Multipliers of the Dirichlet space

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Contents

Chapter 1

The Dirichlet space and its multipliers

The topic of this thesis is the Dirichlet space and the multipliers of this space, to be defined soon. Both the Dirichlet space and its multipliers have been the focus of quite a lot of research and I have endeavoured to present some of the results in this thesis. On the way some extensions and observations by my supervisor Arne Stray and myself are added. The main topics are the elementary properties of the Dirichlet space and its multipliers, characterizations of multipliers by techniques from potential theory, univalent multipliers and the Banach algebra of multipliers.

The reader is assumed to be familiar with the elementary theory of analytic functions and functional analysis. Moreover, some knowledge of the various boundary value results of analytic and harmonic functions in the unit disk will be helpful. A good reference on this subject is [15]. I have tried to introduce most of the capacitary notions needed in Chapter 2 but for Chapter 4 some knowledge of the elementary theory of Banach algebras will be helpful.

1.1 The Dirichlet space and related function spaces

We begin by defining the Dirichlet space and the related Hardy spaces of analytic functions in the unit disk. A general background reference for the Dirichlet space is the article [19]. We shall state and prove some of the elementary properties of this space.

The *Dirichlet space* D is the set of analytic functions on the unit disk U which have a finite Dirichlet integral

$$\mathcal{D}(f) = \frac{1}{\pi}\int_U |f'|^2 dA \tag{1.1}$$

where dA is the two-dimensional Lebesgue measure. An analytic function in the unit disk with finite Dirichlet integral will be called *Dirichlet finite.* If f is univalent and Dirichlet finite we have the following geometric interpretation of the Dirichlet integral.

Proposition 1.1. *Let f be a univalent function in the unit disk. Then $f \in D$ if and only if the area of the range of f, $A(f(U)) = Area(f(U))$, is finite. Moreover*

$$\pi\mathcal{D}(f) = Area(f(U)) \tag{1.2}$$

in this case.

Proof. Since the map $f : U \to f(U)$ is an analytic bijection, a change of variables show that

$$\text{Area}(f(U)) = \int_{f(U)} 1 dA(w) = \int_U |J_f| dA(z) \tag{1.3}$$

where J_f is the Jacobian of the transformation f. There may be points where $J_f = 0$, but by Sard's Theorem the set of such points must have measure zero and can be excluded (see [25], page 72). Since $f = u + iv$ is analytic we can apply the Cauchy-Riemann equations and obtain

$$\begin{aligned} J_f &= \begin{vmatrix} \partial u/\partial x & \partial v/\partial x \\ \partial u/\partial y & \partial v/\partial y \end{vmatrix} \\ &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \\ &= \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x}\Big)^2 \\ &= |f'|^2. \end{aligned} \tag{1.4}$$

From (1.3) and (1.4) we see that

$$\text{Area}(f(U)) = \int_U |f'|^2 dA = \pi \mathcal{D}(f)$$

as was to be proved. □

The expression (1.2) does not in general hold if f is not injective. For example, consider the function $f(z) = z^n$ where $n \in \mathbb{N}$ which maps the unit disk onto itself. Computing the Dirichlet integral in polar coordinates, we see that

$$\begin{aligned} \pi\mathcal{D}(f) &= \int_U |nz^{n-1}|^2 dA \\ &= n^2 \int_0^1 r dr \int_{-\pi}^{\pi} r^{2(n-1)} d\theta \\ &= 2\pi n^2 \int_0^1 r^{2n-1} dr \\ &= n\pi \end{aligned}$$

which is n times the area of U. Thus the Dirichlet integral is sensitive to multiplicity of the range of the function. Indeed, there are bounded analytic functions with infinite Dirichlet integral, for example any infinite Blaschke product (this is a corollary of Carleson's formula for the Dirichlet integral, see [19], Corollary 3.4). If H^∞ denotes the set of bounded analytic functions on the unit disk, we conclude that $H^\infty \not\subseteq D$.

There are unbounded analytic functions with finite Dirichlet integral. To construct an example, let $\triangle$ denote the disk of radius 2 centred at the origin and let

$$E = \{(x, y) : x > 1, -\frac{1}{x^2} < y < \frac{1}{x^2}\}.$$

Evidently, the set $G = \triangle \cup E$ is a simply connected open set and so there exists a univalent function f mapping the unit disk onto G by the Riemann mapping theorem. Clearly, the area of G is finite and so $\pi\mathcal{D}(f) = \text{Area}(G) < \infty$ and $f \in D$ by Proposition 1.1. However, f is unbounded and so $D \not\subseteq H^\infty$.

Let f be any Dirichlet finite function. Since f is analytic, it is representable as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

We can compute the Dirichlet integral from the Taylor coefficients.

Proposition 1.2. *If $f \in D$ then $\mathcal{D}(f) = \sum_{n=1}^{\infty} n|a_n|$ where a_n is the n'th Taylor coefficient of f.*

Proof. Since $\sum a_n z^n$ is a power series, we can differentiate term by term.

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \tag{1.5}$$

Writing (1.5) in polar coordinates, we get

$$f'(re^{i\theta}) = \sum_{n=1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta} \tag{1.6}$$

Since the series (1.5) converges uniformly on compact sets, the series (1.6) converges uniformly on $[-\pi, \pi]$ when r is a fixed number in the interval $(0,1)$. Thus the function $g_r(e^{i\theta}) = f'(re^{i\theta})$ is continuous on T for such r. Since T is compact $C(T) \subset L^\infty(T) \subset L^2(T)$ and so $g_r \in L^2(T)$. Hence (1.6) is the Fourier series for g_r and so its $(n-1)$'th Fourier coefficient is given by the formula

$$\hat{g}_r(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_r(e^{i\theta}) e^{-i(n-1)\theta} d\theta = n a_n r^{n-1}.$$

Computing the L^2-norm of g_r in terms of the Fourier coefficients yields the formula

$$\|g_r\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |g_r(e^{i\theta})|^2 d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(re^{i\theta})|^2 d\theta = \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2(n-1)}. \tag{1.7}$$

We can now compute the Dirichlet integral of f by integrating in polar coordinates and applying Fatou's theorem, (1.7) and the Monotone Convergence Theorem.

$$\begin{aligned}
\mathcal{D}(f) =& \frac{1}{\pi} \int_0^1 r dr \int_{-\pi}^{\pi} |f'(re^{i\theta})|^2 d\theta \\
=& 2 \int_0^1 \left(\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2(n-1)} \right) r dr \\
=& 2 \sum_{n=1}^{\infty} \int_0^1 n^2 |a_n|^2 r^{2n-1} dr \\
=& 2 \sum_{n=1}^{\infty} \left(\frac{n}{2} |a_n|^2 r^{2n} \Big|_0^1 \right) \\
=& \sum_{n=1}^{\infty} n |a_n|^2
\end{aligned}$$

□

The *Hardy space* H^p, $0 < p < \infty$, is the space of analytic functions f in the unit disk for which the functions $f_r(e^{i\theta}) = f(re^{i\theta})$ are bounded in L^p-norm as r tends to 1. To be precise:

$$H^p = \{f : U \longrightarrow \mathbb{C} |\, f \text{ analytic },\ \sup_{0<r<1} \|f_r\|_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty\}.$$

In the definition above, $d\theta$ denotes the Lebesgue measure on T. When $p = \infty$ define H^∞ to be the space of bounded analytic functions on the unit disk. For $1 \le p \le \infty$, the Hardy space is a Banach space with the norm

$$\|f\|_p = \sup_{0<r<1} \|f_r\|_p = \lim_{r\to 1} \|f_r\|_p \qquad (1 < p < \infty) \tag{1.8}$$

$$\|f\|_\infty = \sup_{z\in U} |f(z)| \tag{1.9}$$

The second equality in (1.8) stems from the fact that $\|f_r\|$ is a non-decreasing function of r (see [23] page 338).

Suppose $1 \le p \le \infty$. By a theorem of Fatou the non-tangential limit

$$\tilde{f}(e^{i\theta_0}) = \lim_{re^{i\theta} \to e^{i\theta_0}} f(re^{i\theta})$$

exists for almost all $e^{i\theta_0} \in T$ as the point $re^{i\theta}$ approaches $e^{i\theta_0}$ along any path in the unit disk which is not tangent to the unit circle. Limits of this type are called *non-tangential* limits. The limits define a function $\tilde{f} \in L^p(T)$. Moreover f is the Poisson integral of $\tilde{f}$ and the L^p-norm of the boundary function $\tilde{f}$ is equal to the H^p-norm of f defined above, that is:

$$\|\tilde{f}\|_p = \|f\|_p$$

With this correspondence we can identify H^p with the space of L^p-functions on the unit circle such that the Poisson integral is analytic in the open unit disk. This can also be formulated in terms of Fourier coefficients: H^p is the space of L^p functions on the unit circle whose negative Fourier coefficients vanish. Moreover, if $\{\hat{f}(n)\}_{n=0}^{\infty}$ is the sequence of Fourier coefficients a function $f \in H^p$ then

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

is the Taylor series for f in the unit disk. In this sense the Hardy space is a bridge between analytic function theory and Fourier analysis.

The space H^2 is of special interest. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is the power series representation of a function $f \in H^2$, then

$$\|f\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2 \tag{1.10}$$

The proof is similar to the proof of Proposition 1.2 above.

The *harmonic space* h^p, $0 < p \le \infty$, is the space of harmonic functions f in the unit disk for which the functions $f_r(e^{i\theta}) = f(re^{i\theta})$ are bounded in L^p-norm as r tends to 1. That is, h^p is the harmonic analogue of the Hardy space. The norm on h^p is defined just as for H^p and the various boundary value results mentioned for H^p holds for h^p as well, in particular the theorem of Fatou when $1 \le p \le \infty$. However, the boundary functions in h^p may have non-zero negative Fourier coefficients. In fact, h^p is isomorphic to the space $L^p(T)$ by the same correspondence which identified H^p with the class of L^p functions whose negative Fourier coefficients vanish.

For details and proofs on the boundary behaviour of analytic and harmonic functions in the unit disk, consult [15], chapters 3 and 4.

Another function space we shall consider is the *Bergman space* B, which is the space of all analytic functions f in the unit disk such that the integral $\int_U |f|^2 dA$ is finite. This is a Banach space with the norm

$$\|f\|_B^2 = \frac{1}{\pi}\int_U |f|^2 dA. \tag{1.11}$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the power series representation of $f \in B$. Then

$$\|f\|_B^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} \tag{1.12}$$

The proof of (1.12) is analogous to the proof of Proposition 1.2. From this equation together with Proposition 1.2 one immediately obtains the following relations between the Bergman space and the Dirichlet space:

Proposition 1.3. *1. $g \in D$ if and only if $g' \in B$.*

2. If $f \in D$ then $\|f'\|_B \leq \|f\|_D$.

3. $D \subset B$

Returning to the Dirichlet space, we see from (1.10) and Proposition 1.2 that the Dirichlet space is a subset of H^2. Thus the various boundary value results for H^2 mentioned above apply to D as well. In particular the functions in the Dirichlet space have non-tangential boundary values almost everywhere. In fact, even stronger boundary value results can be obtained for functions in the Dirichlet space. For details, see [19], chapter 5.

We now ask the question: can the Dirichlet integral of a function $f \in D$ be calculated from the boundary values? The answer is affirmative. We have the following formula by Jesse Douglas, see [9].

$$\mathcal{D}(f) = \frac{1}{8\pi^2}\int_{-\pi}^{\pi} dt \int_{-\pi}^{\pi} \frac{|f(\theta) - f(t)|^2}{\sin^2\left(\frac{\theta - t}{2}\right)} d\theta \tag{1.13}$$

We will commonly use the same notation for an analytic function on the unit disk and the corresponding boundary function on the unit circle (whenever it exists). In this case the context should make it clear which function is used. Whenever the need arises to distinguish between an analytic function f and the corresponding boundary function, we will denote the latter by $\tilde{f}$.

The Dirichlet integral is not a norm on D since $D(c) = 0$ for all constants c. But having observed that $D \subset H^2$ we can endow D with the norm

$$\|f\|^2 = \|f\|_2^2 + \mathcal{D}(f) = \sum_{n=0}^{\infty} (1+n)|a_n|^2 \tag{1.14}$$

We shall write $\|.\|_D$ when there is need to be precise. It can be shown that D is complete in this norm and, consequently, it is a Banach space. In fact, D is a Hilbert space with the inner product obtained from the norm (1.14):

$$\langle f, g\rangle = \frac{1}{2\pi}\int_T f\bar{g}d\theta + \frac{1}{\pi}\int_U f'\bar{g}'dA = \sum_{n=0}^{\infty}(1+n)a_n\overline{b_n} \tag{1.15}$$

where the a_n and the b_n are the Taylor coefficients of f and g respectively.

Now, fix $w \in U - \{0\}$ and define the function

$$k_w(z) = \frac{1}{\bar{w}z}\log\frac{1}{1-\bar{w}z} \tag{1.16}$$

If $w = 0$ define k_w to be the constant function with value 1. Let $w \neq 0$ and choose the principal branch of $\log\frac{1}{1-\bar{w}z}$. Then $\lim_{z\to 0} zk_w(z) = 0$ and so the singularity of k_w at $z = 0$ is removable. Consequently k_w can be uniquely extended to an analytic function in the unit disk with the value 1 at $z = 0$. From now k_w will denote this extended function. Note that k_w is a bounded function for fixed w. Moreover k_w produces a point evaluation operator by the inner product, which is the content of the next proposition. For this reason we call k_w a *reproducing kernel* for the Hilbert space D.

Proposition 1.4. *Fix $w \in U$ and let $z \in U$. Then*

1. $k_w(z) = \sum_{n=0}^{\infty}(n+1)^{-1}(\bar{w}z)^n$ *and so* $k_w \in D$
2. $f(w) = \langle f, k_w\rangle$ *for any* $f \in D$*. In particular, point evaluations are bounded linear functionals on* D*.*

Proof. We first prove (1). Choose the principal branch of $\log z$:

$$\log z = \log|z| + i\mathrm{Arg}z, \quad -\pi < \mathrm{Arg}z < \pi$$

Then the function $\log(1+z)$, where $|z| < 1$, has the following power series representation:

$$\log(1+z) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}z^n, \quad |z| < 1$$

Note that $|\bar{w}z| < 1$ since $w, z \in U$. Thus

$$\log(1-\bar{w}z) = -\sum_{n=1}^{\infty}\frac{1}{n}(\bar{w}z)^n, \quad |\bar{w}z| < 1$$

and so

$$k_w(z) = \frac{1}{\bar{w}z}\log\frac{1}{1-\bar{w}z} = \sum_{n=1}^{\infty}\frac{1}{n}(\bar{w}z)^{n-1} = \sum_{n=0}^{\infty}\frac{1}{n+1}(\bar{w}z)^n$$

The second statement of (1) now follows immediately from Proposition 1.2.

To prove (2), let $f(z) = \sum_{n=0}^{\infty}a_nz^n$ be the power series representation of f. From (1.15) we see that

$$\begin{aligned}\langle f, k_w\rangle &= \sum_{n=0}^{\infty}(n+1)a_n\overline{(n+1)^{-1}\bar{w}^n} \\ &= \sum_{n=0}^{\infty}a_nw^n \\ &= f(w).\end{aligned}$$

□

By Proposition 1.4 and the Schwarz inequality we obtain the following pointwise estimate for a function $f \in D$:

$$\begin{aligned}
|f(z)| = |\langle f, k_z\rangle| &\leq \|f\|\|k_z\| \\
&= \|f\|\langle k_z, k_z\rangle^{\frac{1}{2}} \\
&= \|f\|\Big(\sum_{n=0}^{\infty}(n+1)(n+1)^{-1}\bar{z}^n\overline{(n+1)^{-1}\bar{z}^n}\Big)^{\frac{1}{2}} \\
&= \|f\|\Big(\sum_{n=0}^{\infty}(n+1)^{-1}|z|^{2n}\Big)^{\frac{1}{2}} \\
&= \|f\|k_z(z)^{\frac{1}{2}} \\
&= \|f\|\frac{1}{|z|}\Big(\log\frac{1}{1-|z|^2}\Big)^{\frac{1}{2}} \qquad (1.17)
\end{aligned}$$

In the calculation above we have implicitly assumed that $z \neq 0$. If $z = 0$ then of course $k_z = 1$ and we get the estimate $|f(0)| \leq \|f\|$.

Proposition 1.5. *Let $\{f_n\}$ be a sequence of functions in D, converging in norm to some function $f \in D$. Then $\{f_n\}$ converges uniformly to f on compact subsets of U.*

Proof. Let K be a compact subset of U and let z be any non-zero point of K. We replace f by the function $f_n - f$ in (1.17) and deduce the following:

$$\begin{aligned}
|f_n(z) - f(z)| &\leq \|f_n - f\|\frac{1}{|z|}\Big(\log\frac{1}{1-|z|^2}\Big)^{\frac{1}{2}} \\
&= \epsilon\frac{1}{|z|}\Big(\log\frac{1}{1-|z|^2}\Big)^{\frac{1}{2}}, \quad n > N(\epsilon).
\end{aligned}$$

Since K is compact the function $\frac{1}{|z|}\Big(\log\frac{1}{1-|z|^2}\Big)^{\frac{1}{2}}$ is uniformly bounded on $K - 0$. Since $|f_n(0) - f(0)| \leq \|f_n - f\|$ the result now follows. □

The *harmonic Dirichlet space* D_h is the set of all harmonic functions f in the unit disk which have a finite (harmonic) Dirichlet integral.

$$\mathcal{D}(f) = \frac{1}{\pi}\int_U\Big(\Big|\frac{\partial f}{\partial z}\Big|^2 + \Big|\frac{\partial f}{\partial \bar{z}}\Big|^2\Big)dA = \frac{1}{2\pi}\int_U\Big(\Big|\frac{\partial f}{\partial x}\Big|^2 + \Big|\frac{\partial f}{\partial y}\Big|^2\Big)dA \qquad (1.18)$$

If f is analytic then $\dfrac{\partial f}{\partial z}(z) = f'(z)$ and $\dfrac{\partial f}{\partial \bar{z}} = 0$ and so $\mathcal{D}(f) = \frac{1}{\pi}\int_U |f'|^2 dA$. Thus (1.18) is a generalization of the Dirichlet integral (1.1) to harmonic functions.

Now, let u be any real harmonic function in D_h. If v is a harmonic conjugate of u then $f = u + iv$ is analytic and so

$$|f'|^2 = \Big|\frac{\partial f}{\partial x}\Big|^2 = \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x}\Big)^2 = \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2$$

by the Cauchy-Riemann equations. Hence $\mathcal{D}(f) = 2\mathcal{D}(u) < \infty$ and so $f \in D \subset H^2$. Clearly, the real and imaginary parts of a function in H^2 are contained in h^2. Thus $u \in h^2$. Since h^2

is a linear space we conclude that $D_h \subset h^2$. Consequently, if f is any function in D_h we can write

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta} \tag{1.19}$$

where the numbers $\hat{f}(n)$ are the Fourier coefficients for the boundary function $\tilde{f} \in L^2(T)$ corresponding to f.

Proposition 1.6. *If $f \in D_h$ then $\mathcal{D}(f) = \sum_{n=-\infty}^{\infty} |n||\hat{f}(n)|^2$ where $\hat{f}(n)$ is the n'th Fourier coefficient of the boundary function $\tilde{f} \in L^2(T)$ corresponding to f.*

Proof. We will compute the integral (1.18) using polar coordinates. First, suppose u is a real function in D_h. Then the integrand in (1.18) can be written $|\nabla u|^2$ where

$$\nabla u = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j}$$

is the gradient of u. In polar coordinates the gradient at the point $(x, y) = (r\cos(\theta), r\sin(\theta))$ is given by

$$\nabla u = \frac{\partial u}{\partial r}\vec{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\vec{\theta}$$

where $\vec{r}$ is the radial unit vector and $\vec{\theta}$ is the tangential unit vector. Thus

$$|\nabla u|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2. \tag{1.20}$$

Let $f = u + iv \in D_h$. It is easy to see that

$$\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 = \left|\frac{\partial f}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial f}{\partial \theta}\right|^2 \tag{1.21}$$

by applying (1.20) to the real and imaginary parts f separately.

Now, observe that

$$\begin{aligned} f(re^{i\theta}) &= \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta} \\ &= \sum_{n=0}^{\infty} \hat{f}(n) r^n e^{in\theta} + \overline{\sum_{n=1}^{\infty} \overline{\hat{f}(-n)} r^n e^{in\theta}} \\ &= \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \qquad (1.22) \\ &= f_1(z) + \overline{f_2(z)} \end{aligned}$$

where $a_n = \hat{f}(n)$ and $b_n = \overline{\hat{f}(-n)}$. The sequences $\{|a_n|^2\}_{n=0}^{\infty}$ and $\{|b_n|^2\}_{n=1}^{\infty}$ are summable since $\{|\hat{f}(n)|^2\}_{n=-\infty}^{\infty}$ is summable. Thus f_1 and f_2 are in H^2. Consequently, either series to the right of (1.22) converges uniformly on compact subsets of U. Hence the series (1.19) converges uniformly on compact subsets of U and so we can switch limit processes when necessary.

Let $\overline{U_r}$ denote the closed unit disk centred at the origin with radius r. Evidently $\overline{U_r}$ is compact. If $0 < r_0 < 1$ and $z = re^{i\theta} \in \overline{U_{r_0}}$ then

$$\frac{\partial f}{\partial r}(r,\, \theta) = \frac{\partial}{\partial r}\Big(\sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta}\Big) = \sum_{n=-\infty}^{\infty} |n| \hat{f}(n) r^{|n|-1} e^{in\theta}. \tag{1.23}$$

Since f is harmonic, so is $\partial f/\partial r$ and thus (1.23) converges uniformly on compact subsets of U. Hence $\partial f/\partial r \in C(T) \subset L^2(T)$ and so

$$|n|\hat{f}(n) r^{|n|-1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial f}{\partial r}(r,\, \theta) e^{-in\theta} d\theta$$

as in the proof of Proposition 1.2. Thus

$$\Big\|\frac{\partial f}{\partial r}\Big\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|\frac{\partial f}{\partial r}(r,\, \theta)\Big|^2 d\theta = \sum_{n=-\infty}^{\infty} |n|^2 |\hat{f}(n)|^2 r^{2(|n|-1)}. \tag{1.24}$$

Similarly, for any $z = re^{i\theta} \in \overline{U_{r_0}}$

$$\frac{\partial f}{\partial \theta}(r,\, \theta) = \frac{\partial}{\partial \theta}\Big(\sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta}\Big) = \sum_{n=-\infty}^{\infty} in \hat{f}(n) r^{|n|} e^{in\theta}.$$

Since $\partial f/\partial \theta$ is harmonic, we obtain that

$$in\hat{f}(n) r^{|n|} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial f}{\partial \theta}(r,\, \theta) e^{-in\theta} d\theta$$

and so

$$\Big\|\frac{\partial f}{\partial \theta}\Big\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|\frac{\partial f}{\partial \theta}(r,\, \theta)\Big|^2 d\theta = \sum_{n=-\infty}^{\infty} |n|^2 |\hat{f}(n)|^2 r^{2|n|}. \tag{1.25}$$

From (1.21), Fatou's theorem, (1.24), (1.25) and the Monotone Convergence Theorem we can now compute the Dirichlet integral of f.

$$\begin{aligned}
\mathcal{D}(f) &= \frac{1}{2\pi}\int_0^1 r dr \int_{-\pi}^{\pi} \Big(\Big|\frac{\partial f}{\partial r}\Big|^2 + \frac{1}{r^2}\Big|\frac{\partial f}{\partial \theta}\Big|^2\Big) d\theta \\
&= \int_0^1 \Big(\sum_{n=-\infty}^{\infty} |n|^2 |\hat{f}(n)|^2 r^{2(|n|-1)} + \frac{1}{r^2}\sum_{n=-\infty}^{\infty} |n|^2 |\hat{f}(n)|^2 r^{2|n|}\Big) r dr \\
&= 2\int_0^1 \sum_{n=-\infty}^{\infty} |n|^2 |\hat{f}(n)|^2 r^{2|n|-1} \\
&= \sum_{n=-\infty}^{\infty} |n| |\hat{f}(n)|^2.
\end{aligned}$$

The proof is complete. □

The harmonic Dirichlet space can be given a norm similar to the norm on D, that is

$$\|f\|_{D_h} = \|\tilde{f}\|_2^2 + \mathcal{D}(f) = \sum_{n=-\infty}^{\infty} (1 + |n|) |\hat{f}(n)|^2. \tag{1.26}$$

In fact, D_h is a Hilbert space with the inner product

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} (1+|n|)\hat{f}(n)\overline{\hat{g}(n)} \tag{1.27}$$

and the reproducing kernel

$$K_w(z) = k_w(z) + \overline{k_w(z)} - 1 = 2\mathrm{Re}k_w(z) - 1 = \sum_{n=-\infty}^{\infty} [(1+|n|)^{-1}\rho^{|n|}e^{-in\varphi}]r^{|n|}e^{i\phi} \tag{1.28}$$

where $z = re^{i\phi} \in U$ and $w = \rho e^{i\varphi} \in U$.

Finally, we remark that the Douglas formula (1.13) for computing the Dirichlet integral from the boundary values holds for functions in D_h.

1.2 Multipliers of the Dirichlet space

We will now introduce multipliers of the Dirichlet space and deduce some properties of these functions. We will also consider multipliers of some related function spaces introduced in Section 1.1.

An analytic function f on the unit disk is a multiplier of the Dirichlet space D if and only f is bounded and pointwise multiplication of functions in D by the function f is a bounded linear operator M_f of D into D. In other words, the map

$$\begin{aligned} M_f : \quad & D \longrightarrow D \\ & h \longrightarrow fh \end{aligned}$$

is a bounded linear operator. Thus f is a multiplier of D if and only if f is bounded and $\|fh\|_D \leq c\|h\|_D$ for some constant c and for all $h \in D$.

Let $M(D)$ denote the space of multipliers of D. This space might be considered either as a space of analytic functions on the unit disk, or as a subspace of the Banach space of bounded linear operators on the Hilbert space D, denoted by $\mathscr{B}(D)$. This observation makes is clear that $M(D)$ is a normed linear space: the sum of two multipliers is clearly a multiplier and the norm is given by the operator norm on $\mathscr{B}(D)$, namely

$$\|f\| = \|M_f\| = \sup_{f\in D, \|h\|_D \leq 1} \|M_f(h)\|_D = \sup_{f\in D, \|h\|_D \leq 1} \|fh\|_D. \tag{1.29}$$

If $f \in M(D)$ then $f \in D$ since $1 \in D$. Thus $M(D) \subset H_D^\infty = H^\infty \cap D$. In Chapter 3 we will give an example of a bounded analytic function with finite Dirichlet integral which is not a multiplier of D. Another example is to be found, for example, in [31], page 37-39. Thus the inclusion above is strict.

If f is a multiplier of D then $fh \in D$ for every $h \in D$, or written more concisely; $fD \subset D$. For the Dirichlet space D it turns out that this property is sufficient. The following proposition is an adaptation of a more general result; see [10], Lemma 11.

Proposition 1.7. *Let f be an analytic function defined in the unit disk. Then $f \in M(D)$ if and only if $fD \subset D$.*

Proof. If $f \in M(D)$ we have already observed that $fD \subset D$.

Conversely, suppose $fD \subset D$. First we prove that the multiplication operator M_f corresponding to f is bounded. To this end, we shall show that the graph of M_f is closed. Since D is a Banach space, M_f will then be bounded by the closed graph theorem. Let $\{h_n\}$ be a sequence of functions in D, converging in the norm of D to an element $h \in D$, and suppose $\{M_f(h_n)\} = \{fh_n\}$ converges in the norm of D to $g \in D$. Let $z \in U$. By Proposition 1.4 evaluation at a point is a bounded linear functional on D, and so $|(fh_n)(z) - g(z)| \leq c(z)\|fh_n - g\|_D$ where the constant $c(z)$ depends on z but not on the functions. Hence $\{fh_n\}$ converges pointwise to g. But $\{fh_n\}$ evidently converges pointwise to fh, since h_n converges pointwise to h. Since a sequence of functions cannot have two distinct pointwise limits the equality $fh = g$ follows, that is, the graph is closed.

To show that f is bounded, choose $z \in U$. By Proposition 1.4, evaluation at a point is given by the bounded linear functional $\lambda_z(h) = h(z) = \langle h, k_z\rangle$. Hence

$$|\lambda_z(h)||f(z)| = |f(z)h(z)| = |\lambda_z(M_f(h))| \leq \|\lambda_z\|\|M_f(h)\|_D \leq \|\lambda_z\|\|M_f\|\|h\|_D$$

Taking the supremum over all $h \in D$ of norm 1 yields

$$|f(z)| \leq \|M_f\|$$

Since this is true for all $z \in U$, f is bounded. □

This is a very useful result. It allows us to determine if a function f is a multiplier by computing the Dirichlet integral $\mathcal{D}(fh)$ where $h \in D$, instead of having to check whether the operator M_f is bounded or not. Moreover the proposition imply that boundedness could have been omitted from the definition of a multiplier since it is deduced from the boundedness of the corresponding multiplication operator and the boundedness of the point evaluation operator. The proof above also furnishes an estimate which is interesting enough to be stated as a corollary.

Corollary 1.8. *If $f \in M(D)$ then*

$$\sup_{z\in U} |f(z)| \leq \|M_f\| = \sup_{\|h\|_D \leq 1} \|fh\|_D.$$

It is clear that the sum and the pointwise product of two multipliers are multipliers. Indeed, we have the following:

Proposition 1.9. *The space $M(D)$ is a commutative Banach algebra.*

Proof. Let $f, g \in M(D)$. We must show:

1. $\|fg\| \leq \|f\|\|g\|$
2. $M(D)$ is complete

We first prove (1). Observe that

$$M_{fg}(h) = fgh = (M_f \circ M_g)(h).$$

Hence

$$\|M_{fg}(h)\|_D = \|(M_f \circ M_g)(h)\|_D \leq \|M_f\|\|M_g(h)\|_D \leq \|M_f\|\|M_g\|\|h\|_D.$$

Taking supremum over all $h \in D$ of norm 1 we get

$$\|M_{fg}\| \leq \|M_f\|\|M_g\|$$

as required.

To show that $M(D)$ is complete, choose a Cauchy sequence $\{f_n\}$ in $M(D)$. Then

$$\|f_n - f_m\| = \sup_{\|h\|_D \leq 1} \|f_n h - f_m h\|_D \leq \epsilon, \qquad m, n \geq N(\epsilon).$$

Letting $h = 1$ we observe that $\{f_n\}$ is a Cauchy sequence in D and thus converges in the norm of D, as well as pointwise, to a function $f \in D$. Thus $\{f_n h\}$ will converge pointwise to fh for each $h \in D$. We must show that $fh \in D$. Fix a function $h \in D$. Then

$$\|f_n h - f_m h\|_D \leq \|f_n - f_m\|\|h\|_D \leq \epsilon\|h\|_D, \qquad m, n \geq N.$$

Hence $\{f_n h\}$ is a Cauchy sequence in D and consequently converges in the norm of D and pointwise to a function $A(h) \in D$. It follows that $fh = A(h)$. Thus $fh \in D$ for each $h \in D$, and so f is a multiplier of D by Proposition 1.7.

It remains only to show that $\{f_n\}$ converges to f in the operator norm. By the definition of the operator norm, there exist a function $h_0 \in D$ such that $\frac{1}{2}\|f_n - f\| < \|f_n h_0 - f h_0\|_D$. Since $\{f_n h_0\}$ converges in the norm of D to $A(h_0) = f h_0$, the result now follows.

The proof is complete. □

In Chapter 4 we will study the maximal ideal space of the Banach algebra $M(D)$. Presently though, we want to show that the outer part of a multiplier is itself a multiplier. For this we will need the following lemma.

Lemma 1.10. *Let $f \in H^1$. Then f can be written as the difference of two outer functions $f = f_1 - f_2$, where f_1 and f_2 are in H^1. In fact, we can choose $f_1 = (1 + I)F$ and $f_2 = F$ where I is the inner part of f and F is the outer part.*

Proof. Observe that if $(1+I)F$ is an outer function, then $f = (1+I)F - F$ is a representation of f as the difference of two outer functions. Since the product of two outer functions is an outer function, the proof will be complete if we can prove that $1 + I$ is outer.

By the definition of an inner function, I is analytic in U, bounded by 1, and with modulus almost everywhere equal to one on the boundary. Thus, by the maximum modulus principle, $|I(z)| < 1$ in U, and so $\mathrm{Re}(I) > -1$ on U and $\mathrm{Re}(1 + I) > 0$ on U. We shall now show that if g is any analytic function in H^1 such that $\mathrm{Re}(g) > 0$ in U, then g is an outer function.

Since $\mathrm{Re}(g(z)) > 0$ for any $z \in U$ we can define an analytic branch of $\log g(z)$ in U:

$$\log g(z) = \log|g(z)| + i\mathrm{Arg}(g(z))$$

$$\mathrm{Arg}(g(z)) = \tan^{-1}\left(\frac{\mathrm{Im}(g(z))}{\mathrm{Re}(g(z))}\right) \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

Clearly, $\mathrm{Arg}(g(z))$ is a bounded harmonic function, that is, $\mathrm{Arg}(g(z)) \in h^\infty$. In particular $\mathrm{Arg}(g(z)) \in h^2$ since $h^\infty \subset h^2$. Observe that $-i \log g(z) = \mathrm{Arg}(g(z)) - i \log|g(z)|$ is analytic in U. Thus $-\log|g(z)|$ is an harmonic conjugate of $\mathrm{Arg}(g(z))$. But in general, if $u \in h^2$ and v is the harmonic conjugate of u, then $\|v\|_2 \leq \|u\|_2$ where $\|.\|_2$ denotes the h^2-norm (see [11], page 53-54). Thus $-\log|g(z)| \in h^2$. Hence $-\log|g(z)|$ is the Poisson-integral of its boundary values:

$$-\log|g(re^{i\theta})| = -\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\log|g(e^{it})|dt$$

Letting $z = re^{i\theta} = 0$ we obtain

$$\log|g(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|g(e^{it})|dt$$

This is a necessary and sufficient condition for g to be an outer function.

We conclude that $1 + I$ is outer. The proof is complete. □

Proposition 1.11. *Let $f = IF \in M(D)$, where I is the inner part of f and F is the outer part. Then $F \in M(D)$.*

Proof. We must show that $FD \subset D$, that is, $\mathcal{D}(Fh) = \frac{1}{\pi}\int_U |(Fh)'|^2 dA < \infty$ for any $h \in D$. First, assume h is an outer function. If g is any function in the Dirichlet space, and G is the outer part of g, then $G \in D$ and $\mathcal{D}(g) \geq \mathcal{D}(G)$. These results are corollaries of the representation formula for the Dirichlet integral by Lennart Carleson, see [19] Corollary 3.5 and 3.6. From these results, and the fact that $f \in M(D)$ we see that

$$\infty > \mathcal{D}(fh) = \mathcal{D}(IFh) \geq \mathcal{D}(Fh)$$

since the product Fh is outer.

Now, let h be any function in D. Since $D \subset H^1$, h can be written as the difference of two outer functions h_1 and h_2 by Lemma 1.10. Thus

$$\begin{aligned}
\mathcal{D}(Fh) = \mathcal{D}(Fh_1 - Fh_2) &= \frac{1}{\pi}\int_U |(Fh_1 - Fh_2)'|^2 dA \\
&\leq \frac{1}{\pi}\int_U |(Fh_1)'|^2 dA + \frac{1}{\pi}\int_U |(Fh_2)'|^2 dA \\
&\leq \mathcal{D}(Fh_1) + \mathcal{D}(Fh_2) \\
&\leq \mathcal{D}(fh_1) + \mathcal{D}(fh_2) \\
&< \infty
\end{aligned}$$

Hence $F \in M(D)$ as was to be proved. □

We now attempt to define multipliers of the harmonic Dirichlet space D_h in the same manner as for D: $f \in M(D_h)$ if and only if f is bounded, harmonic and the map $M_f(h) = fh$ is a bounded linear operator from D_h into D_h. The functions are considered in the unit disk. But a problem now occurs. When $M(D_h)$ is defined in this manner it turns out to be trivial, that is, consisting only of the constant functions. The reason for this lies with the nature of harmonic functions: unlike analytic functions, the product of two harmonic functions need not be harmonic.

Proposition 1.12. *If f is a multiplier of D_h where the functions are considered in the unit disk, then f is a constant function.*

Proof. The functions $p(x,y) = x$ and $q(x,y) = y$ are evidently harmonic functions in D_h. If f satisfies the hypothesis of the proposition then pf and qf are contained in D_h and, consequently, they are harmonic. Thus

$$\begin{aligned} 0 = \triangle(pf) &= \frac{\partial^2(pf)}{\partial^2 x} + \frac{\partial^2(pf)}{\partial^2 y} = \frac{\partial}{\partial x}\Big(f + x\frac{\partial f}{\partial x}\Big) + x\frac{\partial^2 f}{\partial^2 y} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial^2 x} + x\frac{\partial^2 f}{\partial^2 y} \\ &= 2\frac{\partial f}{\partial x} \end{aligned}$$

since f is harmonic. Thus $\partial f/\partial x = 0$. Similarly $\partial f/\partial y = 0$. We conclude that f is a constant function, as was to be proved. □

This problem can be remedied by considering multipliers only on the unit circle. That is, we consider the boundary functions for the functions in D_h and the corresponding multipliers on the unit circle. This will be done in Chapter 2.

We can define multipliers more successfully on other spaces of analytic functions introduced in Section 1.1. In particular, define multipliers of the Bergman space B and the Hardy space H^2. That is, $f \in M(B)$ if and only if f is bounded, analytic and $\|fh\|_B \le c\|h\|_B$ for some constant c and for each $h \in B$. Define multipliers for H^2 similarly. It turns out that the multipliers of B and H^2 are very easy to characterize.

Proposition 1.13. *The multipliers of B and of H^2 are precisely the bounded analytic functions in the unit disk, that is*

1. $M(B) = H^\infty$
2. $M(H^2) = H^\infty$.

Proof. The inclusions $M(B) \subset H^\infty$ and $M(H^2) \subset H^\infty$ follows by the definition of a multiplier.

Suppose $f \in H^\infty$ and let $h \in B$. Then

$$\|fh\|_B = \Big(\frac{1}{\pi}\int_U |fh|^2 dA\Big)^{\frac{1}{2}} \le \|f\|_\infty \Big(\frac{1}{\pi}\int_U |h|^2 dA\Big)^{\frac{1}{2}} = \|f\|_\infty \|h\|_B$$

and so $f \in M(B)$.

Similarly, suppose $f \in H^\infty$ and let $h \in H^2$. Then

$$\begin{aligned} \|fh\|_2 &= \sup_{0<r<1} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})h(re^{i\theta})|^2 d\theta\right)^{\frac{1}{2}} \\ &\le \|f\|_\infty \sup_{0<r<1} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |h(re^{i\theta})|^2 d\theta\right)^{\frac{1}{2}} \\ &= \|f\|_\infty \|h\|_2 \end{aligned}$$

and so $f \in M(H^2)$. This completes the proof. □

The following question now arise: can we give characterizations of the multipliers of the Dirichlet space? The answer is yes, but the characterizations are not as simple as for the Bergman space and the Hardy space. Characterizations involving notions from potential theory were given by David Stegenga in the article [27]. The results of this article is the content of the next chapter.

We end this section by remarking on the generality of some of the results contained herein. Let H be a Hilbert space of analytic functions in some open and connected subspace $\Omega \subset \mathbb{C}$. Assume further that point evaluations are bounded linear functionals on H. Defining multipliers of H just as for D, it turns out that both the results and the proofs of Proposition 1.7 and Proposition 1.9 remains true if D is replaced by H. The key result in these proofs is precisely the boundedness of the point evaluation operators.

Chapter 2

Characterizations of multipliers of the Dirichlet space

In the article [27] David Stegenga gave two characterizations of multipliers of the Dirichlet space. The first is a characterization in the unit disk and the second is a boundary characterization, that is, a characterization of the boundary functions of the multipliers. Both characterizations involve the notion of a capacity, which is a set function arising in potential theory. The results in this article, and the concepts used in proving them, is the content of this chapter. The presentation follows the article by Stegenga quite closely, but the proofs and explanations are often more detailed as Stegengas style is quite terse, especially the content of Section 2.2 and Section 2.3.

In Section 2.1 the characterization in the unit disk is given. The motivation is that multipliers are closely related to a particular type of measures called Carleson measures.

In Section 2.2 characterizations of the multipliers of two particular function spaces of the real line and the unit circle are given. Denoted by $L^2_{1/2}(\mathbb{R})$ and $L^2_{1/2}(T)$, they are special cases of the so called Bessel potential spaces. We will also investigate the multipliers of the harmonic Dirichlet space.

In Section 2.3 the boundary characterization of the multipliers of D are given by the realization of $M(D)$ as a subset of the multipliers of $L^2_{1/2}(T)$ and a subsequent application of the results of Section 2.2.

2.1 A characterization in the unit disk of multipliers of the Dirichlet space

As mentioned above, we start by defining Carleson measures for D.

Definition 2.1. *A positive Borel measure μ on the unit disk is a Carleson measure for D provided there exists a constant c such that*

$$\int_U |g|^2 d\mu \leq c\|g\|_D^2$$

for all $g \in D$.

Since we shall only work with Carleson measures for D, we will skip the suffix "for D" from now on.

The following proposition reveals that multipliers and Carleson measures are closely related.

Proposition 2.2. *A function f in the unit disk is a multiplier of the Dirichlet space if and only if f is bounded, analytic and the measure*

$$|f'|^2 dA$$

is a Carleson measure.

Proof. Suppose $f \in M(D)$ and let $g \in D$. First, note that $\|g\|_B < \infty$ since $D \subset B$. Now, observe that

$$\pi \|f'g\|_B^2 = \int_U |f'|^2 |g|^2 dA.$$

In light of this equation we must bound $\|f'g\|_B$ by a constant multiple of $\|g\|_D$. From the identity $f'g = (fg)' - fg'$ and the triangle inequality we deduce the following:

$$\|f'g\|_B \leq \|(fg)'\|_B + \|fg'\|_B \tag{2.1}$$

Aided by Proposition 1.3 and the fact that f is a multiplier of D, we obtain:

$$\|(fg)'\|_B \leq \|fg\|_D \leq \|M_f\| \|g\|_D. \tag{2.2}$$

Since f is bounded, we quickly obtain a satisfying estimate for the second number to the right in Equation (2.1):

$$\begin{aligned} \|fg'\|_B &= \Big(\frac{1}{\pi}\int_U |f|^2 |g'|^2 dA\Big)^{\frac{1}{2}} \\ &\leq \|f\|_\infty \Big(\frac{1}{\pi}\int_U |g'|^2 dA\Big)^{\frac{1}{2}} \\ &= \|f\|_\infty \|g'\|_B \\ &\leq \|f\|_\infty \|g\|_D \end{aligned} \tag{2.3}$$

Combining the equations 2.1, 2.2 and 2.3 we obtain the estimate we need:

$$\begin{aligned} \int_U |f'|^2 |g|^2 dA &= \pi \|f'g\|_B^2 \\ &\leq \pi(\|(fg)'\|_B + \|fg'\|_B)^2 \\ &\leq \pi(\|M_f\| + \|f\|_\infty)^2 \|g\|_D^2. \end{aligned}$$

Since g was an arbitrary function in D we conclude that $|f'|^2 dA$ is a Carleson measure.

Conversely, suppose f is bounded, analytic and that $|f'|^2 dA$ is a Carleson measure, that is

$$\int_U |f'|^2 |g|^2 dA \leq c \|g\|_D^2, \quad \text{for all } g \in D$$

We must show that $f \in M(D)$. By Proposition, 1.7 it suffices to prove that $fD \subset D$. To this end, assume $g \in D$. With the aid of our assumptions, we deduce the following estimate:

$$\begin{aligned}\mathcal{D}(fg) &= \frac{1}{\pi}\int_U |(fg)'|^2 dA \\ &\le \frac{1}{\pi}\int_U |f'g|^2 dA + \frac{1}{\pi}\int_U |fg'|^2 dA \\ &\le \frac{c}{\pi}\|g\|_D^2 + \|f\|_\infty^2 \mathcal{D}(g) \\ &< \infty.\end{aligned}$$

Thus f is a multiplier of D. This completes the proof. □

Motivated by the proposition above, the rest of this section is devoted to the characterization of Carleson measures. It is during this endeavour the potential-theoretic concepts come into hand. First though, we shall realize the Dirichlet space as a set of convolutions on the unit circle.

Recall that the boundary function of a function in H^2, and in particular a function in D, is a function in $L^2(T)$ with vanishing negative Fourier coefficients. We now ask two questions:

1. Can we isolate those functions in $L^2(T)$ which are boundary functions of functions in D?

2. And having done so, can this be used to redefine Carleson measures by means of the boundary functions?

The answer to both questions is affirmative. We proceed as follows.

Let $k(x) = |x|^{-\frac{1}{2}}$ for $|x| \le \pi$ and extend it to a 2π-periodic function. It is easily verified that the extended function $k \in L^1(T)$ and if $g \in L^2(T)$ the convolution

$$(k * g)(\theta) = \frac{1}{2\pi}\int_{\pi}^{\pi} k(\theta - t)g(t)dt$$

is in $L^2(T)$. The Fourier coefficients of $k * g$ satisfy $\widehat{(k * g)}(n) = \hat{k}(n)\hat{g}(n)$.

Lemma 2.3. *The Fourier coefficients of $k(x) = |x|^{-\frac{1}{2}}$ are of the form $b_n(1 + |n|)^{-\frac{1}{2}}$ where $0 < \delta^{-1} \le b_n \le \delta$ for all n.*

Proof. The Fourier coefficients of k are given by the improper integral

$$\begin{aligned}\hat{k}(n) =&\frac{1}{2\pi}\int_{-\pi}^{\pi} |x|^{-\frac{1}{2}} e^{-inx} dx \\ =&\frac{1}{2\pi}\int_{-\pi}^{\pi} |x|^{-\frac{1}{2}}\cos(nx)dx - \frac{i}{2\pi}\int_{-\pi}^{\pi} |x|^{-\frac{1}{2}}\sin(nx)dx.\end{aligned} \tag{2.4}$$

Since $|x|^{-\frac{1}{2}}\sin(nx)$ is an odd function, the second integral in (2.4) is zero. On the other hand, $|x|^{-\frac{1}{2}}\cos(nx)$ is even so that

$$\hat{k}(n) = \frac{1}{\pi}\int_0^{\pi} x^{-\frac{1}{2}}\cos(nx)dx. \tag{2.5}$$

Clearly, $\hat{k}(0)$ is finite positive number. Suppose $n \neq 0$. Since x is non-zero in the interval of integration, and since $\cos(-x) = \cos(x)$, we can substitute $x = \frac{1}{|n|}t^2$ and so obtain that

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi} x^{-\frac{1}{2}}\cos(nx)dx &= \lim_{\epsilon\to 0}\frac{2}{\pi\sqrt{|n|}}\int_{\epsilon}^{\sqrt{|n|\pi}}\cos(\frac{n}{|n|}t^2)dt \\ &= \frac{2}{\pi\sqrt{|n|}}\int_0^{\sqrt{|n|\pi}}\cos(t^2)dt. \end{aligned} \tag{2.6}$$

The integral to the right in (2.6) is called a *Fresnel integral.* By means of residues one can show that it converges and that the limit is given by $\int_0^\infty \cos(t^2)dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}$. For a sketch of this calculation, see [6], page 266 and 267.

Now, let $c_n = \frac{2}{\pi}\int_0^{\sqrt{|n|\pi}}\cos(t^2)dt$. Then the sequence $\{c_n\}$ converges, and so it must be bounded. Moreover, $c_n > 0$ for all n. For large $|n|$ this follows by the convergence of the sequence. For small $|n|$ it can be seen by inspection (notice that the function $\cos(t^2)$ oscillates faster and faster as t increases. Since $\cos(t^2)$ is positive for the smallest values of t we conclude that c_n must necessarily be positive for all n). Now, choose a positive number λ so that

$$\lambda^{-1} < \hat{k}(0) < \lambda, \qquad \lambda^{-1} < c_n < \lambda \tag{2.7}$$

for all n. It is easy to see that

$$1 \leq \frac{\sqrt{1+|n|}}{\sqrt{|n|}} \leq \sqrt{2} \tag{2.8}$$

for all nonzero $n \in \mathbb{Z}$, and from (2.5) and (2.6) we obtain the identity

$$\hat{k}(n) = \frac{c_n}{\sqrt{|n|}} = \frac{b_n}{\sqrt{1+|n|}} \tag{2.9}$$

where $b_n = \frac{\sqrt{1+|n|}}{\sqrt{|n|}}c_n$ for all nonzero n. Set $b_0 = \hat{k}(0)$. From (2.7), (2.8) and (2.9) we deduce that

$$0 < (\sqrt{2}\lambda)^{-1} < b_n < \sqrt{2}\lambda.$$

Finally, let $\delta = \sqrt{2}\lambda$. This completes the proof. □

Now, suppose g is a function in $L^2(T)$ whose negative Fourier coefficients vanish, that is, $g \in H^2$ where H^2 is considered as a subset of $L^2(T)$. Then the negative Fourier coefficients of the convolution $k * g$ vanish as well, and so $k * g \in H^2$. Hence the Poisson integral $P[k * g]$ is an analytic function on the unit disk. In fact, this correspondence defines an isomorphism between H^2 and the Dirichlet space, and so we have an answer to question (1) above.

Lemma 2.4. *A function f is in D if and only if $f = P[k * g]$ where g is a function in H^2. Moreover, this correspondence is an isomorphism and the norms $\|f\|_D$ and $\|g\|_2$ are comparable.*

Proof. Suppose $g \in H^2$ and let $g(e^{i\theta}) = \sum_{n=0}^{\infty}\hat{g}(n)e^{in\theta}$ be the Fourier series for g. Then $k * g \in H^2$ and $\widehat{k * g}(n) = \hat{k}(n)\hat{g}(n) = b_n(1+n)^{-\frac{1}{2}}\hat{g}(n)$ where $0 < \delta^{-1} \leq b_n \leq \delta$ for each n

by Lemma 2.3. Thus $f(re^{i\theta}) = P[k*g](re^{i\theta}) = \sum_{n=0}^{\infty} b_n(1+n)^{-\frac{1}{2}}\hat{g}(n)r^n e^{in\theta}$ is a power series representation for the analytic function f and so

$$\begin{aligned}\|f\|_D^2 =&\sum_{n=0}^{\infty}(1+n)|b_n|^2(1+n)^{-1}|\hat{g}(n)|^2\\ \leq&\delta^2\sum_{n=0}^{\infty}|\hat{g}(n)|^2\\ =&\delta^2\|g\|_2^2\\ <&\infty.\end{aligned}$$

Thus $f \in D$. Moreover

$$\|f\|_D^2 \geq \delta^{-2}\sum_{n=0}^{\infty}|\hat{g}(n)|^2 = \delta^{-2}\|g\|_2^2$$

and so $\|f\|_D$ and $\|g\|_2$ are comparable.

Conversely, suppose $f \in D$. Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\{a_n\}$ is the sequence of Fourier coefficients for the boundary function of f. Define $c_n = a_n b_n^{-1}(1+n)^{\frac{1}{2}}$ for $n \geq 0$. Then

$$\sum_{n=0}^{\infty}|c_n|^2 = \sum_{n=0}^{\infty}(1+n)|a_n|^2|b_n|^2 \leq \delta^2\|f\|_D^2$$

and so $\{|c_n|^2\}$ is summable and must be the sequence of Fourier coefficients for some function $g \in H^2$. Then $k*g \in H^2$ and $\widehat{k*g}(n) = \hat{k}(n)\hat{g}(n) = b_n(1+n)^{-\frac{1}{2}}a_n b_n^{-1}(1+n)^{\frac{1}{2}} = a_n$. Thus $k*g$ is equal to the boundary function of f and $f = P[k*g]$. We have now proved that the given correspondence is surjective. Since it is evidently injective, the proof is complete. □

The next lemma answers the second question, (2), above.

Lemma 2.5. *A positive measure μ on the unit disk is a Carleson measure if and only if there exists a constant c such that*

$$\int_U |P[k*g]|^2 d\mu \leq c\|g\|_2^2 \tag{2.10}$$

for all $g \geq 0$ in $L^2(T)$.

Proof. Assume (2.10) holds for all $g \geq 0$ in $L^2(T)$. Now, let h be any function in $L^2(T)$. Writing $h = u + iv$ and observing that $P[k*h] = P[k*u] + iP[k*v]$ is is readily checked that (2.10) holds for h. In particular (2.10) holds for all functions in H^2. By Lemma 2.4, any function $f \in D$ is of the form $f = P[k*g]$ where $g \in H^2$, and the norms $\|f\|_D$ and $\|g\|_2$ are comparable. Thus

$$\begin{aligned}\int_U |f|^2 d\mu = \int_U |P[k*g]|^2 d\mu \leq& c\|g\|_2^2\\ \leq& c\|f\|_D\end{aligned}$$

and so μ is a Carleson measure.

Conversely, suppose μ is a Carleson measure, that is, $\int_U |f|^2 d\mu \leq c\|f\|_D^2$ for all $f \in D$. If $g \in H^2$, then $f = P[k * g] \in D$ and so

$$\begin{aligned}\int_U |P[k*g]|^2 d\mu = \int_U |f|^2 d\mu \leq & c\|f\|_D^2 \\ \leq & c\|g\|_2^2.\end{aligned}$$

This proves (2.10) for functions in H^2. Now let g be any function in $L^2(T)$ and let $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be the Fourier series for g. The partial sums of this series can be written

$$s_N = \sum_{n=-N}^{N} a_n e^{in\theta} = \sum_{n=0}^{N} a_n e^{in\theta} + \overline{\sum_{n=1}^{N} c_n e^{in\theta}} \tag{2.11}$$

where $c_n = \bar{a}_n$. Either series to the right in (2.11) converge in the norm on L^2. Denote the limits by g_1 and $\bar{g}_2$ respectively. Then g_1 and g_2 are in H^2 since either function has vanishing negative Fourier coefficients. It is clear that $g = g_1 + \bar{g}_2$ and $\|g\|_2^2 = \|g_1\|_2^2 + \|g_2\|_2^2$. Finally, since both the Poisson kernel and the function k are nonnegative

$$P[k*g] = P[k*g_1] + P[k*\bar{g}_2] = P[k*g_1] + \overline{P[k*g_2]}$$

and so

$$\begin{aligned}\int_U |P[k*g]|^2 d\mu \leq & \int_U |P[k*g_1]|^2 d\mu + \int_U |P[k*g_2]|^2 d\mu \\ \leq & c(\|g_1\|_2^2 + \|g_2\|_2^2) \\ = & \|g\|_2^2.\end{aligned}$$

Since g was an arbitrary function $L^2(T)$, this completes the proof. □

We must now introduce some concepts related to potential theory. The *Bessel kernel* of order α is the real function on $\mathbb{R}^N$ given by

$$G_\alpha = \mathcal{F}^{-1}((1+|x|^2)^{-\alpha/2}) \tag{2.12}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Thus G_α is the function whose Fourier transform is $\hat{G}_\alpha(x) = (1+|x|^2)^{-\alpha/2}$. For $\alpha > 0$ the function G_α has the integral representation

$$G_\alpha(x) = \frac{1}{(4\pi)^{-\alpha/2}} \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha-N}{2}} e^{-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}} \frac{dt}{t} \tag{2.13}$$

For such α the function G_α is clearly positive. Moreover $G_\alpha \in L^1(\mathbb{R}^N)$ and $G_\alpha(x) = O(e^{-c|x|})$ as $x \to \infty$ for some $c > 0$. The norm is $\|G_\alpha\|_1 = 1$ (see [29] page 132). We are mostly going to consider the case $\alpha = 1/2$, and as such we shall write $G = G_{1/2}$. In this case, $G(x)$ behaves asymptotically as $G(x) \sim c|x|^{-1/2}$ for x near the origin.

We can now define the *Bessel potential spaces* $L_\alpha^p = L_\alpha^p(\mathbb{R}^N)$ by

$$L_\alpha^p(\mathbb{R}^N) = \{f : f = G_\alpha * g,\, g \in L^p(\mathbb{R}^N)\} \tag{2.14}$$

Since $g \in L^p(\mathbb{R}^N)$ and $G_\alpha \in L^1(\mathbb{R}^N)$ it follows that $G_\alpha * g \in L^p(\mathbb{R}^N)$. Thus $L_\alpha^p(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$. In other words $L_\alpha^p(\mathbb{R}^N)$ is a subspace of $L^p(\mathbb{R}^N)$, consisting of all functions $f \in L^p(\mathbb{R}^N)$

that can be written as a convolution $f = G_\alpha * g$. We can place a norm on L^p_α defined by $\|G_\alpha * g\|_{\alpha,p} = \|g\|_p$. The convolution operator is bounded, for

$$\|G_\alpha * g\|_p \leq \|G_\alpha\|_1 \|g\|_p = \|g\|_p = \|G_\alpha * g\|_{\alpha,p} \tag{2.15}$$

since $\|G_\alpha\|_1 = 1$. It is not hard to verify that L^p_α is complete in this norm, and so it is in fact a Banach space. Finally we mention that if α is an integer, $\alpha \geq 1$ and $1 < p < \infty$ then the Bessel potential space L^p_α is equal to the *Sobolev space* $W^{\alpha,p}(\mathbb{R}^N)$ consisting of all functions in L^p whose distribution (weak) derivative $D^\sigma f$ belongs to L^p for each multi-index σ of order $|\sigma| \leq \alpha$ (see [29], page 135-138).

For a detailed exposition of the Bessel kernel and the Bessel potential space, see for example [29], Chapter 5, or [1].

We will now introduce capacities, which is the concept we need for the characterization of the multipliers.

Definition 2.6. *A capacity is a positive set function C given on a σ-additive family of sets $\mathcal{E}$ which contains the compact sets and satisfy the following properties.*

1. $C(\emptyset) = 0$
2. *If $A_1, A_2 \in \mathcal{E}$ and $A_1 \subset A_2$, then $C(A_1) \subset C(A_2)$ (monotonicity)*
3. *If $A_n \in \mathcal{E}$ for $n = 1, 2, \ldots$ then $C(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty C(A_n)$ (countable subadditivity)*

If in addition $C(A) = \sup\{C(K) : K$ compact, $K \subset A\}$ for every $A \in \mathcal{E}$ then C is called an inner capacity, and if $C(A) = \inf\{C(G) : G$ open, $A \subset G\}$ for every $A \in \mathcal{E}$ then C is called an outer capacity. We can define C on all subsets A of $\mathbb{R}^N$ by setting $C(A) = \inf\{C(E) : E \in \mathcal{E}, A \subset E\}$.

Notice the similarity of the properties of a capacity with those of a measure. Indeed the properties 1, 2 and 3 of Definition 2.6 are exactly those defining an *outer measure*, which can be used to construct measures. (see [20], Chapter 12). We say that a property holds *quasi-everywhere* (abbreviated q.e) if it holds everywhere except possibly on some set of capacity zero.

We will now define the capacities we need. The *Bessel capacity* $B_{\alpha,p}$ for $1 < p < \infty$ is defined by

$$B_{\alpha,p}(A) = \inf\{\|G_\alpha * g\|^p_{\alpha,p} = \|g\|^p_p : g \in L^p(\mathbb{R}^N),\ g \geq 0,\ G_\alpha * g \geq 1 \text{ on } A\} \tag{2.16}$$

The fundamental theory of this capacity is given, for example, in the article [17] and the book [1]. We will mention the results we need. First off, the Bessel capacity is an outer capacity defined on all subsets of $\mathbb{R}^N$, see Theorem 1 [17]. Comparing the Bessel capacity with the Lebesgue measure, we find that sets with zero capacity has zero measure, but there are sets with zero measure and positive capacity. Let $\sigma_r(x_0)$ denote the open ball in $\mathbb{R}^N$ with centre at x_o and radius $0 < r < 1$. Suppose $p > 1$ and that $\alpha p = N$. Then we can find a constant c such that

$$c^{-1}\big(\log\frac{1}{r}\big)^{1-p} \leq B_{\alpha,p}(\sigma_r(x_0)) \leq c\big(\log\frac{1}{r}\big)^{1-p}. \tag{2.17}$$

See Lemma 8 [17]. Now, if μ is a positive measure and f is a measurable function, the following identity holds:

$$\int_{\mathbb{R}^N} |f|^p d\mu = \int_0^\infty \mu\{x : |f(x)| > t\} dt^p \tag{2.18}$$

where $dt^p = pt^{p-1}dt$ and $\mu\{x : |f(x)| > t\}$ is the distribution function for $|f|$ (see [23], 172-174). We now ask: is something similar true for the Bessel capacity? The answer is, at least partially, yes. To be precise, we have the following *capacitary strong type inequality*:

$$\int_0^\infty B_{\alpha,p}(\{x : (G_\alpha * f)(x) > t\})dt^p \leq c\|f\|_p^p \tag{2.19}$$

for all $f \geq 0$ in $L^p(\mathbb{R}^N)$. It is required that $1 < p < \infty$, and the constant c depends only on N and p. For the proof of this inequality, see [1] p. 189-191. From now on we will specialize to the situation $\alpha = 1/2$, $p = 2$ and $N = 1$, and write $C = B_{\frac{1}{2},2}$. A nonnegative function $f \in L^2(\mathbb{R})$ such that $G * f \geq 1$ on A will be called a *test function* for $C(A)$.

We will also need a capacity on the circle. If $E \subset T$ let $\tau(E) = \{t \in [-\pi, \pi] : e^{it} \in E\}$. This is clearly an open map from T into the interval $[-\pi, \pi]$. Now define

$$\gamma(E) = \inf\{\|f\|_2^2 : \ f \in L^2(T),\ f \geq 0,\ k * f \geq 1 \text{ on } \tau(E)\}. \tag{2.20}$$

This is an outer capacity defined on all subsets of T, again by Theorem 1 [17]. A nonnegative function $f \in L^2(T)$ such that $k * f \geq 1$ on E will be called a test function for $\gamma(E)$.

Lemma 2.7. *The capacity $\gamma(E)$ is comparable to the Bessel capacity $C(\tau(E))$ for any subset E of the unit circle.*

Proof. We begin by proving that there exists a constant c such that

$$C(\tau(E)) \leq \ c\gamma(E), \quad \text{for any } E \subset T. \tag{2.21}$$

To this end, let f be a test function for $\gamma(E)$. We shall find a test function g for $C(\tau(E))$ such that $\|g\|_2 \leq c_0\|f\|_2$ for some constant c_0, and $(k*f)(x) \leq (G*g)(x)$ for any $x \in [-\pi, \pi]$. Then the inequality (2.21) follows from the definitions of γ and C.

Recall that $G(x) \sim c_1|x|^{-\frac{1}{2}} = c_1k(x)$ where c_1 is some constant and x is near the origin. Hence G and k are comparable on $[-\pi, \pi]$. Define $g_0(x) = f(x)$ for $x \in [-2\pi, 2\pi]$ and zero otherwise. Clearly $g_0 \in L^2(\mathbb{R})$. From these considerations we deduce that

$$\begin{aligned}(k * f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)k(y)dy \leq& c\int_{-\pi}^{\pi} f(x-y)G(y)dy \\ \leq& c\int_{-\infty}^{\infty} g_0(x-y)G(y)dy \\ =& c(G * g_0)(x), \qquad x \in [-\pi, \pi]\end{aligned} \tag{2.22}$$

Setting $g = cg_0$ we get

$$\|g\|_2^2 = c^2\int_{-\infty}^{\infty}|g_0|^2dx = c^2\int_{-2\pi}^{2\pi}|f(x)|^2dx \leq c_0\|f\|_2^2.$$

Evidently, g is nonnegative and $G * g \geq k * f \geq 1$ on $\tau(E)$. Thus g is a test function for $C(\tau(E))$. This completes the proof of (2.21).

We must now show that there exists a constant c such that

$$\gamma(E) \leq cC(\tau(E)), \quad \text{for any } E \subset T. \tag{2.23}$$

First, we will show that it is enough to prove (2.23) for sets with small capacity, and with this in mind suppose (2.23) is true for all sets $E \subset T$ such that $C(\tau(E)) < \epsilon$. Let J be an interval in $[-\pi, \pi]$ with length $m(J) = \delta < 1$ where m is the Lebesgue measure on $[-\pi, \pi]$. By (2.17) $C(J) \leq c(\log \frac{2}{\delta})^{-1}$. Now, choose δ small enough so that $c(\log \frac{2}{\delta})^{-1} < \epsilon$. Then $C(J) < \epsilon$. Since T is compact, it is totally bounded, and so we can find a family of open intervals $\{I_i\}_{i=1}^n$ of the unit circle such that the length $|I_i| = m(\tau(I_i)) = \delta$ for each i and $T = \bigcup_{i=1}^n I_i$. Now, let E be a subset of T such that $C(E) > \epsilon$. Then

$$\begin{aligned}\gamma(E) \leq &\sum_{i=1}^n \gamma(I_i) \qquad \text{since } \gamma \text{ is subadditive}\\ \leq &c\sum_{i=1}^n C(\tau(I_i)) \qquad \text{since } C(\tau(I_i)) < \epsilon,\ 1 \leq i \leq n\\ \leq &cnC(\tau(E)).\end{aligned}$$

Since n only depends on δ and ϵ and not on the subset E, we conclude that (2.23) holds for all $E \subset T$.

Now, let $E \subset T$ with $C(\tau(E)) < \epsilon$. A sufficient upper bound for ϵ will be decided soon. Let h be a test function for $C(\tau(E))$. Evidently $\|h\|_2 \leq \sqrt{\epsilon}$. Recall that G decays exponentially to zero. From these considerations one obtains by the the Schwarz inequality

$$\begin{aligned}\int_{|y|>\pi} h(x-y)G(y)dy \leq &\Big(\int_{|y|>\pi} h^2(x-y)dy\Big)^{\frac{1}{2}}\Big(\int_{|y|>\pi} G^2(y)dy\Big)^{\frac{1}{2}}\\ \leq &\sqrt{\epsilon}\cdot c\\ < &\frac{1}{2}\end{aligned} \tag{2.24}$$

if $\sqrt{\epsilon} < 1/(2c)$. Fix such ϵ. Since $(G * h)(x) = \int_{-\infty}^{\infty} h(x-y)G(y)dy \geq 1$ on $\tau(E)$, we see from (2.24) that $\int_{|y|<\pi} h(x-y)G(y)dy \geq \frac{1}{2}$ for $x \in \tau(E)$. Since k and G are comparable on $[-\pi, \pi]$, this means that $\frac{1}{2\pi}\int_{|y|<\pi} h(x-y)k(y)dy \geq\ c^{-1}$ for some constant c and $x \in \tau(E)$. Let $h|_{[a,\, b]}$ denote the restriction of h to the interval $[a, b]$. Let f_1 be the periodic extension of $h|_{[-2\pi,\, 0]}$ to all of $\mathbb{R}$. Similarly, let f_2 be the periodic extension of $h|_{[0,\, 2\pi]}$ to all of $\mathbb{R}$. Then the function $f = c(f_1 + f_2)$ is 2π-periodic, $f \in L^2(T)$ and $f \geq 0$. Moreover $f_1 + f_2 \geq h$ on $[-2\pi, 2\pi]$ and so

$$\begin{aligned}(k * f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)k(y)dy = &\frac{c}{2\pi}\int_{-\pi}^{\pi}(f_1(x-y) + f_2(x-y))k(y)dy\\ \geq &\frac{c}{2\pi}\int_{-\pi}^{\pi} h(x-y)k(y)dy\\ \geq &1\end{aligned}$$

for $x \in \tau(E)$. We conclude that f is a test function for $\gamma(E)$. Observe that $2\pi\|f_1\|_2^2 = \int_{-\pi}^{\pi} f_1^2(x)dx \leq \int_{-\infty}^{\infty} h^2(x)dx = \|h\|_2^2$. The same inequality holds when f_1 is replaced by f_2.

From our considerations above we obtain that

$$\begin{aligned}\gamma(E) \leq & \|f\|_2^2 = c^2(\|f_1 + f_2\|_2^2) \\ \leq & c^2(\|f_1\|_2 + \|f_2\|_2)^2 \\ \leq & \frac{c^2}{2\pi}(\|h\|_2 + \|h\|_2)^2 \\ = & \frac{2c^2}{\pi}\|h\|_2^2\end{aligned}$$

and so

$$\begin{aligned}\gamma(E) \leq & \frac{2c^2}{\pi}\inf\{\|h\|_2^2 : h \in L^2(\mathbb{R}), h \geq 0, G * h \geq 1 \text{ on } \tau(E)\} \\ = & \frac{2c^2}{\pi}C(\tau(E)).\end{aligned}$$

This completes the proof. □

Corollary 2.8. *The strong type inequality* (2.19) *holds for the capacity* γ*, that is, the inequality*

$$\int_0^\infty \gamma(\{e^{ix} \in T : k * f > t\})dt^2 \leq c\|f\|_2^2$$

holds for all $f \geq 0$ *in* $L^2(T)$*.*

Proof. Let f be a nonnegative function in $L^2(T)$ and let $E_t = \{e^{ix} \in T : k * f > t\}$. Then $\tau(E_t) = \{x \in [-\pi, \pi] : k * f > t\}$. Like in the proof of Lemma 2.7, define $f_0 = f$ on $[-2\pi, 2\pi]$ and zero otherwise. Then $(k * f)(x) \leq c_1(G * f_0)(x)$ for some constant c_1 by (2.22) and $\|f_0\|_2 \leq c_2\|f\|_2$ for some constant c_2. Since $\gamma(E_t)$ and $C(\tau(E_t))$ are comparable we obtain that

$$\begin{aligned}\int_0^\infty \gamma(\{e^{ix} \in T : k * f > t\})dt^2 \leq & c_0\int_0^\infty C(\{x \in [-\pi, \pi] : k * f > t\})dt^2 \\ \leq & c_0\int_0^\infty C(\{x \in [-\pi, \pi] : c_1(G * f_0) > t\})dt^2 \\ \leq & c_0\|c_1 f_0\|_2^2 \quad \text{by (2.19)} \\ \leq & c\|f\|_2^2\end{aligned}$$

as was to be proved. □

Let I be an arc on T and denote the length of I by $|I|$. We define the approximate square with base I by

$$S(I) = \{z \in U : \frac{z}{|z|} \in I, 1 - |I| \leq |z| < 1\} \tag{2.25}$$

If $|I| \geq 1$ let $S(I) = U$. If $z \in U$, let I_z be the arc on T centred at $z/|z|$ and with length $1-|z|$. Similarly, the notation nI_z will denote the arc centred at $z/|z|$ and with length $n(1 - |z|)$.

We now have all the material we need to characterize the Carleson measures.

Theorem 2.9. *A positive measure μ is a Carleson measure if and only if there exists a constant c such that*

$$\mu(\bigcup_{j=1}^{n} S(I_j)) \leq cC(\tau(\bigcup_{j=1}^{n} I_j)) \tag{2.26}$$

for all disjoint collections $\{I_j\}_{j=1}^{n}$ of subarcs on the circle.

Proof. Assume μ is a Carleson measure and set $E = \bigcup_{i=1}^{n} I_j$. We will prove that (2.26) holds for the capacity γ instead of C. It will then be true for the capacity C as well by Lemma 2.7. Accordingly, let f be a test function for $\gamma(E)$, that is, $f \in L^2(T)$, $f \geq 0$ and $k * f \geq 1$ on $\tau(E)$. Clearly $k * f \geq \chi_{I_j}$ for each j.

Suppose $|I_s| \geq 1$ for some $1 \leq s \leq n$. Then $S(I_s) = U$. By (2.17) there exists a constant c_0 such that $0 < c_0 < C(\tau(I))$ for all intervals of length $|I| \geq 1$. Thus we can find another constant c_1 such that

$$\mu(\bigcup_{j=1}^{n} S(I_j)) = \mu(U) \leq c_1 c_0 \leq c_1 C(\tau(I_s)) \leq c_1 C(\tau(\bigcup_{j=1}^{n} I_j))$$

and so (2.26) holds.

Assume $|I_j| < 1$ for all j. Let $z = re^{i\theta} \in S(I_j)$ where $\theta \in [-\pi, \pi]$. Recall that $P_r(\theta - t) = \dfrac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} = \dfrac{1 - |z|^2}{|e^{it} - z|^2} = P_z(t)$ and observe that

$$P[\chi_{I_j}](z) = \frac{1}{2\pi} \int_{\tau(I_j)} P_r(\theta - t)dt. \tag{2.27}$$

First, notice that $P_r(\theta - t)$ is nonnegative. The curve of $P_r(\theta - t)$ (with respect to $\theta - t$) is bell-shaped and assumes its maximum value at $\theta - t = 0$. Since $\theta \in \tau(I_j)$ this maximum is assumed when t runs through the interval $\tau(I_j)$, the interval of integration in (2.27). Notice further that if $0 < \delta < \pi$ then $\lim_{r \to 1} \sup_{|\theta - t| \geq \delta} |P_r(\theta - t)| = 0$. In other words the curve of $P_r(\theta - t)$ becomes slimmer and more concentrated around $\theta = t$ as r increases and so the integral (2.27) increases monotonically with r. Since $\inf\{r : re^{i\theta} \in S(I_j)\} = 1 - |I_j|$ we obtain

$$\int_{\tau(I_j)} P_r(\theta - t)dt \geq \int_{\tau(I_j)} P_{1-|I_j|}(\theta - t)dt = |I_j|(2 - |I_j|) \int_{\tau(I_j)} \frac{dt}{|e^{it} - z|^2}.$$

It is not hard to see that

$$|e^{it} - z| \leq c_0^{-1} |I_j| \tag{2.28}$$

for some constant c_0 and for any $t \in \tau(I_j)$ and $z \in S(I_j)$. The constant c_0 is determined by the geometry of the approximate squares of the form $S(I)$ and so (2.28) holds for any arc I on the unit circle when $t \in \tau(I)$ and $z \in S(I)$. Thus c_0 is independent of j. Consequently,

$$\begin{aligned} P[\chi_{I_j}](z) \geq & \frac{1}{2\pi} |I_j|(2 - |I_j|) \frac{c_0^2}{|I_j|^2} \int_{\tau(I_j)} dt = \frac{c_0^2}{2\pi}(2 - |I_j|) \\ \geq & \frac{c_0^2}{2\pi} = c > 0 \end{aligned}$$

for any $z \in S(I_j)$. Thus $c^{-1}P[k*f](z) \geq c^{-1}P[\chi_{I_j}](z) \geq 1$ for any $z \in S(I_j)$ and for each $1 \leq j \leq n$. Therefore $c^{-1}P[k*f] \geq 1$ on $\bigcup_{j=1}^n S(I_j)$ and so

$$\mu(\bigcup_{j=1}^n S(I_j)) = \int_{\bigcup_{j=1}^n S(I_j)} d\mu \leq c^{-2} \int_{\bigcup_{j=1}^n S(I_j)} |P[k*f]|^2 d\mu$$
$$\leq c^{-2}\|f\|_2^2, \qquad \text{by Lemma 2.5.}$$

Since f was an arbitrary test function for $\gamma(E)$ it follows that $\mu(\bigcup_{j=1}^n S(I_j)) \leq c^{-2}\gamma(E)$ and so μ satisfies (2.26).

Conversely, suppose μ satisfies (2.26). Let f be a nonnegative function in $L^2(T)$ and let $u = P[k*f]$. The *non-tangential approach region* $e^{i\theta}\Omega_r$ with vertex $e^{i\theta}$ is the smallest convex set that contains the disk centred at the origin with radius r, and the point $e^{i\theta}$ on the unit circle. The *non-tangential maximal function* $N_r u$ is defined to be the supremum of $|u(z)|$ on $e^{i\theta}\Omega_r$:

$$(N_r u)(e^{i\theta}) = \sup_{z \in e^{i\theta}\Omega_r} |u(z)| \tag{2.29}$$

Choose r large enough so that $z \in e^{it}\Omega_r$ for all $e^{it} \in 2I_z$.

We will also need the *Hardy-Littlewood maximal function* defined by

$$M[f](e^{i\theta}) = \sup_I \frac{1}{|I|} \int_I |f(t)|dt \tag{2.30}$$

where I is a subarc centred at $e^{i\theta}$. There are three facts concerning this maximal function we are going to need. First, $M[g]$ is bounded on $L^2(T)$, that is

$$\|M[g]\|_2 \leq c\|g\|_2 \tag{2.31}$$

for any $g \in L^2(T)$. See for example Theorem 8.18 [23]. Secondly, if $g \in L^2(T)$ the non-tangential maximal function of $P[g]$ is dominated by the Hardy-Littlewood maximal function corresponding to g:

$$(N_r P[g])(e^{i\theta}) \leq M[g](e^{i\theta}) \tag{2.32}$$

for each $e^{i\theta} \in T$. See Theorem 11.20 [23]. Finally, we want to prove the inequality

$$M[k*f] \leq k * M[f]. \tag{2.33}$$

Let $F(x, y) = f(x-y)k(y)$. Since F is $dxdy$-measurable and $F \geq 0$ we can apply Fubinis theorem. The notation $I(\theta)$ shall denote an arc on T centred at $e^{i\theta}$.

$$\begin{aligned}
\frac{1}{|I(\theta)|}\int_{I(\theta)} |(k*f)(x)|dx =& \frac{1}{2\pi|I(\theta)|}\int_{I(\theta)} dx \int_{-\pi}^{\pi} f(x-y)k(y)dy \\
=& \frac{1}{2\pi|I(\theta)|}\int_{-\pi}^{\pi} k(y)dy \int_{I(\theta)} f(x-y)dx \\
\leq& \frac{1}{2\pi}\int_{-\pi}^{\pi} k(y)dy \sup_{I(\theta)} \frac{1}{|I(\theta)|}\int_{I(\theta)} f(x-y)dx \\
=& \frac{1}{2\pi}\int_{-\pi}^{\pi} k(y)dy \sup_{I(\theta-y)} \frac{1}{|I(\theta-y)|}\int_{I(\theta-y)} f(t)dt \\
=& (k*M[f])(e^{i\theta})
\end{aligned} \tag{2.34}$$

Take the supremum over all arcs centred at θ on the left in (2.34) to complete the proof of (2.33).

Let $A_t = \{z \in U : |u(z)| > t\}$ for $t \geq 0$ and let K be a compact subset of A_t. Then we can find a finite number of points $z_1, z_2, \ldots, z_n \in K$ such that $K \subset \bigcup_{i=1}^n S(2I_{z_i})$. The union $\bigcup_{i=1}^n 2I_{z_i}$ can be expressed as a disjoint union $\bigcup_{j=1}^m J_j$ where each $2I_{z_i}$ is contained in some J_j. Evidently $K \subset \bigcup_{j=1}^m S(J_j)$. Define $B_t = \{e^{i\theta} \in T : (N_r u)(e^{i\theta}) > t\}$ and let $e^{i\theta} \in 2I_{z_i}$. Then $z_i \in e^{i\theta}\Omega_r$ by our choice of r and so $(N_r u)(e^{i\theta}) = \sup_{z \in e^{i\theta}\Omega_r} |u(z)| \geq |u(z_i)| > t$ since $z_i \in K \subset A_t$. Hence $e^{i\theta} \in B_t$ and so $\bigcup_{i=1}^n 2I_{z_i} = \bigcup_{j=1}^m J_j \subset B_t$. From this and the fact that μ satisfies (2.26) we obtain that

$$\begin{aligned}\mu(K) \leq& \mu(\bigcup_{j=1}^m S(J_j)) \\ \leq& c\gamma(\bigcup_{J=1}^m J_j) \\ \leq& c\gamma(B_t).\end{aligned} \tag{2.35}$$

Since μ is a regular measure (2.35) imply that

$$\mu(\{z : |u(z)| > t\}) \leq c\gamma(\{e^{i\theta} : N_r u(e^{i\theta}) > t\}). \tag{2.36}$$

Finally, our considerations yield

$$\begin{aligned}\int_U P[k * f]^2 d\mu =& \int_0^\infty \mu(\{z :\, |u(z)| > t\})dt^2 \qquad \text{by (2.18)} \\ \leq& c\int_0^\infty \gamma(\{e^{i\theta} \in T :\, (N_r u)(e^{i\theta}) > t\})dt^2 \qquad \text{by (2.36)} \\ \leq& c\int_0^\infty \gamma(\{e^{i\theta} \in T :\, M[k * f](e^{i\theta}) > t\})dt^2 \qquad \text{by (2.32)} \\ \leq& c\int_0^\infty \gamma(\{e^{i\theta} \in T :\, (k * M[f])(e^{i\theta}) > t\})dt^2 \qquad \text{by (2.33)} \\ \leq& c\|M[f]\|_2^2 \qquad \text{by Corollary 2.8} \\ \leq& c\|f\|_2^2 \qquad \text{by (2.31).}\end{aligned}$$

Since f was an arbitrary nonnegative function in $L^2(T)$ the measure μ is a Carleson measure by Lemma 2.5. This completes the proof. □

Now that we have classified the Carleson measures, we can formulate a necessary and sufficient condition for a function f to be a multiplier of the Dirichlet space.

Theorem 2.10. *A function $f \in M(D)$ if and only if f is bounded and there is a constant c such that*

$$\int_{\bigcup_{i=1}^n S(I_i)} |f'|^2 dA \leq cC(\tau(\bigcup_{i=1}^n I_i)) \tag{2.37}$$

for each disjoint collection $\{I_i\}_{i=1}^n$ of arcs on the unit circle.

Proof. The result follows immediately from Proposition 2.2 and Theorem 2.9. □

2.2 Multipliers of $L^2_{1/2}$

In this section we shall characterize the multipliers of the Bessel potential space $L^2_{1/2}(\mathbb{R})$ and the related space $L^2_{1/2}(T)$ on the unit circle. This will also yield a characterization of the multipliers of the harmonic Dirichlet space, when properly defined. This work will then aid us in giving a similar boundary characterization of the multipliers of the Dirichlet space in the next section.

Recall that $L^2_{1/2} = L^2_{1/2}(\mathbb{R}) = \{G * f : f \in L^2(\mathbb{R})\}$ is a Banach space with the norm $\|G * f\|_{1/2} = \|f\|_2$. A multiplier of $L^2_{1/2}$ is a function f such that pointwise multiplication of functions in $L^2_{1/2}$ by f is a bounded linear operator from $L^2_{1/2}$ into $L^2_{1/2}$. In other words $f \in M(L^2_{1/2})$ if and only if there is a constant c such that

$$\|fg\|_{1/2} \leq c\|g\|_{1/2} \tag{2.38}$$

for each $g \in L^2_{1/2}$. Multipliers of the general Bessel potential space $L^p_\alpha(\mathbb{R}^N)$ were studied in [30]. In this article it is proved that $M(L^p_\alpha) \subset L^\infty$ for $\alpha \geq 0$ (Proposition 1.1, Chapter 2).

If $f \in L^2$, define the function

$$\mathscr{D}f(x) = \Big(\int_{-\infty}^{\infty} \frac{|f(x-y) - f(x)|^2}{|y|^2} dy\Big)^{1/2}. \tag{2.39}$$

The following characterization is due to E. M. Stein, see [28].

Proposition 2.11. *A function f is in $L^2_{1/2}$ if and only if $f \in L^2$ and $\mathscr{D}f \in L^2$. Moreover $\|f\|_{1/2}$ is comparable to $\|f\|_2 + \|\mathscr{D}f\|_2$.*

Lemma 2.12. *Let μ be a positive Borel measure on $\mathbb{R}$. A necessary and sufficient condition that*

$$\int |f|^2 d\mu \leq c\|f\|^2_{1/2} \tag{2.40}$$

for every $f \in L^2_{1/2}$ is that $\mu(K) \leq cC(K)$ for every compact subset $K \subset \mathbb{R}$.

Proof. Assume (2.40) holds for all $f \in L^2_{1/2}$, and let h be a test function for $C(K)$ where K is a compact subset of $\mathbb{R}$. Then $|G * h|^2 \geq 1$ on K and so

$$\mu(K) = \int_K d\mu \leq \int_K |G * h|^2 d\mu \leq \int_{-\infty}^{\infty} |G * h|^2 d\mu = \|G * g\|_2^2 \leq \|G * h\|^2_{1/2} = \|h\|_2^2$$

by (2.15). Since h was an arbitrary test function for $C(K)$ we conclude that $\mu(K) \leq C(K)$.

Conversely, suppose $\mu(K) \leq cC(K)$ for all compact subsets $K \subset \mathbb{R}$ and let $f \in L^2$. Then $|f| \in L^2$ and so $G * |f| \in L^2_{1/2}$. Let $E_t = \{x : G * |f| > t\}$. With the aid of the capacitary strong type inequality we obtain that

$$\begin{aligned} c\|f\|_2^2 = c\||f|\|_2^2 &\geq \int_0^\infty C(\{x : G*|f| > t\})dt^2 \qquad \text{by (2.19)} \\ &\geq \int_0^\infty \sup\{C(K) : K \subset E_t,\, K \text{ compact}\}dt^2 \qquad \text{since } C \text{ is subadditive} \\ &\geq \frac{1}{c}\int_0^\infty \sup\{\mu(K) : K \subset E_t,\, K \text{ compact}\}dt^2 \qquad \text{by assumption} \\ &= \frac{1}{c}\int_0^\infty \mu(\{x : G*|f| > t\})dt^2 \qquad \text{since } \mu \text{ is regular} \\ &= \frac{1}{c}\int_{-\infty}^\infty |G*f|^2 d\mu \qquad \text{by (2.18).} \end{aligned}$$

Since f was an arbitrary function in L^2, (2.40) holds for all functions in $L^2_{1/2}$. This completes the proof. □

The following result is due to R. S. Strichartz, see Lemma 3.1, Chapter 2, in [30].

Lemma 2.13. *A function f is in $M(L^2_{1/2})$ is and only if $f \in L^\infty$ and*

$$\|g\mathscr{D}f\|_2 \leq c\|g\|_{1/2} \tag{2.41}$$

for all $g \in L^2_{1/2}$.

Proof. Suppose f is bounded and $\mathscr{D}f$ is a multiplier of $L^2_{1/2}$ into L^2. Then $\|fg\|_2 \leq \|f\|_\infty\|g\|_2$ and so $fg \in L^2$. Let $g \in L^2_{1/2}$ and observe that

$$\begin{aligned} \mathscr{D}(fg)(x) &= \Big(\int_{-\infty}^\infty \frac{|f(x-y)g(x-y) - f(x)g(x)|^2}{|y|^2}dy\Big)^{1/2} \\ &= \Big(\int_{-\infty}^\infty \frac{|f(x-y)g(x-y) - f(x-y)g(x) + f(x-y)g(x) - f(x)g(x)|^2}{|y|^2}dy\Big)^{1/2} \\ &\leq \Big(\int_{-\infty}^\infty \frac{|g(x-y) - g(x)|^2|f(x-y)|^2}{|y|^2}dy\Big)^{1/2} \\ &\qquad + \Big(\int_{-\infty}^\infty \frac{|f(x-y) - f(x)|^2|g(x)|^2}{|y|^2}dy\Big)^{1/2} \\ &\leq \|f\|_\infty \mathscr{D}g(x) + |g(x)|\mathscr{D}f(x). \end{aligned} \tag{2.42}$$

Thus $\|\mathscr{D}(fg)\|_2 \leq \|f\|_\infty\|\mathscr{D}g\|_2 + \|g\mathscr{D}f\|_2 \leq \|f\|_\infty\|\mathscr{D}g\|_2 + c\|g\|_{1/2}$ by the hypothesis on $\mathscr{D}f$, and so $\mathscr{D}(fg) \in L^2$. Hence $fg \in L^2_{1/2}$ by Proposition 2.11. Moreover

$$\begin{aligned} \|fg\|_{1/2} &\leq c(\|fg\|_2 + \|\mathscr{D}(fg)\|_2) \\ &\leq c(\|f\|_\infty\|g\|_2 + \|f\|_\infty\|\mathscr{D}g\|_2 + \|g\mathscr{D}f\|_2) \\ &\leq c(c_1\|f\|_\infty\|g\|_{1/2} + c_2\|g\|_{1/2}) \\ &= c\|g\|_{1/2}. \end{aligned}$$

Thus $f \in M(L^2_{1/2})$.

Conversely, suppose $f \in M(L^2_{1/2})$ and let $g \in L^2_{1/2}$. Then f is bounded and

$$\begin{aligned}
|g(x)|\mathscr{D}f(x) &= \Big(\int_{-\infty}^{\infty} \frac{|f(x-y)g(x)-f(x)g(x)|^2}{|y|^2}dy\Big)^{1/2} \\
&\leq \Big(\int_{-\infty}^{\infty} \frac{|f(x-y)|^2|g(x-y)-g(x)|^2}{|y|^2}dy\Big)^{1/2} \\
&\qquad + \Big(\int_{-\infty}^{\infty} \frac{|f(x-y)g(x-y)-f(x)g(x)|^2}{|y|^2}dy\Big)^{1/2} \\
&\leq \|f\|_\infty \mathscr{D}g(x) + \mathscr{D}(fg)(x).
\end{aligned}$$

Hence

$$\begin{aligned}
\|g\mathscr{D}f\|_2 &\leq \|f\|_\infty \|\mathscr{D}g\|_2 + \|\mathscr{D}(fg)\|_2 \\
&\leq \|f\|_\infty(\|g\|_2 + \|\mathscr{D}g\|_2) + \|fg\|_2 + \|\mathscr{D}(fg)\|_2 \\
&\leq c_0\|g\|_{1/2} + c_1\|fg\|_{1/2} \\
&\leq c\|g\|_{1/2} \qquad \text{since } f \in M(L^2_{1/2}).
\end{aligned}$$

Thus (2.41) holds for all $g \in L^2_{1/2}$. This completes the proof. □

Theorem 2.14. *A function f is in $M(L^2_{1/2})$ if and only if $f \in L^\infty$ and*

$$\int_K |\mathscr{D}f|^2 dx \leq cC(K) \tag{2.43}$$

for all compact subsets $K \subset \mathbb{R}$.

Proof. By Lemma 2.13 $f \in M(L^2_{1/2})$ if and only if $f \in L^\infty$ and

$$\|g\mathscr{D}f\|_2^2 = \int_{-\infty}^{\infty} |g\mathscr{D}f|^2 dx \leq c\|g\|^2_{1/2} \tag{2.44}$$

for all $g \in L^2_{1/2}$. Define the positive Borel measure $d\mu = |\mathscr{D}f|^2dx$. By Lemma 2.12, (2.44) holds for all $g \in L^2_{1/2}$ if and only if

$$\mu(K) = \int_K |\mathscr{D}f|^2 dx \leq cC(K)$$

for all compact subsets $K \subset \mathbb{R}$. The proof is complete. □

We now define and study a function space on the unit circle which is closely related to $L^2_{1/2}(\mathbb{R})$, namely the space

$$L^2_{1/2}(T) = \{f : \, f = k * g, \, g \in L^2(T)\}. \tag{2.45}$$

In Section 2.1 functions from this space were used, for example, in the definition of the capacity γ on the unit circle, and in the boundary characterization of D. Indeed, much work in that section centred on moving between the spaces $L^2_{1/2}(\mathbb{R})$ and $L^2_{1/2}(T)$. The key observation was that the kernels G and k are comparable on $[-\pi, \pi]$.

Proposition 2.15. *The space $L^2_{1/2}(T)$ is a Banach space with the norm $\|k * f\|_{1/2} = \|f\|_2$. Moreover $L^2_{1/2}(T) \subset L^2(T)$ and $\|k * f\|_2 \leq \|k\|_1 \|f\|_2 = \|k\|_1 \|k * f\|_{1/2}$.*

Proof. Let $\{k * f_n\}$ be a Cauchy sequence in $L^2_{1/2}(T)$. Then $\{f_n\}$ is clearly a Cauchy sequence in $L^2(T)$ and so converge in the norm on $L^2(T)$ to some function $f \in L^2(T)$. Since $k * f \in L^2_{1/2}(T)$ and

$$\|k * f_n - k * f\|_{1/2} = \|k * (f_n - f)\|_{1/2} = \|f_n - f\|_2$$

we conclude that $\{k * f_n\}$ converges in the norm of $L^2_{1/2}(T)$ to $k * f$. Thus $L^2_{1/2}(T)$ is complete with the given norm.

Since $k \in L^1(T)$ we know from the elementary theory of convolutions that $k * f \in L^2(T)$ for any $f \in L^2(T)$ and that $\|k * f\|_2 \leq \|k\|_1 \|f\|_2$. This completes the proof. □

We define multipliers of $L^2_{1/2}(T)$ just as for $L^2_{1/2}(\mathbb{R})$: $f \in M(L^2_{1/2}(T))$ if and only if $\|fg\|_{1/2} \leq c\|g\|_{1/2}$ for some constant c and for all $g \in L^2_{1/2}(T)$. We will prove in Section 4.3 that $M(L^2_{1/2}(T)) \subset L^\infty(T)$ (Corollary 4.27). With the operator norm

$$\|f\| = \|M_f\| = \sup_{\|g\|_{1/2} \leq 1} \|fg\|_{1/2}$$

the space $M(L^2_{1/2}(T))$ is a Banach algebra. Moreover $M(L^2_{1/2}(T)) \subset L^2_{1/2}(T)$ since $1 \in L^2_{1/2}(T)$. For a proof of these statements, we refer again to Section 4.3 (Proposition 4.28). For now, our interest is to give a characterization of of the multipliers of $L^2_{1/2}(T)$.

Recall from Lemma 2.4 that the Dirichlet space D could be identified with the set of functions $\{k * f : f \in H^2\}$. It turns out that the harmonic Dirichlet space can be identified with the larger space $L^2_{1/2}(T) = \{k * f : f \in L^2(T)\}$. We will prove this shortly.

Let $f \in D_h$. If we define

$$\mathscr{D}f(t) = \Big(\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{|f(\theta) - f(t)|^2}{\sin^2\left(\frac{\theta - t}{2}\right)} d\theta\Big)^{1/2} \tag{2.46}$$

then $\mathcal{D}(f) = \|\mathscr{D}f\|_2^2$ since the Douglas formula (1.13) holds for functions in D_h.

Proposition 2.16. *a function f is in D_h if and only if $f = P[k * g]$ where g is a function in $L^2(T)$. Moreover, this correspondence yields an isomorphism*

$$\begin{aligned} D_h &\longrightarrow L^2_{1/2}(T) \\ f &\longrightarrow \tilde{f} = k * g \end{aligned}$$

and the norms $\|f\|_{D_h}$ and $\|k * g\|_{1/2} = \|g\|_2$ are comparable.

Proof. By Proposition 1.6 the formula $\|f\|_{D_h} = \sum_{n=-\infty}^{\infty} (1 + |n|)|\hat{f}(n)|^2$ holds for any $f \in D_h$ where $\hat{f}(n)$ is the n'th Fourier coefficient for the boundary function of f for any $f \in D_h$. Hence the proof of this proposition is similar to the proof of Lemma 2.4. The only difference is that the bondary functions corresponding to a function in D_h may have nonzero negative Fourier coefficients. □

Recall from Section 1.2 that defining multipliers for the harmonic Dirichlet space in the same manner as for the Dirichlet space turned out to be unsuccessful since products of harmonic functions need not be harmonic. But having identified in the proposition above D_h with the set of boundary functions $L^2_{1/2}(T)$ we may define multipliers of D_h on the boundary. That is, a harmonic function f in the unit disk is a multiplier of D_h if and only if the corresponding boundary function is a multiplier of $L^2_{1/2}(T)$. Denote the set of multipliers of D_h by $M(D_h)$.

We now want to show that Theorem 2.14 holds for multipliers of $L^2_{1/2}(T)$. For this we will use the following Lemma, which is an analogue of Proposition 2.11 for $L^2_{1/2}(T)$.

Lemma 2.17. *A function f is in $L^2_{1/2}(T)$ if and only if $f \in L^2(T)$ and $\mathscr{D}f \in L^2(T)$. Moreover $\|f\|_{1/2}$ is comparable to $\|f\|_2 + \|\mathscr{D}f\|_2$.*

Proof. Suppose $f \in L^2_{1/2}(T)$. Then $f \in L^2(T)$ since $L^2_{1/2}(T) \subset L^2(T)$. Moreover $P[f] \in D_h$ by Proposition 2.16 and so $\mathcal{D}(P[f]) = \|\mathscr{D}f\|_2^2 < \infty$. Thus $\mathscr{D}f \in L^2(T)$.

Conversely, suppose $f \in L^2(T)$ and $\mathscr{D}f \in L^2(T)$. Then $P[f] \in D_h$ since $\mathcal{D}(P[f]) = \|\mathscr{D}f\|_2^2 < \infty$ and so $f \in L^2_{1/2}(T)$ by Proposition 2.16.

To prove the equivalence of $\|f\|_{1/2}$ and $\|f\|_2+\|\mathscr{D}f\|_2$, define the norm $\|g\|' = \|g\|_2+\mathcal{D}(g)^{1/2}$ on D_h. Evidently, $\|g\| \le \|g\|'$ for any $g \in D_h$ where $\|g\| = \|g\|_{D_h}$ is the standard norm (1.26) on D_h. We want to show that when D_h is endowed with the norm $\|g\|'$ the resulting normed space, denoted by $(D_h, \|.\|')$, is complete. To this end, let $\{g_n\}$ be a Cauchy sequence in $(D_h, \|.\|')$. Then $\|g_n - g_m\| \le \|g_n - g_m\|'$ and so $\{g_n\}$ is a Cauchy sequence of $(D_h, \|.\|)$. Thus $\{g_n\}$ converges to some function $g \in D_h$ in the norm $\|.\|$. But then $\|g_n - g\|_2 \to 0$ and $\mathcal{D}(g_n - g) \to 0$ and so $\{g_n\}$ converge to g in the norm $\|.\|'_{D_h}$ as well. Hence $(D_h, \|.\|')$ is complete and so $\|g\|$ and $\|g\|'$ are comparable by a corollary of the open-mapping theorem (see for example Proposition 11 in Chapter 10, [20]).

Since $\|f\|_{1/2}$ is comparable to $\|P[f]\|_{D_h}$ by Proposition 2.16 and $\|P[f]\|_{D_h}$ is comparable to $\|f\|_2+\mathcal{D}(P[f])^{1/2} = \|f\|_2+\|\mathscr{D}f\|_2$, we conclude that $\|f\|_{1/2}$ is comparable to $\|f\|_2+\|\mathscr{D}f\|_2$. This completes the proof. $\square$

We now rewrite $\mathscr{D}f$ in a form more appropriate to our needs. An easy computation shows that $4\sin^2\left(\frac{\theta-t}{2}\right) = |e^{i(\theta-t)} - 1|^2$ and so

$$\begin{aligned}\mathscr{D}f(t) &= \Big(\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{|f(\theta) - f(t)|^2}{|e^{i(\theta-t)} - 1|^2} d\theta\Big)^{1/2} \\ &= \Big(\frac{1}{\pi}\int_{t+\pi}^{t-\pi} \frac{|f(t-s) - f(t)|^2}{|e^{is} - 1|^2} ds\Big)^{1/2} = \Big(\int_{t+\pi}^{t-\pi} d\mu\Big)^{1/2} = \mu(e^{it}T) = \mu(T) \\ &= \Big(\frac{1}{\pi}\int_{\pi}^{\pi} \frac{|f(t-s) - f(t)|^2}{|e^{is} - 1|^2} ds\Big)^{1/2}. \end{aligned} \tag{2.47}$$

Notice the similarity between this expression for $\mathscr{D}$ and (2.39). Indeed, with the aid of this expression, Proposition 2.15 and Lemma 2.17, the proofs of Lemma 2.12 and Lemma 2.13 goes through virtually unchanged. Simply replace the domain of integration by the unit circle, and the denominator of $\mathscr{D}f$ by $|e^{iy} - 1|^2$). In addition, use the capacity γ in Lemma 2.12 instead of the Bessel capacity. Thus we have proved that the characterization of multipliers of $L^2_{1/2}(\mathbb{R})$ holds for $L^2_{1/2}(T)$ as well. Moreover, this will also be the characterization of the multipliers of the harmonic Dirichlet space. Explicitly, we have the following theorem:

Theorem 2.18. *Let f be a function on the unit circle. Then the following are equivalent:*

1. $f \in M(L^2_{1/2}(T))$
2. $P[f] \in M(D_h)$
3. $f \in L^\infty(T)$ *and* $\int_K |\mathscr{D}f|^2 dx \leq cC(K)$ *for all compact subsets* $K \subset T$

2.3 A boundary characterization of multipliers of the Dirichlet space

In this section, we shall give a boundary characterization of multipliers of the Dirichlet space similar to the characterization of $M(L^2_{1/2}(T))$ in the last section. In doing so, we will show that the boundary function of a multiplier of D is a multiplier of $L^2_{1/2}(T)$.

If $\mathscr{D}f$ is the expression (2.47) then $\mathcal{D}(f) = \|\mathscr{D}f\|_2^2$ for any $f \in D$, just as for D_h.

Lemma 2.19. *A function f is in D if and only if $f \in H^2$ and $\mathscr{D}f \in L^2(T)$. Moreover $\|f\|_D$ is comparable to $\|f\|_2 + \|\mathscr{D}f\|_2$.*

Proof. This is just a special case of Lemma 2.17 where f is assumed to be analytic. □

Recall that Lemma 2.12 and Lemma 2.13 remain true if the real line is replaced by the unit circle and $\mathscr{D}f$ is the expression 2.47. We now show that Lemma 2.12 can be improved in this context.

Lemma 2.20. *Let μ be a positive Borel measure on T. A necessary and sufficient condition that*

$$\int_T |\tilde{f}|^2 d\mu \leq c\|f\|_D^2 \tag{2.48}$$

for all $f \in D$ is that $\mu(K) \leq cC(\tau(K))$ for each compact subset $K \subset T$.

Proof. Assume (2.48) holds for all functions in D, and let $f = k * h \in L^2_{1/2}(T)$. Then $h \in L^2(T)$, and $h = h_1 + \bar{h}_2$ where h_1, $h_2 \in H^2$ and $\|h\|_2^2 = \|h_1\|_2^2 + \|h_2\|_2^2$ as in the proof of Lemma 2.5. Clearly $f = k * h = k * h_1 + \overline{k * h_2}$ and so

$$\begin{aligned}
\int_T |f|^2 d\mu &\leq \int_T |k * h_1|^2 d\mu + \int_T |k * h_2|^2 d\mu \\
&\leq c\|P[k * h_1]\|_D^2 + c\|P[k * h_2]\|_D^2 \qquad \text{by (2.48)} \\
&\leq c\|h_1\|_2^2 + c\|h_2\|_2^2 \qquad \text{by Lemma 2.4} \\
&= c\|h\|_2^2 \\
&= c\|f\|_{1/2}^2.
\end{aligned}$$

Since f was an arbitrary function in $L^2_{1/2}(T)$ we conclude that $\mu(K) \leq cC(K)$ for each compact subset of T by Lemma 2.12.

Conversely, suppose $\mu(K) \leq cC(\tau(K))$ for each compact subset of T. Then

$$\int_T |f|^2 d\mu \leq c\|f\|_{1/2}^2$$

for each $f \in L^2_{1/2}(T)$ by Lemma 2.12. Suppose $g = P[f] \in D$. Then $f \in L^2_{1/2}(T)$ and the norms $\|f\|_{1/2}$ and $\|g\|_D$ are comparable by Lemma 2.4. Thus

$$\int_T |\tilde{g}|^2 d\mu = \int_T |f|^2 d\mu \leq c\|f\|^2_{1/2} \leq c\|g\|^2_D.$$

Since g was an arbitrary function in D the proof is complete. □

Lemma 2.21. *A function f is in $M(D)$ if and only if $f \in H^\infty$ and*

$$\|\tilde{g}\mathscr{D}f\|_2 \leq c\|g\|_D \tag{2.49}$$

for each $g \in D$.

Proof. The proof of this lemma is analogous to the proof of Lemma 2.13. Assume f is bounded and that (2.49) holds for all $g \in D$. Let $g \in D$. By an argument similar to the one found in the proof of Lemma 2.13, we get that

$$\begin{aligned}\mathscr{D}(fg)(t) &= \Big(\frac{1}{\pi}\int_{\pi}^{\pi} \frac{|f(t-s)g(t-s) - f(t)g(t)|^2}{|e^{is}-1|^2} ds\Big)^{1/2} \\ &\leq \|f\|_\infty \mathscr{D}g(t) + |g(t)|\mathscr{D}f(t)\end{aligned}$$

and so

$$\begin{aligned}\|\mathscr{D}(fg)\|_2 &\leq \|f\|_\infty \|\mathscr{D}g\|_2 + \|g\mathscr{D}f\|_2 \\ &\leq \|f\|_\infty \|\mathscr{D}g\|_2 + c\|g\|_D \\ &< \infty.\end{aligned}$$

Thus $\mathcal{D}(fg) = \|\mathscr{D}(fg)\|_2^2 < \infty$ and so $fg \in D$. We conclude that $f \in M(D)$.

Conversely, suppose $f \in M(D)$ and let $g \in D$. Then f is bounded, $fg \in D$ and

$$\begin{aligned}|g(t)|\mathscr{D}f(t) &= \Big(\frac{1}{\pi}\int_{\pi}^{\pi} \frac{|f(t-s)g(t) - f(t)g(t)|^2}{|e^{is}-1|^2} ds\Big)^{1/2} \\ &\leq \|f\|_\infty \mathscr{D}g(t) + \mathscr{D}(fg)(t).\end{aligned}$$

Thus

$$\begin{aligned}\|g\mathscr{D}f\|_2 &\leq \|f\|_\infty \|\mathscr{D}g\|_2 + \|\mathscr{D}(fg)\|_2 \\ &\leq \|f\|_\infty(\|g\|_2 + \|\mathscr{D}g\|_2) + \|fg\|_2 + \|\mathscr{D}(fg)\|_2 \\ &\leq c_1\|g\|_D + c_2\|fg\|_D \qquad \text{by Lemma 2.19} \\ &\leq c\|g\|_D \qquad \text{since } f \in M(D).\end{aligned}$$

The proof is complete. □

We now quickly obtain the boundary characterization of functions in $M(D)$. We include the previous characterization for completeness in the following theorem.

Theorem 2.22. *The following are equivalent:*

1. $f \in M(D)$

2. *$f \in H^\infty$ and $\int_{\bigcup_{i=1}^n S(I_i)} |f'|^2 dA \le cC(\tau(\bigcup_{i=1}^n I_i))$ for each disjoint collection $\{I_i\}_{i=1}^n$ of arcs on the unit circle*

3. *$f \in H^\infty$ and $\int_K (\mathscr{D} f)^2 dx \le cC(\tau(K))$ for all compact subsets $K \subset T$*

Proof. We have already proved the equivalence of (1) and (2). We now prove the equivalence of (1) and (3). By Lemma 2.21, $f \in M(D)$ if and only if

$$\|\tilde{g}\mathscr{D} g\|_2 = \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\tilde{g}\mathscr{D} f|^2 dx\Big)^{1/2} \le c\|g\|_2 \tag{2.50}$$

for all $g \in D$. Define the positive Borel measure $d\mu = (\mathscr{D} f)^2 dx$. Then Lemma 2.20 imply that (2.50) holds for all $g \in D$ if and only if $\mu(K) = \int_K (\mathscr{D} f)^2 dx \le cC(K)$ for each compact subset $K \subset T$. This completes the proof. □

The boundary characterization of multipliers of the Dirichlet space together with the characterization of multipliers of the Bessel potential space in the last section furnishes the following interesting result.

Proposition 2.23. *If $f \in M(D)$ then the boundary function of f is contained in $M(L^2_{1/2}(T))$. Equivalently, $f \in M(D_h)$.*

Proof. If $f \in M(D)$ then $f \in H^\infty$ and $\int_K (\mathscr{D}\tilde{f})^2 dx \le cC(K)$ for all compact subsets $K \subset T$ by Theorem 2.22. Thus $\tilde{f} \in M(L^2_{1/2}(T))$ by Theorem 2.18. Hence $f \in M(D_h)$ since $f \in M(D_h)$ if and only if $\tilde{f} \in M(L^2_{1/2}(T))$. This completes the proof. □

Chapter 3

Univalent multipliers of the Dirichlet space

In this chapter we shall investigate a certain class of multipliers of the Dirichlet space, namely the univalent multipliers arising from the application of the Riemann mapping theorem. The discussion is mainly based on the article [5] and most of the results in this section can be found in that article, in particular Theorem 3.5 and Theorem 3.6. However, we will extend some of the results and add some comments to others.

First, we are going to generalize the definition of the Dirichlet space and the Bergman space to include sets different from U. Let G be a connected and open subset of $\mathbb{C}$, which we shall call a *domain.* The Dirichlet space corresponding to G is the set of all analytic functions $f : G \to \mathbb{C}$ such that $f(z_0) = 0$ where z_0 is some distinguished point in G, and

$$\|f\|^2_{D(G)} = \frac{1}{\pi} \int_G |f'|^2 dA < \infty. \tag{3.1}$$

The condition $f(z_0) = 0$ ensures that 0 is the the only constant function contained in $D(G)$ and so (3.1) indeed defines a norm as no non-zero function has zero norm. We can give $D(G)$ the inner product $\langle f, g\rangle = \frac{1}{\pi}\int_G f'\bar{g}' dA$, turning it into a Hilbert space. A multiplier f of $D(G)$ is defined just as for D: f is bounded and the map $M_f(h) = fh$ of pointwise multiplication is a bounded linear operator from $D(G)$ into itself. We will denote the space of multipliers of $D(G)$ by $M(D(G))$ or $M(D_G)$ for brevity.

The Bergman space corresponding to G is the set of all analytic functions $f : G \to \mathbb{C}$ such that

$$\|f\|^2_{B(G)} = \frac{1}{\pi} \int_G |f|^2 dA < \infty. \tag{3.2}$$

The proof of the following proposition is based on the proof of Theorem 12.38 in [21].

Proposition 3.1. *Let G be a domain, and let K be a compact subset of G. Then there is a constant c such that*

$$|f(z)| \leq c\|f\|_{B(G)} \tag{3.3}$$

for all $z \in K$ and $f \in B(G)$. In particular, evaluation at a point is a bounded linear functional on $B(G)$.

Proof. Let $z \in K$ and let δ be a non-zero finite number less the distance from K to the boundary of G. Note that this distance cannot be zero since G is open and K is compact.

Let $\triangle$ be a disk of radius δ centred at z. Then $f(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - z)^n$ for $\zeta \in \triangle$. If $\zeta - z = re^{i\theta}$ then $f(re^{i\theta}) = f_r(e^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$ and $\|f_r\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$. Thus

$$\frac{1}{\pi}\int_{\triangle} |f|^2 dA = \frac{1}{\pi}\int_0^{\delta} rdr \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta = 2\int_0^{\delta} \sum_{n=0}^{\infty} |a_n|^2 r^{2n+1} dr = \sum_{n=0}^{\infty} (n+1)^{-1}|a_n|^2 \delta^{2n+2}.$$

Note that $f(z) = a_0$ and so

$$\begin{aligned} |f(z)| = |a_0| \leq & \delta^{-1}\Big(\sum_{n=0}^{\infty}(n+1)^{-1}|a_n|^2\delta^{2n+2}\Big)^{\frac{1}{2}} \\ = & \delta^{-1}\Big(\int_{\triangle} |f|^2 dA\Big)^{\frac{1}{2}} \\ \leq & \delta^{-1}\|f\|_{B(G)}. \end{aligned}$$

Since δ is independent of z and f, (3.3) holds for all $z \in K$ and $f \in B(G)$. Moreover, point evaluations are bounded since G is locally compact. This completes the proof. □

Corollary 3.2. *Let G be a domain. Evaluation at a point is a bounded linear functional on the Dirichlet space $D(G)$, that is, for each $z \in G$ there exists a constant c such that*

$$|\lambda_z(f)| = |f(z)| \leq c\|f\|_{D(G)} \tag{3.4}$$

for each $f \in D(G)$.

Proof. Let $z \in G$ and let Γ be a rectifiable arc in G from the distinguished point z_0 to z. If $f \in D(G)$ then $f' \in B(G)$ and so

$$\begin{aligned} |f(z)| = \Big|\int_{z_0}^{z} f'(w)dw\Big| \leq & (\text{length of } \Gamma) \sup_{w\in\Gamma} |f'(w)|, && \text{since } \Gamma \text{ is rectifiable} \\ \leq & c\|f'\|_{B(G)}, && \text{by Proposition 3.1 since } \Gamma \text{ is compact} \\ = & c\|f\|_{D(G)}. \end{aligned}$$

This completes the proof. □

Thus D_G is a Hilbert space of analytic functions on the domain G such that evaluation at a point is a bounded linear functional. Hence Proposition 1.7 and Proposition 1.9 holds for D_G as we remarked at the end of Chapter 1. Consequently, $f \in M(D_G)$ if and only if $fD(G) \subset D(G)$. Moreover $M(D_G)$ is a Banach algebra.

Now, consider the special case where $G = U$ and $z_0 = 0$. Evidently, any function $f \in D(U)$ has a power series representation with constant term equal to zero. Moreover Proposition 1.2 holds for $D(U)$. If $w \in U - \{0\}$ define $k_w^*(z) = \log \frac{1}{1-\bar{w}z}$. If $w = 0$ let $k_w^* = 0$. Then k_w^* will be a reproducing kernel for $D(U)$, that is $f(z) = \langle f, k_z^*\rangle$. The proof is similar to the proof of Proposition 1.4. Thus Proposition 1.5 holds for $D(U)$ as well.

Proposition 3.3. *An analytic function f in the unit disk is a multiplier of D if and only it is a multiplier of $D(U)$, that is, $M(D) = M(D_U)$.*

Proof. Suppose $f \in M(D)$. Then f is clearly a multiplier of $D(U)$ since $D(U) \subset D$.

Conversely, if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in M(D_U)$ then $f \in M(D)$ if and only if f multiply the constant functions, that is, $\mathcal{D}(f) < \infty$. But $zf \in D(U)$ since $f \in M(D_U)$ and $z \in D(U)$. Since $(zf)(z) = \sum_{n=0}^{\infty} a_n z^{n+1} = \sum_{n=1}^{\infty} b_n z^n$ where $b_n = a_{n-1}$ we obtain that

$$\infty > \mathcal{D}(zf) = \sum_{n=1}^{\infty} n|b_n|^2 = \sum_{n=1}^{\infty} n|a_{n-1}|^2 = \sum_{n=0}^{\infty}(n+1)|a_n|^2 \geq \sum_{n=1}^{\infty} n|a_n|^2 = \mathcal{D}(f).$$

Thus $f \in M(D)$ and so $M(D) = M(D_U)$. □

If G is a domain and $z_0 \in G$, then the Riemann mapping theorem furnishes an analytic homeomorphism ϕ of U onto G such that $\phi(0) = z_0$. We will call ϕ a *Riemann map*. Let C_ϕ denote the composition operator defined by $C_\phi(g) = g \circ \phi$.

Lemma 3.4. *C_ϕ is an isometry from $D(G)$ onto $D(U)$.*

Proof. C_ϕ is obviously a linear operator. To verify that C_ϕ is surjective, choose $f \in D(U)$. Since $w = \phi(z)$ is an analytic homeomorphism of U onto G, the inverse function $z = \phi^{-1}(w)$ is also analytic. Moreover, the derivative ϕ' can never vanish and $(\phi^{-1}(w))' = \frac{1}{\phi'(\phi^{-1}(w))}$. Then $g = f \circ \phi^{-1}$ is an analytic function and

$$g'(w) = (f(\phi^{-1}(w)))' = \frac{f'(\phi^{-1}(w))}{\phi'(\phi^{-1}(w))}$$

Using the method of changing the integration variables, we deduce that

$$\begin{aligned}\|g\|_{D(G)}^2 &= \int_G |g'(w)|^2 dA(w) = \int_G \left|\frac{f'(\phi^{-1}(w))}{\phi'(\phi^{-1}(w))}\right|^2 dA(w) \\ &= \int_U \left|\frac{f'(z)}{\phi'(z)}\right|^2 |\phi'(z)|^2 dA(z) = \int_U |f'|^2 dA(z) = \|f\|_{D(U)}^2 < \infty.\end{aligned}$$

Hence $g \in D(G)$. Thus C_ϕ is surjective, since $C_\phi(g) = f$. Moreover, the calculation above shows that C_ϕ is norm preserving. It is trivial to verify that it is also injective. Thus C_ϕ is an isometry, as was to be proved. □

For G simply connected we define the integral operator V by the formula

$$(Vg)(z) = \int_{z_0}^{z} g(w)dw$$

Clearly, V maps $B(G)$ onto $D(G)$.

Theorem 3.5. *Let G be a bounded simply connected domain and let ϕ be a Riemann map of the unit disk U onto G. The following are equivalent.*

1. *$D(G) \subset B(G)$*
2. *$zD(G) \subset D(G)$*
3. *$\phi D(U) \subset D(U)$, that is, $\phi \in M(D)$*

4. $\phi' D(U) \subset B(U)$

5. V *maps* $B(G)$ *into* $B(G)$

Proof. For g in $D(G)$, $(zg)' = zg' + g$. Since $g' \in B(G)$ and z is bounded on G, the function zg' is in $B(G)$. Thus the equation above shows that zg is in $D(G)$ if and only if g is in $B(G)$. Hence (1) and (2) are equivalent.

Choose the distinguished point z_0 of G to be $\phi(0)$. Note that the validity of (1) and (2) are independent of the choice of z_0.

To show the equivalence of (2) and (3), suppose that (2) holds. Let f be in $D(U)$. Since $\phi f = C_\phi(zC_\phi^{-1} f)$, we see that ϕf is in $D(U)$ as required.

Conversely, suppose (3) holds. Let g be in $D(G)$. Since $zg = C_\phi^{-1}(\phi C_\phi(g))$ we see that zg is in $D(G)$, and (2) holds.

To see the equivalence of (3) and (4), let f be in $D(U)$. Then $(\phi f)' = \phi' f + \phi f'$. Since ϕ is bounded and f' is in $B(U)$, we see that ϕf is in $D(U)$ if and only if $\phi' f$ is in $B(U)$, and so (3) and (4) are equivalent.

Thus (1) through (4) are all equivalent. Finally, since V maps $B(G)$ onto $D(G)$, (1) is equivalent to (5). This completes the proof. □

A region G is called *star convex* if there is a point w_0 in G such that for each point w in G, the line segment from w_0 to w is included in G. In particular, star convex sets are simply connected.

Theorem 3.6. *Let* G *be a bounded star convex domain and let* ϕ *be a Riemann map of the unit disk* U *onto* G*. Then* $\phi \in M(D)$

Proof. We can assume without loss of generality that that G is star convex with respect to the origin and that the distinguished point z_0 of G is also the origin. We will prove the theorem by showing that condition (1) of theorem 3.5 holds.

Let $g \in D(G)$. Then

$$g(z) = \int_0^z g'(w)dw = z \int_0^1 g'(tz)dt$$

Notice that we used the fact that G is star convex in the last integral above. By the Schwarz inequality we now have

$$\begin{aligned} |g(z)|^2 &= |z|^2 \Big| \int_0^1 g'(tz)dt \Big|^2 \\ &\le |z|^2 \Big(\int_0^1 |g'(tz)|dt \Big)^2 \\ &\le |z|^2 \Big(\int_0^1 1^2 dt \Big) \Big(\int_0^1 |g'(tz)|^2 dt \Big) \\ &\le c \int_0^1 |g'(tz)|^2 dt \end{aligned} \tag{3.5}$$

where c is a constant such that $|z|^2 \leq c$ for all $z \in G$. From (3.5), the Fubini theorem and a change of variables we obtain that

$$\begin{aligned}\pi\|g\|^2_{B(G)} = \int_G |g(z)|^2 dA(z) &\leq c \int_G dA(z) \int_0^1 |g'(tz)|^2 dt \\ &\leq c \int_0^1 dt \int_G |g'(tz)|^2 dA(z) \\ &\leq c \int_0^1 t^{-2} dt \int_{tG} |g'(w)|^2 dA(w). \qquad (3.6)\end{aligned}$$

Since the origin is an interior point of G and G is bounded, there is a positive number $s < 1$ such that the closure of sG is contained in G. Since $|g'|^2$ is continuous, there is a constant c_0 such that $|g'|^2 < c_0$ on sG. If $0 < t < s$ then $tG \subset sG$ and so

$$\begin{aligned}t^{-2} \int_{tG} |g'(w)|^2 dA(w) &< c_0 t^{-2} \int_{tG} dA(w) \\ &= c_0 \int_G dA(z) \\ &= c_0 |G| \qquad (3.7)\end{aligned}$$

where $|G|$ denotes the area of G. So

$$\int_0^s t^{-2} dt \int_{tG} |g'(w)|^2 dA(w) < \int_0^s c_0 |G| dt < \infty \qquad (3.8)$$

Moreover,

$$\begin{aligned}\int_s^1 t^{-2} dt \int_{tG} |g'(w)|^2 dA(w) &\leq \int_s^1 t^{-2} \|g\|^2_{D(G)} dt \qquad (3.9) \\ &= (s^{-1} - 1)\|g\|^2_{D(G)} < \infty \qquad (3.10)\end{aligned}$$

Now Equation (3.6), (3.8) and (3.9) show that g is in $B(G)$, which completes the proof. □

A simply connected domain G of the complex plane shall be called a *piecewise star convex domain* if G is the union of finitely many disjoint star convex domains G_n and a set E of Lebesgue measure zero. One can visualize E as the union of the arcs dividing G into star convex domains. We shall extend Theorem 3.6 to piecewise star convex domains.

Lemma 3.7. *Suppose f is analytic in the piecewise star convex domain $G = G_1 \cup G_2 \cup \ldots \cup G_N \cup E$, where the G_n are pairwise disjoint. Then $f \in D(G)$ if and only if $f \in D(G_n)$ for each $n = 1, 2 \ldots N$.*

Proof. Since E has measure zero, we have

$$\mathcal{D}(f) = \frac{1}{\pi} \int_G |f'|^2 dA = \frac{1}{\pi} \int_{G_1 \cup \ldots \cup G_N} |f'|^2 dA = \frac{1}{\pi} \sum_{n=1}^{N} \int_{G_n} |f'|^2 dA$$

The result now follows. □

Proposition 3.8. *Let G be a bounded piecewise star convex domain, and let ϕ be a Riemann map of the unit disk U onto G. Then $\phi \in M(D)$.*

Proof. We will prove the theorem by showing that condition (2) of Theorem 3.5 holds; $zD(G) \subset D(G)$. Write $G = G_1 \cup G_2 \cup \ldots \cup G_N \cup E$ as in Lemma 3.7 and choose $f \in zD(G)$. Then $f = zg$ where $g \in D(G)$, and so $g \in D(G_n)$ for each $n = 1, 2 \ldots N$ by Lemma 3.7. Thus $f = zg \in zD(G_n)$ for $n = 1, 2 \ldots N$. Since each G_n is a bounded star convex domain, we know that $zD(G_n) \subset D(G_n)$ by Theorem 3.5 and Theorem 3.6. Thus $f \in D(G_n)$ for $n = 1, 2 \ldots N$ and so $f \in D(G)$, again by Lemma 3.7. This completes the proof. □

A *polygon* in the plane is a closed curve composed of a finite number of straight line segments, called *edges*. A point where two edges meet is called a *vertex*. If none of the edges intersect each other except at the vertices, the polygon is called *simple*.

Let P be a simple polygon. By the Jordan curve theorem the simple closed curve P divides the plane into exactly two connected components, one of them being the (bounded) interior of P, denoted by $\text{int}(P)$. We shall show that the interior of a simple polygon is a piecewise star convex domain. Consequently, a Riemann map from U onto $\text{int}(P)$ is a multiplier of the Dirichlet space by Proposition 3.8.

Proposition 3.9. *The interior of a simple polygon P is a piecewise star convex domain.*

Proof. Since P is a closed Jordan curve the interior $\text{int}(P)$ is evidently a simply connected open set. To show that it is a piecewise star convex domain we shall divide the polygon into finitely many triangles. Choose a vertex p such that the interior angle α is less then π. It is easy to see that at least one such angle must exist for simple polygons.

Denote by p_1 and p_2 the vertices connected to p by edges. Let R_p denote the open rectangle with vertices p, p_1 and p_2. We shall draw a line which will be a subset of E; the set dividing P into star convex domains. This will be done in one of two ways, depending on the interior of R_p:

1. $R_p \subset \text{int}(P)$: There are no vertices of P contained in R_p. Draw the straight line between p_1 and p_2. We have now successfully split P into a star convex rectangle and a polygon P_1 having less vertices then P.

2. $R_p \not\subseteq \text{int}(P)$: There are vertices of P contained in R_p. Clearly, a line contained in $\text{int}(P) \cap R_p$ can be drawn from p to one of these vertices. Having done so, we have divided P into two smaller polygons P_1 and P_2, each having less vertices then P.

We proceed inductively in this fashion. With each step the original polygon is split into two smaller polygons, each with less vertices then the polygon we started with. The process only terminates when P has been split into a finite number of star convex rectangles. This completes the proof. □

We will now find another necessary and sufficient condition for a univalent map ϕ to be a multiplier that will relate the problem of finding multipliers to other fields in analysis. First, a lemma.

Lemma 3.10. *Let G be a bounded simply connected domain. If $D(G) \subset B(G)$ then the inclusion map from $D(G)$ into $B(G)$ is a bounded linear operator, that is, there exists a constant c such that*

$$\|f\|_{B(G)} \leq c\|f\|_{D(G)}. \tag{3.11}$$

for all $f \in D(G)$.

Proof. Let I denote the inclusion map. It is evidently a linear operator and we are left with proving that it is also bounded. By the closed graph theorem it is sufficient to show that the graph of I is closed.

Let $\{f_n\}$ be a sequence in $D(G)$ converging in the norm on $D(G)$ to the function $f \in D(G)$ and suppose $I(f_n) = f_n$ converges in the norm on $B(G)$ to the function $g \in B(G)$. We must show that $I(f) = f = g$. But since point evaluations are bounded linear functionals on both $D(G)$ and $B(G)$ we know that $f_n \to f$ and $f_n \to g$ pointwise. Thus $f = g$ and we conclude that I is bounded. This completes the proof. $\square$

The inequality (3.11) is one form the *Poincaré inequality* which arise, for example, in the theory of Sobolev spaces. With this in mind we make the following definitions.

Definition 3.11. *Let G be a domain and fix a point $z_0 \in G$. Then G is called an analytic Poincaré domain if there exists a constant c depending only on G such that*

$$\int_G |f|^2 dA \leq c \int_G |f'|^2 dA \tag{3.12}$$

for all $f \in Hol(G)$ such that $f(z_0) = 0$.

Note that the validity of (3.12) is independent of the choice of z_0. In fact, for G to be an analytic Poincaré domain it is sufficient to check the inequality for all functions $f \in D(G)$ (the point z_0 is then included in the definition of $D(G)$).

Definition 3.12. *A domain G is called a Poincaré domain if there exists a constant c depending only on G such that*

$$\int_G |f|^2 dA \leq c \int_G |\nabla f|^2 dA \tag{3.13}$$

for all $f \in C^1(G)$ such that $\int_G f dA = 0$.

Hamilton proved in [14] that a simply connected domain G with finite area is an analytic Poincaré domain if and only if it is a Poincaré domain. This result was extended by Stanoyevitch and Stegenga in [26]. Combining this result with Theorem 3.5 and Lemma 3.10 we obtain the following extension of Theorem 3.5:

Proposition 3.13. *Let G be a bounded simply connected domain and let ϕ be a Riemann map of the unit disk U onto G. The following are equivalent.*

1. $\phi \in M(D)$
2. $D(G) \subset B(G)$
3. $\int_G |f|^2 dA \leq c \int_G |f'|^2 dA$ *for all* $f \in D(G)$
4. *G is an analytic Poincaré domain*
5. *G is a Poincaré domain*

Proof. The equivalence of (1) and (2) has already been proved. The equivalence of (2) and (3) follows from Lemma 3.10. The equivalence of (3) and (4) follows by definition and the equivalence of (4) and (5) is the result by Hamilton mentioned above. This completes the proof. □

From the observations above we see that finding bounded simply connected Poincaré domains will yield univalent multipliers of the Dirichlet space. In [24] several domains of this type are found and they will all yield multipliers by the corresponding Riemann map from U into the Poincaré domain. In the same article the authors also construct an interesting non-Poincaré domain (see Section 5 of the article). Let ϕ be the Riemann map corresponding to this domain. From the results of Chapter 2 we know that the corresponding measure $|\phi'|^2 dA$ is not a Carleson measure and so the characterization (2) in Theorem 2.22 does not holds for arbitrary collections of disjoint arcs on the unit circle. However, it is shown that the property holds for intervals on the unit circle. We conclude that it is not sufficient for property (2) of Theorem 2.22 to be satisfied on intervals of the unit circle. A counter-example of this sort was also produced in [27] but it was not given by a conformal map as the example in [24].

We now apply the observations above to another concrete example. Let G be the ribbon inside the unit disk spiralling out to the boundary. Then G is a bounded simply connected domain and it is shown in [16] that G is not a Poincaré domain since there exists an analytic function f on G such that

$$\|f\|^2_{D(G)} = \frac{1}{\pi} \int_G |f'|^2 dA < \infty$$

but

$$\|f\|^2_{B(G)} = \frac{1}{\pi} \int_G |f|^2 dA = \infty$$

Thus a Riemann map ϕ from the unit disk onto G is not a multiplier of the Dirichlet space. However, since ϕ is bounded and univalent we know that $\pi\mathcal{D}(\phi) = \text{Area}(\phi(U)) < \infty$ by Proposition 1.1. Thus $\phi \in H^\infty_D$ but $\phi \notin M(D)$.

Chapter 4

$M(D)$ as a Banach algebra

In Chapter 1 we proved that $M(D)$ is a complex commutative Banach algebra. In Section 4.1 we explore the structure of the maximal ideal space of $M(D)$.

In Section 4.2 we change the domain of definition and investigate the maximal ideal space of $M(D_G)$.

In Section 4.3 we discuss the open problem of identifying the Shilov boundary of $M(D)$ with the maximal ideal space of some Banach algebra of functions on the unit circle.

4.1 The maximal ideal space of $M(D)$

The space H^∞ of bounded analytic functions in the unit disk is a Banach algebra with the supremum norm. This Banach algebra is studied extensively in Chapter 10 of the book [15]. The space $H^\infty_D = H^\infty \cap D$ of bounded functions in the Dirichlet space is also a Banach algebra with the norm $\|f\| = \|f\|_\infty + \mathcal{D}(f)^{1/2}$. References on this Banach algebra are the articles [7] and [8]. It will turn out that the structure of the maximal ideal space of $M(D)$ is very similar to that of H^∞_D which in turn is very similar to the structure of the maximal ideal space of H^∞. The key to this observation is Proposition 4.2, which is an analogue of Proposition 2.2 in [7] for $M(D)$. When this result is established several results obtained for H^∞_D in [7] and [8] can be obtained for $M(D)$ in a similar manner. Indeed, all of the propositions in this section are proved for H^∞_D in those articles. However, in some cases the arguments for H^∞_D cannot be reproduced for $M(D)$ because they rely on specific bounded dirichlet finite functions that may not be multipliers. However, this problem will be remedied with the aid of the Corona theorem for multipliers.

First, we consider the following questions: is $M(D)$ a uniformly closed subalgebra of H^∞? And is every function that is analytic in U and continuous on $\overline{U}$ contained in $M(D)$? The answer to both of these questions can be given by considering the function $f(z) = \sum_{n=0}^\infty \frac{z^{n!}}{n^2}$ given in [7]. In this article the same questions were posed for the Banach algebra H^∞_D. First, notice that $z \in M(D)$ since $\mathcal{D}(zf) = \frac{1}{\pi}\int_U |f + zf'|^2 dA \leq \|f\|_B^2 + \mathcal{D}(f)$ for any $f \in D$. Thus all polynomials are contained in $M(D)$ and so the partial sums $f_N(z) = \sum_{n=0}^N \frac{z^{n!}}{n^2} \in M(D)$. If $z \in \overline{U}$ and $\epsilon > 0$ we see that

$$|f(z) - f_N(z)| = \Big|\sum_{n=N+1}^\infty \frac{z^{n!}}{n^2}\Big| \leq \sum_{n=N+1}^\infty \frac{1}{n^2} < \epsilon$$

if $N \geq M(\epsilon)$. Thus f_N converges uniformly to f on $\overline{U}$ and so f is analytic in U and continuous on $\overline{U}$. Observe that $a_{n!} = 1/n^2$ is the $n!$'th Taylor coefficient of f and that $a_m = 0$ if $m \neq n!$ for any $n \in \mathbb{N}$. Thus

$$\mathcal{D}(f) = \sum_{n=1}^{\infty} n|a_n|^2 = \sum_{n=1}^{\infty} n! \frac{1}{n^4}$$

which diverges. Thus $f \notin D$ and so $f \notin M(D)$. We conclude that $M(D)$ is not a uniformly closed subalgebra of H^∞. Moreover, f is an example of a bounded analytic function on the unit disk with continuous boundary values which is not a multiplier of D, nor even in the Dirichlet space.

Lemma 4.1. *If $f \in M(D)$ then $f'D \subset B$.*

Proof. We must show that $\int_U |f'g|^2 dA < \infty$ for every $g \in D$. First, note that $f'g = (fg)' - fg'$. Hence

$$\int_U |f'g|^2 dA = \int_U |(fg)' - fg'|^2 dA \leq \int_U |(fg)'|^2 dA + \int_U |fg'|^2 dA$$

The first integral is finite because $fD \subset D$. The second integral is finite because f is bounded and $g \in D$. Thus $f'D \subset B$, as was to be shown. $\square$

Proposition 4.2. *For $f \in M(D)$, the following are equivalent:*

1. *f is invertible in $M(D)$.*
2. *f is invertible in H^∞_D.*
3. *f is invertible in H^∞*
4. $\inf\{|f(\lambda)| : \lambda \in U\} > 0$

Proof. First, note that (3) and (4) are obviously equivalent. Moreover (2) clearly implies (4). To show that (4) implies (2), note that

$$\mathcal{D}\Big(\frac{1}{f}\Big) = \frac{1}{\pi}\int_U \Big|\Big(\frac{1}{f}\Big)'\Big|^2 dA = \frac{1}{\pi}\int_U \Big| - \frac{f'}{f^2}\Big|^2 dA \leq \frac{1}{\pi \inf_{\lambda\in U} |f(\lambda)|^4}\mathcal{D}(f) < \infty$$

Thus $f^{-1} \in H^\infty_D$, and so (4) implies (2).

We will now show that (1) is equivalent with the other statements. Clearly, (1) implies (2) since $M(D) \subset H^\infty_D$. To show that (2) implies (1), suppose f is invertible in H^∞_D. We must show that $f^{-1} \in M(D)$, that is, $f^{-1}D \subset D$. Since (2) and (4) are equivalent, we have $\inf\{|f(\lambda)| : \lambda \in U\} > 0$. Choose $g \in D$. Then

$$\begin{aligned}\mathcal{D}\Big(\frac{g}{f}\Big) &= \frac{1}{\pi}\int_U \Big|\frac{g'f - gf'}{f^2}\Big|^2 dA \\ &\leq \frac{1}{\pi \inf_{\lambda\in U} |f(\lambda)|^4}\Big[\int_U |g'f|^2 dA + \int_U |gf'|^2 dA\Big]\end{aligned}$$

Since f is bounded and $g \in D$, the first integral above is finite. Since $f \in M(D)$ we know that $gf' \in B$ by Lemma 4.1. Thus the second integral is also finite. Hence $f^{-1} \in M(D)$. This completes the proof. $\square$

Let A be any complex commutative Banach algebra. The *spectrum* of any $f \in A$, denoted by $\mathrm{Sp}(f, A)$, is the set of all complex numbers λ such that $f - \lambda$ is not invertible. The *spectral radius* of f is the number

$$\mathrm{r}(f, A) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(f, A)\}.$$

Let $\mathfrak{M}(A)$ be the set of complex homomorphisms $\phi : A \to \mathbb{C}$. If m is a maximal ideal of A the Gelfand-Mazur theorem states that there is an isomorphism j of the quotient space A/m onto $\mathbb{C}$, and so the canonical surjection k_m of A onto A/m induces a complex homomorphism $\phi = j \circ k_m \in \mathfrak{M}(A)$ of A onto the field of complex numbers whose kernel is m. Conversely, the kernel of every complex homomorphism $\phi \in \mathfrak{M}(A)$ is a maximal ideal. Consequently, the set of complex homomorphisms $\mathfrak{M}(A)$ of A may be identified with the set of maximal ideals of A and, as such, is commonly referred to as the *maximal ideal space*.

If A is equal to H^∞, H^∞_D or $M(D)$ and $\lambda \in U$ define the *the point evaluation homomorphism* at λ by the formula

$$\phi_\lambda(f) = f(\lambda) \tag{4.1}$$

for each $f \in A$. Evidently, $\phi_\lambda \in \mathfrak{M}(A)$.

If $f \in A$ then the *Gelfand transform* of f is the function

$$\begin{aligned} \hat{f} : \quad \mathfrak{M}(A) &\longrightarrow \mathbb{C} \\ \phi &\longrightarrow \hat{f}(\phi) = \phi(f). \end{aligned}$$

Let $\widehat{A}$ be the set of all $\hat{f}$, for $f \in A$. We will sometimes refer to $\widehat{A}$ as the *Gelfand space* corresponding to A, and we shall write $\hat{f}_A$ whenever it is advantageous to specify the Banach algebra to which f belongs. The *weak topology* of $\mathfrak{M}(A)$ is the weakest topology such that all the functions $\hat{f}$ are continuous. With this topology, $\mathfrak{M}(A)$ is a compact Hausdorff space contained in the unit ball of the dual space A^*. Clearly $\widehat{A} \subset C(\mathfrak{M}(A))$ and when $\widehat{A}$ is equipped with the supremum norm

$$\|\hat{f}_A\| = \sup_{\phi \in \mathfrak{M}(A)} |\hat{f}(\phi)| \tag{4.2}$$

is will in fact be a subalgebra of $C(\mathfrak{M}(A))$. The spectral radius theorem asserts that

$$\|\hat{f}_A\| = \mathrm{r}(f, A) \tag{4.3}$$

for any $f \in A$.

A commutative Banach algebra A is called *semisimple* if and only if A is isomorphic to the space of Gelfand transforms $\widehat{A}$. The Banach algebras H^∞, H^∞_D and $M(D)$ are all semisimple. To show this, let f_1 and f_2 be functions in A where A is equal to H^∞, H^∞_D or $M(D)$, and suppose $f_1 \neq f_2$. Then $f_1(\lambda) \neq f_2(\lambda)$ for some point $\lambda \in U$ and so

$$\hat{f}_1(\phi_\lambda) = \phi_\lambda(f_1) = f_1(\lambda) \neq f_2(\lambda) = \phi_\lambda(f_2) = \hat{f}_2(\phi_\lambda).$$

Thus $\hat{f}_1 \neq \hat{f}_2$ and so the map $f \to \hat{f}$ is injective.

It can be shown that H^∞ is isometric to $\widehat{H}^\infty$. This does not hold for H^∞_D or $M(D)$.

Proposition 4.3. *If $f \in M(D)$, then*

1. *$Sp(f, M(D)) = Sp(f, H^\infty_D) = Sp(f, H^\infty)$*

2. $r(f, M(D)) = r(f, H_D^\infty) = r(f, H^\infty)$

Proof. The proof of (1) follows easily from Proposition 4.2. For example, if $\lambda \in \mathrm{Sp}(f, M(D))$, then $f - \lambda$ is not invertible in $M(D)$. But then it cannot be invertible in H_D^∞ either, by Proposition 4.2. Hence $\lambda \in \mathrm{Sp}(f, H_D^\infty)$. The other inclusions are proved in the same manner.

Clearly, (2) follows immediately from (1). □

Proposition 4.4. *If $f \in M(D)$, then $\|\hat{f}_M\| = \|\hat{f}_{H_D^\infty}\| = \|\hat{f}_\infty\|$*

Proof. By the spectral radius theorem, we have $\|\hat{f}_A\| = r(f, A)$ for any commutative Banach algebra A. The result now follows from Proposition 4.3. □

Proposition 4.5. *Let A be $M(D)$, H_D^∞ or H^∞. If $\{f_n\}$ is a sequence of functions in A that converges uniformly to some function $f \in A$ on U, then $\{(\hat{f}_n)_A\}$ converges uniformly to $\hat{f}$ on $\mathfrak{M}(A)$.*

Proof. As we mentioned above, H^∞ is isometrically isomorphic to $\hat{H}^\infty$. From this and Proposition 4.4 we see that

$$\|(\hat{f}_n - \hat{f})_A\| = \|(\hat{f}_n - \hat{f})_\infty\| = \|f_n - f\|_\infty \to 0 \quad \text{as} \quad n \to \infty$$

and so $\{(\hat{f}_n)_A\}$ converges uniformly to $\hat{f}$ on U. □

We will now show that the maximal ideal space of $M(D)$ can be considered as as the unit disk with certain fibers on the boundary, to be defined soon. The interior of the unit disk will consist of the point evaluation homomorphisms. For the proof, we will need the following lemma.

Lemma 4.6. *Let A be $M(D)$, H_D^∞ or H^∞ and let $\lambda \in U$ and $f \in A$. If f has a zero at λ and $f(z) = (z - \lambda)g(z)$, then $g \in A$.*

Proof. First, note that $g \in H^\infty$ and so the statement holds for $A = H^\infty$.

Suppose $f \in H_D^\infty$. We must show that g has finite Dirichlet integral. Let N be a closed disk centred at λ and contained in U. Then

$$\mathcal{D}(g) = \frac{1}{\pi}\int_U |g'|^2 dA = \frac{1}{\pi}\int_N |g'|^2 dA + \frac{1}{\pi}\int_{U-N} |g'|^2 dA.$$

The first integral is finite because N is compact and $|g'|^2$ is continuous. On $U - N$ there exists a constant $c > 0$ such that $|z - \lambda| > c > 0$, and we can write

$$g(z) = \frac{f(z)}{(z - \lambda)}. \tag{4.4}$$

Thus we have

$$g'(z) = \frac{f'(z)}{z - \lambda} - \frac{f(z)}{(z - \lambda)^2} \tag{4.5}$$

and so

$$\int_{U-N} |g'|^2 dA \le c^{-2}\int_{U-N} |f'|^2 dA + c^{-4}\int_{U-N} |f|^2 dA. \tag{4.6}$$

Since $f \in H^\infty_D$ either integral in (4.6) is finite. Thus the proposition holds for $A = H^\infty_D$.

Finally, we must show that $gD \subset D$. We proceed as above. Let N be a closed disk centred at λ and contained in U. For any $h \in D$ we have

$$\begin{aligned}\mathcal{D}(gh) =& \frac{1}{\pi}\int_U |(gh)'|^2 dA = \frac{1}{\pi}\int_U |g'h + gh'|^2 dA \\ \leq& \frac{1}{\pi}\int_N |g'h + gh'|^2 dA + \frac{1}{\pi}\int_{U-N} |g'h + gh'|^2 dA \end{aligned}\tag{4.7}$$

The first integral is finite because $|g'h + gh'|^2$ is continuous and N is compact. On $U - N$ the expressions (4.4) and (4.5) hold and so

$$\begin{aligned}\int_{U-N} |g'h + gh'|^2 dA \leq& \int_{U-N} |g'h|^2 dA + \int_{U-N} |gh'|^2 dA \\ \leq& c^{-2}\int_{U-N} |f'h|^2 dA + c^{-4}\int_{U-N} |fh|^2 dA + \pi \sup_{z\in U} |g(z)|^2 \mathcal{D}(h).\end{aligned}$$

The first integral is finite because f' multiply D into B by Lemma 4.1. The second integral is finite because f is bounded and $h \in D \subset B$. We conclude that $\mathcal{D}(gh)$ is finite for any $h \in D$. Hence $g \in M(D)$. This completes the proof. □

Proposition 4.7. *Let A be $M(D)$, H^∞_D or H^∞. Then the map $\hat{z}_A$ is a continuous mapping of $\mathfrak{M}(A)$ onto the closed unit disk $\bar{U}$. Over the open unit disk U, $\hat{z}_A$ is injective, and $(\hat{z}_A)^{-1}$ maps U homeomorphically onto an open subset $\triangle(A)$ of $\mathfrak{M}(A)$.*

Proof. The cases $A = H^\infty$ and $A = H^\infty_D$ were proved in [15] and [7] respectively. The proof when $A = M(D)$ follows similar lines. Write $\pi = \hat{z}_M = \hat{z}$ and let $\phi \in \mathfrak{M}$. By Proposition 4.4 we have

$$|\pi(\phi)| = |\hat{z}(\phi)| \leq \sup_{\phi\in\mathfrak{M}(M)} |\hat{z}(\phi)| = \|\hat{z}_M\| = \|\hat{z}_\infty\| = \|z\|_\infty = 1$$

Hence π maps $\mathfrak{M}$ into the closed unit disk. Each point λ in the open unit disk is in the range of π, since $\pi(\phi_\lambda) = \phi_\lambda(z) = \lambda$. Moreover, π is continuous by the definition of the weak topology on $\mathfrak{M}(M)$. Since $\mathfrak{M}(M)$ is compact, so is the range of π. Therefore this range must be the entire closed unit disk.

To show that π is one-to-one over U, suppose $\pi(\phi) = \lambda \in U$. Suppose $f \in M(D)$ and $f(\lambda) = 0$. If $f(z) = (z - \lambda)g(z)$ then $g \in M(D)$ by Lemma 4.6. Hence

$$\phi(f) = \phi(z - \lambda)\phi(g) = 0 \cdot \phi(g) = 0$$

Thus $\phi(f) = 0$ for every f which vanish at λ. Let g be any function in $M(D)$. Then the function $g - g(\lambda)$ vanish at λ and so $0 = \phi(g - g(\lambda)) = \phi(g) - g(\lambda)$. Thus $\phi(g) = g(\lambda)$ and since g was an arbitrary function in $M(D)$, we conclude that ϕ is evaluation at λ.

Let $\triangle = \triangle(M) = \pi^{-1}(U)$. It is clearly an open subset of $\mathfrak{M}$. The restriction map $\pi|_\triangle$ is a continuous bijection of $\triangle$ onto U. To show that it is actually a homeomorphism, note first that the set $\triangle$ consists precisely of the evaluation homomorphisms. Fix a point $w \in U$ and the corresponding $\phi_w \in \triangle$. A basis neighbourhood of ϕ_w in the weak topology of $\triangle$ is of the form

$$\begin{aligned}V =& \{\phi_\lambda \in \triangle : \quad |\phi_\lambda(f_i) - \phi_w(f_i)| < \epsilon, \quad 1 \leq i \leq N, \quad f_i \in M(D)\} \\ =& \{\phi_\lambda \in \triangle : \quad |f_i(\lambda) - f_i(w)| < \epsilon, \quad 1 \leq i \leq N, \quad f_i \in M(D)\}\end{aligned}$$

Thus we have

$$\pi(V) = \{\lambda \in U : \quad |f_i(\lambda) - f_i(w)| < \epsilon, \quad 1 \le i \le N, \quad f_i \in M(D)\}$$

This is an open set in the weak topology of U defined by the functions $f \in M(D)$. We conclude that the map $\pi|_\triangle$ from $\triangle$ onto U is open. Hence it is a homeomorphism. The proof is complete. □

We can visualize π as the projection of $\mathfrak{M}(A)$ onto the closed unit disk. The interior of the disk, U, is homeomorphically embedded in $\mathfrak{M}(A)$ by $\lambda \to \phi_\lambda$. The image of the open unit disk by this mapping is the set $\triangle(A)$ introduced above. The rest of $\mathfrak{M}(A)$ is mapped by π onto the unit circle.

If $|\alpha| = 1$ we shall call $\pi^{-1}(\alpha)$ the *fibre* of $\mathfrak{M}(A)$ over α.

$$\mathfrak{M}(\alpha, A) = \pi^{-1}(\alpha) = \{\phi \in \mathfrak{M}(A) : \pi(\phi) = \phi(z) = \alpha)\}$$

It is a closed, hence compact, subset of $\mathfrak{M}(A)$.

The following result was first proved by Lennart Carleson for H^∞. For an extensive discussion of the proof, see [13]. For a proof of the Corona theorem for multipliers, see [32].

Theorem 4.8 (The Corona Theorem). *Let A be $M(D)$, H_D^∞ or H^∞. The open unit disk $\triangle(A)$ is dense in $\mathfrak{M}(A)$.*

For the remainder of this section, we will mostly consider $M(D)$, though the results can be proved for H^∞ and H_D^∞ as well. For brevity, we shall write $\mathfrak{M} = \mathfrak{M}(M)$ and $\mathfrak{M}_\alpha = \mathfrak{M}(\alpha, M)$. Proposition 4.9 and Proposition 4.10 will be obtained independently of the Corona theorem. Proposition 4.11 can be proved independently of the Corona theorem for H_D^∞ but the argument rely on a function that may not be a multiplier.

Proposition 4.9. *Let $f \in M(D)$ and let α be a point of the unit circle. Let $\{\lambda_n\}$ be a sequence of points in U which converges to α and suppose that the limit $\zeta = \lim f(\lambda_n)$ exists. Then there is a $\phi \in \mathfrak{M}_\alpha$ such that $\phi(f) = \zeta$.*

Proof. Let J be the collection of all functions $g \in M(D)$ such that $\lim g(\lambda_n)$ exists. This is an ideal in $M(D)$ and is contained in a maximal ideal, that is, there is a complex homomorphism $\phi \in \mathfrak{M}$ such that $\phi(g) = 0$ for all $g \in J$. But the functions $z - \alpha$ and $f(z) - \zeta$ are both in J. Thus $\phi(z) = \alpha$ and $\phi(f) = \zeta$. □

Proposition 4.10. *Suppose $f \in M(D)$. The function $\hat{f}$ is constant on the fibre $\mathfrak{M}_\alpha$ if and only if f is continuously extendible to $U \cup \{\alpha\}$. If this holds, the constant is $\hat{f}(\phi) = f(\alpha)$ where $\phi \in \mathfrak{M}_\alpha$.*

Proof. Let $\phi \in \mathfrak{M}_\alpha$ and suppose f is continuously extendible to $U \cup \{\alpha\}$ with $f(\alpha) = \zeta$. Since constant functions are multipliers of D, f is a multiplier of D if and only if $f - \zeta$ is a multiplier of D. Thus we can assume that $\zeta = 0$. Define $h \in M(D)$ by $h(z) = \frac{1}{2}(1 + \bar{\alpha}z)$. Clearly $h(\alpha) = 1$ and $|h(z)| < 1$ for all $z \in U$. Since f is continuously extendible to $U \cup \{\alpha\}$ with value 0 at α, $\{h^n f\} \to 0$ uniformly on U as $n \to \infty$. By Proposition 4.5 $\{\widehat{h^n f}\} \to 0$ uniformly on $\mathfrak{M}$. Thus $\lim_{n\to\infty} \phi(h^n f) = 0$. On the other hand, $\phi(h) = 1$ and so $\phi(h^n f) = \phi(f)$ for all n. Hence $\phi(f) = \hat{f}(\phi) = 0$. Since ϕ was an arbitrary homomorphism of $\mathfrak{M}_\alpha$, $\hat{f}$ is constant on $\mathfrak{M}_\alpha$ with value 0.

Conversely, suppose f is not continuously extendible to $U \cup \{\alpha\}$. Then there are sequences $\{\lambda_n\}$ and $\{w_n\}$ in U such that $\lim f(\lambda_n) = \zeta_1$ and $\lim f(w_n) = \zeta_2$. By Proposition 4.9, there are complex homomorphisms ϕ_1, $\phi_2 \in \mathfrak{M}_\alpha$ such that $\phi_1(f) = \zeta_1$ and $\phi_2(f) = \zeta_2$. Hence $\hat{f}$ is not constant on $\mathfrak{M}_\alpha$. □

Proposition 4.11. *Let $f \in M(D)$ and let α be a point of the unit circle. Choose any $\zeta \in U$ and suppose there is a $\phi \in \mathfrak{M}_\alpha$ such that $\phi(f) = \zeta$. Then there exists a sequence $\{\lambda_n\}$ of points in U converging to α such that $\lim_{n\to\infty} f(\lambda_n) = \zeta$.*

Proof. By the Corona theorem, there exists a net of complex homomorphisms $\{\phi_{w_\nu}\}_{\nu \in J}$ in $\triangle$ converging to ϕ, where J is a directed set with partial order $\preceq$, and $w_\nu \in U$ for each $\nu \in J$. For each basis neighbourhood

$$V = \{\varphi \in \mathfrak{M} : \quad |\varphi(f_i) - \phi(f_i)| < \epsilon, \quad 1 \le i \le N, \quad f_i \in M(D)\}$$

of ϕ there exists an $\eta \in J$ such that $\eta \preceq \nu \Rightarrow \phi_{z_\nu} \in V$. In particular, this is true for the neighbourhoods U_ϵ obtained by choosing the functions z and f in the definition of V above:

$$U_\epsilon = \{\varphi \in \mathfrak{M} : \quad |\varphi(f) - \zeta| < \epsilon, \quad |\varphi(z) - \alpha| < \epsilon\}$$

That is, for each $\epsilon > 0$ there is an $\eta(\epsilon) \in J$ such that $|\phi_{w_\nu}(z) - \alpha| = |w_\nu - \alpha| < \epsilon$ and $|\phi_{w_\nu}(f) - \zeta| = |f(w_\nu) - \alpha| < \epsilon$ for $\eta(\epsilon) \preceq \nu$. Let $\epsilon_n = \frac{1}{n}$. Choose one element $\nu_n \succeq \eta(\epsilon_n)$ for each n. We obtain a sequence $\{w_{\nu_n}\} = \{w_n\}$ such that $\lim_{n\to\infty} w_n = \alpha$ and $\lim_{n\to\infty} f(w_n) = \zeta$. This completes the proof. □

Corollary 4.12. *Let $f \in M(D)$ and let α be a point of the unit circle. The range of $\hat{f}$ on the fibre $\mathfrak{M}_\alpha$ consists of all complex numbers ζ for which there is a sequence of points $\{\lambda_n\}$ in U with $\lim_{n\to\infty} \lambda_n = \alpha$ and $\lim_{n\to\infty} f(\lambda_n) = \zeta$.*

Proof. The range of $\hat{f}$ on the fibre $\mathfrak{M}_\alpha$ is $\hat{f}(\mathfrak{M}_\alpha) = \{\phi(f) : \phi \in \mathfrak{M}_\alpha\}$. Choose $\phi \in \mathfrak{M}_\alpha$. Then $\phi(f)$ is in the range of $\hat{f}$ on $\mathfrak{M}_\alpha$ and by Proposition 4.11 we can find a sequence $\{\lambda_n\}$ converging to α such that $\lim f(\lambda_n) = \phi(f)$. Conversely, if there exists a sequence $\{\lambda_n\}$ converging to α such that $\zeta = \lim f(\lambda_n)$ exists, then there is a $\phi \in \mathfrak{M}_\alpha$ such that $\phi(f) = \zeta$ by Proposition 4.9. Thus $\phi(f)$ is in the range of $\hat{f}$ on $\mathfrak{M}_\alpha$. This completes the proof. □

Corollary 4.13. *If $f \in M(D)$ then*

$$\hat{f}_M(\mathfrak{M}(\alpha, M)) = \hat{f}_{H_D^\infty}(\mathfrak{M}(\alpha, H_D^\infty)) = \hat{f}_\infty(\mathfrak{M}(\alpha, H^\infty)).$$

Proof. The characterization of the range of f given in Corollary 4.12 holds for any $f \in H_D^\infty$ and $f \in H^\infty$ as well. Since $M(D) \subset H_D^\infty \subset H^\infty$ the result now follows. □

Since $M(D) \subset H^\infty$, we can define a restriction map from the maximal ideal space of H^∞ to that of $M(D)$ as follows:

$$\begin{aligned} \omega : \quad \mathfrak{M}(H^\infty) &\longrightarrow \mathfrak{M}(M) \\ \phi \quad &\longrightarrow \quad \phi|_M \end{aligned}$$

This restriction map will yield topological properties of the maximal ideal space of $M(D)$ from the corresponding properties of $\mathfrak{M}(H^\infty)$ with minimal effort. Similar restriction maps can be defined from H^∞ to H_D^∞ and from H_D^∞ to $M(D)$.

Proposition 4.14. *The restriction map ω is a continuous surjection from $\mathfrak{M}(H^\infty)$ onto $\mathfrak{M}(M)$. Moreover, ω is bijective over $\triangle(M)$ and maps fibers onto fibers.*

Proof. The restriction map ω is continuous if and only if for every convergent net of complex homomorphisms $(\phi_\nu)_{\nu\in J}$ in (H^∞), converging to ϕ, the net $(\omega(\phi_\nu))_{\nu\in J}$ converges to $\omega(\phi)$. The statement that $(\phi_\nu)_{\nu\in J}$ converges to ϕ in the weak topology on $\mathfrak{M}(H^\infty)$ is equivalent to the following: $(\phi_\nu(f))$ converges to $\phi(f)$ for every $f \in H^\infty$. Since $M(D) \subset H^\infty$, the same statement is true for all $f \in M(D)$. Thus $(\phi_\nu|_M)$ converges to $(\phi|_M)$ in the weak topology on $\mathfrak{M}(M)$, and so $(\omega(\phi_\nu))$ converges to $\omega(\phi)$. Hence ω is continuous.

If $\phi \in \mathfrak{M}(\alpha, H^\infty)$ then $\phi(z) = \phi|_M(z) = \omega(\phi)(z) = \alpha$. Hence $\omega(\phi) \in \mathfrak{M}(\alpha, M)$, and so fibers are sent into fibers.

If $\phi_\lambda \in \triangle(H^\infty)$, then $\omega(\phi_\lambda) = \phi_\lambda|_M$ is the unique evaluation homomorphism at $\lambda \in U$. If $\varphi_\lambda \in \triangle(M)$, then $\omega^{-1}(\varphi_\lambda)$ is the unique evaluation homomorphism in $\mathfrak{M}(H^\infty)$ at λ since $\omega^{-1}(\varphi_\lambda)(z) = \varphi_\lambda(z) = \lambda$. Hence ω is bijective over $\triangle(M)$.

Finally, we must show that ω is surjective on the fibers of $\mathfrak{M}(M)$. Let $\varphi \in \mathfrak{M}(\alpha, M)$. By the Corona theorem, we can find a net $(\varphi_{w_\nu})_{\nu\in J}$ in $\triangle(M)$ converging to φ. The corresponding net $(\omega^{-1}(\varphi_{w_\nu}))_\nu$ in $\triangle(H^\infty)$, which we denote by (ϕ_{w_ν}), must contain a subnet (ϕ_{w_μ}) converging to some $\phi \in \mathfrak{M}(H^\infty)$ since $\mathfrak{M}(H^\infty)$ is compact. We must show that $\phi|_M = \varphi$. To this end, observe that $(\omega(\phi_{w_\mu})) = (\varphi_{w_\mu})$ is a subnet of (φ_{w_ν}). Since (φ_{w_ν}) converges to φ, so must (φ_{w_μ}). Thus, for any $f \in M(D)$ we have

$$\varphi(f) = \lim \varphi_{w_\mu}(f) = \lim \phi_{w_\mu}|_M(f) = \lim \phi_{w_\mu}(f) = \phi(f)$$

But this means that $\phi|M = \omega(\phi) = \varphi$. Hence ω is surjective. The proof is complete. □

Proposition 4.15. *The fibers $\mathfrak{M}(M)$ are connected.*

Proof. We have $\omega(\mathfrak{M}(\alpha, H^\infty)) = \mathfrak{M}(\alpha, M)$ by Proposition 4.14. The fibers $\mathfrak{M}(\alpha, H^\infty)$ are connected; see [15], page 188. Since ω is continuous and continuous images of connected sets are connected, the result now follows. □

Proposition 4.16. *In the maximal ideal space of $M(D)$, the complement of the open unit disk is connected.*

Proof. We have $\mathfrak{M}(M) - \triangle(M) = \omega(\mathfrak{M}(H^\infty) - \triangle(H^\infty))$ by Proposition 4.14. In the maximal ideal space of H^∞, the complement of the open unit disk is connected; see [15], page 188. The result now follows from the from the continuity of ω. □

Let X be a compact Hausdorff space and let A be an algebra of complex-valued continuous functions on X which contains the constant functions and separates the points of X. A boundary for A is a subset $S \subset X$ such that $\sup_{x\in X} |f(x)| = \max_{x\in S} |f(x)|$ for all $f \in A$. In other words, a boundary is a subset of X where all the functions in A attain their maximum modulus. The *Shilov boundary* for A is the intersection of all closed boundaries for A. This is a closed boundary for A and will be denoted by $\mathscr{S}(A)$. For a proof, see Theorem 4.2 of [12].

The Shilov boundary for a semisimple Banach algebra of analytic functions in the open unit disk is defined to be the Shilov boundary of the corresponding algebra of Gelfand transforms, which is an algebra of continuous functions on the compact maximal ideal space. In this sense, the Shilov boundary of H^∞ can be identified with $X = \mathfrak{M}(L^\infty(T))$. The identification is as follows: Since $H^\infty \subset L^\infty$ any $\phi \in X$ can be restricted to H^∞ to yield a complex homomorphism on H^∞. The set of all such restrictions turn out to be the Shilov boundary for $\widehat{H}^\infty$ (and so for H^∞). For a proof of this statement, see Chapter 10 in [15]. Using the map ω we now obtain some information on the Shilov boundary for $M(D)$.

Proposition 4.17. $\mathscr{S}(M(D)) \subset \omega(\mathscr{S}(H^\infty)) = \omega(X)$

Proof. Let $f \in M(D)$ and $\phi \in \mathfrak{M}(H^\infty)$. Then

$$\hat{f}_\infty(\phi) = \phi(f) = \phi|_M(f) = \hat{f}_M(\phi|_M) = (\hat{f}_M \circ \omega)(\phi)$$

and so $\hat{f}_\infty = \hat{f}_M \circ \omega$ for any $f \in M(D)$. Thus

$$\begin{aligned}\sup\{|\hat{f}_M(\phi)| : \phi \in \mathfrak{M}(M)\} = \|\hat{f}_M\| &= \|\hat{f}_\infty\| \\ &= \sup\{|\hat{f}_\infty(\phi)| : \phi \in \mathfrak{M}(H^\infty)\} \\ &= \sup\{|\hat{f}_\infty(\phi)| : \phi \in \mathscr{S}(H^\infty)\} \\ &= \sup\{|\hat{f}_M \circ \omega(\phi)| : \phi \in \mathscr{S}(H^\infty)\} \\ &= \sup\{|\hat{f}_M(\varphi)| : \varphi \in \omega(\mathscr{S}(H^\infty))\}\end{aligned}$$

and so $\omega(\mathscr{S}(H^\infty))$ is a boundary for $M(D)$. The Silov boundary $\mathscr{S}(H^\infty)$ is compact since it is a closed subset of the compact maximal ideal space $\mathfrak{M}(H^\infty)$. Thus $\omega(\mathscr{S}(H^\infty))$ is compact since ω is continuous. Hence $\omega(\mathscr{S}(H^\infty))$ is closed since $\mathfrak{M}(M)$ is Hausdorff. We conclude that $\omega(\mathscr{S}(H^\infty))$ is a closed boundary for $\widehat{M}(D)$ and the result now follows. □

In light of the identification of the Shilov boundary for H^∞, the result above seems somewhat incomplete. Can we identify the Shilov boundary for $M(D)$ with the maximal ideal space of some Banach algebra of functions on the unit circle? This question remains open, see Section 4.3.

We end this section with a remark on representing measures and algebras on the fibers. A *uniform algebra* A on a compact Hausdorff space X is a uniformly closed subalgebra of $C(X)$ which contains the constants and separates points on X. If we endow A with the supremum norm

$$\|f_A\| = \sup_{x\in X} |f(x)|$$

it becomes a Banach algebra. Consequently, if ϕ is a complex homomorphism of A we know that ϕ is continuous:

$$|\phi(f)| \leq \sup_{x\in X} |f(x)|.$$

The map $f \to f|_{\mathscr{S}(A)}$, which restricts each function f to the Shilov boundary $\mathscr{S}(A)$, is an isometric isomorphism of A onto a subalgebra of $C(\mathscr{S}(A))$, and ϕ is a bounded linear functional with norm $\|\phi\| = 1$ on this subalgebra:

$$|\phi(f)| \leq \sup_{x\in\mathscr{S}(A)} |f(x)|.$$

By the Hahn-Banach theorem such a functional can be extended to a bounded linear functional with norm 1 on $C(\mathscr{S}(A))$ and by the subsequent use of the Riesz representation theorem this functional is determined by a complex measure m_ϕ on $\mathscr{S}(A)$. Thus

$$\phi(f) = \int_{\mathscr{S}(A)} f dm_\phi$$

for each $f \in A$. It turns out that m_ϕ must be a positive measure and that $m_\phi(\mathscr{S}(A)) = 1$. Such a measure is called a *representing measure* for ϕ and we see that at least one must exist for each $\phi \in \mathfrak{M}(A)$. The observations above are based on [15] page 180-181, which contains more details.

If we try to apply these concepts to the algebras $\widehat{M}(D)$ and $\widehat{H}^\infty_D$ a problem occurs: neither algebra is uniformly closed and so cannot be Banach algebras when endowed with the supremum norm. We proved this at the start of the section when we observed that the functions $f_N = \sum_{n=0}^N \frac{z^{n!}}{n^2} \in M(D)$ but the uniform limit has infinite Dirichlet integral. Thus $\{\hat{f}_N\}$ is a Cauchy sequence in $\widehat{M}(D)$ and $\widehat{H}^\infty_D$ but the uniform limit $\hat{f}$ is not contained in either algebra.

The solution to this problem is given for H^∞_D in [8], and it works for $M(D)$ as well. We only sketch the solution and refer the reader to the mentioned article for details. Let $X = \mathfrak{M}(M)$. Consider the uniform closure $\mathscr{M}$ of $\widehat{M}(D)$ in $C(X)$. Clearly, $\mathscr{M}$ is a Banach algebra with the supremum norm. It can be shown that the maximal ideal spaces of $M(D)$ and $\mathscr{M}$ are homeomorphic. Moreover $\mathscr{S}(M(D)) \simeq \mathscr{S}(\mathscr{M})$ and $\mathfrak{M}(\alpha, M(D)) \simeq \mathfrak{M}(\alpha, \mathscr{M})$ for each $|\alpha| = 1$ where $\simeq$ denotes a homeomorphism. Let $\mathscr{M}_\alpha$ be the algebra obtained by restricting each function to the fiber $\mathfrak{M}(\alpha, \mathscr{M})$. Since $\mathscr{M}$ is uniformly closed we are now in the same situation as for H^∞ in [15] and the same results and proofs hold.

Proposition 4.18. *1. $\mathscr{M}_\alpha$ is a uniformly closed subalgebra of $C(\mathfrak{M}(\alpha, \mathscr{M}))$.*

2. The maximal ideal space of $\mathscr{M}_\alpha$ is $\mathfrak{M}(\alpha, \mathscr{M})$.

3. The Shilov boundary for $\mathscr{M}_\alpha$ is contained in $X_\alpha = \mathscr{S}(\mathscr{M}) \cap \mathfrak{M}(\alpha, \mathscr{M})$.

Corollary 4.19. *For each $\phi \in \mathfrak{M}(M)$ the representing measures for ϕ are supported on $X_\alpha = \mathscr{S}(\mathscr{M}) \cap \mathfrak{M}(\alpha, \mathscr{M}) \simeq \mathscr{S}(M(D)) \cap \mathfrak{M}(\alpha, M(D))$.*

4.2 The maximal ideal space of $M(D_G)$

Let G be a domain and let $z_0 \in G$ be the distinguished point of the Dirichlet space $D(G)$. Recall that from Chapter 3 that the multipliers of $D(G)$, denoted by $M(D_G)$, is a Banach algebra. We also showed that if $\omega : U \longrightarrow G$ is a Riemann map and $\omega(0) = z_0$, then the composition operator $C_\omega(f) = f \circ \omega$ is an isometry of $D(G)$ onto $D(U)$. This raises several questions. Are $M(D)$ and $M(D_G)$ isometric? Will their maximal ideal spaces be isomorphic, or even homeomorphic? Can we say anything about the structure of the maximal ideal space of $M(D_G)$? The answer to these questions are yes, at least when certain extra conditions are imposed on G.

Proposition 4.20. *The composition operator C_ω is an isometric isomorphism of $M(D_G)$ onto $M(D)$ with respect to the operator norms.*

Proof. If $f \in M(D_G)$ then the number $\|fh\|_{D(G)} = \|C_\omega(fh)\|_{D(U)} = \|(f \circ \omega)(h \circ \omega)\|_{D(U)}$ is finite for each $h \in D(G)$. But every function in $D(U)$ is of the form $h \circ \omega$ for some $h \in D(G)$ and so $f \circ \omega \in M(D) \subset D(U)$. Hence C_ω maps $M(D_G)$ into $M(D)$. Similarly, if $f \in M(D)$ then $C_{\omega^{-1}}(f) = f \circ \omega^{-1} \in M(D_G)$ and so C_ω is an isomorphism of $M(D_G)$ onto $M(D)$.

Now, let $f \in M(D_G)$. Since C_ω is an isometric isomorphism of $D(G)$ onto $D(U)$, we see that

$$\begin{aligned}\|M_f\| &= \sup\{\|fh\|_{D(G)} : \|h\|_{D(G)} \le 1\} \\ &= \sup\{\|C_\omega(fh)\|_{D(U)} : \|C_\omega(h)\|_{D(U)} \le 1\} \\ &= \sup\{\|(f \circ \omega)g\|_{D(U)} : \|g\|_{D(U)} \le 1\} \\ &= \|M_{f\circ\omega}\|.\end{aligned}$$

Thus C_ω is isometric with respect to the operator norms on $M(D_G)$ and $M(D)$. The proof is complete. □

From the proposition above it follows immediately that the maximal ideal spaces of $M(D_G)$ and $M(D)$ are isomorphic. The isomorphism, which we denote by H_ω, sends a maximal ideal $m \subset M(D)$ to the maximal ideal $C_\omega^{-1}(m) = \{g \circ \omega^{-1} : g \in m\} \subset M(D_G)$. Let ϕ be the unique complex homomorphism of $M(D)$ with kernel m. The unique complex homomorphism $\tilde{\phi}$ of $M(D_G)$ with the kernel $C_\omega^{-1}(m)$ is

$$\tilde{\phi}(f) = \phi(f \circ \omega)$$

for $f \in M(G)$. For, if $f \in C_\omega^{-1}(m)$ then $f \circ \omega \in m$ and so $\tilde{\phi}(f) = \phi(f \circ \omega) = 0$. Thus $H_\omega(\phi) = \tilde{\phi} \in \mathfrak{M}(M_G)$ for each $\phi \in \mathfrak{M}(M_U)$.

Lemma 4.21. *The map $H_\omega : \mathfrak{M}(M_U) \longrightarrow \mathfrak{M}(M_G)$ is a homeomorphism.*

Proof. Let $(\phi_\nu)_{\nu\in J}$ be a net in $\mathfrak{M}(M_U)$ converging to $\phi \in \mathfrak{M}(M_U)$. This means that $\lim \phi_\nu(f) = \phi(f)$ for any $f \in M(D)$. We must show that $(H_\omega(\phi_\nu))_{\nu\in J}$ converges to $H_\omega(\phi)$. But $H_\omega(\phi_\nu)(g) = \tilde{\phi}_\nu(g) = \phi_\nu(g \circ \omega)$ and so

$$\lim H_\omega(\phi_\nu)(g) = \lim \phi_\nu(g \circ \omega) = \phi(g \circ \omega) = H_\omega(\phi).$$

By a similar argument the inverse map H_ω^{-1} is also continuous. This completes the proof. □

We will now study the structure of the maximal ideal space of $M(D_G)$, as we did for $M(D)$. Let $\zeta \in G$ and define the point evaluation homomorphism of $M(D_G)$ at ζ by $\varphi_\zeta(f) = f(\zeta)$ for each $f \in M(D_G)$. Let $\triangle(M_G)$ denote the set of point evaluation homomorphisms of $M(D_G)$. Clearly, $\triangle(M_U)$ and $\triangle(M_G)$ are homeomorphic. For, if $\lambda \in U$, then $\omega(\lambda) \in G$ and

$$H_\omega(\phi_\lambda)(f) = \tilde{\phi}_\lambda(f) = \phi_\lambda(f \circ \omega) = (f \circ \omega)(\lambda) = f(\omega(\lambda)) = \varphi_{\omega(\lambda)}(f)$$

for each $f \in M(D_G)$. Conversely, if $\zeta \in G$ then $H_\omega^{-1}(\varphi_\zeta)(f) = \phi_{\omega^{-1}(\zeta)}(f)$ for each $f \in M(D)$.

Now we want to define the fibers of $\mathfrak{M}(M_G)$. But in order for this to make sense, z must be in $M(D_G)$, that is, z must be a multiplier of $D(G)$. Otherwise we cannot consider the map $\hat{z}$ of $\mathfrak{M}(M_G)$ onto some subset of the complex plane. But if G is bounded and simply connected the statement $zD(G) \subset D(G)$ is precisely statement (2) of Theorem 3.5. We will assume that this property holds. In other words, we are assuming that ω is a multiplier of the Dirichlet space or that G is a bounded Poincaré domain.

We are going to need another condition on G as well: it must be a Jordan region (the interior of a closed Jordan curve). For the following is known (a statement without proof can be found in [3], page 232):

Theorem 4.22. *Suppose f is an analytic homeomorphism of Ω onto Ω'. If Ω and Ω' are Jordan regions, then f can be extended to a homeomorphism of the closure of Ω onto the closure of Ω'.*

If $\lambda \in G$ then $\hat{z}(\phi_\lambda) = \phi_\lambda(z) = \lambda$. Thus $\hat{z}$ is a continuous bijection of $\triangle(M_G)$ onto G, given by $\lambda \longrightarrow \phi_\lambda$. In fact, it is a homeomorphism. The proof of this statement is similar to the analogous result for $M(D)$ in Proposition 4.7.

Now, choose $\phi \in \mathfrak{M}(\alpha, M_U)$, that is, a complex homomorphism from the fiber of $\mathfrak{M}(M_U)$ over α. Then $H_\omega(\phi) = \tilde{\phi} \in \mathfrak{M}(M_G)$ and

$$\hat{z}(\tilde{\phi}) = \tilde{\phi}(z) = \phi(z \circ \omega) = \phi(\omega) = \hat{\omega}(\phi). \tag{4.8}$$

By Proposition 4.10, $\hat{\omega}$ is constant on the fiber $\mathfrak{M}(\alpha, M_U)$ if and only if ω is continuously extendible to α. In this case, the value of $\hat{\omega}$ assumed on the fiber is equal to $\omega(\alpha)$. Assuming that G is a Jordan region, we conclude from Theorem 4.22 that for each $\alpha \in \partial U$ the value of $\hat{\omega}$ is constant and equal to $\omega(\alpha) \in \partial G$ on the fiber $\mathfrak{M}(\alpha, M_U)$. Thus $\hat{z}(\tilde{\phi}) = \omega(\alpha)$ if $\phi \in \mathfrak{M}(\alpha, M_U)$ by 4.8. If we define the fiber of $\mathfrak{M}(M_G)$ over $\zeta \in \partial G$ by

$$\mathfrak{M}(\zeta, M_G) = \{\varphi \in \mathfrak{M}(M_G) : \quad \hat{z}(\varphi) = \varphi(z) = \zeta\}$$

we conclude that the map H_ω sends fibers into fibers.

Proposition 4.23. *Let G be a bounded simply connected Jordan region, and suppose $zD(G) \subset D(G)$. If $\omega : U \longrightarrow G$ is a Riemann map, then the map*

$$\begin{aligned} H_\omega : \quad & \mathfrak{M}(M_U) \longrightarrow \mathfrak{M}(M_G) \\ & \phi \quad \longrightarrow \quad \tilde{\phi}(g) = \phi(g \circ \omega) \end{aligned}$$

is a homeomorphism. Moreover, it maps the fibers of $\mathfrak{M}(M_U)$ onto the fibers of $\mathfrak{M}(M_G)$ and $\triangle(M_U)$ onto $\triangle(M_G)$.

Proof. In view of the discussion above, it remains only to show that $H_\omega(\mathfrak{M}(\alpha, M_U)) = \mathfrak{M}(\omega(\alpha), M_G)$. We have already proved the inclusion $\subset$. To prove the reverse inclusion, let $\tilde{\phi} \in \mathfrak{M}(\omega(\alpha), M_G)$. Then

$$\omega(\alpha) = \hat{z}(\tilde{\phi}) = \phi(z \circ \omega) = \phi(\omega) = \hat{\omega}(\phi)$$

Hence $\phi \in \mathfrak{M}(\alpha, M_U)$. For, if $\alpha_0 \neq \alpha$ and $\phi_0 \in \mathfrak{M}(\alpha_0, M_U)$ then $\hat{\omega}(\phi_0) = \omega(\alpha_0) \neq \omega(\alpha)$ since ω is constant on each fiber and maps ∂U bijectively onto ∂G. The proof is complete. □

4.3 The Shilov boundary for $M(D)$ - an open question

At the end of Section 4.2 we asked the question: can we identify the Shilov boundary of $M(D)$ with the maximal ideal space of some Banach algebra of functions on the unit circle? Recall that the boundary function of any multiplier of the Dirichlet space is a multiplier of the Bessel potential space $L^2_{1/2}(T)$. In this sense, $M(D) \subset M(L^2_{1/2}(T))$. If $\phi \in \mathfrak{M}(M(L^2_{1/2}))$ then this inclusion yields the restriction map

$$\begin{aligned} \tau : \quad & \mathfrak{M}(M(L^2_{1/2})) \longrightarrow \mathfrak{M}(M(D)) \\ & \phi \quad \longrightarrow \quad \phi|_{M(D)} \end{aligned}$$

from the maximal ideal space of $M(L^2_{1/2})$, denoted by X, into the maximal ideal space of $M(D)$. The map τ is continuous. The proof of this is analogous to the proof of the continuity of ω in Proposition 4.14. Arne Stray has pointed out that τ maps X homeomorphically onto the Shilov boundary for $M(D)$ if the answer to the following question is yes: if $g \in M(L^2_{1/2}(T))$ is real-valued, $u = P[g]$ and v is the harmonic conjugate of u, is $f = e^{u+iv} \in M(D)$? In other words, is the function

$$f(z) = \exp\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}u(\theta)d\theta\Big]$$

in $M(D)$ for real valued functions u in $M(L^2_{1/2}(T))$? This question remains open and we will refer to it as the *multiplier hypothesis*. Notice that $f \in H^\infty$ even though v may be unbounded.

The goal of this section is the identification of the Shilov boundary for $M(D)$ given that the multiplier hypothesis holds. First, we prove that $M(L^2_{1/2}(T))$ is a Banach algebra and that functions in this space are bounded (independently of the multiplier hypothesis). To this end, we now introduce some tools.

First off, we show an analogue of Proposition 1.7 for $L^2_{1/2}$.

Proposition 4.24. *Let $L^2_{1/2}$ be $L^2_{1/2}(T)$ or $L^2_{1/2}(\mathbb{R})$. A function f is in $M(L^2_{1/2})$ if and only if $fL^2_{1/2} \subset L^2_{1/2}$.*

Proof. If $f \in M(L^2_{1/2})$ then it is clear that $fL^2_{1/2} \subset L^2_{1/2}$.

Conversely, suppose $fL^2_{1/2} \subset L^2_{1/2}$. We must show that the operator M_f of multiplication by f is bounded. To this end, let $\{h_n\}$ be a sequence in $L^2_{1/2}$ converging in the norm on $L^2_{1/2}$ to h. Then $fh_n \in L^2_{1/2}$ for each n and $fh \in L^2_{1/2}$ by assumption. Assume $\{fh_n\}$ converges in the norm on $L^2_{1/2}$ to some function $g \in L^2_{1/2}$. If we can show that $g = fh$, then M_f is a bounded operator by the closed graph theorem.

Recall that $\|g\|_2 \le c\|g\|_{1/2}$ for some constant c and for any $g \in L^2_{1/2}$ ((2.15) and Proposition 2.15). Thus any sequence of functions which converge in $L^2_{1/2}(T)$ will also converge in $L^2(T)$. Thus $\{h_n\}$ has a subsequence $\{h_{n_s}\}$ which converge pointwise almost everywhere to h (this is a general result for L^p spaces, see [23], page 68). Then $\{fh_{n_s}\}$ converges pointwise almost everywhere to fh. But $\{fh_{n_s}\}$ is a subsequence of $\{fh_n\}$ and so converges to g in the norm on L^2. Thus $\{fh_{n_s}\}$ has a subsequence $\{fh_{n_t}\}$ which converges pointwise almost everywhere to g. But $\{fh_{n_t}\}$ is a subsequence of $\{fh_{n_s}\}$ and so converges pointwise almost everywhere to fh. Thus $g = fh$ and we conclude that M_f is a bounded operator. Hence $f \in M(L^2_{1/2})$. This completes the proof. $\square$

Recall that $f \in L^2_{1/2}(\mathbb{R})$ if and only if $f \in L^2(\mathbb{R})$ and

$$\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\frac{|f(x-y)-f(x)|^2}{|y|^2}dy = \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\frac{|f(x)-f(y)|^2}{|x-y|^2}dy < \infty. \tag{4.9}$$

We also showed that $f \in L^2_{1/2}(T)$ if and only if $f \in L^2(T)$ and

$$\int_{-\pi}^{\pi}dx\int_{-\pi}^{\pi}\frac{|f(x-y)-f(x)|^2}{|e^{iy}-1|^2}dy = \int_{-\pi}^{\pi}dx\int_{-\pi}^{\pi}\frac{|f(x)-f(y)|^2}{|e^{i(x-y)}-1|^2}dy < \infty. \tag{4.10}$$

Suppose $\phi \in C^\infty(\mathbb{R})$ have compact support in $(-\pi, \pi)$. Such functions clearly exists, for example

$$\phi(x) = \begin{cases} \exp(\frac{1}{1-x^2}) & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases} . \tag{4.11}$$

Let $\tilde{\phi}$ denote the 2π periodic extension of ϕ. We will often identify ϕ and $\tilde{\phi}$. The context should make it clear which function is referred to.

Proposition 4.25. *Let $\phi \in C^\infty$ have compact support in $(-\pi, \pi)$.*

1. $\phi \in M(L^2_{1/2}(\mathbb{R}))$
2. $\tilde{\phi} \in M(L^2_{1/2}(T))$

Proof. First, we prove (1). Suppose $g \in L^2_{1/2}(\mathbb{R})$. We will show that the integral (4.9) is finite for ϕg. Since ϕg is in $L^2(\mathbb{R})$, this will imply that $\phi g \in L^2_{1/2}(\mathbb{R})$ and so ϕ is a multiplier of $L^2_{1/2}(\mathbb{R})$ by Proposition 4.24. Now

$$\begin{aligned} &\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{|\phi(x)g(x) - \phi(y)g(y)|^2}{|x-y|^2} dy \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{|\phi(x)g(x) - \phi(y)g(x) + \phi(y)g(x) - \phi(y)g(y)|^2}{|x-y|^2} dy \\ &\leq \int_{-\infty}^{\infty} |g(x)|^2 dx \int_{-\infty}^{\infty} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^2} dy \qquad (4.12) \\ &\quad + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |\phi(y)|^2 \frac{|g(x) - g(y)|^2}{|x-y|^2} dy. \qquad (4.13) \end{aligned}$$

The integral (4.13) is finite because ϕ is bounded and $g \in L^2_{1/2}(\mathbb{R})$. For the integral (4.12), observe that

$$\int_{-\infty}^{\infty} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^2} dy = \int_{x-1}^{x+1} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^2} dy + \int_{|y-x| \geq 1} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^2} dy. \tag{4.14}$$

By the mean value theorem

$$\Big| \frac{\phi(x) - \phi(y)}{x-y} \Big|^2 = |\phi'(z)|^2$$

for some z between x and y. Since ϕ' is continuous and has compact support, we know that $|\phi'(z)|^2 \leq M_1$ for some constant M_1 and for all z. Consequently, the the first integral to the right of (4.14) is finite.

For fixed x the expression $|\phi(x) - \phi(y)|^2$ is bounded since ϕ is bounded. Denote the bound by M_2. If $t = y - x$ then $dt = dy$ and

$$\int_{|y-x| \geq 1} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^2} dy \leq M_2 \int_1^{\infty} \frac{dt}{t^2} + M_2 \int_{-\infty}^{-1} \frac{dt}{t^2}$$

which is finite. We conclude that (4.14) is finite and so (4.12) is finite as well since $\int_{-\pi}^{\pi} |g(x)|^2 dx \leq \|g\|_2^2 < \infty$. Hence $\phi g \in L^2_{1/2}(\mathbb{R})$. This proves (1).

Since $|e^{it} - 1|^2$ and t^2 are comparable on $(-\pi, \pi)$ the same arguments as used in the proof of (1) apply for (2) as well. The proof is complete. $\square$

The function ϕ is more then an example of a multiplier; it allows us to move between the spaces $L^2_{1/2}(T)$ and $L^2_{1/2}(\mathbb{R})$ as the following Proposition, based on Lemma 2.2 in [4], shows.

Proposition 4.26. *Let $\phi \in C^\infty(\mathbb{R})$ have compact support on $(-\pi, \pi)$.*

1. *If $f \in L^2_{1/2}(T)$ then $\phi f \in L^2_{1/2}(\mathbb{R})$*
2. *If $f \in L^2_{1/2}(\mathbb{R})$ then $\phi f \in L^2_{1/2}(T)$*
3. *If $f \in M(L^2_{1/2}(T))$ then $\phi f \in M(L^2_{1/2}(\mathbb{R}))$*
4. *If $f \in M(L^2_{1/2}(\mathbb{R}))$ then $\phi f \in M(L^2_{1/2}(T))$*

Proof. We first prove (1). By Proposition 4.25 we know that $\phi f \in L^2_{1/2}(T)$ and so

$$\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} \frac{|\phi(x)f(x) - \phi(y)f(y)|^2}{|e^{i(x-y)} - 1|^2} dy \tag{4.15}$$

is finite. Notice that (4.15) is finite if $|e^{i(x-y)} - 1|^2$ replaced by $|x-y|^2$ since these expressions are comparable on $(-\pi, \pi)$. The function ϕf has compact support in $(-\pi, \pi)$ and is clearly in $L^2(\mathbb{R})$. We must show that the integral

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{|\phi(x)f(x) - \phi(y)f(y)|^2}{|x-y|^2} dy \tag{4.16}$$

is finite. Let $u = \phi f$ and write the integrand as

$$F(x, y) = \frac{|u(x) - u(y)|^2}{|x-y|^2}. \tag{4.17}$$

Notice that (4.17) is zero on $\{(x, y) : |x| \geq \pi, |y| \geq \pi\}$ and so (4.16) reduces to

$$\begin{aligned} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} F(x,y) dy &= \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} F(x,y) dy + \int_{\pi}^{\infty} dx \int_{-\pi}^{\pi} F(x,y) dy \\ &+ \int_{-\infty}^{-\pi} dx \int_{-\pi}^{\pi} F(x,y) dy + \int_{-\pi}^{\pi} dx \int_{\pi}^{\infty} F(x,y) dy + \int_{-\pi}^{\pi} dx \int_{-\infty}^{-\pi} F(x,y) dy. \end{aligned} \tag{4.18}$$

We have already observed that the first integral to the right in (4.18) is finite. The remaining four integrals are all of the same type, that is, integrals over infinite strips in the plane. We consider the second integral:

$$\int_{\pi}^{\infty} dx \int_{-\pi}^{\pi} \frac{|u(x) - u(y)|^2}{|x-y|^2} dy. \tag{4.19}$$

Since ϕ has compact support in $(-\pi, \pi)$ the integral (4.19) is equal to

$$\int_{\pi}^{\infty} dx \int_{-\pi+\epsilon}^{\pi-\epsilon} \frac{|u(y)|^2}{|x-y|^2} dy. \tag{4.20}$$

for some $\epsilon > 0$. By Fubini's theorem we see that (4.20) is equal to

$$\int_{-\pi+\epsilon}^{\pi-\epsilon} |u(y)|^2 dy \int_{\pi}^{\infty} \frac{1}{|x-y|^2} dx. \tag{4.21}$$

Let $t = x - y$. Then $dt = dx$ and the lower bound of integration is $\pi - y \geq \pi - (\pi - \epsilon) = \epsilon$. Thus (4.21) is less then or equal to

$$\int_{-\pi+\epsilon}^{\pi-\epsilon} |u(y)|^2 dy \int_{\epsilon}^{\infty} \frac{dt}{t^2} = \frac{1}{\epsilon} \int_{-\pi+\epsilon}^{\pi-\epsilon} |u(y)|^2 dy \tag{4.22}$$

which is finite since $u = \phi f \in L^2(T)$. Thus (4.19) is finite. The arguments for the remaining three integrals in (4.18) are similar. Thus (4.16) is finite and so $\phi f \in L^2_{1/2}(\mathbb{R})$. This completes the proof of (1).

The proof of (2) is simpler. Suppose $f \in L^2_{1/2}(\mathbb{R})$. Then $\phi f \in L^2_{1/2}(\mathbb{R})$ by Proposition 4.25 and so (4.16) is finite. But this clearly implies that (4.15) is finite as well. Since ϕf has compact support in $(-\pi, \pi)$ we can extend it to a 2π periodic function and so $\phi f \in L^2_{1/2}(T)$.

To prove (3), let $g \in L^2_{1/2}(\mathbb{R})$. We must show that $\phi f g \in L^2_{1/2}(\mathbb{R})$. Then $\phi f \in M(L^2_{1/2}(\mathbb{R}))$ by Proposition 4.24. First, note that $\phi g \in L^2_{1/2}(T)$ by (2). Thus $f \phi g \in L^2_{1/2}(T)$ since f is a multiplier of $L^2_{1/2}(T)$. If $u = \phi f g$ we conclude that the integral

$$\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} \frac{|u(x) - u(y)|^2}{|e^{i(x-y)} - 1|^2} dy \tag{4.23}$$

is finite. But the function u has compact support in $(-\pi, \pi)$ and so the calculations in the proof of (1) shows that the integral (4.23) is finite if and only if the integral

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{|u(x) - u(y)|^2}{|x - y|^2} dy \tag{4.24}$$

is finite. In other words (4.23) and (4.24) are simultaneously bounded for u. Thus $\phi f g \in L^2_{1/2}(\mathbb{R})$ and so $\phi f \in M(L^2_{1/2}(\mathbb{R}))$. This proves (3).

Finally, we prove (4). Let $g \in L^2_{1/2}(T)$. Then $\phi g \in L^2_{1/2}(\mathbb{R})$ by (1) and so $f \phi g \in L^2_{1/2}(\mathbb{R})$ since $f \in M(L^2_{1/2}(\mathbb{R}))$. Writing $u = f \phi g$ we conclude that (4.24) is finite. Thus (4.23) is finite as well and so $f \phi g \in L^2_{1/2}(T)$. Consequently, $f \phi \in M(L^2_{1/2}(T))$ and so (4) holds.

The proof of the proposition is complete. □

Corollary 4.27. *If $f \in M(L^2_{1/2}(T))$ then $f \in L^\infty(T)$.*

Proof. Let $(a, b) \subset (-\pi, \pi)$ where $a, b \notin \{-\pi, \pi\}$. Then (a, b) is contained in the compact set $[a - \epsilon, b + \epsilon]$ where $[a - \epsilon, b + \epsilon] \subset (-\pi, \pi)$ for some $\epsilon > 0$. We can find a function $\phi \in C^\infty$ with compact support K_ϕ containing $[a - \epsilon, b + \epsilon]$ and contained in $(-\pi, \pi)$. If $f \in M(L^2_{1/2}(T))$ then $\phi f \in M(L^2_{1/2}(\mathbb{R}))$ by Proposition 4.26 and so $\phi f \in L^\infty(\mathbb{R})$ since $M(L^2_{1/2}(\mathbb{R})) \subset L^\infty(\mathbb{R})$. Since $(a, b) \subset [a - \epsilon, b + \epsilon] \subset K_\phi$ there exists a constant c such that $|\phi| \geq c > 0$ on (a, b). Thus $f \in L^\infty((a, b))$.

It remains only to verify that no problems occur in neighbourhoods π or $-\pi$. To see this, let I be a small open interval containing π and choose $\delta > 0$ such that $I - \delta \subset (-\pi, \pi)$. The shift $h \to h_\delta = h(x + \delta)$ is an isometric isomorphism of $L^2_{1/2}(T)$ onto itself. For, if $t = x + \delta$ and $\theta = y + \delta$ then

$$\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} \frac{|h(x + \delta) - h(y + \delta)|^2}{|x - y|^2} dy = \int_{-\pi}^{\pi} dt \int_{-\pi}^{\pi} \frac{|h(t) - h(\theta)|^2}{|t - \theta|^2} d\theta \tag{4.25}$$

and so $\|h_\delta\|_{1/2} = \|h\|_{1/2}$. Let $f \in M(L^2_{1/2}(T))$. Then

$$\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} \frac{|f(x+\delta)h(x) - f(y+\delta)h(y)|^2}{|x-y|^2} dy = \int_{-\pi}^{\pi} dt \int_{-\pi}^{\pi} \frac{|f(t)h(t-\delta)) - f(\theta)h(\theta-\delta)|^2}{|t-\theta|^2} d\theta$$

by the same change of variables as in (4.25). Thus $f_\delta \in M(L^2_{1/2}(T))$ since any function in $L^2_{1/2}(T)$ can be written in form $h(x-\delta)$ as we observed above. Thus the shift $f \to f_\delta$ maps multipliers into multipliers. Consequently, $f_\delta \in L^\infty((I-\delta))$ and so $f \in L^\infty(I)$. This completes the proof. □

Proposition 4.28. *The space $M(L^2_{1/2}(T))$ is a Banach algebra with pointwise multiplication and the operator norm*

$$\|f\| = \|M_f\| = \sup_{\|g\|_{1/2} \le 1} \|fg\|_{1/2}.$$

Moreover, $M(L^2_{1/2}(T)) \subset L^2_{1/2}(T)$.

Proof. First, notice that $M(L^2_{1/2}(T))$ is a linear space. Moreover $M(L^2_{1/2}(T)) \subset L^2_{1/2}(T)$ since $1 \in L^2_{1/2}(T)$. To see this, observe that $1 \in L^2(T)$ and (4.10) is finite when $f = 1$:

$$\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} \frac{|1-1|^2}{|e^{iy}-1|^2} dy = 0.$$

If f_1 and f_2 are multipliers of $L^2_{1/2}(T)$ then the pointwise product $f = f_1 f_2$ is multiplier as well. The proof is similar to the proof of the analogous statement for the Banach algebra $M(D)$ in Proposition 1.9.

It remains only to show that $M(L^2_{1/2}(T))$ is complete. Let $\{f_n\}$ be a Cauchy sequence in $M(L^2_{1/2}(T))$ and let $h \in L^2_{1/2}(T)$. Then

$$\|f_n h - f_m h\|_{1/2} \le \|f_n - f_m\| \|h\|_{1/2} < \epsilon$$

when $n, m \ge N(\epsilon, h)$. Thus $\{f_n h\}$ is a Cauchy sequence in $L^2_{1/2}(T)$ for each fixed $h \in L^2_{1/2}(T)$. Consequently $\{f_n h\}$ converges to some function $F(h) \in L^2_{1/2}(T)$ in the norm on $L^2_{1/2}(T)$ since $L^2_{1/2}(T)$ is complete. In particular, $\{f_n\}$ converges to $f = F(1)$. Recall that $\|g\|_2 \le c\|g\|_{1/2}$ for some constant c and for any $g \in L^2_{1/2}(T)$ (Proposition 2.15). Thus any sequence of functions which converge in $L^2_{1/2}(T)$ will also converge in $L^2(T)$. Hence $\{f_n\}$ is a Cauchy sequence in $L^2(T)$ with limit f and so there exists a subsequence $\{f_{n_s}\}$ which converges pointwise almost everywhere to f. Now $\{f_{n_s} h\}$ is a subsequence of $\{f_n h\}$ and, consequently, converge to $F(h)$ in the norm on $L^2_{1/2}(T)$ and the norm on $L^2(T)$. Thus $\{f_{n_s} h\}$ has a subsequence $\{f_{n_t} h\}$ which converge pointwise almost everywhere to $F(h)$. But $\{f_{n_t} h\}$ converges pointwise almost everywhere to fh since $\{f_{n_s}\}$ converges pointwise almost everywhere to f and $\{f_{n_t}\}$ is a subsequence of $\{f_{n_s}\}$. Thus $fh = F(h) \in L^2_{1/2}(T)$ and so

$$f L^2_{1/2}(T) \subset L^2_{1/2}(T). \tag{4.26}$$

Thus $f \in M(L^2_{1/2}(T))$ by Proposition 4.24.

Finally, the proof that $\{f_n\}$ converges to f in the operator norm is similar to the analogous proof in Proposition 1.9. We conclude that $M(L^2_{1/2}(T))$ is complete. This completes the proof. □

We will now state and prove some lemmas concerning the maximal ideal space and Gelfand space of $M(L^2_{1/2}(T))$. They will be needed in the hypothetical identification of the Shilov boundary for $M(D)$, but are not themselves dependent on the multiplier hypothesis.

Lemma 4.29. *If $f \in M(L^2_{1/2}(T))$ is real valued and $\phi \in \mathfrak{M}(M(L^2_{1/2}))$ then $\phi(f)$ is real.*

Proof. Assume $\phi(f) = x + iy$. Then $\phi(f - x) = iy$ and so $\phi((f - x)^2) = -y^2$ and $\phi((f - x)^2 + y^2) = 0$. Set $g = (f - x)^2 + y^2$. Then $g(t) \geq y^2 > 0$ for all $e^{it} \in T$ and $u = \frac{g}{y^2} \geq 1$. Set $v = \frac{1}{u}$. Then $v \leq 1$ and

$$|v(t_1) - v(t_2)| = \Big|\frac{1}{u(t_1)} - \frac{1}{u(t_2)}\Big| = \Big|\frac{u(t_1) - u(t_2)}{u(t_1)u(t_2)}\Big| \leq |u(t_1) - u(t_2)|$$

and so $v \in M(L^2_{1/2}(T))$ by Proposition 2.18. Thus u is invertible in $M(L^2_{1/2}(T))$ which is a contradiction. For $\phi(u) = \phi(\frac{g}{y^2}) = 0$, but if u is invertible then $1 = \phi(1) = \phi(uu^{-1}) = \phi(u)\phi(u^{-1})$ and so $\phi(u) \neq 0$. We conclude that $\phi(f)$ is real. This completes the proof. □

Lemma 4.30. *If $f = u + iv \in M(L^2_{1/2}(T))$ then u, v and $\bar{f}$ are in $M(L^2_{1/2}(T))$ and*

$$\hat{\bar{f}}(\phi) = \overline{\hat{f}(\phi)}$$

for each $\phi \in \mathfrak{M}(M(L^2_{1/2}(T))$.

Proof. Since

$$|f(t_1) - f(t_2)|^2 = |u(t_1) - u(t_2)|^2 + |v(t_1) - v(t_2)|^2$$

we see that $f = u + iv \in M(L^2_{1/2}(T))$ if and only if $u, v \in M(L^2_{1/2}(T))$ by Proposition 2.18. Thus $\bar{f} = u - iv \in M(L^2_{1/2}(T))$ as well.

Let $\phi \in \mathfrak{M}(M(L^2_{1/2}))$. Then $\phi(u)$ and $\phi(v)$ are real numbers by Lemma 4.29 and so

$$\begin{aligned}\hat{\bar{f}}(\phi) &= \phi(\bar{f}) = \phi(u) - i\phi(v)\\ &= \overline{\phi(u) + i\phi(v)}\\ &= \overline{\phi(f)}\\ &= \overline{\hat{f}(\phi)}.\end{aligned}$$

This completes the proof. □

If A is any Banach algebra and $u \in A$ then

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \tag{4.27}$$

is in A where the convergence of the series (4.27) is in the norm on A. To see this, notice that the partial sums $s_N = \sum_{n=0}^{N} \frac{u^n}{n!} \in A$ for any N. Moreover, s_N is a Cauchy sequence in A, for

$$\Big\|\sum_{n=0}^{N} \frac{u^n}{n!} - \sum_{n=0}^{M} \frac{u^n}{n!}\Big\| = \Big\|\sum_{n=M+1}^{N} \frac{u^n}{n!}\Big\| \leq \sum_{n=M+1}^{N} \frac{\|u\|^n}{n!} < \epsilon \tag{4.28}$$

if $N, M > P(\epsilon)$ since $\|u\|$ is a complex number and $\sum_{n=0}^{N} \frac{\|u\|^n}{n!}$ converges to the complex number $e^{\|u\|}$. Thus $s_N \to \sum_{n=0}^{\infty} \frac{u^n}{n!} = e^u$ in the norm on A since A is complete.

From the observations we conclude that $e^u \in M(L^2_{1/2}(T))$ for any $u \in M(L^2_{1/2}(T))$. The same argument cannot be used for the Gelfand space $\widehat{M}(L^2_{1/2}(T))$ since this algebra is not uniformly closed and, consequently, is no Banach algebra with the supremum norm. It is still true though, for $\widehat{e^u} \in \widehat{M}(L^2_{1/2}(T))$ for any $u \in M(L^2_{1/2}(T))$ and if $\phi \in \mathfrak{M}(M(L^2_{1/2}))$ then

$$\begin{aligned}
\widehat{e^u}(\phi) &= \phi(e^u) \\
&= \phi\Big(\sum_{n=0}^{\infty} \frac{u^n}{n!}\Big) \\
&= \lim_{N\to\infty} \phi\Big(\sum_{n=0}^{N} \frac{u^n}{n!}\Big) \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{\phi(u)^n}{n!} = e^{\phi(u)} \\
&= e^{\hat{u}(\phi)}.
\end{aligned}$$

Thus $e^{\hat{u}} \in \widehat{M}(L^2_{1/2}(T))$ and $e^{\hat{u}} = \widehat{e^u}$.

Lemma 4.31. *Suppose $u \in M(L^2_{1/2}(T))$ and*

$$u = \log |F_u|$$

where F_u is some invertible element of $M(D)$. Then

$$\hat{u} = \log |\widehat{F}_u|.$$

Proof. Clearly $\widehat{e^u} = \widehat{|F_u|}$ and from the observations above, we know that $e^{\hat{u}} = \widehat{e^u}$. Moreover, $\overline{F}_u \in M(L^2_{1/2}(T))$ by Lemma 4.30 and if $\phi \in \mathfrak{M}(M(L^2_{1/2}))$ we see that

$$\begin{aligned}
(\widehat{|F_u|}(\phi))^2 &= \phi(|F_u|^2) = \phi(F_u)\phi(\overline{F_u}) \\
&= \phi(F_u)\overline{\phi(F_u)} \\
&= |\widehat{F}_u(\phi)|^2
\end{aligned}$$

and so $\widehat{|F_u|} = |\widehat{F}_u|$. Thus $e^{\hat{u}} = |\widehat{F}_u|$ and $\hat{u} = \log |\widehat{F}_u|$ as was to be shown. □

Lemma 4.32. *The uniform closure of the algebra $\widehat{M}(L^2_{1/2}(T))$ is the set of all complex continuous functions on X.*

Proof. We will show that the algebra $\widehat{M}(L^2_{1/2}(T))$ is self-adjoint, separates points on X and vanishes at no point of X. The result then follows by the Stone-Weierstrass theorem (see for example [22] page 165).

First, $\widehat{M}(L^2_{1/2}(T))$ is self-adjoint by Lemma 4.30.

To show that $\widehat{M}(L^2_{1/2}(T))$ separates points on X, let $\phi_1, \phi_2 \in X$ and suppose $\phi_1 \neq \phi_2$. Then $\phi_1(f) \neq \phi_2(f)$ for some $f \in M(L^2_{1/2}(T))$ and so $\hat{f}(\phi_1) \neq \hat{f}(\phi_2)$.

If $\phi \in X$ and $\phi \neq 0$ then $\phi(f) \neq 0$ for some $f \in M(L^2_{1/2}(T))$. Thus $\hat{f}(\phi) \neq 0$ and so $\widehat{M}(L^2_{1/2}(T))$ vanishes at no point of X. This completes the proof. □

Proposition 4.33. *Suppose the multiplier hypothesis holds. Then τ is a homeomorphism of X into $\mathfrak{M}(M(D))$, and the image set $\tau(X)$ is the Shilov boundary for $M(D)$.*

Proof. First, we prove that $\tau(X)$ is a closed boundary for $M(D)$. But $\tau(X)$ is closed since $\tau(X)$ is compact and compact subsets of Hausdorff spaces are closed. Notice further that since $M(L^2_{1/2}(T)) \subset L^\infty(T)$ there is a restriction map

$$\begin{aligned} \kappa: \quad \mathfrak{M}(L^\infty) &\longrightarrow \mathfrak{M}(M(L^2_{1/2})) \\ \phi \quad &\longrightarrow \quad \phi|_{M(L^2_{1/2})}. \end{aligned}$$

If $f \in M(L^2_{1/2})$ then

$$\sup\{|\hat{f}(\phi)| : \phi \in \mathfrak{M}(M(L^2_{1/2}))\} \geq \sup\{|\hat{f}(\phi)| : \phi \in \mathfrak{M}(M(L^\infty))\}. \tag{4.29}$$

To see this, let $\phi \in \mathfrak{M}(M(L^\infty))$. Then $\kappa(\phi) = \phi|_{M(L^2_{1/2})} \in \mathfrak{M}(M(L^2_{1/2}))$ and $\phi|_{M(L^2_{1/2})}(f) = \phi(f)$. Thus the right side of (4.29) cannot be greater then the left side.

We remark that the map $L^\infty \to \widehat{L}^\infty$ which sends f to its Gelfand transform, $\hat{f}$, is an isometric isomorphism (the norm on L^∞ is the supremum norm). Let $f \in M(D)$. Then

$$\begin{aligned} \sup\{|\hat{f}(\phi)| : \phi \in \tau(X)\} &= \sup\{|\hat{f}(\phi)| : \phi \in \mathfrak{M}(M(L^2_{1/2}))\} \\ &\geq \sup\{|\hat{f}(\phi)| : \phi \in \mathfrak{M}(L^\infty)\} \\ &= \sup_{t\in T} |f(t)| \\ &= \sup_{\lambda\in U} |f(\lambda)| \quad \text{by the maximum modulus principle} \\ &= \sup_{\lambda\in U} |\phi_\lambda(f)| \quad \text{where } \phi_\lambda \text{ is evaluation at } \lambda \\ &= \sup\{|\phi(f)| : \phi \in \triangle(M)\} \\ &= \sup\{|\hat{f}(\phi)| : \phi \in \mathfrak{M}(M(D))\} \end{aligned}$$

since $\triangle(M)$ is dense in $\mathfrak{M}(M(D))$ by the Corona theorem. We conclude that $\tau(X)$ is a closed boundary for $M(D)$.

Since X is compact, $\mathfrak{M}(M(D))$ is Hausdorff and τ is continuous, we can prove that τ is a homeomorphism of X onto its image $\tau(X)$ by showing that it is injective (see [18] page 167). Suppose ϕ_1 and ϕ_2 are in X and that $\phi_1 \neq \phi_2$. Then $\phi_1(u) \neq \phi_2(u)$ for some real valued $u \in M(L^2_{1/2}(T))$. The function

$$F_u(z) = \exp\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} u(\theta)d\theta\Big]$$

is in $M(D)$ by the multiplier hypothesis. Consequently, Lemma 4.31 yields that

$$\begin{aligned} \log|\phi_1(F_u)| &= \log|\hat{F}_u(\phi_1)| \\ &= \hat{u}(\phi_1) \\ &\neq \hat{u}(\phi_2) \\ &= \log|\phi_2(F_u)| \end{aligned}$$

and so $|\phi_1(F_u)| \neq |\phi_2(F_u)|$. In other words $\tau(\phi_1)(F_u) \neq \tau(\phi_2)(F_u)$. Hence τ is injective and so τ is a homeomorphism of X onto its image $\tau(X)$.

Finally, we must show that $\tau(X)$ is the smallest closed boundary for $M(D)$. Suppose $X_0 \subset X$ is a proper closed subset of X. Since X_0 is a closed subset of a compact space, it must be compact. Let $\psi \in X - X_0$. Since X is a compact Hausdorff space, it must be regular (see [18], Exercise 3, page 205) and so we can find disjoint open sets U and V of X containing ψ and X_0 respectively. By Urysohn's Lemma there exists a function $g \in C(X)$ such that

- $0 \leq g(\phi) \leq 1$ for all $\phi \in X$
- $g(\phi) = 1$ for $\phi \in X_0$
- $\mathrm{supp}(g) \subset V$

where $\mathrm{supp}(g)$ denotes the support of g (see [23], page 39). Clearly, $g(\psi) = 0$ since $\psi \in U$.

Let $h = 1 - g$. Then $h \in C(X)$ and $h(\phi) = 0$ for all $\phi \in X_0$, $\mathrm{supp}(g) \subset X - V$ and $h(\psi) = 1$. By Lemma 4.32 the algebra $\widehat{M}(L^2_{1/2})$ is dense in $C(X)$. Thus the real valued functions in $\widehat{M}(L^2_{1/2})$ are dense in the real continuous functions on X. Consequently, for every $\epsilon > 0$ there exists a real valued function $\hat{u} \in \widehat{M}(L^2_{1/2})$ such that

$$\|\hat{u} - h\| = \sup_{\phi \in X} |\hat{u}(\phi) - h(\phi)| < \epsilon.$$

Choosing ϵ small, we get a function $\hat{u}$ which is close to zero on X_0 and close to 1 at ψ. Notice further that $\hat{u} = \bar{\hat{u}}$ since $\hat{u}$ is real. But $\bar{\hat{u}} = \hat{\bar{u}}$ by Lemma 4.30. Thus $\hat{u} = \hat{\bar{u}}$ and so $u = \bar{u}$ since $M(L^2_{1/2}(T))$ is semisimple. We conclude that u is real. Thus

$$F_u(z) = \exp\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u(\theta) d\theta\Big]$$

is in $M(D)$ by the multiplier hypothesis and

- $\log|\phi(F_u)| = |\hat{u}(\phi)| < \epsilon$ for $\phi \in X_0$
- $\big|\log|\psi(F_u)| - 1\big| = |\hat{u}(\psi) - 1| < \epsilon$

by Lemma 4.31. Thus

- $|\phi(F_u)| < e^{\epsilon}$ for $\phi \in X_0$
- $e^{1-\epsilon} < |\psi(F_u)|$.

Choosing ϵ sufficiently small, we see that $\widehat{F}_u$ does not attain its maximum modulus on $\tau(X_0)$ and so $\tau(X_0)$ cannot be a boundary for $M(D)$. Consequently, $\mathscr{S}(M(D)) = \tau(X)$. This completes the proof assuming the multiplier hypothesis is true. $\square$

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