Matching univalent functions and related problems of conformal mappings

Erlend Grong

Master's Thesis in
Mathematical Analysis

Department of Mathematics
University of Bergen
Norway

February 2008

Acknowledgements

First of all, I would like to thank my supervisor, Alexander Vasiliev. He has helped me a great deal to understand this topic, and I am gratefull to have been given interesting problems to work with. I would like to thank him for including me into the scientific community by inviting me to conferences and workshops (or just for discussions over a cup of tea). Apart from that, I owe a great deal of gratitude to to Nils Henry Rasmussen, Henning Lohne and Alexander Lundervold for LaTeX-help, and to Jørn Havfer and Antonella Zanna Munthe-Kaas for helping me make graphs using MatLab. I would also like to thank other people at the Pure Mathematics department, who contributed with valuable explanations and suggestions, in particular, I would like to thank Irina Markina, Bjørn Dundas and Pavel Gumenyuk. In particular, Pavel's lecture notes on the correspondence of $\tilde{S}$ and Kirillov's manifold, have been very usefull.

Contents

Chapter 1

Introduction

One of the oldest topics of complex analysis, is the study of the class S of univalent (holomorphic and injective) functions $f : \mathbb{D} \to \mathbb{C}$ normalised by

$$f(z) = z + \sum_{n=2}^{\infty} a_{n-1} z^n$$

Here, and in the rest of this thesis, $\mathbb{D}$ represents the unit disk $\mathbb{D} = \{z \in \mathbb{C} \big| |z| < 1\}$ and we will denote the exterior of $\mathbb{D}$ in the Riemann sphere $\bar{\mathbb{C}}$, by $\mathbb{D}^*$ (i.e. $\mathbb{D}^* = \bar{\mathbb{C}} \setminus \bar{\mathbb{D}}$, where $\bar{\mathbb{D}}$ is the closure of $\mathbb{D}$). Let f be a map in S, with the image $f(\mathbb{D}) = \Omega$, such that $\partial\Omega$ splits $\bar{\mathbb{C}}$ into two domains Ω and $\Omega^* = \bar{\mathbb{C}} \backslash \bar{\Omega}$, with $\infty \in \Omega^*$. By the well known Riemann mapping theorem, there exists a unique univalent mapping g of $\mathbb{D}^*$ onto Ω^*, fixing the inequality $\lim_{x \to \infty} g(z)/z > 0$. This function has an expansion

$$g(z) = b_1 z + b_0 + \frac{b_{-1}}{z} + ...$$

about ∞, where $b_1 > 0$. We say that the mappings f and g are *matching*. In this thesis, we will be focused on how the properties of f influence properties of the adjoint function g matching it, and vice versa.

The relationship between these two mappings are poorly understood. Knowledge about one of the maps, for instance f, does not give us any clear analytic characteristic of the other one except for the knowledge of the existence of g, and even less hope of constructing g as an explicit function. This problem becomes even more difficult by a general impression that if either one of the functions is given by a seemingly simple formula and is not the identity, then the matching function is rather complicated. Let us consider, as an example, the case where Ω is an ellipse with foci at ± 1. The map from $\mathbb{D}^*$ to the exterior of Ω is given by the Joukowski map

$$z \mapsto \frac{1}{2}\left(cz + \frac{1}{cz}\right) \qquad c > 1$$

The matching function from $\mathbb{D}$ to Ω, however, is given by difficult elliptic integrals, as we will show in section 2.3. Since the problem is so complex, any partial result in this direction would be an asset.

1.1 Conformal radii of matching domains

We consider a property that seems to be obscure by some geometric reasons. Let γ be a simple closed, analytic contour, splitting $\bar{\mathbb{C}}$ into two domains Ω and Ω^*, neighbourhoods of 0 and ∞, respectively. How is the conformal radius of Ω with respect to the origin controlled by the logarithmic capacity of γ or equivalently by the coefficient b_1 in the expansion of $g : \mathbb{D}^* \to \Omega^*$? This question is addressed in section 3.5, where we look at a special case where γ is quasisymmetric, which for example guarantees that Ω and Ω^* are both domains.

1.2 Connections to mathematical physics

Recently several modern developments in the theory of conformal maps in relation with mathematical physics have been seen. This, in particular, applies to conformal field theory (CFT), where conformal maps and Riemann surfaces play an important role in parametrisation of a string worldsheet. Non-linear physics is a natural user of conformal maps. Such sample non-linear processes as Laplacian growth, KdV and mKdV hierarchies use matching functions as 'coordinates' on a phase space. The matching functions appear also in the inverse potential problem of construction of a domain by its harmonic moments. If we calculate the interior moments of Ω, given by

$$C_n = \iint_{\Omega} w^n du\, dv \qquad w = u + iv\,, \quad n \in \mathbb{N},$$

it turns out that the corresponding exterior moments

$$C_{-n} = \iint_{\Omega^*} \bar{w}^n du\, dv \qquad w = u + iv\,, \quad n \in \mathbb{N}$$

can (under some normalisation) be embedded into the Toda (dispersinless) hierarchy, or more precisely, their derivatives with respect to C_n. This integrable hierarchy, solvable by the so-called τ-function, is important in the study of integrable models in CFT. The embedding becomes possible, due to the fact that if γ is an analytic contour, then there exists the Schwarz function $S(w)$, analytic in a neighbourhood of γ and satisfying the condition $S(w) = \bar{w}$ on γ. This function glues Ω with Ω^* through γ and the harmonic moments C_n, are coefficient in the formal Laurent expansion of $S(z)$.

Another formulation of CFT is based on the path integral and leads to the Lagrangian and Hamiltonian formulations. The Virasoro algebra plays an important role in this model. It arises as an operator algebra in the formal Laurent expansion of the analytic component of the momentum-energy tensor. Its mathematical representation is based on the Kirillov homogeneous manifold

$$\mathrm{Diff}_+ S^1 / \mathrm{Rot}\, S^1$$

where $\mathrm{Diff}_+ S^1$ is the Lie group of orientation preserving diffeomorphism of S^1, and $\mathrm{Rot}\, S^1$ is the subgroup of sense preserving rotations. The Virasoro algebra is the central ($\mathbb{C}$ or $\mathbb{R}$) extension of the Lie algebra of the vector fields on S^1, the tangent algebra to $\mathrm{Diff}_+ S^1$.

The Kirillov manifold has many equivalent representations (in the space of unitary probabilistic measures, the Teichmüller space, etc.). We will be focused on the representation given by the matching functions f and g discussed earlier. We have that

$$f^{-1} g|_{S^1} \in \mathrm{Diff}_+ S^1$$

and every element from $\mathrm{Diff}_+S^1/\mathrm{Rot\,S}^1$ can be represented by such a composition of mappings in a unique way. In this model, the link between f and g are given implicitly by positive and negative indexed Virasoro generators considered in section 4.1.

Recently, it became clear that the Virasoro generators appear in the Löwner theory as the first integrals of a partially integrable Hamiltonian system for the coefficients of the Taylor expansion of $f \in S$.

1.3 Precognitions

The reader is assumed to be familiar with theory of real manifolds and constructions on them, as well as some representation theory, in particular the Lie theory. Useful information can be found in, e.g., [8], [32], [17], [13]. Apart from this, and some basic properties taken from topology and complex analysis, we tried to keep the thesis as self-contained as possible. Some language from category theory, and the notion of sheaf will be used, but the thesis is easily understood without any knowledge of these.

1.4 Some notations

We will denote the composition of two functions without $\circ$ (i.e. $f \circ g$ is written fg). When two functions are multiplied, we will explicitly denote it by $f \cdot g$. If U is a subset of a topological space X, then the closure, interior and exterior of U will be denoted by $\bar{U}$, $\mathrm{int}\,U$ and $\mathrm{ext}\,U$. For a mapping $f : X \to Y$, then the composition $U \subseteq X \xrightarrow{f} Y$, is written $f|_U$. If $X = \prod_\alpha X_\alpha$, then pr_{X_α} denotes the projection from X onto X_α. The identity map from X to itself is denoted id_X . If A is a matrix, the transpose is denoted A^T .

If M is a manifold, then we denote the (real) tangent space at $r \in M$ by T_rM , the tangent bundle by TM and cotangent bundle by T^*M. If $f : M \to N$ is a smooth mapping, we write Tf and T^*f for the induced mapping on tangent and contangent bundles respectively. If $\xi : E \to X$ is a smooth vector bundle, then $\Gamma(E)$ is the space of smooth sections on the bundle. If V is a vector space, then V^* is the dual space,

$$\bigwedge^k V = V^{\otimes k}/\{v_1 \otimes \cdots \otimes v_k | v_j = v_l \,\mathrm{for\,some}\, l \neq j\}$$

is the exterior product and

$$\mathrm{Sym}^k V = V^{\otimes k}/\{v_1 \otimes \cdots \otimes v_k - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}\}$$

is the k'th symmetric product, where $v_i \in V$ and σ is a permutation. We will use the same symbols for the corresponding action on bundles.

We denote the integers by $\mathbb{Z}$, positive integers by $\mathbb{N}$, non-negative integers by $\mathbb{N}_0$, real numbers by $\mathbb{R}$, and complex numbers by $\mathbb{C}$. $\mathbb{D}$ will be the unit disk in $\mathbb{C}$ and $\bar{\mathbb{C}}$ will be the Riemann sphere. $S^n = \{x \in \mathbb{R}^{n+1} \big| \, |x| = 1\}$ is the n-sphere, and unless otherwise is mentioned, S^1 is viewed as a subset of $\mathbb{C}$. Apart from that, we define

$$\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | \, x_n \geq 0\}$$

$$\mathbb{C}^n_+ = \{\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{C}^n | \, \mathrm{Im}\,\zeta_n \geq 0\}$$

The projection from either $\mathbb{R}^n$ or $\mathbb{C}^n$ to its k'th coordinate is denoted pr_k . If $\zeta \in \mathbb{C}$, then $\bar{\zeta}$ is its complex conjugate. If $\Omega \subseteq \bar{\mathbb{C}}$, then we use $\Omega^* = \mathrm{ext}\,\Omega$.

A complete index of notation is found at the end of this thesis.

1.5 Purpose and own results

The present thesis contains an overview of some special topics of Complex Analysis and Differential Geometry, e.g., Chapter 2, some advanced and modern known results concerning Hamiltonian systems, Virasoro algebra as well as our own results. In particular, we consider in details the simplest non-trivial model of matching univalent functions which map $\mathbb{D}$ and $\mathbb{D}^*$ onto interior and exterior of an ellipse. The conformal radius of an ellipse with respect to its centre is calculated in terms of elliptic integrals. The problem of the range of the conformal radius of a domain whose exterior has fixed capacity is considered and partial results are obtained for quasiconformal case in sections 3.3 and 3.5. In particular, we used a version of Löwner theory developed by Becker for quasiconformally extendable conformal maps. Then we considered applications of matching univalent functions to the representation of the Virasoro algebra. In particular, we calculate an element of in Diff_+S^1 corresponding to the ellipse. Considering manifolds of coefficients in S, a Hamiltonian interpretation of the Löwner equation is given in section 4.4. Further, the group action of the non-commutative group Diff_+S^1 induces a non-Riemannian geometry on the manifold of univalent functions, which is best described through Sub-Riemannian geometry. Generalising results by Markina, Prokhorov, and Vasiliev [25] we construct a sub-Riemannian manifold based on a distribution generated by more than two of the truncated Kirillov operators (section 4.7).

Chapter 2

Complex manifolds and Riemann surfaces

This chapter is dedicated to some theoretical background about spaces and some constructions on them. Our intention is not to go deeply in the theory of complex manifolds, but only to get some clear definitions and notions for further use. Especially we will be interested in the one-dimensional case, i.e. Riemann surfaces. We end this chapter with a whole section on quadratic differentials. They are connected to problems regarding moduli of families of curves, which will be treated in the next chapter. Using these, we will also construct a conformal mapping from the disk $\mathbb{D}$ onto an ellipse, that will reapear as an example in the later theory, and will also be used in chapter 3.

Thoughout this chapter ζ will denote a point in $\mathbb{C}^n$, r and s will always be points on a manifold, and z and w will denote charts (this reservation of characters will not be kept in later chapters).

2.1 Complex manifolds

We will call a mapping biholomorphic, if it is both holomorphic and bijective (this implies that the inverse is holomorphic too).

Definition 2.1. *Let M be a real, differentiable manifold, on which there exists an open cover $\{U_\alpha\}$ with homeomorphisms $z_\alpha : U_\alpha \to z_\alpha(U_\alpha) \subseteq \mathbb{C}^n$, such that for any pair of these mappings*

$$z_\beta z_\alpha^{-1}|_{z_\alpha(U_\alpha \cap U_\beta)} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$$

is biholomorphic. The collection $\mathcal{B} = \big\{(z_\alpha, U_\alpha)\big\}$ is called a complex atlas. This atlas generates a unique maximal atlas $\mathcal{A}$, and we define the pair $(M, \mathcal{A})$ as a complex manifold.

The maximal atlas of $\mathcal{B}$ is generated in the following way:

$$\mathcal{A} = \{w : V \to w(V) \subseteq \mathbb{C}^n \big| \, w \text{ is a homeomorphism}, wz^{-1}|_{z(U\cap V)} \text{ is biholomorphic for any } (z, U) \in \mathcal{B}\}$$

Such maximal atlases are called *complex structures*. As usual, the manifold is denoted only by M, and we say that the elements of $\mathcal{A}$ are (complex) charts or local coordinate systems.

From the definition it is clear that M must be a real manifold of even dimension, and it also turns out to be orientable. We define a *bordered complex manifold* in a similar way, only with charts given by homeomorphisms $z_\alpha : U_\alpha \to z_\alpha \subseteq \mathbb{C}^n_+$. The boundary of M is defined as

$$\partial M = \{r \in M |\, \text{there is a chart}\, w : M \to \mathbb{C}^n_+,\, \text{such that}\, pr_n w(r) = 0\}$$

Example 2.2.

- $\mathbb{C}^n$ and any open subsets of $\mathbb{C}^n$ are complex manifolds. Generally, any open nonempty subset of an n-dimensional complex manifold is an n-dimensional complex manifold.

- $\mathrm{M}_{m\times n}\mathbb{C}$, the space of all $m\times n$ matrices with complex entries is a manifold by identifying it with $\mathbb{C}^{mn}$. $\mathrm{GL}_n\mathbb{C}$, the space of all invertible $n \times n$ matrices is a manifold because it is an open subset $\mathrm{M}_{n\times n}\mathbb{C}$ (since $\mathrm{GL}_n\mathbb{C} = \det^{-1}(\mathbb{C} \setminus \{0\})$).

- Let
$$\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/ \sim$$
where $r \sim \lambda r, \quad r \in \mathbb{C}^{n+1} \setminus \{0\}$, $\lambda \in \mathbb{C} \setminus \{0\}$. This is the space of all (complex) lines in $\mathbb{C}^{n+1}$. We denote the equivalence class of r by $[r]$. Using the charts $\{z_k : U_k \to \mathbb{C}^n\}_{k=1,2,\dots,n+1}$, with $U_k = \{[r_1, r_2, \dots , r_{n+1}]|r_k \neq 0\}$ and
$$z_k([r]) = \frac{1}{r_k}(r_1, \dots , r_{k-1}, r_{k+1}, \dots , r_{n+1}),$$
$\mathbb{C}\mathrm{P}^n$ becomes a complex manifold of dimension n. In particular, $\mathbb{C}\mathrm{P}^1 = \bar{\mathbb{C}}$ is a complex manifold.

A mapping $f : M \to N$ of two complex manifolds, is *holomorphic* if, for any charts (z, U) and (w, V) on M and N respectively, the mapping

$$wfz^{-1}|_{z(U\cap f^{-1}(V))} : z(U \cap f^{-1}(V)) \to w(f(U) \cap V)$$

is holomorphic in the usual way. Taking complex manifolds as objects, and holomorphic mappings as morphisms, we get a category. Isomorphisms in this category will be biholomorphic mappings, and we write $M \cong N$, when such a mapping exists between M and N. We denote

$$Hol(M, N) = \{f : M \to N \big|\, f\, \text{holomorphic}\},$$

and in particular, we denote holomorphic functions by $\mathscr{O}M = Hol(M, \mathbb{C})$. This is a ring under pointwise addition and multiplication. By definition, for any mapping $f : M \to N$,

$$\mathscr{O}f : \mathscr{O}N \to \mathscr{O}M$$

$$\varphi \mapsto \varphi f,$$

and $\mathscr{O}$ becomes a contravariant functor. Since every open subset of a complex manifold is a complex manifold, $\mathscr{O}$ also forms a sheaf of rings. If $r \in M$, we call

$$\varinjlim_{r\in U} \mathscr{O}U = \mathscr{O}_r$$

the holomorphic function germs at r (the limit is taken over all open sets $U \subseteq M$ containing r). This is a ring, and we denote its field of fractions $\mathscr{M}_r$. These are called the meromorphic function germs at r. We say that a *meromorphic function* on M, is a mapping

$$f : M \to \coprod_{r\in M} \mathscr{M}_r$$
$$r \mapsto [f]_r$$

such that $[f]_r \in \mathscr{M}_r$ for any $r \in M$, and such that any $s \in M$ has an open connected neighbourhood U, with the property that there are functions $\varphi, \psi \in \mathscr{O}U$, $\psi \not\equiv 0$ with images $[\varphi]_r, [\psi]_r$ in $\mathscr{O}_r$, such that $[f]_r = [\varphi]_r/[\psi]_r$ for every $r \in U$. Hence, a meromorphic function on M as a function that locally can be represented by a quotient of two holomorphic functions with the denominator being not identically zero. We denote the set of all meromorphic functions on M, by $\mathscr{M}M$. This is a field if and only if M is connected [28].

A subset $N \subseteq M$ is called a *k-dimensional submanifold*, if about any $r \in N$, there is a chart (z, U), such that $z(U \cap Y) \subseteq \mathbb{C}^k \times \{0\} \subset \mathbb{C}^n$. A difference between the theory of complex manifolds and real manifolds, is that there is no theorem similar to the Whitney embedding theorem. That is, whereas any real manifold may be embedded as a submanifold of $\mathbb{R}^N$, very few complex manifolds may be embedded into $\mathbb{C}^N$. For example, by Liouville's theorem, all holomorphic functions $f : M \to \mathbb{C}$ for a compact manifold M, are constant on each connected component, so an embedding can only exist if M is a discrete set.

Now let us turn to the complex vector bundles, and then in particular, to the tangent bundles. Observe, that they may exist on any differentiable manifold, not just on the complex ones.

Definition 2.3. *Let M be a differentiable manifold. An m-dimensional complex vector bundle of M, is a surjective continuous mapping $\xi : E \to M$, with the following properties*

- *For any $r \in M$, $\xi^{-1}(r)$ has the structure of a complex vector space.*
- *There is an open cover $\mathcal{U}$ of M, such that for any $U \in \mathcal{U}$, there is a homeomorphism $h : \xi^{-1}(U) \to U \times \mathbb{C}^m$ satisfying the following conditions:*
  1. $\xi|_{\xi^{-1}(U)} = pr_U h$.
  2. *The map $h_r := pr_{\mathbb{C}^m} h|_{\xi^{-1}(r)}$ is an isomorphism of complex vector spaces.*
  3. *For any other $V \in \mathcal{U}$, with g being the associated homeomorphism, the mapping*
  $$s \mapsto g_s h_s^{-1}$$
  is a C^∞ map from $U \cap V$ to $GL_m(\mathbb{C})$.

We will often denote a bundle just by E. Similarly to the real case, we call $\xi^{-1}(r)$ the *fibre* over r, the (h, U) and (g, V) are called *(complex) bundle charts* or *trivialisations*. A collection of bundle charts covering M and satisfying condition 1.-3. is called a *(complex) smooth bundle atlas*. A function $s \mapsto g_s h_s^{-1}$, for a chosen pair of bundle charts, is called a *transition function*.

A *complex bundle morphisms* of complex bundles $\xi_1 : E_1 \to M_1$ to $\xi_2 : E_2 \to M_2$, is given by a pair of mappings $F : E_1 \to E_2$, and $f : M_1 \to M_2$, such that $f\xi_1 = \xi_2 F$, and $F|_{\xi^{-1}(r)}$ is a $\mathbb{C}$-linear map for every $r \in M$. If $M_1 = M_2$ and f is the identity, we will just think of bundle morphism $F : E_1 \to E_2$. In general, a complex vector bundle $\xi : E \to M$ over a complex manifold, does not give E the structure of a complex manifold. For that, we need some more requirements.

Definition 2.4. *Let M be a complex manifold. Then a complex vector bundle is called a holomorphic vector bundle if there is a bundle atlas, in which all transition functions are holomorphic.*

The total space E will now be a $n+m$ complex manifold of its own right, by using that if (z,U) and (h,V) is a chart and bundle chart, respectively, then

$$(z \times id_{\mathbb{C}^m})h|_{\xi^{-1}(U\cap V)}$$

is a chart on E. The fact that such charts overlap holomorphically is guaranteed by the holomorphic transition functions. The most interesting vector bundles are, of course, the tangent bundles. If M is a differentiable manifold with TM as a real tangent space, then the complex tangent bundle is simply the complexification of TM. To clear this up, we define

$$T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} (\mathbb{C} \times M),$$

where $\mathbb{C} \times M$ is seen as a one-dimensional trivial bundle over M. Complex structure is given in a usual way (i.e., $T_rM \otimes_{\mathbb{R}} \mathbb{C}$ becomes a complex vector space by defining scalar multiplication with a complex number by $\alpha(X \otimes \zeta) := X \otimes \alpha\zeta$, and introducing conjugation by $\overline{X \otimes \zeta} = X \otimes \bar{\zeta}$). An alternative description of this complexification, is to let $X \in T_rM$ act on complex valued C^∞ function germs about r. If $[\psi]_r : (M,r) \to (\mathbb{C}, \psi(r))$ is such a germ, we define $X(\psi) = X(\operatorname{Re}\psi) + iX(\operatorname{Im}\psi)$, and introduce scalar multiplication in the obvious way.

If M is a complex manifold and (z,U) is a chart, where $z = (z_1, z_2, \ldots, z_n)$, and $z_j = x_j + iy_j$, then TM is spanned locally by the vector fields

$$\Big\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\Big\}.$$

Note that on any complex manifold, there exists a bundle morphism corresponding to multiplication by i

$$J : TM \to TM,$$

such that

$$J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}, \qquad J(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}, \tag{2.1}$$

which means that being applied twice we get the minus identity: $J^2|_{T_rM} = -id_{T_rM}$. In general, any differentiable manifold together with a bundle morphism on its tangent bundle with this property is called an *almost complex manifold*. The operator is called an *almost complex structure*. All almost complex manifolds are even-dimensional (real dimension) and orientable. An almost complex structure may be extended to a complex bundle morphism on $T_{\mathbb{C}}M$ by defining $J(X \otimes \zeta) := J(X) \otimes \zeta$.

The complex tangent bundle is locally spanned by

$$\Big\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial \bar{z}_2}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_n}\Big\},$$

where $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$ and $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$. The complex tangent bundle then splits into two parts

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$$

called the *holomorphic* and *antiholomorphic* tangent bundles, respectively. $T^{1,0}M$ then is spanned by $\{\frac{\partial}{\partial z_j}\}_{j=1}^n$, and $T^{0,1}M$ by $\{\frac{\partial}{\partial \bar{z}_j}\}_{j=1}^n$. The extension of J in (2.1) to $T_{\mathbb{C}}M$ is given by

$$J(\frac{\partial}{\partial z^j}) = i\frac{\partial}{\partial z^j} \qquad J(\frac{\partial}{\partial \bar{z}^j}) = -i\frac{\partial}{\partial \bar{z}^j}$$

Then $T^{1,0}M$ and $T^{0,1}M$ are the eigenspaces for J with respect to i and $-i$ respectively (so that if $X \in \Gamma(T_{\mathbb{C}}M)$, then $\Gamma(T^{1,0}M) = ker\{X \to X + iJ(X)\}$ and $\Gamma(T^{0,1}M) = ker\{X \to X - iJ(X)\}$). $T^{1,0}M$ is a holomorphic vector bundle, and therefore a complex manifold. $T^{0,1}M$ is not in general a holomorphic bundle, so we can only be sure that is $T^{0,1}M$ an almost complex manifold. The dual bundles $(T^{1,0}M)^*$ and $(T^{0,1}M)^*$ are locally spanned by cotangent vector fields $dz_j = dx_j + idy_j$ and $d\bar{z}_j = dx_j - idy_j$ for $j = 1, \dots, n$.

If M is a complex manifold with a complex structure J, then there is a Riemannian metric ρ that satisfies the following relation

$$\rho(JX, JY) = \rho(X, Y), \qquad X, Y \in \Gamma(TM)$$

is called an *Hermitian metric*, and (M, ρ) an *Hermitian manifold*. The metric ρ can be extended to $T_{\mathbb{C}}M$, and may be thought of as an element in $\Gamma((T^{1,0}M \otimes T^{0,1}M)^*)$ (i.e., $\rho = \sum_{j,k=1,\dots n} \rho_{jk} dz_j \otimes d\bar{z}_j$) with the properties

- $\rho(X, \bar{Y}) = \overline{\rho(Y, \bar{X})}$;
- $\rho(X, \bar{X}) > 0$, whenever $X \neq 0$,

for $X, Y \in \Gamma(T^{1,0}M)$. To each Hermitian metric we associate the fundamental form $\Phi(X, Y) = \rho(JX, Y)$. If $d\Phi = 0$, the manifold is called Kählerian.

2.2 Riemann surfaces

We will take a closer look at the one-dimensional case where we can use some results from one dimensional complex analysis.

Definition 2.5. *A Riemann surface is a 1-dimensional, connected complex manifold.*

Since we are now in the one dimensional case, all biholomorphic mappings are conformal, and if $M \cong N$, we say that the Riemann surfaces are *conformally equivalent*.

Example 2.6.

- $\mathbb{D}, \mathbb{C}, \bar{\mathbb{C}}$, and open connected subsets of them, are Riemann surfaces. By the Riemann mapping theorem, we have that if an open simply connected subset $U \subseteq \bar{\mathbb{C}}$ has more than two boundary points, then $U \cong \mathbb{D}$. It is called a *hyperbolic* domain. More about classification of Riemann surfaces will be found later.
- The domains $C_R = \{\zeta \in \mathbb{C} \big| 1 < |\zeta| < R\}$, where $R > 1$, are all Riemann surfaces. Although they are diffeomorphic as real manifolds, $C_{R_1} \ncong C_{R_2}$, when $R_1 \neq R_2$. (see section 3.1, example 3.3).
- Let M be a compact Riemann surface. Since it is a compact orientable surface, it is homeomorphic to S^2 or a sum of tori. We say that a surface homeomorphic to the sphere has genus 0, and that it has genus g if it is the sum of g tori. S^2 admits only

one complex structure, the one given by $\bar{\mathbb{C}}$. However, there are infinitely many compact Riemann surfaces with genus g. If we consider the case of compact Riemann surfaces of genus one (which we will call complex tori), then all such Riemann surfaces are conformally equivalent to a surface on the form

$$\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}),$$

where $\tau \in \mathbb{C}_+$ with $\operatorname{Im}\tau \neq 0$. Futher more, if τ' is another such element, then

$$\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \cong \mathbb{C}/(\mathbb{Z}+\tau'\mathbb{Z})$$

if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$, with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$ (see [12] for more details).

- A Riemann surface M is called *finite* if $\pi_1(M)$, the first homotopy group, is finitely generated. Each finite Riemann surface is conformally equivalent to a domain on a compact Riemann surface. They are classified in the following way. We generalise the notion of genus to non-compact surfaces by saying that a surface is of *genus* g if g is the maximal number, such that there is a simple closed curve γ_1 in M, and curves γ_j in $M \setminus (\gamma_1 \cup \cdots \cup \gamma_{j-1})$, such that $M \setminus (\gamma_1 \cup \cdots \cup \gamma_g)$ is connected (i.e., we can make some g "circular" cuts on M, keeping it connected). An equivalent way of this definition is as follows. If $\bar{M}$ is the closure of M, as a subset of a compact manifold and the boundary of $\bar{M}$ has d components then

$$g := \frac{1}{2}(2 - \chi(M) - d)$$

where χ is the Euler characteristic of M. The orientability of all Riemann surfaces implies that

$$g = \operatorname{rank} H_1(M;\mathbb{Z}) - \operatorname{rank} H_0(\partial\bar{M};\mathbb{Z}),$$

where H_j is the j'th singular homology group. A surface is said to be of type (g, n, l) if

  - M has genus g;
  - M has n punctures: M has n disjoint open neighbourhoods conformally equivalent to $\mathbb{D}\setminus\{0\}$, with only one of its boundary components in M, and this one is a curve;
  - M has l hyperbolic boundary components: M has l disjoint open neighbourhoods conformally equivalent to C_R (for some R) with only one of its boundary components in M.

So $\bar{\mathbb{C}}$ is of type (0,0,0), $\mathbb{C}$ is of type (0,1,0), $\mathbb{D} \setminus \{0\}$ is of type (0,1,1) and the C_R is of type (0,0,2).

The Riemann mapping theorem has the following generalisation to Riemann surfaces.

Theorem 2.7 ([9]). *Let M be a simply connected Riemann surface. Then M is conformally equivalent to either $\bar{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{D}$.*

We want to use this theorem to classify all Riemann surfaces. First we need some concepts from Topology.

Definition 2.8. *Let X and Y be topological spaces. A continuous map $p : Y \to X$ is called covering if, for any $x \in X$, there is an open neighbourhood U of x such that $p^{-1}(U) = \coprod_{j\in J} V_j$ and $p|_{V_j}$ is a homeomorphism for any j.*

The covering map $p : Y \to X$ is called *universal* if it fulfils the following property. If $q : Z \to X$ is another covering map, then there is a covering map $f : Y \to Z$, such that $p = qf$. The space Y is then called the *universal cover* of X, and is unique up to a homeomorphism. If in addition X is Hausdorff and connected, then it is sufficient to require that Y is simply connected for p to be a universal covering map. Such a cover always exists for any Riemann surface (see [11]), and the surface can be given a complex structure in order to guarantee that it is a simply connected Riemann surface and p is holomorphic. By theorem 2.7, the following definition is consistent.

Definition 2.9. *Let M be a Riemann surface and let N be its universal cover. Then M is called*

- *elliptic if N is conformally equivalent to $\bar{\mathbb{C}}$.*
- *parabolic if N is conformally equivalent to $\mathbb{C}$.*
- *hyperbolic if N is conformally equivalent to $\mathbb{D}$.*

If we have two coverings $p : N \to M$ and $q : L \to M$, then a continuous map $f : N \to L$ is fibre preserving if $p = qf$. We will denote by $Deck(N/M)$ the group of fibre preserving automorphism on N under composition, with respect to some covering map $p : N \to M$. The elements of $Deck(N/M)$ are called deck transformations. If N is connected, then any nontrivial element of $Deck(N/M)$ has no fix points. If N is the universal covering of a Riemann surface M, then $Deck(N/M)$ is isomorphic to $\pi_1(M)$, and

$$N/Deck(N/M) \cong M.$$

Denote by $Aut(M) = \{f \in Hol(M, M) \big| \, f \text{ is bijective}\}$. If M is an elliptic Riemann surface, then $Deck(\bar{\mathbb{C}}/M)$ is a subgroup of $Aut(\bar{\mathbb{C}})$. $Aut(\bar{\mathbb{C}})$ is the group of Möbius transforms

$$\zeta \mapsto \frac{a\zeta + b}{c\zeta + d},$$

where $ad - bc = 1$, which may be identified with $\mathrm{PGL}_n\mathbb{C} = \mathrm{GL}_n\mathbb{C}/\sim$, where the equivalence relation is given by $A \sim \lambda A$, for $A \in \mathrm{GL}_n\mathbb{C}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Since all nontrivial Möbius transforms has at least one fixed point, there is no possibility for any nontrivial deck transformation to exist and hence, $\bar{\mathbb{C}}$ is the only elliptic Riemann surface.

Similar observations for

$$Aut(\mathbb{C}) = \{\zeta \mapsto \lambda\zeta + \kappa \big| \, \lambda, \kappa \in \mathbb{C}, \, \lambda \neq 0\},$$

show that the only interesting subgroup of deck transformations is the group generated by $\zeta \mapsto \zeta + 1$ and the ones generated by $\zeta \mapsto \zeta + 1$ and $\zeta \mapsto \zeta + \tau$, with $\tau \in \mathbb{C}_+$, $\operatorname{Im}\tau \neq 0$ (and groups generated by a constant times these elements). It follows that $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$, and the torus with different complex structures are the only parabolic Riemann surfaces.

The rest are hence hyperbolic Riemann surfaces. $Aut(\mathbb{D})$ consists of Möbius transformations on the form

$$\zeta \mapsto \lambda\frac{\zeta - \kappa}{1 - \bar{\kappa}\zeta}$$

where $|\lambda| = 1$ and $\kappa \in \mathbb{D}$. This group may be identified with $\mathrm{PSL}_n\mathbb{R} = \mathrm{SL}_n\mathbb{R}/\sim$, where $A \sim -A$ for $A \in \mathrm{SL}_n\mathbb{R}$. Discrete subgroups G of $Aut(\mathbb{D})$ are called *Fuschian*. It turns out that if G is a torsion free Fuschian group, then $\mathbb{D}/G$ is a hyperbolic Riemann surface, and any hyperbolic Riemann surface is of this form. See [9] for more details.

Example 2.10. For the simplest case, where $\pi_1(M)$ is abelian, we only have the following possibilities.

- If $\pi_1(M) = 0$, then M is $\mathbb{D}$, $\mathbb{C}$ or $\bar{\mathbb{C}}$.
- If $\pi_1(M) = \mathbb{Z}$, then M is $\mathbb{D} \setminus \{0\}$, $\mathbb{C} \setminus \{0\}$ or C_R for some value of R. Note that $\mathbb{D} \setminus \{0\}$ and C_R for different values of R, are all given by the quotient of $\mathbb{D}$ by a group isomorphic to $\mathbb{Z}$, yet none of them are conformally equivalent.
- If $\pi_1(M) = \mathbb{Z} \times \mathbb{Z}$, then M is a complex torus.

2.3 Quadratic differentials on a Riemann surface

We remark that for a Riemann surface M, $\mathscr{M}M$ may be identified with $Hol(M, \bar{\mathbb{C}}) \setminus \{\infty\}$, where ∞ denotes a function constantly equal to infinity.

Definition 2.11. *A holomorphic (resp. meromorphic) quadratic differential on a Riemann surface $(M, \mathcal{A})$, is a mapping ϕ on $\mathcal{A}$, such that for every chart (z_α, U_α), a mapping $\phi_{z_\alpha} \in \mathscr{O}z_\alpha(U_\alpha)$ (resp. $\mathscr{M}z_\alpha(U_\alpha)$) is associated, so that*

$$\phi_{z_\beta}(z_\beta(r)) = \phi_{z_\alpha}(z_\alpha(r)) \cdot \Big(\frac{dz_\alpha}{dz_\beta}(r)\Big)^2, \quad \forall\, r \in U_\alpha \cap U_\beta.$$

We often denote a quadratic differential by $\phi(z)dz^2$ with respect to some chart z. A holomorphic quadratic differential may also be seen as an element in $\Gamma\big(\mathrm{Sym}^2(T^{1,0}M)^*\big)$. Unless otherwise mentioned, the quadratic differentials below are meromorphic. We call points on the Riemann surface *regular* regarding to a differential ϕ, if they are neither poles nor zeroes of ϕ.

Definition 2.12. *A trajectory of a quadratic differential ϕ, is a maximal regular (i.e., C^1 with non-vanishing derivative) curve γ on M, such that $\phi(z)dz^2 > 0$ on γ, i.e., for any local representation ϕ_{z_α} of ϕ, we have*

$$\phi_{z_\alpha} z_\alpha \gamma(t) \cdot \big((z_\alpha\gamma)'(t)\big)^2 > 0.$$

A maximal regular curve η, such that $\phi(z)dz^2 < 0$ on η, is called an orthogonal trajectory

Geometric interpretation of the behaviour of trajectories is expressed in the following theorem.

Theorem 2.13. *Let ϕ be a quadratic differential on M. For any regular point $r \in M$, there exists a chart (w, V) about r, such that $\phi_w \equiv 1$. Moreover, if $\tilde{w}$ is another such chart, then $\tilde{w} = \pm w + constant$.*

Proof. Let $\phi_z(z)dz^2$ be a local representation in a neighbourhood around r. Since r is regular, there is a neighbourhood V, where we may choose a single valued branch of $\sqrt{\phi_z(z)}$ (the sign $\pm$ comes from the choice of a branch). Let $w = \int \sqrt{\phi_z(z)}dz$ be the natural parameter, which is defined up to some choice of constant, and independent of the local parameter z chosen (by the definition of the quadratic differential). We know that

$$\phi_w(w) = \frac{\phi_z(z)}{(\frac{dw}{dz})^2} = \frac{\phi_z(z)}{\phi_z(z)} \equiv 1.$$

□

We call a θ-arc a maximal regular curve along which

$$\arg dw^2 = \arg \phi_z(z) dz^2 = \theta \text{ (constant)}.$$

A trajectory is clearly a 0-arc, and an orthogonal trajectory a π-arc, and it is easy to see that using w as a local parameter, trajectories and orthogonal trajectories form horizontal and vertical straight lines respectively. Since the change of charts is conformal, we get that no matter what local parametrisation we use, the orthogonal trajectories are indeed orthogonal to trajectories, whenever they intersect. We also remark that orthogonal trajectories become trajectories when we consider $-\phi$.

Trajectories may be also seen as solutions to the differential equation $\phi(z)\left(\frac{dz}{du}\right)^2 = 1$ for a real parameter u.

The following example is important in our further considerations.

Example 2.14. Let $E_{a;\lambda} = \{\zeta \in \mathbb{C} \big| \, (\frac{\mathrm{Re}}{a})^2 + (\frac{\mathrm{Im}}{a\sqrt{1-\lambda^2}})^2 = 1\}$ be the ellipse with the major axis a along the real line, and with the eccentricity λ. We find a conformal map from the unit disk onto this ellipse. Let us consider the quadratic differentials

$$\frac{dz^2}{(z^2 - r^{-2})(z^2 - r^2)} \qquad \text{and} \qquad \frac{-dw^2}{w^2 - 1},$$

where $0 < r < 1$, so that the unit circle and ellipses with foci at ± 1, respectively, are trajectories. In order to verify this, let us take $z = e^{i\theta}$, $\theta \in [0, 2\pi]$. Then

$$\frac{dz^2}{(z^2 - r^{-2})(z^2 - r^2)} = \frac{-e^{2i\theta} d\theta^2}{(e^{2i\theta} - r^{-2})(e^{2i\theta} - r^2)} = \frac{d\theta^2}{(r^2 + r^{-2}) - (e^{2i\theta} + e^{-2i\theta})}$$

$$= \frac{d\theta^2}{r^2 + r^{-2} - 2\cos 2\theta} \geq 0$$

due to $r^2 + r^{-2} - 2\cos 2\theta \geq r^2 + r^{-2} - 2 = (1/r - r)^2 > 0$. For the other differential, let us take $w = a\cos\theta + ib\sin\theta$, where $a^2 - b^2 = 1$. Then

$$\frac{-dw^2}{w^2 - 1} = \frac{(-a\sin\theta + ib\cos\theta)^2 d\theta^2}{1 - (a\cos\theta + ib\sin\theta)^2} = \frac{(a^2\sin^2\theta - b^2\cos^2\theta - 2abi\sin\theta\cos\theta) d\theta^2}{a^2 - b^2 - a^2\cos^2\theta + b^2\sin^2\theta - 2abi\sin\theta\cos\theta}$$

$$= \frac{(a^2\sin^2\theta - b^2\cos^2\theta - 2abi\sin\theta\cos\theta) d\theta^2}{a^2\sin^2\theta - b^2\cos^2\theta - 2abi\sin\theta\cos\theta} = d\theta^2$$

Hence we may solve the equation

$$\frac{\kappa^2 dz^2}{(z^2 - r^{-2})(z^2 - r^2)} = \frac{-dw^2}{w^2 - 1},$$

where $\kappa \in \mathbb{R}_+ \setminus \{0\}$ and the solution will be the desirable conformal mapping of the unit disk onto the ellipse. The reason for the constant κ appearing, is trajectory structure of a quadratic differential is invariant under multiplication by such positive constants. It will be found when satisfying additional normalising conditions. We solve the equation for $w = f(z)$ with initial conditions $f(z) = 0, f(r) = 1$. We will let $\mathbf{F}(x, k)$ denote the elliptic integral

$$\mathbf{F}(x, k) = \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}},$$

and let $\mathbf{F}(1,k)=\mathbf{K}(k)$. We get

$$\int_0^{f(z)}\frac{idw}{\sqrt{w^2-1}}=i\log(-i(f(z)+\sqrt{f(z)^2-1})=\int_0^z\frac{\kappa dz}{\sqrt{(z^2-r^{-2})(z^2-r^2)}}$$

$$=\kappa\int_0^z\frac{dz}{\sqrt{(r^2z^2-1)(\frac{z^2}{r^2}-1)}}=\kappa r\int_0^{z/r}\frac{dt}{\sqrt{(1-t^2)(1-r^4t^2)}}=\kappa r\mathbf{F}(\frac{z}{r},r^2)$$

Applying the condition $f(r)=1$, we get that $\kappa r=\frac{\pi}{2\mathbf{K}(r^2)}$. Let $\varsigma(z)=\frac{\pi\mathbf{F}(\frac{z}{r},r^2)}{2\mathbf{K}(r^2)}$. Then we obtain the equation $f(z)+\sqrt{f(z)^2-1}=ie^{-i\varsigma(z)}$, solving which we get

$$f(z)^2-1=-e^{-2i\varsigma(z)}-2ie^{-i\varsigma(z)}f(z)+f(z)^2,$$

and finally,

$$f(z)=\frac{1}{2i}(e^{i\varsigma(z)}-e^{-i\varsigma(z)})=\sin\Big(\frac{\pi\mathbf{F}(\frac{z}{r},r^2)}{2\mathbf{K}(r^2)}\Big)$$

This can be easily generalised to that an arbitrary ellipse. Explicitly, let

$$\mathfrak{r}^{-1}(x)=\frac{1}{\sin(\frac{\pi\mathbf{F}(\frac{1}{x},x^2)}{2\mathbf{K}(x^2)})}.$$

Then the univalent mapping of the unit disk onto an ellipse with the major axis a, and eccentricity λ is given by

$$\mathcal{E}_{a;\lambda}(z)=a\lambda\sin\Big(\frac{\pi\mathbf{F}(\frac{z}{r},r^2)}{2\mathbf{K}(r^2)}\Big),\qquad\text{where}\quad r=\mathfrak{r}(\lambda)$$

The difficult part of this formula, is that, there is no simple way to derive r as a function of λ by an explicit formula. This mapping will be used in further examples.

The local trajectory structure about a regular point is trivial, but can be more complex about the critical points of our differential (its zeroes and poles). The natural parameter proved to be useful for study of the trajectory structure about a regular point. We continue to use it in a similar way for singular points.

Theorem 2.15 ([33]). *Let $s\in M$ be a critical point of order $n\in\mathbb{Z}$ of the quadratic differential ϕ.*

a) *Let n be odd. Then there exists a chart $\tilde{w}$ about s, such that ϕ has the local representation*

$$\phi_{\tilde{w}}(\tilde{w})=\big(\frac{n+2}{2}\big)^2\tilde{w}^n.$$

b) *Let n be even and positive. Then there exists a chart $\tilde{w}$ about s, such that ϕ has the local representation*

$$\phi_{\tilde{w}}(\tilde{w})=\big(\frac{n+2}{2}\big)^2\tilde{w}^n.$$

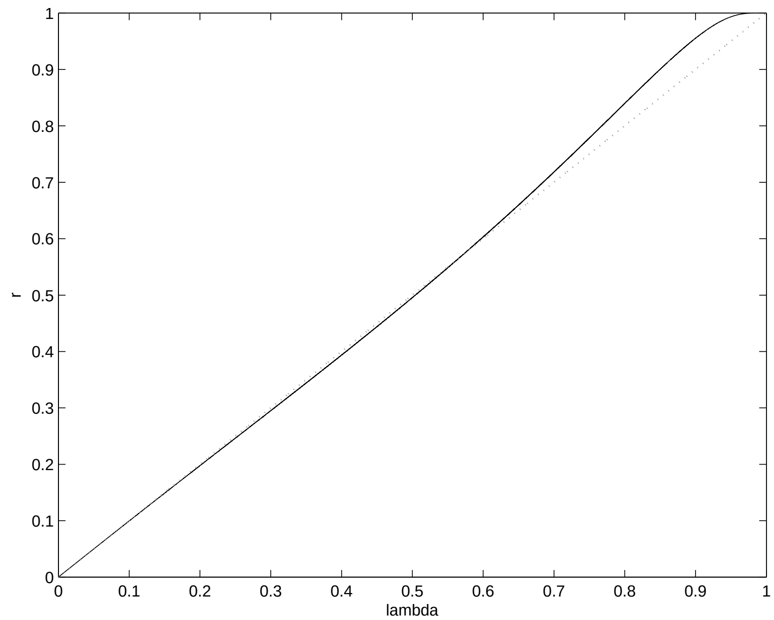

Figure 2.1: A graph of $\mathfrak{r}(\lambda)$ (black) compared to the identity (grey)

c) *Let $n = -2$. Let z be any chart about s, and let $\phi_z(z) = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + \cdots$. Then there exists a chart $\tilde{w}$ about s, such that ϕ has the local representation*

$$\phi_{\tilde{w}}(\tilde{w}) = \frac{a_{-2}}{\tilde{w}^2}.$$

d) *Let $n \leq -4$ and even. Then there exists a chart $\tilde{w}$ about s, such that ϕ has the local representation*

$$\phi_{\tilde{w}}(\tilde{w}) = (\frac{n+2}{2}\,\tilde{w}^{n/2} + \frac{b}{\tilde{w}})^2$$

for some constant b.

Proof. We will use the natural parameter $w = \int \sqrt{\phi_z(z)}dz$. In the following proof, w will not always be a single valued function, and never a chart, but we will still use it for practical reasons, since formally $dw^2 = \phi_z(z)dz^2$ is still valid.

a) Let z be some chart about the singular point s. We may assume $z(s) = 0$. Then ϕ may be represented as

$$\phi_z(z) = z^n(a_n + a_{n+1}z + \cdots)$$

where n is odd and $a_n \neq 0$. We may locally choose a branch of the square root of $a_n + a_{n+1}z + \ldots$. Let it be equal to

$$(a_n + a_{n+1}z + \ldots)^{1/2} = b_0 + b_1 z + \cdots$$

Let us put $\sqrt{\phi_z(z)} = z^{n/2}(b_0 + b_1 z + \cdots)$, with a chosen branch of $z^{n/2}$. We integrate and put

$$w = z^{\frac{n+2}{2}}(c_0 + c_1 z + \cdots), \qquad c_k = \frac{2b_k}{n + 2(k+1)}.$$

Let $d_0 + d_1 z + d_2 z^2 + \cdots = (c_0 + c_1 z + c_2 z^2 + \cdots)^{\frac{2}{n+2}}$, and let $\tilde{w} = z(d_0 + d_1 z + \cdots)$. Then $w = \tilde{w}^{\frac{n+2}{2}}$, for the chosen branch of the square root. After differentiating and squaring, we end up with a single valued expression

$$dw^2 = (\frac{n+2}{2})^2 \tilde{w}^n d\tilde{w}^2$$

b) We proceed in a similar way to a) without branching problems on the way.

c) Assume $\sqrt{\phi_z(z)} = z^{-1}(a_{-2} + a_{-1}z + a_0 z^2 + \cdots)^{1/2} = z^{-1}(b_0 + b_1 z + b_2 z^2 + \cdots)$, (since we may choose a branch of $(a_{-2} + a_{-1}z + a_0 z^2 + \cdots)^{1/2}$ locally). Integrating, we get $w = b_0 \log z + b_1 z + \frac{b_2}{2} z^2 + \cdots$. Let $\tilde{w} = z \cdot \exp\left(\frac{b_1}{b_0} z + \frac{b_2}{2b_0} z^2 + \cdots\right)$. Then

$$dw^2 = (\frac{b_0}{\tilde{w}} d\tilde{w})^2 = \frac{a_{-2}}{\tilde{w}^2} d\tilde{w}.$$

d) We may define a single valued branch as before $\sqrt{\phi_z(z)} = z^{n/2}(b_0 + b_1 z + \cdots)$. Integrating, we get $w = z^{n/2+1}(c_0 + c_1 z + \cdots) + b \log z$, where $(b_{-n/2-1} = b)$. Choose $\tilde{w} = z(d_0 + d_1 z + \cdots)$, such that $w = \tilde{w}^{n/2+1} + b\log(\tilde{w})$. This ends the proof.

$\square$

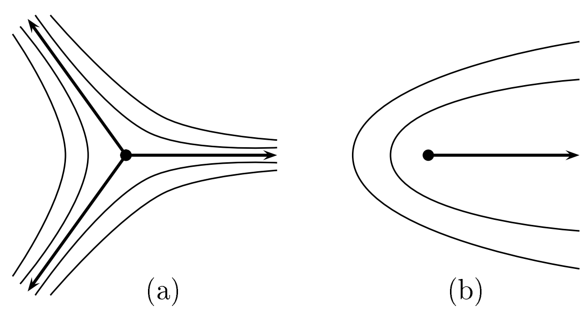

Figure 2.2: The local trajectory structure near (a) simple zero, (b) simple pole

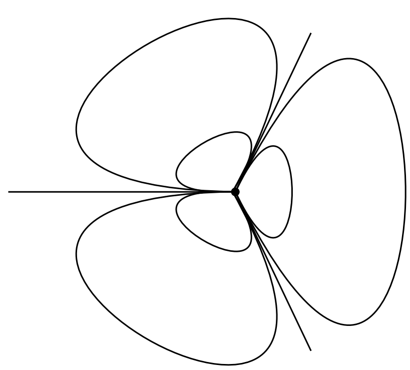

Figure 2.3: The local trajectory structure near a pole of 5-th order

Now let us proceed with the global trajectory structure.

Definition 2.16. *A conformally invariant metric on a Riemann surface $(M, \mathcal{A})$, is a mapping ρ on $\mathcal{A}$, that to every chart (z_α, U_α), associates a measurable function $\rho_{z_\alpha} : z_\alpha(U_\alpha) \to [0, \infty)$, satisfying the relation*

$$\rho_{z_\beta}(z_\beta(r)) = \rho_{z_\alpha}(z_\alpha(r))\left|\frac{dz_\alpha}{dz_\beta}(r)\right| \quad \forall\, r \in U_\alpha \cap U_\beta$$

We often denote a conformally invariant metric by $\rho(z)|dz|$ with respect to some chart z. To a quadratic differential ϕ, we associate a conformally invariant metric $\sqrt{|\phi(z)|}|dz|$. A trajectory is called *finite*, if it has finite length in this metric.

We say that an oriented trajectory is a *trajectory ray.*

Definition 2.17. *For a trajectory ray $\gamma^+ : I \to M$, where I is an interval with $b = \sup I$, we denote $I_t = \{u \in I | u > t\}$ and associate the limit set*

$$A_{\gamma^+} = \lim_{t \to b} \overline{\gamma^+(I_t)}$$

Let γ be a trajectory of a quadratic differential on M, with trajectory rays γ^+ and γ^-.

- *If $A_{\gamma^+} = \emptyset$, then γ^+ is called a boundary ray. If both γ^+ and γ^- are boundary rays, γ is called a cross cut.*

- *If $A_{\gamma^+} = s$ for some $s \in M$, then γ^+ is called a critical ray and s is then either a zero or a pole. If γ^+ or γ^- are critical rays, γ is called a critical trajectory.*

- *If A_{γ^+} consists of more than one point, then γ^+ is called a divergent ray. If $A_{\gamma^+} = \overline{\gamma(I)}$, then γ is called a spiral*

These critical trajectories form different kinds of global structure on the Riemann surface.

Definition 2.18. *Let ϕ be a quadratic differential on a bordered Riemann surface M, and let U be a domain in M. Then we define the following types of domains associated to ϕ (all trajectories and singularities described are with respect to ϕ).*

Ring Domain: *U is a maximal doubly connected domain, such that any non-critical closed trajectory intersecting U, is contained in U.*

Circular Domain: *U is a maximal hyperbolic domain which contains a second order pole s_0, such that $U \setminus \{s_0\}$ is a ring domain.*

Strip Domain: *U is a maximal hyperbolic domain such that ∂U contains two second order poles s_1 and s_2, and any trajectory passing through some point $r \in U$, connects s_1 and s_2.*

Ending Domain: *U is a maximal hyperbolic domain such that ∂U contains a third or higher order pole s_0, and any trajectory that starts and ends at s_0, and intersects U is contained in U.*

Spiral Domain: *U can be described in the following way. Let γ be a spiral which is not closed. Then $U = \operatorname{int} A_{\gamma^+}$.*

Quadrangle: *U is a maximal hyperbolic domain such that it contains no singularities, any trajectory that intersects U is in U, and ∂U contains two non-intersecting connected components that lie in ∂M*

Triangle: *U is a maximal hyperbolic domain such that ∂U contains a second order pole s_0, and it intersects ∂M. This intersection is connected, and any trajectory that intersects U is in U and connects s_0 and ∂M.*

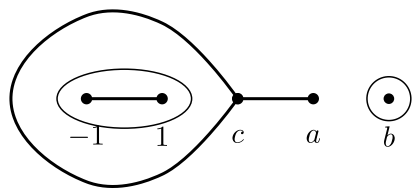

Figure 2.4: The trajectory structure of the differential $\frac{(c-z)dz^2}{(z-b)^2(z^2-1)(z-a)}$ on the Riemann surface $S_0 = \overline{\mathbb{C}} \setminus \{-1, 1, a, b\}$, $1 < c < a < b$

From the definitions we know that

- Except for parts that coincide with ∂M, ∂U consists of critical trajectories for any of the domains above.
- Ring domains contain no singularities.
- Ring domains, Circular domains, Strip domains, and Quadrangles may be conformally mapped on to an annulus, a circle, a strip, and a rectangle respectively.

If the trajectory structure of a quadratic differential gives spiral domains, we say that it possesses dense structure. Dense structure happens more often than other types.

Example 2.19. Consider quadratic differential

$$\phi_\alpha(z) = \frac{e^{i\alpha} dz^2}{(z^2-1)(z^2-2)}$$

Then it is known that the set $\{\alpha \in \mathbb{R} | \phi_\alpha \, \text{does not have dense structure}\}$ is countable [34].

For a general Riemann surface, we have very little control over the global structure. However, for the case of compact and finite Riemann surfaces (i.e., compact Riemann surfaces and their subsurfaces) we can, by some extra requirement, get some knowledge of what kind of domains may occur and how many they are.

Theorem 2.20 ([34])**.** *If ϕ is a quadratic differential on a compact Riemann surface M, then all domains mentioned in definition 2.18 (except for quadrangles and triangles) may occur. However, if we require that the differential*

- *ϕ has a finite L^1-norm, then only ring and spiral domains may occur.*

- *ϕ has only finite trajectories, then only ring, strip, circular, and ending domains may occur.*

- *ϕ has a finite L^1-norm and have only finite trajectories, then only ring domains may occur.*

Theorem 2.21 (Basic Structure Theorem)**.** *Let M be a finite Riemann surface, and $\bar{M}$ its closure, when seen as a subset of a compact manifold. Let ϕ be a quadratic differential on M, such that its extension to $\bar{M}$ satisfies $\phi_z(z(s)) \geq 0$ for any $s \in \partial\bar{M}$ and some (hence any) chart z. Let $\mathcal{C}$ be the union of all critical trajectories. If (M, ϕ) is not conformally equivalent to the following cases*

- *$M = \bar{\mathbb{C}}$ and $\phi_z = 1$, where z is the identity chart on $\mathbb{C}$.*

- *$M = \bar{\mathbb{C}}$ and $\phi_z = \frac{\kappa e^{i\alpha}}{z^2}$, $\alpha, \kappa \in \mathbb{R}$, $\kappa > 0$, where z is the identity chart on $\mathbb{C} \setminus \{0\}$.*

- *M is a torus, and ϕ is regular on M (ϕ has no singular points).*

Then

1. *M - $\bar{\mathcal{C}}$ consists of a finite number of ending, strip, circular, and ring domains. Any such a domain, is bounded by the boundary components of M and a finite number of critical trajectories. Each boundary component of critical trajectories contains a critical point.*

2. *Every pole of order $m > 2$ has a neighbourhood covered by the inner closure of $m - 2$ end domains and a finite number (may be zero) of possible strip domains. A second order pole has a neighbourhood which is contained in a circular domain or it is covered by $\operatorname{int} \bigcup_{j=1}^{k} \bar{V}_j$, where the V_j are strip domains.*

3. *$\operatorname{int} \bar{\mathcal{C}}$ consists of a finite number of domains, with a finite number of boundary components.*

Proof of theorem may be found in [18].

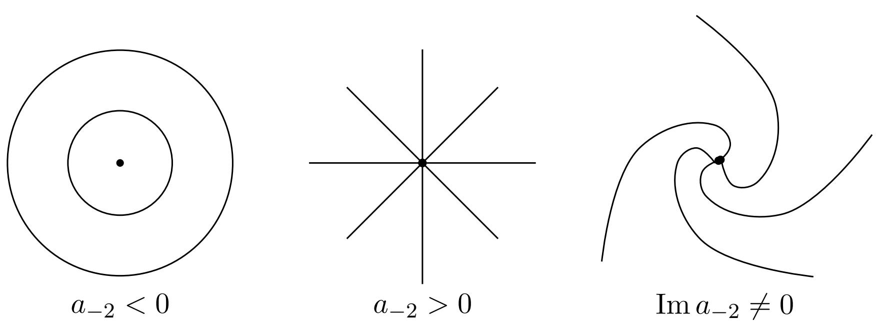

Figure 2.5: The local trajectory structure near a double pole illustrates part of 2. in theorem 2.21

Chapter 3

Quasiconformal mappings and the reduced modulus

The famous length-area principle (see. e.g., [15]) permits us to consider families of curves taking into account both the length of these curves and the area of the surface they sweep out. This principle implicitly formulated by Grötzsch in earlier 30's then took a form of extremal length by Ahlfors and Beurling and of the modulus of a family of curves by Jenkins. The latter considered a dual problem of the modulus of a family of curves and the extremal partitioning of a Riemann surface by non-intersecting domains of certain size. In order to define such size one uses the concepts of the conformal modulus of an annulus and the conformal radius of a simply connected domain. First they were defined for plane domains and then, developed for arbitrary Riemann surfaces. The conformal radius (or the reduced modulus) is not a conformal invariant unlike conformal modulus. But its distortion under conformal map is well defined. It changes proportionally to the derivative of the mapping at the point where we measure this conformal radius. The distortion of the conformal radius under a quasiconformal map is not known and we will try to clear up this case. First we will consider the general case of moduli of families of curves and conformal maps, and then, turn to the quasiconformal mappings admitting conformal extension. In particular, we shall consider quasiconformally extendable conformal maps given by the Löwner-Kufarev equations. These equations where originally introduced by Löwner in order to study the behaviour of slit mappings. They were further generalised by Kufarev and later by Pommerenke to study general subordination chains.

3.1 Modulus of a family of curves

Definition 3.1. *Let N be a Riemann surface and let Γ be a family of rectifiable curves in N. Let P be a collection of conformally invariant metrics that are L^2 on N and such that, for any $\rho \in P$, we have that $\int_\gamma \rho \geq 1$ for any $\gamma \in \Gamma$. Then the modulus of N with respect to Γ, is*

$$m(N, \Gamma) = \inf_{\rho \in P} \iint_N \rho^2(z) d\sigma_z \tag{3.1}$$

where $d\sigma_z$ is an element of area.

An alternative way to define the modulus is the following. Let P' be the set of conformally invariant metrics which are L^2. Define $L_\rho(\Gamma) := \inf_{\gamma\in\Gamma}\int_\gamma \rho(z)|dz|$. Then the modulus is

$$m(N,\Gamma) = \inf_{\rho\in P'} \frac{\iint_X \rho^2(z)d\sigma_z}{\big(L_\rho(\Gamma)\big)^2}.$$

The first definition is more convenient for us to work with, but we mention the second because it illustrates how the length of curves and the area in some metric interplay in the definition of the modulus.

Moduli defined on Riemann surfaces have the following properties.

Proposition 3.2. *Let Γ be a family of curves on N, and let P be the collection of admissible metrics on Γ*

1. *If $\Gamma \subseteq \Gamma'$, then $m(N,\Gamma) \le m(N,\Gamma')$.*

2. *If $N \subseteq N'$, then $m(N,\Gamma) = m(N',\Gamma)$. In this sense, the modulus only depends on Γ.*

3. *If there is an extremal metric $\rho^* \in P$ such that*

$$m(N,\Gamma) = \iint_N \rho^{*2} d\sigma_z, \tag{3.2}$$

then ρ^* is almost everywhere unique, and $L_{\rho^*}(\Gamma) = 1$.

4. *If $f : N \to L$ is an injective conformal mapping, then $m(N,\Gamma) = m(f(N), f(\Gamma))$.*

Proof. 1. From the simple observation that if P and P' are the sets of admissible metrics for Γ and Γ', then $\Gamma \subseteq \Gamma'$ implies $P' \subseteq P$.

2. Since all curves are contained in N, all metrics in P can be extended to admissible metrics in N' with respect to Γ, by letting them vanish on $N' \setminus N$. Clearly, for determining the infimum over all integrals $\iint_{N'} \rho^2$, it is sufficient to consider metrics that vanish on $N' \setminus N$.

3. Assume that there are two admissible metrics ρ_1 and ρ_2 satisfying (3.2). Then $\frac{1}{2}(\rho_1 + \rho_2)|dz|$ is admissible, and we have that $\iint_N (\frac{\rho_1(z)+\rho_2(z)}{2})^2 d\sigma_z \ge m(N,\Gamma)$, but also

$$0 \le \iint_N (\frac{\rho_1(z)-\rho_2(z)}{2})^2 d\sigma_z = \iint_N \frac{\rho_1^2(z)+\rho_2^2(z)}{2} d\sigma_z - \iint_N (\frac{\rho_1(z)+\rho_2(z)}{2})^2 d\sigma_z$$

$$= m(N,\Gamma) - \iint_N (\frac{\rho_1(z)+\rho_2(z)}{2})^2 d\sigma_z \le 0$$

which can only hold if $\rho_1 = \rho_2$ almost everywhere.

In order to prove the second statement, let $L_{\rho^*}(\Gamma) = l \ge 1$. Then $\frac{1}{l}\rho^*(z)|dz|$ is admissible, and

$$m(N,\Gamma) \le \frac{1}{l^2}\iint_N (\rho^*(z))^2 d\sigma_z = \frac{1}{l^2} m(N,\Gamma) \le m(N,\Gamma)$$

implies that $l = 1$.

4. Let Q be the set of admissible metrics on $f(\Gamma)$. Then for every $\mu \in Q$ we associate a metric $\rho(z)|dz| = \mu(f(z))|f'(z)||dz|$. Since

$$\int_{\gamma\in\Gamma} \rho(z)|dz| = \int_{f(\gamma)\in f(\Gamma)} \mu(f(z))|df(z)| \geq 1$$

by the change of variable formula, $\rho \in P$. Since also

$$\iint_N \left(\rho(z)\right)^2 d\sigma_z = \iint_{f(N)} \left(\mu(f(z))\right)^2 d\sigma_{f(z)},$$

we have $m(N,\Gamma) \leq m(f(N), f(\Gamma))$. Reciprocal statement follows from similar argument, by considering f^{-1}.

□

We present here some important examples.

Example 3.3. • Let $Q_l = \{\zeta \in \mathbb{C} \big| \, 0 < \operatorname{Re}\zeta < l, 0 < \operatorname{Im}\zeta < 1\}$ be a rectangle, and let Γ be the set of curves that connect its horizontal sides. Then $m(Q_l, \Gamma) = l$.

Proof. The metric $|dz|$ is admissible, so $l = \iint_{Q_l} dx\,dy \geq m(Q_l,\Gamma)$. We know that for any $\rho \in P$, $\int_0^1 \rho(x+iy)dy \geq 1$, so we have

$$\iint_{Q_l} \rho(z)dx\,dy = \int_0^l \Big(\int_0^1 \rho(x+iy)dy\Big)dx \geq l.$$

Finally,

$$\begin{aligned} 0 \leq \iint_{Q_l} (\rho-1)^2 dx\,dy &= \iint_{Q_l} \rho^2 dx\,dy - 2\iint_{Q_l} \rho\,dx\,dy + l \\ &\leq \iint_{Q_l} \rho^2 dx\,dy - l, \end{aligned}$$

so we conclude that $l \leq m(Q_l, \Gamma)$. □

- More generally, let V be a simply connected domain on a Riemann surface, with ∂V to be a simple closed curve. Let r_1, r_2, r_3, r_4 be four different points on ∂V, indexed in the counterclockwise order. Let γ_{ij} be the part of ∂V connecting r_i and r_j. We pair γ_{12} with γ_{34}. Then $\big(V, (\gamma_{12}, \gamma_{34})\big)$ is called a *quadrilateral*. We will call $(\gamma_{12}, \gamma_{34})$ the *primary sides* of the quadrilateral (remark that this is not a standard term). To any quadrilateral, there is only one Q_l, such that there is an injective conformal function mapping V onto Q_l, and the primary sides onto the horizontal edges of the rectangle. If Γ is the family of all curves connecting these primary sides, then $m(V,\Gamma) = l$. Since the modulus of quadrilateral is often in use we shall denote it by $M(V)$.

- Let Γ be the set of curves in C_R, that separate the boundaries. Then

$$m(C_R, \Gamma) = \frac{1}{2\pi} \log R \tag{3.3}$$

Proof. The metric $\frac{|dz|}{2\pi|z|}$ is admissible, so $\frac{1}{2\pi}\log R = \iint_{C_R} \frac{1}{(2\pi|z|)^2} dxdy \geq m(C_R, \Gamma)$. For any $\rho \in P$ and $1 < r < R$, we have $\frac{1}{r} < 1 \leq \int_0^{2\pi} \rho(re^{i\theta})d\theta$. So

$$\log R \leq \int_1^R \Big(\int_0^{2\pi} \rho(re^{i\theta})d\theta \Big) dr = \iint_{C_R} \frac{\rho(z)}{|z|} d\sigma_z,$$

and

$$0 \leq \iint_{C_R} \Big(\frac{1}{2\pi|z|} - \rho(z)\Big) d\sigma_z = \frac{1}{2\pi}\log R - \frac{1}{\pi}\iint_{C_R} \frac{\rho(z)}{|z|} d\sigma_z + \iint_{C_R} \rho^2(z) d\sigma_z$$

$$\leq \iint_{C_R} \rho^2(z) d\sigma_z - \frac{1}{2\pi}\log R.$$

Therefore, $\frac{1}{2\pi}\log R \leq m(C_R, \Gamma)$. □

We observe that $C_{R_1} \ncong C_{R_2}$ whenever $R_1 \neq R_2$, because the modulus is a conformal invariant by Prop 3.2.

- More generally, for any doubly connected hyperbolic Riemann surface V, there is a unique $R \in (1, \infty]$, such that V may be mapped conformally onto C_R (note that $C_\infty \cong \mathbb{D} \setminus \{0\}$). Then if Γ is the set of curves that separate the boundaries, then $m(V, \Gamma) = \frac{1}{2\pi}\log R$. This is also one the most used modulus for doubly connected domains, we will denote it by $M(V)$.

Definition 3.4. *Let Ω be a simply connected hyperbolic domain in $\bar{\mathbb{C}}$. Let $f : \Omega \to \mathbb{D}$ be the Riemann mapping of Ω with respect to $s \in \Omega$, $s \neq \infty$ (i.e. f is bijective and conformal, $f(s) = 0$, $f'(s) > 0$). Then we define the conformal radius of Ω with respect to s by*

$$R(\Omega, s) = \frac{1}{f'(s)}$$

In other words, $R(\Omega, s)$ is defined such that there is a bijective conformal map $g : \Omega \to \{\omega \in \mathbb{C} \big| \, |\omega| < R(\Omega, s)\}$, with the Taylor expansion $g(\zeta) = (\zeta - s) + a_2(\zeta - s)^2 + \cdots$. We may extend this definition to the case $s = \infty$ by defining $R(\Omega, \infty)$ so that there exists a bijective conformal map $g : \Omega \to \{\omega \in \bar{\mathbb{C}} \big| \, |\omega| > R(\Omega, \infty)\}$ with the Taylor expansion $g(\zeta) = \zeta + b_0 + \frac{b_1}{\zeta} + \cdots$. Conformal radius is connected with a limiting case of the modulus of a doubly connected domain in the following way.

Definition 3.5. *Let Ω be a simply connected hyperbolic domain in $\bar{\mathbb{C}}$. and let $s \in \Omega$, $|s| < \infty$. Let $\Omega_\varepsilon = \Omega \setminus \{\zeta \in \mathbb{C} \big| \, |s - \zeta| \leq \varepsilon\}$ be defined for ε small enough, such that Ω_ε is doubly connected. We define the reduced modulus of Ω with respect to s by*

$$m(\Omega, s) = \lim_{\varepsilon \to 0} \Big(M(\Omega_\varepsilon) + \frac{1}{2\pi}\log \varepsilon \Big).$$

If $s = \infty$, then we define

$$m(\Omega, \infty) = m(\varphi(\Omega), 0)$$

where $\varphi : \Omega \to \bar{\mathbb{C}}$ is given by $\zeta \mapsto \frac{1}{\zeta}$.

Theorem 3.6. *Let Ω be a simply connected hyperbolic domain, $s \in \Omega$, $|s| < \infty$. If $R(\Omega, s)$ is the conformal radius of Ω with respect to s, then the reduced modulus $m(\Omega, s)$ exists and*

$$m(\Omega, s) = \frac{1}{2\pi} \log R(\Omega, s).$$

If $s = \infty$, then

$$m(\Omega, \infty) = -\frac{1}{2\pi} \log R(\Omega, \infty).$$

Proof. We have to prove the theorem only for $|s| < \infty$. Let $\zeta = f(z) = z - s + b_2(z-s)^2 + \cdots$ be the Riemann mapping from Ω onto the disk of radius $R(\Omega, s)$. Then

$$\{\zeta \in \mathbb{C} \big| \, \varepsilon(1+|b_2|\varepsilon+o(\varepsilon)) < |\zeta| < R(\Omega, s)\} \subseteq f(\Omega_\varepsilon) \subseteq \{\zeta \in \mathbb{C} \big| \, \varepsilon(1-|b_2|\varepsilon-o(\varepsilon)) < |\zeta| < R(\Omega, s)\}.$$

From (3.3) we have that

$$\frac{1}{2\pi} \log \frac{R(\Omega, s)}{\varepsilon(1 + |b_2|\varepsilon + o(\varepsilon))} \leq M(\Omega_\varepsilon) = M(f(\Omega_\varepsilon)) \leq \frac{1}{2\pi} \log \frac{R(\Omega, s)}{\varepsilon(1 - |b_2|\varepsilon - o(\varepsilon))}.$$

Adding $\frac{1}{2\pi} \log \varepsilon$ to all sides of this inequality and taking the limit as $\varepsilon \to 0$, this leads to the result. $\square$

From these results, it follows immediately that the reduced modulus is not conformally invariant.

Corollary 3.7. *Let $f : \Omega \to \bar{\mathbb{C}}$ be an injective conformal map. Let $s \in \Omega$ be such that $|s| < \infty$, $|f(s)| < \infty$. Then,*

$$m(f(\Omega), f(s)) = m(\Omega, s) + \frac{1}{2\pi} \log |f'(s)|.$$

We use this formula to define the reduced modulus for a general simply connected domain on a Riemann surface.

Example 3.8.

- Clearly $m(\mathbb{D}, 0) = 0$. For any other $s \in \mathbb{D}$, we can use a Möbius transform $\tau_s(z) = \frac{z+s}{1+\bar{s}z}$ of the unit disk, mapping 0 to s. Since $\tau_s(z)' = \frac{1-|s|^2}{(z\bar{s}+1)^2}$, we conclude that

$$m(\mathbb{D}, s) = \frac{1}{2\pi} \log |\tau_s'(0)| = \frac{1}{2\pi} \log(1 - |s|^2). \tag{3.4}$$

- Let us consider the reduced modulus of the ellipse with the major axis a, and the eccentricity λ. From (3.4) we get that

$$m(E_{a;\lambda}, s) = \frac{1}{2\pi} \log(1 - |\omega|^2)\mathcal{E}'_{a;\lambda}(\omega)$$

$$= \frac{1}{2\pi} \log a + \frac{1}{2\pi} \log \frac{\pi\lambda(1 - |\omega|^2)}{2r\mathbf{K}(r^2)} \left| \frac{\cos\left(\frac{\pi \mathbf{F}(\frac{\omega}{r}, r^2)}{2\mathbf{K}(r^2)}\right)}{\sqrt{(1 - (r^{-2} + r^2)\omega^2 + \omega^4)}} \right|,$$

where $\omega = \mathcal{E}^{-1}_{a;\lambda}(s)$ and $r = \mathfrak{r}(\lambda)$. Note that given s in the expression above, the first term depends only on a and the second one only on λ. In particular, the reduced modulus with respect to the centre and foci (ω equal to respectively 0 and r) becomes

$$m(\Omega, 0) = \frac{1}{2\pi} \log a + \frac{1}{2\pi} \log \frac{\pi\lambda}{2r\mathbf{K}(r^2)},$$

and

$$m(\Omega, a\lambda) = \frac{1}{2\pi}\log a + \frac{1}{2\pi}\log\left(\frac{\pi}{2\mathbf{K}(r^2)}\right)^2 \frac{\lambda}{r(1+r^2)}.$$

We include here, one of the most important tools when working with moduli.

Theorem 3.9 (Grötsch Lemmas). *1. Let $V_1, ..., V_n$ be non-intersecting quadrilaterals in Q_l with primary sides on the opposite horizontal sides of V. Then,*

$$M(Q_l) \geq \sum_{j=1}^{n} M(V_j).$$

The equality is attained if and only if $\bigcup_{j=1}^{n} \overline{V_j} = Q_l$ and every V_j is a rectangle.

2. Let $V_1, ..., V_n$ be non-intersecting doubly connected domains in C_R, separating the boundary components of C_R. Then

$$M(C_R) \geq \sum_{j=1}^{n} M(V_j).$$

The equality is attained if and only if $\bigcup_{j=1}^{n} \overline{V_j} = C_R$, and all ∂V_j are concentric circles for any j.

The proof can found, e.g., in [34] . Letting $R \to \infty$ in the second Grötsch Lemma, we get a corollary.

Corollary 3.10. *Let V be a disk centred at the origin, split into domains $V_1, ..., V_n$ by curves separating the boundary and 0. Let V_1 be the domain which contains 0. Then,*

$$m(V, 0) \geq m(V_1, 0) + \sum_{j=2}^{n} M(V_j).$$

The equality is attained if and only if ∂V_j are concentric circles for $2 < j < n$.

The definition of the modulus can also be extended to several families of curves. We will use the sign $\simeq$ to denote free homotopy of curves.

Definition 3.11. *Let N be a (bordered) Riemann surface, with ∂N consisting of hyperbolic boundaries. Then:*

- *A curve γ_j in N is said to be of type I if it is a simple loop not null homotopic.*
- *A curve γ_j in N is said to be of type II if it is a simple arc ending on boundary and not null homotopic.*
- *A collection of curves $\gamma = \{\gamma_1, ...\gamma_n\}$ is said to be an admissible system of curves, if all curves are disjoint, $\gamma_j \not\simeq \gamma_k$ for any $j \neq k$, and all curves are either of type I or type II.*

Given such an admissible system, and a non-zero vector $\alpha = (\alpha_1, \ldots, \alpha_m)$, let $\Gamma_j = \{\eta | \eta \simeq \gamma_j\}$, and let $\Gamma = \{\Gamma_1, \ldots, \Gamma_m\}$. Admissible metrics consist of all L^2 metrics which, for all j and any $\gamma \in \Gamma_j$, satisfy the inequality

$$\int_\gamma \rho(z)|dz| \geq \alpha_j.$$

Let P be the collection of all admissible metrics. Then the modulus of Γ with respect to N and α is defined by

$$m(N, \Gamma, \alpha) = \inf_{\rho \in P} \iint_N \rho^2(z) d\sigma_z.$$

Observe that if we let α be a vector with 1 at the j'th place, and the resting coordinates zero, then $m(N, \Gamma, \alpha) = m(N, \Gamma_j)$.

It is known that in these types of problems, there is always a unique extremal metric. This metric is on the form $\rho^*(z)|dz| = \sqrt{\phi(z)}|dz|$, where $\phi(z)dz^2$ is a quadratic differential with finite trajectories.

To each such problem, we may associate collections of domains. If N be a Riemann surface and if Γ is a collection of homotopy classes, as above, then we may associate a collection of non-intersecting connected domains $\mathscr{V} = \{V_1, V_2, \ldots, V_n\}$ which satisfies the following properties:

- if Γ_j is of type I, then V_j is a doubly connected domain with ∂V consisting of curves from Γ_j;
- if Γ_j is of type II, then V_j is a quadrilateral with primary sides on the boundary of N and the other pair of sides are from Γ_j.

The following important result make a connection between two dual problems: the modulus problem as the infimum of certain reduced area, and the problem of extremal partition of a Riemann surface by non-intersecting domains of certain type.

Theorem 3.12 ([34]). *If $\mathscr{V}$ is any collection of domains associated with Γ and if $\alpha = (\alpha_1, ..., \alpha_n)$ is a non-zero vector, then*

$$\sum_{j=1}^{n} \alpha_j M(V_j) \leq m(N, \Gamma, \alpha).$$

Equality is given by an extremal collection $\mathscr{V}^*$, consisting of domains which are either ring domains or quadrangles of the extremal quadratic differential.

3.2 Quasiconformal mappings

We will give two equivalent definitions of a K-quasiconformal (shortened to K-q.c.) mapping, where $K \in [1, \infty)$.

Definition 3.13. *(geometric definition) A homeomorphism $f : \Omega \to \Omega'$, where Ω and Ω' are domains in $\bar{\mathbb{C}}$, is called K-q.c. if, for every quadrilateral $Q \subseteq \Omega$, the following inequality*

$$M(f(Q)) \leq KM(Q) \tag{3.5}$$

holds.

This definition is nice for determining some properties of quasiconformal maps.

- Composition of a K_1-q.c. map and a K_2-q.c. map, is a K_1K_2-q.c. map.
- The inequality (3.5) implies that $K^{-1}M(Q) \le M(f(Q)) \le KM(Q)$. This follows from the fact that if Q is any quadrilateral and $\tilde{Q}$ is the same quadrilateral, with permutated primary sides, then $M(\tilde{Q}) = \big(M(Q)\big)^{-1}$. From this, it follows that the inverse to a K-q.c. map is K-q.c.
- A 1-q.c. map is conformal.
- If a mapping f is such that for every point in Ω there is a neighbourhood where f is K-q.c., then f is a K-q.c. map on Ω.
- If V is a doubly connected domain, then $K^{-1}M(V) \le M(f(V)) \le KM(V)$.

However, this geometric definition can often in practice, be hard to check. Let us therefore present the notion of quasiconformality from a more analytic point of view.

Definition 3.14. *A function $g(x)$ is absolutely continuous on an interval $I = [a, b]$, if for every $\epsilon > 0$, there exists a $\delta > 0$, such that for any finite set of disjoint intervals $\{(a_i, b_i)\}_{i=1}^n$ in I, we have*

$$\sum_{i=1}^{n}(b_i - a_i) < \delta \quad \Rightarrow \quad \sum_{i=1}^{n}|f(b_i) - f(a_i)| < \epsilon.$$

Clearly absolute continuity implies continuity. Apart from this definition, absolute continuity can be checked in the following way.

Theorem 3.15 ([30])**.** *Let $I = [a, b]$, and let $f : I \to \mathbb{R}$ be continuous and non-decreasing. Then the following statements are equivalent:*

a) f is absolutely continuous on I;

b) If $K \subset I$ has measure 0, then $f(K)$ has measure 0 (both with respect to the Lebesgue measure);

c) f is differentiable a.e. on I, $f' \in L^1(I)$, and for any $x \in I$,

$$f(x) - f(a) = \int_a^x f'(t)dt$$

A function $u(x, y)$ is called *absolutely continuous on lines* (ACL) in Ω, if for every closed rectangle $R \subseteq \Omega$, with the sides parallel to the x- and y-axes, then $u(x, y)$ is absolutely continuous on a.e. horizontal and vertical lines in R. Note that ACL implies the existence of partial derivatives a.e. in Ω.

Definition 3.16. *(Analytic definition) Let Ω and Ω' are domains. A homeomorphism $f : \Omega \to \Omega'$ is a K-q.c. map if*

- *f is ACL in Ω;*
- $|f_{\bar{z}}| \overset{a.e.}{\le} \frac{K-1}{K+1}|f_z|$.

The equivalence of these definitions is shown in [1]. It is also common to call such a mapping k-q.c., where $k = \frac{K-1}{K+1}$, instead of K-q.c. Obviously, $k \in [0,1)$, and $k = 0$ implies that the mapping is conformal. If a mapping is C^1, it is quasiconformal if $\frac{|f_{\bar{z}}|}{|f_z|} \leq k$. We say that $\mu_f = \frac{f_{\bar{z}}}{f_z}$ (which is almost everywhere defined for a quasiconformal mapping) is the Beltrami coefficient for the quasiconformal mapping. For the composition of two quasiconformal mappings we have the following relation for the Beltrami coefficient:

$$\mu_g f = \frac{\frac{\partial}{\partial z} f}{\frac{\partial}{\partial \bar{z}} \bar{f}} \cdot \frac{\mu_{gf} - \mu_f}{1 - \mu_f \cdot \mu_{gf}}. \tag{3.6}$$

If g is conformal ($\mu_g = 0$), then $\mu_{gf} = \mu_f$, so the solution to the Beltrami equation

$$h_{\bar{z}} = \mu_f \cdot h_z \tag{3.7}$$

is equal to f composed with a comformal map. Moreover, for any measurable complex valued function μ on the domain Ω, with $\|\mu\|_\infty \leq k < 1$, there is a k-q.c mapping of Ω, satisfying (3.7). Quasiconformal maps transform infinitesimal circles to infinitesimal ellipses with the ratio of the major and minor axis less than K.

3.3 Distortion of the reduced modulus under q.c. mappings

Let f be a C^1 K-q.c. injective mapping from the unit disk $\mathbb{D}$, onto Ω, with $f(0) = 0$. We know that it is possible to find estimates for the modulus of doubly connected domains under q.c. maps, and we also know, that if f is conformal, it is sufficient to know $f'(0)$, to define $m(\Omega, 0)$. It seems to be reasonable to expect that knowing $|f_z|(0)$ or perhaps the Jacobian $\mathcal{J}_f(0) = |f_z|^2(0) + |f_{\bar{z}}|^2(0)$, we will be able to determine an interval for $m(\Omega, 0)$, depending only on K.

We will try to use the Grötzsch Lemma to estimate $m(\Omega, 0)$. Let us split $\mathbb{D}$ into two parts $\mathbb{D}_\varepsilon = \{z \in D \big| \, |z| > \varepsilon\}$ and $E_{\varepsilon;0}$ as defined in example 2.14. Let $\Omega_\varepsilon = f(\mathbb{D}_\varepsilon)$. The domain $f(E_{\varepsilon;0})$ approaches an ellipse for small ε. Let $K_0 = \frac{|f_z|(0)+|f_{\bar{z}}|(0)}{|f_z|(0)-|f_{\bar{z}}|(0)}$. As ε becomes smaller, $f(E_{\varepsilon;0})$ approaches the ellipse with the major axis $a_0(\varepsilon)$ and the eccentricity $\lambda_0 = \sqrt{1 + K_0^{-2}}$. We can use the fact that the Jacobian is a local enlargement factor to determine $a_0(\varepsilon)$. This gives us

$$\mathcal{J}_f(0)\pi\epsilon^2 = (|f_z|^2(0) + |f_{\bar{z}}|^2(0))\pi\epsilon^2 = \frac{\pi a_0^2}{K_0},$$

so that $a_0 = (|f_z|(0) + |f_{\bar{z}}|(0))\varepsilon$ and $\lambda_0 = 2\frac{\sqrt{|f_z|(0)|f_{\bar{z}}|(0)}}{|f_z|(0)+|f_{\bar{z}}|(0)}$. Using the Grötzsch Lemma for small ε, we get

$$m(\Omega, 0) \geq M(\Omega_\varepsilon) + m(E_{a_0;\lambda_0}, 0) = M(\Omega_\varepsilon) + \frac{1}{2\pi}\log\varepsilon + \frac{1}{2\pi}\log\frac{\pi\lambda_0(|f_z|(0) + |f_{\bar{z}}|(0))}{2\mathfrak{r}(\lambda_0)\mathbf{K}((\mathfrak{r}(\lambda_0))^2)}.$$

We see that the last part is independent of ε. The problem is that $\log\varepsilon^{-K^{-1}} \leq M(\Omega_\varepsilon) \leq \log\varepsilon^{-K}$ and we can not control the distortion as $\varepsilon \to 0$.

Proposition 3.17. *For any non-zero $a \in \mathbb{R}_+$, any $K > 1$, and for any $\kappa \in \mathbb{R}$, there is a K-q.c. map f, with $|f_z|(0) = a$ such that $m(f(\mathbb{D}), f(0)) = \kappa$.*

Proof. Assume $a = 1$. The K q.c. mapping

$$f(z;\varepsilon) = z(\frac{|z|}{\varepsilon})^{K-1} \qquad z \in \mathbb{D}_\varepsilon,$$

is such that the image of the domain $\mathbb{D}_\varepsilon$ has modulus $-\frac{K}{2\pi}\log\varepsilon$. It may be extended to the whole unit disk by

$$f(z;\varepsilon) = \begin{cases} z(\frac{|z|}{\varepsilon})^{K-1} & \text{if } 1 > |z| > \varepsilon, \\ z & \text{if } |z| \leq \varepsilon, \end{cases}$$

for $0 < \varepsilon < 1$. We further define $f(z;1) = id_{\mathbb{D}}(z)$, and $f(z;\frac{1}{\varepsilon}) = \varepsilon^{K-1}f^{-1}(z\varepsilon^{1-K};\varepsilon)$, such that for any $\varepsilon > 0$

$$m(f(\mathbb{D},\varepsilon),0) = \frac{1-K}{2\pi}\log\varepsilon$$

So we only need to show that all these mapping are quasiconformal, and it is sufficient to prove this for $0 < \varepsilon < 1$. $f(z;\varepsilon)$ is a homeomorphism (may be extended to a bijective mapping in the closed unit disk), and $|\frac{\partial f}{\partial \bar{z}}(z;\varepsilon)| \leq \frac{K-1}{K+1}|\frac{\partial f}{\partial z}(z;\varepsilon)|$ where they are defined (in $\mathbb{D}\backslash\{z\big|\,|z| = \varepsilon)$. The function $f(z;\varepsilon)$ is clearly absolutely continuous on line segments lying entirely in $\mathbb{D}_\varepsilon$ or $E_{\varepsilon,0}$. Let $I = [a,b] + ic$ be a horizontal line segment in first quadrant, such that $\sqrt{\varepsilon^2 - c^2} \in [a,b]$, that is, it crosses the circle $|z| = \varepsilon$. Let $\phi_1(x) = \operatorname{Re}(f|_I)$ and $\phi_2(x) = \operatorname{Im}(f|_I)$. We have

$$\phi_1(x) = \begin{cases} (\frac{\sqrt{x^2+c^2}}{\varepsilon})^{K-1}x & \text{if } x > \sqrt{\varepsilon^2 - c^2}, \\ x & \text{if } x \leq \sqrt{\varepsilon^2 - c^2}, \end{cases}$$

and

$$\phi_1'(x) = \begin{cases} \frac{(x^2+c^2)^{\frac{K-3}{2}}}{\varepsilon^{K-1}}(Kx^2 + c^2) & \text{if } x > \sqrt{\varepsilon^2 - c^2}, \\ 1 & \text{if } x < \sqrt{\varepsilon^2 - c^2}. \end{cases}$$

Since the derivative of ϕ_1 is positive, ϕ_1 is increasing. We clearly have that $\phi_1(x) - \phi_1(a) = \int_a^x \phi_1'(t)dt$, so $\phi_1(x)$ is absolutely continuous on I. Similar argument works for

$$\phi_2(x) = \begin{cases} c(\frac{\sqrt{x^2+c^2}}{\varepsilon})^{K-1} & \text{if } x > \sqrt{\varepsilon^2 - c^2}, \\ c & \text{if } x \leq \sqrt{\varepsilon^2 - c^2}, \end{cases}$$

and

$$\phi_2'(x) = \begin{cases} cx(K-1)\frac{(x^2+c^2)^{\frac{K-3}{2}}}{\varepsilon^{K-1}} & \text{if } x > \sqrt{\varepsilon^2 - c^2}, \\ 0 & \text{if } x < \sqrt{\varepsilon^2 - c^2}. \end{cases}$$

For I crossing $|z| = \varepsilon$ and lying in another quadrant we use the same arguments (switching orientation on I when necessary). The arguments for vertical lines are similar. Hence all the mappings are K-quasiconformal. ☐

Observe that these maps can even be made C^∞ by using a bump function.

3.4 The Löwner-Kufarev equation

We consider a chain of simply connected hyperbolic domains $\{\Omega(t)\}_{t=0}^{\tau}$ in $\mathbb{C}$, with $\tau \in (0,\infty]$, such that

- $t_1 < t_2 \Rightarrow \Omega(t_1) \subsetneq \Omega(t_2)$ (that is $\{\Omega(t)\}_{t=0}^{\tau}$ is a subordination chain)
- $0 \in \Omega(0)$
- $R(\Omega(t), 0) = e^t$ (for an arbitrary subordination chain, this can be achieved by a reparametrisation)

From Riemann mapping theorem we know, that for any fixed t, there exists a unique holomorphic univalent $f(\zeta, t) : \mathbb{D} \to \mathbb{C}$,

$$\zeta \mapsto e^t(\zeta + \alpha_2(t)\zeta^2 + \alpha_3(t)z^3 + \cdots),$$

such that $f(\mathbb{D}, t) = \Omega(t)$. A necessary condition for for a function to represent such a subordination chain, is that there exists an analytic regular function

$$p(\zeta, t) = 1 + p_1(t)\zeta + p_2(t)\zeta^2 + ... \qquad \zeta \in \mathbb{D},$$

with $\operatorname{Re} p(\zeta, t) > 0$ for almost every $(\zeta, t) \in \mathbb{D} \times [0, \tau)$, such that f is a solution to the equation

$$\frac{\partial f(\zeta, t)}{\partial t} = \zeta \frac{\partial f(\zeta, t)}{\partial \zeta} p(\zeta, t) \qquad \text{for } \zeta \in \mathbb{D} \text{ and for almost all } \ t \in [0, \tau). \tag{3.8}$$

This equation is called *the Löwner-Kufarev equation*. Let us denote the class of all such $p(z, t)$ by C . We introduce a parameter s, to solve the the above equation by the characteristic method with the initial condition $f(\zeta, 0) = f_0(\zeta)$, associated to some initial domain $\Omega(0) = \Omega_0$. Then

$$\frac{dt}{ds} = 1, \qquad \frac{d\zeta}{ds} = -\zeta\, p(\zeta, t), \qquad \frac{df_0}{ds} = 0, \tag{3.9}$$

with initial conditions $t(0) = 0$, $\zeta(0) = z$, $f(z, 0) = f_0(z)$ where $z \in \mathbb{D}$. Clearly $t = s$. However, we may only get solutions for ζ in some subdomain of $\mathbb{D}$. We therefore use $w(z.t)$ instead of ζ for solutions of (3.9). We end up with the following differential equation

$$\frac{dw}{dt} = -wp(w, t) \qquad w(z, 0) = z \tag{3.10}$$

Solving this for w, we get that $f_0(w^{-1}(\zeta, t))$, as a solution for (3.8) in parts on the unit disk. In the attempts to extend this solutions to the entire unit disk, we may loose injectivity. However, is was shown by Pommerenke [26], that for every function $f \in S$, there is a $p(z, t) \in C$, such that the solution $w(z, t)$ to (3.10) with this $p(z, t)$, we have

$$f(z) = \lim_{t \to \infty} e^t w(z, t) \tag{3.11}$$

We say in this case that f is generated by p (this is not a standard term). The choice of p is not unique, but every $p \in C$ generate a function in S. If a function f generated by $p(z, t)$, is taken as the initial condition in (3.8) with the same $p(z, t)$, then the solution is univalent for any t.

Example 3.18. Let S' be the subclass of S, whose image is $\mathbb{C} \setminus \gamma$, where γ is a Jordan curve starting at a point in the complex plane and ending at infinity. These mappings are called slit mappings and they are dense in S with respect to the local uniform convergence. Any $f \in S'$ is generated by $p(z,t) \in C$ in the form

$$p(z,t) = \frac{e^{iu(t)} + z}{e^{iu(t)} - z},$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function. (3.8) was originally studied by Löwner [24] with p on this form, and then later generalised Kufarev [22]. If f is a multislit mapping, whose image in the complex plane has a tree with $m-1$ tips as complement, then it f is generated by elements from C in the form

$$p(z,t) = \sum_{k=1}^{m} \lambda_k(t) \frac{e^{iu_k(t)} + z}{e^{iu_k(t)} - z}.$$

Here, $u_k(t) : \mathbb{R}_+ \to \mathbb{R}$ are continuous, and $\lambda_k(t) : \mathbb{R}_+ \to [0,1]$ are measurable functions with $\sum_{k=1}^{m} \lambda_k(t) = 1$. For a general mapping f, it is difficult to know which $p \in C$ that generate it.

We define, for $k \in [0,k)$ a subclass $S_k \subseteq S$, extracting the class of functions generated by some $p(z,t) \in C$ satisfying

$$\left| \frac{p(z,t) - 1}{p(z,t) + 1} \right| \leq k \tag{3.12}$$

for all $(z,t) \in \mathbb{D} \times \mathbb{R}_+$. We denote by $\tilde{S}$ the subclass of S consisting of functions whose image is bounded by a C^∞ Jordan curve. If $f \in \tilde{S}$, then both $f(\mathbb{D})$, and $\operatorname{ext} f(\mathbb{D})$ are domains with the common boundary. Note that $S_k \subsetneq \tilde{S}$ for any $k \in [0,1)$.

By the analogy with S, we can define the class of univalent functions f in $\mathbb{D}^*$ normalised by $g(z) = z + b_0 + \frac{b_1}{z} + \dots$ by Σ . We denote the subclass of functions with $b_0 = 0$ by Σ_0, and the subclass of functions such that $0 \notin g(\mathbb{D}^*)$ by Σ'. . Any $g \in \Sigma$ is equal to a constant plus some element in Σ' (or Σ_0). There is a one-to-one correspondence mapping $\Xi : S \to \Sigma'$, which, for f with the expansion $f(z) = z + a_1 z^2 + a_2 z^3 \dots$, is given by

$$\Xi : f(z) \mapsto \frac{1}{f(\frac{1}{z})}.$$

For this class we have a similar way to generate its elements. Let $C^* = \{p(z,t) : \mathbb{D}^* \to \mathbb{C} \big| \, p(\frac{1}{z}, t) \in C\}$. Then we have that for any $g \in \Sigma$, there is $p \in C^*$, such that

$$g(z) = \lim_{t \to \infty} e^{-t} w(z,t), \tag{3.13}$$

where w is a solution to

$$\frac{dw}{dt} = w p(w,t) \tag{3.14}$$

We also define the subclass $\Sigma_k = \Xi(S_k)$.

3.5 Conformal maps with quasiconformal extension

Since there is no consistent estimate for the distortion of the conformal radius under general quasiconformal map, we will look at quasiconformal functions in $\mathbb{D}$, that have a univalent extension into the exterior $\mathbb{D}^*$, and use the properties of the exterior function, to determine properties of the interior one.

Definition 3.19. *A curve γ is called a k-quasicircle, if $\gamma = f(\partial\mathbb{D})$ for a k-q.c. mapping $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$*

In particular, a C^∞ closed Jordan curve is a quasicircle. We consider a quasiconformal mapping $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that $f|_{\mathbb{D}^*}$ is univalent. To normalise, we require that $f|_{\mathbb{D}^*} \in \Sigma_k$ and $f(0) = 0$. For any $g_0 \in \Sigma_k$, there is a k-q.c. mapping $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, with $f(0) = 0$ and $f|_{\mathbb{D}^*} = g_0$:

Theorem 3.20. *Let $g(z,t)$ be a univalent solution to*

$$\frac{\partial g}{\partial t} = -zp(z,t)\frac{\partial g}{\partial z}, \tag{3.15}$$

with the initial condition $g(z,0) = g_0(z)$, and let

$$\Big|\frac{p(z,t)-1}{p(z,t)+1}\Big| \leq k < 1.$$

Then $g(\mathbb{D},t)$ is a Jordan domain bounded by a k-quasicirle for each $t \geq 0$, and f defined by

$$f(z) = \begin{cases} g(z,0) & \text{if } \ z \in \mathbb{D}^*, \\ g(\frac{z}{|z|}, -\log|z|) & \text{if } \ z \in \bar{\mathbb{D}}, \end{cases}$$

is a k-q.c. extension of $g(z,0)$ into $\bar{\mathbb{C}}$ with $f(0) = 0$.

This follows from similar theorem for functions in S_k, found in [4]. It is important to notice, that although all elements in Σ_k admits an quasiconformal extension, that does not necessarily mean that all functions that have a k-quasiconformal extension is in Σ_k. For example, the mappings later considered (3.17), has a k^2-q.c. extension, but there is no obvious choice for p satisfying inequality (3.12) for k^2. The reason we mention this, is to avoid confusion, since in much literature on Complex Analysis, the symbol Σ_k denotes functions in Σ or Σ_0 that admit k-q.c. extension.

We will try to look at the distortion of the reduced modulus under mapping given by the above theorem. From the classical area theorem we have that the area of the complement to $f(\mathbb{D}^*)$ is less than π. Of all domains of constant area, the disk has the maximal conformal radius. Actually, for any simply connected domain Ω, with $R = \max_{s\in\Omega} R(\Omega, s)$,

$$\pi R^2 \leq Area(\Omega),$$

with equality if and only if Ω is a disk (see e.g. [16]). Therefore, $m(f(\mathbb{D}), 0) \leq 0$. Let $f|_{\mathbb{D}^*}$ be generated by $p(z,t)$, with $\Big|\frac{p(z,t)-1}{p(z,t)+1}\Big| \leq k$ for almost every $(z,t) \in \mathbb{D}^* \times \mathbb{R}_+$. From Swartz Lemma, we have that

$$\Big|\frac{p(w,t)-1}{p(w,t)+1}\Big| \leq \frac{k}{|w|}, \tag{3.16}$$

which gives

$$\frac{|w|-k}{|w|+k} \leq |p(w,k)| \leq \frac{|w|+k}{|w|-k}.$$

Using the latter inequality, we get

$$\operatorname{Re} p(z,t) = \operatorname{Re}\big(\frac{\frac{\partial}{\partial t}w}{w}\big) = \operatorname{Re}\big(\frac{\partial}{\partial t}\log w\big) = \frac{\partial}{\partial t}\log|w| = \frac{\frac{\partial}{\partial t}|w|}{|w|} \leq \frac{|w|+k}{|w|-k}.$$

Rearanging and integrating the inequality with respect to t, we get

$$\int_0^t dt = t \geq \int_{|z|}^{|w|} \frac{x-k}{(x+k)x} dx = \log \frac{(|w|+k)^2|z|}{|w|(|z|+k)^2},$$

and further

$$0 \geq \log \frac{(|w|+k)^2|z|}{|w|(|z|+k)^2} - \log e^t = \log \frac{(|w|+k)^2|z|}{e^t|w|(|z|+k)^2} = \log \frac{e^{-2t}(|w|+k)^2|z|}{e^{-t}|w|(|z|+k)^2}.$$

Letting $t \to \infty$ we get $\left(|f(z)| = \lim_{t\to\infty} e^{-t}|w(z,t)|\ \right)$

$$|f(z)| \leq \frac{(|z|+k)^2}{|z|}$$

and doing similar operations for the lower estimate, we get

$$\frac{(|z|-k)^2}{|z|} \leq |f(z)| \leq \frac{(|z|+k)^2}{|z|} \qquad z \in \mathbb{D}^*.$$

From this, we have that $f(\mathbb{D})$ will always contain $(1-k)^2\mathbb{D}$, and it is always contained in $(1+k)^2\mathbb{D}$. If we have equality in (3.16), then, if $\alpha \in \mathbb{R}$

$$f(z) = \frac{(z+e^{i\alpha}k)^2}{z} \qquad z \in \mathbb{D}^* \tag{3.17}$$

and we see, that in this case, $f(\mathbb{D})$ in an ellipse, with the centre at $2ke^{i\alpha}$, with the major axis $1+k^2$, the minor axis $1-k^2$, rotated by an angle α. These mappings do appear as extremal functions in other areas of complex analysis (see e.g. [23]).

The mappings have an obvious k^2-q.c. extension

$$f(z) = \begin{cases} \frac{(z+e^{i\alpha}k)^2}{z} & \text{if } z \in \mathbb{D}^*, \\ z + 2ke^{i\alpha} + k^2e^{2i\alpha}\bar{z} & \text{if } z \in \bar{\mathbb{D}}, \end{cases}$$

but we do not have the normalisation $f(0) = 0$. We try to find another extension by the theorem 3.20, and solving (3.15) with the initial condition $g_0(z) = \frac{(z+e^{i\alpha}k)^2}{z}$, $z \in \mathbb{D}^*$. The solution is $f(z,t) = \frac{(z+e^{i\alpha}k)^2}{z}e^{-t}$, and we get the following extension

$$f(z) = \begin{cases} \frac{(z+e^{i\alpha}k)^2}{z} & \text{if } z \in \mathbb{D}^* \\ z + 2ke^{i\alpha}|z| + k^2e^{2i\alpha}\bar{z} & \text{if } z \in \bar{\mathbb{D}}, \end{cases}$$

which as we can see satisfies the equality

$$\frac{|f_{\bar{z}}|}{|f_z|} \overset{a.e.}{=} k\left|\frac{k+\frac{z}{|z|}}{1+k\frac{\bar{z}}{|z|}}\right| = k,$$

and $f(0) = 0$.

The investigation above ends up with the following result.

Theorem 3.21. *Let $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a quasiconformal automorphism with $f(0) = 0$, $f(\infty) = \infty$. Let $f|_{\mathbb{D}^*} \in \Sigma_k$ and let $f|_{\mathbb{D}}$ be k -q.c. Then,*

$$\ell_k \le m(f(\mathbb{D}, 0) \le 0,$$

where ℓ_k is the uniform lower boundary of the modulus of all such maps and

$$\frac{1}{2\pi} \log(1-k)^2 \le \ell_k \le \frac{1}{2\pi} \log\Big(\frac{\pi}{K(r^2)}\Big)^2 \frac{k}{2r(1+r^2)},$$

where $r = \mathfrak{r}(\frac{2k}{1+k^2})$.

Proof. For ℓ_k, the left inequality follows because all such maps contain $(1-k)^2\mathbb{D}$, and the right-hand inequality is from the formula for the reduced modulus of the ellipse. $\square$

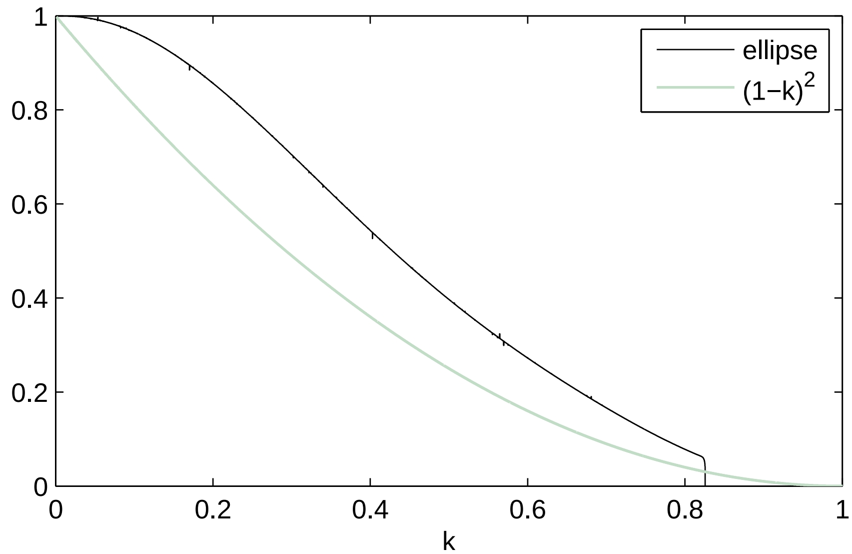

Figure 3.1: The bounds for the conformal radius. The "step down" at the end of the upper bound (ellipse), is because of numerical error.

Chapter 4

Class S and the coefficient manifold

We will now study some properties of the class S of univalent functions $f : \mathbb{D} \to \mathbb{C}$ normalised by

$$f(z) = z + c_1 z^2 + c_2 z^3 + \cdots$$

One of the most important result for this class is the de Branges theorem proving the famous Bieberbach conjecture [7]. This result states that for any n, $|c_n| \leq n + 1$ with the equality only for rotations of the Koebe function. A natural next step is to find out what elements of $\mathbb{C}^{\mathbb{N}}$ being inside the bounded domain given by the de Branges theorem can be the coefficient sequence of a univalent function. To do this we study manifold of coefficients of S and $\tilde{S}$. We will begin this chapter with the definition of the Witt and Virasoro algebras, as they will later appear in section 4.2, when working on a manifold of coefficients. Then we shall deal in more details with the coefficient manifolds by constructing different Hamiltonian systems for the coefficients in S based on the Löwner-Kufarev equation.

4.1 Virasoro algebra

Definition 4.1. *The Virasoro algebra* $\mathfrak{vir}$ *is a Lie algebra consisting of a vector space over* $\mathbb{C}$ *spanned by basis elements* $\{L_n\}_{n\in\mathbb{Z}}$ *and* c *with Lie brackets defined by the relations*

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{-m,n},$$

$$[c, L_n] = 0 \qquad \forall n \in \mathbb{Z},$$

where $\delta_{m,n}$ *is the Kronecker delta.*

The element c is called the central charge, and in most representations it works as a multiplication by scalar. This algebra usually appears as a central extension of the Witt algebra.

Definition 4.2. *If* $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ *are Lie algebras, then* $\mathfrak{h}$ *is the central extension of* $\mathfrak{g}$ *by* $\mathfrak{f}$, *if there is an exact sequence of Lie algebras*

$$0 \to \mathfrak{f} \to \mathfrak{h} \to \mathfrak{g} \to 0,$$

such that $\mathfrak{f}$ *is contained in the centre of* $\mathfrak{h}$.

Treated as vector spaces we can write $\mathfrak{h} \cong \mathfrak{g} \oplus \mathfrak{f}$. If $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{f}$ as Lie algebras, (the Lie algebra $\mathfrak{g} \oplus \mathfrak{f}$ is defined to be the vector space $\mathfrak{g} \oplus \mathfrak{f}$ with brackets given by

$$[(g_1, f_1), (g_2, f_2)] = \big([g_1, g_2], [f_1, f_2]\big)$$

and the last brackets will always be 0 in case of a central extension, since $\mathfrak{f}$ has to be contained in the centre) we say that $\mathfrak{h}$ is a trivial central extension of $\mathfrak{g}$ by $\mathfrak{f}$.

A vector field on a Riemannian manifold, is called a Killing vector field if it preserves the metric (i.e. given a Riemannian metric ρ, then $\mathcal{L}_X \rho = 0$, where $\mathcal{L}_X$ is the Lie derivative). The Witt algebra is the Lie algebra of holomorphic Killing vector fields on $\mathbb{C} \setminus \{0\}$, with brackets given by the commutators. This has basis elements $\{L_n = -z^{n+1}\frac{\partial}{\partial z}\}_{n\in\mathbb{Z}}$ of this algebra, and the brackets of these two vector fields are given by

$$[L_m, L_n] = (n-m)z^{m+n+1}\frac{\partial}{\partial z} = (m-n)L_{m+n}.$$

The Virasoro algebra is then a nontrivial central extension of the Witt algebra by $\mathbb{C}$ (actually, there exists no other nontrivial extension by $\mathbb{C}$, see, e.g., [14]).

To get some details on this extension, we first need some definitions. A Fréchet space V is a Hausdorff topological complete vector space, in which its topology may be induced by a countable family of seminorms $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ (i.e., the sets $U_{x,k,\varepsilon} = \{y \in V \,\big|\, \|x-y\|_k < \varepsilon\}$, where $x \in V, k \in \mathbb{N}, \varepsilon > 0$ form a basis for the topology). Fréchet spaces are always locally convex [10]. Derivation of a function between two Fréchet spaces is defined by the Gâteaux derivative. A function $f : U \to W$, where V and W are locally convex topological vector spaces, and U is open in V, is said to have Gâteaux derivative in $x \in U$ in the direction $y \in V$ if

$$df_x(y) = \lim_{t\to 0}\frac{f(x+ty) - f(x)}{t}$$

exists. If $df_x(y)$ exists for any $y \in V$, then f is said to be differentiable at x. A function f is continuously differentiable (a C^1-mapping) in an open subset $U \subseteq V$, if $df : U \times V \to W$ is continuous. Note that $df_x(y)$ needs not, in general, be linear with respect to y. This definition can easily be extended to C^n -mappings for any $n \in \mathbb{N} \cup \{\infty\}$. A Fréchet manifold M (over a model space V, where V is a Fréchet space) is then a topological Hausdorff space M with a maximal atlas $\mathcal{A} = \{x_\alpha : U_\alpha \to V\}$, (as usual U_α are open sets covering M, and each x_α is homeomorphic on its image) where $x_\beta^{-1}x_\alpha|_{x(U_\alpha\cap U_\beta)}$ now is required to be C^∞ as mappings of Fréchet spaces for any $x_\alpha, x_\beta \in \mathcal{A}$. We further define differentiable mappings between Fréchet manifolds and tangent spaces in an analogous way of what is done for real manifolds. A Lie-Fréchet group G, is a topological group on a Fréchet manifold, such that multiplication and inversion are C^∞-mappings.

An important subclass of the Lie-Fréchet groups, are the Lie-Banach groups, defined similarly, only with "Fréchet space" interchanged with "Banach space", and the Gâteaux derivative interchanged with the Fréchet derivative. If V, W are Banach spaces, and U is open in V, then $f : U \to V$, is Fréchet differentiable at x if there exists a bounded linear operator $A_x : V \to W$, such that

$$\lim_{y\to 0}\frac{\|f(x+y) - f(x) - A_x(y)\|}{\|y\|} = 0,$$

and continuously differentiable in U, if $df_x(y) = A_x y$ is continuous in $U \times V$.

Going back to the general case, if G is a Lie Fréchet group (over model space V), TG is a Fréchet manifold (over model space $V \times V$), and if e is the identity element in G, the multiplication on G, induces a bracket operation on $\mathfrak{g} = T_eG$ in the usual way, and $\mathfrak{g}$ is called the Lie algebra of G. A Fréchet (resp. Banach) space with a skew symmetric bilinear continuous map $[\cdot,\cdot]$, satisfying the Jacobi identity is called a Lie-Fréchet (resp. Lie-Banach) algebra. While the Lie algebra of a Lie-Fréchet or Lie-Banach group, is a Lie-Fréchet or Lie-Banach algebra, respectively, the converse is not necessarily true (both in Banach and Fréchet case). For instance, if M is a real noncompact manifold, then $\Gamma(TM)$ is a Lie-Fréchet algebra, but it is not the Lie algebra of any Lie-Fréchet Group.

The case of Lie-Banach groups is mostly studied, since its Lie-algebra always have an exponential map to a local Lie-Banach group. We have no such knowledge of the existence of a exponential map from any general Lie algebra of a Lie-Fréchet group. Studies are often therefore restricted to Lie-Fréchet group that are regular, where some sort of exponential map may be defined. For the definition of a regular Lie-Fréchet group, and more details, see [21].

If M is any real compact manifold, then $\Gamma(TM)$ is a Fréchet space with seminorms given by supremum over partial derivatives. The space of diffeomorphism of M, $\operatorname{Diff} M$, is then a Fréchet manifold over model space $\Gamma(TM)$, and it is a Lie-Fréchet group under composition (in fact, a regular one). Its Lie algebra will be $\Gamma(TM)$. The simplest case of this, but also one of the most important, is Diff_+S^1 , the collection of C^∞-sense preserving diffeomorphisms of the unit circle. These are all of the form

$$\gamma : e^{i\theta} \mapsto e^{i\alpha(\theta)},$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a monotonically increasing function, with $\alpha(\theta + 2\pi)$. We denote its Lie algebra by $\operatorname{Vect} S^1$, which is just $\Gamma(TS^1)$ with usual commutator brackets. This Lie algebra has a non-trivial unique central extension by $\mathbb{R}$ with brackets given by

$$[X_1 + ac, X_2 + bc] = [X_1, X_2] + \frac{c}{12}\omega(\varphi_1, \varphi_2)\,,$$

where $X_i(\theta) = \varphi_i(\theta)\frac{d}{d\theta} \in \operatorname{Vect} S^1$, $a, b \in \mathbb{R}$, c is a basis vector for $\mathbb{R}$, and ω is the so-called Gelfand-Fuchs cocycle

$$\omega(\varphi_1, \varphi_2) = \frac{1}{2\pi}\int_0^{2\pi} (\varphi_1' \cdot \varphi_2'' - \varphi_1'' \cdot \varphi_2')d\theta.$$

Note that integration by parts, gives the 2-cycle condition:

$$\omega(\varphi_1, [\varphi_2, \varphi_2]) + \omega(\varphi_2, [\varphi_3, \varphi_1]) + \omega(\varphi_3, [\varphi_1, \varphi_2]) = 0$$

and a reformulation of ω

$$\omega(\varphi_1, \varphi_2) = -\frac{1}{4\pi}\int_0^{2\pi} (\varphi_1' + \varphi_1''') \cdot \varphi_2 d\theta.$$

This will be a real version of the Virasoro algebra which we denote $\mathfrak{vir}_{\mathbb{R}}$. Complexification of $\mathfrak{vir}_{\mathbb{R}}$ gives us $\mathfrak{vir}$. To see that this is indeed the case, we write $\operatorname{Vect} S^1$ as $S^1 \times \operatorname{Vect}_0 S^1$, where here S^1 represent the constant vector fields on S^1, $\operatorname{Vect}_0 S^1$ is the quotient of all vector fields out by these contant elements. All elements in $\operatorname{Vect}_0 S^1$ we may identify with with

C^∞ functions $\varphi : S^1 \to \mathbb{R}$ of vanishing mean value over S^1. This gives for each function an expansion

$$\varphi(\theta) = \sum_{n=1}^{\infty} \big(a_n \cos n\theta + b_n \sin n\theta\big).$$

We can define almost complex structure by operator

$$J : \varphi = \sum_{n=1}^{\infty} \big(a_n \cos n\theta + b_n \sin n\theta\big) \mapsto \sum_{n=1}^{\infty} \big(- a_n \sin n\theta + b_n \cos n\theta\big).$$

Clearly, $J^2 = -id_{\mathrm{Vect}\, S^1}$. Holomorphic elements are are of the form

$$\frac{1}{2}(\varphi - iJ(\varphi)) = \sum_{n=1}^{\infty}(a_n - ib_n)e^{in\theta}\,,$$

which can be extended to holomorphic functions into the unit disk. The antiholomorphic elements are of the form $\frac{1}{2}(\varphi + iJ(\varphi)) = \sum_{n=1}^{\infty}(a_n + ib_n)e^{-in\theta}$.

We choose the basis $\{L_n = -ie^{in\theta}\frac{\partial}{\partial\theta}\}_{n\in\mathbb{Z}}$ of $\mathrm{Vect}_{\mathbb{C}}\, S^1$, given by the restriction of the basis elements in the Witt algebra to the unit circle. By the above discussion, we know that these elements does in fact span $\mathrm{Vect}\, S^1$ ($\{L_n\}_{n\geq 1}$ for the holomorphic elements, $\{L_n\}_{n\leq -1}$ for the antiholomorphic elements, and L_0 for the constant elements). Computing the brackets for this basis, we have the following

$$[L_m + ac, L_n + bc] = [L_m, L_n] + \frac{c}{12}\omega(-ie^{im\theta}, -ie^{in\theta})$$
$$= (m-n)L_{n+m} + \frac{c}{12}m(m^2-1)\delta_{-m,n},$$

which shows that this is indeed the Virasoro algebra.

$\mathrm{Diff}_+ S^1$ also has a unique nontrivial central extension by $\mathbb{R}$, which has the Virasoro algebra as Lie algebra (the definition of central extension for Lie groups, or any group, is defined similar to that of Lie algebras). This central extension is called the Virasoro-Bott group Vir and it is given by the manifold $\mathrm{Diff}_+ S^1 \times \mathbb{R}$ equipped with the product

$$(\gamma_1, a)(\gamma_2, b) = \big(\gamma_1\gamma_2, a + b + \frac{c}{12}\Omega(\gamma_1, \gamma_2)\big)$$

where $\gamma_1, \gamma_2 \in \mathrm{Diff}^+ S^1$, $a, b \in \mathbb{R}$, and Ω is the Thurston-Bott cocycle

$$\Omega(\gamma_1, \gamma_2) = \frac{1}{2\pi}\int_0^{2\pi} \log((\gamma_1\gamma_2)')\, d\log(\gamma_2').$$

Vir is a regular Lie-Fréchet group, so there exists an exponential map $\exp : \mathfrak{vir}_{\mathbb{R}} \to Vir$, however, this is not a local diffeomorphism (in fact, there are points arbitrary close to identity element, which are not in the image of the exponential map, see [6] Appendix C).

4.2 The Coefficient manifold and connections to Kirillov's manifold

Let us consider the class S and its subclass $\tilde{S}$ from a more geometrical point of view. For any $f \in S$ and for $0 < \kappa < 1$, $\frac{1}{\kappa}f(\kappa z)$ is in $\tilde{S}$. We may therefore look at any f, as a limit of functions in $\tilde{S}$, by letting $\kappa \to 1$. Let us embed S and $\tilde{S}$ in $\mathbb{C}^{\mathbb{N}}$ by identifying $f(\zeta) = \zeta + \sum_{n=2}^{\infty} c_{n-1}\zeta^n$ with the point $(c_1, c_2, c_3, \dots)$. We will denote these embeddings by $\mathcal{N}$ and $\mathcal{M}$ respectively. They can be seen as limits of coefficient bodies

$$\mathcal{M}_n = \big\{(c_1, c_2, \dots, c_n) | \exists f \in \tilde{S}, \text{ with } f = \zeta + \sum_{n=2}^{\infty} c_{n-1}\zeta^n\big\}$$

$$\mathcal{N}_n = \big\{(c_1, c_2, \dots, c_n) | \exists f \in S, \text{ with } f = \zeta + \sum_{n=2}^{\infty} c_{n-1}\zeta^n\big\}$$

Naturally, $\mathcal{M}_n$ is dense in $\mathcal{N}_n$. Generally, for coefficient bodies it is known that

- $\mathcal{N}_n$ homeomorphic to $D^{2n-2} = \{r \in \mathbb{R}^{2n-2} \big| \, |r| \le 1\}$.
- $\partial\mathcal{N}_n$ homeomorphic to S^{2n-3}.
- Any $x \in \partial\mathcal{N}_n$ corresponds to exactly one $f \in S$, which is called a boundary function for $\mathcal{N}_n$. All boundary functions map. $\mathbb{D}$ onto $\mathbb{C}$ minus a piecewise analytic Jordan arcs forming a tree with a root at infinity and having at most n tips. It follows that $\partial\mathcal{N}_n \cap \mathcal{M}_n = \emptyset$.
- With exception for a set of smaller dimension, for any $x \in \mathcal{N}_n$, there is a normal vector satisfying Lipschitz condition .
- There is a connected open subset X_1 on $\partial\mathcal{N}_n$, such that $\partial\mathcal{N}_n$ is an analytic hypersurface at every point of X_1. Points of $\partial\mathcal{N}_n$, correspond to functions giving extremum to a linear functional belonging to $\bar{X}_1$.
- By the de Branges theorem $\mathcal{N}_n \subseteq \prod_{i=1}^{n}(i+1)\bar{\mathbb{D}}$.

Not very much else is known except of the simplest cases:

- $\mathcal{N}_1 = 2\bar{\mathbb{D}}$, that is an closed disk of radius 2. $\mathcal{M}_1 = 2\mathbb{D}$.
- $\mathcal{N}_2$ was in 1950 completely described by Schaffer and Spencer [31].

We also have an alternative description of the manifold $\mathcal{M}$. Given any mapping $f \in \tilde{S}$, it has a continuation to the unit circle. Since the exterior of $f(\mathbb{D}) = \Omega$ is also a domain, by the Riemann mapping theorem, there is a conformal map of $\mathbb{D}^*$ onto Ω^* with $g(\infty) = \infty$, and this mapping also has an extension to the unit circle. Let

$$f^{-1}g|_{S^1} \in \mathrm{Diff}_+ S^1 \tag{4.1}$$

and we will denote this diffeomorphism by γ. Given any mapping $f \in \tilde{S}$, the different choices of g differ only by rotation (in the sense that $\tilde{g}$ is any other such function, then $\tilde{g}(z) = g(e^{i\alpha}z)$, with $\alpha \in \mathbb{R}$), thus, by different choices of g, f generates a subgroup $\gamma\mathrm{Rot}\, S^1$. $\mathrm{Rot}\, S^1$ here denotes the subgroup of $\mathrm{Diff}_+ S^1$ by sense preserving rotations. Define function

$$\mathcal{K} : \tilde{S} \to \mathrm{Diff}_+ S^1 / \mathrm{Rot}\, S^1$$

$$f \mapsto [\gamma] = \gamma\mathrm{Rot}\, S^1$$

where γ is constructed as in (4.1). Kirillov [19] showed that $\mathcal{K}$ bijection, and this gives identification

$$\mathcal{M} = \mathrm{Diff}_{+}S^1/\mathrm{Rot}\,S^1.$$

The inverse map may be constructed in the following way.

- Given equivalence class $\gamma\mathrm{Rot}\,S^1$, pick a representative. Define $w : \mathbb{D}^* \to \mathbb{D}^*$ by

$$w(\zeta) = |\zeta|\gamma(\frac{\zeta}{|\zeta|}).$$

- We construct a quasiconformal mapping $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ in the following way
  - $f|_{\mathbb{D}}$ and fw has to be conformal.
  - $f|_{\mathbb{D}} \in \tilde{S}$ and $f(\infty) = \infty$.
- $[\gamma] \mapsto f|_{\mathbb{D}}$ is then the inverse of F (the function matching $f|_{\mathbb{D}}$ will be fw).

The above mapping f is a quasiconformal automorphism of $\bar{\mathbb{C}}$. Using that fw is conformal, the relation (3.6) for the mapping $f = fww^{-1}$, tells us that

$$\mu_f = \mu_{w^{-1}}.$$

in $\mathbb{D}^*$. Note that, if we denote $z = re^{i\theta}$, then $w^{-1}(re^{i\theta}) = r\gamma^{-1}(e^{i\theta})$, and from $\frac{\partial}{\partial z} = \frac{\bar{z}}{2|z|}(\frac{\partial}{\partial r} - \frac{i}{|z|}\frac{\partial}{\partial \theta})$ and $\frac{\partial}{\partial \bar{z}} = \frac{z}{2|z|}(\frac{\partial}{\partial r} + \frac{i}{|z|}\frac{\partial}{\partial \theta})$ we get that f is the solution to a Beltrami equation, with Beltrami coefficient

$$\mu_f(z) = \begin{cases} \frac{\frac{\partial}{\partial \bar{z}}w^{-1}}{\frac{\partial}{\partial z}w^{-1}} = \frac{z^2}{|z|^2} \cdot \frac{1-\mathfrak{V}\gamma^{-1}(\frac{z}{|z|})}{1+\mathfrak{V}\gamma^{-1}(\frac{z}{|z|})} & z \in \overline{\mathbb{D}^*} \\ 0 & z \in \mathbb{D} \end{cases}$$

and initial conditions $f(0) = 0$, $f(\infty) = \infty$ and $f_z(0) = 0$. Here we define the operator $\mathfrak{V}$ by $\mathfrak{V}\gamma(e^{i\theta}) = \frac{1}{i\gamma(e^{i\theta})}\frac{d}{d\theta}\gamma(e^{i\theta})$. Note that if $\gamma(e^{i\theta}) = e^{i\alpha(\theta)}$, then $\mathfrak{V}\gamma(e^{i\theta}) = \alpha'(\theta)$.

This manifold $\mathrm{Diff}_{+}S^1/\mathrm{Rot}\,S^1$ is called Kirillov's manifold, and it is a homogeneous (homogeneous means that $\mathrm{Diff}_{+}S^1$ act transitively) Kählerian complex manifold.

Example 4.3. Let us consider, as an example, the mapping from the unit disk $\mathbb{D}$ onto an ellipse. From earlier, we know that

$$\mathcal{E}'_{a;\lambda}(0) = \frac{\pi a\lambda}{2r\mathbf{K}(r^2)}\,.$$

To get a normalised mapping, let $a = \mathfrak{a}(\lambda) = \frac{2\mathfrak{r}(\lambda)\mathbf{K}(\mathfrak{r}^2(\lambda))}{\pi\lambda}$. $f_\lambda := \mathcal{E}_{\mathfrak{a}(\lambda);\lambda}$ is then a mapping in S, and the matching mapping is given by

$$g_\lambda(z) = \frac{\lambda\mathfrak{a}(\lambda)}{2}\Big(\frac{1+\sqrt{1-\lambda^2}}{\lambda}z + \frac{\lambda}{1+\sqrt{1-\lambda^2}}z^{-1}\Big).$$

Since these are one of the few nontrivial examples of matching functions, we try to do the correspondence with diffeomorphisms of the circle in this case. We will look at the image $\gamma_\lambda\mathrm{Rot}\,\mathrm{S}^1$

of f_λ under $\mathcal{K}$, and then try to use the construction above for $\mathcal{K}^{-1}$ on the diffeomorphism γ_λ. We note that $g_\lambda(z)^{-1} = \frac{z+\sqrt{z^2-\lambda^2\mathfrak{a}(\lambda)^2}}{\mathfrak{a}(\lambda)(1+\sqrt{1-\lambda^2})}$, so if we let $\varsigma(z) = \frac{\pi \mathbf{F}(z/r,r^2)}{2\mathbf{K}(r^2)}$ then

$$\gamma_\lambda^{-1}(z) := (g_\lambda^{-1} f_\lambda)(z) = \frac{\lambda\mathfrak{a}(\lambda)\sin\varsigma(z) + \sqrt{\lambda^2\mathfrak{a}(\lambda)\sin^2\varsigma(z) - \lambda^2\mathfrak{a}(\lambda)^2}}{\mathfrak{a}(\lambda)(1+\sqrt{1-\lambda^2})}.$$

$$= \frac{\lambda}{1+\sqrt{1-\lambda^2}} e^{i(\varsigma(z)-\pi/2)}.$$

The diffeomorphisms are therefore of the form

$$\gamma_\lambda^{-1}(e^{i\theta}) = \frac{i\lambda}{1+\sqrt{1-\lambda^2}} e^{i\varsigma(e^{i\theta})}.$$

Using that $\gamma_\lambda(1) = 1$, we get

$$\gamma_\lambda^{-1}(e^{i\theta}) = \exp\Big(\frac{i\pi}{2}\frac{\tilde{\mathbf{F}}(\theta+\frac{\pi}{2},\frac{2r}{1+r^2}) - \mathbf{K}(\frac{2r}{1+r^2})}{\mathbf{K}(r^2)(1+r^2)}\Big),$$

where $\tilde{\mathbf{F}}(\theta,\kappa) = \int_0^\theta \frac{d\theta}{\sqrt{1-\kappa^2\sin^2\theta}} = \mathbf{F}(\sin\theta,\kappa)$. In the above formula, we have used that $\tilde{\mathbf{F}}(\frac{\pi}{2},\kappa) = \mathbf{K}(\kappa)$. Using relation $\mathbf{K}(\frac{2\sqrt{\kappa}}{1+\kappa}) = (1+\kappa)\mathbf{K}(\kappa)$ and calculating the inverse, we find

$$\gamma_\lambda^{-1}(e^{i\theta}) = \exp\Big(\frac{i\pi}{2}\big(\frac{\tilde{\mathbf{F}}(\theta+\frac{\pi}{2},\frac{2r}{1+r^2})}{\mathbf{K}(r^2)(1+r^2)} - 1\big)\Big) = -i\exp\Big(\frac{i\pi\tilde{\mathbf{F}}(\theta+\frac{\pi}{2},\frac{2r}{1+r^2})}{2(1+r^2)\mathbf{K}(r^2)}\Big),$$

$$\gamma_\lambda(e^{i\theta}) = \exp\Bigg(-i\cos^{-1}\mathbf{sn}\Big((1+r^2)(\frac{2}{\pi}\theta+1)\mathbf{K}(r^2), \frac{2r}{1+r^2}\Big)\Bigg).$$

$\mathbf{sn}(\omega,k)$ is here the inverse function of $\omega = \mathbf{F}(x,k)$ with respect to x ($\mathbf{sn}$ is called the Jacobi's elliptic sine). Now assume that we know γ_λ, and try to find the Beltrami coefficient of the quasiconformal extension of f_λ

$$\mathfrak{V}\gamma_\lambda^{-}1 = \frac{\pi}{2(1+r^2)\mathbf{K}(r^2)\sqrt{1-(\frac{2r}{1+r^2})^2\cos^2\theta}},$$

$$\mu_{f_\lambda}(z) = \frac{z}{\bar{z}}\cdot\frac{\sqrt{(1+r^2)-(\frac{2rz+2r\bar{z}}{|z|})^2} - \frac{\pi^2}{2K(r^2)}}{\sqrt{(1+r^2)-(\frac{2rz+2r\bar{z}}{|z|})^2} + \frac{\pi^2}{2K(r^2)}} \qquad z\in\overline{\mathbb{D}^*}.$$

To in practice solve this Beltrami equation with this coefficient (which has solution $f_\lambda(z)$ in $\mathbb{D}$ and $g_\lambda(|z|\gamma_\lambda(\frac{z}{|z|})$ outside) is very hard, and we can probably not expect that it is much easier for a general diffeomorphism of the circle. However, if the Beltrami equation was solved for some $\gamma\in\mathrm{Diff}_+S^1$, it has the advantage, that the solution would give us a pair of matching functions simultaneously.

The action of Diff_+S^1 on itself, induces a left action on $\tilde{S}$. That is, if we denote $\mathcal{K}^{-1}(\gamma\mathrm{Rot}\,S^1) = f_\gamma$, we can define $\gamma_1\cdot f_{\gamma_2} = f_{\gamma_1\gamma_2}$. Again, it is hard to see what this action does explicitly, but the corresponding infinitesimal action is given by variation

$$\delta_\nu f(z) = \frac{f(z)}{2\pi i}\int_{S^1}\Big(\frac{\zeta f'(\zeta)}{f(\zeta)}\Big)^2\frac{\nu(\zeta)d\zeta}{\zeta(f(\zeta)-f(z))}$$

for any $\nu(e^{i\theta})\frac{d}{d\theta} \in \mathrm{Vect}_{\mathbb{C}}S^1$ [20]. The infinitesimal action of the basis vectors $\{L_k\}_{k\in\mathbb{Z}}$, which we will denote by the same letter, is for $k > 0$

$$L_k(f)(z) = \delta_{-iz^k} f(z) = z^{k+1} f'(z)$$

which means that L_k work on $\mathcal{M}$ with the action

$$L_k = \frac{\partial}{\partial c_k} + \sum_{j=1}^{\infty} (j+1) c_j \frac{\partial}{\partial c_{k+j}} \in \Gamma(T^{1,0}\mathcal{M}). \tag{4.2}$$

For $k = 0$

$$L_0(f)(z) = zf'(z) - f(z) \qquad \text{or} \qquad L_0 = \sum_{j=1}^{\infty} j c_j \frac{\partial}{\partial c_j} \in \Gamma(T^{1,0}\mathcal{M})$$

reflecting that L_0 is a rotation (we have variation $e^{i\varepsilon} f(e^{-i\varepsilon}z) = f(z) + \varepsilon(zf'(z) + f(z)) + o(\varepsilon)$). The action of L_k for $k < 0$ is much more complicated [2]. As an example

$$L_{-1}(f)(z) = f'(z) - 1 - 2c_1 f(z)$$

$$L_{-2}(f)(z) = z^{-1} f'(z) - \frac{1}{f(z)} - 3c_1 + (c_1^2 - 4c_2) f(z)$$

However, they all may be described by elements in $\Gamma(T^{1,0}\mathcal{M})$. $\{L_k\}_{k\geq 1}$ forms a basis for this space. Their action restricted to $\mathcal{M}_n$ is given by the truncated Kirillov's operators

$$L_k = \frac{\partial}{\partial c_k} + \sum_{j=1}^{n-k} (j+1) c_j \frac{\partial}{\partial c_{k+j}} \in \Gamma(T^{1,0}\mathcal{M}_n). \tag{4.3}$$

4.3 Hamiltonian mechanics on a complex manifold

We will first recall the definition of a Hamiltonian system for real manifolds based on [3], and then generalise it to complex manifolds. *A motion* in a real manifold M is a smooth function $\zeta : I \to M$, where I is a interval in $\mathbb{R}$. Dot will indicate partial derivative with respect to real variable t (i.e. $\dot{\zeta} = \frac{\partial \zeta}{\partial t}$), which we will call time. To avoid confusion in the following presentation, the image of a section η at r, is denoted $\eta|_r$ (so for example, the cotangent vector field $dx_j \in \Gamma(T^*M)$ evaluating tangent vector $X \in T_r(X)$, is denoted $dx_j|_r(X)$). If M is an n-dimensional real manifold, then it has local coordinates $x = (x_1, \ldots, x_n)$. This induces local coordinates (x, p) on T^*M, that is $\eta \in T^*M$ has coordinates $(\tilde{x}, \tilde{p})$ if $\eta = \sum_{j=1}^n \tilde{p}_j dx_j|_{\tilde{x}} \in T_{\tilde{x}}M$. Define one-form on T^*M by

$$\vartheta = \sum_{j=1}^{n} p_j dx_j.$$

By the change of variable formula this is independent the chart x, and is therefore ϑ may be defined as a global section on the bundle $T^*T^*M \to T^*M$. ϑ is called the canonical one-form. If we let $\omega = d\vartheta \in \Gamma(\bigwedge^2 T^*T^*M)$, then the pair (T^*M, ω) becomes a *symplectic manifold* (an even-dimensional manifold with a closed nondegenerate two-form). For any two functions

$f, g : T^*M \to \mathbb{R}$, we may define the Lie-Poisson brackets, by $[f, g] = \omega(Tf, Tg)$. Explicitely, in local coordinates

$$\omega = \sum_{j=1}^{n} dp_j \wedge dx_j \,,$$

$$[f, g] = \sum_{j=1}^{n} \Big(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j}\Big).$$

ω also induces an isomorphism Ψ given by

$$\Psi : TT^*M \overset{\cong}{\to} T^*T^*M$$

$$X \to \eta_X$$

where $\eta_X(Y) = \omega(X, Y)$.

Definition 4.4. *A (real) Hamiltonian function on a manifold M is a differentiable function $H : T^*M \to \mathbb{R}$. A motion $\zeta : I \to M$ is the solution to a Hamiltonian system if*

$$\dot{\zeta} = \Psi^{-1}(TH)|_{\zeta}$$

for any $t \in I$. We call such a curve a geodesic.

If we again look at this locally, we notice that

$$\Psi^{-1}\Big(\sum_{j=1}^{n}(a_j dx_j + b_j dp_j)\Big) = \sum_{j=1}^{n}(b_j \frac{\partial}{\partial x_j} - a_j \frac{\partial}{\partial p_j})$$

and letting $\zeta = (x_1, \ldots, x_n, p_1, \ldots, p_n)$ and $\dot{\zeta} = \sum_{j=1}^{n}(\dot{x}_j \frac{\partial}{\partial x_j} + \dot{p}_j \frac{\partial}{\partial p_j})$, we get the more classical formulation

$$\dot{x}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial x_j} \qquad j = 1, \ldots, n\,. \tag{4.4}$$

If M is an n-dimensional complex manifold, then M is a $2n$-dimensional real manifold, and we may define a Hamiltonian function $H : T^*M \to \mathbb{R}$ mentioned above. We give T^*M complex coordinates formally, by denoting $(x_{n+1}, \ldots, x_{2n}) = (y_1, \ldots, y_n)$ and similarly $p_{n+j} = q_j$, and let $z_j = x_j + iy_j$, and $\psi_j = p_j + iq_j$. Then the equations above becomes

$$\dot{z} = 2\frac{\partial H}{\partial \bar{\psi}} \qquad \dot{\bar{\psi}} = -2\frac{\partial H}{\partial z} \tag{4.5}$$

and, off course also, $\dot{\bar{z}} = 2\frac{\partial H}{\partial \psi}$, $\dot{\psi} = -2\frac{\partial H}{\partial \bar{z}}$.

A complex Hamiltonian function, is a function $\mathcal{H} \in \mathscr{O}(T^{1,0}M)^*$. If we let $(z, \bar{\psi})$ be coordinates on the holomorphic cotangent bundle (to denote the last n coordinates by $\bar{\psi}_j$, is just a matter of convention, we could also have used ψ). Defining complex symplectic 2-form form by $\omega = \sum_{j=1}^{n} \bar{\psi}_j \wedge dz_j$, and using similar technique as in the real case, we can find an analogous definition of geodesic. In local coordinates

$$\dot{z}_j = \frac{\partial \mathcal{H}}{\partial \bar{\psi}_j} \qquad \dot{\bar{\psi}}_j = -\frac{\partial \mathcal{H}}{\partial z_j}\,, \tag{4.6}$$

(and off course, $\frac{\partial \mathcal{H}}{\partial \psi} = \frac{\partial \mathcal{H}}{\partial \bar{z}} = 0$). We call z generalised coordinates and $\bar{\psi}$ generalised momenta. The Lie-Poisson brackets associated to ω locally given by

$$[f,g] = \sum_{k=1}^{n} \big(\frac{\partial f}{\partial \zeta_k}\frac{\partial g}{\partial \bar{\psi}_k} - \frac{\partial f}{\partial \bar{\psi}_k}\frac{\partial g}{\partial \zeta_k}\big).$$

A function $I : (T^{1,0}M)^* \to \mathbb{C}$ is called *a first integral* if $[I, \mathcal{H}] = 0$. From (4.6) Hamiltonian function will always be a first integral.

Definition 4.5. *A Hamiltonian system is completely integrable (in sense of Louville) if we can find n functionally independent first integrals $I_1, I_2 \ldots, I_n$ which are pairwise involuntary (i.e. $[I_i, I_j] = 0$).*

If the system admits only $1 \leq k < n$ independent involuntary integrals, then it is called *partially integrable.*

The Hamiltonian functions we have described above are independent of time. In this case, the value of the Hamiltonian function is constant along geodesics. We will later also consider Hamiltonian functions on the form $\mathcal{H}(z, \bar{\psi}, t)$, and in this case, $\mathcal{H}$ changes with $\frac{\partial}{\partial t}\mathcal{H}$ along geodesics.

4.4 Hamiltonian interpretation of the Löwner equation

In section 3.4 we looked at the Löwner-Kufarev equation, and how every function in S is represented as a limit

$$f(z) = z + a_1 z^2 + a_3 z^3 + \cdots = \lim_{t\to\infty} e^t w(z,t),$$

where $w(z,t)$ is the solution to

$$\frac{dw}{dt} = -wp(w,t), \qquad w(z,0) = 0 \tag{4.7}$$

for some $p \in C$. Surprisingly, there is a connection with this more "classical" method and the action of the Kirillov operators on $\mathcal{M}$. This connection first appeared in an article on slit mappings (p as in example 3.18) by Prokhorov and Vasiliev [27], and a more general case was considered by Markina, Prokhorov and Vasiliev in [25]. The connection comes from Hamiltonian systems on the coefficient bodies, in which geodesics give solutions to (4.7). Explicitely, let $g(z,t) = e^t w(z,t) = z + c_1(t)z^2 + c_2(t)z^3 + \cdots$. This will be a curve in S, with $g(z,0) = z$ and $\lim_{t\to\infty} g(z,t) = f(z)$, and inserted in (4.7), with $p(z,t) = 1 + p_1(t)z + p_2(t)z^2 + \cdots$, we have the following

$$\dot{g} = g - gp(e^{-t}g, t) \tag{4.8}$$

or as a series

$$\sum_{m=1}^{\infty} \dot{c}_m z^{m+1} = -\sum_{m=1}^{\infty} p_m e^{-mt} g^{m+1} = -\sum_{m=1}^{\infty} \big(\sum_{j=1}^{m} p_j e^{-jt} d_{j+1,m+1}\big) z^{m+1}$$

where $g^m = \left(\sum_{j=1}^{\infty} c_{j-1} z^j\right)^m = \sum_{j=1}^{\infty} d_{m,j} z^j$. If we define the coefficients are given by:

$$d_{m,j} = \sum_{k_1+k_2+\cdots+k_m=j-m-1} c_{k_1} c_{k_2} \dots c_{k_m} ,$$

where we define $c_0 \equiv 1$ and $k_1, \dots, k_m$ are all required to be nonnegative integers (note that the sum over the empty set is 0, so this equation holds also for $m > j$). For the first coefficients, for example

$$\dot{c}_1 = -p_1 e^{-t}$$

$$\dot{c}_2 = -2c_1 p_1 e^{-t} - p_2 e^{-2t}$$

$$\dot{c}_3 = -(2c_2 + c_1^2) p_1 e^{-t} - 3c_1 p_2 e^{-2t} - p_3 e^{-3t} .$$

If we identify z with

$$Z = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

then

$$G := \sum_{n=1}^{\infty} c_{n-1} Z^n = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ c_1 & 1 & 0 & 0 & \dots \\ c_2 & c_1 & 1 & 0 & \dots \\ c_3 & c_2 & c_1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.9}$$

we get a matrix formulation

$$\dot{G} = -\sum_{m=1}^{\infty} p_m e^{-mt} G^{m+1} .$$

Both sides are uniquely determined by their first column, so we get

$$\dot{c} = -\Big(\sum_{m=1}^{\infty} p_m e^{-mt} G^m\Big) c$$

where $c^T = (0, 1, c_1, c_2, \dots)$. The latter formula are practical for computations in concrete cases.

We introduce formal variables $\{\psi_k\}_{k\in\mathbb{N}}$ as generalised momenta, and make an Hamiltonian system, such that the equations for $\dot{c}$ are solutions. We restrict ourself to a finite number of coefficient, and define the following Hamiltonian on $\mathcal{M}_n$:

$$\mathcal{H}(c, \bar{\psi}, t) = -\sum_{j=1}^{n} \big(\sum_{m=1}^{j} p_m e^{-mt} d_{m+1,j+1}\big) \bar{\psi}_j = -\bar{\psi}^T \Big(\sum_{m=1}^{\infty} p_m e^{-mt} G^m\Big) c \tag{4.10}$$

and we define a vector $\psi^T = (0, 0, \psi_1, \psi_2, \dots, \psi_n)$.

The Hamiltonian give the following derivatives of the generalised momenta

$$\dot{\bar{\psi}}_k = \sum_{j=1}^{n}\Big(\sum_{m=1}^{j} p_m e^{-mt}(m+1)d_{m,j+1-k}\Big)\bar{\psi}_j = \bar{\psi}^T\Big(\sum_{m=1}^{n}(m+1)p_m e^{-mt}G^{m-1}Z^k\Big)c\,. \quad (4.11)$$

For instance, for three coefficients

$$\dot{\bar{\psi}}_1 = 2e^{-t}p_1\bar{\psi}_2 + (2e^{-t}p_1c_1 + 3e^{-2t}p_2)\bar{\psi}_3$$

$$\dot{\bar{\psi}}_2 = 2e^{-t}p_1\bar{\psi}_2$$

$$\dot{\bar{\psi}}_3 = 0\,.$$

Now let us define a series in the following way

$$\sum_{j=0}^{n-1} v_{n-j}z^j = g'\Big(\sum_{j=0}^{n-1}\bar{\psi}_{n-j}z^j + \sum_{j=n}^{\infty} b_j(t)z^j\Big) \quad (4.12)$$

where the coefficients b_k depends on t, such that all derivatives of higher order than $n-1$ on the right-hand term vanish. Using that

$$\dot{g}' = g'(1 - p(e^{-t}g,t) - e^{-t}gp'(e^{-t}g,t) = -g'\sum_{m=1}^{\infty}(m+1)p_m e^{-mt}g^m,$$

we get that

$$\frac{d}{dt}\sum_{j=0}^{n-1} v_{n-j}z^j = g'\Big(-(\sum_{m=1}^{\infty}(m+1)p_m e^{-mt}g^m)(\sum_{j=0}^{n-1}\bar{\psi}_{n-j}z^j + \sum_{j=n}^{\infty}b_j(t)z^j) + \sum_{j=0}^{n-1}\dot{\bar{\psi}}_{n-j}z^j + \sum_{j=n}^{\infty}\dot{b}_j(t)z^j\Big)$$

$$= g'\Big(\sum_{j=0}^{n-1}(\dot{\bar{\psi}}_{n-j} - \bar{\psi}_{n-j}\sum_{m=1}^{\infty}(m+1)p_m e^{-mt}g^m)z^j + \sum_{j=n}^{\infty}(\dot{b}_j(t) - b_j\sum_{m=1}^{\infty}(m+1)p_m e^{-mt}g^m)z^j\Big)$$

From (4.11) we know that the terms of degree less or equal to the $n-1$ first terms vanish, and the rest vanishes from the definition of the series. If we view the v_k as functions of $c, \bar{\psi}$ and t, we see that terms 0 to $n-1$ depend only on c and $\bar{\psi}$, so $\frac{\partial}{\partial t}v_k = 0$. From this we get that

$$[v_k, \mathcal{H}] = \sum_{j=1}^{n}\Big(\frac{\partial v_k}{\partial c_j}\frac{\partial \mathcal{H}}{\partial \bar{\psi}_j} - \frac{\partial v_k}{\partial \bar{\psi}_j}\frac{\partial \mathcal{H}}{\partial c_j}\Big) = \sum_{j=1}^{n}\Big(\frac{\partial v_k}{\partial c_j}\dot{c}_j + \frac{\partial v_k}{\partial \bar{\psi}_j}\dot{\bar{\psi}}_j\Big) = \dot{v}_k = 0$$

So all v_k are the first integrals of the Hamiltonian system (4.10). Explicitly, they are given by

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} = (\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3, \dots, \bar{\psi}_n)\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 2c_1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & 0 \\ (n-1)c_{n-2} & (n-2)c_{n-3} & (n-3)c_{n-4} & \dots & 0 \\ nc_{n-1} & (n-1)c_{n-2} & (n-2)c_{n-3} & \dots & 1 \end{pmatrix} \quad (4.13)$$

Note that the matrix given here is the restriction to the n first rows and columns of the matrix corresponding to the function g' in the way of (4.9). If we take the brackets for different combinations of v_k, we obtain

$$[v_j, v_k] = \begin{cases} (j-k)v_{k+j} & \text{when } k+j \le n \\ 0 & \text{else} \end{cases} , \tag{4.14}$$

similar to that of the Witt algebra. This makes the Hamiltonian system partially integrable since $(v_{[n+1/2]}, \dots, v_n)$ are pairwise involuntary ($[\cdot]$ means integer part). The brackets from $(v_1, \dots v_{[n-1]/2})$ generate all the other first integrals. Notice that if we identify $\bar{\psi}_j$ with $\frac{\partial}{\partial c_j}$, then v_i correspond to the truncated Kirillov operators in (4.3). It is not known why there is such a connection.

Generally, it is very hard to find any solutions to this system, because of the complicated formula for $\dot{\bar{\psi}}$.

4.5 Hamiltonian with p_j as generalised momenta

Part of the reason for introducing the Hamiltonian system mentioned above, was to get a better understanding of the problem of, for a given $f \in S$, finding $p \in C$ which generate f. The idea that this should appear as geodesics of a Hamiltonian systems, comes from the connections of univalent functions with $\mathrm{Diff}_+ S^1$, which is important in mathematical physics. Another motivation is that curves in $\mathcal{N}$ that arises from the Löwner-Kufarev equation, never leaves coefficient manifold, so if a Hamiltonian system was constructed with such curves as solutions, we would be able to study $\mathcal{N}$ through geodesics. The Hamiltonian system in the previous section has the weakness that it depends on p_k, however, these are neither generalised coordinates or momenta. In some sense, we get one Hamiltonian for every choice of p. It would be better if we could find a system, in which the coefficients of p are generalised momenta. Off course, it will not be possible to put $p_j = \psi_j$ in (4.10), since $\mathcal{H} = -\bar{p}\big(\sum_{m=1}^{\infty} p_m e^{mt} G^m\big)$ is obviously not holomorphic with respect to $\bar{p}$.

We will make an attempt to make another Hamiltonian system, by choose different coordinates on $\mathcal{M}_n$. In simplest case when $n = 2$, the system above is actually integrable, and we take advantage of this. Let $q_1 = c_1$ and $q_2 = c_2 - c_1^2$. This gives derivatives

$$\dot{q}_1 = -p_1 e^{-t} ,$$

$$\dot{q}_2 = -p_2 e^{-2t} ,$$

and taking p as generalised momenta in a real Hamiltonian system, we obtain the Hamiltonian

$$H(c, \bar{p}, t) = -\frac{1}{2}\big(e^{-t}|p_1|^2 + e^{-2t}|p_2|^2\big) .$$

This gives $\dot{p}_1 = \dot{p}_2 = 0$. If we denote $\dot{p}_j = b_j$, and solve the system for initial conditions $c(0) = 0$, $\lim_{t\to\infty} c(t) = (a_1, a_2)$, we obtain

$$q_1 = -b_1(1 - e^{-t})$$

and then taking the limit, we get that $b_1 = -a_1$, and similar calculation gives that $q_2 = -\frac{b_2}{2}(1 - e^{-2t})$, with $b_2 = -2(a_2 - a_1^2)$. Hence the solutions to the system is given by

$$c_1 = a_1(1 - e^{-t})$$

$$c_2 = a_2(1 - e^{-2t}) - 2a_1^2 e^{-t}(1 - e^{-t})$$

which if $a_1 \neq 0$, gives $c_2 = (2 - \frac{a_2}{a_1})c_1^2 - 2(a_1 - \frac{a_2}{a_1})c_1$

Does these solutions correspond to an actually choice of p? Even if this is the case, we have no direct way continue this to more than two coefficients. For instance, there exist no choice of $q = (q_1, q_2, q_3)$ as combinations of c_1, c_2, c_3 (and not their derivatives), such that $\dot{q}_j = -e^{jt}p_j$. However, we will investigate if this is possible to extend this Hamiltonian to more coefficients using Sub-Riemannian geometry.

4.6 Sub-Riemannian geometry

Let M be a n-dimensional smooth manifold. We say that a distribution D of a M (i.e. a subbundle of it's tangent bundle) fulfils the bracket generating condition if there exist vector fields $X_1, \ldots, X_k$, with $k \leq n$ in D, such that they and their brackets generate TM. By "their brackets", we mean not only $[X_i, X_j]$, but also $[X_i, [X_j, X_k]]$ and so on.

Definition 4.6. *Let M be an n-dimensional differentiable manifold, then a sub-Riemannian manifold is a triple (M, D, ρ), where D is a distribution of M satisfying the bracket generating condition, and ρ is a Riemannian metric on D (i.e. a positive definite section on the bundle $Sym^2 D^* \to M$)*

As usual, we will only denote the sub-Riemannian manifold by M. We have the following notions connected to sub-Riemannian manifolds:

- ρ is called a sub-Riemannian metric. We will use the usual inner product notation $\langle \cdot, \cdot \rangle$ for ρ.
- If the D is a k-dimensional distribution, we say that M has dimension (k, n). If M has dimension (n, n), then it is off course just a Riemannian manifold.
- A horizontal path is an absolutely continuous path $\gamma : [0, 1] \to M$ such that $\dot{\gamma}$ is in D
- If $r, s \in M$, we define the Carnot-Caratheodory distance by

$$d(r, s) = \inf_{\gamma \in \Gamma_{r,s}} \int_{[0,1]} \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} dt$$

where $\Gamma_{r,s} = \{\gamma : [0, 1] \to M \,|\, \gamma \,\text{is a horizontal path}, \gamma(0) = r, \; \gamma(1) = s\}$.

If a manifold is connected, then there always exist at least one smooth horizontal path connecting any two points (this is called Chow's theorem, see [5] and [29]).

4.7 Results for sub-Riemannian Hamiltonian

We consider the truncated Kirillov's operators

$$L_j = \frac{\partial}{\partial c_j} + \sum_{m=1}^{n-m} (m+1) c_m \frac{\partial}{\partial c_{j+m}} \in \Gamma(T^{1,0}\mathcal{M}_n).$$

From the relations in (4.14) and the fact that $\{L_j\}_{j=1}^n$ form a basis for $T^{1,0}\mathcal{M}_n$, we get that $D = \text{span}(L_1, L_2, \ldots, L_k)$ is a bracket generating distribution for $k = 2, 3, \ldots, n$. This will enable us to to denfine sub-Riemannian geometry on $\mathcal{M}_n$, which in turn will be used this to define a Hamiltonian, requirering that any geodesic has to be horizontal, with respect to this geometry. Some results for $k = 2$ was done in [25], and we try to generalise this to arbitrary k, and look at the geodesics for some values (k, n).

Proposition 4.7. *A path $\gamma(t) = (c_1(t), \dots, c_n(t))$ in $\mathcal{M}_n$ is horizontal with respect to distribution D as described above, if and only if*

$$\dot{c}_j = \sum_{m=1}^{k} (j + 1 - m) c_{j-m} u_m \qquad j = k+1, k+2, \dots, n\,,$$

where

$$u_j = \dot{c}_j - \sum_{m=1}^{j-1} (j + 1 - m) c_{j-m} u_m \qquad j = 1, 2, \dots k\,.$$

Proof. The whole result is done by changing basis from $\partial_j = \frac{\partial}{\partial c_j}$ to L_j by formula

$$\partial_m = L_m - \sum_{j=m+1}^{n} (j - m + 1) c_{j-m} \partial_j\,.$$

We obtain

$$\begin{aligned}
\dot{\gamma} &= \sum_{j=1}^{n} \dot{c}_j \partial_j \\
(u_1 = \dot{c}_1) \qquad &= u_1 L_1 + \sum_{j=2}^{n} (\dot{c}_j - j c_{j-1} u_1) \partial_j \\
(u_2 = \dot{c}_2 - 2 c_1 u_1) \qquad &= u_1 L_1 + u_2 L_2 + \sum_{j=3}^{n} (\dot{c}_j - j c_{j-1} u_1 - (j-1) c_{j-2} u_2) \partial_j \\
&\cdots
\end{aligned}$$

$$= u_1 L_1 + u_2 L_2 + \dots + u_k L_k + \sum_{j=k+1}^{n} (\dot{c}_j - j c_{j-1} u_1 - (j-1) c_{j-2} u_2 - \dots - (j-k+1) c_{j-k} u_k) \partial_j\,.$$

In order to be horizontal, every term in the last sum must be 0. $\square$

We make $\mathcal{M}_n$ a sub-Riemannian manifold, by restricting the usual Hermitian innerproduct to D, multiplied by $\frac{1}{2}$ for simplicity. We let $\bar{\xi}_j = \frac{\partial}{\partial c_j}$ be the generalised momenta and for $j = 1, \dots, k$ let $l_j(\bar{\xi}, c)$ be the function of these variables corresponding to L_j, that is

$$l_j = \bar{\xi}_j + \sum_{m=1}^{n-j} (m+1) c_m \bar{\xi}_{j+m}\,.$$

We obtained the following real Hamiltonian

$$H(\bar{\xi}, c) = \frac{1}{2} \left(|l_1|^2 + |l_2|^2 + \dots + |l_k|^2 \right). \tag{4.15}$$

Solutions is then given by

$$\dot{c}_1 = \bar{l}_1$$

$$\dot{c}_2 = 2c_1\bar{l}_1 + \bar{l}_2$$

$$\vdots$$

$$\dot{c}_{k-1} = (k-1)c_{k-2}\bar{l}_1 + (k-2)c_{k-3}\bar{l}_2 + \cdots + 2c_1\bar{l}_{k-2} + \bar{l}_{k-1}$$

$$\dot{c}_k = kc_{k-1}\bar{l}_1 + (k-1)c_{k-2}\bar{l}_2 + (k-2)c_{k-3}\bar{l}_3 + \cdots + 2c_1\bar{l}_{k-1} + \bar{l}_k$$

and further

$$\dot{c}_{k+1} = (k+1)c_k\bar{l}_1 + kc_{k-1}\bar{l}_2 + \cdots + 3c_2\bar{l}_{k-1} + 2c_1\bar{l}_k$$

$$\vdots$$

$$\dot{c}_n = nc_{n-1}\bar{l}_1 + (n-1)c_{n-2}\bar{l}_2 + \cdots + (n-k+2)c_{n-k+1}\bar{l}_{k-1} + (n-k+1)c_{n-k}\bar{l}_k\,.$$

As expected, the equations for $\dot{c}_j$ is just the horizontally conditions. From the first k terms we obtain that $l_j = u_j$, and the $n-k$ next terms just give the statement in proposition 4.7. For the generalised momenta, we have

$$\dot{\xi}_j = -(j+1)(\xi_{j+1}l_1 + \xi_{j+2}l_2 + \cdots + \xi_{j+k}l_k) \qquad j = 1,2,\ldots,n-k$$

and further

$$\dot{\xi}_{n-k+1} = -(n-k+2)(\xi_{n-k+2}l_1 + \xi_{n-k+3}l_2 + \cdots + \xi_{n-1}l_{k-2} + \xi_n l_{k-1})$$

$$\dot{\xi}_{n-k+2} = -(n-k+3)(\xi_{n-k+3}l_1 + \xi_{n-k+4}l_2 + \cdots + \xi_n l_{k-2})$$

$$\vdots$$

$$\dot{\xi}_{n-2} = -(n-1)(\xi_{n-1}l_1 + \xi_n l_2)$$

$$\dot{\xi}_{n-1} = -n\xi_n l_1$$

$$\dot{\xi}_n = 0\,.$$

In addition we have the following relations with the derivatives of the l_j.

Proposition 4.8. *Let*

$$l_j = \sum_{m=0}^{n-j}(m+1)c_m\bar{\xi}_{j+m}$$

for $j = 1,\ldots,n$, where the derivatives of ξ_j and c_j are given by Hamiltonian system (4.15), and let $l_j \equiv 0$ for $j > n$. Then

$$\dot{l}_j = \sum_{\nu=1}^{k}(\nu - j)\bar{l}_\nu l_{\nu+j}$$

for any $j \in \mathbb{N}$

Proof. To simplify notation, we will say that any sum $\sum_{\nu=a}^{b}\beta_\nu$, with $b<a$, will be zero (a sum over the empty set). Then for any $j\in\mathbb{N}$

$$l_j=\sum_{n=0}^{n-j}(m+1)c_m\bar{\xi}_{j+m}$$

Differentiating we get

$$\dot{l}_j=\sum_{m=1}^{\min(k,n-j)}(m+1)\bar{\xi}_{j+m}\sum_{\nu=1}^{m}(m-\nu+1)c_{m-\nu}\bar{l}_\nu+\sum_{m=k+1}^{n-j}(m+1)\bar{\xi}_{j+m}\sum_{\nu=1}^{m}(m-\nu+1)c_{m-\nu}\bar{l}_\nu$$

$$-\sum_{m=0}^{n-j-k}(m+1)c_m(j+m+1)\sum_{\nu=1}^{k}\bar{\xi}_{m+j+\nu}\bar{l}_\nu-\sum_{m=\max(n-j-k+1,0)}^{n-j-k}(m+1)c_m(j+m+1)\sum_{\nu=1}^{k}\bar{\xi}_{m+j+\nu}\bar{l}_\nu$$

$$=\sum_{\nu=1}^{k}\bar{l}_\nu\Big(\sum_{m=\nu}^{n-j}(m+1)(m-\nu+1)\bar{\xi}_{j+m}c_{m-\nu}-\sum_{m=\nu}^{n-j}(m-\nu+1)(m-\nu+j+1)\bar{\xi}_{j+m}c_{m-\nu}\Big)$$

$$=\sum_{\nu=1}^{k}(\nu-j)\bar{l}_\nu\Big(\sum_{m=\nu}^{n-j}(m-\nu+1)\bar{\xi}_{j+m}c_{m-\nu}\Big)=\sum_{\nu=1}^{k}(\nu-j)\bar{l}_\nu l_{\nu+j}$$

□

Although this system might be interresting, it has its problems. Most importantly, there is nothing insuring us that a solution to this system will stay in $\mathcal{M}_n$. The description above is therefore merely local.

Example 4.9. We try to get some sense into what the geodesics are for this Hamiltonian when $\mathcal{M}_n$ is is a (k,n)-dimensional sub-Riemannian manifold. To simplify, we will look at a geodesic c in $\mathcal{M}_n$, from $c(0)=0$ to $c(\tau)=a$. We will denote $\xi(0)=\bar{l}(0)=b$, and if we look at geodesics with H constantly equal to 1, this means that $|b_1|^2+\cdots+|b_k|^2=1$.

- For $k=2, n=2$, we have the solutions for H given by

$$c_1=b_1t$$

$$c_2=b_1^2t^2+b_2t$$

$$\xi_1=b_1-2b_2\bar{c}_1$$

$$\xi_2=b_2\,.$$

  Geodesics are hence on the form $c_2=c_1^2+\frac{b_2}{b_1}c_1^2$ or straight lines $c_2=b_2t$. From $H=1$, we get that $b_1=\frac{a_1}{\sqrt{|a_1|^2+|a_2-a_1^2|}}$, $b_2=\frac{a_2-a_1^2}{\sqrt{|a_1|^2+|a_2-a_1^2|}}$ and $\tau=\sqrt{|a_1|^2+|a_2-a_1^2|}$. There is hence a unique choice of geodesics in this case.

- More generally, geodesics in the case $k=n$, c_j is given by a j'th order polynomial, and the there is a unique geodesic from $c(0)=0$ to $c(1)=a$.

- For $k=2, n=3$, the solutions to H are

$$\xi_3 = b_3$$

$$\xi_2 = b_2 - 3b_3\bar{c}_1$$

$$\xi_1 = b_1 - 2b_2\bar{c}_1 + 5b_3\bar{c}_1^2 - 2b_3\bar{c}_2\,.$$

From this and the fact that $\ddot{c}_1 = \dot{\bar{l}}_1 = l_2\bar{l}_3 = \bar{b}_2 b_3 - |b_3|^2 c_1$, we get that for $b_3 \neq 0$

$$c_1 = \frac{1}{\bar{b}_3}(\bar{b}_2 - \bar{b}_2\cos|b_3|^2 t + \frac{\bar{b}_1}{b_3}\sin|b_3|^2 t)$$

$$c_2 = c_1^2 + \int_0^t \bar{l}_2 dt = c_1^2 - \frac{1}{|b_3|^2}(b_2\sin|b_3|^2 t + \frac{b_1}{\bar{b}_3}\cos|b_3|^2 t - \frac{b_1}{\bar{b}_3})$$

$$c_3 = 3c_1c_2 - c_1^3 - \int_0^t c_1\bar{l}_2 dt$$

$$= 3c_1c_2 - c_1^3 + \frac{1}{\bar{b}_3|b_3|^2}\Big(\frac{1}{4}(\frac{\bar{b}_1 b_2}{b_3} - \frac{3b_1\bar{b}_2}{\bar{b}_3}) - \frac{|b_2b_3|^2+|b_1|^2}{2}t + |b_2|^2\sin|b_3|^2 t + \frac{b_1\bar{b}_2}{\bar{b}_3}\cos|b_3|^2 t$$
$$- \frac{|b_2b_3|^2 - |b_1|^2}{4|b_3|^2}\sin 2|b_3|^2 t - \frac{1}{4}(\frac{\bar{b}_1 b_2}{b_3} + \frac{b_1\bar{b}_2}{\bar{b}_3})\cos 2|b_3|^2 t\Big)$$

and for $b_3 = 0$

$$c_1 = b_1 t$$

$$c_2 = b_1^2 t^2 + b_2 t$$

$$c_3 = b_1^3 t^3 + \frac{5}{2}b_1 b_2 t^2$$

To find a general geodesic is hard, so let us pick a point $(0,0,a_3)$, with $a_3 \neq 0$, as an example. $c_1(\tau) = c_2(\tau) = 0$ just gives us the information that $|b_3|^2\tau = 2\pi\nu$ for some $\nu \in \mathbb{N}$ (note that b_3 must be nonzero). From $c_3(\tau) = a_3$, we get

$$4a_3\bar{b}_3|b_3|^2 = \frac{\bar{b}_1 b_2}{b_3} - \frac{b_1\bar{b}_2}{\bar{b}_3} - 2|b_2|^2|b_3|^2\tau - 2|b_1|^2\tau + \frac{4b_1\bar{b}_2}{\bar{b}_3} - \frac{\bar{b}_1 b_2}{b_3} - \frac{b_1\bar{b}_2}{\bar{b}_3}$$

$$= -2|b_2|^2|b_3|^2\tau - 2|b_1|^2\tau$$

$a_3\bar{b}_3$ must therefore be real negative. Inserting $b_3 = -\frac{a_3}{|a_3|}\sqrt{\frac{2\pi\nu}{\tau}}$ and $|b_2|^2 = 1 - |b_1|^2$, we have the relation

$$|b_1|^2 = \frac{2\pi\nu(\tau^{3/2} - 2|a_3|\sqrt{2\pi\nu})}{(2\pi\nu - \tau)\tau^{3/2}}\,, \tag{4.16}$$

which have a solution for choices of ν, τ such that the right hand equation is in the interval $[0,1]$. From this we get that there are uncountable many geodesics:

  - The solution is completely independent of the arguments of b_1 and b_2.

- We may also freely choose any value of $\nu > 1$ and $0 \leq |b_1| \leq 1$. To see this, let us denote the right hand side of (4.16) by
$$\mathfrak{b}_\nu(\tau) = \frac{2\pi\nu(\tau^{3/2} - 2|a_3|\sqrt{2\pi\nu})}{(2\pi\nu - \tau)\tau^{3/2}}.$$
This is a continuous functions with respect to τ in $(0, 2\pi\nu)$. Since, we know that $|a_3| \leq 4 < 2\pi$, we get that for any choice of $\nu \geq 2$
$$\lim_{\tau \to 0+} \mathfrak{b}_\nu(\tau) = -\infty \qquad \text{and} \qquad \lim_{t \to 2\pi\nu-} \mathfrak{b}_\nu(\tau) = \infty.$$
From this we get that the image of the interval $(0, 2\pi)$ under $\mathfrak{b}_\nu$ is the real line, and in particular, for any choice of $\nu > 1$ and $|b_1|$, there is a τ such that $|b_1|^2 = \mathfrak{b}_\nu(\tau)$. (if $|a_3| < \pi$, this is also guaranteed when $\nu = 1$).
- From the discussion above, it also follows that the geodesic that reach $(0, 0, a_3)$ in minimal time, for fixed ν, has $b_1 = 0$ (which imply $b_2 = e^{i\alpha}$). In this case
$$c_1 = -\frac{a_3 e^{i\alpha}}{|a_3|}\sqrt{\frac{\tau}{2\pi\nu}}(1 - \cos\frac{2\pi\nu t}{\tau})$$
$$c_2 - c_1^2 = -\frac{\tau e^{i\theta}}{2\pi\nu}\sin\frac{2\pi\nu t}{\tau}$$
$$c_3 - 3c_1c_2 + c_1^3 = \frac{a_3}{|a_3|}\sqrt{\frac{\pi\nu}{2\tau}}t - \frac{4a_3}{|a_3|}\sqrt{\frac{\tau}{2\pi\nu}}(4\sin\frac{2\pi\nu t}{\tau} - \sin\frac{4\pi\nu t}{\tau})$$
and it follows that $\tau = \frac{2|a_3|^2}{\pi\nu}$ is the minimal time to reach (0,0,$|a_3|$).

The general case is much more difficult, since $|b_3|$ appear both inside and outside the trigonometric functions. We can however expect there to be fewer geodesics in this case, since we get more "information" from c_1 and c_2 in this case.

Since there have been very few investigations into this sub-Riemannian Hamiltonian system, in is still uncertain how to interpret solutions. We end this chapter discussing some possible connections.

- Starting with the Löwner-Kufarev equation for subordinationchains
$$\frac{\partial}{\partial t}f(z,t) = zp(z,t)\frac{\partial}{\partial z}f(z,t)$$
for $f(z,t) = e^t(z + \cdots)$. Let $\eta(z,t) = e^{-t}f(z,t) = z + c_1z^2 + c_2z^3 + \cdots$ be a curve in S. Assume that its image is contained in $\tilde{S}$. Then, if we denote $\frac{\partial}{\partial z}\eta(z,t) = \eta'(z,t)$
$$\dot{\eta} = z\eta' p(z,t) - \eta = L_0(\eta) + p_1L_1(\eta) + p_2L_2(\eta) + \cdots$$
and $p_j = u_j$, where u_j are the coefficients in the horizontally condition (both follow from the same change of basis). If η is a also a solution to the (k,n) sub-Riemannian Hamiltonian system, this means that $\sum_{j=1}^k |p_j|^2$ is constant.

- The Hamiltonian system $\tilde{H} = e^{-t}|l_1|^2 + e^{-2t}|l_2|^2$ give the same solutions as the Hamiltonian described in 4.5 for $n = 2$. This means that, in some sense, we can extend this simple Hamiltonian using sub-Riemannian geometry. The simplicity will not be kept, thought. For instance, for $n = 3$, c_1 is given by solution
$$\ddot{c}_1 - \dot{c}_1 - b_3\bar{b}_2e^{-t}c_1 = |b_3|^2.$$

Index of Notation

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