4) are black, of the ASMP system (1.3)–(1.4) are blue, of the Hur–Pandey system (1.5)–(1.6) are green, of the Hamiltonian Hur–Pandey system (3.16)–(3.17) are purple, and of the right–left system (3.19)–(3.20) are red. (See Table 1.)

In order to quantitatively compare the accuracy of each approximate model we calculate the differences between Euler solutions and solutions of each system correspondingly. These errors are measured in the integral L^2-norm normalized by initial condition as follows

$$\mathcal{E} = \frac{\|\eta_{Euler} - \eta_{model}\|}{\|\eta_{initial}\|}$$

where

$$\|\eta_{Euler} - \eta_{model}\| = \max_t \sqrt{\int (\eta_{Euler}(x,t) - \eta_{model}(x,t))^2 dx}$$

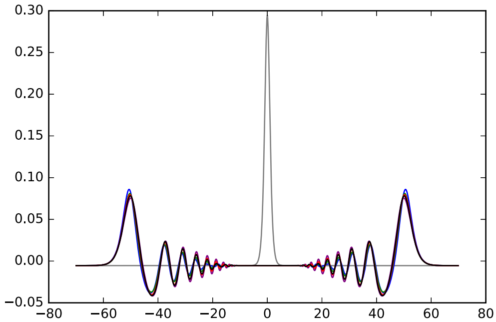

Fig. 7. Experiment (E3). The thin grey curve represents the initial data. The black curve is the approximate solution of the full Euler system at $t = 50$. The color coding is as follows: purple – Hamiltonian HP system; red – right–left system; blue – ASMP system; green – HP system.

Table 1
Hamiltonians $\mathcal{H}$ for various systems, evaluated at $t = 50$.

Experiment	A	B	C	E1	E2	E3
Euler	0.1316	0.03290755955585	0.03291	0.1481	0.1398	0.0740610317118
ASMP	0.1440	0.0329075170851	0.03136	0.1686	0.1569	0.0740419134333
Hamiltonian HP	0.1405	0.0329075170854	0.03180	0.1626	0.1524	0.0740419134422
Right–Left	0.1420	0.0329075170854	0.03162	0.1651	0.1543	0.0740419134422

Table 2
Errors $\mathcal{E}$, evaluated at $t = 50$.

Experiment	A	B	C	E1	E2	E3
ASMP	0.488	0.109	0.149	0.883	0.768	0.153
Hur–Pandey	0.253	0.085	0.126	0.339	0.315	0.082
Hamiltonian HP	0.167	0.130	0.106	0.231	0.207	0.061
Right–Left	0.167	0.089	0.128	0.240	0.218	0.048

and

$$\|\eta_{initial}\| = \sqrt{\int \eta(x,0)^2 dx}.$$

Here $\eta_{Euler}(x,t)$ is the solution for the Euler system and $\eta_{model}(x,t)$ corresponds either to ASMP, Hur–Pandey, Hamiltonian Hur–Pandey or Right–Left system. The corresponding results are represented in Table 2.

As was stated above some models work better in the sense of numerical stability. There were many discussions about ill-posedness of ASMP model [4]. In the next experiment we provide an example with initial data satisfying the condition for local well posedness. One can see that the initial data is lifted over the real axis so the mean value is approximately 0.35. It is known from Ehrnström, Pei, Wang [14] that we are in a locally well posed situation, however, the obtained solution seems very unstable as one can see in Fig. 8. This experiment was repeated with different time integrators, including the symplectic first-order Euler method described in Section 4. The results were always the same, pointing to doubts about the long-time well posedness of the ASMP system.

In order to systematize our experiments regarding the well posedness and stability of the Whitham systems, we used the following initial data:

Experiment 5.5. *Suppose we have a trough with amplitude $a = 0.3$. Let $x_0 = 0$. Solve System (1.3)–(1.4) with the initial surface*

$$\eta(x,0) = -\eta_0(x) + 0.35$$

and the initial velocity

$$u(x,0) = \mathcal{W}^{-1}\eta(x,0).$$

Problems with the HP system (1.5)–(1.6) may occur if an initial trough is deep enough. In the example shown on Fig. 9 we have to filter half of the high Fourier modes to make computations stable. The resulting noisy solution continues its

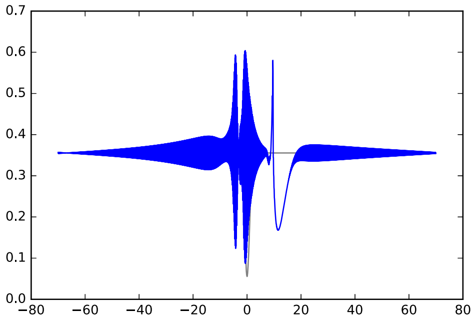

Fig. 8. Approximate solution of the ASMP Whitham system with initial data satisfying the condition $\inf \eta_0 > 0$.

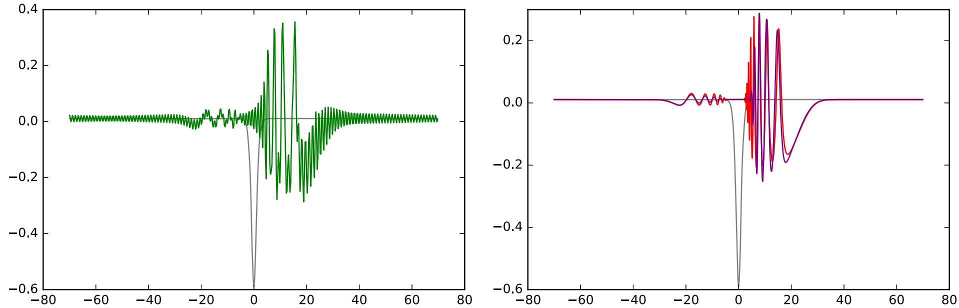

Fig. 9. Left panel: De-aliased solution of the HP system with N = 512 and time step = 0.05. Snapshot is taken at $t = 25$. Right panel: The same for Hamiltonian version of the HP system and the Right–Left system.

propagation and one can notice that all the oscillations happen around some reasonable mean curve that can be obtained easily by solving either the system (3.16)–(3.17) or the system (3.19)–(3.20) without any filtration. The results are represented on Fig. 9.

Experiment 5.6. *Suppose* $a = 0.6$ *and* $x_0 = 0$. *Solve System* (1.5)–(1.6) *with initial surface*

$$\eta(x, 0) = -\eta_0(x)$$

and initial velocity

$$v(x, 0) = \mathcal{K}\mathcal{W}^{-1}\eta(x, 0).$$

As to numerical stability of the Right–Left system (3.19)–(3.20), we can notice that this system encountered problems only in extreme non-physical situations, as for example, with an initial deep trough of amplitude $a = 1.2$ and increasing number of harmonics up to $N = 2^{15}$. The Hamiltonian version of the Hur–Pandey system (3.16)–(3.17) is numerically stable even in such a physically absurd problem.

Finally, let us look at the development of the Hamiltonian in two cases. First, an example of self-stabilization in the Matsuno system:

Experiment 5.7. *Suppose* $a = 0.2$ *and* $x_0 = 0$. *Solve Matsuno System* (3.12)–(3.13) *with initial surface* $\eta(x, 0) = \eta_0(x)$ *and initial velocity* $u(x, 0) = \mathcal{K}\mathcal{W}^{-1}\eta(x, 0)$. *We take the time step* $\delta t = 0.1$ *and the number of grid points* $N = 512$.

One might think that a numerical method conserving the total energy could remove the instabilities in the solution. Unfortunately this is not the case. We applied a simple projection method [15] to obtain a conservative method. With this method, energy was indeed conserved, and we managed to get a constant instead of the time-varying energy shown in Fig. 11. However, the solutions itself remained noisy such as in Fig. 10, and the computational cost is substantially higher than in the nonconservative method.

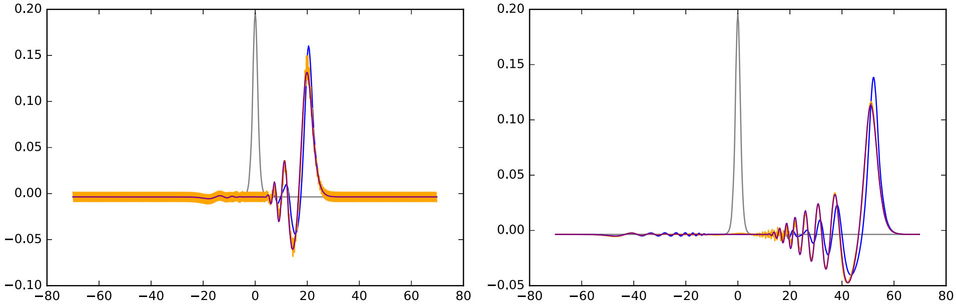

Fig. 10. Self-stabilized solution of the ASMP, Hamiltonian HP and Matsuno systems with $N = 512$ and time step $\delta t = 0.1$. Left panel: $t = 20$, right panel: $t = 50$.

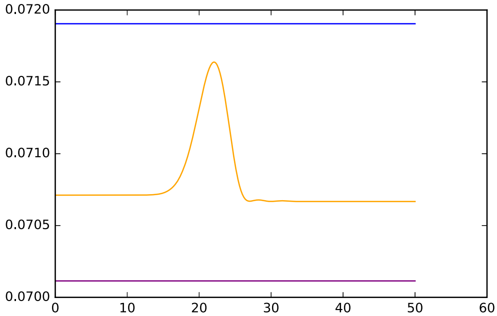

Fig. 11. Total energy of solutions of ASMP and Hamiltonian HP systems, and self-stabilized solution of the Matsuno system as a function of time t, with $N = 512$ and time step $\delta t = 0.1$.

Acknowledgements

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Paper V

3.5 On well-posedness of a dispersive system of the Whitham–Boussinesq type

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On well-posedness of a dispersive system of the Whitham–Boussinesq type

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The initial-value problem for a particular bidirectional Whitham system modelling surface water waves is under consideration. This system was recently introduced in Dinvay (2018). It is numerically shown to be stable and a good approximation to the incompressible Euler equations. Here we prove local in time well-posedness. Our proof relies on an energy method and a compactness argument. In addition some numerical experiments, supporting the validity of the system as an asymptotic model for water waves, are carried out.

1. Introduction

We regard the Cauchy problem for the system that in non-dimensional variables has the form

$$\eta_t = -v_x - i \tanh D(\eta v), \tag{1.1}$$

$$v_t = -i \tanh D\eta - i \tanh D v^2/2 \tag{1.2}$$

where $D = -i\partial_x$ and so $\tanh D$ is a bounded self-adjoint operator in $L_2(\mathbb{R})$. The system models the two-dimensional water wave problem for an inviscid incompressible flow. As usual η denotes the surface elevation. Its dual variable v roughly speaking has the meaning of the surface fluid velocity. In the Boussinesq regime it coincides with the horizontal fluid velocity at the surface.

Eqs. (1.1)–(1.2) appeared in literature recently as an alternative to other linearly fully dispersive models able to describe two-wave propagation [1]. Those models capture many interesting features of the full water wave problem and are in a good agreement with experiments [2]. As to well-posedness, the existing results for them are not satisfactory. For example, the system regarded in [3] is locally well posed if only an additional non-physical condition $\eta \geqslant C > 0$ is imposed. This system is probably ill-posed for large data if one removes the assumption $\eta > 0$. An heuristic argument is given in [4]. This is not a problem for System (1.1)–(1.2).

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Another important property of System (1.1)–(1.2) is its Hamiltonian structure. Indeed, regarding the functional

$$\mathcal{H}(\eta, v) = \frac{1}{2}\int_{\mathbb{R}} \left(\eta^2 + v\frac{D}{\tanh D}v + \eta v^2 \right) dx$$

Eqs. (1.1)–(1.2) can be rewritten in the form

$$\partial_t(\eta, v)^T = J\nabla\mathcal{H}(\eta, v)$$

with the skew-adjoint matrix

$$J = \begin{pmatrix} 0 & -i\tanh D \\ -i\tanh D & 0 \end{pmatrix}.$$

In particular, $\mathcal{H}$ is a conserved quantity. Thus Eqs. (1.1)–(1.2) provide an example of a nonlinear Hamiltonian system that is locally well posed. It is worth to notice that this model can be obtained in the long wave framework from Zakharov's system [5] also known to be Hamiltonian.

In some sense (1.1)–(1.2) can be regarded as a regularisation of the system introduced in [6]. Indeed, if one formally admits that $\tanh D \sim D$ for small frequencies, then substituting D instead of $\tanh D$ to the nonlinear part of System (1.1)–(1.2) one arrives to the system regarded in [6]. Such approximation is in line with the long wave framework, when we keep all dispersive terms in the linear part and exactly first dispersive term untouched in the nonlinear part. That is formally justified due to smallness of regarded water waves. Changing variables and admitting $\tanh D \sim D$ in nonlinear part, as explained in [1], one can arrive to the system studied in [3]. It is also worth to notice that for the system regarded in [6] the Benjamin–Feir instability of periodic travelling waves is proved. If one in addition formally discards the term $\eta\partial_x u$ in the system given in [6], then a new alternative system turns out to be locally well-posed and features wave breaking [7]. However, the latter does not belong to the class of Boussinesq–Whitham models since nonlinear non-dispersive terms have been neglected.

In addition it is worth to notice that System (1.1)–(1.2) outperforms other bidirectional Whitham models both in the sense of numerical stability and accuracy of approximation of Euler equations [1]. This is might not be surprising since in the nonlinear part of Eqs. (1.1)–(1.2) we have a bounded operator. However, if one tries to diagonalise the system then one will encounter a fractional derivative $|D|^{1/2}$ both in the linear and nonlinear parts. So further considerations turn out to be not completely straightforward.

Finally, let us formulate the main result. We stick to the usual notations of Sobolev spaces $H^s = H^s(\mathbb{R})$ with the norm defined via Fourier transform.

Theorem 1.1. *Let $s \geqslant 0$. For any $\eta_0 \in H^{s+1/2}(\mathbb{R})$ and $v_0 \in H^{s+1}(\mathbb{R})$ there exists a positive time $T > 0$ depending only on the norm $\|\eta_0\|_{H^{s+1/2}} + \|v_0\|_{H^{s+1}}$ such that there exists unique solution $(\eta, v) \in C([0,T]; H^{s+1/2}(\mathbb{R}) \times H^{s+1}(\mathbb{R}))$ of System (1.1)–(1.2) with the initial data (η_0, v_0). Moreover, it depends continuously on the initial data.*

In the following section a priori bound is established. The complete proof of the existence would result from a standard compactness argument implemented on a regularised version of the system. In the third section, we derive an estimate for the difference of two solutions. With this estimate in hand, one can prove the uniqueness as well the continuity of the flow map. For the complete proof one can follow Bona and Smith [8], for example. In the fourth section consistency is analysed. In the end, the relevance of (1.1)–(1.2) as an asymptotic model for water waves is supported by numerical calculations. The latter demonstrates a good agreement with the Euler equations. It is worth to notice that Theorem 1.1 does not rely on the non-cavitation hypothesis $1 + \eta > 0$, since smallness of waves is implied in the model.

2. A priori estimate

For the notational convenience we stick to the case of the low regularity with $s = 0$ in the theorem formulation. One can see that it changes insignificantly the derivation of the estimate. Introduce a functional of the form

$$B(\eta, v) = \frac{1}{2} \int_{\mathbb{R}} \left(\eta |D| \eta + v |D|^2 v \right) dx \tag{2.1}$$

and a norm $E(\eta, v)$ of the view

$$E^2(\eta, v) = \frac{1}{2} \|\eta\|_{L_2}^2 + \frac{1}{2} \|v\|_{L_2}^2 + B(\eta, v) \tag{2.2}$$

that is obviously equivalent to $\|\eta\|_{H^{1/2}} + \|v\|_{H^1}$. Here the pair $\eta(x, t)$, $v(x, t)$ represents a possible solution of System (1.1)–(1.2).

Lemma 2.1 (*A Priori Estimate*)**.** *Suppose $\eta(t) \in H^{1/2}(\mathbb{R})$ and $v(t) \in H^1(\mathbb{R})$ solving System (1.1)–(1.2) are defined on some interval including zero. Then there exist constants $C > 0$ and $T > 0$ such that*

$$E(t) \leqslant \frac{E_0 e^{Ct}}{1 - E_0(e^{Ct} - 1)}$$

for any $t \in [0, T)$. Here $E(t)$ stands for $E(\eta(t), v(t))$ and $E_0 = E(0)$.

Proof. Firstly, calculate the obvious derivative

$$\frac{1}{2} \frac{d}{dt} \|\eta\|_{L_2}^2 = \int \eta \eta_t = - \int \eta v_x - i \int \eta \tanh D(\eta v) \leqslant \|\eta\|_{L_2} \|\partial_x v\|_{L_2} + \|\eta\|_{L_2}^2 \|v\|_{L_\infty}$$

that follows from Hölder's inequality and boundedness of operator $\tanh D$ in $L_2(\mathbb{R})$. Similarly

$$\frac{1}{2} \frac{d}{dt} \|v\|_{L_2}^2 \leqslant \|\eta\|_{L_2} \|v\|_{L_2} + \|v\|_{L_2}^2 \|v\|_{L_\infty}.$$

Hence derivative of the first two terms in (2.2) is bounded as

$$\frac{1}{2} \frac{d}{dt} \left(\|\eta\|_{L_2}^2 + \|v\|_{L_2}^2 \right) \leqslant \left(\|\eta\|_{L_2} + \|\eta\|_{L_2}^2 + \|v\|_{L_2}^2 \right) \|v\|_{H^1}. \tag{2.3}$$

Differentiating (2.1) with respect to t obtain

$$\begin{aligned} \frac{d}{dt} B(\eta, v) &= \int_{\mathbb{R}} \left[\eta |D| \eta_t + v |D|^2 v_t \right] dx = \\ &= \int_{\mathbb{R}} \left[-\eta |D| \partial_x v - i\eta |D| \tanh D(\eta v) - iv |D|^2 \tanh D\eta - iv |D|^2 \tanh D v^2/2 \right] dx \end{aligned} \tag{2.4}$$

Note that $i|D|^2 \tanh D = \partial_x |D| |\tanh D|$ and so combining the first and the third integral in (2.4) gives

$$- \int \eta |D| \partial_x v - \int v \partial_x |D| |\tanh D| \eta = \int v \partial_x |D| (1 - |\tanh D|) \eta \leqslant C \|\eta\|_{L_2} \|v\|_{L_2}$$

since operator $\partial_x |D| (1 - |\tanh D|)$ is obviously bounded. Again applying $i|D| \tanh D = \partial_x (|\tanh D| - 1) + \partial_x$ to the second part of the Integral (2.4) obtain

$$- i \int \eta |D| \tanh D(\eta v) = \int \eta \partial_x (1 - |\tanh D|)(\eta v) - \int \eta \partial_x (\eta v)$$

where the first integral is bounded by $\|\eta\|_{L_2}^2 \|v\|_{L_\infty}$ up to a constant. The second integral

$$-\int \eta\partial_x(\eta v) = \frac{1}{2}\int v\partial_x\eta^2 = -\frac{1}{2}\int \eta^2\partial_x v \leqslant \frac{1}{2}\|\eta^2\|_{L_2}\|\partial_x v\|_{L_2} \leqslant C\|\eta\|_{H^{1/2}}^2 \|v\|_{H^1}$$

where the L_4-norm was controlled by $H^{1/2}$-norm as in Theorem 3.3 of the book by Linares and Ponce [9]. Noticing again $i|D|^2\tanh D = iD|D|(|\tanh D| - 1) + \partial_x|D|$ one can treat the last part of Integral (2.4) as

$$-i\int v|D|^2\tanh Dv^2/2 = i\int vD|D|(1-|\tanh D|)v^2/2 - \int v\partial_x v|D|v \leqslant$$
$$\leqslant C\|v\|_{L_2}^2\|v\|_{L_\infty} + \|\partial_x v\|_{L_2}^2\|v\|_{L_\infty} \leqslant C\|v\|_{H^1}^3 .$$

Thus combining all these inequalities in Identity (2.4) one arrives at

$$\frac{d}{dt}B(\eta,v) \leqslant C\left(\|\eta\|_{L_2}\|v\|_{L_2} + \|\eta\|_{L_2}^2\|v\|_{H^1} + \|\eta\|_{H^{1/2}}^2\|v\|_{H^1} + \|v\|_{H^1}^3\right)$$

that is together with (2.2) and (2.3) results in

$$\frac{d}{dt}E \leqslant C(E+E^2) \tag{2.5}$$

where equivalence of $E(\eta,v)$ to $\|\eta\|_{H^{1/2}} + \|v\|_{H^1}$ was used. Integration of (2.5) proves the lemma. □

3. Uniqueness

Suppose on some time interval we have two solution pairs η_1, v_1 and η_2, v_2 of System (1.1)–(1.2). Introduce functions $\theta = \eta_1 - \eta_2$, $w = v_1 - v_2$ and $\zeta = (\eta_1+\eta_2)/2$, $u = (v_1+v_2)/2$. Then θ and w satisfy the following system

$$\theta_t = -w_x - i\tanh D(u\theta + \zeta w), \tag{3.1}$$
$$w_t = -i\tanh D\theta - i\tanh D(uw). \tag{3.2}$$

The idea is to obtain an estimate for this system similar to the priori bound given in the above lemma. For this purpose one calculates derivative of the square norm $E^2(\theta,w)$. Calculations are similar

$$\frac{1}{2}\frac{d}{dt}\left(\|\theta\|_{L_2}^2 + \|w\|_{L_2}^2\right) \leqslant \sqrt{2}\|\theta\|_{L_2}\|w\|_{H^1} + (\|\zeta\|_{L_2} + \|u\|_{H^1})(\|\theta\|_{L_2} + \|w\|_{H^1})^2 \tag{3.3}$$

and for the derivative of the rest part of E^2 obtain

$$\frac{d}{dt}B(\theta,w) = \int_{\mathbb{R}}[-\theta|D|\partial_x w -$$
$$-i\theta|D|\tanh D(u\theta+\zeta w) - iw|D|^2\tanh D\theta - iw|D|^2\tanh D(uw)\Big]\,dx. \tag{3.4}$$

The first and the third integral in (3.4) together are estimated exactly as the corresponding part in (2.4) by $\|\theta\|_{L_2}\|w\|_{L_2}$ up to some constant. Similarly also estimate the fourth integral in (3.4) by $\|u\|_{H^1}\|w\|_{H^1}^2$ up to a constant. Due to identity $i|D|\tanh D = \partial_x(|\tanh D| - 1) + \partial_x$, instead of regarding the second integral in (3.4) it is enough to estimate the following integral

$$\int|\theta\partial_x(\zeta w)| \leqslant \||\partial_x|^{1/2}\theta\|_{L_2}\||\partial_x|^{1/2}(\zeta w)\|_{L_2} \leqslant$$
$$\leqslant C\||\partial_x|^{1/2}\theta\|_{L_2}\left(\||\partial_x|^{1/2}\zeta\|_{L_2}\|w\|_{L_\infty} + \|\zeta\|_{L_4}\||\partial_x|^{1/2}w\|_{L_4}\right) \leqslant C\|\zeta\|_{H^{1/2}}\|\theta\|_{H^{1/2}}\|w\|_{H^1}$$

which finishes the estimation of Derivative (3.4). Firstly, the fractional Leibniz rule was used here, that was derived by Kenig, Ponce, and Vega [10]. For the exact form we apply, one can look on page 52 of the book by Linares and Ponce [9]. Secondly, L_4-norms were estimated via $H^{1/2}$-norms. This estimate follows also from product estimates in Sobolev spaces.

The resulting inequality has the form

$$\frac{d}{dt}E(\theta, w) \leqslant C\left(1 + \|\zeta\|_{H^{1/2}} + \|u\|_{H^1}\right) E(\theta, w). \tag{3.5}$$

Taking into account boundedness of the norm $\|\zeta\|_{H^{1/2}} + \|u\|_{H^1}$ on the regarded time interval one can deduce uniqueness from the obtained inequality (3.5).

4. Consistency

In this section we will show that System (1.1)–(1.2) is at least as precise as any of Boussinesq systems [11] in the sense of approximation of water waves. Denoting by h the mean depth of the fluid layer, by a a typical amplitude and by λ a typical wave length we assume

$$\varepsilon = a/h = (h/\lambda)^2 \ll 1$$

where ε is a long wave parameter in this Boussinesq regime. In this framework the regarded Whitham–Boussinesq model has the form

$$\eta_t = -v_x - i\sqrt{\varepsilon}\tanh(\sqrt{\varepsilon}D)(\eta v), \tag{4.1}$$

$$v_t = -\frac{i}{\sqrt{\varepsilon}}\tanh(\sqrt{\varepsilon}D)\eta - i\sqrt{\varepsilon}\tanh(\sqrt{\varepsilon}D)v^2/2 \tag{4.2}$$

that is consistent with the following Boussinesq system

$$\eta_t = -v_x - \varepsilon(\eta v)_x, \tag{4.3}$$

$$v_t = -\left(1 + \frac{\varepsilon}{3}\partial_x^2\right)\eta_x - \varepsilon v v_x \tag{4.4}$$

rigorously studied in [12,13]. In fact the latter is an ill-posed almost fully justified model [5].

Lemma 4.1 (*Consistency*). *Let $s \geqslant 0$. Suppose that there exist $T > 0$ and a pair $\eta \in L^\infty([0,T); H^{s+5}(\mathbb{R}))$, $v \in L^\infty([0,T); H^{s+3}(\mathbb{R}))$ solving System (1.1)– (1.2). Then this pair satisfies* Eq. (4.3) *up to a reminder r bounded as*

$$\|r\|_{L_T^\infty H^s} \leqslant C\varepsilon^2 \|\eta\|_{L_T^\infty H^{s+3}} \|v\|_{L_T^\infty H^{s+3}},$$

and Eq. (4.4) *up to a reminder q bounded as*

$$\|q\|_{L_T^\infty H^s} \leqslant C\varepsilon^2 \left(\|\eta\|_{L_T^\infty H^{s+5}} + \|v\|^2_{L_T^\infty H^{s+3}}\right).$$

Proof. The proof follows from a straightforward estimation of reminders r and q. □

The energy method reapplied to System (4.1)–(4.2) can guarantee the time of existence only of order $\mathcal{O}(1/\sqrt{\varepsilon})$. Thus for the full justification one needs to prove long-time existence.

5. Computation of solitary waves

In this section we calculate numerically solitary waves corresponding to the Whitham–Boussinesq system (1.1)–(1.2) and compare them with the Euler solitary waves. We also regard evolution of Euler solitary waves with respect to System (1.1)–(1.2). This comparison supports relevance of System (1.1)–(1.2) for water waves theory. It is just an additional justification to what has been done in [1]. However, these new calculations give a complete quantitative information on the accuracy of the model when it comes to solitary wave problems. The numerical scheme implemented here is relatively simple, so one should anticipate that a similar analysis has been done with most other justified models either unidirectional or bidirectional. On the other hand accuracy calculated below is impressive and deserves to be pointed out.

For notational convenience, we use the same notations η, v for solitary waves profiles corresponding to (1.1)–(1.2). In other words, we write $\eta(x,t) = \eta(x - ct)$ and $v(x,t) = v(x - ct)$. Here c stands for a Froude number coinciding with the speed of a soliton in our non-dimensional framework. The corresponding solitary waves system has the view

$$c\eta = v + \mathcal{K}(\eta v), \tag{5.1}$$

$$cv = \mathcal{K}\eta + \mathcal{K}v^2/2 \tag{5.2}$$

where $\mathcal{K} = \tanh D/D$ is a bounded self-adjoint operator in $L_2(\mathbb{R})$. A simple heuristic analysis shows that solutions of System (5.1)–(5.2) are smooth and expected to exist for any $c > 1$. Indeed, expressing η via v by (5.2) and substituting to (5.1) one obtains

$$v = \frac{1}{c^2}\mathcal{K}v + \frac{1}{2c}\mathcal{K}v^2 + \frac{1}{c}\mathcal{K}^2(v\mathcal{K}^{-1}v) - \frac{1}{2c^2}\mathcal{K}^2v^3.$$

Clearly, operator $\mathcal{K}^{-1} - |D|$ is bounded and the operator $\mathcal{K}$ improves the smoothness of its operand by one order. So if one takes $v \in H^1$ and substitutes it to the right part of the last identity then one obviously gets $v \in H^2$, which results in the fact that both solutions v and η are infinitely smooth and there is no restriction on their amplitudes.

A use of the Petviashvili iteration method is made to calculate solitary waves [14]. Applicability of the method, as well as existence of such solutions, is out of scope of this note. The essence of the method is to split the linear $\mathcal{L}$ and the nonlinear $\mathcal{N}$ parts as follows

$$\mathcal{L}(\eta, v) = \begin{pmatrix} c & -1 \\ -\mathcal{K} & c \end{pmatrix}\begin{pmatrix} \eta \\ v \end{pmatrix}, \quad \mathcal{N}(\eta, v) = \begin{pmatrix} \mathcal{K}(\eta v) \\ \mathcal{K}v^2/2 \end{pmatrix}$$

and so System (5.1)–(5.2) can be rewritten as $\mathcal{L}(\eta, v) = \mathcal{N}(\eta, v)$. Clearly, the operator $\mathcal{L}$ is invertible if and only if $c^2 > 1$. The Petviashvili iterative scheme is defined by

$$(\eta_{n+1}, v_{n+1})^T = S_n^2\mathcal{L}^{-1}(\mathcal{N}(\eta_n, v_n))$$

where S_n is a stabilisation factor computed by

$$S_n = \frac{\int(\eta_n, v_n)\mathcal{L}(\eta_n, v_n)dx}{\int(\eta_n, v_n)\mathcal{N}(\eta_n, v_n)dx}.$$

An analogous splitting is applied to the Babenko equation describing Euler gravity solitary surface waves [14]. This is implemented in the code [15]. For time evolution performance of System (1.1)–(1.2), it is treated by the numerical scheme thoroughly described in [1].

For comparison with the fully nonlinear model we introduce the relative difference between waves η_1 and η_2 as

$$d(\eta_1, \eta_2) = \frac{\|\eta_1 - \eta_2\|_{L_2}}{\|\eta_1\|_{L_2}}. \tag{5.3}$$

As is pointed out above, solutions of System (5.1)–(5.2) are defined for any Froude number $c > 1$, whereas for the fully nonlinear solitary waves it does not exceed $c \approx 1.29421$ that is the highest possible speed

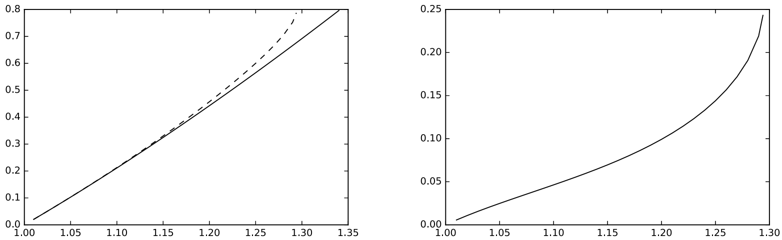

Fig. 1. Amplitude versus speed relation $a(c)$ for the Whitham system (solid line) and the Euler system (dashed line) on the left. Relative difference $\epsilon(c)$ on the right.

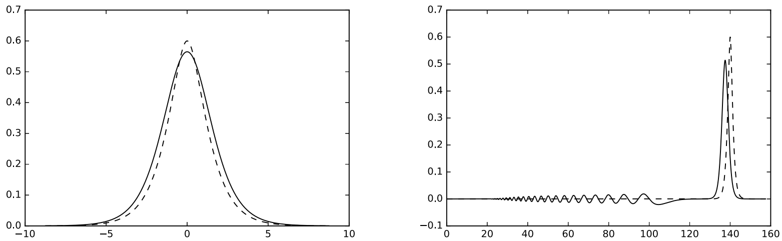

Fig. 2. Solitary waves corresponding the Froude number $c = 1.25$ for the Whitham system (solid line) and the Euler system (dashed line) on the left. Evolution of the Euler solitary wave due to the Whitham system on the right.

for solitary waves. In Fig. 1 solitons for different models are compared. In the left picture one can see the dependence of amplitude $a = \eta(0)$ on speed c. The black line corresponds to the Whitham–Boussinesq model and the dashed line to the full Euler model. In the right picture one can see the dependence on speed of the relative difference $\epsilon(c) = d(\eta_0, \eta)$, where Euler η_0 and Whitham η solitons correspond to the same speed c. It is worth to notice that even for solitary waves with amplitude of order $a = 0.4$ the error of approximation does not exceed 10%. It approaches zero when amplitudes are taken small.

In Fig. 2 approximation of relatively high solitary waves is examined. In the left picture solitons corresponding to $c = 1.25$ for different models are represented. The dashed line is for the Euler solitary wave $\eta_0(x)$. The latter is taken as an initial condition for numerical integration of System (1.1)–(1.2). Thus one can look at the time evolution of the fully nonlinear solitary wave with respect to the approximate model System (1.1)–(1.2) in the right picture in Fig. 2. The shot is taken at the moment $t = 112$. The corresponding initial data has the form

$$\eta(x, 0) = \eta_0(x), \quad v(x, 0) = \mathcal{K}(u_1 + u_2 \partial_x \eta_0)$$

where elevation η_0, horizontal u_1 and vertical u_2 velocities are associated to the Euler solitary wave moving with the speed $c = 1.25$ (the dashed line in the picture). One can see that the initial wave is diminishing leaving a dispersive tail behind. It is worth to notice that after some time this leading wave turns out to be a solitary solution of (5.1)–(5.2). More precisely, if one excludes the tail from the solution $\eta(x, t)$ then at the moment $t = 10$ minutes (according to our nondimensional settings) we have the difference $d(\eta_s, \eta) = 2.2 \cdot 10^{-5}$. Here η_s is the solution of (5.1)–(5.2) corresponding to the Froude number $c = 1.22957$. This allows us to make a conjecture about asymptotic stability of solitary waves for the regarded model (1.1)–(1.2).

6. Conclusions

The dispersive Boussinesq system (1.1)–(1.2) was derived using Hamiltonian perturbation theory by Dinvay, Dutykh and Kalisch [1]. In the current paper this system has been proved to be locally well-posed. Its accuracy as of an asymptotic model was tested with solitary waves, the latter admit a complete characterisation via speed–amplitude relation.

There are many possibilities for further study of System (1.1)–(1.2). First, it is desirable to prove long time existence and so complete the full justification of the model. Second, it is of interest to check if the model features modulational instability and wave breaking. Third, it would be interesting to try to extend the local result of the paper to a global well-posedness and possibly to prove asymptotic stability of solitary waves. Existence of the latter should be also shown rigorously.

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Paper VI

3.6 Well-posedness for a dispersive system of the Whitham-Boussinesq type

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WELL-POSEDNESS FOR A DISPERSIVE SYSTEM OF THE WHITHAM–BOUSSINESQ TYPE

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Abstract. We regard the Cauchy problem for a particular Whitham–Boussinesq system modelling surface waves of an inviscid incompressible fluid layer. We are interested in well-posedness at a very low level of regularity. We derive dispersive and Strichartz estimates, and implement them together with a fixed point argument to solve the problem locally. Hamiltonian conservation guarantees global well-posedness for small initial data in the one dimensional settings.

1. Introduction

We consider the following Whitham-type system posed on $\mathbb{R}^{1+1}$

$$\begin{cases} \partial_t\eta + \partial_x v = -K_1^2\partial_x(\eta v) \\ \partial_t v + K_1^2\partial_x\eta = -K_1^2\partial_x(v^2/2), \end{cases} \tag{1.1}$$

where

$$K_1 := K_1(D) = \sqrt{\tanh(D)/D} \quad \text{with } D = -i\partial_x. \tag{1.2}$$

The operator K_1 is a Fourier multiplier operator with the symbol $\xi \mapsto \sqrt{\tanh\xi/\xi}$. It is bounded and invertible in $L^2(\mathbb{R})$, more precisely, it is a linear isomorphism from $L^2(\mathbb{R})$ to $H^{1/2}(\mathbb{R})$. Its inverse K_1^{-1} is equivalent to the Bessel potential $J^{1/2}$ defined by the symbol $\xi \mapsto \left(1+\xi^2\right)^{1/4}$. Functions η, v are assumed to be real valued. Note that $K_1^2\partial_x = i\tanh D$ and so System (1.1) has a semilinear nature.

We complement (1.1) with the initial data

$$\eta(0) = \eta_0 \in H^s(\mathbb{R}), \qquad v(0) = v_0 \in H^{s+1/2}(\mathbb{R}), \tag{1.3}$$

where $H^s = \left(1-\partial_x^2\right)^{-s/2} L^2(\mathbb{R})$ is the standard notation for the Sobolev space of order s. Such initial value problem describes evolution with time of surface waves of a liquid layer. The model approximates the two-dimensional water wave problem for an inviscid incompressible potential flow. The variables η and v denote the surface elevation and fluid velocity, respectively. For some discussion on its precise physical meaning we refer the reader to the work by Dinvay, Dutykh and Kalisch [8], where the system (1.1) appeared for the first time. Formally, v equals $i\tanh D$-derivative of the velocity potential trace on surface associated with the irrotational velocity field. In the long wave Boussinesq regime v coincides with the horizontal fluid velocity at the surface.

The system (1.1) possesses a Hamiltonian structure [8]. To our knowledge, there are at least two conserved quantities associated with this system. The first one

$$\mathcal{H}(\eta, v) = \frac{1}{2}\int_{\mathbb{R}} \left(\eta^2 + vK_1^{-2}v + \eta v^2\right) dx \tag{1.4}$$

has the meaning of total energy. The second one

$$\mathcal{I}(\eta, v) = \int_{\mathbb{R}} \eta K_1^{-2} v dx$$

has the meaning of momentum. The system (1.1) has a Hamiltonian structure of the form

$$\partial_t(\eta, v)^T = \mathcal{J}\nabla\mathcal{H}(\eta, v)$$

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with the skew-adjoint matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -i\tanh D \\ -i\tanh D & 0 \end{pmatrix},$$

which in particular guarantees conservation of the energy functional $\mathcal{H}$. It is worth to notice that System (1.1) can be derived at least formally in the long wave asymptotic regime from the Zakharov-Craig-Sulem formulation of the water wave problem [18] also known to be Hamiltonian. The Hamiltonian structure of the Zakharov-Craig-Sulem formulation is canonical, in the sense that the corresponding skew-adjoint matrix $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It is interesting to notice that Model (1.1) also enjoys a canonical Hamiltonian structure, which is directly comparable with the one of the full water wave system, when using variables (η, ψ) where ψ is such that $v = i\tanh D\psi$. Numerical simulations done in [8] show how insignificantly values of functional $\mathcal{H}$ differ from the corresponding energy levels of the full water problem.

We also consider a system posed on $\mathbb{R}^{2+1}$ of the following Whitham-Boussinesq type

$$\begin{cases} \partial_t\eta + \nabla\cdot\mathbf{v} = -K_2^2\nabla\cdot(\eta\mathbf{v}), \\ \partial_t\mathbf{v} + K_2^2\nabla\eta = -K_2^2\nabla\left(|\mathbf{v}|^2/2\right), \end{cases} \tag{1.5}$$

where $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ is a curl free vector field, i.e., $\nabla\times\mathbf{v} = 0$, and

$$K_2 := K_2(D) = \sqrt{\tanh|D|/|D|} \quad (D = -i\nabla)$$

with the corresponding symbol $K_2(\xi) = \sqrt{\tanh(|\xi|)/|\xi|}$. We complement (1.5) with the initial data

$$\eta(0) = \eta_0 \in H^s\left(\mathbb{R}^2\right), \qquad \mathbf{v}(0) = \mathbf{v}_0 \in \left[H^{s+1/2}\left(\mathbb{R}^2\right)\right]^2. \tag{1.6}$$

This is a two dimensional analogue of System (1.1) describing evolution with time of surface waves of a liquid layer in the three dimensional physical space. As above the variables η and $\mathbf{v}$ denote the surface elevation and the fluid velocity, respectively. The system enjoys the Hamiltonian structure

$$\partial_t(\eta, \mathbf{v})^T = \mathcal{J}\nabla\mathcal{H}(\eta, \mathbf{v})$$

with the skew-adjoint matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -K_2^2\partial_{x_1} & -K_2^2\partial_{x_2} \\ -K_2^2\partial_{x_1} & 0 & 0 \\ -K_2^2\partial_{x_2} & 0 & 0 \end{pmatrix},$$

which in particular guarantees conservation of the energy functional

$$\mathcal{H}(\eta, \mathbf{v}) = \frac{1}{2}\int_{\mathbb{R}^2}\left(\eta^2 + \left|K_2^{-1}\mathbf{v}\right|^2 + \eta|\mathbf{v}|^2\right)dx. \tag{1.7}$$

Equations (1.1) were firstly proposed and studied numerically in [8]. Later in [7] the first proof of local well-posedness based on an energy method and a compactness argument was given. System (1.1) is an alternative to other weakly nonlinear dispersive models describing two-wave propagation [8]. Those models are in a good agreement with experiments [5]. They also have many peculiarities of the full water wave problem. The existing results on well-posedness theory, however, are not completely satisfactory. To our knowledge, apart from the model under consideration, there is only one local well-posedness result so far for the regarded system in [8] that have been proved by Ehrnström, Pei and Wang [11]. To achieve this the authors imposed an additional non-physical condition $\eta \geqslant C > 0$. The initial value problem regarded in [11] is probably ill-posed for large data if one removes the positivity assumption $\eta > 0$, as an heuristic argument given in [17] shows. Recently, Kalisch and Pilod [16] have proved local well posedness for a surface tension regularisation of the system from [11]. They were able to exclude the positivity assumption $\eta > 0$. However, the maximal time of existence for their regularisation is bounded by the capillary parameter. One does not need any regularisation or special non-physical conditions to claim the well posedness for (1.1), (1.3).

In fact (1.1) can be regarded itself as a regularization of the system introduced by Hur and Pandey [14]. The latter was also investigated numerically in [8] and compared with other models of Whitham-Boussinesq type. Admitting formally $\tanh D \sim D$ for small frequencies and substituting D instead of $\tanh D$ to the nonlinear part of Equations (1.1), one comes to the system regarded in [14]. Hur and Pandey have proved the Benjamin–Feir instability [14] of periodic travelling waves for their system, which makes it valuable. If one in addition formally discards the term $\eta \partial_x u$ in the system given in [14], then a new alternative system turns out to be locally well-posed and features wave breaking [15]. However, the latter does not belong to the class of Boussinesq–Whitham models since nonlinear non-dispersive terms have been neglected.

We would like to pay special attention to a system that was not considered in [8] but was introduced by Duchêne, Israwi and Talhouk [9]. They modified the bi-layer Green-Naghdi model improving the frequency dispersion. In fact, their system is also linearly fully dispersive, which makes it a close relative to System (1.1). Note that their system is Hamiltonian as well. Moreover, they have justified the Green-Naghdi modification proving well-posedness, consistency and convergence to the full water wave problem in the Boussinesq regime [9]. In addition, consistency of Hamiltonian structure is shown, so that energy levels of the approximate model can be compared with the full water energy. Existence of solitary waves for their system is also proved in [10]. Returning to the system regarded by Ehrnström, Pei and Wang [11], we should notice that a question of existence of solitary waves for it, is closed as well [20]. Finally, we point out that well-posedness of the modified Green-Naghdi model is satisfactory, in the sense that it needs neither surface tension nor any non-physical initial condition. All this together makes it a promising system. And indeed, as noticed in [9], their modification gives more reliable results when it comes to large-frequency Kelvin-Helmholtz instabilities than other models of the Green-Naghdi type.

On the contrary, System (1.1) has a couple of advantages compared with the modified Green-Naghdi model [9]. Firstly, it is derived, though not rigorously, from the Zakharov-Craig-Sulem formulation, and as a result one knows the relation between variables (η, v) and those describing the full potential fluid flow [8]. As to the modification discussed, it is presented in variables where the first one has the meaning of the surface elevation and so coincides with η. Its dual variable is called the layer-averaged horizontal velocity [9]. In the Boussinesq regime it definitely coincides with the same object associated with the full Euler equations. However, one cannot guarantee that it will be the case in shorter wave regimes. Whereas for Whitham type models one might anticipate a good agreement which is confirmed by experiments [5]. Here we must admit that neither the Whitham-Boussinesq system (1.1) nor the modified Green-Naghdi system are tested by Carter [5]. So it might be only a matter of time before the modified Green-Naghdi velocity is given an exact physical meaning. In other words, we expect that this velocity will be associated with the full water problem notions. The second issue is that it does not seem obvious how the modified Green-Naghdi system can be generalized to a three-dimensional model, whereas for System (1.1) it is straightforward.

Let us formulate the main results. The first one is an improvement of the local existence claimed in [7].

Theorem 1 (Local existence in 1d)**.** *Let $s > -1/16$. Given any $R > 0$ there exists a time $T = T(R) > 0$ such that for any initial data $(\eta_0, v_0) \in X^s := H^s(\mathbb{R}) \times H^{s+1/2}(\mathbb{R})$ with norm $\|\eta_0\|_{H^s} + \|v_0\|_{H^{s+1/2}} \le R$, there exists a solution (η, v) in the space $X^s_T := C([0,T]; H^s(\mathbb{R}) \times H^{s+1/2}(\mathbb{R}))$ of the Cauchy problem (1.1), (1.3). Moreover, the solution is unique in a subspace of X^s_T and it depends continuously on the initial data.*

Theorem 2 (Local existence in 2d)**.** *Let $s > 1/4$. Given any $R > 0$ there exists a time $T = T(R) > 0$ such that for any initial data $(\eta_0, \mathbf{v}_0) \in X^s := H^s(\mathbb{R}^2) \times \left(H^{s+1/2}(\mathbb{R}^2)\right)^2$ with $\nabla \times \mathbf{v}_0 = 0$ and with norm $\|\eta_0\|_{H^s} + \|\mathbf{v}_0\|_{(H^{s+1/2})^2} \le R$, there exists a solution $(\eta, \mathbf{v})$ in the space $X^s_T := C\left([0,T]; H^s(\mathbb{R}^2) \times \left(H^{s+1/2}(\mathbb{R}^2)\right)^2\right)$ of the Cauchy problem (1.5), (1.6). Moreover, the solution is unique in a subspace of X^s_T and it depends continuously on the initial data.*

Remark 1. For $s > 0$ in 1d and $s > 1/2$ in 2d the solution is unique in the whole space X^s_T. Moreover, the flow map is real analytic for such values of s.

Theorem 1 does not rely on the non-cavitation hypothesis $1 + \eta > 0$, since smallness of waves is implied in the model. It can be seen as a drawback comparing with the model from [9]. However, as mentioned above, it is difficult to say for now which one of these two competing models is a better approximation to the Euler equations. Instead of the non-cavitation, there is another condition that we have to impose to prove the following global result. The meaning of this new condition is that the total energy should be positive and not too big. We point out that this condition is imposed at the energy level of regularity and is independent on the regularity s of the initial data.

Theorem 3 (Global existence in 1d)**.** *Assume that* $s \geqslant 0$ *and consider the local solution from Theorem 1. There exists* $\delta > 0$ *such that if*

$$\|\eta_0\|_{L^2(\mathbb{R})} + \|v_0\|_{H^{1/2}(\mathbb{R})} \leqslant \delta$$

then the solution extends to a global-in-time solution

$$(\eta, v) \in C\left(\mathbb{R}; H^s(\mathbb{R}) \times H^{s+1/2}(\mathbb{R})\right).$$

In the sections below, we first diagonalize Systems (1.1) and (1.5) and reformulate the local theorems in the new variables. Then we demonstrate how the local result can be obtained in less general settings applying an elegant classical PDE technique based on the standard Sobolev embedding. This also demonstrates the necessity of dispersive estimates for going down to the energy level of regularity $s = 0$ in 1d. Note that the domain of the Hamiltonian functional (1.4) is $L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R})$. After that we obtain estimates of Strichartz type studying asymptotic behaviour of a particular oscillatory integral (see Lemma 9 and its proof below). This is an improvement comparing with dispersive estimates obtained in [1]. In fact we have L^∞-norm decay dominated by L^1-norm locally in frequency, which gives us localised Strichartz estimates. Whereas the decay in [1] is dominated by weighted Sobolev spaces, though frequency independent. With the new estimates in hand we can apply the fixed point argument in a ball of the Bourgain space associated with the water wave dispersion. This gives us the local existence theorems, Theorems 1 and 2.

The last step is to prove the global well-posedness theorem 3. For $s = 0$ it comes straightforwardly from the energy (1.4) conservation via the continuity argument and the local result. For $s > 0$ we prove the persistence of regularity. Surprisingly, it is not enough just to have the dispersive Strichartz estimates to claim the persistence. Thankfully, our velocity variable v is bounded in $H^{1/2}$-norm and so we are able to use the following limiting case of the Sobolev embedding theorem.

Lemma 1 (Brezis-Gallouet inequality)**.** *Suppose* $f \in H^s(\mathbb{R}^d)$ *with* $s > d/2$*. Then*

$$\|f\|_{L^\infty} \leqslant C_{s,d}\left(1 + \|f\|_{H^{d/2}}\sqrt{\log(2 + \|f\|_{H^s})}\right). \tag{1.8}$$

Inequality (1.8) was firstly put forward and proved for a domain in $\mathbb{R}^d$ with $d = 2$ in the work by Brezis, Gallouet [3]. It was extended to the other Sobolev spaces in [4]. An implementation of this inequality for deriving a global a priori estimate can be found, for example, in the work by Ponce [21] on the global well-posedness of the Benjamin-Ono equation. We apply a similar trick here, and so that we repeat the formulation of Lemma 1 as it is given in [21]. This provides us with the persistence of regularity that in turn concludes the proof of Theorem 3.

Let us finally give some explanations for the choice of strategy, focusing on the one dimensional case. The local well-posedness for $s > 0$ follows from the standard technique related to semilinear equations. It requires only Duhamel's formula and suitable product estimates for the right hand side of (1.1) in the Sobolev-based space $X^s = H^s \times H^{s+1/2}$. Global bound in X^0 follows from the Hamiltonian conservation, since $\mathcal{H}(\eta, v) \approx \|(\eta, v)\|^2_{X^0}$ provided $\|(\eta, v)\|_{X^0}$ is small. Hence the global well-posedness in X^s with $s > 0$ follows from the local result and an a priori bound obtained from the persistence of regularity and the Brezis-Gallouet inequality.

The main focus of the work is on lowering the regularity threshold for the local well-posedness through the use of dispersive estimates. One anticipates that even the weak dispersive properties of System (1.1) can lower the threshold at least to the limit case $s = 0$. This together with the global bound automatically gives us the global well-posedness in X^0. However, the weakness of dispersion means that the time-decaying $L^1 \to L^\infty$-boundedness of the semigroup, associated with the linearised system, does not hold. As a result the standard strategy based on Strichartz estimates is unavailable. So instead, we obtain the decay estimate on each component of the dyadic Littlewood-Paley decomposition with a sharp dependence on the dyadic number. From this local decay we deduce bilinear estimates in the Bourgain space associated with the water wave dispersion relation. The local well-posedness is deduced from Duhamel's formula with the help of these bilinear estimates.

The main peculiarity of the two dimensional case is that with this technique we are able to prove the local well-posedness in $X^s = H^s \times H^{s+1/2} \times H^{s+1/2}$ only for $s > 1/4$. It still leaves a gap from the energy space X^0, too big to claim global existence. Moreover, even in 1d it is not clear so far if the problem is globally well-posed for some $s \in (-1/16, 0)$.

2. Diagonalization of (1.1) and (1.5), and reformulations of the local existence theorems

We diagonalize (1.1) as follows. Defining the new variables

$$u_1^+ = \frac{K_1\eta + v}{2K_1}, \qquad u_1^- = \frac{K_1\eta - v}{2K_1}$$

we have

$$\eta = u_1^+ + u_1^-, \quad v = K_1(u_1^+ - u_1^-). \tag{2.1}$$

Then we can write the equation for $u_1^\pm$ as follows:

$$\begin{aligned} 2K_1\partial_t u_1^\pm &= K_1\eta_t \pm v_t \\ &= -K_1\partial_x v - K_1^3\partial_x(\eta v) \mp K_1^2\partial_x\eta \mp K_1^2\partial_x(v^2/2) \\ &= \mp iDK_1(K_1\eta \pm v) - iDK_1^2[K_1(\eta v) \pm v^2/2]. \end{aligned}$$

Thus,

$$i\partial_t u_1^\pm = \pm DK_1 u_1^\pm + \frac{DK_1}{2}[K_1(\eta v) \pm v^2/2]. \tag{2.2}$$

The nonlinear terms can also be written in terms of $u_1^\pm$ as

$$\eta v = (u_1^+ + u_1^-)K_1(u_1^+ - u_1^-), \qquad v^2 = [K_1(u_1^+ - u_1^-)]^2. \tag{2.3}$$

Now let

$$m_1(D) = DK_1(D).$$

From (2.2)–(2.3) we see that the system (1.1) transforms to

$$\begin{cases} (i\partial_t - m_1(D))u_1^+ = B_1^+(u_1^+, u_1^-), \\ (i\partial_t + m_1(D))u_1^- = B_1^-(u_1^+, u_1^-), \end{cases} \tag{2.4}$$

where

$$4B_1^\pm(u_1^+, u_1^-) = DK_1\left[2K_1\left\{(u_1^+ + u_1^-)K_1(u_1^+ - u_1^-)\right\} \pm [K_1(u_1^+ - u_1^-)]^2\right], \tag{2.5}$$

The initial data (1.3) transforms to

$$u_1^\pm(0) = f_1^\pm := \frac{K_1\eta_0 \pm v_0}{2K_1} \in H^s(\mathbb{R}), \tag{2.6}$$

where we used the fact that $K_1(\xi) \sim \langle\xi\rangle^{-1/2}$, and hence

$$\|K_1^{-1}v_0\|_{H^s(\mathbb{R})} \sim \|\langle D\rangle^{1/2}v_0\|_{H^s(\mathbb{R})} = \|v_0\|_{H^{s+1/2}(\mathbb{R})}.$$

Here and below we use the notation $\langle\xi\rangle = \sqrt{1+\xi^2}$, so $\langle D\rangle = J$ is the Bessel potential of order -1.

To diagonalize (2.9) we define

$$u_2^\pm = \frac{K_2|D|\eta \mp i\nabla\cdot\mathbf{v}}{2K_2|D|}$$

Hence

$$\eta = u_2^+ + u_2^- \quad \mathbf{v} = -i|D|^{-1}K_2\nabla(u_2^+ - u_2^-), \tag{2.7}$$

where we used the fact that $\mathbf{v}$ is curl free which in turn implies $\nabla\nabla\cdot\mathbf{v} = \Delta\mathbf{v} = -|D|^2\mathbf{v}$. Then the equations for $u_2^\pm$ are written as follows:

$$\begin{aligned} 2K_2|D|\partial_t u_2^\pm &= K_2|D|\eta_t \mp i\nabla\cdot\mathbf{v}_t \\ &= \mp iK_2|D|(K_2|D|\eta \mp i\nabla\cdot\mathbf{v}) + i|D|^2K_2^2[K_2|D|^{-1}(i\nabla)\cdot(\eta\mathbf{v}) \pm (|\mathbf{v}|^2)/2]. \end{aligned}$$

Thus,

$$i\partial_t u_2^\pm = \pm|D|K_2u_2^\pm - \frac{|D|K_2}{2}[iK_2R\cdot(\eta\mathbf{v}) \mp |\mathbf{v}|^2/2)], \tag{2.8}$$

where $R = (R_1, R_2)$ with $R_j = \partial_j/|D|$ being the Riesz transforms. Now setting

$$m_2(D) := |D|K_2(D)$$

and combining the equations (2.7)–(2.8) we see that the system (2.9) transforms to

$$\begin{cases} (i\partial_t - m_2(D))u_2^+ = B_2^+(u_2^+, u_2^-), \\ (i\partial_t + m_2(D))u_2^- = B_2^-(u_2^+, u_2^-), \end{cases} \tag{2.9}$$

where

$$4B_2^\pm(u_2^+, u_2^-) = -|D|K_2\left[2K_2R\left\{(u_2^+ + u_2^-)K_2R(u_2^+ - u_2^-)\right\} \mp \left|K_2R(u_2^+ - u_2^-)\right|^2\right]. \tag{2.10}$$

The initial data (1.6) transforms to

$$u_2^\pm(0) = f_2^\pm := \frac{K_2|D|\eta_0 \mp i\nabla\cdot\mathbf{v}_0}{2K_2|D|} \in H^s(\mathbb{R}), \tag{2.11}$$

where we used the fact that $K_2(\xi) \sim \langle\xi\rangle^{-1/2}$.

Now let us reformulate Theorem 1 and Theorem 2 in terms of the new variables as follows.

Theorem 4. *Let $s > -1/16$. Given any $R > 0$ there exists a time $T = T(R) > 0$ such that for any initial data $(f_1^+, f_1^-) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with norm $\|f_1^+\|_{H^s(\mathbb{R})} + \|f_1^-\|_{H^s(\mathbb{R})} \le R$, the Cauchy problem (2.4)–(2.6) has a solution*

$$(u_1^+, u_1^-) \in C\left([0,T]; H^s(\mathbb{R}) \times H^s(\mathbb{R})\right).$$

Moreover, the solution is unique in a subset of this space and depends continuously on the data.

Theorem 5. *Let $s > 1/4$. Given any $R > 0$ there exists a time $T = T(R) > 0$ such that for any initial data $(f_2^+, f_2^-) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ with norm $\|f_2^+\|_{H^s(\mathbb{R}^2)} + \|f_2^-\|_{H^s(\mathbb{R}^2)} \le R$, the Cauchy problem (2.9)–(2.11) has a solution*

$$(u_2^+, u_2^-) \in C\left([0,T]; H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)\right).$$

Moreover, the solution is unique in a subset of this space and depends continuously on the data.

The system (2.4)–(2.6) can be written in the form of integral equations as

$$u_1^\pm(t) = e^{\mp itm_1(D)}f_1^\pm \mp i\int_0^t e^{\mp i(t-s)m_1(D)}B_1^\pm(u_1^+, u_1^-)(s)\,ds. \tag{2.12}$$

Similarly, the system (2.9)–(2.11) can be written in the form of integral equations as

$$u_2^\pm(t) = e^{\mp itm_2(D)}f_2^\pm \mp i\int_0^t e^{\mp i(t-s)m_2(D)}B_2^\pm(u_2^+, u_2^-)(s)\,ds. \tag{2.13}$$

Applying the contraction argument to (2.12) together with the Sobolev embedding one can prove Theorem 4 for $s > 0$ and Theorem 5 for $s > 1/2$, as shown in the next section. However, to prove

Theorem 4 for $s > -1/16$ and Theorem 5 for $s > 1/4$ we need to derive dispersive estimates on the semigroups $S_{m_d}(\pm t) := e^{\mp itm_d(D)}$, where

$$m_1(\xi) = \xi K_1(\xi) = \xi\sqrt{\frac{\tanh \xi}{\xi}} \qquad (\xi \in \mathbb{R}),$$

$$m_2(\xi) = |\xi| K_2(\xi) = |\xi|\sqrt{\frac{\tanh |\xi|}{|\xi|}} \qquad (\xi \in \mathbb{R}^2).$$

3. Non-dispersive estimates

3.1. **Local well-posedness for $s > 0$ in 1d.** In this section we prove the local well-posedness in $H^s \times H^{s+1/2}$ with $s > 0$ for System (1.1) applying a fixed-point argument. It is only a particular case of Theorem 1 (or of the equivalent theorem 4). In this sense, the section has mainly an illustrative character. However, the proof is elegant and does not need any use of dispersive techniques. The idea is close to the one used in [2], for instance. This allows us to think about System (1.1) as a fully dispersive bi-directional relative to the BBM equation.

Regard the Whitham operator $K = \sqrt{\tanh D/D}$ and introduce the space $X^s = H^s \times H^{s+1/2}$ equipped with the norm

$$\|(f,g)\|_{X^s}^2 = \|f\|_{H^s}^2 + \|K^{-1}g\|_{H^s}^2 \tag{3.1}$$

that is obviously equivalent to the standard one. Denote by X_T^s the space of continuous functions defined on $[0,T]$ with values in X^s, equipped with the supremum-norm. Define matrices

$$\mathcal{K} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ K & -K \end{pmatrix}, \quad \mathcal{K}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & K^{-1} \\ 1 & -K^{-1} \end{pmatrix}.$$

Clearly, that $\mathcal{K}$ is isometric from $H^s \times H^s$ to X^s for any $s \in \mathbb{R}$, i. e. $\|\mathcal{K}(f,g)^T\|_{X^s} = \|(f,g)\|_{H^s \times H^s}$. Regard the unitary group

$$\mathcal{S}(t) = \mathcal{K}\begin{pmatrix} e^{-itm} & 0 \\ 0 & e^{itm} \end{pmatrix}\mathcal{K}^{-1}$$

where $m = m(D) = \sqrt{D\tanh D}\,\mathrm{sgn}\, D$. Note that for any $s,t \in \mathbb{R}$, $u \in X^s$ holds $\|\mathcal{S}(t)u\|_{X^s} = \|u\|_{X^s}$ and consequently $\|\mathcal{S}(t)u\|_{X_T^s} = \|u\|_{X_T^s}$ for any $T > 0$. These follow from isometricity of operators $\mathcal{K}$, $\mathcal{K}^{-1}$ and that symbols of eigenvalues of $\mathcal{S}(t)$ have absolute value equal to one. For any fixed $u_0 = (\eta_0, v_0)^T \in X^s$ function $\mathcal{S}(t)u_0$ solves the linear initial-value problem associated with (1.1). Regard a mapping $\mathcal{A} : X_T^s \to X_T^s$ defined by

$$\mathcal{A}(\eta,v) = \mathcal{A}(\eta,v;u_0)(t) = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t-t')(-i\tanh D)\begin{pmatrix} \eta v \\ v^2/2 \end{pmatrix}(t')dt'. \tag{3.2}$$

Then the Cauchy problem for System (1.1) with the initial data u_0 may be rewritten equivalently as an equation in X_T^s of the form

$$u = \mathcal{A}(u;u_0) \tag{3.3}$$

where $u = (\eta,v)^T \in X_T^s$. Below the latter integral equation is solved locally in time by making use of Picard iterations.

Lemma 2 (Particular case of Theorem 1)**.** *Let $s > 0$, $u_0 = (\eta_0, v_0)^T \in X^s$ and $T = (7C_s\|u_0\|_{X^s})^{-1}$ with some constant $C_s > 0$ depending only on s. Then there exists a unique solution $u = (\eta, v)^T \in X_T^s$ of Problem (3.3).*

Moreover, for any $R > 0$ there exists $T = T(R) > 0$ such that the flow map associated with Equation (3.3) is a real analytic mapping of the open ball $B_R(0) \subset X^s$ to X_T^s.

Proof. The idea is to show that the restriction of $\mathcal{A}$ on some closed ball B_M centered at $\mathcal{S}(t)u_0$ is a contraction mapping. The key ingredient is the product estimate $\|\eta v\|_{H^s} \lesssim \|\eta\|_{H^s}\|v\|_{H^{s+1/2}}$ that can be found, for example in [13]. Obviously, there exists a positive constant C_s such that

$$\|(\eta v, v^2/2)\|_{X^s} \leqslant C_s\|(\eta,v)\|_{X^s}^2$$

and

$$\|(\eta_1 v_1 - \eta_2 v_2, v_1^2/2 - v_2^2/2)\|_{X^s} \leqslant C_s \|(\eta_1 - \eta_2, v_1 - v_2)\|_{X^s} (\|(\eta_1, v_1)\|_{X^s} + \|(\eta_2, v_2)\|_{X^s}).$$

Thus for any $T, M > 0$ and $u, u_1, u_2 \in B_M \subset X_T^s$ hold

$$\|\mathcal{A}(u) - \mathcal{S}(t)u_0\|_{X_T^s} \leqslant \int_0^T \|(\eta v, v^2/2)\|_{X^s} \leqslant C_s T \|u\|_{X_T^s}^2,$$

$$\|\mathcal{A}(u_1) - \mathcal{A}(u_2)\|_{X_T^s} \leqslant C_s T \|u_1 - u_2\|_{X_T^s} (\|u_1\|_{X_T^s} + \|u_2\|_{X_T^s}),$$

and so taking $M = 2\|u_0\|_{X^s}$ and T as in the lemma formulation we conclude that $\mathcal{A}$ is a contraction in the closed ball B_M. The first statement of the lemma follows from the contraction mapping principle.

We turn our attention to smoothness of the flow map. Let $R > 0$, $T = (7C_s R)^{-1}$ and $B = B_R(0)$ be an open ball in X^s. Define $\Lambda : B \times X_T^s \to X_T^s$ as

$$\Lambda(u_0, u) = u - \mathcal{A}(u; u_0)$$

that is obviously a smooth map. Its Fréchet derivative with respect to the second variable is defined by

$$d_u \Lambda(u_0, u)h = h + i \int_0^t \mathcal{S}(t - t') \tanh D \begin{pmatrix} v & \eta \\ 0 & v \end{pmatrix} h(t') dt'$$

where $u = (\eta, v)^T$ and $h \in X_T^s$. If $u_1 \in X_T^s$ is the solution of Problem (3.3) corresponding the initial data $u_0 \in B$ then $\Lambda(u_0, u_1) = 0$. Moreover, it satisfies the following estimate

$$\|u_1(t)\|_{X^s} \leqslant \|u_0\|_{X^s} + C_s \int_0^t \|u_1(t')\|_{X^s}^2 dt'$$

and so

$$\int_0^t \|u_1(t')\|_{X^s}^2 dt' \leqslant \frac{t\|u_0\|_{X^s}^2}{1 - C_s t \|u_0\|_{X^s}}$$

for any t. The latter is used to estimate operator $I - d_u \Lambda(u_0, u_1)$ as follows

$$\begin{aligned} \|h - d_u \Lambda(u_0, u_1)h\| &\leqslant C_s \sup_{t \in [0,T]} \int_0^t \|u_1(t')\|_{X^s} \|h(t')\|_{X^s} dt' \\ &\leqslant C_s \sup_{t \in [0,T]} \left(t \int_0^t \|u_1(t')\|_{X^s}^2 dt' \right)^{1/2} \|h\|_{X_T^s} \leqslant \frac{C_s T \|u_0\|_{X^s}}{\sqrt{1 - C_s T \|u_0\|_{X^s}}} \|h\|_{X_T^s} \leqslant \frac{1}{\sqrt{42}} \|h\|_{X_T^s} \end{aligned}$$

which is true for any $h \in X_T^s$. As a result operator $d_u \Lambda(u_0, u_1)$ is invertible and so the second assertion of the lemma follows from the Implicit Function Theorem. □

The next and most difficult step is to extend the statement of the lemma to the case $s \leqslant 0$ as well. Even extension to the limiting case $s = 0$ is not trivial. On the one hand, it seems possible to do it without the dispersive estimates, applying the energy method, for example. Indeed, we have the Hamiltonian conservation that can provide us with a necessary a priori bound (see Lemma 4 below). However, at such level of regularity with $s = 0$ the regularization of System (1.1) can be a serious issue. In other words, one cannot guarantee that the a priori estimate will be still valid for the regularised problem. Moreover, we can hardly hope for more than a weak solution after implementing the compactness argument. So we turn our attention to the Harmonic Analysis methods, since we can eventually achieve a more general result with the dispersive estimates obtained below in the next sections.

3.2. **Local well-posedness for $s > 1/2$ in 2d.** The proof is essentially the same. Now the change of variables has the form

$$\mathcal{K} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -iKR_1 & iKR_1 \\ -iKR_2 & iKR_2 \end{pmatrix},$$

where $K = \sqrt{\tanh|D|/|D|}$. Then $\mathcal{K}$ is an isometric operator from $H^s \times H^s$ to the subspace X^s of $H^s \times \left(H^{s+1/2}\right)^2$ with the curl free second coordinate and endowed with the norm $\left\|\mathcal{K}^{-1}(\eta, \mathbf{v})^T\right\|_{H^s \times H^s}$. This $\mathcal{K}$ defines a continuous group $\mathcal{S}(t)$ as above. For any fixed $u_0 = (\eta_0, \mathbf{v}_0)^T \in X^s$ function $\mathcal{S}(t)u_0$ solves the linear initial-value problem associated with (1.5) in $X_T^s = C([0,T]; X^s)$. Considering the map $\mathcal{A} : X_T^s \to X_T^s$ defined by

$$\mathcal{A}(\eta, \mathbf{v}; u_0)(t) = \mathcal{S}(t)u_0 - \int_0^t \mathcal{S}(t-t') \begin{pmatrix} K^2\nabla \cdot (\eta \mathbf{v}) \\ K^2\nabla\left(|\mathbf{v}|^2/2\right) \end{pmatrix} (t')dt' \tag{3.4}$$

we reduce the Cauchy problem for System (1.5) with the initial data u_0 to Equation (3.3) in X_T^s again, with the only difference that now $u = (\eta, \mathbf{v})^T \in X_T^s$ is a three component vector.

Lemma 3 (Particular case of Theorem 2)**.** *Let $s > 1/2$, $u_0 \in X^s$ and $T = (7C_s\|u_0\|_{X^s})^{-1}$ with some constant $C_s > 0$ depending only on s. Then there exists a unique solution $u \in X_T^s$ of Problem (3.3).*

Moreover, for any $R > 0$ there exists $T = T(R) > 0$ such that the flow map associated with Equation (3.3) is a real analytic mapping of the open ball $B_R(0) \subset X^s$ to X_T^s.

As above the key ingredient is the same product estimate that in the case $d = 2$ is valid only provided $s > 1/2$, and so we omit the proof.

3.3. **A priori estimates for $s \geqslant 0$ in 1d.** Firstly, we prove the following global bound in the energy space X^0.

Lemma 4. *There exists a constant $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, if a pair $u(t) = (\eta(t), v(t)) \in L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ having initial condition $\|u_0\|_{L^2 \times H^{1/2}} \leqslant \epsilon/2$, solves System (1.1), then its norm remains bounded $\|u(t)\|_{L^2 \times H^{1/2}} \leqslant \epsilon$ for any time t.*

Proof. We use a continuity argument. Without loss of generality we prove the statement with the X^0-norm defined in (3.1), which is equivalent to the $L^2 \times H^{1/2}$-norm. For $u = (\eta, v)$, define

$$\|u\|^2 := \frac{1}{2}\|u\|_{X^0}^2 = \frac{1}{2}\|\eta\|_{L^2}^2 + \frac{1}{2}\|K^{-1}v\|_{L^2}^2.$$

Then there exists $C > 0$ such that

$$\|u\|^2(1 - C\|u\|) \leqslant \mathcal{H}(u) \leqslant \|u\|^2(1 + C\|u\|)$$

where $u = u(t)$ is a solution of (1.1) defined on some interval. Take $\epsilon_0 = (2C)^{-1}$, any $0 < \epsilon \leqslant \epsilon_0$ and a solution with $u_0 = u(0)$ having $\|u_0\| \leqslant \epsilon/2$. By continuity $\|u\| \leqslant \epsilon$ on some $[0, T_\epsilon]$ and so

$$\|u\| \leqslant \sqrt{2\mathcal{H}(u)} = \sqrt{2\mathcal{H}(u_0)} \leqslant \sqrt{\frac{1 + C\epsilon/2}{2}}\epsilon < \epsilon$$

which means that the continuous function $\|u(t)\|$ cannot touch the level ϵ with time. □

Proving the next lemma we will employ a sharper variant of the bilinear estimates used at the beginning of the proof of Lemma 2. Recall the notation $\|(\eta, v)\|_{X^s}$ defined by (3.1).

Lemma 5 (Persistence of regularity)**.** *Suppose $s > 0$ and a pair $\eta(t) \in H^s$, $v(t) \in H^{s+1/2}$ solves Problem (1.1), (1.3). Then if $s < 1/2$ the following holds true*

$$\|(\eta, v)(t)\|_{X^s} \leqslant \|(\eta_0, v_0)\|_{X^s} + C_s \int_0^t \left(\|v\|_{H^{1/2}} + \|v\|_{L^\infty}\right) \|(\eta, v)\|_{X^s},$$

and if $s \geqslant 1/2$ then

$$\|(\eta, v)(t)\|_{X^s} \leqslant \|(\eta_0, v_0)\|_{X^s} + C_s \int_0^t \|v\|_{H^{s+1/4}} \|(\eta, v)\|_{X^s}$$

where constant C_s depends only on s.

Proof. Estimating $\mathcal{A}(t)$ given by (3.2) in X^s-norm defined by (3.1), one deduces from (3.3) the following inequality

$$\|(\eta, v)(t)\|_{X^s} \leqslant \|(\eta_0, v_0)\|_{X^s} + \int_0^t \left\| \begin{pmatrix} \tanh D(\eta v) \\ \tanh D(v^2/2) \end{pmatrix} (t') \right\|_{X^s} dt'.$$

It is left to calculate the integrand. Provided $s \in (0, 1/2)$ by the Leibniz rule [19] we have

$$\|J^s \tanh D(\eta v)\|_{L^2} \lesssim \|J^s \eta\|_{L^{p_1}} \|v\|_{L^{q_1}} + \|\eta\|_{L^{p_2}} \|J^s v\|_{L^{q_2}} \tag{3.5}$$

where setting $p_1 = 2$, $q_1 = \infty$, $p_2 = 2/(1-2s)$, $q_2 = 1/s$ and using the Sobolev embedding obtain

$$\|J^s \tanh D(\eta v)\|_{L^2} \lesssim \|\eta\|_{H^s} \left(\|v\|_{L^\infty} + \|v\|_{H^{1/2}}\right).$$

Similarly, but now for any $s \in (0, \infty)$ we have

$$\left\|J^s K^{-1} \tanh D v^2\right\|_{L^2} \lesssim \left\|J^{s+1/2} v^2\right\|_{L^2} \lesssim \|v\|_{L^\infty} \left\|J^{s+1/2} v\right\|_{L^2} \lesssim \|v\|_{L^\infty} \left\|K^{-1} v\right\|_{H^s}. \tag{3.6}$$

This implies the first inequality in the statement valid for $s \in (0, 1/2)$.

Regarding the case $s = 1/2$ and setting $p_2 = q_2 = 4$ with the same $p_1 = 2$, $q_1 = \infty$ in the Leibniz inequality (3.5), after implementation the Sobolev embedding, obtain

$$\|J^s \tanh D(\eta v)\|_{L^2} \lesssim \|\eta\|_{H^s} \|v\|_{H^{s+1/4}}.$$

This inequality is obvious for $s > 1/2$ since H^s is an algebra under the point-wise product, and so is true for any $s \geqslant 1/2$. Taking into account (3.6) we deduce the second inequality of the lemma. □

In order to use the persistence of regularity lemma 5 one needs two Gronwall inequalities. One of them is considered to be standard. For the completeness, we give here a proof of the other Gronwall type inequality, which is less standard and will be used below.

Lemma 6 (Gronwall inequality)**.** *Let $y(t) > 1$ be a continuous function defined on some interval $[0, T]$ with $y(0) = y_0$. Suppose that for any $t \in [0, T]$ hold*

$$y(t) \leqslant y_0 + C \int_0^t y \log y.$$

Then

$$y(t) \leqslant \exp\left(e^{Ct} \log y_0\right).$$

Proof. One can easily calculate

$$\frac{d}{dt} \log\log\left(y_0 + C\int_0^t y \log y\right) = \frac{C y \log y}{\left(y_0 + C\int_0^t y\log y\right) \log\left(y_0 + C\int_0^t y \log y\right)} \leqslant C$$

where we have used the dominance of $y(t)$ by the integral expression. The fundamental theorem of calculus provides us with the claim.

□

The persistence of regularity based on the energy estimate lemma 5 transforms to the following a priori estimates.

Lemma 7. *Suppose $s > 0$ and a pair $u(t) = (\eta(t), v(t)) \in X^s$ solves System (1.1) on some time interval with $u(0) = u_0$ small enough with respect to X^0-norm in the sense of Lemma 4. Then if $s < 1/2$ the following holds true*

$$\|u(t)\|_{X^s} \leqslant \exp\left(C e^{Ct}\right),$$

and if $s \geqslant 1/2$ *then*

$$\|u(t)\|_{X^s} \leqslant \|u_0\|_{X^s} \exp\left(C\int_0^t \|v\|_{H^{s+1/4}}\right)$$

where constant C *depends only on* s, $\|u(0)\|_{X^0}$ *and* $\|u(0)\|_{X^s}$.

Proof. Suppose $s \in (0,1/2)$ and $u(t) = (\eta(t), v(t)) \in X^s$ solves System (1.1) on some time interval. Let its initial data u_0 be small with respect to X^0-norm in the sense of Lemma 4. Then $u(t)$ stays bounded in X^0, and so $\|v(t)\|_{H^{1/2}}$ is bounded by the same constant independent on the time interval. Hence from the Brezis-Gallouet limiting embedding (1.8) one deduces

$$\|v(t)\|_{L^\infty} \lesssim 1 + \log\left(2 + \|v(t)\|_{H^{s+1/2}}\right)$$

and applying Lemma 5 obtain

$$\|u\|_{X^s} \leqslant \|u_0\|_{X^s} + C\int_0^t \left(1 + \log\left(2 + \|u\|_{X^s}\right)\right)\|u\|_{X^s}.$$

Introducing $y(t) = 2 + \|u(t)\|_{X^s}$ we arrive at the assumption of the Gronwall inequality lemma 6. As a result we have the estimate

$$2 + \|u\|_{X^s} \leqslant \exp\left(e^{2Ct}\log\left(2 + \|u_0\|_{X^s}\right)\right)$$

that is the first claim.

In the case $s \geqslant 1/2$ we make use of the second inequality in Lemma 5 and a more standard Gronwall inequality [23]. □

4. Dispersive estimate for $S_{m_d}(\pm t)f$

First we establish a lower bound for the first and second derivatives of the function $m(r) = r\sqrt{\tanh(r)/r}$. These estimates will be used later to derive dispersive estimates for the free waves $S_{m_d}(\pm t)f$ using a stationary phase method.

Throughout the next three sections we use the following notation: The Greek letter λ denotes a dyadic number, i.e., this variable ranges over numbers of the form 2^k for $k \in \mathbb{Z}$. In estimates we use $A \lesssim B$ as shorthand for $A \le CB$ and $A \ll B$ for $A \le C^{-1}B$, where $C \gg 1$ is a positive constant which is independent of dyadic numbers such as λ and time T, whereas $A \sim B$ means $B \lesssim A \lesssim B$.

Lemma 8. *Set* $m(r) = rK(r)$, *where* $K(r) = \sqrt{\tanh(r)/r}$. *Then for* $r > 0$,

$$0 < m'(r) \sim \langle r\rangle^{-1/2}, \tag{4.1}$$

$$0 < -m''(r) \sim r\langle r\rangle^{-5/2}. \tag{4.2}$$

Proof. First note that

$$K'(r) = \frac{r\mathrm{sech}^2(r) - \tanh(r)}{2r^2K(r)},$$

$$K''(r) = -\frac{\tanh(r)\mathrm{sech}^2(r)}{rK(r)} - \frac{\left(r\mathrm{sech}^2(r) - \tanh(r)\right)}{r^3K(r)} - \frac{\left(r\mathrm{sech}^2(r) - \tanh(r)\right)^2}{4r^4K^3(r)}$$

which imply

$$\begin{aligned} m'(r) &= K(r) + rK'(r) = \frac{K(r)}{2} + \frac{\mathrm{sech}^2(r)}{2K(r)} > 0, \\ m''(r) &= 2K'(r) + rK''(r) \\ &= -\frac{\tanh(r)\mathrm{sech}^2(r)}{K(r)} - \frac{\left(r\mathrm{sech}^2(r) - \tanh(r)\right)^2}{4r^3K^3(r)} \\ &= -\frac{1}{4r}\left[4r^2K\mathrm{sech}^2(r) + K^{-3}(r)\left(K^2(r) - \mathrm{sech}^2(r)\right)^2\right]. \end{aligned}$$

Now let us estimate $m'(r)$. One can assume without loss of generality that $r > 0$. Since

$$K(r) = \sqrt{\tanh(r)/r} \sim \langle r\rangle^{-1/2} \quad \text{and} \quad \operatorname{sech}(r) \sim e^{-r}$$

we have

$$m'(r) \sim \langle r\rangle^{-1/2} + \langle r\rangle^{1/2} e^{-2r} \sim \langle r\rangle^{-1/2}. \tag{4.3}$$

Next we estimate $m''(r)$. We can write

$$K^2(r) - \operatorname{sech}^2(r) = E(r)\operatorname{sech}^2(r),$$

where

$$E(r) = \frac{e^{2r} - e^{-2r} - 4r}{4r}.$$

Now if $0 < r < 1/2$ we have

$$E(r) = \frac{1}{2r}\sum_{n=0}^{\infty} \frac{(2r)^{2n+3}}{(2n+3)!} = 4Cr^2,$$

where $C := C(r) = \sum_{n=0}^{\infty} \frac{(2r)^{2n}}{(2n+3)!} < \infty$. If $r \geq 1/2$ we have

$$E(r) = \frac{e^{2r}}{4r}[1 - e^{-4r} - 4re^{-2r}] \sim \frac{e^{2r}}{r}.$$

Therefore,

$$E(r) \sim \begin{cases} r^2 & \text{if } 0 < r < 1/2, \\ r^{-1}e^{2r} & \text{if } r \geq 1/2. \end{cases} \tag{4.4}$$

Then using (4.3) and (4.4) we obtain

$$\begin{aligned} |m''(r)| &= \frac{1}{4|r|}\left[4r^2 K(r)\operatorname{sech}^2(r) + K^{-3}(r)E^2(r)\operatorname{sech}^4(r)\right] \\ &\sim |r|^{-1}\left[r^2\langle r\rangle^{-\frac{1}{2}} e^{-2r} + \langle r\rangle^{\frac{3}{2}} E^2(r) e^{-4r}\right] \\ &\sim |r|\langle r\rangle^{-5/2}. \end{aligned}$$

□

Next we use the estimates on the derivatives of $m(r)$ in Lemma 8 and stationary phase method to derive a frequency localized dispersive estimate for the free waves $S_m(\pm t)f$. To this end, we consider an even function $\chi \in C_0^\infty((-2, 2))$ such that $\chi(s) = 1$ if $|s| \leq 1$. Let

$$\beta(s) = \chi(s) - \chi(2s), \qquad \beta_\lambda(s) := \beta(s/\lambda),$$

where $\lambda \in 2^{\mathbb{Z}}$ is dyadic. Thus, $\operatorname{supp}\beta_\lambda \subset \{s \in \mathbb{R} : \lambda/2 \leqslant |s| \leq 2\lambda\}$. Now define the frequency projection P_λ by

$$\widehat{P_\lambda f}(\xi) = \begin{cases} \chi(|\xi|)\widehat{f}(\xi) & \text{if } \lambda = 1, \\ \beta_\lambda(|\xi|)\widehat{f}(\xi) & \text{if } \lambda > 1. \end{cases}$$

We write $f_\lambda := P_\lambda f$. Then $f = \sum_{\lambda \geq 1} f_\lambda$.

The following is the key dispersive estimate that will be crucial in the proof of Theorem 4 and Theorem 5.

Lemma 9 (Localized dispersive estimate)**.** *Let $\lambda > 1$ and $d \in \{1, 2\}$. Then we have the estimate*

$$\|S_{m_d}(\pm t)f_\lambda\|_{L_x^\infty(\mathbb{R}^d)} \lesssim \lambda^{3d/4}|t|^{-d/2}\|f\|_{L_x^1(\mathbb{R}^d)}.$$

Interpolating this with the trivial bound (by Plancherel)

$$\|S_{m_d}(\pm t)f_\lambda\|_{L_x^2(\mathbb{R}^d)} \leq \|f\|_{L_x^2(\mathbb{R}^d)},$$

we obtain the following.

Corollary 1. *Assuming $\lambda > 1$, $d \in \{1, 2\}$ and $2 \le r \le \infty$, we have*

$$\|S_{m_d}(\pm t) f_\lambda\|_{L^r_x(\mathbb{R}^d)} \lesssim \left(\lambda^{3d/4}|t|^{-d/2}\right)^{1-2/r} \|f\|_{L^{r'}_x(\mathbb{R}^d)}.$$

The remainder of this section is devoted to the proof of Lemma 9. It suffices to prove the estimate for positive times:

$$\|S_{m_d}(t) f_\lambda\|_{L^\infty_x(\mathbb{R}^d)} \lesssim \lambda^{3d/4} t^{-d/2} \|f\|_{L^1_x(\mathbb{R}^d)} \quad (t > 0). \tag{4.5}$$

One can write

$$[S_{m_d}(t) f_\lambda](x) = \mathcal{F}_x^{-1}\left[e^{itm_d(\xi)}\beta_\lambda(|\xi|)\hat{f}\right](x) = (I_{\lambda,t} * f)(x),$$

where

$$I_{\lambda,t}(x) = \mathcal{F}_x^{-1}\left[e^{itm_d(\xi)}\beta_\lambda(|\xi|)\right](x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi + itm_d(\xi)}\beta_\lambda(|\xi|)\, d\xi = \lambda^d \int_{\mathbb{R}^d} e^{i\lambda x\cdot\xi + itm_d(\lambda\xi)}\beta(|\xi|)\, d\xi. \tag{4.6}$$

Then by Young's inequality

$$\|S_{m_d}(t) f_\lambda\|_{L^\infty_x(\mathbb{R}^d)} \le \|I_{\lambda,t}\|_{L^\infty_x(\mathbb{R}^d)} \|f\|_{L^1_x(\mathbb{R}^d)}, \tag{4.7}$$

so (4.5) reduces to proving

$$\|I_{\lambda,t}\|_{L^\infty_x(\mathbb{R}^d)} \lesssim \lambda^{3d/4} t^{-d/2}. \tag{4.8}$$

But clearly,

$$\|I_{\lambda,t}\|_{L^\infty_x(\mathbb{R}^d)} \lesssim \lambda^d,$$

so in view of (4.8) it is enough to consider the case where

$$\lambda^{3d/4} t^{-d/2} \ll \lambda^d \Leftrightarrow t \gg \lambda^{-1/2}. \tag{4.9}$$

The proof of (4.8) in this case is given in the following two subsections, first for space dimension $d = 1$ and then for $d = 2$.

4.1. **Proof of (4.8) when $d = 1$.** In one dimension we have

$$I_{\lambda,t}(x) = \lambda \int_{\mathbb{R}} e^{it\phi_\lambda(\xi)}\beta(|\xi|)\, d\xi,$$

where

$$\phi_\lambda(\xi) := \lambda\xi x/t + m_1(\lambda\xi) = \lambda\xi x/t + \lambda\xi K_1(\lambda\xi).$$

Note that $m_1(\xi) = m(\xi)$, where m is as in Lemma 8. Now since the function ϕ_λ is odd we can write

$$I_{\lambda,t}(x) = 2\lambda \int_0^\infty \cos(t\phi_\lambda(\xi))\beta(\xi)\, d\xi = 2\lambda \int_{1/2}^2 \cos(t\phi_\lambda(\xi))\beta(\xi)\, d\xi.$$

Since

$$\phi'_\lambda(\xi) = \lambda\left[x/t + m'(\lambda\xi)\right], \tag{4.10}$$

$$\phi''_\lambda(\xi) = \lambda^2 m''(\lambda\xi), \tag{4.11}$$

we see from Lemma 8 that

$$0 < -\phi''_\lambda(\xi) = -\lambda^2 m''(\lambda\xi) \sim \lambda^3\langle\lambda\rangle^{-5/2} \sim \lambda^{1/2} \tag{4.12}$$

for $\xi \in [1/2, 2]$. Here we used also the assumption $\lambda \ge 1$.

4.1.1. *Non-stationary contribution.* This is the case when either (i) $x \geq 0$ or (ii) $x < 0$ and $-x/t \ll \lambda^{-1/2}$ or $-x/t \gg \lambda^{-1/2}$. Then since $m'(\lambda\xi))$ is positive and comparable to $\langle\lambda\xi\rangle^{-1/2}$ (Lemma 8), we see from (4.10) that

$$|\phi'_\lambda(\xi)| \gtrsim \lambda^{1/2} \tag{4.13}$$

for $\xi \in [1/2, 2]$. Integration by parts yields

$$\begin{aligned} I_{\lambda,t}(x) &= 2\lambda t^{-1} \int_{1/2}^{2} \frac{d}{d\xi}\left[\sin(t\phi_\lambda(\xi))\right] \left[\phi'_\lambda(\xi)\right]^{-1} \beta(\xi)\, d\xi \\ &= -2\lambda t^{-1} \int_{1/2}^{2} \sin\left(t\phi_\lambda(\xi)\right) \left[\phi'_\lambda(\xi)\right]^{-2} \left[\beta'(\xi)\phi'_\lambda(\xi) - \beta(\xi)\phi''_\lambda(\xi)\right]\, d\xi, \end{aligned} \tag{4.14}$$

hence (4.12) and (4.13) allow us to estimate

$$\begin{aligned} |I_{\lambda,t}(x)| &\leq 2\lambda t^{-1} \int_{1/2}^{2} |\phi'_\lambda(\xi)|^{-2} \left[|\beta'(\xi)||\phi'_\lambda(\xi)| + |\beta(\xi)||\phi''_\lambda(\xi)|\right]\, d\xi \\ &\lesssim \lambda t^{-1}\left[\lambda^{-1/2} + \lambda^{-1}\cdot\lambda^{1/2}\right] \\ &\sim \lambda^{1/2} t^{-1} \\ &\ll \lambda^{3/4} t^{-1/2}, \end{aligned} \tag{4.15}$$

where the last step follows by the assumption (4.9). This concludes the proof of the desired estimate (4.8) with $d = 1$ in the non-stationary case.

4.1.2. *Stationary contribution: $x < 0$ and $-x/t \sim \lambda^{-1/2}$.* In this case, we see from (4.10) that $\phi'_\lambda(\xi)$ may vanish, but this can happen for at most one point $\xi \in [1/2, 2]$, since $\xi \mapsto \phi'_\lambda(\xi)$ is strictly decreasing for $\xi > 0$ (indeed, $\phi''_\lambda(\xi)$ is negative, by Lemma 8). We consider first the case where there exists such a point in $[1/2, 2]$.

So suppose first that $\phi'_\lambda(\xi_0) = 0$ for some $\xi_0 \in [1/2, 2]$. Define

$$\delta = t^{-1/2}\lambda^{-1/4}.$$

Note that $\delta \ll 1$ by (4.9). Assuming for the moment that $1/2 \leq \xi_0 - \delta$ and $\xi_0 + \delta \leq 2$, we decompose the integral as

$$I_{\lambda,t}(x) = 2\lambda\left(\int_{1/2}^{\xi_0-\delta} + \int_{\xi_0-\delta}^{\xi_0+\delta} + \int_{\xi_0+\delta}^{2}\right)\cos(t\phi_\lambda(\xi))\beta(\xi)\, d\xi. \tag{4.16}$$

To estimate the first integral, we use integration by parts to get

$$\begin{aligned} &\left|\int_{1/2}^{\xi_0-\delta} \cos(t\phi_\lambda(\xi))\beta(\xi)\, d\xi\right| \\ &\leq t^{-1}\left|\left[\sin(t\phi_\lambda(\xi))\frac{\beta(\xi)}{\phi'_\lambda(\xi)}\right]_{\xi=1/2}^{\xi=\xi_0-\delta}\right| + t^{-1}\left|\int_{1/2}^{\xi_0-\delta} \sin(t\phi_\lambda(\xi))\left(\frac{\beta'(\xi)}{\phi'_\lambda(\xi)} - \frac{\beta(\xi)\phi''_\lambda(\xi)}{[\phi'_\lambda(\xi)]^2}\right) d\xi\right|. \end{aligned}$$

Since ϕ'_λ is positive and decreasing in the interval $[1/2, \xi_0 - \delta]$, and since ϕ''_λ is negative, we can continue the estimate by

$$\begin{aligned} &\lesssim t^{-1}\left(\frac{1}{\phi'_\lambda(\xi_0-\delta)} + \int_{1/2}^{\xi_0-\delta} \frac{-\phi''_\lambda(\xi)}{[\phi'_\lambda(\xi)]^2}\, d\xi\right) \\ &= t^{-1}\left(\frac{1}{\phi'_\lambda(\xi_0-\delta)} + \int_{1/2}^{\xi_0-\delta} \frac{d}{d\xi}\left(\frac{1}{\phi'_\lambda(\xi)}\right) d\xi\right) \\ &\leq 2t^{-1}\frac{1}{\phi'_\lambda(\xi_0-\delta)}. \end{aligned}$$

But by the mean value theorem and (4.12),

$$|\phi_\lambda'(\xi)| = |\phi_\lambda'(\xi) - \phi_\lambda'(\xi_0)| \sim \lambda^{1/2}|\xi - \xi_0| \quad \text{for } \xi \in [1/2, 2],$$

so we we conclude that

$$\left| \int_{1/2}^{\xi_0 - \delta} \cos(t\phi_\lambda(\xi))\beta(\xi)\, d\xi \right| \lesssim t^{-1}\lambda^{-1/2}\delta^{-1} = t^{-1/2}\lambda^{-1/4},$$

by the definition of δ above. The third integral in (4.16) can be estimated in a similar way, and satisfies the same estimate, while the second integral (4.16) is trivially estimated as

$$\int_{\xi_0-\delta}^{\xi_0+\delta} \cos(t\phi_\lambda(\xi))\beta(\xi)\, d\xi \lesssim \delta = t^{-1/2}\lambda^{-1/4}.$$

Summing up the three contributions, we conclude that the desired estimate holds,

$$|I_{\lambda,t}(x)| \lesssim \lambda^{3/4}t^{-1/2},$$

in the stationary case under the assumptions that $\phi_\lambda'(\xi_0) = 0$ for some $\xi_0 \in [1/2, 2]$, and that $1/2 \le \xi_0 - \delta$ and $\xi_0 + \delta \le 2$. If $1/2 > \xi_0 - \delta$ or $\xi_0 + \delta > 2$, the above argument is easily modified. For example, if $\xi_0 + \delta > 2$, we split the integral as $\int_{1/2}^{\xi_0-\delta} + \int_{\xi_0-\delta}^{2}$ instead; the first integral is then treated as above and the second is trivially $O(\delta)$.

It remains to prove the estimate when the function ϕ_λ' has no zero in $[1/2, 2]$, so it is either positive or negative everywhere in that interval. Since the arguments for these two cases are similar, we just treat the case where $\phi_\lambda' < 0$ in $[1/2, 2]$. Then we split the integral as

$$I_{\lambda,t}(x) = 2\lambda \left(\int_{1/2}^{1/2+\delta} + \int_{1/2+\delta}^{2-\delta} + \int_{2-\delta}^{2} \right) \cos(t\phi_\lambda(\xi))\beta(\xi)\, d\xi.$$

The first and third integrals are trivially dominated in absolute value by δ, while for the second integral we use integration by parts, estimating

$$\begin{aligned}
\left| \int_{1/2+\delta}^{2-\delta} \cos(t\phi_\lambda(\xi))\beta(\xi)\, d\xi \right| &\lesssim t^{-1} \left(\frac{1}{-\phi_\lambda'(1/2+\delta)} + \int_{1/2+\delta}^{2-\delta} \frac{-\phi_\lambda''(\xi)}{[\phi_\lambda'(\xi)]^2}\, d\xi \right) \\
&= t^{-1} \left(\frac{1}{-\phi_\lambda'(1/2+\delta)} + \int_{1/2+\delta}^{2-\delta} \frac{d}{d\xi}\left(\frac{1}{\phi_\lambda'(\xi)} \right) d\xi \right) \\
&\le 2t^{-1} \frac{1}{-\phi_\lambda'(1/2+\delta)}.
\end{aligned}$$

Here we used the fact that ϕ_λ' is negative and decreasing in the interval $[1/2, 2]$, and that ϕ_λ'' is negative. Using the mean value theorem and the estimate (4.12) on the second derivative, we find moreover that

$$-\phi_\lambda'(1/2+\delta) \ge \phi_\lambda'(1/2) - \phi_\lambda'(1/2+\delta) \sim \lambda^{1/2}\delta,$$

so we conclude that

$$\left| \int_{1/2+\delta}^{2-\delta} \cos(t\phi_\lambda(\xi))\beta(\xi)\, d\xi \right| \lesssim t^{-1}\lambda^{-1/2}\delta^{-1} = t^{-1/2}\lambda^{-1/4},$$

as desired.

4.2. **Proof of** (4.8) **when** $d = 2$**.** In two dimensions we have

$$I_{\lambda,t}(x) = \lambda^2 \int_{\mathbb{R}^2} e^{i\lambda x\cdot\xi + itm_2(\lambda\xi)}\beta(\xi)\, d\xi$$

which is the inverse Fourier transform of the radial function $\lambda^2 e^{itm_2(\lambda\xi)}\beta(\xi)$, and hence $I_{\lambda,t}(x)$ is also radial. So we may set $x = (|x|, 0)$. Then in polar coordinates we have

$$I_{\lambda,t}(x) = \lambda^2 \int_0^\infty \int_0^{2\pi} e^{i\lambda r|x|\cos\theta} e^{itm_2(\lambda r)} r\beta(r)\, d\theta dr.$$

We can write

$$\begin{aligned}\int_0^{2\pi} e^{i\lambda r|x|\cos\theta}\, d\theta &= \int_0^{\pi} \left(e^{i\lambda r|x|\cos\theta} + e^{-i\lambda r|x|\cos\theta}\right) d\theta \\ &= 2\int_{-1}^{1} e^{i\lambda r|x|s}\left(1-s^2\right)^{-1/2} ds \\ &= 2\pi J_0(\lambda r|x|),\end{aligned}$$

where $J_k(r)$ is the Bessel function:

$$J_k(r) = \frac{(r/2)^k}{\Gamma(k+1/2)\sqrt{\pi}} \int_{-1}^{1} e^{irs}\left(1-s^2\right)^{k-1/2} ds \quad \text{for } k > -1/2.$$

Thus,

$$I_{\lambda,t}(x) = 2\pi\lambda^2 \int_{1/2}^{2} e^{itm(\lambda r)} J_0(\lambda r|x|)\tilde{\beta}(r)\, dr, \tag{4.17}$$

where $\tilde{\beta}(r) = r\beta(r)$ and $m(r) = m_2(r)$.

We shall use the following properties of $J_k(r)$ for $k > -1/2$ and $r > 0$ (See [12, Appendix B] and [22].)

$$J_k(r) \le Cr^k, \tag{4.18}$$

$$J_k(r) \le Cr^{-1/2}, \tag{4.19}$$

$$\partial_r\left[r^{-k}J_k(r)\right] = -r^{-k}J_{k+1}(r) \tag{4.20}$$

Moreover, we can write

$$J_0(s) = e^{is}h(s) + e^{-is}\bar{h}(s) \tag{4.21}$$

for some function h satisfying the estimate

$$|\partial_r^j h(r)| \le C_j\langle r\rangle^{-1/2-j} \quad \text{for all } j \ge 0. \tag{4.22}$$

We treat the cases $|x| \lesssim \lambda^{-1}$ and $|x| \gg \lambda^{-1}$ separately.

4.2.1. *Case 1:* $|x| \lesssim \lambda^{-1}$. By (4.18) and (4.20) we have for all $r \in (1/2, 2)$ the estimate

$$\left|\partial_r^j J_0(\lambda r|x|)\right| \lesssim 1 \quad \text{for } j = 0, 1. \tag{4.23}$$

We integrate by parts (4.17) to obtain

$$\begin{aligned}I_{\lambda,t}(x) &= -2\pi i\lambda t^{-1} \int_{1/2}^{2} \frac{d}{dr}\left\{e^{itm(\lambda r)}\right\}[m'(\lambda r)]^{-1} J_0(\lambda r|x|)\tilde{\beta}(r)\, dr \\ &= 2\pi i\lambda t^{-1} \int_{1/2}^{2} e^{itm(\lambda r)}[m'(\lambda r)]^{-1}\partial_r\left[J_0(\lambda r|x|)\tilde{\beta}(r)\right] dr \\ &\quad - 2\pi i\lambda t^{-1}\int_{1/2}^{2} e^{itm(\lambda r)}[m'(\lambda r)]^{-2}\lambda m''(\lambda r) J_0(\lambda r|x|)\tilde{\beta}(r)\, dr.\end{aligned}$$

Then applying Lemma 8 and (4.23) we obtain

$$|I_{\lambda,t}(x)| \lesssim \lambda t^{-1}\left(\lambda^{1/2} + \lambda^2\cdot\lambda^{-3/2}\right) \lesssim \lambda^{3/2}t^{-1}. \tag{4.24}$$

4.2.2. *Case 2:* $|x| \gg \lambda^{-1}$. Using (4.21) in (4.17) we write

$$I_{\lambda,t}(x) = 2\pi\lambda^2 \left\{ \int_{1/2}^{2} e^{it\phi_\lambda^+(r)} h(\lambda r|x|)\tilde{\beta}(r)\,dr + \int_{1/2}^{2} e^{-it\phi_\lambda^-(r)} \bar{h}(\lambda r|x|)\tilde{\beta}(r)\,dr \right\},$$

where

$$\phi_\lambda^\pm(r) = \lambda r|x|/t \pm m(\lambda r).$$

Set $F_\lambda(|x|, r) = h(\lambda r|x|)\tilde{\beta}(r)$. In view of (4.22) we have

$$|F_\lambda(|x|, r)| + |\partial_r F_\lambda(|x|, r)| \lesssim (\lambda|x|)^{-1/2} \tag{4.25}$$

for all $r \in (1/2, 2)$, where we also used the fact $\lambda|x| \gg 1$.

Now we write

$$I_{\lambda,t}(x) = I_{\lambda,t}^+(x) + I_{\lambda,t}^-(x),$$

where

$$I_{\lambda,t}^+(x) = 2\pi\lambda^2 \int_{1/2}^{2} e^{it\phi_\lambda^+(r)} F_\lambda(|x|, r)\,dr,$$

$$I_{\lambda,t}^-(x) = 2\pi\lambda^2 \int_{1/2}^{2} e^{-it\phi_\lambda^-(r)} \bar{F}_\lambda(|x|, r)\,dr.$$

Observe that

$$\partial_r\phi_\lambda^\pm(r) = \lambda\left[|x|/t \pm m'(\lambda r)\right], \qquad \partial_r^2\phi_\lambda^\pm(r) = \pm\lambda^2 m''(\lambda r),$$

and hence by Lemma 8,

$$|\partial_r\phi_\lambda^+(r)| \gtrsim \lambda^{1/2}, \qquad |\partial_r^2\phi_\lambda^\pm(r)| \sim \lambda^{1/2} \tag{4.26}$$

for all $r \in (1/2, 2)$, where we also used the fact that m' is positive.

Estimate for $I_{\lambda,t}^+(x)$. It is easy to estimate $I_{\lambda,t}^+(x)$ since $\partial_r\phi_\lambda^+(r)$ is never zero. Indeed, using integration by parts we have

$$\begin{aligned} I_{\lambda,t}^+(x) &= -2\pi i t^{-1}\lambda^2 \int_{1/2}^{2} \partial_r\left[e^{it\phi_\lambda^+(r)}\right]\left[\partial_r\phi_\lambda^+(r)\right]^{-1} F_\lambda(|x|, r)\,dr \\ &= 2\pi i t^{-1}\lambda^2 \int_{1/2}^{2} e^{it\phi_\lambda^+(r)} \left\{ \frac{\partial_r F_\lambda(|x|, r)}{\partial_r\phi_\lambda^+(r)} - \frac{\partial_r^2\phi_\lambda^+(r) F_\lambda(|x|, r)}{\left[\partial_r\phi_\lambda^+(r)\right]^2} \right\} dr. \end{aligned}$$

Then using (4.25) and (4.26) we have

$$|I_{\lambda,t}^+(x)| \lesssim t^{-1}\lambda^2 \cdot \lambda^{-1/2} \cdot (\lambda|x|)^{-1/2} \lesssim \lambda^{3/2} t^{-1}. \tag{4.27}$$

Estimate for $I_{\lambda,t}^-(x)$. We treat the the non-stationary and stationary cases separately. In the non-stationary case, where $|x|/t \ll \lambda^{-1/2}$ or $|x|/t \gg \lambda^{-1/2}$, we have

$$|\partial_r\phi_\lambda^-(r)| \gtrsim \lambda^{1/2},$$

and hence $I_{\lambda,t}^-(x)$ can be estimated in exactly the same way as $I_{\lambda,t}^+(x)$ above. It satisfies

$$|I_{\lambda,t}^-(x)| \lesssim \lambda^{3/2} t^{-1}. \tag{4.28}$$

It remains to consider the stationary case, where $|x|/t \sim \lambda^{-1/2}$. Note that $\partial_r\phi_\lambda^-(r)$ is strictly increasing for $r > 0$, since $\partial_r^2\phi_\lambda^-(r) = -\lambda^2 m''(\lambda r)$ is strictly positive, by Lemma 8. Thus there is at most one point $r_0 \in [1/2, 2]$ at which $\partial_r\phi_\lambda^-$ vanishes. Setting as before

$$\delta = t^{-1/2}\lambda^{-1/4},$$

we limit our attention to the case where there is such a point r_0 in $[1/2 + \delta, 2 - \delta]$; the remaining cases are treated by straightforward modifications of the following argument, much as in the 1d case in subsection 4.1.2.

We decompose

$$I^-_{\lambda,t}(x) = 2\pi\lambda^2\left(\int_{1/2}^{r_0-\delta} + \int_{r_0-\delta}^{r_0+\delta} + \int_{r_0+\delta}^{2}\right) e^{-it\phi^-_\lambda(r)} \bar{F}_\lambda(|x|,r)\,dr. \tag{4.29}$$

Integrating by parts we write the first integral as

$$\begin{aligned}
&\int_{1/2}^{r_0-\delta} e^{-it\phi^-_\lambda(r)} \bar{F}_\lambda(|x|,r)\,dr \\
&= \underbrace{it^{-1}\left[e^{-it\phi^-_\lambda(r)} \frac{\bar{F}_\lambda(|x|,r)}{\partial_r\phi^-_\lambda(r)}\right]_{r=1/2}^{r_0-\delta}}_{=:A} \underbrace{- it^{-1}\int_{1/2}^{r_0-\delta} e^{-it\phi^-_\lambda(r)}\partial_r\left(\frac{\bar{F}_\lambda(|x|,r)}{\partial_r\phi^-_\lambda(r)}\right) dr}_{=:B}.
\end{aligned}$$

Using (4.25) and noting that for $r \in [1/2, r_0-\delta]$, $\partial_r\phi^-_\lambda(r)$ is negative and increasing, while $\partial_r^2\phi^-_\lambda(r)$ is positive, we find

$$|A| \lesssim t^{-1}(\lambda|x|)^{-1/2}\frac{1}{|\partial_r\phi^-_\lambda(r_0-\delta)|}$$

and

$$\begin{aligned}
|B| &\lesssim t^{-1}(\lambda|x|)^{-1/2}\left[\int_{1/2}^{r_0-\delta}\frac{1}{|\partial_r\phi^-_\lambda(r)|}\,dr + \int_{1/2}^{r_0-\delta}\frac{\partial_r^2\phi^-_\lambda(r)}{[\partial_r\phi^-_\lambda(r)]^2}\,dr\right] \\
&= t^{-1}(\lambda|x|)^{-1/2}\left[\int_{1/2}^{r_0-\delta}\frac{1}{|\partial_r\phi^-_\lambda(r)|}\,dr + \int_{1/2}^{r_0-\delta}\partial_r\left(\frac{1}{-\partial_r\phi^-_\lambda(r)}\right) dr\right] \\
&\lesssim t^{-1}(\lambda|x|)^{-1/2}\frac{1}{|\partial_r\phi^-_\lambda(r_0-\delta)|}
\end{aligned}$$

But using (4.26) and the mean value theorem, we see that

$$|\partial_r\phi^-_\lambda(r_0-\delta)| = |\partial_r\phi^-_\lambda(r_0-\delta) - \partial_r\phi^-_\lambda(r_0)| \sim \lambda^{1/2}\delta = \lambda^{1/4}t^{-1/2}.$$

Using also the assumption $|x|/t \sim \lambda^{-1/2}$, we conclude that

$$\begin{aligned}
\left|\int_{1/2}^{r_0-\delta} e^{it\phi^-_\lambda(r)}\bar{F}_\lambda(|x|,r)\,dr\right| &\le |A| + |B| \\
&\lesssim t^{-1}(\lambda|x|)^{-1/2}\lambda^{-1/4}t^{1/2} \\
&\sim t^{-1}(\lambda^{1/2}t)^{-1/2}\lambda^{-1/4}t^{1/2} \\
&= t^{-1}\lambda^{-1/2}.
\end{aligned}$$

The third integral in (4.29) can also be estimated in a similar way, and satisfies the same estimate, while the second integral can be simply estimated as

$$\left|\int_{r_0-\delta}^{r_0+\delta} e^{it\phi^-_\lambda(r)}\bar{F}_\lambda(|x|,r)\,dr\right| \lesssim (\lambda|x|)^{-1/2}\delta \lesssim t^{-1}\lambda^{-1/2}.$$

Therefore, combining the above computations with (4.29) we have

$$\left|I^-_{\lambda,t}(x)\right| \lesssim \lambda^{3/2}t^{-1}, \tag{4.30}$$

concluding the stationary case.

In summary, from (4.27), (4.28) and (4.30) we obtain

$$|I_{\lambda,t}(x)| \le \sum_\pm |I^\pm_{\lambda,t}(x)| \lesssim \lambda^{3/2}t^{-1}$$

which is the desired estimate (4.8) with $d = 2$.

5. Function spaces, Linear and bilinear estimates

5.1. **Function spaces.** The mixed space-time Lebesgue space $L_t^q L_x^r$ on $\mathbb{R}^{d+1}$ is defined with the norm

$$\|u\|_{L_t^q L_x^r} = \|\|u(t,\cdot)\|_{L_x^r}\|_{L_t^q} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |u(t,x)|^r \, dx\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}}$$

for $1 \le q, r < \infty$ with an obvious modification when $q = \infty$ or $r = \infty$. We write $L_T^q L_x^r$ when the time variable is restricted to the interval $[0,T]$.

Define the Bourgain space $X_\pm^{s,b}$ on $\mathbb{R}^{d+1}$ by the norm

$$\|u\|_{X_\pm^{s,b}} = \left\| \langle\xi\rangle^s \langle\tau \pm m_d(\xi)\rangle^b \widetilde{u}(\xi,\tau) \right\|_{L^2_{\tau,\xi}},$$

where $\widetilde{u}$ denotes the space-time Fourier transform given by

$$\widetilde{u}(\tau,\xi) = \int_{\mathbb{R}^{d+1}} e^{-i(t\tau + x\cdot\xi)} u(t,x) \, dtdx.$$

The restriction to the time slab $(0,T) \times \mathbb{R}^d$ of the Bourgain space, denoted by $X_\pm^{s,b}(T)$, is a Banach space when equipped with the norm

$$\|u\|_{X_\pm^{s,b}(T)} = \inf\left\{ \|v\|_{X_\pm^{s,b}} : \ v = u \text{ on } (0,T) \times \mathbb{R}^d \right\}.$$

5.2. **Linear estimates.** Let us recall some of the properties of these spaces. We have

$$\sup_{0\le t\le T} \|u(t)\|_{H^s} \le C \|u\|_{X_\pm^{s,b}(T)} \quad \text{for } b > 1/2. \tag{5.1}$$

For $-1/2 < b' \le b < 1/2$ and $0 < T < 1$ we have

$$\|u\|_{X_\pm^{s,b'}(T)} \le C T^{b-b'} \|u\|_{X_\pm^{s,b}(T)}, \tag{5.2}$$

where C is independent on T. The proof for (5.1) and (5.2) can for instance be found in [23]. We recall also that for $2 \le q \le \infty$ and $b > 1/2$,

$$\|u\|_{L_t^q L_x^2} \le C \|u\|_{X_\pm^{0,b}}, \tag{5.3}$$

as can be seen by writing the left hand side as $\left\|e^{\pm itm_d(D)} u\right\|_{L_T^q L_x^2}$, applying Plancherel in x, then using Minkowski's integral inequality to switch the order of the norms to $L_\xi^2 L_t^q$, and finally applying Sobolev embedding in t.

It is well known (see, e.g., [6]) that for any $s \in \mathbb{R}$ and $b > 1/2$ one has

$$\|S_{m_d}(\pm t) f\|_{X_\pm^{s,b}(T)} \le C \|f\|_{H^s}, \tag{5.4}$$

$$\left\| \int_0^t S_{m_d}(\pm(t-t')) F(t') \, dt' \right\|_{X_\pm^{s,b}(T)} \le C \|F\|_{X_\pm^{s,b-1}(T)}, \tag{5.5}$$

where the constant $C > 0$ depends only on b.

We need the following Bernstein inequality which is valid for $1 \le p \le r \le \infty$ (see for instance [23, Appendix A]):

$$\|P_\lambda f\|_{L^r(\mathbb{R}^d)} \le C \lambda^{\frac{d}{p} - \frac{d}{r}} \|P_\lambda f\|_{L^p(\mathbb{R}^d)} \tag{5.6}$$

Another useful tool is the Hardy-Littlewood-Sobolev inequality (see [23, Appendix A]) which asserts that

$$\left\| |\cdot|^{-\alpha} * f \right\|_{L^a(\mathbb{R})} \le C \|f\|_{L^b(\mathbb{R})} \tag{5.7}$$

whenever $1 < b < a < \infty$ and $0 < \alpha < 1$ obey the scaling condition

$$\frac{1}{b} = \frac{1}{a} + 1 - \alpha.$$

Lemma 10 (Localized Strichartz estimates). *Let $\lambda > 1$ and $d \in \{1, 2\}$. Assume that $2 < q < \infty$ and $2 \le r \le \infty$ satisfy*

$$\frac{2}{q} = \frac{d}{2}\left(1 - \frac{2}{r}\right).$$

Then we have the estimate

$$\|S_{m_d}(\pm t) f_\lambda\|_{L_t^q L_x^r(\mathbb{R}^{d+1})} \lesssim \lambda^{(3d/8)(1-2/r)} \|f_\lambda\|_{L_x^2(\mathbb{R}^d)}. \tag{5.8}$$

Moreover, if $b > 1/2$, we have

$$\|u_\lambda\|_{L_t^q L_x^r(\mathbb{R}^{d+1})} \lesssim \lambda^{(3d/8)(1-2/r)} \|u_\lambda\|_{X_\pm^{0,b}}. \tag{5.9}$$

Proof. By the standard TT^*-argument, (5.8) is equivalent to the estimate

$$\|K_\lambda \star F\|_{L_t^q L_x^r(\mathbb{R}^{d+1})} \lesssim \lambda^{(3d/4)(1-2/r)} \|F\|_{L_t^{q'} L_x^{r'}(\mathbb{R}^{d+1})}, \tag{5.10}$$

where $1/q + 1/q' = 1$ and $1/r + 1/r' = 1$, and where

$$K_\lambda(x, t) = \int_{\mathbb{R}^d} e^{ix\cdot\xi \pm itm_d(\xi)} \tilde{\beta}_\lambda(\xi)\, d\xi \tag{5.11}$$

with $\tilde{\beta}_\lambda = \beta_\lambda^2$. Here $\star$ denotes the space-time convolution. Then by Corollary 1 (with β replaced by β^2, which does not affect the validity of the corollary) we have the estimate

$$\|K_\lambda(\cdot, t) * f\|_{L_x^r(\mathbb{R}^d)} \lesssim \left(\lambda^{3d/4}|t|^{-d/2}\right)^{1-2/r} \|f\|_{L_x^{r'}(\mathbb{R}^d)}.$$

Combining this with the Hardy-Littlewood-Sobolev inequality in the t variable, with $(a, b) = (q, q')$ and $\alpha = (d/2)(1 - r/2)$, we estimate

$$\begin{aligned}
\|K_\lambda \star F\|_{L_t^q L_x^r} &= \left\| \int_{\mathbb{R}} \int_{\mathbb{R}^d} K_\lambda(x - y, t - s) F(y, s)\, dy\, ds \right\|_{L_t^q L_x^r} \\
&\le \left\| \int_{\mathbb{R}} \left\| \int_{\mathbb{R}^d} K_\lambda(x - y, t - s) F(y, s)\, dy \right\|_{L_x^r} ds \right\|_{L_t^q} \\
&\lesssim \left\| \int_{\mathbb{R}} \left(\lambda^{3d/4}|t - s|^{-d/2}\right)^{1-2/r} \|F(\cdot, s)\|_{L_x^{r'}}\, ds \right\|_{L_t^q} \\
&\lesssim \lambda^{(3d/4)(1-2/r)} \left\| \|F\|_{L_x^{r'}} \right\|_{L_t^{q'}},
\end{aligned}$$

proving (5.10) and hence (5.8). By a standard argument, the latter implies (5.9) (see, for example, the proof of Lemma 2.9 in [23]). □

5.3. **Bilinear estimates.** Set $K = K(D) = K_d(D)$ for $d = 1, 2$, and note that the Fourier symbol equals in both dimensions

$$K(\xi) = \sqrt{\frac{\tanh|\xi|}{|\xi|}} \sim \langle\xi\rangle^{-1/2},$$

hence

$$\|K f_\lambda\|_{L_x^2} \lesssim \lambda^{-\frac{1}{2}} \|f_\lambda\|_{L_x^2}, \quad \||D| K^2 f_\lambda\|_{L_x^2} \lesssim \|f_\lambda\|_{L_x^2}, \quad \||D| K f_\lambda\|_{L_x^2} \lesssim \lambda^{\frac{1}{2}} \|f_\lambda\|_{L_x^2}. \tag{5.12}$$

We first note the following consequence of the localized Strichartz estimates in Lemma 10.

Lemma 11. *Let $b > 1/2$ and dyadic $\lambda_1, \lambda_2 \ge 1$. For $d = 1$ we have the estimate*

$$\|u_{\lambda_1} v_{\lambda_2}\|_{L_t^2 L_x^2} \lesssim \min(\lambda_1, \lambda_2)^{3/8} \|u_{\lambda_1}\|_{X_\pm^{0,b}} \|v_{\lambda_2}\|_{X_\pm^{0,b}}.$$

For $d = 2$ and $2 < r < \infty$, we have for all $T > 0$,

$$\|u_{\lambda_1} v_{\lambda_2}\|_{L_T^2 L_x^2} \lesssim T^{1/r} \min(\lambda_1, \lambda_2)^{3/4 + 1/(2r)} \|u_{\lambda_1}\|_{X_\pm^{0,b}} \|v_{\lambda_2}\|_{X_\pm^{0,b}}.$$

In both estimates, the signs in the $X_\pm$ norms can be chosen independently on each other.

Proof. By symmetry we may assume $1 \le \lambda_1 \le \lambda_2$. Consider first the case $d = 1$. By Hölder's inequality and (5.3),

$$\|u_{\lambda_1} v_{\lambda_2}\|_{L_t^2 L_x^2} \le \|u_{\lambda_1}\|_{L_t^4 L_x^\infty} \|v_{\lambda_2}\|_{L_t^4 L_x^2} \lesssim \|u_{\lambda_1}\|_{L_t^4 L_x^\infty} \|v_{\lambda_2}\|_{X_\pm^{0,b}},$$

so it only remains to check that

$$\|u_{\lambda_1}\|_{L_t^4 L_x^\infty} \lesssim \lambda_1^{3/8} \|u_{\lambda_1}\|_{X_\pm^{0,b}},$$

but this holds by Lemma 10 if $\lambda_1 > 1$, while if $\lambda_1 = 1$ we can use the Bernstein inequality (5.6) followed by (5.3) to obtain

$$\|u_{\lambda_1}\|_{L_t^4 L_x^\infty} \lesssim \|u_{\lambda_1}\|_{L_t^4 L_x^2} \lesssim \|u_{\lambda_1}\|_{X_\pm^{0,b}}.$$

Now consider the case $d = 2$. We apply Hölder's inequality and (5.3) to write

$$\|u_{\lambda_1} v_{\lambda_2}\|_{L_T^2 L_x^2} \le \|u_{\lambda_1}\|_{L_T^2 L_x^\infty} \|v_{\lambda_2}\|_{L_T^\infty L_x^2} \lesssim \|u_{\lambda_1}\|_{L_T^2 L_x^\infty} \|v_{\lambda_2}\|_{X_\pm^{0,b}}.$$

To estimate $\|u_{\lambda_1}\|_{L_T^2 L_x^\infty}$ we want to use Lemma 10, so we let $2 < r < \infty$ and define q by $2/q = 1 - 2/r$. Thus $1/2 = 1/q + 1/r$, so applying Hölder in t, the Bernstein inequality in x, and finally Lemma 10, we get

$$\|u_{\lambda_1}\|_{L_T^2 L_x^\infty} \le T^{1/r} \lambda_1^{2/r} \|u_{\lambda_1}\|_{L_T^q L_x^r} \lesssim T^{1/r} \lambda_1^{2/r} \lambda_1^{(3/4)(1-2/r)} \|u_{\lambda_1}\|_{X_\pm^{0,b}},$$

proving the claimed estimate in the case $\lambda_1 > 1$. If $\lambda_1 = 1$, we can apply the Bernstein inequality and (5.3), instead of Lemma 10, and again we get the desired estimate. □

We now present the key bilinear space-time estimates needed for the proof of local well-posedness.

Lemma 12. *Let $1/2 < b < 1$ and $0 < T < 1$. Assume that $s_d > -1/16$ if $d = 1$ and $s_d > 1/4$ if $d = 2$. Then we have the estimates*

$$\left\| |D| K^2 (u \cdot Kv) \right\|_{X_\pm^{s_d, b-1}(T)} \lesssim T^{1-b} \|u\|_{X_\pm^{s_d, b}} \|v\|_{X_\pm^{s_d, b}}, \tag{5.13}$$

$$\| |D| K (Ku \cdot Kv) \|_{X_\pm^{s_d, b-1}(T)} \lesssim T^{1-b} \|u\|_{X_\pm^{s_d, b}} \|v\|_{X_\pm^{s_d, b}}, \tag{5.14}$$

where the signs in all the $X_\pm$ norms can be chosen independently on each other.

Proof of (5.13). In view of (5.2) the estimate (5.13) reduces to proving

$$\left\| |D| K^2 (u \cdot Kv) \right\|_{L_T^2 H_x^{s_d}} \lesssim \|u\|_{X_\pm^{s_d, b}} \|v\|_{X_\pm^{s_d, b}},$$

which by duality can be reduced to

$$\left| \int_0^T \int_{\mathbb{R}^d} |D| K^2 \langle D \rangle^{s_d} \left(\langle D \rangle^{-s_d} u \cdot \langle D \rangle^{-s_d} Kv \right) w \, dx dt \right| \lesssim \|u\|_{X_\pm^{0,b}} \|v\|_{X_\pm^{0,b}} \|w\|_{L_{t,x}^2}. \tag{5.15}$$

Decomposing $u = \sum_{\lambda_1 \ge 1} u_{\lambda_1}$ and $v = \sum_{\lambda_2 \ge 1} v_{\lambda_2}$ we have

$$\text{LHS (5.15)} \lesssim \sum_{\lambda, \lambda_1, \lambda_2 \ge 1} \left| \int_0^T \int_{\mathbb{R}^d} |D| K^2 \langle D \rangle^{s_d} P_\lambda \left(\langle D \rangle^{-s_d} u_{\lambda_1} \cdot \langle D \rangle^{-s_d} K v_{\lambda_2} \right) w_\lambda \, dx dt \right|. \tag{5.16}$$

Setting

$$a_{\lambda_1} := \|u_{\lambda_1}\|_{X_\pm^{0,b}}, \quad b_{\lambda_2} := \|v_{\lambda_2}\|_{X_\pm^{0,b}}, \quad c_\lambda := \|w_\lambda\|_{L_{t,x}^2}$$

we have

$$\|u\|_{X_\pm^{0,b}} \sim \|(a_{\lambda_1})\|_{l_{\lambda_1}^2}, \quad \|v\|_{X_\pm^{0,b}} \sim \|(b_{\lambda_2})\|_{l_{\lambda_2}^2}, \quad \|w\|_{L_{t,x}^2} \sim \|(c_\lambda)\|_{l_\lambda^2},$$

hence the estimate (5.15) reduces to proving

$$\text{RHS (5.16)} \lesssim \|(a_{\lambda_1})\|_{l_{\lambda_1}^2} \|(b_{\lambda_2})\|_{l_{\lambda_2}^2} \|(c_\lambda)\|_{l_\lambda^2}. \tag{5.17}$$

To this end, we note that by Lemma 11 we have, for $\varepsilon > 0$ arbitrarily small,

$$\|P_\lambda (u_{\lambda_1} v_{\lambda_2})\|_{L_T^2 L_x^2} \lesssim \min(\lambda_1, \lambda_2)^{3d/8 + \varepsilon} \|u_{\lambda_1}\|_{X_\pm^{0,b}} \|v_{\lambda_2}\|_{X_\pm^{0,b}}. \tag{5.18}$$

We remark that in dimension $d = 1$, the lemma would actually allow us to take $\varepsilon = 0$, but the proof below works for sufficiently small, positive $\varepsilon > 0$ in both dimensions.

Then using Cauchy-Schwarz, (5.12) and (5.18) we obtain

$$\begin{aligned} \text{RHS (5.16)} &\lesssim \sum_{\lambda,\lambda_1,\lambda_2\geq 1} \left\| |D| K^2 \langle D\rangle^{s_d} P_\lambda \left(\langle D\rangle^{-s_d} u_{\lambda_1} \cdot \langle D\rangle^{-s_d} K v_{\lambda_2} \right) \right\|_{L^2_{T,x}} \|w_\lambda\|_{L^2_{T,x}} \\ &\lesssim \sum_{\lambda,\lambda_1,\lambda_2\geq 1} \lambda^{s_d} \min(\lambda_1,\lambda_2)^{3d/8+\varepsilon} \lambda_1^{-s_d} \lambda_2^{-1/2-s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda \\ &\lesssim I_1(d) + I_2(d) + I_3(d), \end{aligned} \tag{5.19}$$

where

$$I_1(d) = \sum_{\substack{\lambda,\lambda_1,\lambda_2\geq 1 \\ \lambda\lesssim\lambda_1\sim\lambda_2}} \lambda^{s_d} \lambda_2^{3d/8+\varepsilon-1/2-2s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda,$$

$$I_2(d) = \sum_{\substack{\lambda,\lambda,\lambda_2\geq 1 \\ \lambda_1\ll\lambda_2\sim\lambda}} \left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} \lambda_1^{3d/8+\varepsilon-1/2-s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda,$$

$$I_3(d) = \sum_{\substack{\lambda,\lambda_1,\lambda\geq 1 \\ \lambda_2\ll\lambda_1\sim\lambda}} \lambda_2^{3d/8+\varepsilon-1/2-s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda.$$

We first estimate $I_1(1)$. If $s_1 \geq 0$ then we have $\lambda^{s_1}\lambda_2^{\varepsilon-1/8-2s_1} \lesssim \lambda^{\varepsilon-1/8}$. Consequently, we can apply the Cauchy-Schwarz inequality first in $\lambda_1 \sim \lambda_2$ and then in λ to estimate $I_1(1)$ as

$$I_1(1) \lesssim \left(\sum_{\lambda\geq 1} \lambda^{\varepsilon-1/8} c_\lambda \right) \|(a_{\lambda_1})\|_{l^2_{\lambda_1}} \|(b_{\lambda_2})\|_{l^2_{\lambda_2}} \lesssim \|(a_{\lambda_1})\|_{l^2_{\lambda_1}} \|(b_{\lambda_2})\|_{l^2_{\lambda_2}} \|(c_\lambda)\|_{l^2_\lambda}.$$

If $s_1 < 0$, we have $\lambda^{s_1}\lambda_2^{\varepsilon-1/8-2s_1} \lesssim \lambda^{s_1}$ (taking $\varepsilon > 0$ small enough) since $s_1 > -1/16$ by assumption, and hence this case can be handled in the same way.

Next we estimate $I_1(2)$. Since $s_2 > 1/4$ by assumption, we have $\lambda^{s_2}\lambda_2^{\varepsilon+1/4-2s_2} \lesssim (\lambda/\lambda_2)^{s_2}$. Then we apply the Cauchy-Schwarz inequality first in λ and then in $\lambda_1 \sim \lambda_2$ to obtain the desired estimate:

$$I_1(2) \lesssim \sum_{\lambda_1\sim\lambda_2} \left(\sum_{\lambda\lesssim\lambda_2} (\lambda/\lambda_2)^{s_2} c_\lambda \right) a_{\lambda_1} b_{\lambda_2} \lesssim \|(a_{\lambda_1})\|_{l^2_{\lambda_1}} \|(b_{\lambda_2})\|_{l^2_{\lambda_2}} \|(c_\lambda)\|_{l^2_\lambda}.$$

Next we estimate $I_3(d)$. Since $s_d > 3d/8 - 1/2$, we have $\varepsilon + 3d/8 - 1/2 - s_d < 0$, for $\varepsilon > 0$ small enough. Applying the Cauchy-Schwarz inequalityfirst in $\lambda_1 \sim \lambda$ and then in λ_2, we get

$$I_3(d) \lesssim \left(\sum_{\lambda_2\geq 1} \lambda_2^{3d/8+\varepsilon-1/2-s_d} b_{\lambda_2} \right) \|(a_{\lambda_1})\|_{l^2_{\lambda_1}} \|(c_\lambda)\|_{l^2_\lambda} \lesssim \|(a_{\lambda_1})\|_{l^2_{\lambda_1}} \|(b_{\lambda_2})\|_{l^2_{\lambda_2}} \|(c_\lambda)\|_{l^2_\lambda}.$$

Finally, we note that in $I_2(d)$, we can discard the small factor $(\lambda_1/\lambda_2)^{1/2}$ and reduce to the same estimate as for $I_3(d)$. This completes the proof of (5.13). □

Proof of (5.14). We follow the same argument as in the proof of (5.13). By duality and dyadic decomposition, (5.14) reduces to proving

$$S \lesssim \|(a_{\lambda_1})\|_{l^2_{\lambda_1}} \|(b_{\lambda_2})\|_{l^2_{\lambda_2}} \|(c_\lambda)\|_{l^2_\lambda},$$

where

$$S = \sum_{\lambda,\lambda_1,\lambda_2\geq 1} \left| \int_0^T \int_{\mathbb{R}^d} |D| K \langle D\rangle^{s_d} P_\lambda \left(\langle D\rangle^{-s_d} K u_{\lambda_1} \cdot \langle D\rangle^{-s_d} K v_{\lambda_2} \right) w_\lambda \, dxdt \right|.$$

By Cauchy-Schwarz, (5.12) and (5.18) we obtain

$$
\begin{aligned}
S &\lesssim \sum_{\lambda,\lambda_1,\lambda_2\geq 1} \lambda^{\frac{1}{2}+s_d} \min(\lambda_1,\lambda_2)^{3d/8+\varepsilon}(\lambda_1\lambda_2)^{-1/2-s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda \\
&= \sum_{\lambda,\lambda_1,\lambda_2\geq 1} \left(\frac{\lambda}{\lambda_1}\right)^{1/2} \lambda^{s_d} \min(\lambda_1,\lambda_2)^{3d/8+\varepsilon} \lambda_1^{-s_d}\lambda_2^{-1/2-s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda,
\end{aligned}
$$

and comparing with the corresponding sum (5.19) from the proof of (5.13), we see that the only difference is that we now have an extra factor $(\lambda/\lambda_1)^{1/2}$. This factor is small except in the case $\lambda_1 \ll \lambda_2 \sim \lambda$, so it is enough to consider $I_2(d)$ with this factor inserted:

$$
I_2'(d) = \sum_{\substack{\lambda,\lambda,\lambda_2\geq 1\\ \lambda_1\ll\lambda_2\sim\lambda}} \left(\frac{\lambda}{\lambda_1}\right)^{1/2}\left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} \lambda_1^{3d/8+\varepsilon-1/2-s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda = \sum_{\substack{\lambda,\lambda,\lambda_2\geq 1\\ \lambda_1\ll\lambda_2\sim\lambda}} \lambda_1^{3d/8+\varepsilon-1/2-s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda.
$$

But the right hand side was already estimated in the proof of (5.13) (the estimate for $I_3(d)$). This completes the proof of (5.14). □

6. Proof of Theorem 4, Theorem 5 and Theorem 3

6.1. **Proof of Theorems 4 and Theorem 5.** We solve the integral equations (2.12) and (2.13) by contraction mapping techniques as follows. Define the mapping

$$
(u_d^+, u_d^-) \mapsto (\Phi_+(u_d^+, u_d^-), \Phi_-(u_d^+, u_d^-))
$$

by

$$
\Phi_\pm(u_d^+, u_d^-)(t) := S_{m_d}(\pm t) f_d^\pm - i\int_0^t S_{m_d}(\pm(t-s)) B_d^\pm(u_d^+, u_d^-)(s)\, ds.
$$

Let

$$
R_d = \|f_d^+\|_{H^{s_d}} + \|f_d^-\|_{H^{s_d}}.
$$

We look for a solution in the set

$$
\mathcal{D}(R_d) = \left\{ (u_d^+, u_d^-) \in X_+^{s_d,b}(T)\times X_-^{s_d,b}(T) \colon \|u_d^+\|_{X_+^{s_d,b}(T)} + \|u_d^-\|_{X_-^{s_d,b}(T)} \leq 4CR_d \right\}
$$

where $b \in (1/2, 1)$ and C is as in (5.4), (5.5). Now for $(u_d^+, u_d^-) \in \mathcal{D}(R_d)$ we have by (5.4), (5.5) and Lemma 12,

$$
\|\Phi_+(u_d^+, u_d^-)\|_{X_+^{s_d,b}(T)} + \|\Phi_-(u_d^+, u_d^-)\|_{X_-^{s_d,b}(T)} \leq 2CR_d + C'T^{1-b}R_d^2 \leq 4CR_d,
$$

where the last inequality certainly holds provided that

$$
T = \left(\frac{1}{16CC'(1+R_d)}\right)^{\frac{1}{1-b}}.
$$

Moreover, for (u_d^+, u_d^-) and (v_d^+, v_d^-) in $\mathcal{D}(R_d)$ with the same data, one can show similarly the difference estimate

$$
\begin{aligned}
&\sum_\pm \|\Phi_\pm(u_d^+, u_d^-) - \Phi_\pm(v_d^+, v_d^-)\|_{X_\pm^{s_d,b}(T)} \\
&\quad\leq C'T^{1-b}\left(\sum_\pm \|u_d^\pm - v_d^\pm\|_{X_\pm^{s_d,b}(T)}\right)\left(\sum_\pm \left(\|u_d^\pm\|_{X_\pm^{s_d,b}(T)} + \|v_d^\pm\|_{X_\pm^{s_d,b}(T)}\right)\right) \\
&\quad\leq 8CC'R_dT^{1-b}\left(\sum_\pm \|u_d^\pm - v_d^\pm\|_{X_\pm^{s_d,b}(T)}\right).
\end{aligned}
$$

With T chosen as above, the constant $8CC'R_dT^{1-b}$ is strictly less than one, hence (Φ_+, Φ_-) is a contraction on $\mathcal{D}(R_d)$ and therefore it has a unique fixed point $(u_d^+, u_d^-) \in \mathcal{D}(R_d)$ solving the integral equation on $\mathbb{R}^d \times [0, T]$. Uniqueness in the whole space $X_+^{s_d,b}(T) \times X_-^{s_d,b}(T)$ and continuous dependence on the initial data can be shown in a similar way, by the difference estimates. This concludes the proof of Theorems 4 and 5.

Then we use the transformation (2.1) to obtain the solution

$$(\eta, v) \in C\left([0,T]; H^{s_1}(\mathbb{R}) \times H^{s_1+1/2}(\mathbb{R})\right)$$

of the original system (1.1)–(1.3). Similarly, we use the transformation (2.7) to obtain the solution

$$(\eta, \mathbf{v}) \in C\left([0,T]; H^{s_2}\left(\mathbb{R}^2\right) \times \left(H^{s_2+1/2}\left(\mathbb{R}^2\right)\right)^2\right)$$

of the original system (1.5)–(1.6). Thus we obtain also Theorems 1 and 2.

6.2. **Proof of Theorem 3.** Here we assume $d = 1$. For $s = 0$ one can easily extend the local result globally making use of Lemma 4. With the global bound of the lemma we can reapply the local result, Theorem 1, as many times as we want, thus proving Theorem 3 with $\delta = \epsilon_0/2$ for $s = 0$. The proof for positive s is done iteratively. In other words, assuming the result for some $s' \geq 0$ we prove for $s \in (s', s' + 1/4]$. The argument is essentially the persistence of regularity based on the a priori estimate lemma 7, where we use the notation $\|(\eta, v)\|_{X^s}$ defined by (3.1). Indeed, the first estimate in Lemma 7 allows to reapply the local result and extend the solution to any time interval if $0 < s < 1/2$. In the case $s \geqslant 1/2$ extension is carried out iteratively making use of the second inequality in Lemma 7.

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Paper VII

3.7 Solitary wave solutions of a Whitham-Boussinesq system

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Solitary wave solutions of a Whitham-Boussinesq system

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Abstract

The travelling wave problem for a particular bidirectional Whitham system modelling surface water waves is under consideration. This system firstly appeared in [9], where it was numerically shown to be stable and a good approximation to the incompressible Euler equations. In subsequent papers [8, 10] the initial-value problem was studied and well-posedness in classical Sobolev spaces was proved. Here we prove existence of solitary wave solutions and provide their asymptotic description. Our proof relies on a variational approach and a concentration-compactness argument. The main difficulties stem from the fact that in the considered Euler-Lagrange equation we have a non-local operator of positive order appearing both in the linear and non-linear parts.

1 Introduction

1.1 Motivation and background

We consider the system

$$\eta_t = -v_x - \mathrm{i}\tanh(\mathrm{D})(\eta v), \tag{1.1}$$

$$v_t = -\mathrm{i}\tanh(\mathrm{D})\eta - \mathrm{i}\tanh(\mathrm{D})\left(\frac{v^2}{2}\right), \tag{1.2}$$

with $\mathrm{D} = -\mathrm{i}\partial_x$ and $\mathcal{F}(\tanh(\mathrm{D})f)(\xi) = \tanh(\xi)\widehat{f}(\xi)$, where $\mathcal{F}$ is the Fourier transform

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x)\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x.$$

We are interested in solitary wave solutions of (1.1)–(1.2) and so we search for solutions of the form

$$\eta(x,t) = \eta(x+ct),\ v(x,t) = v(x+ct), \tag{1.3}$$

with $\eta(x+ct),\ v(x+ct) \to 0$, as $|x+ct| \to \infty$. Here η denotes surface elevation and v is the fluid velocity at the surface. Such systems describes permanent progressive surface waves of a fluid layer. The model (1.1)-(1.2) approximates the two-dimensional water wave problem for an inviscid incompressible potential flow.

System (1.1)–(1.2) was introduced in [9] as a fully dispersive model for two-way wave propagation. In recent years several fully dispersive two-way systems have been paid close attention to, and for a survey we again refer to [9], where they compare some of these models, and in particular, find that the system (1.1)–(1.2) approximates the full water-wave problem better than some of the other fully dispersive bidirectional models. It worth to point out that those demonstrate a good agreement with experiments [7]. In addition, System (1.1)–(1.2) has been recently shown to be well-posed in [8, 10]. Moreover, the result is global if the initial data is sufficiently small. The latter is the main advantage of Equations (1.1)–(1.2) comparing with other models regarded in [9]. Indeed, there is a local well-posedness result for another system regarded in [9] obtained by Ehrnström, Pei and Wang [13]. However, they impose an additional non-physical condition $\eta \geqslant C > 0$. Kalisch and Pilod [16] have proved local well posedness for a surface tension regularisation of the system from [13] without the positivity assumption $\eta > 0$. However, the maximal time of existence for their regularisation is bounded by the capillary parameter. Whereas one does not need any regularisation or special non-physical conditions to claim the well posedness for (1.1)–(1.2). In fact Model (1.1)–(1.2) can be regarded itself as a regularization, arising naturally from the Hamiltonian formulation of the water wave problem, for the system introduced by Hur and Pandey [15]. There is another Whitham-Boussinesq type model known to be well-posed that was not considered in [9] and was introduced by Duchêne, Israwi and Talhouk [11]. For more discussion on the Cauchy problem and rigorous justification of the various Whitham related equations we refer to [17]. Note that for systems regarded in [11] and [13] existence of solitary waves was proved in [6] and [22], respectively. The next natural step is to show the solitary wave existence for Equations (1.1)–(1.2). This is the main aim of the current paper.

We use a variational approach together with Lion's method of concentration-compactness [19] to establish the existence of solitary wave solutions of (1.1)–(1.2). This approach has been used extensively to prove existence of solitary wave solutions to equations of the form

$$u_t + Lu_x + n(u)_x = 0, \tag{1.4}$$

where L is a Fourier multiplier operator of order s and $n(u)$ is a homogeneous nonlinear term. Under the travelling wave ansatz $u = u(x+ct)$, equation (1.4) becomes

$$cu + Lu + n(u) = 0. \tag{1.5}$$

In [23] the author studied long wave model equations of the form (1.4), with $s \geq 1$, and proved existence and stability of solitary wave solutions. This approach was later used in [3] to prove existence of solitary waves for an equation used to model stratified fluids, with $s = 1$, and was later generalized in [1] to $s \geq 1$. A class of Whitham type equations of the form (1.4) was studied in [12], with a Fourier multiplier operator of negative order. In this case the resulting functional in the constrained minimization problem is not coercive. This makes the application of the concentration compactness theorem a lot more technical, requiring the authors to use a strategy developed in [4, 14] and first consider a related penalized functional acting on periodic functions. In the recent work [21] an entirely different approach to proving the existence of solitary wave solutions of the Whitham equation, based on the implicit function theorem instead, was presented. Arnesen proved existence of solitary wave solutions to two different classes of model equations [2], one of them of the form (1.4), for $s > 0$. Results, similar and previous to those of Arnesen, were obtained in [18] in application to two particular cases, namely, the fractional Korteweg-de Vries and the fractional Benjamin-Bona-Mahony equations. The case when the nonlinearity n is allowed to be inhomogeneous was considered in [20], where the author proved the existence of solitary wave solutions of (1.4), for operators of positive order and with weak assumptions on the regularity of the symbol.

These methods have also been applied to bidirectional Whitham type equations. As mentioned above, in [6] the authors established the existence of solitary waves for the class of modified Green–Naghdi equations introduced in [11], and in [22] the authors proved the existence of solitary waves for the Whitham–Boussinesq system regarded in [9, 13]. Just as in [12], both of the functionals appearing in [6, 22] are noncoercive, so the minimization arguments adapted to noncoercive functionals developed in [4, 14] are used in order to obtain the existence of minimizers. In addition, the Fourier multiplier operator is entangled with the nonlinearity in [6, 22], which makes the proofs more technical.

1.2 The minimization problem

We formulate the problem in the variational settings. A Hamiltonian structure [8] of System (1.1)–(1.2) allows us to do this in a straightforward way. Indeed, under the travelling wave ansatz (1.3), Equations (1.1)–(1.2) can be written as

$$Kv + \eta v + cK\eta = 0, \tag{1.6}$$

$$\eta + \frac{v^2}{2} + cKv = 0, \tag{1.7}$$

where we have introduced a Fourier multiplier of the form

$$K = \frac{D}{\tanh(D)}. \tag{1.8}$$

Note that this operator is of order one. It is equivalent to the Bessel potential $J = (1 - \partial_x^2)^{1/2}$ associated with the symbol $\langle \xi \rangle = \sqrt{1 + \xi^2}$, since $\xi / \tanh \xi \simeq \langle \xi \rangle$. Regarding the Hamiltonian and momentum

$$\mathcal{H}(\eta, v) = \frac{1}{2} \int_{\mathbb{R}} \eta^2 + vKv + \eta v^2 \, \mathrm{d}x,$$
$$\mathcal{I}(\eta, v) = \int_{\mathbb{R}} \eta K v \, \mathrm{d}x,$$

one can notice that Equation (1.6) can be written as

$$\mathrm{d}_v \mathcal{H} + c\mathrm{d}_v \mathcal{I} = 0,$$

and Equation (1.7) as

$$\mathrm{d}_\eta \mathcal{H} + c\mathrm{d}_\eta \mathcal{I} = 0.$$

Our aim is to obtain a single travelling wave equation that can in turn be interpreted as a constrained minimization problem. We can derive a travelling wave equation in the following way. In (1.6)–(1.7) we make the change of variable $v = K^{-1/2}\tilde{v}$, which yields the new system

$$K^{1/2}\tilde{v} + \eta(K^{-1/2}\tilde{v}) + cK\eta = 0, \tag{1.9}$$

$$\eta + \frac{(K^{-1/2}\tilde{v})^2}{2} + cK^{1/2}\tilde{v} = 0. \tag{1.10}$$

From (1.10) we get that

$$\eta = -\frac{(K^{-1/2}\tilde{v})^2}{2} - cK^{1/2}\tilde{v}, \tag{1.11}$$

and inserting this into (1.9) yields

$$\tilde{v} - K^{-1/2}\left(\frac{(K^{-1/2}\tilde{v})^3}{2}\right) - cK^{-1/2}(K^{1/2}\tilde{v}K^{-1/2}\tilde{v}) - cK^{1/2}\left(\frac{(K^{-1/2}\tilde{v})^2}{2}\right) - c^2K\tilde{v} = 0. \tag{1.12}$$

Here we make the change of variables $\tilde{v} = cu$ so that (1.12) becomes

$$\frac{1}{c^2}u - K^{-1/2}\left(\frac{(K^{-1/2}u)^3}{2}\right) - K^{-1/2}(K^{\frac{1}{2}}uK^{-1/2}u) - K^{1/2}\left(\frac{(K^{-1/2}u)^2}{2}\right) - Ku = 0. \tag{1.13}$$

Now let us show that Equation (1.13) represents an Euler-Lagrange equation for some functional. Indeed, regard the surface elevation and velocity defined by u as follows

$$\eta_u = -c^2\left(\frac{(K^{-1/2}u)^2}{2} + K^{1/2}u\right), \tag{1.14}$$

$$v_u = cK^{-1/2}u, \tag{1.15}$$

and note that

$$\mathcal{H}(\eta_u, v_u) + c\mathcal{I}(\eta_u, v_u) = c^4\left[-\frac{1}{2}\int_{\mathbb{R}} uKu + K^{1/2}u(K^{-1/2}u)^2 + \frac{(K^{-1/2}u)^4}{4}\,\mathrm{d}x + \frac{1}{2c^2}\int_{\mathbb{R}} u^2\,\mathrm{d}x\right],$$

which leads us to define

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}} uKu + K^{1/2}u(K^{-1/2}u)^2 + \frac{(K^{-1/2}u)^4}{4}\,\mathrm{d}x,$$

$$\mathcal{Q}(u) = \frac{1}{2}\int_{\mathbb{R}} u^2\,\mathrm{d}x.$$

We then note that equation (1.13) can be written as

$$\mathrm{d}\mathcal{E}(u) + \lambda\mathrm{d}\mathcal{Q}(u) = 0,$$

where $\lambda = -1/c^2$. Hence, in order to find solutions of (1.13) we can consider the constrained minimization problem

$$\inf_{u\in U_q}\mathcal{E}(u) \quad \text{with} \quad U_q = \left\{u \in H^{1/2}(\mathbb{R}) : \mathcal{Q}(u) = q\right\}. \tag{1.16}$$

Instead of working with the specific Fourier multiplier K, we will work with a more general class of Fourier multipliers, and thus a more general constrained minimization problem.

Definition 1.1 (Admissible Fourier multipliers)**.** *Let operator L be a Fourier multiplier, with symbol m, i.e.*

$$\mathcal{F}(Lf)(\xi) = m(\xi)\widehat{f}(\xi).$$

We say that L is admissible if m is even, $m(0) > 0$ and for some $s' > 1$ and $s > 1/2$ the symbol satisfies the following restrictions.

(i). The function $\xi \mapsto \frac{m(\xi)}{\langle\xi\rangle^s}$ is uniformly continuous, and

$$m(\xi) - m(0) \simeq |\xi|^{s'} \text{ for } |\xi| \leqslant 1,$$
$$m(\xi) - m(0) \simeq |\xi|^{s} \text{ for } |\xi| > 1.$$

(ii). For each $\varepsilon > 0$ the kernel of operator $L^{-1/2}$ satisfies

$$\mathcal{F}^{-1}\left(m^{-1/2}\right) \in L^2(\mathbb{R} \setminus (-\varepsilon, \varepsilon)). \tag{1.17}$$

There exists $p \in (1,2) \cap [2/(s+1), 2)$ such that

$$\mathcal{F}^{-1}\left(m^{-1/2}\right) \in L^p(-1, 1). \tag{1.18}$$

The symbol $m(\xi) = \xi/\tanh(\xi)$ satisfies the conditions of Definition 1.1 with $s = 1$ and $s' = 2$ [5]. We have the corresponding functional

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}} \left(L^{1/2}u + \frac{1}{2}(L^{-1/2}u)^2\right)^2 \, \mathrm{d}x \tag{1.19}$$

defined on $H^{s/2}(\mathbb{R})$. Our main goal is then to obtain a solution of the minimization problem

$$\inf_{u \in U_q} \mathcal{E}(u) \quad \text{with} \quad U_q = \left\{u \in H^{s/2}(\mathbb{R}) : \mathcal{Q}(u) = q\right\}. \tag{1.20}$$

For convenience we separate $\mathcal{E}$ into the functionals

$$\mathcal{L}(u) = \frac{1}{2}\int_{\mathbb{R}} uLu \, \mathrm{d}x,$$
$$\mathcal{N}_c(u) = \frac{1}{2}\int_{\mathbb{R}} L^{1/2}u(L^{-1/2}u)^2 \, \mathrm{d}x,$$
$$\mathcal{N}_r(u) = \frac{1}{2}\int_{\mathbb{R}} \frac{(L^{-1/2}u)^4}{4} \, \mathrm{d}x$$

so that

$$\mathcal{E}(u) = \mathcal{L}(u) + \mathcal{N}(u)$$

where

$$\mathcal{N}(u) = \mathcal{N}_c(u) + \mathcal{N}_r(u).$$

We are now ready to state our main results.

Theorem 1.2. *Let D_q be the set of minimizers of $\mathcal{E}$ over U_q. There exists $q_0 > 0$ such that for each $q \in (0, q_0)$, the set D_q is nonempty and $\|u\|^2_{H^{\frac{s}{2}}} \lesssim q$ uniformly for $u \in D_q$. Each element of D_q is a solution of the Euler–Lagrange equation*

$$\lambda u + L^{-1/2}\left(\frac{(L^{-1/2}u)^3}{2}\right) + L^{-1/2}(L^{1/2}uL^{-1/2}u) + L^{1/2}\left(\frac{(L^{-1/2}u)^2}{2}\right) + Lu = 0. \quad (1.21)$$

The Lagrange multiplier λ satisfies

$$\frac{m(0)}{2} < -\lambda < m(0) - Dq^{\beta}, \quad (1.22)$$

where $\beta = \frac{s'}{2s'-1}$ and D is a positive constant.

Our other main result concerns the asymptotic behavior of travelling wave solutions of (1.1)–(1.2).

Theorem 1.3. *If $L = K$ then there exists $q_0 > 0$ such that for any $q \in (0, q_0)$ each minimizer $u \in D_q$ belongs to $H^r(\mathbb{R})$ for any $r \geqslant 0$ with $\|u\|^2_{H^r} \lesssim q$, and moreover, it satisfies the following long wave asymptotics*

$$\sup_{u\in D_q} \inf_{x_0\in\mathbb{R}} \left\|q^{-2/3}u(q^{-1/3}\cdot) - \psi_{KdV}(\cdot - x_0)\right\|_{H^1(\mathbb{R})} \lesssim q^{1/6},$$

whereas the corresponding surface elevation (1.14) *and speed* (1.15) *satisfy*

$$\sup_{u\in D_q} \inf_{x_0\in\mathbb{R}} \left\|q^{-2/3}\eta_u(q^{-1/3}\cdot) + \psi_{KdV}(\cdot - x_0)\right\|_{H^{1/2}(\mathbb{R})} \lesssim q^{1/6},$$
$$\sup_{u\in D_q} \inf_{x_0\in\mathbb{R}} \left\|q^{-2/3}v_u(q^{-1/3}\cdot) + \psi_{KdV}(\cdot - x_0)\right\|_{H^{3/2}(\mathbb{R})} \lesssim q^{1/6},$$

where

$$\psi_{KdV}(x) = -\lambda_0 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{3\lambda_0}x\right)$$

and $\lambda_0 = 3/\sqrt[3]{16}$. In addition, the Lagrange multiplier λ satisfies

$$\lambda = -1 + \lambda_0 q^{2/3} + \mathcal{O}(q^{5/6}).$$

We discuss here briefly how to prove Theorems 1.2, 1.3.

The main ingredient in proving Theorem 1.2 is Lion's concentration compactness theorem [19]:

Theorem 1.4 (Concentration-compactness). *Any sequence $\{e_n\}_{n\in\mathbb{N}} \subset L^1(\mathbb{R})$ of non-negative functions such that*

$$\lim_{n\to\infty} \int_{\mathbb{R}} e_n \, \mathrm{d}x = I > 0$$

admits a subsequence, denoted again $\{e_n\}_{n\in\mathbb{N}}$, for which one of the following phenomena occurs.

- *(Vanishing) For each* $r > 0$*, one has*

$$\lim_{n\to\infty}\left(\sup_{x\in\mathbb{R}}\int_{x-r}^{x+r} e_n \,\mathrm{d}x\right) = 0.$$

- *(Dichotomy) There are real sequences* $\{x_n\}_{n\in\mathbb{N}}, \{M_n\}_{n\in\mathbb{N}}, \{N_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ *and* $I^* \in (0, I)$ *such that* $M_n, N_n \to \infty$*,* $M_n/N_n \to 0$*, and*

$$\int_{x_n-M_n}^{x_n+M_n} e_n \,\mathrm{d}x \to I^* \quad \textit{and} \quad \int_{x_n-N_n}^{x_n+N_n} e_n \,\mathrm{d}x \to I^*,$$

  as $n \to \infty$*.*

- *(Concentration) There exists a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ *with the property that for each* $\epsilon > 0$*, there exists* $r > 0$ *with*

$$\int_{x_n-r}^{x_n+r} e_n \, x \geq I - \epsilon,$$

  for all $n \in \mathbb{N}$*.*

We will apply this theorem to $e_n = u_n^2$, where $\{u_n\}_{n=1}^{\infty}$ is a minimizing sequence, and show that the vanishing and dichotomy scenarios cannot occur. Then we obtain a convergent subsequence of $\{u_n\}_{n=1}^{\infty}$ using the concentration scenario. The functional $\mathcal{E}$ is similar to the corresponding functionals appearing in [6, 22], in the sense that the Fourier multiplier and the nonlinerity are entangled. However, in contrast with [6, 22], our functional $\mathcal{E}$ is coercive, hence the penalization argument of [4, 14] is not necessary in our case.

In [6], the exclusion of dichotomy gets more technical due to the entanglement of the Fourier multiplier with the nonlinearity, and this is true for the present work as well. In contrast, the exclusion of the vanishing scenario is straightforward in [6], while this is not the case in the present work. This is due to the fact that in [6] the constrained minimization problem is formulated in $H^s(\mathbb{R})$, $s > 1/2$, allowing the use of the embedding $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, while our problem is formulated in $H^{s/2}(\mathbb{R})$, preventing us to make use of this embedding. Instead we show that if $\{u_n\}_{n=1}^{\infty}$ is vanishing, then $L^{-1/2}u_n$ is vanishing as well, which leads to a contradiction. In order to show that $L^{-1/2}u_n$ is vanishing we make use of the integrability assumptions (1.17), (1.18) imposed on the kernel of L, and this is the only instance where these assumptions are used. Apart from (1.17), (1.18) we have precisely the same assumptions on L as in [20], and we are able to adopt many of the methods used in that paper to our present work. Also, we refer to [20] for a discussion on the necessity of assumptions (i), (ii) in Definition 1.1.

Theorem 1.3 is established using standard arguments, see for example [6, 12].

2 Technical results

The current section is devoted to the general properties of the functionals introduced above. We start with a useful proposition on continuity of symbol $m(\xi)$ described by Definition 1.1.

Lemma 2.1. *There is a function* $\omega : \mathbb{R} \to [0, \infty)$*, bounded above by a polynomial, with* $\lim_{\lambda\to 0}\omega(\lambda) = 0$*, such that*

$$|m(\xi) - m(\eta)| \leq \omega(\xi - \eta)\langle\xi\rangle^{\frac{s}{2}}\langle\eta\rangle^{\frac{s}{2}}.$$

Proof. See [20, Proposition 2.1]. □

The following functional estimates will be used a lot in the text below, sometimes without references.

Proposition 2.2. *For any $u \in H^{s/2}(\mathbb{R})$ one has*

$$\mathcal{L}(u) \simeq \|u\|^2_{H^{s/2}}\,.$$

Proof. This is immediate from Definition 1.1. □

Proposition 2.3. *For any $s > 1/2$ and $u \in H^{s/2}(\mathbb{R})$ one has*

$$|\mathcal{N}_c(u)| \lesssim \|u\|^2_{L^2}\,\|u\|_{H^{s/2}}\,, \tag{2.1}$$

$$|\mathcal{N}_r(u)| \lesssim \|u\|^4_{L^2}\,. \tag{2.2}$$

Proof. Inequality (2.2) follows from the Sobolev embedding

$$|\mathcal{N}_r(u)| = \frac{1}{8}\left\|L^{-1/2}u\right\|^4_{L^4} \lesssim \left\||\partial_x|^{1/4}L^{-1/2}u\right\|^4_{L^2} \lesssim \left\|J^{1/4-s/2}u\right\|^4_{L^2} \lesssim \|u\|^4_{L^2}\,.$$

Inequality (2.1) follows from (2.2) and Hölder's inequality. □

Proposition 2.4. *For $s > 1/2$ and $u, h \in H^{\frac{s}{2}}(\mathbb{R})$ the Fréchet derivative of $\mathcal{E}$ satisfies*

$$|\mathrm{d}\mathcal{E}(u)(h)| \lesssim \|u\|_{H^{\frac{s}{2}}}\,(1 + \|u\|_{L^2} + \|u\|^2_{L^2})\,\|h\|_{H^{\frac{s}{2}}}$$

Proof. We first note that

$$|\mathrm{d}\mathcal{L}(u)(h)| \lesssim \|u\|_{H^{\frac{s}{2}}}\,\|h\|_{H^{\frac{s}{2}}}\,.$$

Next consider

$$\mathrm{d}\mathcal{N}_c(u)(h) = \frac{1}{2}\int_{\mathbb{R}} L^{1/2}h(L^{-1/2}u)^2 + 2uL^{1/2}(L^{-1/2}uL^{-1/2}h)\,\mathrm{d}x, \tag{2.3}$$

where

$$\left\|L^{1/2}h(L^{-1/2}u)^2\right\|_{L^1} \le \left\|L^{1/2}h\right\|_{L^2}\left\|L^{-1/2}u\right\|^2_{L^4} \lesssim \|u\|^2_{L^2}\,\|h\|_{H^{\frac{s}{2}}}\,,$$

$$\begin{aligned}\left\|uL^{1/2}(L^{-1/2}uL^{-1/2}h)\right\|_{L^1} &\le \|u\|_{L^2}\left\|L^{1/2}(L^{-1/2}uL^{-1/2}h)\right\|_{L^2}\\ &\lesssim \|u\|_{L^2}\left\|L^{-1/2}uL^{-1/2}h\right\|_{H^{\frac{s}{2}}}\\ &\lesssim \|u\|_{L^2}\left\|L^{-1/2}u\right\|_{H^{\frac{s}{2}}}\left\|L^{-1/2}h\right\|_{H^s}\\ &\lesssim \|u\|^2_{L^2}\,\|h\|_{H^{\frac{s}{2}}}\,.\end{aligned}$$

Using the above estimates in (2.3), we immediately get that

$$|\mathrm{d}\mathcal{N}_c(u)(h)| \lesssim \|u\|^2_{L^2}\,\|h\|_{H^{\frac{s}{2}}}\,.$$

In a similar way we find that

$$|\mathrm{d}\mathcal{N}_r(u)(h)| \lesssim \|u\|^3_{L^2}\,\|h\|_{H^{\frac{s}{2}}}\,,$$

which concludes the proof. □

We next record a decomposition result for $\mathcal{N}_c$.

Lemma 2.5. *Let* $u \in H^{s/2}(\mathbb{R})$. *Then*

$$\mathcal{N}_c(u) = \frac{1}{2\sqrt{m(0)}} \int_{\mathbb{R}} u^3 \, \mathrm{d}x + \mathcal{N}_{c1}(u) + \mathcal{N}_{c2}(u) + \mathcal{N}_{c3}(u),$$

where

$$\begin{aligned}
\mathcal{N}_{1c}(u) &= \frac{\sqrt{m(0)}}{2} \int_{\mathbb{R}} u \left((L^{-1/2} - m^{-1/2}(0))u\right)^2 \, \mathrm{d}x \\
\mathcal{N}_{2c}(u) &= \int_{\mathbb{R}} u^2 (L^{-1/2} - m^{-1/2}(0))u \, \mathrm{d}x \\
\mathcal{N}_{3c}(u) &= \frac{1}{2} \int_{\mathbb{R}} (L^{-1/2}u)^2 (L^{1/2} - m^{1/2}(0))u \, \mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{N}_{2c}(u)| &\leq \|u\|_{L^4}^2 \left\|(L^{-1/2} - m^{-1/2}(0))u\right\|_{L^2} \\
|\mathcal{N}_{3c}(u)| &\lesssim \|u\|_{L^2}^2 \left\|(L^{-1/2} - m^{-1/2}(0))u\right\|_{L^2}.
\end{aligned}$$

Proof. The proof is straightforward and is therefore omitted. □

Before we continue we want to make a remark on the convolution theorem. According to our choice of the Fourier transform normalisation, for any two functions f and g we have

$$\mathcal{F}(fg) = \frac{1}{2\pi} \widehat{f} * \widehat{g}$$

where star stands for convolution.

Lemma 2.6. *The functional* $\mathcal{E}$ *defined by* (1.19) *is translation invariant. In other words, for any* $u \in H^{s/2}(\mathbb{R})$ *then* $\mathcal{E}(u_h) = \mathcal{E}(u)$, *where* $u_h(x) = u(x-h)$ *denotes translation by* $h \in \mathbb{R}$.

Proof. Due to the property $\widehat{u_h}(\xi) = e^{-\mathrm{i}h\xi}\widehat{u}(\xi)$ and the Plancherel theorem we have

$$\begin{aligned}
\mathcal{E}(u_h) &= \frac{1}{4\pi} \int_{\mathbb{R}} \left| \sqrt{m(\xi)} \widehat{u_h}(\xi) + \frac{1}{2} \mathcal{F}\left(\left(L^{-1/2}u_h\right)^2 \right)(\xi) \right|^2 \, \mathrm{d}\xi \\
&= \frac{1}{4\pi} \int_{\mathbb{R}} \left| e^{-\mathrm{i}h\xi} \sqrt{m(\xi)} \widehat{u}(\xi) + \frac{1}{2} e^{-\mathrm{i}h\xi} \mathcal{F}\left(\left(L^{-1/2}u\right)^2 \right)(\xi) \right|^2 \, \mathrm{d}\xi = \mathcal{E}(u)
\end{aligned}$$

where we have also used the fact that the Fourier transform of multiplication is convolution of Fourier transforms up to a normalization constant. □

In the following lemma we provide a slightly sharper estimate for $\mathcal{N}_c$. It will be the first step towards the non-vanishing proof given below.

Lemma 2.7. *For* $s > 1/2$ *the following estimate hold true*

$$|\mathcal{N}_c(u)| \lesssim \|u\|_{L^2(\mathbb{R})}^2 \|L^{-1/2}u\|_{L^\infty(\mathbb{R})}$$

Proof. Clearly, $L^{-1/2}u \in L^\infty(\mathbb{R})$ and so applying a Kato–Ponce type estimate obtain

$$\begin{aligned} |\mathcal{N}_c(u)| &\lesssim \|u\|_{L^2}\|L^{1/2}(L^{-1/2}u)^2\|_{L^2} \lesssim \|u\|_{L^2}\|J^{s/2}(L^{-1/2}u)^2\|_{L^2} \\ &\lesssim \|u\|_{L^2}\|J^{s/2}L^{-1/2}u\|_{L^2}\|L^{-1/2}u\|_{L^\infty} \lesssim \|u\|_{L^2}^2\|L^{-1/2}u\|_{L^\infty}. \end{aligned}$$

□

We finish this section with a lemma which will be used when ruling out the dichotomy scenario.

Lemma 2.8. *Let $\varphi \in \mathcal{S}(\mathbb{R})$, and let $A_r\colon H^{s/2}(\mathbb{R}) \to H^{s/2}(\mathbb{R})$, $B_r\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the operators*

$$\begin{aligned} A_r f &= [L, \varphi\left(\frac{\cdot}{r}\right)]f, \\ B_r f &= [L^{-\frac{1}{2}}, \varphi\left(\frac{\cdot}{r}\right)]f. \end{aligned}$$

Then the operator norms

$$\|A_r\|, \|B_r\| \to 0 \ \text{ as } \ r \to \infty.$$

Proof. We follow the proof of [20, Lemma 6.2]. Let $\varphi_r(x) = \varphi(x/r)$. Using Lemma 2.1, we find that for $f, g \in H^{s/2}(\mathbb{R})$

$$\begin{aligned} |\langle A_r f, g\rangle| &= \frac{1}{2\pi}\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{\varphi_r}(\eta)\widehat{f}(\xi-\eta)(m(\xi)-m(\xi-\eta))\overline{\widehat{g}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi\right| \\ &\lesssim \int_{\mathbb{R}} |\widehat{\varphi_r}(\eta)\omega(\eta)| \int_{\mathbb{R}} \langle \xi-\eta\rangle^{\frac{s}{2}}|\widehat{f}(\xi-\eta)|\langle\eta\rangle^{\frac{s}{2}}|\widehat{g}(\eta)|\,\mathrm{d}\eta\mathrm{d}\xi \\ &\lesssim \int_{\mathbb{R}} |\widehat{\varphi}(\eta)\omega(\eta/r)|\,\mathrm{d}\eta\,\|f\|_{H^{s/2}}\|g\|_{H^{s/2}}. \end{aligned}$$

Hence $\|A_r\| \lesssim \int_{\mathbb{R}}|\widehat{\varphi}(\eta)\omega(\eta/r)|\,\mathrm{d}\eta$ and this last integral tends to zero by the dominated convergence theorem as $r \to \infty$, since ω is bounded above by a polynomial and $\lim_{\eta\to 0}\omega(\eta) \to 0$.

Similarly, for $f, g \in L^2(\mathbb{R})$ we have

$$\begin{aligned} |\langle B_r f, g\rangle| &= \frac{1}{2\pi}\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{\varphi_r}(\eta)\widehat{f}(\xi-\eta)(m^{-1/2}(\xi)-m^{-1/2}(\xi-\eta))\overline{\widehat{g}(\xi)}\,\mathrm{d}\eta\mathrm{d}\xi\right| \\ &= \frac{1}{2\pi}\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{\varphi_r}(\eta)\widehat{f}(\xi-\eta)\left(\frac{m(\xi-\eta)-m(\xi)}{m^{1/2}(\xi-\eta)m^{1/2}(\xi)(m^{1/2}(\xi-\eta)+m^{1/2}(\xi))}\right)\overline{\widehat{g}(\xi)}\,\mathrm{d}\eta\mathrm{d}\xi\right| \\ &\lesssim \int_{\mathbb{R}} |\widehat{\varphi_r}(\eta)\omega(\eta)| \int_{\mathbb{R}} \frac{\langle\xi-\eta\rangle^{s/2}}{m^{1/2}(\xi-\eta)}|\widehat{f}(\xi-\eta)|\frac{\langle\eta\rangle^{s/2}}{m^{1/2}(\eta)}|\widehat{g}(\eta)|\,\mathrm{d}\eta\mathrm{d}\xi \\ &\lesssim \int_{\mathbb{R}} |\widehat{\varphi}(\eta)\omega(\eta/r)|\,\mathrm{d}\eta\,\|f\|_{L^2}\|g\|_{L^2}, \end{aligned}$$

and we can conclude in the same way as before that $\|B_r\| \to 0$ as $r \to \infty$. □

3 Near minimizers

In this section we provide necessary estimates for the infimum

$$I_q = \inf_{u \in U_q} \mathcal{E}(u) \tag{3.1}$$

and for those $u \in U_q$ that give values $\mathcal{E}(u)$ close to this infimum. The regarded functional (1.19) is non-negative and so the same is true for the infimum. However, we also need an upper bound for I_q and this is addressed in the next result.

Proposition 3.1. *There exist constants $D, q_0 > 0$ such that for $q \in (0, q_0)$ holds*

$$0 \leqslant I_q < m(0)q - Dq^{1+\beta},$$

with $\beta = \frac{s'}{2s'-1}$.

Proof. It is immediate that $0 \leqslant I_q$. To establish the other inequality we consider $\varphi \in C^\infty(\mathbb{R})$, with $\text{supp}(\hat{\varphi}) \subseteq (-1, 1)$, $\varphi(x) \leq 0$, $x \in \mathbb{R}$ and $\mathcal{Q}(\varphi) = 1$. We rescale and define $\varphi_{q,\alpha}(x) = \sqrt{q/\alpha}\varphi(x/\alpha)$, $\alpha > 1$, so that $\mathcal{Q}(\varphi_{q,\alpha}) = q$.

We first note that

$$\mathcal{L}(\varphi_{q,\alpha}) \leq m(0)q + C_1 q\alpha^{-s'},\ C_1 > 0, \tag{3.2}$$

and using Proposition 2.3

$$|\mathcal{N}_r(\varphi_{q,\alpha})| \leq C_2 q^2,\ C_2 > 0. \tag{3.3}$$

In order to estimate $\mathcal{N}_c(\phi_{q,\alpha})$ we begin by estimating

$$0 \leq m^{1/2}(\xi) - m^{1/2}(0) \leq \frac{m(\xi) - m(0)}{2\sqrt{m(0)}},$$
$$|m^{-1/2}(\xi) - m^{-1/2}(0)| \leq \frac{m(\xi) - m(0)}{2m(0)\sqrt{m(0)}},$$

and then, using Lemma 2.5, we find that

$$|\mathcal{N}_{2c}(\varphi_{q,\alpha})| \lesssim q^{3/2}\alpha^{-s'-1/2},$$
$$|\mathcal{N}_{3c}(\varphi_{q,\alpha})| \lesssim q^{3/2}\alpha^{-s'}.$$

Moreover, since $\varphi(x) \leq 0$, we have that

$$\frac{1}{2\sqrt{m(0)}} \int_{\mathbb{R}} \varphi_{q,\alpha}(x)^3 \, \mathrm{d}x = -2C_0 q^{3/2}\alpha^{-1/2},\ C_0 > 0$$
$$\mathcal{N}_{1c}(\varphi_{q,\alpha}) \leq 0.$$

Hence, it follows from the above estimates that there exists $\alpha_0 > 1$, such that for $\alpha \geq \alpha_0$,

$$\mathcal{N}_c(\varphi_{q,\alpha}) \leq -C_0 q^{3/2}\alpha^{-1/2},$$

and combining this with (3.2), (3.3), yields

$$\mathcal{E}(\varphi_{q,\alpha}) \leq m(0)q - \left(C_0 q^{3/2}\alpha^{-1/2} - C_1 q\alpha^{-s'}\right) + C_2 q^2, \tag{3.4}$$

and by choosing $\alpha^{-s'} = Bq^{\beta}$, with $0 < B \leq \alpha_0^{-s'} q^{-\beta}$, so that $\alpha \geq \alpha_0$, we get from (3.4) that

$$\mathcal{E}(\varphi_{q,\alpha}) \leq m(0)q - \underbrace{(C_0 B^{1/(2s')} - C_1 B)}_{=:2D} q^{1+\beta} + C_2 q^2, \tag{3.5}$$

By choosing B small enough we have that $D > 0$, and if we in addition choose q_0 sufficiently small, we find that

$$I_q \leq \mathcal{E}(\varphi_{q,\alpha}) < m(0)q - Dq^{1+\beta}.$$

□

We now define a near minimizer to be an element u of U_q such that

$$\mathcal{E}(u) < m(0)q - Dq^{1+\beta}. \tag{3.6}$$

By the previous proposition, there exist such elements $u \in U_q$.

Proposition 3.2. *A near minimizer $u \in U_q$ satisfies*

$$\|u\|^2_{H^{s/2}} \lesssim q.$$

Proof. Using propositions 2.2, 2.3 and 3.1, we find that

$$\begin{aligned}
\|u\|^2_{H^{\frac{s}{2}}(\mathbb{R})} &\simeq \mathcal{L}(u) \\
&= \mathcal{E}(u) - \mathcal{N}(u) \\
&\lesssim m(0)q - Dq^{1+\beta} + \|u\|^2_{L^2(\mathbb{R})} \|u\|_{H^{\frac{s}{2}}(\mathbb{R})} + \|u\|^4_{L^2(\mathbb{R})} \\
&\lesssim m(0)q - Dq^{1+\beta} + q\|u\|_{H^{\frac{s}{2}}(\mathbb{R})} + q^2.
\end{aligned}$$

Hence, it follows that for q sufficiently small

$$\|u\|^2_{H^{\frac{s}{2}}(\mathbb{R})} \lesssim m(0)q - Dq^{1+\beta} \lesssim q.$$

□

We next show that I_q is strictly subadditive as a function of q. This is essential when proving that dichotomy cannot occur.

Proposition 3.3. *For any $q_1, q_2 \in (0, q_0)$ such that $q_1 + q_2 \in (0, q_0)$, holds*

$$0 < I_{q_1+q_2} < I_{q_1} + I_{q_2}. \tag{3.7}$$

Proof. We show that I_q is strictly subhomogeneous, i.e

$$I_{aq} < aI_q,\ a > 1, q < aq < q_0, \tag{3.8}$$

from which the strict subadditivity follows from a standard argument. First we show that (3.8) holds for $a \in (1, 2]$. Let $\{u_n\}_{n=1}^{\infty}$ be a minimizing sequence. From (3.6) we have that

$$\mathcal{L}(u_n) + \mathcal{N}_c(u_n) + \mathcal{N}_r(u_n) < m(0)q - Dq^{1+\beta}, \tag{3.9}$$

and since $\mathcal{L}(u_n) \geq m(0)q, \mathcal{N}_r(u) \geq 0$, we get from (3.9) that

$$\mathcal{N}_c(u) < -Dq^{1+\beta}. \tag{3.10}$$

We also note that $\sqrt{a} - 1 \geq (a-1)/(1+\sqrt{2})$. With this in mind we see that

$$\begin{aligned}
I_{aq} &\leq \mathcal{E}(a^{1/2}u_n) \\
&= \mathcal{L}(a^{1/2}u_n) + \mathcal{N}(a^{1/2}u_n) \\
&= a\mathcal{L}(u_n) + a^{3/2}\mathcal{N}_c(u_n) + a^2\mathcal{N}_r(u_n) \\
&= a\mathcal{E}(u_n) - a(\mathcal{N}_c(u_n) + \mathcal{N}_r(u_n)) + a^{3/2}\mathcal{N}_c(u_n) + a^2\mathcal{N}_r(u_n) \\
&= a\mathcal{E}(u_n) + (a^{3/2} - a)\mathcal{N}_c(u_n) + (a^2 - a)\mathcal{N}_r(u_n) \\
&\leq a\mathcal{E}(u_n) - (a^{3/2} - a)Dq^{1+\beta} + (a^2 - a)C_3q^2 \\
&\leq a\mathcal{E}(u_n) - (a^2 - a)\left(\frac{Dq^{1+\beta}}{1+\sqrt{2}} - C_3q^2\right).
\end{aligned}$$

Hence, for q_0 sufficiently small

$$I_{aq} + (a^2 - a)\frac{Dq^{1+\beta}}{2\sqrt{2}} < aI_q,$$

which implies (3.8) for $a \in (1, 2]$, but also that $I_q > 0$, for $q \in (0, q_0)$, proving the first inequality in (3.7). For the general case when $a > 1$, we choose $l \in \mathbb{N}$ sufficiently big so that $a \in (1, 2^l]$. Then $a^{1/l} \in (1, 2]$, and so

$$I_{aq} = I_{a^{1/l}a^{(l-1)/l}q} < a^{1/l}I_{a^{(l-1)/l}q} = a^{1/l}I_{a^{1/l}a^{(l-2)/l}q} < a^{2/l}I_{a^{(l-2)/l}q} < \ldots < aI_q.$$

□

4 Existence of minimizers

In order to establish the existence of minimizers, we will apply the concentration-compactness principle (Theorem 1.4) to $e_n = u_n^2$, where $\{u_n\}_{n=1}^{\infty}$ is a minimizing sequence. The idea is to show that the vanishing and dichotomy scenarios cannot occur and then prove the existence of a minimizer using concentration. We start by excluding the vanishing scenario.

Proposition 4.1. *Vanishing does not occur.*

Proof. Let $\{u_n\}_{n=1}^\infty \subseteq U_q$ be a minimizing sequence of $\mathcal{E}$. By Lemma 2.7 we have

$$|\mathcal{N}_c(u)| \lesssim \|u\|_{L^2(\mathbb{R})}^2 \|L^{-1/2}u\|_{L^\infty(\mathbb{R})}$$

and so for a minimizing sequence

$$q^\beta \lesssim \|L^{-1/2}u_n\|_{L^\infty(\mathbb{R})}.$$

Arguing as in the proof of [6, Lemma 4.5], we have for any $x \in \mathbb{R}$ that

$$\begin{aligned}\|L^{-1/2}u_n\|_{L^\infty(x-1,x+1)} &\lesssim \|L^{-1/2}u_n\|_{L^2(x-1,x+1)}^{1-1/(2s)} \|L^{-1/2}u_n\|_{H^s(\mathbb{R})}^{1/(2s)} \\ &\lesssim \|L^{-1/2}u_n\|_{L^2(x-1,x+1)}^{1-1/(2s)} \|u_n\|_{H^{\frac{s}{2}}(\mathbb{R})}^{1/(2s)} \lesssim q^{1/(4s)} \|L^{-1/2}u_n\|_{L^2(x-1,x+1)}^{1-1/(2s)},\end{aligned}$$

and hence

$$q^{\beta-1/(4s)} \lesssim \sup_{x\in\mathbb{R}} \|L^{-1/2}u_n\|_{L^2(x-1,x+1)}^{1-1/(2s)},$$

which means that $L^{-1/2}u_n$ cannot vanish. Now we show that $L^{-1/2}u_n$ is vanishing if one assumes that u_n is vanishing. In order to do this we start by decomposing

$$\begin{aligned}(L^{-1/2}u_n)(x) &= (\mathcal{F}^{-1}(m^{-1/2}) * u_n)(x) \\ &= \int_{\mathbb{R}} \mathcal{F}^{-1}(m^{-1/2})(y) u_n(x-y)\,\mathrm{d}y \\ &= \underbrace{\int_{|y|<\epsilon} \mathcal{F}^{-1}(m^{-1/2})(y) u_n(x-y)\,\mathrm{d}y}_{=:I_1} + \underbrace{\int_{\epsilon\le|y|\le R} \mathcal{F}^{-1}(m^{-1/2})(y) u_n(x-y)\,\mathrm{d}y}_{=:I_2} \\ &\quad + \underbrace{\int_{|y|\ge R} \mathcal{F}^{-1}(m^{-1/2})(y) u_n(x-y)\,\mathrm{d}y}_{=:I_3},\end{aligned}$$

and so

$$\left\|L^{-1/2}u_n\right\|_{L^2(\tilde{x}-1,\tilde{x}+1)} \le \|I_1\|_{L^2(\tilde{x}-1,\tilde{x}+1)} + \|I_2\|_{L^2(\tilde{x}-1,\tilde{x}+1)} + \|I_3\|_{L^2(\tilde{x}-1,\tilde{x}+1)}.$$

The goal is then to show that each of the above integrals can be made arbitrarily small.

By assumption there exists $p \in (1,2) \cap [2/(s+1), 2)$ such that (1.18) holds, and so $\left\|\mathcal{F}^{-1}\left(m^{-1/2}\right)\right\|_{L^p(-\varepsilon,\varepsilon)} = o(1)$ as $\varepsilon \to 0$. On the other hand its dual number p' satisfies condition $1/2 - 1/p' \leqslant s/2$ resulting in the embedding $H^{\frac{s}{2}}(\mathbb{R}) \hookrightarrow L^{p'}(\mathbb{R})$. Thus applying Hölder's inequality to I_1 yields

$$\|I_1\|_{L^2(\tilde{x}-1,\tilde{x}+1)}^2 \leqslant \int_{\tilde{x}-1}^{\tilde{x}+1} \left\|\mathcal{F}^{-1}\left(m^{-1/2}\right)\right\|_{L^p(-\varepsilon,\varepsilon)}^2 \|u_n\|_{L^{p'}(\mathbb{R})}^2 \,\mathrm{d}x = o(1) \text{ as } \varepsilon \to 0.$$

For I_3 we apply the Cauchy–Schwarz inequality as follows

$$\|I_3\|_{L^2(\tilde{x}-1,\tilde{x}+1)}^2 \leqslant \int_{\tilde{x}-1}^{\tilde{x}+1} \left\|\mathcal{F}^{-1}\left(m^{-1/2}\right)\right\|_{L^2(\mathbb{R}\setminus(-R,R))}^2 \|u_n\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}x = o(1) \text{ as } R \to \infty.$$

After choosing ε, R we turn our attention to I_2

$$\|I_2\|^2_{L^2(\tilde{x}-1,\tilde{x}+1)} \leqslant \int_{\tilde{x}-1}^{\tilde{x}+1} \left\|\mathcal{F}^{-1}\left(m^{-1/2}\right)\right\|^2_{L^2((-R,R)\setminus(-\varepsilon,\varepsilon))} \|u_n(x-y)\|^2_{L^2(\varepsilon<|y|<R)}\,\mathrm{d}x$$
$$\leqslant C(\varepsilon,R)\|u_n\|^2_{L^2(-1-|\tilde{x}|-R,1+|\tilde{x}|+R)} \to 0 \text{ as } n\to\infty,$$

if one assumes vanishing of u_n. □

We next turn our attention to the dichotomy scenario.

Proposition 4.2. *Dichotomy cannot occur.*

Proof. Let $\chi : \mathbb{R} \to [0,1]$ be a smooth cutoff function with $\chi(x) = 1$, for $|x| \le 1$ and $\chi(x) = 0$, for $|x| \ge 2$, and such that

$$\chi = \chi_1^2, \quad 1-\chi = \chi_2^2,$$

where χ_1, χ_2 are smooth. Next, let $w_n(x) = u_n(x - x_n)$ and

$$w_n^{(1)}(x) = \underbrace{\chi_1\left(\frac{x}{M_n}\right)}_{=:\chi_{1n}(x)} w_n(x), \quad w_n^{(2)}(x) = \underbrace{\chi_2\left(\frac{x}{M_n}\right)}_{=:\chi_{2n}(x)} w_n(x),$$

Note that from the dichotomy assumption

$$\begin{aligned}\frac{1}{2}\int_{M_n\le|x|\le 2M_n} w_n^2\,\mathrm{d}x &\le \frac{1}{2}\int_{M_n\le|x|\le N_n} w_n^2\,\mathrm{d}x\\ &= \frac{1}{2}\int_{-N_n}^{N_n} w_n^2\,\mathrm{d}x - \frac{1}{2}\int_{-M_n}^{M_n} w_n^2\,\mathrm{d}x\\ &\to q^* - q^*\\ &= 0.\end{aligned}$$

Since $|w_n^i(x)| \le |w_n(x)|$, $i = 1,2$, it follows directly that $\int_{M_n\le|x|\le 2M_n}(w_n^{(i)})^2\,\mathrm{d}x \to 0$, as $n\to\infty$. From this we can then deduce

$$\frac{1}{2}\int_{\mathbb{R}}(w_n^{(1)})^2\,\mathrm{d}x = \frac{1}{2}\int_{-M_n}^{M_n} w_n^2\,\mathrm{d}x - \frac{1}{2}\int_{M_n\le|x|\le 2M_n}(w_n^{(1)})^2\,\mathrm{d}x \to q^*,$$

and similarly

$$\frac{1}{2}\int_{\mathbb{R}}(w_n^{(2)})^2\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} w_n^2\mathrm{d}x - \frac{1}{2}\int_{-2M_n}^{2M_n} w_n^2\,\mathrm{d}x + \frac{1}{2}\int_{M_n\le|x|\le 2M_n}(w_n^{(2)})^2\,\mathrm{d}x \to q - q^*.$$

We next show that

$$\mathcal{E}(w_n^{(1)}) + \mathcal{E}(w_n^{(2)}) - \mathcal{E}(w_n) \to 0,\ n\to\infty. \tag{4.1}$$

As a first step towards this, we show that

$$\mathcal{L}(w_n^{(1)}) + \mathcal{L}(w_n^{(2)}) - \mathcal{L}(w_n) \to 0,\ n\to\infty \tag{4.2}$$

Indeed, note that

$$\mathcal{L}(w_n^{(1)}) + \mathcal{L}(w_n^{(2)}) - \mathcal{L}(w_n) = \frac{1}{2}\int_{\mathbb{R}} w_n^{(1)} L w_n^{(1)} + w_n^{(2)} L w_n^{(2)} - (\chi_{1n}^2 + \chi_{2n}^2) w_n L w_n \,\mathrm{d}x,$$

and using Lemma 2.8 we find that

$$\begin{aligned}\int_{\mathbb{R}} w_n^{(1)} L w_n^{(1)} - \chi_{1n}^2 w_n L w_n \,\mathrm{d}x &= \int_{\mathbb{R}} \chi_{1n} w_n (L(\chi_{1n} w_n) - \chi_{1n} L w_n) \,\mathrm{d}x \\ &= \int_{\mathbb{R}} \chi_{1n} w_n [L, \chi_{1n}] \,\mathrm{d}x \\ &\to 0,\ n \to \infty\end{aligned}$$

In the same way we find that

$$\int_{\mathbb{R}} w_n^{(2)} L w_n^{(2)} - \chi_{2n}^2 w_n L w_n \,\mathrm{d}x = \int_{\mathbb{R}} \chi_{2n} w_n [L, \chi_{2n} - 1] w_n \,\mathrm{d}x \to 0,\ n \to \infty,$$

hence, (4.2) holds. The next step is to show that

$$\mathcal{N}(w_n^{(1)}) + \mathcal{N}(w_n^{(2)}) - \mathcal{N}(w_n) \to 0,\ n \to \infty, \tag{4.3}$$

and for this we use the decomposition $\mathcal{N} = \mathcal{N}_c + \mathcal{N}_r$, and show that

$$\mathcal{N}_c(w_n^{(1)}) + \mathcal{N}_c(w_n^{(2)}) - \mathcal{N}_c(w_n) \to 0\ n \to \infty, \tag{4.4}$$

$$\mathcal{N}_r(w_n^{(1)}) + \mathcal{N}_r(w_n^{(2)}) - \mathcal{N}_r(w_n) \to 0\ n \to \infty. \tag{4.5}$$

Starting with (4.4), we note that

$$\begin{aligned}\mathcal{N}_c(w_n^{(1)}) + \mathcal{N}_c(w_n^{(2)}) - \mathcal{N}_c(w_n) = \frac{1}{2}\int_{\mathbb{R}} \Big(& L^{1/2} w_n^{(1)} (L^{-1/2} w_n^{(1)})^2 + L^{1/2} w_n^{(2)} (L^{-1/2} w_n^{(2)})^2 \\ & - (\chi_{1n}^2 + \chi_{2n}^2) L^{1/2} w_n (L^{-1/2} w_n)^2 \Big) \,\mathrm{d}x,\end{aligned}$$

furthermore

$$\begin{aligned}&\int_{\mathbb{R}} L^{1/2} w_n^{(1)} (L^{-1/2} w_n^{(1)})^2 - \chi_{1n}^2 L^{1/2} w_n (L^{-1/2} w_n)^2 \,\mathrm{d}x \\ &\quad = \int_{\mathbb{R}} L^{1/2} w_n^{(1)} (L^{-1/2} w_n^{(1)})^2 - \chi_{1n}^2 L^{1/2} w_n^{(1)} (L^{-1/2} w_n)^2 \,\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}} \chi_{1n}^2 L^{1/2} w_n^{(1)} (L^{-1/2} w_n)^2 - \chi_{1n}^2 L^{1/2} w_n (L^{-1/2} w_n)^2 \,\mathrm{d}x,\end{aligned}$$

and using Lemma 2.8 we find that

$$\begin{aligned}
&\int_{\mathbb{R}} L^{\frac{1}{2}} w_n^{(1)} (L^{-1/2} w_n^{(1)})^2 - \chi_{1n}^2 L^{1/2} w_n^{(1)} (L^{-1/2} w_n)^2 \, \mathrm{d}x \\
&= \int_{\mathbb{R}} L^{1/2} w_n^{(1)} ((L^{-1/2} w_n^{(1)})^2 - \chi_{1n}^2 (L^{-1/2} w_n)^2) \, \mathrm{d}x \\
&= \int_{\mathbb{R}} L^{1/2} w_n^{(1)} (L^{-1/2} w_n^{(1)} + \chi_{1n} L^{-1/2} w_n)(L^{-1/2} w_n^{(1)} - \chi_{1n} L^{-1/2} w_n) \, \mathrm{d}x \\
&= \int_{\mathbb{R}} L^{1/2} w_n^{(1)} (L^{-1/2} w_n^{(1)} + \chi_{1n} L^{-1/2} w_n)[L^{-1/2}, \chi_{1n}] w_n \, \mathrm{d}x \\
&\to 0, \ n \to 0,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}} \chi_{1n}^2 L^{1/2} w_n^{(1)} (L^{-1/2} w_n)^2 - \chi_{1n}^2 L^{1/2} w_n (L^{-1/2} w_n)^2 \, \mathrm{d}x \\
&= \int_{\mathbb{R}} \chi_{1n}^2 L^{1/2} (w_n^{(1)} - w_n)(L^{-1/2} w_n)^2 \, \mathrm{d}x \\
&= \int_{\mathbb{R}} L^{1/2} (\chi_{1n} (w_n^{(1)} - w_n)) \chi_{1n} (L^{-1/2} w_n)^2 \, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}} [L^{1/2}, \chi_{1n}] (w_n^{(1)} - w_n) \chi_{1n} (L^{-1/2} w_n)^2 \, \mathrm{d}x,
\end{aligned}$$

where

$$\begin{aligned}
&\left| \int_{\mathbb{R}} L^{1/2} (\chi_{1n} (w_n^{(1)} - w_n)) \chi_{1n} (L^{-1/2} w_n)^2 \, \mathrm{d}x \right| \\
&= \left| \int_{\mathbb{R}} \chi_{1n} (\chi_{1n} - 1) w_n L^{1/2} (\chi_{1n} (L^{-1/2} w_n)^2) \, \mathrm{d}x \right| \\
&\leq \| \chi_{1n} (\chi_{1n} - 1) w_n \|_{L^2} \left\| L^{1/2} (\chi_{1n} (L^{-1/2} w_n)^2) \right\|_{L^2} \\
&\lesssim \| w_n \|_{L^2([-2M_n, -M_n] \cup [M_n, 2M_n])} \| w_n \|_{H^{s/2}}^2 \\
&\to 0, \ n \to \infty,
\end{aligned}$$

and $\int_{\mathbb{R}} [L^{1/2}, \chi_{1n}] (w_n^{(1)} - w_n) \chi_{1n} (L^{-1/2} w_n)^2 \, \mathrm{d}x \to 0, \ n \to \infty$, according to Lemma 2.8. Hence $\lim_{n\to\infty} \int_{\mathbb{R}} L^{1/2} w_n^{(1)} (L^{-1/2} w_n^{(1)})^2 - \chi_{1n}^2 L^{1/2} w_n (L^{-1/2} w_n)^2 \, \mathrm{d}x = 0$, and in the same way we can show that $\lim_{n\to\infty} \int_{\mathbb{R}} L^{1/2} w_n^{(2)} (L^{-1/2} w_n^{(2)})^2 - \chi_{2n}^2 L^{1/2} w_n (L^{-1/2} w_n)^2 \, \mathrm{d}x = 0$, which implies (4.4). The limit (4.5) can be shown using similar techniques as (4.4) and we therefore omit the details.

We conclude that (4.3) holds, which together with (4.2) implies (4.1). Since $\{w_n\}_{n=1}^{\infty}$ is a minimizing sequence, we get that

$$\lim_{n\to\infty} \mathcal{E}(w_n^{(1)}) + \mathcal{E}(w_n^{(1)}) \to I_q. \tag{4.6}$$

However,

$$\lim_{n\to\infty} \left(\mathcal{E}\left(v_n^{(i)}\right) - \mathcal{E}\left(w_n^{(i)}\right) \right) = 0, \ i = 1, 2, \tag{4.7}$$

where $v_n^{(1)} = \sqrt{q^*/Q(w_n^{(1)})}w_n^{(1)}$, $v_n^{(2)} = \sqrt{(q-q^*)/Q(w_n^{(2)})}w_n^{(2)}$. By construction $v_n^{(1)} \in U_{q^*}$, $v_n^{(2)} \in U_{q-q^*}$, and so using (4.6), (4.7), we find that

$$I_q = \lim_{n\to\infty} \mathcal{E}(v_n^{(1)}) + \mathcal{E}(v_n^{(1)}) \geq I_{q^*} + I_{q-q^*},$$

which contradicts Proposition 3.3. □

Proposition 4.3. *There exists $u \in U_q$ solving minimization problem $\mathcal{E}(u) = I_q$.*

Proof. By the concentration-compactness principle our minimizing sequence $e_n = u_n^2 \in L^1(\mathbb{R})$, $n \in \mathbb{N}$, concentrates. Moreover, due to the translation invariance one can assume that it concentrates around zero, and so

$$\int_{|x|>r} u_n^2(x)\,\mathrm{d}x \to 0 \text{ uniformly with respect to } n \in \mathbb{N} \text{ as } r \to \infty.$$

In addition, $\{u_n\}_{n=1}^\infty$ is a bounded sequence in $H^{\frac{s}{2}}(\mathbb{R})$ due to Proposition 3.2, and so

$$\|(u_n)_h - u_n\|_{L^2}^2 \lesssim q \left\|\xi \mapsto |e^{i\xi h} - 1|\langle\xi\rangle^{-s/2}\right\|_{L^\infty}^2$$

that tends to zero uniformly with respect to $n \in \mathbb{N}$ as $h \to 0$. Taking into account the boundedness of $\{u_n\}_{n=1}^\infty$ in $L^2(\mathbb{R})$ one deduces from the Frechet–Kolmogorov theorem that $\{u_n\}_{n=1}^\infty$ is relatively compact in $L^2(\mathbb{R})$. Thus we can assume that $\{u_n\}_{n=1}^\infty$ converges to some u in $L^2(\mathbb{R})$. Again using that $\{u_n\}_{n=1}^\infty$ is bounded in $H^{\frac{s}{2}}(\mathbb{R})$, we may in addition assume that u_n converges weakly in $H^{\frac{s}{2}}(\mathbb{R})$ to u. Hence $u \in U_q$ and it is left to check that it solves the minimization problem.

Firstly, applying the weak lower semi-continuity argument we deduce

$$\mathcal{L}(u) \leqslant \liminf_{n\to\infty} \mathcal{L}(u_n).$$

Indeed, the square root of $\mathcal{L}(u)$ defines a norm in $H^{\frac{s}{2}}(\mathbb{R})$, equivalent to the standard Sobolev norm. By the Mazur theorem a closed ball is weakly closed. The latter property implies the weak lower semi-continuity of the functional $\mathcal{L}$.

It is left to show that $\mathcal{N}(u_n)$ tends to $\mathcal{N}(u)$ as $n \to \infty$. The cubic part is estimated as

$$\begin{aligned} |\mathcal{N}_c(u) - \mathcal{N}_c(u_n)| &\leqslant \frac{1}{2}\left|\int_{\mathbb{R}} \left(L^{-1/2}u\right)^2 L^{1/2}(u-u_n)\,\mathrm{d}x\right| \\ &+ \frac{1}{2}\left|\int_{\mathbb{R}} \left(\left(L^{-1/2}u\right)^2 - \left(L^{-1/2}u_n\right)^2\right) L^{1/2}u_n\,\mathrm{d}x\right| = \frac{1}{2}\left|\int_{\mathbb{R}} (u-u_n)L^{1/2}\left(L^{-1/2}u\right)^2\,\mathrm{d}x\right| \\ &+ \frac{1}{2}\left|\int_{\mathbb{R}} \left(L^{-1/2}(u-u_n)\right)\left(L^{-1/2}(u+u_n)\right) L^{1/2}u_n\,\mathrm{d}x\right| \lesssim \|u-u_n\|_{L^2}\left\|L^{1/2}\left(L^{-1/2}u\right)^2\right\|_{L^2} \\ &+ \left\|L^{-1/2}(u-u_n)\right\|_{H^{\frac{s}{2}}}\left\|L^{-1/2}(u+u_n)\right\|_{H^{\frac{s}{2}}}\left\|L^{1/2}u_n\right\|_{L^2} \lesssim q\|u-u_n\|_{L^2} \end{aligned}$$

which tends to zero as $n \to \infty$. For the remainder we have

$$\begin{aligned}
&|\mathcal{N}_r(u) - \mathcal{N}_r(u_n)| \\
&\quad = \frac{1}{8} \left| \int_{\mathbb{R}} \left(L^{-1/2}(u-u_n)\right) \left(L^{-1/2}(u+u_n)\right) \left(\left(L^{-1/2}u\right)^2 + \left(L^{-1/2}u_n\right)^2 \right) \, \mathrm{d}x \right| \\
&\quad \lesssim \left\| L^{-1/2}(u-u_n) \right\|_{H^{s/2}} \left\| L^{-1/2}(u+u_n) \right\|_{H^{s/2}} \left(\left\| L^{-1/2}u \right\|_{L^4}^2 + \left\| L^{-1/2}u_n \right\|_{L^4}^2 \right) \\
&\qquad \lesssim q^{3/2} \left\| u - u_n \right\|_{L^2}
\end{aligned}$$

that tends to zero as $n \to \infty$. Summing up we obtain

$$I_q \leqslant \mathcal{E}(u) \leqslant \liminf_{n \to \infty} \mathcal{E}(u_n) = I_q$$

which concludes the proof. □

We finish the proof of Theorem 1.2 by proving the estimate. Let u be a minimizer. We know that u satisfies the Euler–Lagrange equation

$$\lambda u + \mathrm{d}\mathcal{E}(u) = 0.$$

Taking the inner product in this equation with u yields

$$\begin{aligned}
-2\lambda q &= \mathrm{d}\mathcal{E}(u)(u) \\
&= 2\mathcal{L}(u) + 3\mathcal{N}_c(u) + 4\mathcal{N}_r(u) \\
&= -\mathcal{L}(u) + 3\mathcal{E}(u) + 4\mathcal{N}_r(u) \qquad (4.8)
\end{aligned}$$

Since $\mathcal{L}(u) \geq m(0)q$ and $|\mathcal{N}_c(u)| = \mathcal{O}(q^{3/2})$, $|\mathcal{N}_r(u)| = \mathcal{O}(q^2)$ by Proposition 2.3, it is easy to see from the second inequality in (4.8) that for q sufficiently small

$$-\lambda > \frac{m(0)}{2}.$$

For the upper bound we use (4.8) together with propositions 2.3, 3.1, 3.2 to deduce that

$$\begin{aligned}
-2\lambda q &= -\mathcal{L}(u) + 3\mathcal{E}(u) + 4\mathcal{N}_r(u) \\
&= -\mathcal{L}(u) + 3I_q + 4\mathcal{N}_r(u) \\
&\leq -m(0)q + 3(m(0)q - Dq^{1+\beta}) + \mathcal{O}(q^2) \\
&= 2m(0)q - 3Dq^{1+\beta} + \mathcal{O}(q^2),
\end{aligned}$$

hence, for q sufficiently small

$$-\lambda < m(0) - Dq^{\beta}.$$

5 Long wave approximation

In this section we return to the initial variational problem for the Whitham–Boussinesq system. So from now on $L = K$. We will show that all minimizers are infinitely smooth and refine existing estimates for them.

Lemma 5.1. *There exists* $q_0 > 0$ *such that for each* $r \geqslant 0$ *holds* $\|u\|^2_{H^r} \lesssim q$ *uniformly for* $q \in (0, q_0)$ *and* $u \in D_q$.

Proof. Firstly, one can notice that the statement holds for $r \in [0, 1/2]$, due to Proposition 3.2. We will extend the result by induction to bigger values of r applying Formula (1.13).

Let $r \geqslant 1/2$, then from the equivalence of operators K, J and product estimates in Sobolev spaces we deduce

$$\left\| K^{-1/2} \left(K^{-1/2} v \right)^3 \right\|_{H^r} \lesssim \|v\|^3_{H^r},$$

$$\left\| K^{-1/2} \left(K^{1/2} v K^{-1/2} v \right) \right\|_{H^r} \lesssim \|v\|^2_{H^r},$$

$$\left\| K^{1/2} \left(K^{-1/2} v \right)^2 \right\|_{H^r} \lesssim \|v\|^2_{H^r}$$

for any $v \in H^r(\mathbb{R})$. All three constants here depend only on r.

Now for any minimizer $u \in D_q$ calculate Ku by Formula (1.13) and obtain

$$\|u\|_{H^{r+1}} \lesssim \|Ku\|_{H^r} \leqslant |\lambda| \|u\|_{H^r} + \frac{1}{2} \left\| K^{-1/2} \left(K^{-1/2} u \right)^3 \right\|_{H^r} + \left\| K^{-1/2} \left(K^{1/2} v K^{-1/2} u \right) \right\|_{H^r} + \frac{1}{2} \left\| K^{1/2} \left(K^{-1/2} u \right)^2 \right\|_{H^r} \lesssim \sqrt{q}$$

for any $r \geqslant 1/2$. We have used $|\lambda| \leqslant 1$ according to Theorem 1.2. This concludes the proof by induction. □

Lemma 5.2. *There exist* $q_0 > 0$ *and* $C > 0$ *such that the following estimates hold*

$$\|u\|_{L^\infty} \leqslant C q^{2/3}, \tag{5.1}$$

$$\|\partial_x u\|^2_{L^2} \leqslant C q^{5/3}, \tag{5.2}$$

$$\|\partial^2_x u\|^2_{L^2} \leqslant C q^{7/3} \tag{5.3}$$

uniformly for $q \in (0, q_0)$ *and* $u \in D_q$.

Proof. Introducing the notation

$$M(u) = \frac{1}{2} K^{-1/2} \left(K^{-1/2} u \right)^3 + K^{-1/2} \left(K^{1/2} u K^{-1/2} u \right) + \frac{1}{2} K^{1/2} \left(K^{-1/2} u \right)^2$$

one can rewrite Equation (1.13) in the form

$$(\lambda + K) u = -M(u).$$

Note that $-\lambda \in \left(0, 1 - D q^{2/3}\right)$ according to Theorem 1.2 and so $\lambda + 1 > D q^{2/3}$. The Fourier transform of minimizer u can be estimated as

$$|\widehat{u}(\xi)| = \left| \frac{\mathcal{F}(M(u))}{\lambda + m(\xi)} \right| \leqslant \frac{|\mathcal{F}(M(u))|}{D q^{2/3} + m(\xi) - 1} \lesssim |\mathcal{F}(M(u))(\xi)| \left(\frac{\chi_{|\xi| \leqslant 1}(\xi)}{q^{2/3} + \xi^2} + \frac{\chi_{|\xi| > 1}(\xi)}{q^{2/3} + |\xi|} \right)$$

where $\chi_A(\xi)$ stands for the characteristic function of a set A. As was shown in the proof of Lemma 5.1 $M(u)$, is smooth and its H^s-norm is bounded by q for any non-negative s. Hence $\mathcal{F}(M(u))$ multiplied by any power of ξ is bounded by q with respect to L^2-norm.

Let us show that the L^∞-norm of $\mathcal{F}(M(u))$ is bounded by q. Indeed, we have

$$\left|\mathcal{F}\left(K^{1/2}\left(K^{-1/2}u\right)^2\right)(\xi)\right| \lesssim \int_{\mathbb{R}} \frac{\sqrt{m(\xi)}|\widehat{u}(\xi-\zeta)\widehat{u}(\zeta)|}{\sqrt{m(\xi-\zeta)m(\zeta)}}\,\mathrm{d}\zeta \lesssim \|u\|_{L^2}^2 \lesssim q,$$

$$\left|\mathcal{F}\left(K^{-1/2}\left(K^{1/2}uK^{-1/2}u\right)\right)(\xi)\right| \lesssim \int_{\mathbb{R}} \frac{\sqrt{m(\xi-\zeta)}|\widehat{u}(\xi-\zeta)\widehat{u}(\zeta)|}{\sqrt{m(\xi)m(\zeta)}}\,\mathrm{d}\zeta \lesssim \|u\|_{L^2}^2 \lesssim q$$

and similarly

$$\left|\mathcal{F}\left(K^{-1/2}\left(K^{-1/2}u\right)^3\right)(\xi)\right| \lesssim \|u\|_{L^2}\left\|\left(K^{-1/2}u\right)^2\right\|_{L^2} \lesssim \|u\|_{L^2}^3 \lesssim q^{3/2}.$$

Thus $\|\mathcal{F}(M(u))\|_{L^\infty} \lesssim q$. So we are in a position to prove (5.1), indeed,

$$\begin{aligned}\|u\|_{L^\infty} \lesssim \|\widehat{u}\|_{L^1} \lesssim \int_{|\xi|\leqslant 1} \frac{|\mathcal{F}(M(u))(\xi)|}{q^{2/3}+\xi^2}d\xi + \int_{|\xi|>1} \frac{|\mathcal{F}(M(u))(\xi)|}{q^{2/3}+|\xi|}\,\mathrm{d}\xi \\ \lesssim q^{-1/3}\|\mathcal{F}(M(u))\|_{L^\infty} + \|\mathcal{F}(M(u))\|_{L^2} \lesssim q^{2/3}.\end{aligned}$$

Estimate (5.2) is proved as follows

$$\begin{aligned}\|\partial_x u\|_{L^2}^2 = \|\xi \mapsto \xi\widehat{u}(\xi)\|_{L^2}^2 \lesssim \int_{|\xi|\leqslant 1} \frac{\xi^2|\mathcal{F}(M(u))(\xi)|^2}{(q^{2/3}+\xi^2)^2}\,\mathrm{d}\xi + \int_{|\xi|>1} \frac{\xi^2|\mathcal{F}(M(u))(\xi)|^2}{(q^{2/3}+|\xi|)^2}\,\mathrm{d}\xi \\ \lesssim q^{-1/3}\|\mathcal{F}(M(u))\|_{L^\infty}^2 + \|\mathcal{F}(M(u))\|_{L^2}^2 \lesssim q^{5/3}.\end{aligned}$$

A straightforward repetition of the last argument for the second derivative of the minimizer gives

$$\begin{aligned}\|\partial_x^2 u\|_{L^2}^2 = \|\xi \mapsto \xi^2\widehat{u}(\xi)\|_{L^2}^2 \lesssim \int_{|\xi|\leqslant 1} \frac{\xi^4|\mathcal{F}(M(u))(\xi)|^2}{(q^{2/3}+\xi^2)^2}\,\mathrm{d}\xi + \int_{|\xi|>1} \frac{\xi^4|\mathcal{F}(M(u))(\xi)|^2}{(q^{2/3}+|\xi|)^2}\,\mathrm{d}\xi \\ \lesssim q^{1/3}\|\mathcal{F}(M(u))\|_{L^\infty}^2 + \|\mathcal{F}(\partial_x M(u))\|_{L^2}^2 \lesssim q^{7/3} + \|\partial_x M(u)\|_{L^2}^2\end{aligned} \tag{5.4}$$

that is only $\mathcal{O}(q^2)$ and so weaker than (5.3). However, Estimate (5.2) is a refinement compared with Lemma 5.1, so it can be used for more delicate estimate of the square norm $\|\partial_x M(u)\|_{L^2}^2$ as follows

$$\begin{aligned}\left\|\partial_x K^{1/2}\left(K^{-1/2}u\right)^2\right\|_{L^2} \lesssim \left\|K^{-1/2}uK^{-1/2}\partial_x u\right\|_{H^{1/2}} \\ \lesssim \left\|K^{-1/2}u\right\|_{H^1}\left\|K^{-1/2}\partial_x u\right\|_{H^{1/2}} \lesssim q^{4/3},\end{aligned}$$

where product estimates were used. To continue, first note that the estimate of the derivative (5.2), will not be spoiled if one changes L^2-norm to H^s-norm with any $s \geqslant 0$. In other words, $\|\partial_x u\|_{H^s} \lesssim q^{5/6}$, and so

$$\begin{aligned}\left\|\partial_x K^{-1/2}\left(K^{1/2}uK^{-1/2}u\right)\right\|_{L^2} \lesssim \left\|K^{1/2}\partial_x uK^{-1/2}u\right\|_{L^2} + \left\|K^{1/2}uK^{-1/2}\partial_x u\right\|_{L^2} \\ \lesssim \|\partial_x u\|_{H^{3/2}}\|u\|_{L^2} + \|u\|_{H^{3/2}}\|\partial_x u\|_{L^2} \lesssim q^{4/3}.\end{aligned}$$

The last remaining term is estimated similarly

$$\left\| \partial_x K^{-1/2} \left(K^{-1/2} u \right)^3 \right\|_{L^2} \lesssim \left\| (K^{-1/2} u)^2 K^{-1/2} \partial_x u \right\|_{L^2} \lesssim \|u\|_{H^{1/2}}^2 \|\partial_x u\|_{L^2} \lesssim q^{11/6}.$$

Thus

$$\|\partial_x M(u)\|_{L^2} \lesssim q^{4/3}$$

that together with (5.4) conclude the proof of Estimate (5.3). □

Remark 5.3. *Lemmas 5.1, 5.2 remain valid with the surface elevation η_u and velocity v_u defined by* (1.14), (1.15) *substituted instead of the minimizer $u \in D_q$.*

We now turn to the task of approximating the solutions found in Theorem 1.2 with solutions of the KdV-equation. For this part we follow [6] closely.

We introduce the long-wave scaling $S_{\mathrm{KdV}}(f)(x) = q^{2/3} f(q^{1/3}x)$ and note that when making the ansatz $u = S_{\mathrm{KdV}}(\psi)$ in (1.13), the leading order part of the equation as $q \to 0$ is, with $\lambda = -1 + \lambda_0 q^{2/3}$,

$$\lambda_0 \psi + \frac{3}{2}\psi^2 - \frac{\psi_{xx}}{3} = 0. \tag{5.5}$$

Equation (5.5) is the travelling wave version of the KdV-equation, which has the up to translation the following unique solution

$$\psi_{\mathrm{KdV}}(x) = -\lambda_0 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{3\lambda_0}x\right).$$

We note that (5.5) is the Euler-Lagrange equation of the minimization problem

$$I_{\mathrm{KdV}} := \min_{\psi \in V_1} \mathcal{E}_{KdV}(\psi),$$

where

$$\mathcal{E}_{KdV}(\psi) := \frac{1}{2}\int_{\mathbb{R}} \frac{\psi_x^2}{3} + \psi^3 \,\mathrm{d}x,$$

and $V_1 := \{\psi \in H^1(\mathbb{R}) \ : \ \mathcal{Q}(\psi) = 1\}$. The constraint $\mathcal{Q}(\psi_{\mathrm{KdV}}) = 1$ requires that $\lambda_0 = 3/16^{\frac{1}{3}}$. The relation between $\mathcal{E}$ and $\mathcal{E}_{\mathrm{KdV}}$ is now established.

Lemma 5.4. *For $u \in H^2(\mathbb{R})$ hold*

$$\mathcal{E}(u) = \mathcal{Q}(u) + \mathcal{E}_{KdV}(u) + \mathcal{E}_{rem}(u), \tag{5.6}$$

with

$$|\mathcal{E}_{rem}(u)| \lesssim \left\|\partial_x^2 u\right\|_{L^2}^2 + \|u\|_{L^\infty} \|\partial_x u\|_{L^2}^2 + \|u\|_{L^2}^2 \|\partial_x u\|_{L^2} + \|u\|_{L^4}^2 \left\|\partial_x^2 u\right\|_{L^2} + \|u\|_{L^2}^4, \tag{5.7}$$

$$|\langle \mathrm{d}\mathcal{E}_{rem}(u), u\rangle| \lesssim \left\|\partial_x^2 u\right\|_{L^2}^2 + \|u\|_{L^\infty} \|\partial_x u\|_{L^2}^2 + \|u\|_{L^2}^2 \|\partial_x u\|_{L^2} + \|u\|_{L^4}^2 \|\partial_x u\|_{L^2} + \|u\|_{L^2}^4 \tag{5.8}$$

Proof. We note that

$$\begin{aligned}
\mathcal{E}(u) &= \frac{1}{2}\int_{\mathbb{R}} uKu \, \mathrm{d}x + \mathcal{N}_c(u) + \mathcal{N}_r(u) \\
&= Q(u) + \frac{1}{2}\int_{\mathbb{R}} u(K-1)u \, \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} u^3 \, \mathrm{d}x + \mathcal{N}_{1c}(u) + \mathcal{N}_{2c}u + \mathcal{N}_{3c}(u) + \mathcal{N}_r(u) \\
&= Q(u) + \mathcal{E}_{\mathrm{KdV}}(u) \\
&\quad + \underbrace{\frac{1}{2}\int_{\mathbb{R}} \left(m(\xi) - 1 - \frac{\xi^2}{3} \right) |\hat{u}|^2 \, \mathrm{d}\xi + \mathcal{N}_{1c}(u) + \mathcal{N}_{2c}(u) + \mathcal{N}_{3c}(u) + \mathcal{N}_r(u)}_{=:\mathcal{E}_{\mathrm{rem}}(u)}
\end{aligned}$$

Since $m(\xi) = \xi/\tanh(\xi)$, we have that $|m(\xi) - 1 - \frac{\xi^2}{3}| \lesssim \xi^4$, so that

$$\int_{\mathbb{R}} \left| m(\xi) - 1 - \frac{\xi^2}{3} \right| |\hat{u}|^2 \, \mathrm{d}\xi \lesssim \left\| \partial_x^2 u \right\|_{L^2}^2 .$$

From Lemma 2.5 we have

$$|\mathcal{N}_{1c}(u)| \lesssim \int_{\mathbb{R}} |u((K^{-1/2} - 1)u)^2| \, \mathrm{d}x \lesssim \|u\|_{L^\infty} \|\partial_x u\|_{L^2}^2 .$$

Similarly we find that

$$\begin{aligned}
|\mathcal{N}_{2c}(u)| &\lesssim \|u\|_{L^4}^2 \|\partial_x u\|_{L^2} , \\
|\mathcal{N}_{3c}(u)| &\lesssim \|u\|_{L^2}^2 \|\partial_x u\|_{L^2} .
\end{aligned}$$

The term $\mathcal{N}_r(u)$ is estimated in Proposition 2.3, hence (5.7) is established. The estimate (5.8) is proved in a similar way and we therefore omit the details. □

Lemma 5.5. *There exists $q_0 > 0$ such that*

$$I_q = q + \mathcal{E}_{KdV}(u) + \mathcal{O}(q^2), \text{ uniformly over } u \in D_q, \tag{5.9}$$

$$I_q = q + q^{\frac{5}{3}} I_{KdV} + \mathcal{O}(q^2). \tag{5.10}$$

Proof. Let $u \in D_q$. From Lemma 5.1 we know that $u \in H^r(\mathbb{R})$ for any $r \geq 0$. In particular $u \in H^2(\mathbb{R})$, hence by Lemma 5.4

$$\mathcal{E}(u) = q + \mathcal{E}_{\mathrm{KdV}}(u) + \mathcal{E}_{\mathrm{rem}}(u).$$

Using (5.7) together with Lemma 5.2, we get $|\mathcal{E}_{\mathrm{rem}}(u)| \lesssim q^2$. Hence, (5.9) follows.

Turning now to (5.10) we let $\psi = S_{\mathrm{KdV}}^{-1}(u)$ and note that $\psi \in V_1$ and

$$\mathcal{E}_{\mathrm{KdV}}(u) = q^{5/3}\mathcal{E}_{\mathrm{KdV}}(\psi) \geq q^{5/3} I_{\mathrm{KdV}},$$

so this together with (5.9) implies

$$I_q \geq q + q^{5/3} I_{\mathrm{KdV}} + \mathcal{O}(q^2).$$

On the other hand, $\tilde{u} := S_{\text{KdV}}(\psi_{\text{KdV}}) \in U_q$, so again using (5.9) obtain

$$\begin{aligned} I_q &\le \mathcal{E}(\tilde{u}) \\ &= q + \mathcal{E}_{\text{KdV}}(\tilde{u}) + \mathcal{O}(q^2) \\ &= q + q^{5/3}\mathcal{E}_{\text{KdV}}(\psi_{\text{KdV}}) + \mathcal{O}(q^2) \\ &= q + q^{5/3} I_{\text{KdV}} + \mathcal{O}(q^2), \end{aligned}$$

which concludes the proof of (5.10). □

The statement of Theorem 1.3 is a summary of the following lemmas.

Lemma 5.6. *There exists $q_0 > 0$ such that for any $q \in (0, q_0)$ and $u \in D_q$ there exists $x_u \in \mathbb{R}$ such that*

$$\left\| S_{KdV}^{-1}(u) - \psi_{KdV}(\cdot - x_u) \right\|_{H^1} \lesssim q^{1/6},$$

uniformly with respect to $q \in (0, q_0)$ and $u \in D_q$.

The proof of Lemma 5.6 is identical to the proof of [6, Theorem 5.5] and is therefore omitted. We next relate the two Lagrange multipliers λ and λ_0.

Lemma 5.7. *The Lagrange multipliers related to the minimization problem* (1.20)*, satisfy*

$$\lambda = -1 + \lambda_0 q^{2/3} + \mathcal{O}(q^{5/6}).$$

Proof. Let $u \in D_q$. From Lemma 5.4 we have

$$\langle \mathrm{d}\mathcal{E}(u), u \rangle = 2q + \langle \mathrm{d}\mathcal{E}_{\text{KdV}}(u), u \rangle + \mathcal{O}(q^2). \tag{5.11}$$

Moreover, $\langle \mathrm{d}\mathcal{E}_{\text{KdV}}(u), u \rangle = q^{5/3} \langle \mathrm{d}\mathcal{E}_{\text{KdV}}(S_{\text{KdV}}^{-1}(u)), S_{\text{KdV}}^{-1}(u) \rangle$, and by Lemmas 5.2, 5.6

$$\langle \mathrm{d}\mathcal{E}_{\text{KdV}}(S_{\text{KdV}}^{-1}(u)), S_{\text{KdV}}^{-1}(u) \rangle - \langle \mathrm{d}\mathcal{E}_{\text{KdV}}(\psi_{\text{KdV}}), \psi_{\text{KdV}} \rangle = \mathcal{O}(q^{1/6}).$$

Combining this with (5.11), we obtain

$$\langle \mathrm{d}\mathcal{E}(u), u \rangle = 2q + q^{5/3} \langle \mathrm{d}\mathcal{E}_{\text{KdV}}(\psi_{\text{KdV}}), \psi_{\text{KdV}} \rangle + \mathcal{O}(q^{11/6}). \tag{5.12}$$

On the other hand, from the Euler-Lagrange equations we have

$$\begin{aligned} 2\lambda q + \langle \mathrm{d}\mathcal{E}(u), u \rangle &= 0, \\ 2\lambda_0 + \langle \mathrm{d}\mathcal{E}_{\text{KdV}}(\psi_{\text{KdV}}), \psi_{\text{KdV}} \rangle &= 0, \end{aligned}$$

and when we combine this with (5.12), we get

$$-2\lambda q = 2q - 2\lambda_0 q^{5/3} + \mathcal{O}(q^{11/6}),$$

and dividing with $-2q$ yields

$$\lambda = -1 + \lambda_0 q^{2/3} + \mathcal{O}(q^{5/6}).$$

□

For each solution u of (1.13), we have the corresponding physical parameters η_u, v_u defined by (1.14), (1.15) where $-1/c^2 = \lambda = -1 + \lambda_0 q^{2/3} + \mathcal{O}(q^{5/6})$ by Lemma 5.7. We have the following estimates for η_u, v_u that are similar to the one given in Lemma 5.6.

Lemma 5.8. *There exists $q_0 > 0$ such that for $q \in (0, q_0)$ and $u \in D_q$ there exists $x_u \in \mathbb{R}$ such that*

$$\left\| S_{KdV}^{-1}(\eta_u) + \psi_{KdV}(\cdot - x_u) \right\|_{H^{1/2}} \lesssim q^{1/6},$$
$$\left\| S_{KdV}^{-1}(v_u) + \psi_{KdV}(\cdot - x_u) \right\|_{H^{3/2}} \lesssim q^{1/6}$$

uniformly with respect to $q \in (0, q_0)$ and $u \in D_q$.

Proof. We will prove the first inequality. The second one can be proved analogously. Firstly, one can notice that due to $1/2 < -\lambda < 1$ in accordance with to Estimate (1.22), it is enough to prove

$$\left\| \lambda S_{\mathrm{KdV}}^{-1}(\eta_u) + \lambda \psi_{\mathrm{KdV}}(\cdot - x_u) \right\|_{H^{1/2}} \lesssim q^{1/6}, \tag{5.13}$$

where x_u is taken as in Lemma 5.6. The first term under the norm in (5.13) has the form

$$\lambda S_{\mathrm{KdV}}^{-1}(\eta_u) = \frac{q^{-2/3}}{2} \left(K^{-1/2} u\right)^2 (q^{-1/3}\cdot) + q^{-2/3}(K^{1/2}u)(q^{-1/3}\cdot)$$

where the first element of the sum is negligible in view of the straightforward estimate

$$\left\| (K^{-1/2}u)^2 (q^{-1/3}\cdot) \right\|_{H^{1/2}} \lesssim q.$$

The second element of the sum can be rewritten as follows. We note that

$$(K^{1/2}u)(q^{-1/3}x) = (K_q^{1/2} u(q^{-1/3}\cdot))(x),$$

where we used K_q to denote the Fourier multiplier operator with symbol $m(q^{1/3}\cdot)$. We then get that $q^{-2/3}(K^{1/2}u)(q^{-1/3}\cdot) = K_q^{1/2} S_{\mathrm{KdV}}^{-1}(u)$. Using this, we find that

$$\begin{aligned} q^{-2/3}(K^{1/2}u)(q^{-1/3}\cdot) - \psi_{\mathrm{KdV}}(\cdot - x_u) &= K_q^{1/2} S_{\mathrm{KdV}}^{-1}(u) - \psi_{\mathrm{KdV}}(\cdot - x_u) \\ &= K_q^{1/2}(S_{\mathrm{KdV}}^{-1}(u) - \psi_{\mathrm{KdV}}(\cdot - x_u)) + (K_q^{1/2} - 1)\psi_{\mathrm{KdV}}(\cdot - x_u). \end{aligned}$$

Here the last term is estimated as

$$\left\| (K_q^{1/2} - 1)\psi_{\mathrm{KdV}}(\cdot - x_u) \right\|_{H^{1/2}} \lesssim q^{1/3} \left\| \psi_{\mathrm{KdV}} \right\|_{H^{3/2}}.$$

Finally, we have

$$\begin{aligned} \left\| \lambda S_{\mathrm{KdV}}^{-1}(\eta_u) + \lambda \psi_{\mathrm{KdV}}(\cdot - x_u) \right\|_{H^{1/2}} &\lesssim \left\| K_q^{1/2}(S_{\mathrm{KdV}}^{-1} - \psi_{\mathrm{KdV}}(\cdot - x_u)) \right\|_{H^{1/2}} \\ &+ \left\| (K_q^{1/2} - 1)\psi_{\mathrm{KdV}}(\cdot - x_u) \right\|_{H^{1/2}} + \left\| (1+\lambda)\psi_{\mathrm{KdV}}(\cdot - x_u) \right\|_{H^{1/2}} + q^{1/3} \end{aligned}$$

that gives (5.13) by Lemma 5.6 and 5.7.

□

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Paper VIII

3.8 Well-posedness for a Whitham-Boussinesq system with surface tension

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WELL-POSEDNESS FOR A WHITHAM–BOUSSINESQ SYSTEM WITH SURFACE TENSION

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Abstract. We regard the Cauchy problem for a particular Whitham–Boussinesq system modelling surface waves of an inviscid incompressible fluid layer. The system can be seen as a weak nonlocal dispersive perturbation of the shallow water system. The proof of well-posedness relies on energy estimates. However, due to the symmetry lack of the nonlinear part, in order to close the a priori estimates one has to modify the traditional energy norm in use. Hamiltonian conservation provides with global well-posedness at least for small initial data in the one dimensional settings.

1. Introduction

Consideration is given to the following one-dimensional Whitham-type system

$$\begin{cases} \partial_t \eta = -\partial_x v - i \tanh D(\eta v), \\ \partial_t v = -i \tanh D(1 + \varkappa D^2)\eta - i \tanh D v^2/2 \end{cases} \tag{1.1}$$

where $D = -i\partial_x$ and $\tanh D$ are Fourier multiplier operators in the space of tempered distributions $\mathcal{S}'(\mathbb{R})$. The positive parameter $\varkappa$ stands for the surface tension here. The space variable is $x \in \mathbb{R}$ and the time variable is $t \in \mathbb{R}$. The unknowns η, v are real valued functions of these variables. We pick the initial values $\eta(0)$, $v(0)$ corresponding to the time moment $t = 0$ in Sobolev spaces as follows

$$\eta_0 \in H^{s+1/2}(\mathbb{R}), \qquad v_0 \in H^s(\mathbb{R}) \tag{1.2}$$

where $s \geqslant 1/2$. System (1.1) has the Hamiltonian structure

$$\partial_t(\eta, v)^T = \mathcal{J}\nabla\mathcal{H}(\eta, v)$$

with the skew-adjoint matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -i\tanh D \\ -i\tanh D & 0 \end{pmatrix}$$

and the energy functional

$$\mathcal{H}(\eta, v) = \frac{1}{2}\int_{\mathbb{R}} \left(\eta^2 + \varkappa(\partial_x\eta)^2 + v\frac{D}{\tanh D}v + \eta v^2\right) dx \tag{1.3}$$

well defined on $H^1 \times H^{1/2}$. The latter conserves on solutions together with momentum $\mathcal{I}(\eta, v)$ that has the same view as in the pure gravity case

$$\mathcal{I}(\eta, v) = \int_{\mathbb{R}} \eta\frac{D}{\tanh D}v dx.$$

In case of the trivial surface tension $\varkappa = 0$, System (1.1) was proposed in [6] as an approximate model for the study of water waves to provide a two-directional alternative to the well-known Whitham equation [22]. The latter was proved to be consistent with the KdV equation [18] in the long wave regime [19]. We also refer to [10] for another version of the fully-dispersive Boussinesq type. Importance of such models is supported by experiments [4]. The unknown η denotes the deflection of the free surface from its equilibrium position, corresponding to the vertical level $z = 0$. The bottom is assumed to be flat and located at the level $z = -1$. The variable v is associated with the free surface velocity as explained in [6].

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The initial value problem for Model (1.1) was studied in [5, 9] in the case of vanishing surface tension $\varkappa = 0$. In the same framework existence of solitary waves was proved in [8]. A natural extension of the existing results is to consider the case of non-trivial capillarity $\varkappa > 0$. Note that usually the term $1 + \varkappa D^2$ is applied to $-v_x$ in the first equation as it is done in [12], for example, to regularise the system regarded in [11]. However, the case regarded here is physically more relevant [7]. It turns out that surface tension spoils regularity. Indeed, the multiplication operator $\eta \mapsto v\eta$ is not bounded in our problem. We have 1/2 loss of regularity here, which means that System (1.1) is quasilinear. As a result the proof of well-posedness demands a technique different from the one used in [9].

As to additional initial conditions, apart from inclusions given in (1.2), one has to impose a restriction essentially similar to the one used in [9], namely, smallness of the $H^1 \times H^{1/2}$-norm of (η_0, v_0). This is important for the global-in-time existence. The meaning of this condition is that the total energy $\mathcal{H}(\eta_0, v_0)$ should be positive and not too big. We point out that this condition cannot be significantly weakened even for the proof of the local result, which is also different from the non-capillarity situation. More precisely, for the local regular (s is large enough in (1.2)) well-posedness result it is enough to assume non-cavitation instead.

Definition 1. We say that elevation $\eta \in C([0,T]; L^\infty(\mathbb{R}))$ satisfies the non-cavitation condition if there exist $h, H > 0$ such that $H \geqslant \eta \geqslant h - 1$ on $\mathbb{R} \times [0,T]$. Analogously, one defines non-cavitation at a particular time moment.

The non-cavitation condition is a physical condition meaning that the elevation of the wave should not touch the bottom of the fluid for System (1.1) to be a relevant model. For convenience we have also included boundedness from above in this definition. We exploit the definition for providing with more general local existence formulation at high regularity level. However, in the low regularity case this condition cannot be controlled without imposing a stronger assumption, as we shall see below. We turn now to the formulation of the main result.

Theorem 1. *Let $s > 1/2$. For any $\eta_0 \in H^{s+1/2}(\mathbb{R})$ and $v_0 \in H^s(\mathbb{R})$ having sufficiently small $H^1 \times H^{1/2}$-norm there exists a unique global solution $(\eta, v) \in C([0,\infty); H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R}))$ of System* (1.1) *with the initial data (η_0, v_0). Moreover, the solution depends continuously on the initial data on any finite time interval $[0,T]$.*

Remark 1. Assuming instead of smallness of $H^1 \times H^{1/2}$-norm the noncavitation condition for η_0 one obtains local well-posedness for $s > 3/2$.

The proof is essentially based on the energy method, that is natural to apply to quasilinear equations. The scaling $H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})$ is needed to rule out the linear terms. The main difficulty lies in the lack of symmetry of the nonlinearity. Indeed, a direct time differentiation of the norm $\|\eta, v\|_{H^{s+1/2} \times H^s}$ leads to the term $\int \left(J^{s-1/2}\partial_x \eta\right) \eta J^{s+1/2} v$, where J^σ stands for the Bessel potential of order $-\sigma$ (see the proof of Lemma 6 below). Note that this term cannot be handled by integration by parts or commutator estimates, and so cannot be estimated via the energy norm. To overcome this difficulty we modify the energy norm adding the cubic term $\int \eta \left(J^{s-1/2} v\right)^2$. The linear contribution of the derivative of this term will cancel out the mentioned inconvenient term. Meanwhile, the contribution coming from the nonlinear terms can easily be controlled. As we point out below a hint on the choice of the modifier comes from Hamiltonian (1.3). Note that after adding the cubic term the energy loses coercivity, and so one has to impose an additional condition. Either the noncavitation for big s or the smallness for small s of the initial data, both propagating through the flow of System (1.1), is enough to ensure that the modified energy is coercive.

Additionally, consideration is also given to a system posed on $\mathbb{R}^{2+1}$ of the form

$$\begin{cases} \partial_t \eta + \nabla \cdot \mathbf{v} = -K^2 \nabla \cdot (\eta \mathbf{v}), \\ \partial_t \mathbf{v} + K^2 \nabla (1 + \varkappa |D|^2) \eta = -K^2 \nabla \left(|\mathbf{v}|^2/2\right) \end{cases} \tag{1.4}$$

that is a direct two dimensional extension of Model (1.1). Here $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ is a curl free vector field, that is $\nabla \times \mathbf{v} = 0$, and

$$K = \sqrt{\tanh |D|/|D|}$$

with $D = -i\nabla$. So the corresponding symbol $K(\xi) = \sqrt{\tanh(|\xi|)/|\xi|}$. We complement (1.4) with the initial data

$$\eta(0) = \eta_0 \in H^{s+1/2}\left(\mathbb{R}^2\right), \qquad \mathbf{v}(0) = \mathbf{v}_0 \in H^s\left(\mathbb{R}^2\right) \times H^s\left(\mathbb{R}^2\right). \tag{1.5}$$

As above the variables η and $\mathbf{v}$ stand for the surface elevation and the surface fluid velocity, respectively. The system enjoys the Hamiltonian structure

$$\partial_t(\eta, \mathbf{v})^T = \mathcal{J}\nabla\mathcal{H}(\eta, \mathbf{v})$$

with the skew-adjoint matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -K^2\partial_{x_1} & -K^2\partial_{x_2} \\ -K^2\partial_{x_1} & 0 & 0 \\ -K^2\partial_{x_2} & 0 & 0 \end{pmatrix},$$

which in particular, guarantees conservation of the energy functional

$$\mathcal{H}(\eta, \mathbf{v}) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\eta^2 + \varkappa|\nabla\eta|^2 + \left|K^{-1}\mathbf{v}\right|^2 + \eta|\mathbf{v}|^2\right) dx. \tag{1.6}$$

The noncavitation definition in the two dimensional problem has exactly the same view as in Definition 1 with the real line $\mathbb{R}$ substituted by the plane $\mathbb{R}^2$.

Theorem 2. *Let $s > 1$. Suppose that the initial data (1.5) has curl free velocity $\nabla \times \mathbf{v}_0 = 0$ and either has small enough $H^1 \times H^{1/2} \times H^{1/2}$-norm if $s \leqslant 2$ or satisfies the noncavitation condition if $s > 2$. Then there exist $T > 0$ depending only on the initial data and a unique solution $(\eta, \mathbf{v}) \in C\left([0, T]; H^{s+1/2}(\mathbb{R}) \times (H^s(\mathbb{R}))^2\right)$ of System (1.4) associated with this initial data. Moreover, the solution depends continuously on the initial data.*

Note that the theorem has the local character, in the opposite of the one dimensional case.

Remark 2. The same results hold in the periodic case as well. The proof is similar up to some small changes in the commutator estimates [15].

In the next section some important inequalities are recalled. In Section 3 we introduce the modified energy and obtain the corresponding energy estimate for System (1.1). In Section 4 we obtain the energy estimate for the difference of two solutions of System (1.1). In Section 5 the parabolic regularisation is studied and the corresponding energy estimate is deduced. In Section 6 a priori estimates are obtained. Finally, in Section 7 we comment on the last steps in the proof of Theorem 1, omitting only the thorough discussion of the initial data regularisation. In Section 8 we discuss some peculiarities of the two dimensional problem.

2. Preliminary estimates

We start this section by recalling all the necessary standard notations. For any positive numbers a and b we write $a \lesssim b$ if there exists a constant C independent on a, b such that $a \leqslant Cb$. The Fourier transform is defined by the formula

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}dx$$

on Schwartz functions. By the Fourier multiplier operator $\varphi(D)$ with symbol φ we mean the line $\mathcal{F}(\varphi(D)f) = \varphi(\xi)\widehat{f}(\xi)$. In particular, $D = -i\partial_x$ is the Fourier multiplier associated with the symbol $\varphi(\xi) = \xi$. For any $\alpha \in \mathbb{R}$ the Riesz potential of order $-\alpha$ is the Fourier operator $|D|^\alpha$ and the Bessel potential of order $-\alpha$ is the Fourier operator $J^\alpha = \langle D\rangle^\alpha$, where we exploit the notation

$\langle\xi\rangle = \sqrt{1+\xi^2}$. The L^2-based Sobolev space H^α is defined by the norm $\|f\|_{H^\alpha} = \|J^\alpha f\|_{L^2}$, whereas the homogeneous Sobolev space $\dot{H}^\alpha$ is defined by $\|f\|_{\dot{H}^\alpha} = \||D|^\alpha f\|_{L^2}$.

Introduce the operator

$$K_\varkappa = \sqrt{(1+\varkappa D^2)\frac{\tanh D}{D}} \tag{2.1}$$

where $\varkappa$ is the surface tension. Note that $\varkappa > 0$ is a fixed constant. We implement the notation $K = K_0 = \sqrt{\tanh D/D}$ used in [9]. Its inverse K^{-1} and $K_\varkappa$ both have the domain $H^{1/2}(\mathbb{R})$ and are equivalent to the Bessel potential $J^{1/2}$. Below we will need to compare J, $|D|$ and K^{-2} and so we prove the following simple estimates.

Lemma 1. *For any $f \in L^2(\mathbb{R})$ hold*

$$\left\|\left(J - K^{-2}\right) Df\right\|_{L^2} \leqslant \|(J-|D|)Df\|_{L^2} \leqslant \frac{1}{2}\|f\|_{L^2}.$$

Proof. By the Plancherel identity it is enough to check the following inequalities

$$0 \leqslant \langle\xi\rangle - \frac{\xi}{\tanh\xi} \leqslant \langle\xi\rangle - |\xi| \leqslant \frac{1}{2|\xi|}$$

where the middle one is trivial. The rightmost inequality follows from

$$\langle\xi\rangle - |\xi| = \frac{1}{\langle\xi\rangle + |\xi|} \leqslant \frac{1}{2|\xi|}.$$

The leftmost one follows from the tanh-definition via exponents and the obvious

$$e^{2\xi} + e^{-2\xi} \geqslant 2 + 4\xi^2.$$

□

Throughout the text we make an extensive use of the following bilinear estimates. Firstly, we state the Kato-Ponce commutator estimate [14].

Lemma 2 (Kato-Ponce commutator estimate)**.** *Let $s \geqslant 1$, $p,\ p_2,\ p_3 \in (1,\infty)$ and $p_1,\ p_4 \in (1,\infty]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. Then*

$$\|[J^s, f]g\|_{L^p} \lesssim \|\partial_x f\|_{L^{p_1}}\|J^{s-1}g\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}}\|g\|_{L^{p_4}} \tag{2.2}$$

for any f, g defined on $\mathbb{R}$.

By the commutator $[A,B]$ between operators A and B we mean the operator $[A,B]f = ABf - BAf$. Secondly, we state the fractional Leibniz rule proved in the appendix of [16].

Lemma 3. *Let $\sigma = \sigma_1 + \sigma_2 \in (0,1)$ with $\sigma_i \in (0,\sigma)$ and $p,\ p_1,\ p_2 \in (1,\infty)$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then*

$$\||D|^\sigma(fg) - f|D|^\sigma g - g|D|^\sigma f\|_{L^p} \lesssim \||D|^{\sigma_1} f\|_{L^{p_1}}\||D|^{\sigma_2} g\|_{L^{p_2}} \tag{2.3}$$

for any f, g defined on $\mathbb{R}$. Moreover, the case $\sigma_2 = 0$, $p_2 = \infty$ is also allowed.

We also state an estimate, firstly appeared in [17] in a weaker form, and later sharpened in [21].

Lemma 4. *Suppose $a, b, c \in \mathbb{R}$. Then for any $f \in H^a(\mathbb{R})$, $g \in H^b(\mathbb{R})$ and $h \in H^c(\mathbb{R})$ hold*

$$\|fgh\|_{L^1} \lesssim \|f\|_{H^a}\|g\|_{H^b}\|h\|_{H^c} \tag{2.4}$$

provided that

$$a + b + c > \frac{1}{2},$$
$$a + b \geqslant 0, \quad a + c \geqslant 0, \quad b + c \geqslant 0.$$

Proving a global-in-time a priori estimate we will use the following limiting case of the Sobolev embedding theorem.

Lemma 5 (Brezis-Gallouet inequality)**.** *Suppose $f \in H^s(\mathbb{R}^n)$ with $s > n/2$. Then*

$$\|f\|_{L^\infty} \leqslant C_{s,n}\left(1 + \|f\|_{H^{n/2}}\sqrt{\log(2 + \|f\|_{H^s})}\right). \tag{2.5}$$

Inequality (2.5) was firstly put forward and proved for a domain in $\mathbb{R}^n$ with $n = 2$ in the work by Brezis, Gallouet [2]. It was extended to the other Sobolev spaces in [3].

3. Modified energy

For $s \geqslant 1/2$ define the modified energy

$$E^s(\eta, v) = \frac{\varkappa}{2}\|\eta\|^2_{H^{s+1/2}} + \frac{1}{2}\|v\|^2_{H^s} + \frac{1}{2}\int \eta\left(J^{s-\frac{1}{2}}v\right)^2 \tag{3.1}$$

where the pair $\eta(x,t)$, $v(x,t)$ represents a possible solution of System (1.1). Note that in the limit case $s = 1/2$ this quantity almost coincides with the Hamiltonian

$$E^{1/2}(\eta, v) = \mathcal{H}(\eta, v) + \frac{\varkappa - 1}{2}\|\eta\|^2_{L^2} + \frac{1}{2}\int v\left(J - K^{-2}\right)v.$$

Lemma 6. *Suppose $s \geqslant 1/2$ and $\eta(t) \in H^{s+1/2}(\mathbb{R})$, $v(t) \in H^s(\mathbb{R})$ solve System (1.1). Then*

$$\frac{d}{dt}E^s(\eta, v) \lesssim \|\eta\|^2_{H^{s+1/2}} + \|v\|^2_{H^s} + \left(\|\eta\|^2_{H^{s+1/2}} + \|v\|^2_{H^s}\right)^2.$$

Proof. Firstly, we regard the limit case $s = 1/2$. Taking into account energy conservation derive

$$\frac{d}{dt}E^{1/2}(\eta, v) = (\varkappa - 1)\int \eta\eta_t + \int v\left(J - K^{-2}\right)v_t.$$

Substituting the right hand side of the first equation (1.1) to the first integral obtain

$$\int \eta\eta_t = -\int \eta v_x - i\int \eta \tanh D(\eta v) \leqslant \|\partial_x\eta\|_{L^2}\|v\|_{L^2} + \|\eta\|_{L^2}\|\eta\|_{L^\infty}\|v\|_{L^2}$$

following from Hölder's inequality and boundedness of operator $\tanh D$. Similarly, the second integral

$$\int v\left(J - K^{-2}\right)v_t = -i\int v\left(J - K^{-2}\right)\tanh D\eta + i\varkappa\int\left(D\tanh D\eta\right)\left(J - K^{-2}\right)Dv$$
$$- \frac{i}{2}\int v\left(J - K^{-2}\right)\tanh Dv^2 \leqslant \frac{1}{2}\|\eta\|_{L^2}\|v\|_{L^2} + \frac{\varkappa}{2}\|\partial_x\eta\|_{L^2}\|v\|_{L^2} + \frac{1}{4}\|v\|_{L^2}\|v\|^2_{L^4}$$

is estimated using Hölder's inequality and boundedness of operators $\tanh D$, $\left(J - K^{-2}\right)D$ in $L^2(\mathbb{R})$. Applying standard Sobolevs' embeddings one proves the statement for $s = 1/2$.

Assuming $s > 1/2$ calculate the derivative

$$\frac{\varkappa}{2}\frac{d}{dt}\|\eta\|^2_{H^{s+1/2}} = -\varkappa\int\left(J^{s+\frac{1}{2}}\eta\right)J^{s+\frac{1}{2}}\partial_x v - i\varkappa\int\left(J^{s+\frac{1}{2}}\eta\right)J^{s+\frac{1}{2}}\tanh D(\eta v)$$
$$= i\varkappa\int\left(J^{s+1}D\eta\right)J^s v + i\varkappa\int\left(J^{s-\frac{1}{2}}D\eta\right)J^{s+\frac{1}{2}}(\eta v) + I_1$$

where the rest

$$I_1 = i\varkappa\int\left(J^{s+1}\tanh D\eta - J^s D\eta\right)J^s(\eta v)$$
$$= i\varkappa\int J^s\left((J - |D|)\tanh D\eta + D(|\tanh D| - 1)\eta\right)J^s(\eta v) \lesssim \|\eta\|^2_{H^s}\|v\|_{H^s}.$$

The derivative of velocity norm

$$\frac{1}{2}\frac{d}{dt}\|v\|_{H^s}^2 = -i\int (J^s v) J^s \tanh D\eta - i\varkappa \int (J^s v) J^s D^2 \tanh D\eta - \frac{i}{2}\int (J^s v) J^s \tanh D v^2$$
$$= -i\varkappa \int (J^s v) J^{s+1} D\eta + I_2 + I_3$$

where the first and the third integrals are put in I_2. They can be estimated straightforwardly

$$I_2 \lesssim \|\eta\|_{H^s}\|v\|_{H^s} + \|v\|_{H^s}^3$$

and the rest

$$I_3 = i\varkappa \int (J^s v) J^s (J - D\tanh D) D\eta$$
$$= i\varkappa \int (J^s v) J^s (J - |D| - |D|(|\tanh D| - 1)) D\eta \lesssim \|\eta\|_{H^s}\|v\|_{H^s}.$$

Summing up these derivatives obtain

$$\frac{\varkappa}{2}\frac{d}{dt}\|\eta\|_{H^{s+1/2}}^2 + \frac{1}{2}\frac{d}{dt}\|v\|_{H^s}^2 = +i\varkappa \int \left(J^{s-\frac{1}{2}} D\eta\right) \eta J^{s+\frac{1}{2}} v + I_1 + I_2 + I_3 + I_4$$

where the last part

$$I_4 = i\varkappa \int \left(J^{s-\frac{1}{2}} D\eta\right) \left[J^{s+\frac{1}{2}}, \eta\right] v \lesssim \|\eta\|_{H^{s+1/2}} \left(\|\partial_x \eta\|_{L^{p_1}} \|J^{s-\frac{1}{2}} v\|_{L^{p_2}} + \|J^{s+\frac{1}{2}}\eta\|_{L^2}\|v\|_{L^\infty}\right)$$

is estimated applying Hölder's inequality and the Kato–Ponce commutator estimate. Taking $p_1(s) = \frac{1}{1-s}$, $p_2(s) = \frac{2}{2s-1}$ for $s \in (\frac{1}{2}, 1)$ and $p_1 = p_2 = 4$ in case $s \geqslant 1$ one deduces to

$$I_4 \lesssim \|\eta\|_{H^{s+1/2}}^2 \|v\|_{H^s}$$

after implementing the Sobolev embedding. Differentiation of the last summand of energy E^s gives

$$\frac{1}{2}\frac{d}{dt}\int \eta \left(J^{s-\frac{1}{2}} v\right)^2 = -i\int \eta \left(J^{s-\frac{1}{2}} v\right) J^{s-\frac{1}{2}} \tanh D\eta - i\varkappa \int \eta \left(J^{s-\frac{1}{2}} v\right) J^{s-\frac{1}{2}} D^2 \tanh D\eta$$
$$- \frac{i}{2}\int \eta \left(J^{s-\frac{1}{2}} v\right) J^{s-\frac{1}{2}} \tanh D v^2 - \frac{1}{2}\int \partial_x v \left(J^{s-\frac{1}{2}} v\right)^2 - \frac{i}{2}\int \tanh D(\eta v) \left(J^{s-\frac{1}{2}} v\right)^2 \quad (3.2)$$

where the first summand, that we notate by I_5, is estimated easily

$$I_5 \leqslant \|\eta\|_{L^\infty}\|v\|_{H^{s-1/2}}\|\eta\|_{H^{s-1/2}}$$

via Hölder inequality. The third integral in (3.2), notated by I_6, is estimated in a similar way

$$I_6 \leqslant \frac{1}{2}\|\eta\|_{L^\infty}\|v\|_{H^{s-1/2}}\|v^2\|_{H^{s-1/2}} \lesssim \|\eta\|_{H^{s+1/2}}\|v\|_{H^s}^3$$

where the last bound follows from the fact that H^s is an algebra. The fourth integral in (3.2) equals

$$I_7 = -\frac{i}{2}\int \left(\operatorname{sgn} D|D|^{\frac{1}{2}} v\right) |D|^{\frac{1}{2}} \left(J^{s-\frac{1}{2}} v\right)^2 = -i\int \left(\operatorname{sgn} D|D|^{\frac{1}{2}} v\right)\left(J^{s-\frac{1}{2}} v\right) J^{s-\frac{1}{2}}|D|^{\frac{1}{2}} v$$
$$- \frac{i}{2}\int \left(\operatorname{sgn} D|D|^{\frac{1}{2}} v\right)\left[|D|^{\frac{1}{2}}\left(J^{s-\frac{1}{2}} v\right)^2 - 2\left(J^{s-\frac{1}{2}} v\right) J^{s-\frac{1}{2}}|D|^{\frac{1}{2}} v\right]$$

where the first integral can be treated with interpolation in Sobolev spaces and the second integral by the fractional Leibniz rule as follows

$$I_7 \lesssim \|\operatorname{sgn} D|D|^{\frac{1}{2}} v\|_{H^{s-1/2}}\|J^{s-\frac{1}{2}} v\|_{H^{1/2}}\|J^{s-\frac{1}{2}}|D|^{\frac{1}{2}} v\|_{L^2} + \|\operatorname{sgn} D|D|^{\frac{1}{2}} v\|_{L^2}\|J^{s-\frac{1}{2}}|D|^{\frac{1}{2}} v\|_{L^2}^2 \lesssim \|v\|_{H^s}^3.$$

The last integral in (3.2), that we notate by I_8, is bounded by

$$I_8 \leqslant \frac{1}{2}\|\eta\|_{L^\infty}\|v\|_{L^2}\|J^{s-\frac{1}{2}} v\|_{L^4}^2 \lesssim \|\eta\|_{H^{s+1/2}}\|v\|_{H^s}^3.$$

It is left to calculate the second integral in (3.2). For this we approximate $D \tanh D$ by J in exactly the same way as was done for estimating integral I_3 so that

$$- i\varkappa \int \eta \left(J^{s-\frac{1}{2}} v\right) J^{s-\frac{1}{2}} D^2 \tanh D\eta = -i\varkappa \int \eta \left(J^{s-\frac{1}{2}} v\right) J^{s+\frac{1}{2}} D\eta + I_9$$

$$= -i\varkappa \int \eta \left(J^{s+\frac{1}{2}} v\right) J^{s-\frac{1}{2}} D\eta + I_9 + I_{10}$$

where the first rest

$$I_9 = i\varkappa \int \eta \left(J^{s-\frac{1}{2}} v\right) J^{s-\frac{1}{2}} D(J - D\tanh D)\eta \lesssim \|\eta\|_{L^\infty} \|v\|_{H^{s-1/2}} \|\eta\|_{H^{s-1/2}}$$

follows from Hölder's inequality together with boundedness of operator $D(J - D\tanh D)$ and the last part

$$I_{10} = -i\varkappa \int \left(J^{s-\frac{1}{2}} D\eta\right) [J, \eta]\, J^{s-\frac{1}{2}} v \lesssim \|\eta\|_{H^{s+1/2}} \left(\|\partial_x \eta\|_{L^{p_1}} \|J^{s-\frac{1}{2}} v\|_{L^{p_2}} + \|J\eta\|_{L^{p_3}} \|J^{s-\frac{1}{2}} v\|_{L^{p_4}}\right)$$

follows from the Hölder and Kato–Ponce inequalities. Again taking $p_1 = p_3 = \frac{1}{1-s}$, $p_2 = p_4 = \frac{2}{2s-1}$ for $s \in (\frac{1}{2}, 1)$ and $p_1 = p_2 = p_3 = p_4 = 4$ for $s \geqslant 1$ one deduces

$$I_{10} \lesssim \|\eta\|^2_{H^{s+1/2}} \|v\|_{H^s}$$

after implementing the Sobolev embedding. Finally, summing Derivative (3.2) with the derivative of square of $H^{s+1/2} \times H^s$-norm according to Definition (3.1) obtain

$$\frac{d}{dt} E^s(\eta, v) = I_1 + \ldots + I_{10} \lesssim \|\eta\|^2_{H^{s+1/2}} + \|v\|^2_{H^s} + \left(\|\eta\|^2_{H^{s+1/2}} + \|v\|^2_{H^s}\right)^2$$

which concludes the proof. □

In the following obvious statement the non-cavitation condition plays a crucial role.

Lemma 7 (Coercivity). *Let $s \geqslant 1/2$ and $(\eta, v) \in C\left([0,T]; H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})\right)$ be a solution of System* (1.1) *for some $T > 0$. If in addition η satisfies the non-cavitation condition then*

$$E^s(\eta, v) \sim \|\eta\|^2_{H^{s+1/2}} + \|v\|^2_{H^s}.$$

Corollary 1 (Energy estimate). *In the conditions of the previous lemma holds true*

$$\frac{d}{dt} E^s(\eta, v) \lesssim E^s(\eta, v) + E^s(\eta, v)^2.$$

As we shall see below, the non-cavitation condition is convenient to work with only in the case of high regularity $s > 3/2$. Then the time interval on which the condition holds true can be easily estimated through the first equation in (1.1). Our goal is to study well-posedness in spaces of low regularity. So in case of $s \leqslant 3/2$ we will have to impose a stronger condition, instead of non-cavitation, namely smallness of the initial data norm to control it in time with the help of Hamiltonian conservation.

Lemma 8. *There exists a constant $H > 0$ depending only on the surface tension $\varkappa > 0$ such that for any $\epsilon \in (0, H]$ if a pair $u(t) = (\eta(t), v(t)) \in H^1(\mathbb{R}) \times H^{1/2}(\mathbb{R})$, having initial condition $\|u_0\|_{H^1 \times H^{1/2}} \leqslant \epsilon/2$, solves System* (1.1) *then $\|u(t)\|_{H^1 \times H^{1/2}} \leqslant \epsilon$ for any time t.*

Proof. We use a continuity argument. Without loss of generality we prove the statement with a norm equivalent to $H^1 \times H^{1/2}$-norm instead. Regard the norm defined by

$$\|u\|^2 = \frac{\varkappa}{2} \|\partial_x \eta\|^2_{L^2} + \frac{1}{2} \|\eta\|^2_{L^2} + \frac{1}{2} \|K^{-1} v\|^2_{L^2}.$$

Then there exists $C > 0$ such that

$$\|u\|^2 (1 - C\|u\|) \leqslant \mathcal{H}(u) \leqslant \|u\|^2 (1 + C\|u\|)$$

where $u = u(t)$ is a solution of (1.1) defined on some interval. Take $H = (2C)^{-1}$, any $0 < \epsilon \leqslant H$ and a solution with $u_0 = u(0)$ having $\|u_0\| \leqslant \epsilon/2$. By continuity $\|u\| \leqslant \epsilon$ on some $[0, T_\epsilon]$ and so

$$\|u\| \leqslant \sqrt{2\mathcal{H}(u)} = \sqrt{2\mathcal{H}(u_0)} \leqslant \sqrt{\frac{1 + C\epsilon/2}{2}}\epsilon < \epsilon$$

which means that the continuous function $\|u(t)\|$ cannot touch the level ϵ with time. □

As a consequence of the lemma we can control $\|\eta\|_{L^\infty}$ for any $s \geqslant 1/2$ admitting only small initial data. Paying this price we can guarantee non-cavitation, in particular.

4. Uniqueness type estimate

Suppose that we have two solution pairs η_1, v_1 and η_2, v_2 of System (1.1) on some time interval. Define functions $\theta = \eta_1 - \eta_2$, $w = v_1 - v_2$. Then θ and w satisfy the following system

$$\theta_t = -w_x - i \tanh D(\theta v_2 + \eta_1 w), \tag{4.1}$$

$$w_t = -i \tanh D(1 + \varkappa D^2)\theta - i \tanh D((v_1 + v_2)w)/2. \tag{4.2}$$

We need an a priori estimate similar to one obtained in the previous section for the difference of solutions. For this purpose we introduce the difference energy

$$E^r(\eta_1, v_1, \eta_2, v_2) = \frac{\varkappa}{2}\|\theta\|^2_{H^{r+1/2}} + \frac{1}{2}\|w\|^2_{H^r} + \frac{1}{2}\int \eta_1 \left(J^{r-\frac{1}{2}}w\right)^2. \tag{4.3}$$

Note that $E^s(\eta, v) = E^s(\eta, v, 0, 0)$ and so this new notation is in line with (3.1).

Lemma 9. *Let $(\eta_1, v_1), (\eta_2, v_2) \in C\left([0,T]; H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})\right)$ be solutions of System (1.1) for some $T > 0$ and $s > 1/2$. Their difference is denoted by (θ, w). Let $0 < r \leqslant s - 1/2$. Then*

$$\frac{d}{dt}E^r(\eta_1, v_1, \eta_2, v_2) \lesssim \left(1 + \|\eta_1\|^2_{H^{s+1/2}} + \|v_1\|^2_{H^s} + \|v_2\|^2_{H^s}\right)\left(\|\theta\|^2_{H^{r+1/2}} + \|w\|^2_{H^r}\right).$$

Proof. We follow the same arguments as in the proof of Lemma 6. The derivative of squared norm

$$\begin{aligned}
\frac{\varkappa}{2}\frac{d}{dt}\|\theta\|^2_{H^{r+1/2}} + \frac{1}{2}\frac{d}{dt}\|w\|^2_{H^r} &= -\varkappa\int\left(J^{r+\frac{1}{2}}\theta\right)J^{r+\frac{1}{2}}\partial_x w - i\varkappa\int\left(J^{r+\frac{1}{2}}\theta\right)J^{r+\frac{1}{2}}\tanh D(\theta v_2) \\
&\quad - i\varkappa\int\left(J^{r+\frac{1}{2}}\theta\right)J^{r+\frac{1}{2}}\tanh D(\eta_1 w) - i\int(J^r w)J^r \tanh D\theta \\
&\quad - i\varkappa\int(J^r w)J^r D^2 \tanh D\theta - \frac{i}{2}\int(J^r w)J^r \tanh D(v_1 + v_2)w \\
&= I_1 + \mathcal{O}\left(\|\theta\|_{H^r}\|w\|_{H^r} + \|v_2\|_{H^{r+1/2}}\|\theta\|^2_{H^{r+1/2}} + \|\eta_1\|_{H^r}\|\theta\|_{H^r}\|w\|_{H^r} + \|v_1 + v_2\|_{H^s}\|w\|^2_{H^r}\right)
\end{aligned}$$

where

$$I_1 = i\varkappa\int\left(J^{r-\frac{1}{2}}D\theta\right)J^{r+\frac{1}{2}}(\eta_1 w).$$

In the case $r \geqslant 1/2$ we have the commutator estimate

$$\left\|\left[J^{r+\frac{1}{2}}, \eta_1\right]w\right\|_{L^2} \lesssim \|\partial_x\eta_1\|_{L^4}\left\|J^{r-\frac{1}{2}}w\right\|_{L^4} + \left\|J^{r+\frac{1}{2}}\eta_1\right\|_{L^4}\|w\|_{L^4} \lesssim \|\eta_1\|_{H^{s+1/2}}\|w\|_{H^r}$$

and so

$$I_1 = i\varkappa\int\left(J^{r-\frac{1}{2}}D\theta\right)\eta_1 J^{r+\frac{1}{2}}w + \mathcal{O}\left(\|\eta_1\|_{H^{s+1/2}}\|\theta\|_{H^{r+1/2}}\|w\|_{H^r}\right). \tag{4.4}$$

For $r \in (0, 1/2)$ we apply the Leibniz rule

$$\left\||D|^{r+\frac{1}{2}}(\eta_1 w) - w|D|^{r+\frac{1}{2}}\eta_1 - \eta_1|D|^{r+\frac{1}{2}}w\right\|_{L^2} \lesssim \||D|^{\sigma_1}\eta_1\|_{L^{p_1}}\||D|^{\sigma_2}w\|_{L^{p_2}} \lesssim \|\eta_1\|_{H^1}\|w\|_{H^r}$$

where $p_2 > 2$ is such that $\sigma_2 = r - 1/2 + 1/p_2 > 0$. The last estimate is due to Sobolev's embedding. Operator $J^{r+\frac{1}{2}} - |D|^{r+\frac{1}{2}}$ is bounded in L^2. Thus

$$I_1 = i\varkappa \int \left(J^{r-\frac{1}{2}} D\theta\right) |D|^{r+\frac{1}{2}}(\eta_1 w) + \mathcal{O}\left(\|\eta_1\|_{H^{s+1/2}} \|\theta\|_{H^{r+1/2}} \|w\|_{H^r}\right)$$
$$= i\varkappa \int \left(J^{r-\frac{1}{2}} D\theta\right) w|D|^{r+\frac{1}{2}}\eta_1 + i\varkappa \int \left(J^{r-\frac{1}{2}} D\theta\right) \eta_1 |D|^{r+\frac{1}{2}} w + \mathcal{O}\left(\|\eta_1\|_{H^{s+1/2}} \|\theta\|_{H^{r+1/2}} \|w\|_{H^r}\right)$$

where the first integral can be estimated by interpolation in Sobolev spaces. In the second integral the fractional derivative $|D|^{r+\frac{1}{2}}$ can be approximated by $J^{r+\frac{1}{2}}$ to come again to (4.4) now for $0 < r < 1/2$.

Differentiation of the energy modifier gives

$$\frac{1}{2}\frac{d}{dt} \int \eta_1 \left(J^{r-\frac{1}{2}} w\right)^2 = -i \int \eta_1 \left(J^{r-\frac{1}{2}} w\right) J^{r-\frac{1}{2}} \tanh D\theta - i\varkappa \int \eta_1 \left(J^{r-\frac{1}{2}} w\right) J^{r-\frac{1}{2}} D^2 \tanh D\theta$$
$$-\frac{i}{2} \int \eta_1 \left(J^{r-\frac{1}{2}} w\right) J^{r-\frac{1}{2}} \tanh D(v_1 + v_2) w - \frac{1}{2} \int \partial_x v_1 \left(J^{r-\frac{1}{2}} w\right)^2 - \frac{i}{2} \int \tanh D(\eta_1 v_1) \left(J^{r-\frac{1}{2}} w\right)^2$$
$$= I_2 + \mathcal{O}\left(\|\eta_1\|_{H^s} \|\theta\|_{H^{r-1/2}} \|w\|_{H^{r-1/2}} + (1 + \|\eta_1\|_{H^s}) (\|v_1\|_{H^s} + \|v_2\|_{H^s}) \|w\|_{H^r}^2\right)$$

where

$$I_2 = -i\varkappa \int \left(J^{r-\frac{1}{2}} D\theta\right) J(\eta_1 J^{r-\frac{1}{2}} w) = -i\varkappa \int \left(J^{r-\frac{1}{2}} D\theta\right) \eta_1 J^{r+\frac{1}{2}} w$$
$$+ \|\theta\|_{H^{r+1/2}} \mathcal{O}\left(\|\partial_x \eta_1\|_{L^{p_1}} \|J^{r-\frac{1}{2}} w\|_{L^{p_2}} + \|J\eta_1\|_{L^{p_3}} \|J^{r-\frac{1}{2}} w\|_{L^{p_4}}\right)$$
$$= -i\varkappa \int \left(J^{r-\frac{1}{2}} D\theta\right) \eta_1 J^{r+\frac{1}{2}} w + \mathcal{O}\left(\|\eta_1\|_{H^{s+1/2}} \|\theta\|_{H^{r+1/2}} \|w\|_{H^r}\right)$$

following from the Kato–Ponce inequality with $p_1 = p_3 = \frac{1}{1-s}$, $p_2 = p_4 = \frac{2}{2s-1}$ for $s \in \left(\frac{1}{2}, 1\right)$ and $p_1 = p_2 = p_3 = p_4 = 4$ for $s \geqslant 1$. Summing I_2 together with I_1 calculated in (4.4) we conclude the proof. □

Corollary 2 (Energy estimate for difference)**.** *If in addition to the conditions of the previous we assume non-cavitation for η_1 then*

$$\frac{d}{dt} E^r(\eta_1, v_1, \eta_2, v_2) \lesssim \left(1 + \|\eta_1\|_{H^{s+1/2}}^2 + \|v_1\|_{H^s}^2 + \|v_2\|_{H^s}^2\right) E^r(\eta_1, v_1, \eta_2, v_2).$$

Proof. Non-cavitation implies coercivity for E^r and the rest is obvious. □

Remark 3. The restriction $s > 1/2$ appeared in the lemma and its corollary is inconvenient. It comes from the loss of Hamiltonian structure of System (4.1)-(4.2). This results in the fact that we can obtain only a weak solution in case $s = 1/2$ and probably not unique.

5. Parabolic regularisation

For application of the energy method we need to do a parabolic regularisation of the view

$$\begin{cases} \eta_t + v_x + i \tanh D(\eta v) = -\mu |D|^p \eta, \\ v_t + i \tanh D(1 + \varkappa D^2)\eta + i \tanh D v^2/2 = -\mu |D|^p v \end{cases} \tag{5.1}$$

where $\mu \in (0, 1)$. We want to prove solution existence for (5.1) for any given μ, by the contraction mapping principal and so p should be big enough. However, we also do not want to spoil our energy estimates, and so p should be small enough. As we shall see below, this bounds us to $p \in (1/2, 1]$. Here the left number comes from the following lemma.

Lemma 10. *For any $s \geqslant 1/2$, $\mu \in (0,1)$ and $p > 1/2$ there exists a finite positive bound $C(T)$, tending to zero as $T \to 0$, such that*

$$\int_0^T \left\| e^{-\mu t|D|^p}(f(t)g(t)) \right\|_{H^r} dt \leqslant C(T) \|f\|_{C_T H^r} \|g\|_{C_T H^s}$$

for any functions f, g defined on $[0,T]$. Here either $r = s + 1/2$ or $r = s$.

Proof. In the case $r = s > 1/2$ the statement is obvious due to boundedness of $\exp(-\mu t|D|^p)$ and the algebraic property $\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s}$. Hence $C(T) = c_s T$ with some constant c_s depending only on s.

Otherwise we use

$$\left\| e^{-\mu t|D|^p}(fg) \right\|_{H^r} \leqslant \left\| \xi \mapsto e^{-\mu t|\xi|^p} \langle \xi \rangle^{1/2} \right\|_{L^\infty} \|fg\|_{H^{r-1/2}}$$

where in the case $r = s = 1/2$ by the Hölder inequality we have

$$\|fg\|_{H^{r-1/2}} = \|fg\|_{L^2} \leqslant \|f\|_{L^4} \|g\|_{L^4} \lesssim \|f\|_{H^{1/4}} \|g\|_{H^{1/4}} \lesssim \|f\|_{H^s} \|g\|_{H^s}$$

and in the case $r = s + 1/2$ we obviously have

$$\|fg\|_{H^{r-1/2}} \lesssim \|f\|_{H^r} \|g\|_{H^s}.$$

It is left to check that the L^∞-norm above is locally integrable. Indeed, we can estimate the function at $\xi \in [0,1]$ and at $\xi \geqslant 1$ separately

$$e^{-\mu t|\xi|^p} \langle \xi \rangle^{1/2} \leqslant \max\left\{ 2^{1/4}, \sup_{\xi \geqslant 1} 2^{1/4} \xi^{\frac{1}{2p}} e^{-\mu t|\xi|} \right\} \leqslant 2^{1/4} \max\left\{ 1, (2pe\mu t)^{-\frac{1}{2p}} \right\}$$

that is an integrable function with respect to time over any bounded interval for $p > 1/2$. The integral of this function over $[0,T]$ defines the bound $C(T)$. □

With Lemma 10 in hand we can prove the local well-posedness in $H^{s+1/2} \times H^s$ with $s \geqslant 1/2$ for System (5.1) by the fixed-point argument. Diagonalization has the matrix form

$$\mathcal{S}(t) = \exp(-\mu t|D|^p) \mathcal{K} \begin{pmatrix} \exp(-itK_\varkappa D) & 0 \\ 0 & \exp(itK_\varkappa D) \end{pmatrix} \mathcal{K}^{-1}$$

where

$$\mathcal{K} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ K_\varkappa & -K_\varkappa \end{pmatrix}, \quad \mathcal{K}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & K_\varkappa^{-1} \\ 1 & -K_\varkappa^{-1} \end{pmatrix}$$

with $K_\varkappa$ defined by (2.1). For any fixed $u_0 = (\eta_0, v_0)^T \in X^s = H^{s+1/2} \times H^s$ function $\mathcal{S}(t)u_0$ solves the linear initial-value problem associated with (5.1). Let $X_T^s = C([0,T]; X^s)$ and regard a mapping $\mathcal{A} : X_T^s \to X_T^s$ defined by

$$\mathcal{A}(\eta, v; u_0)(t) = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t - t')(-i \tanh D) \begin{pmatrix} \eta v \\ v^2/2 \end{pmatrix} (t') dt'. \tag{5.2}$$

Then the Cauchy problem for System (5.1) with the initial data u_0 may be rewritten equivalently as an equation in X_T^s of the form

$$u = \mathcal{A}(u; u_0) \tag{5.3}$$

where $u = (\eta, v)^T \in X_T^s$.

Lemma 11. *Let $s \geqslant 1/2$, $p > 1/2$, $\mu \in (0,1)$ and $u_0 = (\eta_0, v_0)^T \in X^s$. Then there is $T = T(s, p, \mu, \|u_0\|_{X^s}) > 0$, decreasing to zero with increase of the norm of u_0, such that there exists a unique solution $u = (\eta, v)^T \in X_T^s$ of Problem (5.3).*

Moreover, for any $R > 0$ there exists $T = T(s, p, \mu, R) > 0$ such that the flow map associated with Equation (5.3) is a real analytic mapping of the open ball $B_R(0) \subset X^s$ to X_T^s.

Proof. We need to show that the restriction of $\mathcal{A}$ on some closed ball B_M with the center at point $\mathcal{S}(t)u_0$ is a contraction mapping. Note that $\|\mathcal{S}(t)u\|_{X^s} \lesssim \|\exp(-\mu t|D|^p)u\|_{X^s}$. Hence by Lemma 10 for any $T, M > 0$ and $u, u_1, u_2 \in B_M \subset X_T^s$ hold

$$\|\mathcal{A}(u) - \mathcal{S}(t)u_0\|_{X_T^s} \leqslant C(T)\|u\|_{X_T^s}^2 \leqslant C(T)(M + \|u_0\|_{X^s})^2,$$

$$\|\mathcal{A}(u_1) - \mathcal{A}(u_2)\|_{X_T^s} \leqslant C(T)\|u_1 - u_2\|_{X_T^s}(\|u_1\|_{X_T^s} + \|u_2\|_{X_T^s}) \leqslant 2C(T)(M + \|u_0\|_{X^s})\|u_1 - u_2\|_{X_T^s},$$

and so taking $M = \|u_0\|_{X^s}$ one can find such T that $\mathcal{A}$ will be a contraction in the closed ball B_M. The first statement of the lemma follows from the contraction mapping principle. Smoothness of the flow map can be proved in the same spirit applying the implicit function theorem instead, and so we omit it. Some details can be found in [9]. □

Lemma 12. *Suppose $s \geqslant 1/2$ and $\eta(t) \in H^{s+1/2}(\mathbb{R})$, $v(t) \in H^s(\mathbb{R})$ solve System (5.1) with $\mu \in (0,1)$ and $p \in (1/2, 1]$. Then*

$$\frac{d}{dt}E^s(\eta, v) \lesssim \|\eta\|_{H^{s+1/2}}^2 + \|v\|_{H^s}^2 + \left(\|\eta\|_{H^{s+1/2}}^2 + \|v\|_{H^s}^2\right)^2.$$

In other words, the parabolic regularisation (5.1) does not spoil the energy estimate.

Proof. Following the proof of Lemma 6 one arrives at

$$\frac{d}{dt}E^s(\eta, v) \lesssim \widetilde{I}_1 + \widetilde{I}_2 + \dots$$

where

$$\widetilde{I}_1 = -\mu\varkappa \left\| |D|^{p/2}\eta \right\|_{H^{s+1/2}}^2 - \mu \left\| |D|^{p/2}v \right\|_{H^s}^2 \leqslant 0,$$

$$\widetilde{I}_2 = -\frac{\mu}{2}\int \left(J^{s-1/2}v\right)^2 |D|^p\eta - \mu\int \eta \left(J^{s-1/2}v\right) J^{s-1/2}|D|^p v \lesssim \|\eta\|_{H^{s+1/2}}\|v\|_{H^s}^2$$

for $p \leqslant 1$ and the rest is the same as in Lemma 6. □

As was noticed at the end of Section 3, one has to make sure that the modified energy is coercive. An effective way to do it at the low level of regularity is to control $\|\eta\|_{L^\infty}$ via the energy conservation. One can get the same controllability for the regularised problem via the energy dissipation due to the following result.

Lemma 13. *Suppose $\eta(t) \in H^1(\mathbb{R})$, $v(t) \in H^{1/2}(\mathbb{R})$ solve System (5.1) with $\mu \in (0,1)$ and $p \in (1/2, 1]$. Then there exists $\delta > 0$ independent on the viscosity μ such that $\mathcal{H}(\eta, v)$ is a non-increasing function of time t provided $\|\eta(t)\|_{H^1} + \|v(t)\|_{H^{1/2}} \leqslant \delta$ holds for any t.*

Proof. Hamiltonian (1.3) has the derivative

$$\frac{1}{\mu}\frac{d}{dt}\mathcal{H}(\eta, v) = -\|\eta\|_{\dot{H}^{p/2}}^2 - \varkappa\|\eta\|_{\dot{H}^{p/2+1}}^2 - \left\|K^{-1}v\right\|_{\dot{H}^{p/2}}^2 - I_1 - I_2,$$

where the rest integrals

$$I_1 = \int \eta v|D|^p v, \quad I_2 = \frac{1}{2}\int v^2|D|^p\eta$$

are of no definite sign. One has to check that I_1, I_2 are absorbed by the first three norms. The main difficulty arising here is that two different homogeneous Sobolev spaces cannot be compared for inclusion, however, there is interpolation between them.

Using the Hölder inequality, the fractional Leibniz rule for $|D|^{p/2}$, the Sobolev embeddings $\dot{H}^{1/4} \hookrightarrow L^4$, $H^1 \hookrightarrow L^\infty$ and $H^{1/2} \hookrightarrow H^{1/4} \hookrightarrow L^2$ obtain

$$\begin{aligned} |I_1| = \left| \int |D|^{p/2}(\eta v)|D|^{p/2}v \right| &\leqslant \left\| |D|^{p/2}(\eta v) - v|D|^{p/2}\eta - \eta|D|^{p/2}v \right\|_{L^2} \left\| |D|^{p/2}v \right\|_{L^2} \\ &+ \left\| v|D|^{p/2}\eta \right\|_{L^2} \left\| |D|^{p/2}v \right\|_{L^2} + \left\| \eta|D|^{p/2}v \right\|_{L^2} \left\| |D|^{p/2}v \right\|_{L^2} \\ &\lesssim \|\eta\|_{L^\infty} \left\| |D|^{p/2}v \right\|_{L^2}^2 + \left\| |D|^{p/2}\eta \right\|_{L^4} \|v\|_{L^4} \left\| |D|^{p/2}v \right\|_{L^2} \\ &\lesssim \|\eta\|_{H^1} \left\| K^{-1}v \right\|_{\dot{H}^{p/2}}^2 + \|v\|_{H^{1/2}} \|\eta\|_{\dot{H}^{p/2+1/4}} \left\| K^{-1}v \right\|_{\dot{H}^{p/2}} \end{aligned}$$

where $\|\eta\|_{\dot{H}^{p/2+1/4}}$ can be interpolated between $\dot{H}^{p/2}$-norm and $\dot{H}^{p/2+1}$-norm. Hence I_1 can be absorbed provided $H^1 \times H^{1/2}$-norm of the solution is small.

The second integral I_2 can be treated similarly for $p = 1$ exploiting $|D| = D \operatorname{sgn} D$. Indeed,

$$I_2 = -\int |D|^{1/2}(v \operatorname{sgn} D\eta)|D|^{1/2} \operatorname{sgn} Dv$$

and so it can be estimated by the same chain of inequalities since $\operatorname{sgn} D$ preserves Sobolev norms. For $p \in (1/2, 1)$ we have

$$2|I_2| \leqslant \| |D|^p\eta \|_{L^{\frac{4}{1+p}}} \|v\|_{L^{\frac{4}{1+p}}} \|v\|_{L^{\frac{2}{1-p}}} \lesssim \|\eta\|_{\dot{H}^{\frac{1+3p}{4}}} \|v\|_{\dot{H}^{\frac{1-p}{4}}} \|v\|_{\dot{H}^{p/2}}$$

by the Hölder inequality and the Sobolev embeddings $\dot{H}^{\frac{1-p}{4}} \hookrightarrow L^{\frac{4}{1+p}}$, $\dot{H}^{p/2} \hookrightarrow L^{\frac{2}{1-p}}$. Again we interpolate the norm of η between $\dot{H}^{p/2}$-norm and $\dot{H}^{p/2+1}$-norm. Estimate $\|v\|_{\dot{H}^{\frac{1-p}{4}}} \leqslant \|v\|_{H^{1/2}}$ and $\|v\|_{\dot{H}^{p/2}} \leqslant \left\| K^{-1}v \right\|_{\dot{H}^{p/2}}$. Eventually we obtain

$$|I_1| + |I_2| \lesssim \left(\|\eta\|_{\dot{H}^{p/2}}^2 + \varkappa \|\eta\|_{\dot{H}^{p/2+1}}^2 + \left\| K^{-1}v \right\|_{\dot{H}^{p/2}}^2 \right) \max\left\{ \|\eta\|_{H^1}, \|v\|_{H^{1/2}} \right\}$$

that concludes the proof.

□

As a simple corollary with the proof similar to that of Lemma 8 one obtains the following.

Corollary 3. *There exists a constant $\delta > 0$ depending only on the surface tension $\varkappa > 0$ and the parabolic regularisation power p such that if a pair $u(t) = (\eta(t), v(t)) \in H^1(\mathbb{R}) \times H^{1/2}(\mathbb{R})$, having initial condition $\|u_0\|_{H^1 \times H^{1/2}} \leqslant \delta/2$, solves System* (5.1) *then $\|u(t)\|_{H^1 \times H^{1/2}} \leqslant \delta$ for any time t.*

The dependence of δ on the parabolic regularisation power p is unimportant since below we stick only to the case $p = 1$.

6. A priori estimate

We have an a priori global bound for solutions of both systems (1.1) and (5.1) in $H^1 \times H^{1/2}$ due to Lemma 8 and Corollary 3, respectively. Our aim is it to obtain estimates in $H^{s+1/2} \times H^s$ with $s > 1/2$.

Lemma 14 (A priori estimate)**.** *Suppose $s > 1/2$. Let $(\eta, v) \in C\left([0, T^\star); H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})\right)$ be a solution of System* (1.1) *(or of the regularised system* (5.1) *with $\mu \in (0, 1)$ and $p = 1$) defined on its maximal time of existence and satisfying the blow-up alternative*

$$T^\star < +\infty \text{ implies } \lim_{t \to T^\star} \|\eta(t), v(t)\|_{H^{s+1/2} \times H^s} = +\infty. \tag{6.1}$$

Suppose that its initial data (1.2) *either satisfies the non-cavitation condition for $s > 3/2$ or has small enough $H^1 \times H^{1/2}$-norm for $s \leqslant 3/2$. Then there exists $T_0 = T_0\left(\|\eta_0, v_0\|_{H^{s+1/2} \times H^s}\right) < T^\star$ such that*

$$\sup_{t \in [0, T_0]} \|\eta(t), v(t)\|_{H^{s+1/2} \times H^s} \leqslant C \|\eta_0, v_0\|_{H^{s+1/2} \times H^s} \tag{6.2}$$

for some $C > 0$ independent on μ.

Proof. We closely follow the arguments in [12] since we have essentially the same energy estimates. The main difference lies in the control of coercivity of the modified energy (3.1) for small s. Let h_0, H_0 define non-cavitation of η_0 according to Definition 1. Regard $h = h_0/2$ and $H = H_0 + h_0/2$. If the wave η satisfies the non-cavitation condition associated with h, H then there exist positive constants $c_0(h)$, $C_0(H)$ such that

$$c_0 \|\eta, v\|^2_{H^{s+1/2}\times H^s} \leqslant E^s(\eta, v) \leqslant C_0 \|\eta, v\|^2_{H^{s+1/2}\times H^s}$$

by coercivity of the energy. These constants depend only on h_0, H_0. They are used to define the time set

$$\mathcal{T} = \left\{ T \in (0, T^\star) \ : \ \sup_{t\in[0,T]} \|\eta(t), v(t)\|_{H^{s+1/2}\times H^s} \leqslant 3\sqrt{C_0/c_0}\, \|\eta_0, v_0\|_{H^{s+1/2}\times H^s} \right\}$$

that is non-empty and closed in $(0, T^\star)$ by the solution continuity. Moreover, for $\widetilde{T} = \sup \mathcal{T}$ we have either $\widetilde{T} < T^\star$ and so $\widetilde{T} \in \mathcal{T}$ or $\widetilde{T} = T^\star = +\infty$ by the blow-up alternative (6.1). Introduce $T_0 = \min\{T_1, T_2\}$ with

$$T_1 = \frac{1}{C_1} \log\left(1 + \frac{1}{1 + C_1 C_0 \|\eta_0, v_0\|^2_{H^{s+1/2}\times H^s}}\right),$$

$$T_2 = \begin{cases} \dfrac{h_0}{C_2\left(\|\eta_0, v_0\|_{H^{s+1/2}\times H^s} + \|\eta_0, v_0\|^2_{H^{s+1/2}\times H^s}\right)} & \text{for } s > 3/2 \\ 1 & \text{otherwise} \end{cases}$$

where C_1, C_2 are two big positive constants to be fixed below in the proof. The idea is to show that these constants can be chosen, independently on the initial data, in such a way that $T_0 \in \mathcal{T}$ or equivalently $T_0 \leqslant \widetilde{T}$.

Assume the opposite $\widetilde{T} < T_0$. Firstly, we will check that the non-cavitation condition holds on $[0, \widetilde{T}]$. Indeed, in the low regularity case $s \in (1/2, 3/2]$ it is assumed smallness of the initial data and so $H^1 \times H^{1/2}$-norm of the solution stays small with time evolution by Lemma 8 and Corollary 3. In particular, the wave satisfies the non-cavitation condition. For $s > 3/2$ one can estimate η using the first equation in System (1.1) (or in System (5.1)) as follows

$$\eta(x, t) = \eta_0(x) + \int_0^t \partial_t \eta(x, t')dt'$$

where

$$\|\partial_t \eta\|_{L^\infty} \leqslant \|\partial_x v\|_{L^\infty} + \|\tanh D(\eta v)\|_{L^\infty} + \mu \||D|\eta\|_{L^\infty} \lesssim \|\eta\|_{H^s} + \|v\|_{H^s} + \|\eta\|_{H^s}\|v\|_{H^s}$$

with the implicit constant independent on $\mu \in (0, 1)$, obviously. Hence

$$\|\partial_t \eta\|_{L^\infty} \lesssim \|\eta_0, v_0\|_{H^{s+1/2}\times H^s} + \|\eta_0, v_0\|^2_{H^{s+1/2}\times H^s}$$

uniformly on $(0, \widetilde{T}] \subset \mathcal{T}$. Thus we have

$$\left\| \int_0^t \partial_t \eta(x, t')dt' \right\|_{L^\infty} \leqslant \widetilde{T} \sup_{t\in(0,\widetilde{T}]} \|\partial_t \eta(t)\|_{L^\infty} \leqslant \frac{h_0}{2}$$

for big enough C_2 since $\widetilde{T} < T_2$. As a result the non-cavitation

$$h - 1 = h_0/2 - 1 \leqslant \eta \leqslant H_0 + h_0/2 = H$$

holds on $\mathbb{R} \times [0, \widetilde{T}]$. Without loss of generality one can assume that for $s \leqslant 3/2$ the non-cavitation of η is governed by the same h, H.

Let $E(t) = E^s(\eta, v)(t)$ be the energy defined by (3.1) and $E_0 = E(0)$. For System (1.1) (or for System (5.1)) we have the a priori energy estimate given in its differential form by Corollary 1. A straightforward integration gives

$$E(t)\left(1 - \frac{E_0}{1+E_0}e^{ct}\right) \leqslant \frac{E_0}{1+E_0}e^{ct}$$

for any $t \in [0, \widetilde{T}]$ with c depending only on h. Note that

$$e^{ct} \leqslant 1 + \frac{1}{1 + C_1 E_0}$$

for any $C_1 \geqslant c$ and $0 \leqslant t \leqslant \widetilde{T} < T_1$. In particular,

$$\frac{E_0}{1+E_0}e^{ct} \leqslant \frac{1/C_1 + E_0}{1+E_0} < 1$$

if in addition $C_1 > 1$. Thus

$$E(t) \leqslant \frac{1}{\left(\frac{E_0}{1+E_0}e^{ct}\right)^{-1} - 1} \leqslant E_0 \frac{2 + C_1 E_0}{1 + (C_1 - 1)E_0} \leqslant 2E_0$$

if in addition $C_1 \geqslant 2$. As a result we have

$$\|\eta(t), v(t)\|_{H^{s+1/2}\times H^s} \leqslant \sqrt{2C_0/c_0}\, \|\eta_0, v_0\|_{H^{s+1/2}\times H^s}$$

for all $t \in [0, \widetilde{T}]$. Taking into account $\widetilde{T} < T^\star$ and continuity of the solution one can find $\widetilde{T} < T' < T^\star, T_0$ such that on $[0, T']$ holds

$$\|\eta(t), v(t)\|_{H^{s+1/2}\times H^s} \leqslant 2\sqrt{C_0/c_0}\, \|\eta_0, v_0\|_{H^{s+1/2}\times H^s}$$

which contradicts the definition of $\widetilde{T}$. Therefore, we showed that $T_0 \leqslant \widetilde{T}$ concluding the proof. □

Lemma 15. *Suppose $s > 1/2$ and a pair $\eta(t) \in H^{s+1/2}$, $v(t) \in H^s$ solves System* (1.1) *(or the regularised system* (5.1) *with $\mu \in (0,1)$ and $p = 1$). Then if $s < 1$ the following holds true*

$$\frac{d}{dt}E^s(\eta, v) \lesssim \left(1 + \|v\|_{L^\infty} + \|\eta, v\|^2_{H^1\times H^{1/2}}\right) \|\eta, v\|^2_{H^{s+1/2}\times H^s},$$

and if $s \geqslant 1$ then

$$\frac{d}{dt}E^s(\eta, v) \lesssim \left(1 + \|\eta, v\|^2_{H^{s+1/4}\times H^{s-1/4}}\right) \|\eta, v\|^2_{H^{s+1/2}\times H^s}.$$

Moreover, the implicit constants do not depend on μ.

Proof. The estimates obtained while proving Lemmas 6, 12 need to be refined for $s > 1/2$ as follows. We stick to the notations used in the corresponding proofs. Note that by Lemma 4 we have

$$\widetilde{I}_2 \lesssim \left\|J^{s-1/2}v\right\|_{L^2} \left\|J^{s-1/2}v\right\|_{H^{1/2}} \||D|\eta\|_{H^{s-1/2}} + \|\eta\|_{H^s} \left\|J^{s-1/2}v\right\|_{H^{1/2}} \left\|J^{s-1/2}|D|v\right\|_{H^{-1/2}}$$
$$\lesssim \|\eta, v\|_{H^s\times H^{s-1/2}} \|\eta, v\|^2_{H^{s+1/2}\times H^s}.$$

Since $H^s \cap L^\infty$ is an algebra under the point-wise product one obtains

$$I_2 \lesssim \|\eta\|_{H^s}\|v\|_{H^s} + \|v\|_{L^\infty}\|v\|^2_{H^s} \lesssim (1 + \|v\|_{L^\infty}) \|\eta, v\|^2_{H^{s+1/2}\times H^s}.$$

The integral I_4 is essentially estimated already as

$$I_4 \lesssim \|\eta\|^2_{H^{s+1/2}} \begin{cases} \|v\|_{H^{1/2}} + \|v\|_{L^\infty} & \text{for } s \in (1/2, 1) \\ \|v\|_{H^{s-1/4}} & \text{for } s \geqslant 1 \end{cases}.$$

In order to refine I_7 we need to estimate

$$\left\|\left(\operatorname{sgn} D|D|^{1/2}v\right) J^{s-1/2}v\right\|_{L^2} \lesssim \left\|\left(\operatorname{sgn} D|D|^{1/2}v\right)\right\|_{L^{p_1}} \left\|J^{s-1/2}v\right\|_{L^{p_2}}$$

following from Hölder's inequality with $p_1(s) = \frac{1}{1-s}$, $p_2(s) = \frac{2}{2s-1}$ for $s \in (\frac{1}{2}, 1)$ and $p_1 = p_2 = 4$ in case $s \geqslant 1$. Implementing the Sobolev embedding and gathering the rest of I_7 obtain

$$I_7 \lesssim \|v\|_{H^s}^2 \begin{cases} \|v\|_{H^{1/2}} & \text{for } s \in (1/2, 1) \\ \|v\|_{H^{s-1/4}} & \text{for } s \geqslant 1 \end{cases}.$$

The integral I_{10} is also estimated already as

$$I_{10} \lesssim \|\eta\|_{H^{s+1/2}}^2 \begin{cases} \|v\|_{H^{1/2}} & \text{for } s \in (1/2, 1) \\ \|v\|_{H^{s-1/4}} & \text{for } s \geqslant 1 \end{cases}$$

Thus gathering all the parts obtain

$$\widetilde{I}_1 + \widetilde{I}_2 + I_1 + \ldots + I_{10} \lesssim \|\eta, v\|_{H^{s+1/2}\times H^s}^2 \begin{cases} 1 + \|v\|_{L^\infty} + \|\eta, v\|_{H^1\times H^{1/2}}^2 & \text{for } s \in (1/2, 1) \\ 1 + \|\eta, v\|_{H^{s+1/4}\times H^{s-1/4}}^2 & \text{for } s \geqslant 1 \end{cases}$$

which are the desired estimates. □

Knowing coercivity of the energy E^s, controlled either by the smallness or by the non-cavitation of the initial data, one can deduce from the lemma that the time of existence depends only on $\|\eta_0, v_0\|_{H^{s'+1/2}\times H^{s'}}$ where $1/2 < s' < s$. Taking into account the boundedness of $\|\eta, v\|_{H^1\times H^{1/2}}$, holding true at least for small initial data, one can get a stronger result thanks to the Brezis-Gallouet limiting embedding (2.5). In order to exploit it we need the following Gronwall inequality.

Lemma 16 (Gronwall inequality)**.** *Let y be an absolutely continuous positive function defined on some interval $[0, T]$. Suppose that almost everywhere*

$$y' \leqslant Ay \log y$$

where $A > 0$ is constant. Then there exists $C > 0$ independent on T such that

$$y(t) \leqslant \exp\left(Ce^{At}\right).$$

Proof. Denote the right hand side by $z(t) = \exp\left(Ce^{At}\right)$, where we take $C > 0$ such that $z(0) > y(0)$. Regard the derivative

$$\left(\frac{y}{z}\right)' = \frac{y'z - yz'}{z^2} \leqslant A\frac{y}{z}\log\frac{y}{z}$$

where the latter is less than zero at least for $t = 0$. So the fraction y/z decreases and stays always below the unity.

□

Corollary 4 (Persistence of regularity)**.** *In the conditions of the a priori estimate lemma 14 the following holds true*

$$\|\eta(t), v(t)\|_{H^{s+1/2}\times H^s} \leqslant \exp\left(Ce^{Ct}\right)$$

provided $s < 1$, and if $s \geqslant 1$ then

$$\|\eta(t), v(t)\|_{H^{s+1/2}\times H^s} \leqslant \|\eta_0, v_0\|_{H^{s+1/2}\times H^s} \exp\left(Ct + C\int_0^t \|\eta, v\|_{H^{s+1/4}\times H^{s-1/4}}^2\right)$$

where the constant $C > 0$ does not depend on μ. In particular, the maximal time of existence $T^\star = +\infty$ provided $\|\eta_0, v_0\|_{H^1\times H^{1/2}}$ is small enough.

Proof. The statement is obvious for $s \geqslant 1$. Suppose $s \in (1/2, 1)$. By Lemma 8 and Corollary 3 the norm $\|\eta(t), v(t)\|_{H^1\times H^{1/2}}$ stays bounded with time. Hence from the Brezis-Gallouet inequality (2.5) one deduces

$$\|v(t)\|_{L^\infty} \lesssim 1 + \log\left(2 + \|v(t)\|_{H^s}\right).$$

Thus applying Lemma 15 and taking into account that E^s is coercive obtain

$$\frac{d}{dt}E^s \lesssim \left(1 + \log\left(2 + E^s\right)\right) E^s.$$

As a result, after the application of the previous lemma with $y = 2 + E^s$, we have the estimate

$$E^s \leqslant \exp\left(Ce^{Ct}\right),$$

which again due to coercivity of E^s leads to the first inequality of the corollary after renaming the constant.

□

7. Proof of Theorem 1

With the a priori estimate (6.2) in hand we can reapply the local existence lemma 11 for the regularised problem (5.1) with $\mu \in (0,1)$ and $p = 1$ in order to obtain solution $u^\mu = (\eta^\mu, v^\mu)$ on the time interval $[0, T_0]$ defined by Lemma 14. Convergence of u^μ as $\mu \to 0$ follows from an adaptation of Lemma 9 to the difference energy (4.3) with $\eta_j = \eta^{\mu_j}$, $v_j = v^{\mu_j}$ $(j = 1, 2)$ and $0 < \mu_2 < \mu_1 < 1$. The proof repeats the arguments of Lemma 9 and Lemma 13. Moreover, using the Gagliardo–Nirenberg interpolation one can obtain that u^μ converges to some u in $C([0, T_0]; H^{r+1/2} \times H^r)$ as $\mu \to 0$ for any $0 < r < s$. This u is a solution of (1.1) in the distributional sense. Furthermore, to prove persistence $u \in C([0, T_0]; H^{s+1/2} \times H^s)$, justify all the previous steps and obtain continuity of the flow map one has to regularise the initial data (1.2) as $u_0^\epsilon = (\eta_0 * \rho_\epsilon, v_0 * \rho_\epsilon)$, where ρ_ϵ is an approximation of the identity parametrised by $0 < \epsilon < 1$ [1, 13]. An application of the Bona–Smith argument in a straightforward standard way [1, 15, 20] results in the persistence and continuous dependence. We omit further details.

8. The two-dimensional problem

In this section we comment briefly on adaptation of the proof for the two dimensional case. Firstly, we define the modified energy

$$E^s(\eta, v) = \frac{\varkappa}{2}\|\eta\|^2_{H^{s+1/2}} + \frac{1}{2}\|\mathbf{v}\|^2_{H^s \times H^s} + \frac{1}{2}\int \eta \left|J^{s-1/2}\mathbf{v}\right|^2 \tag{8.1}$$

and notice that it is coercive provided the wave η satisfies the noncavitation condition or has small H^1-norm. Note that the latter does not imply the first one, since now we do not have embedding of H^1 to L^∞. The smallness of $H^1 \times H^{1/2} \times H^{1/2}$-norm can be controlled by the energy conservation. Indeed, by Hölder's inequality and the Sobolev embedding the cubic part of Hamiltonian (1.6) is estimated as

$$\int_{\mathbb{R}^2} \eta|\mathbf{v}|^2 dx \lesssim \|\eta\|_{L^2} \|\mathbf{v}\|^2_{H^{1/2}\times H^{1/2}}$$

and so repeating the arguments given in the proof of Lemma 8 we arrive at the conclusion that the small enough initial data stays small through the flow. For $s > 2$ the noncavitation preserves locally-in-time due to the first equation in (1.4). The energy estimates and the rest of the proof of Theorem 2 can be done in exactly the same manner as in the one dimensional case, and so we omit further details.

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