Higher order space-time elements for a non-linear Biot model

Manuel Borregales[1] and Florin Adrian Radu[1]

[1] Department of Mathematics, University of Bergen, Norway
Email: Manuel.Borregales@uib.no, Florin.Radu@uib.no

Abstract. In this work, we consider a non-linear extension of the linear, quasi-static Biot's model. Precisely, we assume that the volumetric strain and the fluid compressibility are non-linear functions. We propose a fully discrete numerical scheme for this model based on higher order space-time elements. We use mixed finite elements for the flow equation, (continuous) Galerkin finite elements for the mechanics and discontinuous Galerkin for the time discretization. We further use the L-scheme for linearising the system appearing on each time step. The stability of this approach is illustrated by a numerical experiment.

1 Introduction

Flow in deformable porous media appears to be relevant for several important applications including groundwater hydrology, CO_2 sequestration, geothermal energy, and subsidence phenomena. A commonly used mathematical model for flow in deformable porous media is the linear, quasi-static Biot model [7]. In this work, a generic, non-linear extension of the linear Biot model is studied. The volumetric strain in the mechanics deformation model and the fluid compressibility are assumed to be now non-linear. For a discussion concerning the considered model we refer to [8], where a discretization based on lowest order mixed finite elements, Galerkin finite elements (FE) and backward Euler was proposed and analysed.

The non-linear Biot model consists on two fully coupled, non-linear partial differential equations. As a linearization we will use the robust, linearly convergent L-scheme [15,18,19]. The well-known fixed stress and fixed strain splitting schemes [1,6,9,12,13,17] can be interpreted as particular cases of the L-scheme for coupled problems [10]. For the discretization in space and time we propose a higher order space-time method [4,5]. We use the mixed finite element method (MFEM) for the flow equation, continuous Galerkin (cG) FE for the mechanics equation and discontinuous Galerkin (dG) FE for the discretization in time. We refer to [4] for a similar approach for the linear Biot model. In the future we plan to extend the methodology for more complex non-linear models and to the fully-dynamic Biot-Allard system [16].

The paper is structured as follows. In the next section we briefly introduce the considered non-linear Biot model. In Sec. 3 we present the space-time discretization and announce a convergence result. Numerical simulations are shown in Sec. 4. Finally, concluding remarks are given in Sec. 5.

2 A non-linear Biot's model

We consider flow of a slightly compressible fluid in a non-linear elastic, homogeneous, isotropic, porous medium $\Omega \subset \mathbb{R}^d$, with $d = 2, 3$ being the spatial dimension. The medium is assumed to be saturated. We extend the linear Biot consolidation model by considering a non-linear volumetric strain and non-linear fluid compressibility. On the space-time domain $\Omega \times (0, T]$, where T denotes the final time, the governing equations read as follows:

$$-\nabla \cdot [2\mu\varepsilon(\mathbf{u}) + \mathfrak{h}(\nabla \cdot \mathbf{u}) - \alpha(pI)] = \rho_b \mathbf{g}, \tag{1}$$

$$\partial_t \left(\mathfrak{b}(p) + \alpha \nabla \cdot \mathbf{u}\right) + \nabla \cdot \mathbf{q} = f, \tag{2}$$

$$\mathbf{q} = -\mathbf{K}\left(\nabla p - \rho_f \mathbf{g}\right), \tag{3}$$

where $\mathbf{u}$ is the displacement, p is the fluid pressure and $\mathbf{q}$ is the Darcy flux. We denoted by $\mathfrak{h}(\cdot)$ the non-linear volumetric stress in the mechanics equation (1). Further, $\varepsilon(\mathbf{u}) := \frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^t}{2}$ is the linearised strain tensor, μ is the shear modulus, α is the Biot's coefficient, ρ_b is the bulk density and $\mathbf{g}$ the gravity vector. The non-linear compressibility is denote by $\mathfrak{b}(\cdot)$, f is a volume source term, $\mathbf{K}$ is the permeability tensor divided by fluid viscosity and ρ_f is the fluid density. For simplicity, we assume homogeneous Dirichlet boundary conditions $u = 0$, $p = 0$ on $\partial\Omega \times (0, T]$. At the initial time we assume $\mathbf{u} = \mathbf{u}_0$, $p = p_0$ in $\Omega \times \{0\}$.

3 A fully discrete higher order numerical scheme

Throughout our paper we use common notations of functional analysis. Let $L^2(\Omega)$ be the space of Lebesque measurable and square integrable functions on Ω and $H^m(\Omega)$, $m \geq 1$ be the space of L^2-functions having weak derivates up to order m in $L^2(\Omega)$. We denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the inner product and norm in $L^2(\Omega)$. We are using bold letters for variables, functions or spaces which are vectors or tensors. For rank 2 tensors $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^{d,d}$ $\langle \mathbf{A} : \mathbf{B} \rangle := \int_\Omega \sum_{i,j=1}^d A_{ij} B_{ij} d\mathbf{x}$. Further, we consider the spaces

$$\mathbf{H}_0^1(\Omega) := \{\mathbf{u} \in \mathbf{H}^1(\Omega) |\ \mathbf{u} = 0 \text{ on } \partial\Omega\},$$

$$\mathbf{H}(\text{div}; \Omega) := \{\mathbf{q} \in L^2(\Omega) \mid \nabla \cdot \mathbf{q} \in L^2(\Omega)\}.$$

Let $X_0 \subset X \subset X_1$ be three reflexive Banach spaces with continuous embedding and let $I = (0, T)$ be the time interval. Following [4,5] we consider the following set of Bochner spaces

$$L^2(I; X) = \left\{ w : (0, T) \to X \,\middle|\, \int_0^T \|w(t)\|_X^2 dt < \infty \right\},$$

$$H^1(I;X_0,X_1)=\{w\in L^2(I;X_0)\mid \partial_t w\in L^2(I,X_1)\},$$

that are equipped with their natural norms and where the time derivative ∂_t is understood in the sense of distributions. In particular, every function in $H^1(I;X_0,X_1)$ is continuous on I with values in X. For $X_0=X=X_1$ we simply write $H^1(I;X)$.

3.1 Variational formulation of the non-linear Biot model

We can now proceed and state the variational formulation of the considered non-linear Biot model (1)-(3):

Find $\mathbf{u}\in H^1\left(I;\mathbf{H}^1(\Omega)\right)\cap L^2\left(I;\mathbf{H}_0^1(\Omega)\right)$, $\mathbf{q}\in L^2\left(I;\mathbf{H}(\mathrm{div};\Omega)\right)$ and $p\in H^1\left(I;L^2(\Omega)\right)$ such that:

$$\begin{aligned}2\mu\int_I\langle\varepsilon(\mathbf{u}):\varepsilon(\mathbf{v})\rangle d\tau+\int_I\langle\mathfrak{h}(\nabla\cdot\mathbf{u}),\nabla\cdot\mathbf{v}\rangle d\tau&\\ -\alpha\int_I\langle p,\nabla\cdot\mathbf{v}\rangle d\tau&=\int_I\langle\rho_b\mathbf{g},\mathbf{v}\rangle d\tau, \qquad (4)\\ \int_I\langle\mathbf{K}^{-1}\mathbf{q},\mathbf{z}\rangle d\tau-\int_I\langle p,\nabla\cdot\mathbf{z}\rangle d\tau&=\int_I\langle\rho_f\mathbf{g},\mathbf{z}\rangle d\tau, \qquad (5)\\ \int_I\langle\partial_t(\mathfrak{b}(p)+\alpha\nabla\cdot\mathbf{u}),w\rangle d\tau+\int_I\langle\nabla\cdot\mathbf{q},w\rangle d\tau&=\int_I\langle f,w\rangle d\tau, \qquad (6)\end{aligned}$$

for all $\mathbf{v}\in L^2\left(I;\mathbf{H}_0^1(\Omega)\right)$, $\mathbf{z}\in L^2\left(I;\mathbf{H}(\mathrm{div};\Omega)\right)$ and $w\in L^2\left(I;L^2(\Omega)\right)$. We refer to [4] for the linear case of the above system.

3.2 A L-scheme type linearization

The non-linear system above (4)-(6) can be solved monolithically [8] or by a splitting approach [4]. Following [8,15], a monolithic version of the L-scheme applied to the system (4)-(6) reads as:

Given $\mathbf{u}^0=\mathbf{u}_0$, $\mathbf{q}^0=\mathbf{q}_0$ and $p^0=p_0$, for $s\geq 1$, find $\mathbf{u}^s\in H^1\left(I;\mathbf{H}^1(\Omega)\right)\cap L^2\left(I;\mathbf{H}_0^1(\Omega)\right)$, $\mathbf{q}^s\in L^2\left(I;\mathbf{H}(\mathrm{div};\Omega)\right)$ and $p^s\in H^1\left(I;L^2(\Omega)\right)$ such that there holds

$$\begin{aligned}2\mu\int_I\langle\varepsilon(\mathbf{u}^s):\varepsilon(\mathbf{v})\rangle d\tau+\int_I\langle L_2(\nabla\cdot\delta\mathbf{u}^s)-\alpha p^s,\nabla\cdot\mathbf{v}\rangle d\tau&\\ +\langle\mathfrak{h}(\nabla\cdot\mathbf{u}^{s-1}),\nabla\cdot\mathbf{v}\rangle d\tau&=\int_I\langle\rho_b\mathbf{g},\mathbf{v}\rangle,\\ \int_I\langle\mathbf{K}^{-1}\mathbf{q}^s,\mathbf{z}\rangle d\tau-\int_I\langle p^s,\nabla\cdot\mathbf{z}\rangle d\tau&=\int_I\langle\rho_f\mathbf{g},\mathbf{z}\rangle d\tau,\\ \int_I\langle\partial_t\left(L_1\delta p^s+\alpha\nabla\cdot\mathbf{u}^s\right),w\rangle d\tau+\int_I\langle\nabla\cdot\mathbf{q}^s,w\rangle d\tau&=\int_I\langle f-\partial_t\mathfrak{b}(p^{s-1}),w\rangle d\tau, \qquad (7)\end{aligned}$$

for all $\mathbf{v} \in L^2\left(I; \mathbf{H}_0^1(\Omega)\right)$, $\mathbf{z} \in L^2\left(I; \mathbf{H}(\text{div}; \Omega)\right)$ and $w \in L^2\left(I; L^2(\Omega)\right)$, where $\delta(\cdot)^s := (\cdot)^s - (\cdot)^{s-1}$.

The monolithic L-scheme introduced above can be modified by replacing $\nabla \cdot \mathbf{u}^s \approx \nabla \cdot \mathbf{u}^{s-1}$ in (7) to obtain a fixed stress type of splitting scheme. We refer to [8] for the details.

3.3 Discretization in time: discontinuous Galerking dG(r)

The time interval $(0, T]$ is decomposed in N subintervals $I_n = (t_{n-1}, t_n]$, where n=$1, ..., N$, $0 = t_0 < t_1 < ... < t_{n-1} < t_n = T$ and $\tau_n = t_n - t_{n-1}$. Moreover $\tau = max_{1\leq n\leq N}\tau_n$ denotes the time discretizations parameter. In order to define a higher order dG scheme in time we need to introduce the space of piecewise polynomials of order r in time with coefficients in a Banach space X and the associated Bochner space $\mathcal{X}^r(X)$:

$$\mathbb{P}_r(I_n; X) := \left\{ \psi_n : I_n \to X \,\middle|\, \psi_n(t) = \sum_{j=0}^{r} \xi_n^j t^j,\ \xi_n^j \in X,\ j = 0, ..., r \right\},$$
$$\mathcal{X}^r(X) := \left\{ \psi_\tau \in L^2(I; X) \ \middle|\ \psi_{\tau|_{I_n}} = \psi_n \in \mathbb{P}_r(I_n; X); \forall n \in \{1, ..., N\} \right\}.$$

We can now state a semi-discrete variational form of the system (1)-(3). We mention that the test functions ψ_n are vanishing on $I \setminus I_n$. The semi-discrete scheme reads as:

Find $\mathbf{u}_\tau^s \in \mathcal{X}^r\left(\mathbf{H}^1(\Omega)\right)$, $\mathbf{q}_\tau^s \in \mathcal{X}^r\left(\mathbf{H}(\text{div}; \Omega)\right)$ and $p_\tau^s \in \mathcal{X}^r\left(L^2(\Omega)\right)$, such that

$$2\mu \int_{I_n} \langle \varepsilon(\mathbf{u}_\tau^s) : \varepsilon(\mathbf{v}_\tau) \rangle d\tau + \int_{I_n} \langle L_2(\nabla \cdot \delta \mathbf{u}_\tau^s) + \alpha p_\tau^s, \nabla \cdot \mathbf{v}_\tau \rangle d\tau = \int_{I_n} \langle \rho_b \mathbf{g}, \mathbf{v}_\tau \rangle d\tau - \int_{I_n} \langle \mathfrak{h}(\nabla \cdot \mathbf{u}_\tau^{s-1}), \nabla \cdot \mathbf{v}_\tau \rangle d\tau,$$

$$\int_{I_n} \langle \mathbf{K}^{-1} \mathbf{q}_\tau^s, \mathbf{z}_\tau \rangle d\tau - \int_{I_n} \langle p_\tau^s, \nabla \cdot \mathbf{z}_\tau \rangle d\tau = \int_{I_n} \langle \rho_f \mathbf{g}, \mathbf{z}_\tau \rangle d\tau,$$

$$\int_{I_n} \langle \partial_t \left(L_2 \delta p_\tau^s + \alpha \nabla \cdot \mathbf{u}_\tau^s \right), w_\tau \rangle d\tau + \int_{I_n} \langle \nabla \cdot \mathbf{q}_\tau^s, w_\tau \rangle d\tau + \langle [L_2 p_\tau^s + \alpha \nabla \cdot \mathbf{u}_\tau^s]_{n-1}, w_\tau(t_n^+) \rangle = -\int_{I_n} \langle \partial_t \mathfrak{b}(p_\tau^{s-1}), w_\tau \rangle d\tau - \langle \left[\mathfrak{b}(p_\tau^{s-1})\right]_{n-1}, w_\tau^+(t_{n-1}) \rangle - \int_{I_n} \langle f, w_\tau \rangle d\tau,$$

for all $\mathbf{v}_\tau \in \mathcal{X}^r\left(\mathbf{H}_0^1(\Omega)\right)$, $\mathbf{z}_\tau \in \mathcal{X}^r\left(\mathbf{H}(\text{div}; \Omega)\right)$, $w_\tau \in \mathcal{X}^r\left(L^2(\Omega)\right)$. We also used the notations $[w_\tau]_{n-1} = w_\tau^+(t_{n-1}) - w_\tau^-(t_{n-1})$, $w_\tau^+(t_{n-1}) = w_{\tau|_{I_n}}(t_{n-1})$ and $w_\tau^-(t_{n-1}) = w_{\tau|_{I_{n-1}}}(t_{n-1})$.

In the next we represent $\mathbf{u}^s_{\tau|I_n}$, $\mathbf{q}^s_{\tau|I_n}$ and $p^s_{\tau|I_n}$, in terms of the basis functions with respect to the time variable of $\mathcal{X}^r\left(\mathbf{H}^1(\Omega)\right)$, $\mathcal{X}^r\left(\mathbf{H}(\mathrm{div};\Omega)\right)$ and $\mathcal{X}^r\left(L^2(\Omega)\right)$, respectively. For this, let t^j_n, for $j=0,\dots,r$ be the $(r+1)$ Gauss quadrature points on I_n. We define ψ^j_n to be the Lagrange polynomial of degree r, which satisfies $\psi^j_n(t^i_n)=\hat{\delta}_{ji}$, with $\hat{\delta}$ being the Kronecker symbol. Now we can write

$$\mathbf{u}^s_{\tau|I_n}(t)=\sum_{j=0}^{r}\mathbf{u}^{s,j}_n\psi^j_n(t),\quad \mathbf{q}^s_{\tau|I_n}(t)=\sum_{j=0}^{r}\mathbf{q}^{s,j}_n\psi^j_n(t),\quad p^s_{\tau|I_n}(t)=\sum_{j=0}^{r}p^{s,j}_n\psi^j_n(t).$$

Then, by taking $\mathbf{v}_\tau=\mathbf{v}\psi^i_n$, $\mathbf{z}_\tau=\mathbf{z}\psi^i_n$ and $w_\tau=w\psi^i_n$,$i=0,\dots,r$ in the semi-discrete problem above, we get the equivalent formulation on each time interval I_n:

Find $\mathbf{u}^{s,j}_n\in H^1_0(\Omega)$, $\mathbf{q}^{s,j}_n\in\mathbf{H}(\mathrm{div};\Omega)$ and $p^{s,j}_n\in L^2(\Omega)$ for every $j=0,...,r$ such that

$$\begin{aligned}
2\mu\langle\varepsilon(\mathbf{u}^{s,i}_n),\varepsilon(\mathbf{v})\rangle+\langle L_2\nabla\cdot\delta\mathbf{u}^{s,i}_n-\alpha p^{s,i}_n,\nabla\cdot\mathbf{v}\rangle&=\langle\rho_b\mathbf{g},\mathbf{v}\rangle-\langle\mathfrak{h}(\nabla\cdot\mathbf{u}^{s-1}_n(t^i_n)),\nabla\cdot\mathbf{v}\rangle,\\
\langle\mathbf{K}^{-1}\mathbf{q}^{s,i}_n,\mathbf{z}\rangle-\langle p^{s,i}_n,\nabla\cdot\mathbf{z}\rangle&=\langle\rho_f\mathbf{g},\mathbf{z}\rangle,\\
\sum_{j=0}^{r}\left\{\alpha_{ij}\langle L_1\delta p^{s,j}_n+\alpha\nabla\cdot\mathbf{u}^{s,j}_n,w\rangle\right\}+\beta_{ii}\langle\nabla\cdot\mathbf{q}^{s,i}_n,w\rangle&=\beta_{ii}\langle f(t^i_n),w\rangle\\
-\sum_{j=0}^{r}\left\{\alpha_{ij}\langle\mathfrak{b}'(p^{s-1}_n(t^j_n))p^{s-1}_n(t^j_n),w\rangle\right\}&-\psi^i_n(t^+_{n-1})\left\langle\alpha\nabla\cdot\mathbf{u}^-_{n-1}+\mathfrak{b}(p^-_{n-1}),w\right\rangle,
\end{aligned}$$

holds true $\forall i=0,..,r$ and for all $\mathbf{v}\in\mathbf{H}^1_0(\Omega)$, $\mathbf{z}\in\mathbf{H}(\mathrm{div};\Omega)$, $w\in L^2(\Omega)$. The coefficients above are defined by $\alpha_{ij}:=\int_{I_n}\partial_t(\psi^j_n)\psi^i_n dt+\psi^{j+}_n(t_{n-1})\psi^{i+}_n(t_{n-1})$ and $\beta_{ii}:=\int_{I_n}\phi^i_n\psi^i_n dt$, see [4,5] for more details.

3.4 Discretization in Space by cG(p+1)-MFEM(p)

We proceed by formulating now a fully discrete scheme for solving (1)-(3). Let $\mathcal{K}_h$ be a regular decomposition of Ω into d-simplices. We denote by h the mesh diameter. The lowest order of the discrete spaces cG(1)-MFEM(0) are given by $\mathbf{V}_h:=\{\mathbf{v}_h\in H^1(\Omega)^d\,;\,\mathbf{v}_{h|K}\in\mathbb{P}^d_1,\,\forall K\in\mathcal{K}_h\}$, $W_h:=\{w_h\in L^2(\Omega)\,;\,w_{h|K}\in\mathbb{P}_0,\,\forall K\in\mathcal{K}_h\}$ and $\mathbf{Z}_h:=\{\boldsymbol{z}_h\in H(\mathrm{div};\Omega)\,;\,\boldsymbol{z}_{h|K}(\boldsymbol{x})=\boldsymbol{a}+b\boldsymbol{x},\,\boldsymbol{a}\in\mathbb{R}^d,\,b\in\mathbb{R},\,\forall K\in\mathcal{K}_h\}$. Further higher order cG(p+1)-MFEM(p) for $p>1$ are described in [3,5]. This space discretization is not uniformly inf-sup stable with respect to the physical parameters. In this regard, a small enough h is needed to avoid oscillations [20]. Then the fully discrete scheme for solving (1)-(3) on each time interval I_n reads as follows:

For every $i \in \{0, ..., r\}$, find $\mathbf{u}_{n,h}^{s,i}, \in \mathbf{Z}_h$, $\mathbf{q}_{n,h}^{s,i} \in \mathbf{V}_h$ and $p_{n,h}^{s,i} \in W_h$, such that there holds for all $\mathbf{v}_h \in \mathbf{V}_h$, $\mathbf{z}_h \in \mathbf{Z}_h$ and $w_h \in Q_h$:

$$
\begin{aligned}
2\mu\langle \varepsilon(\mathbf{u}_{n,h}^{s,i}), \varepsilon(z_h)\rangle + \langle L_2 \nabla\cdot\delta\mathbf{u}_h^{s,i} - \alpha p_{n,h}^{s,i}, \nabla\cdot\mathbf{v}_{n,h}\rangle &= \langle \rho_b \mathbf{g}, \mathbf{v}_h\rangle \\
&\quad - \langle \mathfrak{h}(\nabla\cdot\mathbf{u}_\tau^{s-1}), \nabla\cdot\mathbf{v}_h\rangle, \\
\langle \mathbf{K}^{-1}\mathbf{q}_{n,h}^{s,i}, \mathbf{z}_h\rangle - \langle p_{n,h}^{s,i}, \nabla\cdot\mathbf{z}_h\rangle &= \langle \rho_f \mathbf{g}, \mathbf{z}_h\rangle, \\
\sum_{j=0}^{r}\left\{\alpha_{ij}\langle L_1 \delta p_{n,h}^{s,j} + \alpha\nabla\cdot\mathbf{u}_h^{s,j}, w_h\rangle\right\} + \beta_{ii}\langle \nabla\cdot\mathbf{q}_{n,h}^{s,i}, w_h\rangle &= \beta_{ii}\langle f(t_{n,i}), w_h\rangle \\
-\sum_{j=0}^{r}\left\{\alpha_{ij}\langle \mathfrak{b}'(p_{n,h}^{s-1}(t_n^j))p_{n,h}^{s-1}(t_n^j), w_h\rangle\right\} &- \psi_n^i(t_{n-1}^+)\left\langle \alpha\nabla\cdot\mathbf{u}_{n-1}^-, w_h\right\rangle \\
&- \psi_n^i(t_{n-1}^+)\left\langle \mathfrak{b}(p_{n-1}^-), w_h\right\rangle.
\end{aligned}
$$

We end this section by postulating the following convergence result.

Theorem 1. *Assuming that the functions $\mathfrak{h}(\cdot)$ and $\mathfrak{b}(\cdot)$ are Lipschitz continuous and that the time step is small enough, then the fully discrete scheme above converges for any $L_1 \geq L_b$ and $L_2 \geq L_h$.*

The proof combines the ideas in [4] with the ones in [8].

4 Numerical results

We solve the non-linear Biot problem (1)-(3) in the unit-square $\Omega = (0,1)^2$ and time interval $I = [0,1]$. We consider $K = \nu_f = M = \alpha = \lambda = \mu = 1.0$. The mesh size is $h = 0.1$ and the time step size is $\tau = 0.1$ For all cases, we use as stopping criterion for the iterations $\|\delta p^i\| + \|\delta\mathbf{q}^i\| + \|\delta\mathbf{u}^i\| \leq 10^{-8}$. For dG(0) and dG(1), we investigated a range of values for L_1 and L_2 to assess the sensitivity of the proposed L-scheme with respect to these parameters. All numerical experiments were implemented using the open-source finite element library deal.II [2] and the DTM++ framework [4].

Fig. 1 illustrates the number of iterations for the L-scheme for different values of L_1 and L_2 at the last time step. For both dG(0) and dG(1), the L-scheme is more sensitive on the choice of the stabilizing parameter L_2. The fastest convergence for dG(0) scheme is obtained when $L_1 \sim L_b$ and $L_2 \sim L_h$. Nevertheless, for dG(1) the fastest convergence is obtained for a different value of $L_1 \sim 0.5L_b$.

Fig. 2 shows the performance of the L-scheme for different discretizations in space and time. We observe, in accordance with the theoretical results [15,8], that a larger time step leads to a faster convergence. Furthermore, the convergence seems not to be affected by the mesh size h, but slighlty depends on the order of the spatial discretization.

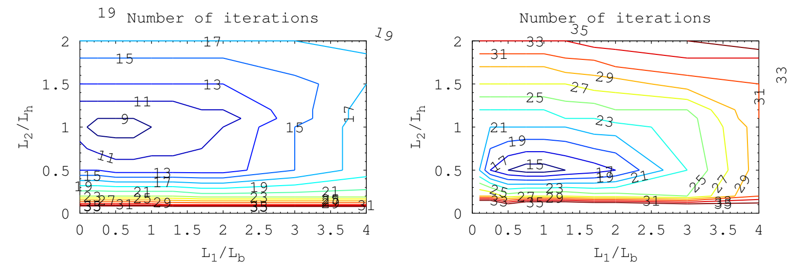

Fig. 1. Performance of L-scheme for different values of L_1 and L_2 for test problem 1, $\mathfrak{b}(p) = e^p$; $\mathfrak{h}(\nabla \cdot \mathbf{u}) = \sqrt[3]{(\nabla \cdot \mathbf{u})^5} + \nabla \cdot \mathbf{u}$, dG(0) to the left and dG(1) to the right.

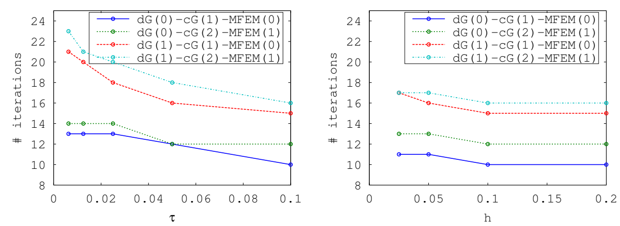

Fig. 2. Performance of L-scheme for different mesh size h, time step τ and order of the elements at the last time step.

We remark that the algebraic systems obtained by using the higher-order space-time discretizaton are more challenging to solve, see also [4]. A preconditioner based on the splitting L-scheme is a promising choice to solve the algebraic system efficiently, see [8,14,21].

5 Conclusions

We considered a generic non-linear Biot model to be used for simulation of flow in deformable porous media. We proposed a higher order space-time numerical scheme. Numerical results were shown to illustrate the performance of the scheme. A convergence result has been stated, a rigorous proof will be the subject of a follow-up paper.

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References

1. T. Almani, K. Kumar, A. H. Dogru, G. Singh, and M. F. Wheeler, *Convergence Analysis of Multirate Fixed-Stress Split Iterative Schemes for Coupling*

Flow with Geomechanics, Comput. Methods. Appl. Mech. Eng. **311** (2016), 180–207.

2. W. Bangerth, G. Kanschat, T. Heister, L. Heltai, G. Kanschat, *The deal.II library version 8.4* J. Numer. Math. **24** (2016), 135–141.
3. M. Bause and U. Köcher, *Variational time discretization for mixed finite element approximations of nonstationary diffusion problems* J. Comput. Appl. Math. **289** (2015), 208–224.
4. M. Bause, F. Radu, and U. Köcher, *Space–time finite element approximation of the Biot poroelasticity system with iterative coupling*, Comput. Methods. Appl. Mech. Eng. **320** (2017), 745–768.
5. M. Bause, F. A. Radu, and U. Köcher, *Error analysis for discretizations of parabolic problems using continuous finite elements in time and mixed finite elements in space*, Numerische Mathematik **137**:4 (2017), 773–818.
6. M. Bause, *Iterative coupling of mixed and discontinuous Galerkin methods for poroelasticity*, arXiv:1802.03230 (2018).
7. M. A. Biot, *General theory of three-dimensional consolidation*, J. Appl. Phys. **12**:2 (1941), 155–164.
8. M. Borregales, J. M. Nordbotten, K. Kumar, and F. A. Radu, *Robust iterative schemes for non-linear poromechanics*, Comput. Geosci. 2018, to be appear (also arXiv:1702.00328).
9. M. Borregales, K. Kumar, F. A. Radu, C. Rodrigo and F. J. Gaspar, *A parallel-in-time fixed-stress splitting method for Biot's consolidation model*, arXiv:1802.00949 (2018).
10. J. W. Both, M. Borregales, J. M. Nordbotten, K. Kumar, and F. A. Radu, *Robust fixed stress splitting for Biot's equations in heterogeneous media*, Appl. Math. Letters **68** (2017), 101–108.
11. J. Carmeliet and K. Van Den Abeele, *Poromechanical approach describing the moisture influence on the non-linear quasi-static and dynamic behaviour of porous building materials*, Materials and Structures **37**:4 (2004), 271–280.
12. F. J. Gaspar and C. Rodrigo, *On the fixed-stress split scheme as smoother in multigrid methods for coupling flow and geomechanics*, Comput. Methods. Appl. Mech. Eng. **326** (2017), 526–540.
13. J. Kim, H. Tchelepi, and R. Juanes, *Stability and convergence of sequential methods for coupled flow and geomechanics: Fixed-stress and fixed-strain splits*, Comput. Methods. Appl. Mech. Eng. **200**:13–16 (2011), 1591–1606.
14. U. Köcher, *Space-Time-Parallel Poroelasticity Simulation*, arXiv:1801.04984 (2018).
15. F. List and F. A. Radu, *A study on iterative methods for solving Richards' equation*, Comput. Geosci. **20**:2 (2016), 341–353.
16. A. Mikelić and M. F. Wheeler, *Theory of the dynamic Biot-Allard equations and their link to the quasi-static Biot system*, J. Math. Phys. **53**:12 (2012).
17. ———, *Convergence of iterative coupling for coupled flow and geomechanics*, Comput. Geosci. **18**:3-4 (2013), 325–341.

18. I. Pop, F. Radu, and P. Knabner, *Mixed finite elements for the Richards' equation: linearization procedure*, J. Comput. Appl. Math. **168**:1–2 (2004), 365–373.

19. F. A. Radu, J. M. Nordbotten, I. S. Pop, and K. Kumar, *A robust linearization scheme for finite volume based discretizations for simulation of two-phase flow in porous media.*, J. Comput. Appl. Math. **289** (2015), 134–141.

20. C. Rodrigo, X. Hu, P. Ohm, J. H. Adler, F. J. Gaspar, and L. Zikatanov, *New stabilized discretizations for poroelasticity and the Stokes' equations*, arXiv:1706.05169 (2017).

21. J. A. White, N. Castelletto, and H. A. Tchelepi, *Block-partitioned solvers for coupled poromechanics: A unified framework*, Comput. Methods. Appl. Mech. Eng. **303** (2016), 55–74.