Non-Slit Solutions to the Loewner Equation

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Contents

Introduction

Loewner's parametric method based on the Loewner equation proved to be one of the most powerful tools for solution of extremal problems in the theory of univalent functions. It was first applied in 1923 for proving the Bieberbach conjecture for the third coefficient (namely, that if $f = z + a_2z^2 + a_3z^3 + \dots$ is univalent in $\mathbb{D}$, then $|a_3| \leq 3$). Later, in 1984 it was used to prove finally the Bieberbach conjecture for all coefficients, and after that interest in the Loewner equation waned.

In 2000 Oded Schramm revived the interest when he considered a stochastic version of the Loewner equation. This allowed to find solutions to several problems of mathematical physics which previously seemed to be insoluble. The solution to this equation is now known as the Schramm-Loewner evolution. Despite considerable interest in this equation now, very little is known about geometric properties of its solutions. In 1940's Kufarev constructed an example in which the Loewner equation generated a non-slit solution. He also formulated a condition for the driving term guaranteeing that the Loewner equation generates slit solutions. The next sufficient condition was found only in 2005 and was given in terms of the Lipschitz-1/2 norm of the driving term, and the non-slit example remained the only known of its kind.

In this thesis we consider examples of non-slit solutions given by Prokhorov and Vasil'ev and study Lipschitz properties of the corresponding driving terms. We also construct a new family of non-slit examples which generalize Kufarev's example, and in which the local Lipschitz characteristic of the driving term takes values from $3\sqrt{2}\frac{\sqrt{5}-1}{\sqrt{3\sqrt{5}-5}} \approx 4.01$ to $+\infty$. This should be compared to the results by Marshall, Rohde and Lind, which say that driving terms with local characteristic less than 4 always generate slits.

The first chapter contains basic prerequisites from the theory of univalent functions and also several other definitions and results that we will use in this thesis. In the second chapter we consider different types of the Loewner and Loewner-Kufarev equations and describe Loewner's parametric method. In the third chapter we focus attention on relations between geometric properties of solutions and analytic properties of the driving term. The last chapter contains the main results of this thesis.

Chapter 1

Preliminaries

In this chapter we give a brief summary of fundamental facts from complex and real analysis and, in particular, from the theory of univalent functions that we will use further in this work. Some of them (e. g., properties of the class S and the Carathéodory convergence theorem) are not mentioned that often in basic complex analysis courses, whereas others (normal families and the Riemann mapping theorem) are covered in most textbooks.

Regarding the Lipschitz condition, only the case of order 1 is usually mentioned in courses of ordinary differential equations, but the general case of $\alpha \leq 1$ and the local Lipschitz conditions are used quite infrequently. So, it might be useful to collect all this information in one chapter for future reference. In our presentation we mostly follow the classical texts [Ahl78], [Dur83] and [Rud87].

1.1 Local Uniform Convergence and Normal Families

One of the most widely used notions of functional convergence in the theory of analytic functions is the so-called local uniform convergence.

Definition 1.1. *A sequence $\{f_j\}$ of functions in a domain D is said to converge to f locally uniformly if to every $z_0 \in D$ and to every $\epsilon > 0$ there corresponds $\delta > 0$ and an integer N such that $|f_j(z) - f(z)| < \epsilon$ for all z in the δ-neighborhood of z_0 and all $j > N$.*

Since the complex plane is a locally compact topological space, local uniform convergence in D is equivalent to compact convergence in D, that is, uniform convergence on all compact subsets of D.

The topology of local uniform convergence is metrizable (an example of a distance function inducing it can be found in [Ahl78, p. 220]), and thus, sequential compactness is equivalent to compactness in this topology.

An important property of local uniform convergence is the fact that it preserves analyticity. The precise statement of this is given in the following theorem.

Theorem 1.2 (Weierstrass). *Suppose that the functions f_n are analytic in a domain D, and that the sequence $\{f_n\}$ converges to f locally uniformly in D. Then f is analytic in D. Moreover, f_n' converges uniformly to f' locally uniformly in D.*

Hurwitz's theorem, given below, or rather its corollary, can be used for showing univalence of the locally uniform limit of a sequence of univalent functions (recall that a function is called *univalent* in a domain D if it is meromorphic and injective there).

Theorem 1.3 (Hurwitz). *If the functions f_n are analytic and nonvanishing in a domain D, and if $f_n \to f$ as $n \to \infty$ locally uniformly in D, then f is either identically zero or never equal to zero in D.*

Corollary 1.4. *Let f_n be analytic and univalent in a domain D, and suppose $f_n \to f$ as $n \to \infty$ locally uniformly in D. Then f is either univalent or constant in D.*

The notion and properties of the local uniform convergence lead to the definition of normal families.

Definition 1.5. *A family $\mathfrak{F}$ of functions analytic in a domain D is called normal if every sequence $\{f_n\} \subset \mathfrak{F}$ contains a subsequence converging locally uniformly in D.*

In other words, a normal family is a family whose closure is compact in the topology of local uniform convergence.

Montel's theorem gives a simple characterization of normal families.

Theorem 1.6 (Montel). *A family of functions $\mathfrak{F}$ analytic in a domain D is normal if and only if it is uniformly bounded on each compact subset of D.*

We conclude this section with the Vitali theorem, which states that for normal families pointwise convergence is equivalent to local uniform convergence.

Theorem 1.7 (Vitali). *Let the functions f_n be analytic and uniformly bounded on compact subsets in a domain D, and suppose that $\{f_n(z)\}$ converges at each point of a set which has a limit point in D. Then $\{f_n\}$ converges locally uniformly in D.*

1.2 The Riemann Mapping Theorem

We mention first a simple, but nevertheless a useful result by Schwarz. By $\mathbb{D}$ we denote the open unit disc $\{z : |z| < 1\}$.

Theorem 1.8 (Schwarz's Lemma). *If f is analytic in $\mathbb{D}$ and satisfies the conditions $|f(z)| \le 1$, $f(0) = 0$, then $|f(z)| \le |z|$ and $|f'(0)| \le 1$. If $|f(z)| = |z|$ for some $z \ne 0$, or if $|f'(0)| = 1$, then $f(z) = cz$ with a constant c of absolute value 1.*

This statement, as well as properties of normal families, are used substantially in the proof of the following theorem, which is in fact, one of the main results of complex analysis.

Theorem 1.9 (The Riemann Mapping Theorem). *Let D be a simply connected domain which is not the whole complex plane, and let $a \in D$. Then there exists a unique univalent map f of D onto the unit disc $\mathbb{D}$ normalized in D by the conditions $f(a) = 0$, $f'(a) > 0$.*

The first formulation of this theorem (though essentially different from the one above) was given by Riemann in 1851 in his inaugural dissertation [Rie51], but his proof was incomplete. It took over fifty years of continuous efforts of many mathematicians to produce the formulation and the proof of what is now known as the Riemann mapping theorem. Although the works by Osgood [Osg00], Koebe[Koe15] and Carathéodory[Car12] were particularly important, contributions of Schwarz, Poincaré, Hilbert, Courant, Bieberbach, Gronwall, Lindelöf, Montel, Fejér, Riesz and Ostrowski were also valuable. More on the theorem's history can be found in [Hil62, p. 320–321] and [SZ65, p. 230].

Note, that if we replace $f(z)$ in the theorem's formulation by $g(z) = \frac{f(z)}{f'(a)}$, the new function satisfies the conditions $g(a) = 0$, $g'(a) = 1$ and maps D onto the disc $\{w : |w| < \frac{1}{f'(a)}\}$. The number $\frac{1}{f'(a)}$ is called *the conformal radius* of the domain D and denoted by $\mathrm{rad}(D)$.

Conformal radius is monotone with respect to the set-theoretical inclusion. This is stated in the following corollary of Schwarz's lemma.

Theorem 1.10 (The Subordination Principle). *If f is analytic and univalent in $\mathbb{D}$ and if g is a function analytic in $\mathbb{D}$ with $g(0) = f(0)$ and $g(\mathbb{D}) \subset f(\mathbb{D})$, then $|g'(0)| \leq |f'(0)|$ and $g(\mathbb{D}_r) \subset f(\mathbb{D}_r)$ for every $r < 1$, where $\mathbb{D}_r = \{z : |z| < r\}$.*

We are often interested in conditions which guarantee that the conformal map $f : D \to \mathbb{D}$ or $f^{-1} : \mathbb{D} \to D$ can be continuously extended to the boundary. This question is thoroughly discussed, e. g., in the books [Pom75] and [Pom92]. We will need the following two theorems. The first one gives a necessary and sufficient condition for the continuous extension. It uses the notion of local connectedness; we remind that a closed set A is said to be *locally connected* if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any two points $z_1, z_2 \in \mathbb{A}$, $|z_1 - z_2| < \delta$ it is possible to find a compact connected set $B \subset A$ containing both z_1 and z_2, such that $\operatorname{diam} B < \epsilon$ ($\operatorname{diam} B$ is defined as $\sup_{a,b \in B} |a - b|$).

Theorem 1.11 (Continuity Theorem, [Pom75, p. 279]). *The function $f^{-1} : \mathbb{D} \to D$ has a continuous extension to $\overline{\mathbb{D}}$ if and only if the boundary of D is locally connected.*

The question about injectivity of this extension is answered by the Carathéodory theorem.

Theorem 1.12 (Carathéodory Extension Theorem [Car13]). *The function $f^{-1} : \mathbb{D} \to D$ has a continuous and injective extension to $\overline{\mathbb{D}}$ if and only if the boundary of D is a Jordan curve.*

The notion of *half-plane capacity* is relevant to the study of the chordal Loewner equation, and plays a role similar to the role of the conformal radius in the study of the radial Loewner equation. Let A be a closed subset of the upper half-plane $\mathbb{H}$, such that its complement $\mathbb{H} \setminus A$ is simply connected. Such sets are sometimes called compact $\mathbb{H}$-hulls. One can show that there exists a unique conformal map f_A of $\mathbb{H} \setminus A$ onto $\mathbb{H}$ such that $\lim_{z\to\infty}(f_A(z) - z) = 0$, and this map has expansion

$$f_A = z + \frac{a}{z} + \dots$$

near infinity. The number $a = \lim_{z\to\infty}(f_A(z) - z)$ is called the half-plane capacity (from infinity) of A. In other words,

$$f_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty.$$

The unique map f_{rA} corresponding to the $\mathbb{H}$-hull rA which is obtained by scaling A by $r > 0$ has the form $f_{rA} = rf_A(z/r)$, and thus $\mathrm{hcap}(rA) = r^2\mathrm{hcap}(A)$. For the set $A + x$ obtained by translation of A, we have $f_{A+x} = f_A(z - x) + x$, and $\mathrm{hcap}(A + x) = \mathrm{hcap}(A)$.

1.3 Some Properties of the Class S

We denote by S the class of functions which are analytic and univalent in the unit disc $\mathbb{D}$, and which are normalized by the conditions

$$f(0) = 0, \quad \text{and} \quad f'(0) = 1.$$

It follows that each $f \in S$ has a Taylor expansion of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \dots, \quad z \in \mathbb{D}. \tag{1.1}$$

The "S" stands for the word "schlicht" of German origin which means the same as "univalent" and was in use until 1960's.

In view of the Riemann mapping theorem one can find, of course, numerous functions from S, with the identity map being the simplest one. There is, however, one important family of functions from this class, namely the Koebe function and its rotations, which at the same time solves several extremal problems in S. It is defined as

$$k_\theta(z) = \frac{z}{(1 - ze^{i\theta})^2} = z + 2e^{i\theta}z^2 + 3e^{2i\theta}z^3 + \dots, \quad \theta \in [0, 2\pi).$$

The Koebe function $k(z)$ is obtained when $\theta = 0$, $k(z) = \frac{z}{(1-z)^2}$. $k(z)$ maps $\mathbb{D}$ conformally onto $\mathbb{C} \setminus (-\infty, -1/4]$.

Let $\mathfrak{F}$ be some class of complex functions in the unit disc. The intersection

$$\mathfrak{K}_{\mathfrak{F}} = \bigcap_{f\in\mathfrak{F}} f(\mathbb{D})$$

is called the Koebe set of $\mathfrak{F}$. The Koebe one-quarter theorem states that $\mathfrak{K}_S = \{w : |w| < 1/4\}$.

Theorem 1.13 (Koebe One-Quarter Theorem)**.** *The range of every function of class S contains the disc $\{w : |w| < \frac{1}{4}\}$.*

This class is not closed under, e. g., addition or multiplication, but there is still a number of simple operations which preserve the class S. For example, if $f \in S$ then the functions $\overline{f(\bar{z})}$, $e^{-i\theta} f(e^{i\theta} z)$ and $r^{-1} f(rz)$ also belongs to the class S (here $\theta \in \mathbb{R}$, $0 < r < 1$).

A less trivial fact is that $\sqrt{f(z^2)} \in S$. The resulting function is odd, and it should be remarked that every odd function in S can be obtained this way.

We now state two classical inequalities for the class S.

Theorem 1.14 (Distortion Theorem)**.** *For each $f \in S$,*

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |f'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}.$$

For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

Theorem 1.15 (Growth Theorem)**.** *For each $f \in S$,*

$$\frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2}.$$

For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

It follows immediately from the growth theorem that the family S is uniformly bounded on compact subsets of $\mathbb{D}$ and thus, by Montel's theorem (Theorem 1.6), S is a normal family. Moreover, S is compact in the local uniform convergence topology. Indeed, by Weierstrass' theorem (Theorem 1.2) the limit f of a convergent sequence $\{f_n\} \subset S$ is analytic and $f'(0) = \lim_{n \to \infty} f_n'(0) = 1$. Thus, f is not constant and in view of Corollary 1.4 $f \in S$.

The coefficients a_n can be considered as complex-valued functionals $a_n[f]$ on S. In view of the Weierstrass theorem, these functionals are continuous in the topology of local uniform convergence. Since S is compact, $|a_n[f]|$ should attain their minima and maxima in S, and since S is connected (see the remark after Theorem 2.7), they should take all intermediate values. It is clear that the minimum is 0 for all n (consider the identity function). Finding their maxima turned out to be a far more complicated problem.

The first step in this direction was made by Bieberbach [Bie16] in 1916.

Theorem 1.16 (Bieberbach)**.** *If $f \in S$, then $|a_2| \leq 2$, with equality if and only if f is a rotation of the Koebe function.*

This leads to the so-called Bieberbach conjecture, namely that $|a_n| \leq n$ for all $f \in S$.

In 1923 Loewner [Löw23] used his parametric method, which will be introduced in Chapter 2, to prove the Bieberbach conjecture for $|a_3|$. In 1955 Garabedian and Schiffer proved that $|a_4| \leq 4$ by combining variational and Loewner's method. The Bieberbach conjecture for $|a_6|$ was proved in 1968 by Pederson [Ped69] and, independently, by Ozawa [Oza69]. In 1972 Pederson and Schiffer [PS72] proved the Bieberbach conjecture for $|a_5|$. Finally, in 1984 de Branges [dB84] presented his proof for all coefficients. The final version of the proof was published in 1985 [dB85]. Again, de Branges' proof made considerable use of Loewner's parametric method.

Recall that a set $E \subset \mathbb{C}$ is called starlike with respect to a point $w_0 \in E$ if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E. The following result was first obtained by Grunsky [Gru34], and then reproved by using Loewner's method by Goluzin [Gol36a], [Gol36b].

Theorem 1.17 (Radius of starlikeness)**.** *For every radius $r \leq \rho = \tanh(\pi/4)$, each function $f \in S$ maps the disc $|z| < r$ onto a domain starlike with respect to the origin. This is false in general for every $r > \rho$.*

At the end of this section we mention one more result obtained with Loewner's method.

Theorem 1.18 (Rotation Theorem, Goluzin [Gol36b])**.** *For each $f \in S$,*

$$|\arg f'(z)| \leq \begin{cases} 4\arcsin|z|, & |z| \leq 1/\sqrt{2}, \\ \pi + \log\frac{|z|^2}{1-|z|^2}, & |z| > 1/\sqrt{2}. \end{cases}$$

The bound is sharp for each $z \in \mathbb{D}$.

1.4 Carathéodory Convergence Theorem

The Carathéodory theorem [Car12] establishes the connection between convergent sequences of univalent functions and the geometric properties of their image domains in the complex plane.

This theorem uses a special notion of domain convergence, so-called Carathéodory kernel convergence , which we will define now.

Let $D_1, D_2, \ldots$ be a sequence of simply connected domains in the complex plane, such that each of them contains zero. If there exists a disc $|w| < \rho$ belonging to all of the domains D_n, then we define *the kernel of the sequence* $\{D_n\}$ to be the largest domain D containing zero and having the property that each compact subset of D belongs to all but a finite number of the domains $\{D_n\}$. On the other hand, if there is no such disc, then we define the kernel to be $D = \{0\}$ ("degenerate kernel"). The sequence $\{D_n\}$ is then said to *converge* to D in the Carathéodory sense if every its subsequence has the same kernel D.

The Carathéodory theorem states, roughly speaking, that the Carathéodory kernel convergence of domains corresponds to the local uniform convergence of the associated Riemann maps. The precise formulation is given below.

Theorem 1.19 (Carathéodory Convergence Theorem)**.** *Let $\{D_n\}$ be a sequence of simply connected domains with $0 \in D_n \subsetneqq \mathbb{C}, n = 1, 2, \ldots$. Let f_n map the unit disc $\mathbb{D}$ conformally onto D_n and satisfy $f_n(0) = 0$ and $f_n'(0) > 0$. Let D be the kernel of $\{D_n\}$. Then $f_n \to f$*

locally uniformly in $\mathbb{D}$ if and only if $D_n \to D \neq \mathbb{C}$. In the case of convergence there are two possibilities. If $D = \{0\}$, then $f \equiv 0$. If $D \neq \{0\}$, then D is a simply connected domain, f maps $\mathbb{D}$ conformally onto D, and the inverse functions $f_n^{-1} \to f^{-1}$ converge locally uniformly in D.

One can also introduce the notion of Carathéodory kernel convergence of continuous-parameter families of domains $\{D_t\}$, $t \in [a,b]$, rather than discrete sequences D_n, $n = 1, 2, \dots$. A family $\{D_t\}$, $t \in [a,b]$ of domains is said to converge to D in the Carathéodory sense as $t \to t_0$, if for any sequence $\{t_n\} \subset [a,b]$, $t_n \to t_0$, D is the kernel of D_{t_n}.

Using this notion we can formulate the continuous version of the Carathéodory theorem (see, e. g., [Ale76] for details):

Theorem 1.20. *Let $\{D_t\}$, $t \in [a,b]$ be a family of simply connected domains with $0 \in D_t \subsetneqq \mathbb{C}, t \in [a,b]$. Let $f(z,t)$ map the unit disc $\mathbb{D}$ conformally onto D_t and satisfy $f(0,t) = 0$ and $f'(0,t) > 0$. Let D_t converge to D_{t_0} in the Carathéodory sense as $t \to t_0$. Then $f(z,t) \to f(z,t_0)$ locally uniformly in z in $\mathbb{D}$ as $t \to t_0$ if and only if $D_t \to D_{t_0} \neq \mathbb{C}$ in the Carathéodory sense as $t \to t_0$. In the case of convergence there are two possibilities. If $D_{t_0} = \{0\}$ then $f(z,t_0) \equiv 0$. If $D_{t_0} \neq \{0\}$, then D_{t_0} is a simply connected domain, $f(z,t_0)$ maps $\mathbb{D}$ conformally onto D_{t_0}, and $f^{-1}(w,t) \to f^{-1}(w,t_0)$ converges locally uniformly in D_{t_0}.*

A *single-slit map* is a function which maps a domain conformally onto the complex plane minus a Jordan arc extended to ∞. An important consequence of the Carathéodory theorem which underlies the Loewner parametric method is density of single-slit maps in the class S, which can be formulated as a theorem.

Theorem 1.21. *To each function $f \in S$ there corresponds a sequence of single-slit maps $\{f_n\} \subset S$, such that $f_n \to f$ uniformly on each compact subset of $\mathbb{D}$.*

This fact makes it possible to reduce the problem of finding extremal values of a real functional over the whole class S to the problem of finding its infimum or supremum over the class of single-slit maps. In particular, this was used in the proof of the Bieberbach conjecture.

1.5 Conformal maps of $\mathbb{D}$ onto $\mathbb{H}$

Simple formulas from this section will be used several times in Chapter 4.

Conformal maps of the upper half-plane $\mathbb{H}$ onto $\mathbb{D}$ which send $a \in \mathbb{H}$ to 0 have the form

$$w = e^{i\alpha}\frac{z-a}{z-\bar{a}}.$$

The inverse map is

$$g(z) = \frac{a - \bar{a}ze^{-i\alpha}}{1 - ze^{-i\alpha}}.$$

The derivative is

$$g'(z) = \frac{2e^{i\left(\frac{\pi}{2}-\alpha\right)} \operatorname{Im} a}{(1 - ze^{-i\alpha})^2}.$$

Finally,

$$g'(0) = 2e^{i\left(\frac{\pi}{2}-\alpha\right)} \operatorname{Im} a, \tag{1.2}$$

$$|g'(0)| = 2 \operatorname{Im} a. \tag{1.3}$$

$$\arg g'(0) = \frac{\pi}{2} - \alpha. \tag{1.4}$$

1.6 Lipschitz Continuity

The condition described in this section was first introduced by Lipschitz [Lip64] in 1864 as a sufficient condition for convergence of a Fourier series of a given function. Proofs of the properties listed here can be found, for example, in [Nat64].

A function $f : [a, b] \to \mathbb{R}$ is said to be *Lipschitz continuous* of order $\alpha > 0$ (or, sometimes, Hölder continuous of order $\alpha > 0$) if there exists a real positive constant c such that

$$|f(s) - f(t)| \leq c|s - t|^{\alpha}, \quad \text{for all} \quad a < s, t < b.$$

We denote the smallest c by $\|f\|_{\alpha}$. Note, that this is a seminorm rather than a norm, since $\|f\|_{\alpha} = 0$ for any constant function.

We denote the class of functions which are Lipschitz continuous of order α by $\operatorname{Lip}(\alpha)$.

Lipschitz continuity can be characterized in terms of modulus of continuity. Recall, that the *modulus of continuity* or *oscillation* of a function $f : [a, b] \to \mathbb{R}$ is defined as

$$\operatorname{osc}(f, \delta) = \sup_{|x-y|\leq\delta} \{|f(x) - f(y)|\}.$$

It is not hard to show that f is Lipschitz continuous of order α if and only if $\operatorname{osc}(f, \delta) \leq c\delta^{\alpha}$.

If a function is Lipschitz continuous of order $\alpha \geq 1$, then it follows that it is everywhere differentiable and its derivative is zero, thus, the function is constant. That is why only the case $0 < \alpha \leq 1$ is usually considered.

If $\alpha < \beta$ and f is given on a finite interval, then $f \in \operatorname{Lip}(\beta) \Rightarrow f \in \operatorname{Lip}(\alpha)$. It is not true if the interval is infinite, for example, $f(x) = x$, $x \in \mathbb{R}$ is in $\operatorname{Lip}(1)$ but not in $\operatorname{Lip}(1/2)$.

If f is everywhere differentiable and $|f'(x)| \leq M$ then $f \in \operatorname{Lip}(1)$ and $\|f\|_1 \leq M$.

If a function is Lipschitz continuous of some order α, then it is uniformly continuous. Functions in $\operatorname{Lip}(1)$ are also absolutely continuous.

We also use the so-called *local Lipschitz characteristic* of order α, defined by

$$\|f\|_{\alpha_{loc}} = \inf_{\epsilon>0} \sup_{|t-s|<\epsilon} \frac{|f(t) - f(s)|}{|t - s|^{\alpha}},$$

and the local Lipschitz condition of order α, namely $\|f\|_{\alpha_{loc}} \leq M$. We can replace $\inf_{\epsilon>0}$ in the definition by $\lim_{\epsilon\to 0^+}$, since the set over which we take the supremum shrinks as ϵ decreases.

This local condition is weaker then the global Lipschitz condition, since if $\|f\|_\alpha \leq C$, then

$$\sup_{|t-s|<\epsilon} \frac{|f(t)-f(s)|}{|t-s|^\alpha} \leq C.$$

If f is differentiable and its derivative is bounded, then $\|f\|_{\alpha_{loc}} = 0$ for $\alpha < 1$. Indeed, by the mean-value theorem,

$$\inf_{\epsilon>0} \sup_{|t-s|<\epsilon} \frac{|f(t)-f(s)|}{|t-s|^\alpha} = \inf_{\epsilon>0} \sup_{|t-s|<\epsilon} \frac{|f'(\theta)||t-s|}{|t-s|^\alpha} \leq \inf_{\epsilon>0} \sup_{|t-s|<\epsilon} M|t-s|^{1-\alpha} = \inf_{\epsilon>0} M\sqrt{\epsilon} = 0.$$

We will be particularly interested in the case when f is differentiable, but the derivative is unbounded at one point, say at 0, i. e., $\lim_{t\to 0} |f'(t)| = \infty$, but, at the same time $\lim_{t\to 0} \frac{|f(t)-f(0)|}{|t|^\alpha} = C < \infty$. By l'Hôpital's rule, this is equivalent to $\lim_{t\to 0} \frac{1}{\alpha}|f'(t)|t^{1-\alpha} = C$. Note, that for each fixed $s > 0$ by the mean-value theorem

$$\lim_{\epsilon\to 0} \sup_{|t-s|<\epsilon} \frac{|f(t)-f(s)|}{|t-s|^\alpha} = 0.$$

On the other hand, if $s = 0$,

$$\lim_{\epsilon\to 0} \sup_{|t|<\epsilon} \frac{|f(t)-f(0)|}{|t|^\alpha} = \lim_{t\to 0} \frac{|f(t)-f(0)|}{|t|^\alpha} = C.$$

We conclude that

$$\inf_{\epsilon>0} \sup_{|t-s|<\epsilon} \frac{|f(t)-f(s)|}{|t-s|^\alpha} = \lim_{t\to 0} \frac{|f(t)-f(0)|}{|t|^\alpha} = C.$$

Chapter 2

The Loewner Equation of Different Types

In this chapter we give a general overview of the Loewner theory. We start by defining radial Loewner chains, and then we introduce the Loewner-Kufarev ODE and PDE. We also present derivation of its special case, namely the Loewner equation for single-slit chains. After that we introduce a related equation, known as the chordal Loewner ODE. Finally, we give a quite informal introduction to the stochastic Loewner equation (SLE), a topic, which has gained immense popularity recently.

In the first two sections we follow primarily the classical monograph by Pommerenke [Pom75]. The discussion of the radial Loewner equation is mostly based on [Dur83], and the sections on the chordal equation and SLE are based on [Law05].

2.1 Radial Loewner Chains

We begin with considering families of domains $\{D_\tau\}$, $0 \leq \tau < \infty$ satisfying the following three conditions

- $0 \in D_\sigma \subsetneqq D_\tau \quad (0 \leq \sigma < \tau < \infty)$,
- $D_{\tau_n} \to D_{\tau_0}$ as $\tau_n \to \tau_0 < +\infty$, $D_{\tau_n} \to \mathbb{C}$ as $\tau_n \to +\infty$ (convergence of domains is understood in the Carathéodory sense),
- D_0 has conformal radius 1 with respect to 0.

We define the function $F(z,\tau)$ for each τ to be the Riemann map of the unit disc onto D_τ (i. e., these are conformal maps normalized by the conditions $F(0,\tau) = 0$, $F'(0,\tau) > 0$). The first coefficient in the expansion of $F(z,\tau)$, $a_1(\tau) = F'(0,\tau)$ is a strictly increasing function by the subordination principle (Theorem 1.10). By the Carathéodory convergence theorem (Theorem 1.19) and the Weierstrass theorem (Theorem 1.2) all the coefficients $a_n(\tau)$ are continuous functions of τ, in particular, $a_1(\tau)$. We can thus make a reparametrization $e^t = a_1(\tau)$ and with this new parameter t we have the following Taylor expansion for

$F(z,t)$

$$F(z,t) = e^t z + a_2(t)z^2 + \dots .$$

Now we can give the definition of a radial Loewner chain.

Definition 2.1. *A function $F(z,t) : \mathbb{D} \times [0,+\infty) \to \mathbb{C}$ is called a radial Loewner chain if*

- *for each fixed $t \in [0,+\infty)$, $F(z,t)$ is analytic and univalent in $\mathbb{D}$ as a function of z,*
- *$F(\mathbb{D},s) \subset F(\mathbb{D},t)$ for $0 \le s < t < +\infty$,*
- *for each $t \ge 0$*
$$F(z,t) = e^t z + a_2(t)z^2 + \dots, \quad z \in \mathbb{D}.$$

So far, it is not clear why these three conditions suffice for the domains $F(\mathbb{D},t)$ to behave as described in the beginning of the section. This is, however, an easy consequence of the following lemma.

Lemma 2.2. *Let $F(z,t)$ be a radial Loewner chain. Then, for $0 \le s \le t < \infty$*

$$|F(z,s) - F(z,t)| \le \frac{8|z|}{(1-|z|)^4}(e^t - e^s), \quad |z| < 1.$$

This means that $F(z,s) \to F(z,t)$ as $s \to t$ locally uniformly in $\mathbb{D}$ as functions of z, and thus $F(\mathbb{D},s) \to F(\mathbb{D},t)$ in the Carathéodory sense. Also, from the Koebe one-quarter theorem it follows that $F(\mathbb{D},t) \to \mathbb{C}$ as $t \to \infty$. Indeed, $e^{-t}F(z,t) \in S$, and thus, $F(\mathbb{D},t)$ contains the disc $\{w : |w| < e^t/4\}$.

Loewner chains form a compact family. In other words,

Theorem 2.3. *Every sequence $F_n(z,t)$ of radial Loewner chains contains a subsequence which converges to a radial Loewner chain $F(z,t)$ uniformly on compact subsets of $\mathbb{D}$ for each $t \ge 0$.*

This theorem leads to the following result.

Theorem 2.4. *For every function $f \in S$ there exists a radial Loewner chain $F(z,t)$, such that $F(z,0) \equiv f(z)$, $z \in \mathbb{D}$.*

We will also use the two-parametric family of conformal maps $\phi(z,s,t) := F^{-1}(F(z,s),t)$ $0 \le s \le t < +\infty$. We will be particularly interested in $f(z,t) := \phi(z,0,t)$. It is easy to see that $\phi(z,s,t)$ maps the unit disc conformally onto some subset of the unit disc, i. e., $|\phi(z,s,t)| < |z|$, $|z| < 1$.

Due to the normalization of $F(z,t)$, we have

$$\phi(z,s,t) = e^{s-t}z + \dots .$$

It is important to note that the family $\phi(z,s,t)$ possesses the semigroup property, i. e.,

1. $\phi(z,s,\tau) = \phi(\phi(z,s,t),t,\tau), \quad 0 \le s \le t \le \tau < \infty,$
2. $\phi(z,s,s) \equiv z, \quad z \in \mathbb{D}.$
3. $\phi(z,s,t) \to z$ locally uniformly in $\mathbb{D}$, as $t \to s$.

2.2 The Radial Loewner-Kufarev ODE and PDE

The cornerstone of the Loewner theory is the fact that Loewner chains can be represented as the solutions to certain differential equations. This was first noted by Loewner [Löw23] in 1923 for the special case of single-slit subordination chains (see Chapter 2.3), then generalized by Kufarev [Kuf43] in 1943 for general Loewner chains and studied further by Pommerenke [Pom65] and others. We will formulate this characterization in several theorems.

We will say that a function $p(z)$ belongs to the Carathéodory class ($p \in C$) if p is analytic in $\mathbb{D}$, normalized as $p(z) = 1 + p_1 z + p_2 z^2 + \dots$, $z \in \mathbb{D}$, and $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$.

Theorem 2.5. *Let $F(z,t)$ be a Loewner chain. Then there exists a function $p(z,t)$, measurable in t for every fixed $z \in \mathbb{D}$ and such that for every fixed $t \geq 0$, $p(z,t) \in C$ and for almost all $t \geq 0$*

$$\dot{F}(z,t) = zF'(z,t)p(z,t). \tag{2.1}$$

We will say that $p(z,t)$ generates the subordination chain $F(z,t)$. The equation (2.1) is known as the *Loewner-Kufarev PDE*.

Theorem 2.6. *Suppose $p(z,t) \in C$ for each $t \geq 0$ and is measurable in t. Then for each $z \in \mathbb{D}$ and $s \geq 0$ fixed, the Cauchy problem*

$$\frac{dw}{dt} = -wp(w,t) \quad \text{for almost all } t \in [s, +\infty), \tag{2.2}$$

$$w(s) = z,$$

has a unique absolutely continuous solution $w(t)$ denoted by $\phi(z,s,t)$. The functions $\phi(z,s,t)$ are univalent in $z \in \mathbb{D}$, and the limit

$$F(z,s) = \lim_{t\to\infty} e^t \phi(z,s,t), \quad z \in \mathbb{D}, s \geq 0$$

exists as the local uniform limit in $\mathbb{D}$. The function $F(z,t)$ is a Loewner chain and it satisfies the Loewner-Kufarev PDE

$$\dot{F}(z,t) = zF'(z,t)p(z,t)$$

with the same function $p(z,t)$.

The equation (2.2) is called the Loewner-Kufarev ODE. The existence and uniqueness of its solution can be proved by the usual Picard-Lindelöf iterations.

The converse is also true.

Theorem 2.7. *Let $F(z,t)$ be a Loewner chain, $p(z,t)$ be the function generating it. Put $\phi(z,s,t) := F^{-1}(F(z,s),t)$, $0 \leq s \leq t < +\infty$. Then, for almost all $t \in [s,\infty)$,*

$$\frac{\partial}{\partial t}\phi(z,s,t) = -\phi(z,s,t)p(\phi(z,s,t),t), \quad z \in \mathbb{D},$$

and

$$F(z,s) = \lim_{t\to\infty} e^t \phi(z,s,t), \quad z \in \mathbb{D}, \quad s \geq 0.$$

One of the consequences of this theorem is path-connectedness of S. Indeed, let $f \in S$. By Theorem 2.4, there exists a subordination chain $F(z,t)$ such that $F(z,0) \equiv f(z)$. Define $\phi(z,s,t) = F^{-1}(F(z,s),t)$. Theorem 2.7 says that $f(z) = \lim_{t\to\infty} e^t \phi(z,0,t)$. All the functions $e^t\phi(z,0,t)$ belong to S, and they actually form a curve in S, one of the endpoints of which is f, the other is the identity function. Continuity of the curve is a consequence of the Growth theorem (Theorem 1.15). Thus, we have just shown that every function in S can be connected to the identity function by a continuous curve, and S is thus connected in the local uniform convergence topology.

Finally, we mention a theorem which gives a necessary and sufficient condition for the existence, uniqueness and univalence of the solution to the Loewner-Kufarev PDE (see [Pom75], [PV06]).

Theorem 2.8. *The solution to the Loewner-Kufarev PDE*

$$\dot{F}(z,t) = zF'(z,t)p(z,t)$$

exists, is unique and univalent for all $t \geq 0$, if and only if, the initial condition is given as

$$F(z,0) = \lim_{t\to\infty} e^t f(z,t),$$

where $f(z,t) = \phi(z,0,t)$ is the solution to the Loewner-Kufarev ODE

$$\frac{df}{dt} = -fp(f(z,t),t),$$

$$f(z,0) = z.$$

2.3 The Radial Loewner Equation

In this section we consider one of the simplest and most important examples of Loewner chains, known as single-slit subordination chains. They were also historically the first example of Loewner chains and were studied by Loewner himself [Löw23] in 1923. The proofs and other details can be found in [Ahl73], [Dur83], [Gol69], [Hay94] or [Law05].

We define single-slit subordination chains geometrically. First, we take a Jordan arc $\Gamma : [0,+\infty) \to \mathbb{C}$ that does not pass through 0 and such that $\Gamma(t) \to \infty$ as $t \to \infty$. We can assume that $\mathbb{C} \setminus \Gamma$ has conformal radius 1 with respect to zero, otherwise we can just rescale it appropriately. We denote by Γ_t the restriction of Γ to $[t,+\infty)$ ("partially erased slit") and put $D_t = \mathbb{C} \setminus \Gamma_t$. We can also assume that Γ is parameterized in such a way that D_t has conformal radius e^t, otherwise we can make a reparameterization, as it was remarked at the beginning of Section 2.1.

The family of conformal maps $F(z,t)$ associated with the domains $\{D_t\}$ constructed as above is called a single-slit subordination chain (Figure 2.1).

Each of the maps $F(\cdot,t)$ can be continuously extended to the closed disc $\overline{\mathbb{D}}$. We denote by $e^{iu(t)}$ the point on the unit circle which is mapped to the tip of the slit by the extended

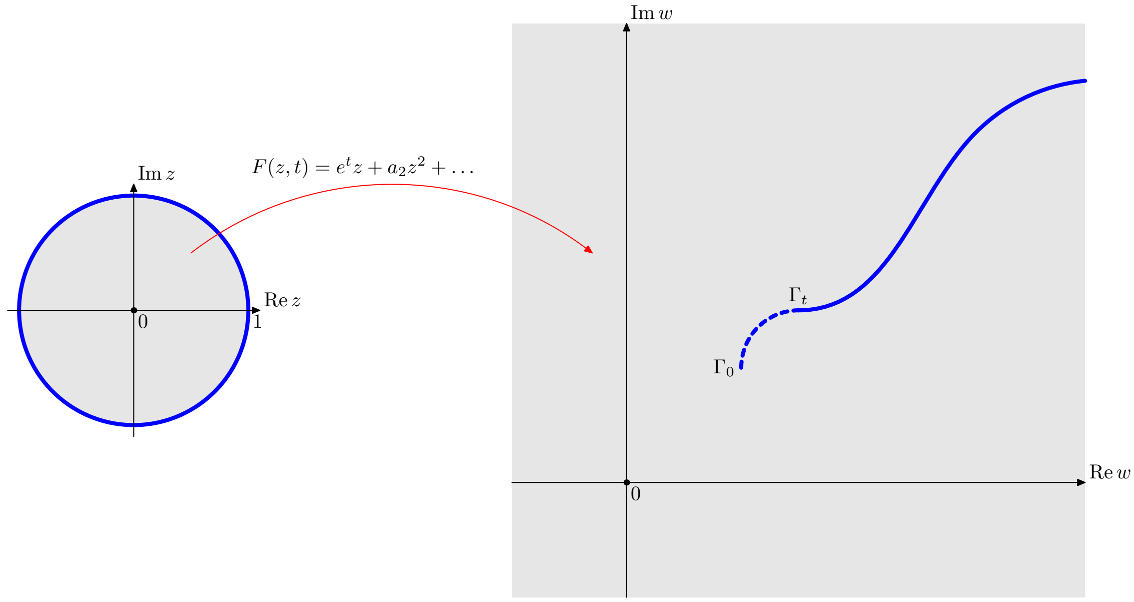

Figure 2.1: A single-slit subordination chain

map, i. e., $F(e^{iu(t)}, t) = \Gamma(t)$. The real-valued function $u(t)$ is called *the driving term* (or the driving function, or, sometimes, *forcing*).

It can be shown that $u(t)$ is a continuous function. Moreover, it turns out that the function $p(z,t)$ generating the corresponding single-slit subordination chain is of the form

$$p(z,t) = \frac{e^{iu(t)} + z}{e^{iu(t)} - z}.$$

We thus have the following two differential equations for $F(z,t)$ and $f(z,t) = F^{-1}(F(z,0),t)$.

$$\dot{F}(z,t) = zF'(z,t)\frac{e^{iu(t)} + z}{e^{iu(t)} - z}, \tag{2.3}$$

$$\dot{f}(z,t) = -f(z,t)\frac{e^{iu(t)} + f(z,t)}{e^{iu(t)} - f(z,t)}. \tag{2.4}$$

We outline here the derivation of (2.4). In order to do this, one should consider the function $\phi(z,s,t) = F^{-1}(F(z,s),t)$, $s < t$, where $F(z,t)$ are the maps constituting the Loewner chain, extended continuously to $\overline{\mathbb{D}}$. We denote $B_{st} = F^{-1}(\Gamma([s,t]),s)$, $J_{st} = F^{-1}(\Gamma([s,t]),t)$. J_{st} is a closed Jordan arc in the unit circle, containing $e^{iu(t)}$. B_{st} is a closed Jordan arc lying inside the closed unit disc and meeting the unit circle at $e^{iu(t)}$. Of course, $J_{st} = \phi(B_{st},s,t)$. The unit disc $\mathbb{D}$ is mapped onto $\mathbb{D} \setminus J_{st}$ by $\phi(z,s,t)$.

We introduce the function

$$\Phi(z) = \log\frac{\phi(z,s,t)}{z},$$

where the branch of the logarithm is chosen so that $\Phi(0) = s - t$. This function is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. Note, that $\operatorname{Re}\Phi(z) = 0$ everywhere on the unit circle except B_{st}. By the Poisson formula,

$$\Phi(z) = \frac{1}{2\pi}\int_{\alpha}^{\beta} \operatorname{Re}(\Phi(e^{i\theta}))\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta, \tag{2.5}$$

where $e^{i\alpha}$ and $e^{i\beta}$ are the endpoints of B_{st}. When $z = 0$, we have

$$s - t = \frac{1}{2\pi}\int_{\alpha}^{\beta} \operatorname{Re}\Phi(e^{i\theta})d\theta. \tag{2.6}$$

Due to the property $\phi(\phi(z,s,t),t,\tau) = \phi(z,s,\tau)$, substituting $f(z,s)$ for z in (2.5), we obtain

$$\log\frac{f(z,t)}{f(z,s)} = \frac{1}{2\pi}\int_{\alpha}^{\beta} \operatorname{Re}(\Phi(e^{i\theta}))\frac{e^{i\theta}+f(z,s)}{e^{i\theta}-f(z,s)}d\theta.$$

We apply then the mean-value theorem to the real and imaginary parts of the integral, and obtain

$$\log\frac{f(z,t)}{f(z,s)} = \frac{1}{2\pi}\left[\operatorname{Re}\frac{e^{i\sigma}+f(z,s)}{e^{i\sigma}-f(z,s)} + i\operatorname{Im}\frac{e^{i\tau}+f(z,s)}{e^{i\tau}-f(z,s)}\right]\int_{\alpha}^{\beta}\operatorname{Re}\Phi(e^{i\theta})d\theta,$$

where $e^{i\sigma}, e^{i\tau} \in B_{st}$. We divide then this equality by $t-s$ and let $t \downarrow s$. Since B_{st} contracts to $\lambda(s)$, we obtain

$$\lim_{t\to s+0}\frac{1}{t-s}\log\frac{f(z,t)}{f(z,s)} = -\frac{\lambda(s)+f(z,s)}{\lambda(t)-f(z,t)},$$

and finally,

$$\frac{\partial}{\partial s}g(z,s) = -\frac{\lambda(s)+f(z,s)}{\lambda(t)-f(z,t)}.$$

The derivative in the last equation is the right-hand side derivative, but the same reasoning works also for the left-hand side derivatives.

General results from Section 2.2 imply that every single-slit map can be represented as $\lim_{t\to\infty} e^t f(z,t)$, where $f(z,t)$ solves (2.4) with some continuous driving term $u(t)$. Together with the fact that single-slit maps form a dense subclass of the class S, this constitutes the essence of the Loewner parametric method.

We will study connections between analytic properties of driving terms and geometry of corresponding Loewner chains in more details later, in Chapter 3.

2.4 The Chordal Loewner Equation

Early works where the half-plane version of the Loewner equation was studied include [Pop49], [Pop54], [Sob70], [KSS68]. The interest to this version of the Loewner equation grew considerably after the Schramm-Loewner evolution was introduced [Sch00] in 2000.

This equation turned out to be a natural continuous model for studying loop-erased random walks and percolation.

There are several differences between the half-plane (chordal) and the disc (radial) versions of the Loewner equation. The most essential one is that in the chordal case normalization is given at the domain's boundary point, instead of internal one in the radial version. Strictly speaking, the chordal Loewner equation is not a special case of the Loewner-Kufarev equation. A generalization of the Loewner-Kufarev equation which includes both the radial and the chordal cases was constructed [BCDM08, BCDM09] in 2008.

In this section we follow Lawler's book [Law05], where the reader can find the details omitted.

Let $\Gamma : [0, +\infty) \to \mathbb{C}$ be a simple curve with $\Gamma(0) \in \mathbb{R}$ and $\Gamma(0, \infty) \subset \mathbb{H}$. For each $t \geq 0$ put $D_t = \mathbb{H} \setminus \Gamma[0, t]$. That is, the domains D_t shrink as t increases, not expand, as it was in the radial case.

As it was mentioned in Section 1.2, there exists a unique conformal map g_t of D_t onto $\mathbb{H}$ such that

$$g_t(z) = z + \frac{b(t)}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty.$$

Here $b(t)$ is the half-plane capacity $\operatorname{hcap}(\Gamma[0, t])$. We will assume that Γ is parametrized in such a way that $b(t) \in C^1$.

The map g_t can be continuously extended to the tip of the slit $\Gamma(t)$, and the function $\lambda(t) = \lim_{z \to \Gamma(t), z \in \mathbb{H} \setminus \Gamma[0,t]} g_t(z)$ is called, by analogy with the radial case, the driving term. Again, it can be proved that the driving term is a continuous function. The main result is that $g_t(z)$ solves the Cauchy problem

$$\dot{g}_t(z) = \frac{\dot{b}(t)}{g_t(z) - \lambda(t)}, \quad g_0(z) = z. \tag{2.7}$$

If $z \notin \Gamma$, (2.7) holds for all $t \geq 0$. If $z = \Gamma(t_0)$, this holds for $t < t_0$.

It is usual to parameterize Γ in such a way that $b(t) = 2t$ ("hydrodynamic normalization"), and in this case (2.7) becomes

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z. \tag{2.8}$$

Now, let us proceed in the opposite direction. Let $\lambda(t)$ be some continuous driving term. We solve the initial-value problem (2.8) for each $z \in \mathbb{H}$ and denote by τ_z the supremum of all t, such that the solution is well defined up to t, $g_t(z) \in \mathbb{H}$. Then, we denote $D_t = \{z : \tau_z > t\}$. It can be proved that g_t maps D_t conformally onto $\mathbb{H}$.

Note, that D_t does not have to be of the form $\mathbb{H} \setminus \Gamma[0, t]$, where Γ is some curve growing from the real line. In any case, $\{D_t\}$ is a shrinking family of domains. This is similar to what we have seen in the radial case.

2.5 An Example: Calculating the Driving Term for the Given Slit in $\mathbb{H}$

Let us give an example showing how the driving term corresponding to the given slit can be found in the chordal case.

Take the slit Γ to be the ray going from the origin at the angle $\pi\theta$ to the positive direction of the real line (Figure 2.2). The function $g_t(z)$ maps $D_t = \mathbb{H} \setminus \Gamma[0,t]$ conformally onto $\mathbb{H}$. Parameterization of Γ will be chosen later, in such a way that the normalization condition

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty$$

is satisfied.

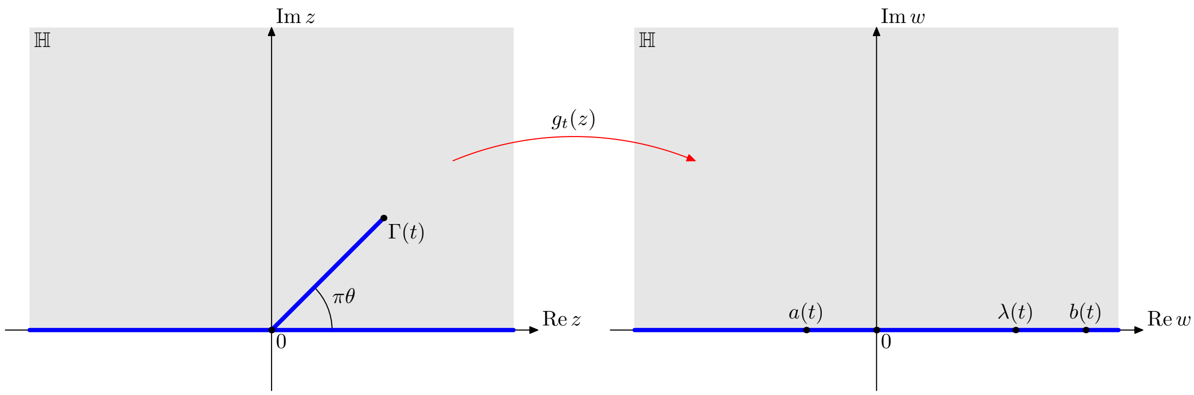

Figure 2.2: An illustration for the example

We denote by $\chi_t(w)$ the inverse map $g_t^{-1}(w)$. By matching coefficients in the identity $g_t(\chi_t(w)) = w$ we find the following normalization for $\chi_t(w)$

$$\chi_t(w) = w - \frac{2t}{w} + O\left(\frac{1}{|w|^2}\right), \quad w \to \infty. \tag{2.9}$$

If we differentiate the identity $\chi_t(g_t(z)) \equiv z$ with respect to t we can get the following expression for the driving term

$$\lambda(t) = w + 2\frac{\chi_t'(w)}{\dot{\chi}_t(w)}. \tag{2.10}$$

In order to find the map $\chi_t(w)$ we use the Schwarz-Christoffel formula which gives us the map

$$\chi_{a,b}(w) = (w-a)^{1-\theta}(w-b)^{\theta}, \quad a < b$$

of $\mathbb{H}$ onto D_t for some t. Here $a, b \in \mathbb{R}$ are the inverse images of the starting point of the slit. The inverse image of slit's tip is

$$\chi_{a,b}^{-1}(\Gamma(t)) = b - \theta(b-a).$$

The tip of the slit is $\Gamma(t) = (b-a)(1-\theta)^{1-\theta}\theta^{\theta}e^{\pi i\theta}$. We choose $a = a(t)$ and $b = b(t)$ so that (2.9) is satisfied. That is,

$$a(t) = -2\sqrt{\frac{t\theta}{1-\theta}}, \quad b(t) = 2\sqrt{\frac{t(1-\theta)}{\theta}},$$

$$\chi_t(w) = \left(w + 2\sqrt{\frac{t\theta}{1-\theta}}\right)^{1-\theta}\left(w - 2\sqrt{\frac{t(1-\theta)}{\theta}}\right)^{\theta}.$$

The correct parameterization of the slit is thus $\Gamma(t) = 2\sqrt{t}\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{2}-\theta} e^{\pi i\theta}$. The driving term can be found as $\lambda(t) = g_t(\Gamma(t)) = \chi_t^{-1}(\Gamma(t))$. That is, $\lambda(t) = 2\sqrt{t}\frac{1-2\theta}{\sqrt{\theta(1-\theta)}}$. We get the same result using the formula (2.10).

2.6 Schramm-Loewner Evolution

Schramm-Loewner evolution (SLE) is the main motivation behind the revived interest in the Loewner equation in the 21st century. Even though we do not study SLE in details in this thesis, we still include a brief section providing basic facts about the stochastic version of the Loewner equation. More information on this topic, including prerequisites from stochastic calculus, the reader can find in Lawler's book [Law05], which is an excellent introduction to SLE.

There is a number of two-dimensional discrete random models widely used in statistical physics: Ising's model, percolation, loop-erased random walk, to name a few. For quite a long time physicists were unsuccessfully looking for continuum (scaling) limits of those models, but this problem seemed to be quite hard.

In 2000 Schramm [Sch00] introduced a stochastic version of the Loewner equation. Namely, he took $\sqrt{k}B_t$ as a driving term (B_t here denotes the standard Brownian motion) and considered the equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{k}B_t}, \quad g_0(z) = z, \quad z \in \mathbb{H}. \tag{2.11}$$

$$Z_t + \sqrt{k}B_t - z = \int_0^t \frac{2}{Z_t}dt,$$

or in the Itô form

$$dZ_t = \frac{2}{Z_t}dt - \sqrt{k}dB_t, \quad Z_0 = z.$$

This stochastic differential equation is known as the Bessel equation .

Solving (2.11) one gets a random collection of conformal maps $g_t(z) : D_t \to \mathbb{H}$ denoted as SLE_k. It can be shown that D_t are the unbounded components of $\mathbb{H} \setminus \Gamma[0,t]$ with probability one, where Γ is some curve, not necessarily Jordan.

Properties of Γ depend on k qualitatively. If $0 < k \leq 4$, Γ is simple and $\Gamma(0, +\infty) \subset \mathbb{H}$. If $4 < k < 8$, Γ has self-intersections but not self-crossings and it does not fill up the plane. If $k \geq 8$, it fills $\overline{\mathbb{H}}$. In 2008 Beffara [Bef08] showed that the Hausdorff dimension of Γ for $k \leq 8$ is given by $\dim(\Gamma[0, t]) = 1 + \frac{k}{8}$. For $k \geq 8$ the Hausdorff dimension is 2, since Γ is space-filling.

Schramm proved that if some discrete process has a conformally invariant scaling limit, the limit must be SLE_k for some k. It was found later that for the loop-erased random walk $k = 2$, for percolation $k = 6$, for uniform spanning trees $k = 8$, for interfaces for the Gaussian free field and for the harmonic explorer $k = 4$. It is conjectured that the interfaces for the Ising model converge to SLE_3, and that the scaling limit of the self-avoiding walk is $SLE_{8/3}$. It was also shown [LSW01a] that the frontier of the Brownian motion is a version of $SLE_{8/3}$, and this proved Mandelbrot's conjecture about its Hausdorff dimension.

A recent overview of the current state of the SLE theory and a description of physical models motivating its development can be found in [Law09].

Chapter 3

The Driving Term of the Loewner Equation

Recall that in Section 2.3 we defined the driving term $u(t)$ as the argument of the inverse image under $F(z,t)$ of the tip of the slit Γ_t, i. e., $F(e^{iu(t)},t) = \Gamma(t)$. As we have seen, this information is sufficient for recovering the whole subordination chain $F(z,t)$, and, as a consequence, the geometry of the slit Γ by means of the Loewner equation. The real-valued function $u(t)$ thus encodes the geometric properties of the slit Γ in a way which is still not understood completely.

Every single-slit subordination chain is generated by some continuous driving term $u(t)$, but is the converse also true? Given a continuous driving term $u(t)$, does the Loewner equation always generate a single-slit subordination chain? General results from Section 2.2 guarantee that $F(\mathbb{D},t)$ is a Loewner chain. However, in 1947 Kufarev found a counterexample showing that it does not have to be a single-slit subordination chain, and up to now there is no known necessary and sufficient condition for $u(t)$ to generate a single-slit subordination chain.

In this chapter we summarize the results known up to now about the relations between analytic properties of the driving term and geometric properties of the corresponding Loewner chain.

3.1 Smoothness Properties and Simple Transformations

3.1.1 Analyticity and Smoothness

The driving term preserves in a certain sense analyticity of the slit Γ. It is known that if the slit Γ is analytic, then the driving term $u(t)$ is real analytic. Different versions of this fact were proved by Komatu [Kom41], Schiffer [Sch45], Brickman, Leung and Wilken [BLW83].

In 2001 Earle and Epstein [EE01] gave a new proof of this fact and also showed that if Γ is C^n then $u(t)$ is at least C^{n-1}.

These results are also valid for the chordal case.

3.1.2 Rotations around the origin (Radial)

Let $u(t)$ be the driving term corresponding to the slit $\Gamma(t)$. Let us find the driving term $\tilde{u}(t)$ corresponding to the slit $\tilde{\Gamma}(t) = e^{i\theta}\Gamma(t)$. The corresponding Loewner chain is given by $\tilde{F}(z,t) = e^{i\theta}F(e^{-i\theta}z,t)$, the inverse maps are $\tilde{F}^{-1}(w,t) = e^{i\theta}F^{-1}(e^{-i\theta}w,t)$. The corresponding driving term is thus given by

$$e^{i\tilde{u}(t)} = \tilde{F}^{-1}(\tilde{\Gamma}(t),t) = e^{i\theta}F^{-1}(e^{-i\theta}e^{i\theta}\Gamma(t),t) = e^{i\theta}e^{iu(t)}.$$

That is, $\tilde{u}(t) = u(t) + \theta$.

3.1.3 Reflection (Radial)

Let $\tilde{\Gamma} = \bar{\Gamma}$ be the reflection of Γ. The corresponding Loewner chain is $\tilde{F}(z,t) = \overline{F(\bar{z},t)}$, the inverse maps are $\tilde{F}^{-1}(w,t) = \overline{F^{-1}(\bar{w},t)}$. The driving term is defined by

$$e^{i\tilde{u}(t)} = \tilde{F}^{-1}(\tilde{\Gamma}(t),t) = \overline{F^{-1}\left(\bar{\tilde{\Gamma}}_t, t\right)} = \overline{F^{-1}\left(\Gamma_t, t\right)} = \overline{e^{iu(t)}}.$$

Thus, $\tilde{u}(t) = -u(t)$. As a consequence, the slits $[1/4, +\infty)$ and $(-\infty, -1/4]$ are generated by $u(t) \equiv 0$ and $u(t) \equiv \pi$, correspondingly.

3.1.4 Truncation (Radial)

Let us truncate Γ, that is, consider the slit $\Gamma[T,\infty)$, $T > 0$. In order to preserve normalization, we scale it by e^{-T}. Thus, we consider the slit $\tilde{\Gamma}(t) = e^{-T}\Gamma(t+T)$. The Loewner chain is given then by $\tilde{F}(z,t) = e^{-T}F(z,t+T)$, and the inverse maps by $\tilde{F}^{-1}(w,t) = F^{-1}(e^{T}w,t+T)$. Then,

$$e^{i\tilde{u}(t)} = F^{-1}(e^{T}\tilde{\Gamma}_t, t+T) = F^{-1}(\Gamma(t+T), t+T) = e^{iu(t+T)}.$$

That is, $\tilde{u}(t) = u(t+T)$.

It follows that the slits of the form $\Gamma(t) = \frac{1}{4}e^{i\theta}e^{t}$ are generated by constant driving terms. It also follows that if the driving term is periodic, i.e. $u(t) = u(t+T)$, $t > 0$, then the slit is self-similar, in the sense that $\Gamma(t+T) = e^{T}\Gamma(t)$, $t > 0$.

3.1.5 Scaling (Chordal)

Let $\Gamma(t)$ be a growing slit in $\mathbb{H}$. We consider another slit $\tilde{\Gamma}(t)$, which is obtained from $\Gamma(t)$ by scaling by r. Since $\mathrm{hcap}(rK) = r^2\mathrm{hcap}(K)$, where K is a compact hull, we need to make a time reparameterization, that is, $\tilde{\Gamma}(t) = r\Gamma\left(\frac{t}{r^2}\right)$. If $g_t(z)$ are the maps corresponding to

$\Gamma(t)$, then the maps corresponding to $\tilde{\Gamma}(t)$ are given by $\tilde{g}_t = rg_{\frac{t}{r^2}}\left(\frac{z}{r}\right)$, and the corresponding driving term is

$$\tilde{\lambda}(t) = \tilde{g}_t(\tilde{\Gamma}(t)) = rg_{\frac{t}{r^2}}\left(\frac{\tilde{\Gamma}(t)}{r}\right) = rg_{\frac{t}{r^2}}\left(\Gamma\left(\frac{t}{r^2}\right)\right) = r\lambda\left(\frac{t}{r^2}\right).$$

3.1.6 Truncation (Chordal)

As it is remarked in [LMR09], if $\lambda(t)$ is the driving term corresponding to $\Gamma(t)$, then the slit $\Gamma(t) = g_T(\Gamma(t))$ is generated by $\tilde{\lambda(t)} = \lambda(T+t)$.

It follows that the growing vertical slits in the chordal case are generated by constant driving terms.

3.1.7 Reflection (Chordal)

Let R_I denote reflection with respect to the imaginary axis,and $\tilde{\Gamma}(t) = R_I(\Gamma(t))$. Then $\tilde{g}_t(z) = R_I(g(R_I(z)))$, so $\tilde{\lambda}(t) = R_I(g(\Gamma_t)) = R_I(\lambda(t)) = -\lambda(t)$.

3.1.8 Translation (Chordal)

Let $\tilde{\Gamma}(t) = \Gamma(t) + x$, $x \in \mathbb{R}$. Then $\tilde{g}_t = g_t(z-x) + x$ and

$$\tilde{\lambda}(t) = \tilde{g}_t(\tilde{\Gamma}(t)) = g_t(\Gamma(t) + x - x) + x = \lambda(t) + x.$$

3.2 Kufarev's Sufficient Condition and Example

The first known nontrivial sufficient condition for a driving term to generate a single-list subordination chain was given by Kufarev [Kuf46] in 1946.

Theorem 3.1. *Let $u(t)$ have bounded derivative on a segment $[a,b]$, and let $\phi(z,s,t)$ denote the solution to the problem*

$$\frac{dw}{dt} = -w\frac{e^{iu(t)} + w}{e^{iu(t)} - w},$$
$$w(s) = z.$$

Then, for all s,t such that $a \le s \le t \le b$, the domain $\phi(\mathbb{D},s,t)$ is of the form $\mathbb{D} \setminus \gamma_{s,t}$, where $\gamma_{s,t}$ is a C^1-smooth Jordan arc.

Kufarev remarks that similar results can be proved for the chordal case.

Later, in 1947 Kufarev [Kuf47] gave the first known example of a non-slit solution to the Loewner equation. He integrated the Loewner ODE

$$\frac{dw}{dt} = -w\frac{e^{iu(t)} + w}{e^{iu(t)} - w}, \quad w(0) = z, \tag{3.1}$$

with the driving term, given by $e^{iu(t)} = \kappa^3$, where $\kappa = e^{-t} + i\sqrt{1-e^{-2t}}$, that is $u(t) = 3\arctan\sqrt{e^{2t}-1}$. Note, that $u(t) = 3\sqrt{2}\sqrt{t} - \frac{\sqrt{2}}{2}t\sqrt{t} + O(t^{\frac{5}{2}})$ and that $u'(t) = \frac{3}{\sqrt{e^{2t}-1}}$ is unbounded at $t = 0$. As t varies in $[0, +\infty)$, $u(t)$ increases from 0 to $\frac{3\pi}{2}$.

The solution to (3.1)

$$w = f(z,t) = \frac{\kappa}{\kappa^2+1}\left(z + \kappa^2 - \sqrt{(1-z)(\kappa^4 - z)}\right)$$
$$= \frac{1}{2}e^t\left(z + e^{\frac{2}{3}iu(t)} - \sqrt{(1-z)\left(e^{\frac{4}{3}iu(t)} - z\right)}\right),$$

maps the unit disc $\mathbb{D}$ onto $\mathbb{D}$ minus part of the disc bounded by the circular arc orthogonal to the unit circle and joining the points $e^{iu(t)}$ and $e^{\frac{1}{3}iu(t)} = f(e^{iu(0)}, t)$ (Figure 3.1).

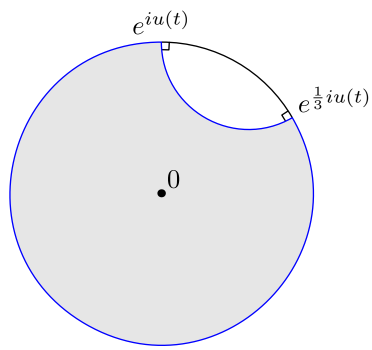

Figure 3.1: An image of the unit disc in Kufarev's example

The function $F(z,0) = \lim_{t\to\infty} e^t f(z,t) = \frac{z}{1-z}$ maps the unit disc conformally onto the half-plane $\left\{w : \operatorname{Re} w > -\frac{1}{2}\right\}$. As it was remarked in [PV08], we can reconstruct the whole subordination chain by extending the map

$$F(f^{-1}(w,t),0) = \frac{e^t w - e^{-2i\alpha(t)}w^2}{(1-e^{-i\alpha(t)}w)^2}, \quad \alpha = \arccos(e^{-t}) \in [0, \pi/2)$$

by reflection into the whole disc $\mathbb{D}$. The extended map $F(z,t)$ maps $\mathbb{D}$ onto the complex plane minus the slit along the ray $\{w : w = -\frac{1}{2} + iy, y \in (-\infty, \frac{1}{2}\cot 2\alpha(t)]\}$ (Figure 3.2).

3.3 Pommerenke's Geometric Characterization

Recall, that the *spherical metric* in the extended complex plane $\bar{\mathbb{C}}$ is defined by the formula

$$d^\sharp(z,w) = \frac{|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}, \quad z,w \in \mathbb{C},$$
$$d^\sharp(z,\infty) = \frac{1}{\sqrt{1+|z|^2}}, \quad z \in \mathbb{C},$$
$$d^\sharp(\infty,\infty) = 0.$$

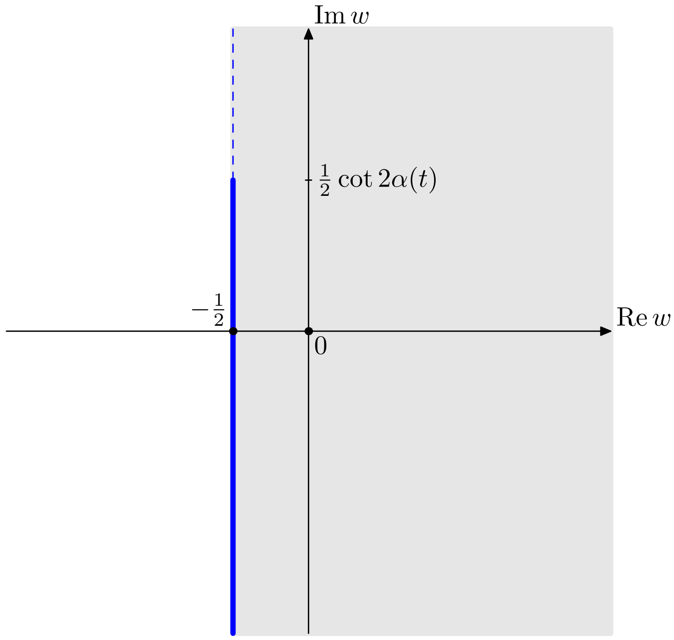

Figure 3.2: The subordination chain of domains in Kufarev's example

The *spherical diameter* of a set E is defined as $\operatorname{diam}^\sharp E = \sup\{d^\sharp(z,w) : z, w \in E\}$

Let $D \subset \mathbb{C}$ be a domain in the complex plane. An open Jordan arc $C \subset D$ is said to be a *crosscut* of D, if $\bar{C} = C \cup \{a, b\}$ with $a, b \in \partial D$ (a and b may coincide).

Two points in a domain $D \subset \bar{\mathbb{C}}$ are said to be *separated* in D by the closed set A if they lie in different components of $D \setminus A$.

In 1966 Pommerenke [Pom66] gave the following necessary and sufficient condition for radial Loewner chains to satisfy the radial Loewner PDE.

Theorem 3.2. *Let $F(z,t)$ be a radial Loewner chain, and $\{D_t = F(\mathbb{D}, t)\}_{t\geq 0}$ be the associated family of domains in the complex plane. $F(z,t)$ satisfies for all $t \geq 0$ the radial Loewner PDE*

$$\dot{F}(z,t) = zF'(z,t)\frac{e^{iu(t)} + z}{e^{iu(t)} - z},$$

with a continuous driving term $u(t)$, if and only if, for every $\epsilon > 0$ there exists a $\delta > 0$, such that whenever $0 \leq t - s \leq \delta$, some cross-cut C of D_t with $\operatorname{diam}^\sharp C < \epsilon$ separates 0 from $D_t \setminus D_s$ in D_t.

We are more interested in the following simple corollary of this theorem, which, as we will see soon, gives us an easy way of constructing examples of Kufarev type.

Corollary 3.3. *Let $F(z,t)$ be a Loewner chain. Suppose that $D_t = F(\mathbb{D}, t)$ is the component containing 0 of the $\bar{\mathbb{C}} \setminus \Gamma_t$, where $\Gamma : [0, \infty) \to \bar{\mathbb{C}}$ is an arc, and $\Gamma_t = \Gamma([t, +\infty))$ is its restriction to $[t, +\infty)$. Then, for all $t \geq 0$ $F(z,t)$ satisfies the radial Loewner PDE with a continuous driving term.*

Note, that the arc in the corollary above does not have to be Jordan (see Figure 3.3). The subordination chain in Kufarev's example also satisfies the corollary's conditions, and the corresponding arc is in fact a closed curve (both its endpoints are infinite).

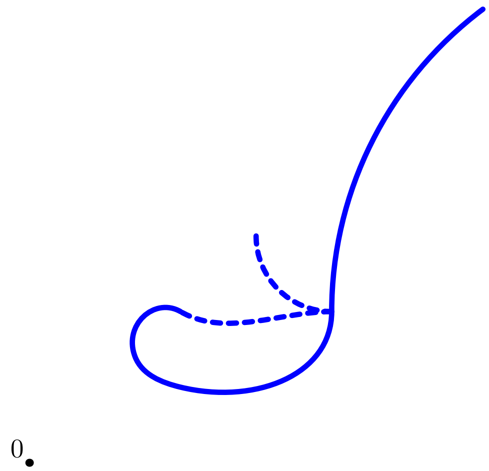

Figure 3.3: An example of a subordination chain for the Corollary 3.3

An analogous geometrical characterization for chordal Loewner chains can be found in [LSW01b].

Theorem 3.4 ([LSW01b, Theorem 2.6])**.** *Let K_t, $t \geq 0$ be a family of growing hulls in $\mathbb{H}$, such that* $\operatorname{hcap}(K_t) = 2t$, *and let $D_t = \mathbb{H} \setminus K_t$. Then the following conditions are equivalent:*

1. *D_t is generated by the chordal Loewner equation*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z,$$

   where $\lambda(t)$ is a continuous driving term.

2. *For every $\epsilon > 0$ there exists a $\delta > 0$, such that whenever $0 \leq t - s \leq \delta$, some bounded connected set C separates $D_s \setminus D_t = K_t \setminus K_s$ from infinity in D_s.*

3.4 A Sufficient Condition by Marshall and Rohde

For a long time, the only one known nontrivial sufficient condition guaranteeing that the Loewner equation generates a single-slit subordination chain, was Kufarev's. Next step towards the solution of the problem that we stated in the beginning of the chapter was done in 2005 by Marshall and Rohde [MR05].

Before we formulate their result, we first give some necessary definitions. We will say that a curve is a *quasiarc* if it is the image of $[0, \infty)$ under a quasiconformal homeomorphism (for the definitions and fundamental facts of the theory of quasiconformal mappings, see, e.g., [LV73]) of the complex plane. For example, piecewise smooth curves without zero-angle cusps are quasislits.

We reformulate slightly the result of Marshall and Rohde for our notation.

Theorem 3.5 ([MR05, Theorem 1.1])**.** *If $\Gamma(t)$ is a quasiarc, then $e^{iu(t)} \in \mathrm{Lip}(\frac{1}{2})$. Conversely, there is a constant $C > 0$, such that if $e^{iu(t)} \in \mathrm{Lip}(\frac{1}{2})$ with $\|e^{iu(t)}\|_{\frac{1}{2}} < C$, then $\mathbb{C} \setminus F(\mathbb{D}, t)$ is a quasiarc for all t.*

It is possible to replace the global Lipschitz condition $\|e^{iu(t)}\|_{\frac{1}{2}} < C$ by a weaker local Lipschitz condition $\|e^{iu(t)}\|_{\frac{1}{2}_{loc}} < C$. Note, that the local Lip(1/2)-norms of $e^{iu(t)}$ and $u(t)$ coincide.

The exact value of the constant for the chordal case ($C = 4$) was found by Lind [Lin05] the same year. Prokhorov and Vasil'ev [PV09] showed that the constants should coincide in the chordal and radial cases. They also found that in the chordal case, circular slits, tangent to the real axis are generated by Lip(1/3)-continuous driving terms.

Kager, Nienhuis and Kadanoff [KNK04] gave multiple examples of solutions to the chordal Loewner equation, corresponding to driving terms with various Lip(1/2)-norms. On one hand, they found examples of slit solutions with arbitrarily large norms of the driving term, on the other hand, they found examples of slit solutions where the norm of the driving term approaches 4 arbitrarily close from above.

Chapter 4

Examples of Non-Slit Solutions

This chapter contains the main results of this thesis. We construct several examples of Kufarev type and then study analytic properties of the corresponding driving terms. The local Lipschitz characteristic of the driving terms is of particular interest for us, in view of the sufficient condition by Marshall and Rohde described in the previous chapter.

4.1 The Method

For a long time Kufarev's example represented the only known case where the radial Loewner equation generated a non-slit solution. Attempts to modify the driving term slightly in order to obtain new non-slit solutions failed, and it was not clear why that driving term behaved so strange.

In order to construct new non-slit solutions we use the technique described in [PV08]. It uses the simple observation that if the subordination chain $F(z,t)$ is constructed as in the Corollary 3.3, and if the arc Γ is not Jordan, then starting from some t_0 the domains $f(\mathbb{D},t)$ will not be slit discs .

We will use the following scheme. Let $F_0(z) = z + a_2 z^2 + \dots$ map the unit disc $\mathbb{D}$ conformally onto the domain D_0. Let Γ be an open Jordan arc, more precisely, a homeomorphic image of $(0,+\infty)$, and let $\Gamma(t) \to \infty$ as $t \to \infty$. Let the closure $\bar{\Gamma}$ meet itself once (by construction, it cannot meet itself more than once), possibly at infinity. Let Γ be parameterized in such a way that $\mathrm{rad}(\mathbb{C} \setminus \Gamma_t) = e^t$, $t \in (0,\infty)$. Let $\partial D_0 \subset \bar{\Gamma}$. We construct the subordination chain $F(z,t)$ corresponding to the domains $D_t = \mathbb{C} \setminus \Gamma_t$. On the one hand, it satisfies the Loewner equation by Corollary 3.3. On the other hand, after a non-zero area is added to D_0 after the initial moment, so $\mathbb{D} \setminus f(\mathbb{D},t) = F^{-1}(F(\mathbb{D},t) \setminus F(\mathbb{D},0),t)$ has non-zero area, and $f(\mathbb{D},t)$ is thus not a slit disc for any $t > 0$.

One of the examples illustrating this scheme is Kufarev's. Several other examples are described in this chapter.

4.2 The First Example

The description of the domains and the corresponding mapping functions in this example are taken from [PV08].

4.2.1 The Subordination Chain

The initial domain D_0 in this example is simply the unit disc $\mathbb{D}$, and F_0 is just the identity function. The arc Γ consists of the circular arc $\{e^{i\theta} : 0 < \theta \leq 2\pi\}$ and the ray $[1, +\infty)$. After the initial moment $t = 0$ we start erasing the circular arc in the positive direction (Figure 4.1). Since $\mathrm{rad}(\mathbb{C} \setminus [1, +\infty)) = 4$, the circular arc is completely erased by the time $t = \ln 4$. It follows from the concatenation property that the driving term $u(t) \equiv 0$, $t \geq \ln 4$, and therefore the only problem is to find the values of $u(t)$ for $t \in (0, \ln 4)$.

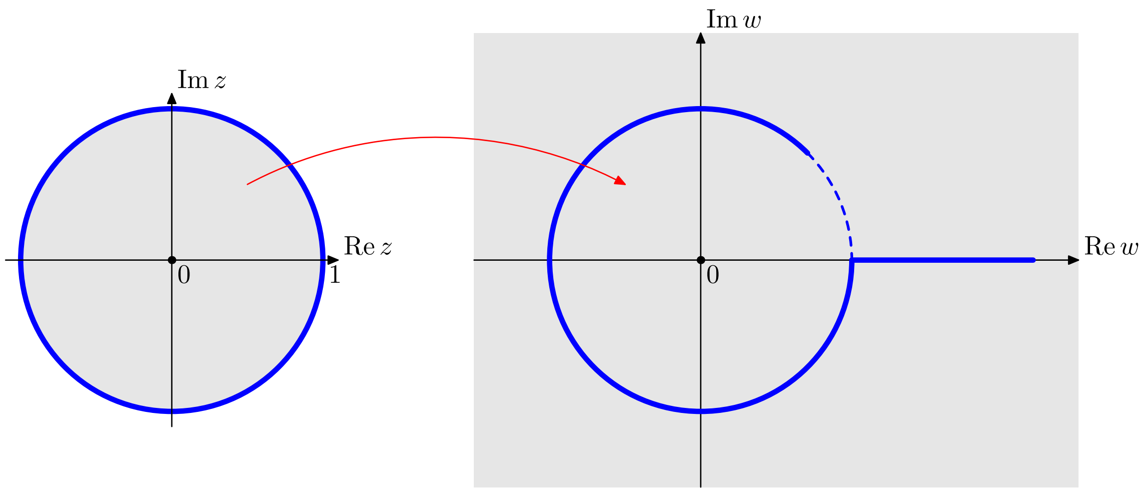

Figure 4.1: The subordination chain in the first example

First, we need to find the form of the conformal maps $F(z,t) : \mathbb{D} \to D_t$, $0 < t < \ln 4$. We map the unit disc onto the upper half-plane by the function

$$\zeta(z,t) = \frac{\zeta_0(t) - \overline{\zeta_0(t)} z e^{-i\alpha(t)}}{1 - z e^{-i\alpha(t)}}$$

(zero is mapped to $\zeta(0,t) = \zeta_0(t)$). The function

$$w(\zeta, t) = \frac{1}{2}\left(1 - \frac{3\zeta^2 + 3(1+\lambda(t))\zeta - \lambda(t)}{2(3+\lambda(t))}\zeta^{-3/2}\right), \quad \lambda \in (-3, 0)$$

which can be found in [vKS59], maps the upper half plane onto $\mathbb{C}$ minus the negative real axis $(-\infty, 0]$ and the vertical ray

$$\left\{ w : w = \frac{1}{2} + iy, y \in \left(-\infty, \frac{1 + 3\lambda(t)}{2(3+\lambda(t))}(-\lambda(t))^{-1/2}\right]\right\}.$$

Finally, the function

$$g(w) = 1 - \frac{1}{w}$$

maps the above domain onto $\mathbb{C}$ minus $[1, +\infty)$ and the circular arc

$$\left\{ z : z = e^{i\theta}, \quad \theta \in \left[2 \arctan \frac{3+\lambda}{1+3\lambda} \sqrt{-\lambda}, 2\pi \right] \right\},$$

that is, onto D_t for some t. We define $F(z,t) = g(w(\zeta(z,t),t))$. In order to make $F(z,t)$ a Loewner chain we should choose the parameters $\alpha(t)$, $\zeta_0(t)$, $\lambda(t)$ in such a way that $F(0,t) = 0$, $F'(0,t) = e^t$.

Note, that $g^{-1}(0) = 1$, and $w^{-1}(1)$ solves the equation

$$\left(3\zeta^2 + 3(1+\lambda)\zeta - \lambda\right) \zeta^{-3/2} = -2(3+\lambda).$$

In order to satisfy the condition $F(0,t) = 0$, we must thus have $\zeta_0(t) = w^{-1}(1)$, i. e., $\zeta_0(t)$ must be the unique solution with the positive imaginary part to the equation above.

The condition $F'(0,t) = e^t$ can be split into two: $|F'(0,t)| = e^t$ and $\arg F'(0,t) = 0$.

By the chain rule,

$$F'(z,t)_{z=0} = g'(w)_{w=w(\zeta_0)=1} \cdot w'(\zeta)_{\zeta=\zeta(0,t)=\zeta_0} \cdot \zeta'(z,t)_{z=0},$$

$$|F'(z,t)|_{z=0} = |g'(w)|_{w=w(\zeta_0)=1} \cdot |w'(\zeta)|_{\zeta=\zeta(0,t)=\zeta_0} \cdot |\zeta'(z,t)|_{z=0},$$

$$\arg F'(z,t)_{z=0} = \arg g'(w)_{w=1} + \arg w'(\zeta)_{\zeta=\zeta_0} + \arg \zeta'(z,t)_{z=0},$$

We have seen in Section 1.5 that $|\zeta'(z,t)|_{z=0} = 2 \operatorname{Im} \zeta_0$, $\arg \zeta'(z,t)_{z=0} = \frac{\pi}{2} - \alpha$. Since $g'(w) = \frac{1}{w^2}$, $|g'(1)| = 1$ and $\arg g'(1) = 0$. The ζ-derivative of $w(\zeta,t)$ is

$$w'(\zeta,t) = -\frac{3}{8(3+\lambda)} \zeta^{-\frac{5}{2}} \left(\zeta^2 - (1+\lambda)\zeta + \lambda\right) = -\frac{3}{8(3+\lambda)} \zeta^{-\frac{5}{2}} (\zeta - 1)(\zeta - \lambda),$$

so

$$|w'(\zeta_0,t)| = \frac{3}{8(3+\lambda)} |\zeta_0|^{-\frac{5}{2}} |\zeta_0 - 1| |\zeta_0 - \lambda|,$$

$$\arg w'(\zeta_0,t) = \pi - \frac{5}{2} \arg \zeta_0 + \arg(\zeta_0 - 1) + \arg(\zeta_0 - \lambda).$$

Hence, if the parameters $\alpha(t)$, $\zeta_0(t)$, $\lambda(t)$ are defined from the equations

$$\left(3\zeta_0^2 + 3(1+\lambda)\zeta_0 - \lambda\right) \zeta_0^{-3/2} = -2(3+\lambda), \tag{4.1}$$

$$\frac{3}{4(3+\lambda)} |\zeta_0|^{-\frac{5}{2}} |\zeta_0 - 1| |\zeta_0 - \lambda| \operatorname{Im} \zeta_0 = e^t, \tag{4.2}$$

$$-\frac{\pi}{2} - \alpha - \frac{5}{2} \arg \zeta_0 + \arg(\zeta_0 - 1) + \arg(\zeta_0 - \lambda) = 0, \tag{4.3}$$

then $F(z,t)$ is the Loewner chain corresponding to the domains on the Figure 4.1.

4.2.2 The Driving Term

Recall that the driving term is argument of the inverse image of the tip of the slit, i.e., $u(t) = \arg F^{-1}(\Gamma(t), t)$. In this example, $\Gamma(t) = \exp\left(i2\arctan\frac{3+\lambda(t)}{1+3\lambda(t)}\sqrt{-\lambda(t)}\right)$ for $0 < t < \ln 4$. Its inverse image under $g(w)$ is

$$g^{-1}(\Gamma(t)) = \frac{1+3\lambda(t)}{2(3+\lambda(t))}(-\lambda(t))^{-\frac{1}{2}}.$$

Further,

$$w^{-1}\left(\frac{1+3\lambda(t)}{2(3+\lambda(t))}(-\lambda(t))^{-\frac{1}{2}}, t\right) = \lambda(t),$$

and

$$\zeta^{-1}(\lambda(t), t) = e^{i\alpha(t)}\frac{\lambda(t)-\zeta_0(t)}{\lambda(t)-\overline{\zeta_0(t)}}.$$

Thus,

$$F^{-1}(\Gamma(t), t) = e^{i\alpha(t)}\frac{\lambda(t)-\zeta_0(t)}{\lambda(t)-\overline{\zeta_0(t)}},$$

and

$$u(t) = \alpha(t) + 2\arg(\zeta_0(t) - \lambda(t)),$$

or, in view of (4.3),

$$u(t) = -\frac{\pi}{2} - \frac{5}{2}\arg\zeta_0 + \arg(\zeta_0 - 1) + 3\arg(\zeta_0 - \lambda). \tag{4.4}$$

4.2.3 Determining the Parameters and Calculating $\|u\|_{\frac{1}{2}loc}$

We can express ζ_0 from (4.1). The equation has two solutions,

$$\left[\begin{array}{l} \zeta_0 = \frac{2}{9}\lambda^2 + \frac{1}{3}\lambda - i\frac{2}{9}\lambda\sqrt{-\lambda^2 - 3\lambda}, \\ \zeta_0 = \frac{2}{9}\lambda^2 + \frac{1}{3}\lambda + i\frac{2}{9}\lambda\sqrt{-\lambda^2 - 3\lambda}, \end{array}\right.$$

and, since $\lambda \in (-3, 0)$, $\lambda^2 + 3\lambda < 0$, only the first solution has positive imaginary part, and thus

$$\zeta_0 = \frac{2}{9}\lambda^2 + \frac{1}{3}\lambda - i\frac{2}{9}\lambda\sqrt{-\lambda^2 - 3\lambda}. \tag{4.5}$$

It follows that

$$\operatorname{Im}\zeta_0 = \frac{2}{9}(-\lambda)\sqrt{-\lambda}\sqrt{\lambda+3}, \tag{4.6}$$

$$|\zeta_0| = -\frac{\lambda}{3}, \tag{4.7}$$

$$|\zeta_0 - 1| = \frac{1}{\sqrt{3}}\sqrt{1-\lambda}\sqrt{\lambda+3}, \tag{4.8}$$

and

$$|\zeta_0 - \lambda| = \frac{2}{3}(-\lambda)\sqrt{1-\lambda}. \tag{4.9}$$

Substituting these identities into (4.2), we can see that

$$|F'(z,t)| = 1 - \lambda = e^t,$$

and, thus

$$\lambda(t) = 1 - e^t. \tag{4.10}$$

This allows us to find explicit expressions for $\zeta_0(t)$ (by (4.5)):

$$\zeta_0(t) = \frac{1}{9}(e^t - 1)(2e^t - 5 + 2i\sqrt{e^t-1}\sqrt{4-e^t}),$$

for $\alpha(t)$ (by (4.3)):

$$\begin{aligned}\alpha(t) = -\frac{\pi}{2} - \frac{5}{2}\arg(2e^t - 5 + 2i\sqrt{e^t-1}\sqrt{4-e^t})\\ + \arg(2e^{2t} - 7e^t - 4 + 2i(e^t-1)^{3/2}\sqrt{4-e^t}) + \arg(e^t + 2 + i\sqrt{e^t-1}\sqrt{4-e^t}),\end{aligned}$$

and for $u(t)$ (by (4.4)):

$$\begin{aligned}u(t) = -\frac{\pi}{2} - \frac{5}{2}\arg(2e^t - 5 + 2i\sqrt{e^t-1}\sqrt{4-e^t})\\ + \arg(2e^{2t} - 7e^t - 4 + 2i(e^t-1)^{3/2}\sqrt{4-e^t}) + 3\arg(e^t + 2 + i\sqrt{e^t-1}\sqrt{4-e^t}).\end{aligned}$$

The function $u(t)$ is differentiable everywhere on the segment $[0, \ln 4]$ except for its endpoints. As $t \to 0$ we have $u(t) = \frac{8\sqrt{3}}{3}\sqrt{t} + O(t)$. As $t \to \ln 4$, $u(t) = 2\pi - \frac{4\sqrt{3}}{3}\sqrt{\ln 4 - t} + O(\ln 4 - t)$. Thus we get, finally, that

$$\|u\|_{\frac{1}{2}loc} = \frac{8\sqrt{3}}{3} \approx 4.62.$$

4.3 The Second Example

4.3.1 The Initial Moment

In this example, the arc Γ consists of two rays which are symmetric with respect to the negative real axis, have angle $\pi\theta$ between them ($\theta \in (0,2)$) and start at the point $-c(\theta) \in \mathbb{R}$. Γ is parameterized in such a way that $\lim_{t\to 0}\Gamma(t) = \lim_{t\to\infty}\Gamma(t) = \infty$, $\operatorname{Im}\Gamma(t) > 0$ for small t, and $\operatorname{rad}(\mathbb{C}\setminus\Gamma_t) = e^t$. The closure $\overline{\Gamma}$ meets itself at infinity, where it forms an angle of $\pi\theta$ (Figure 4.2).

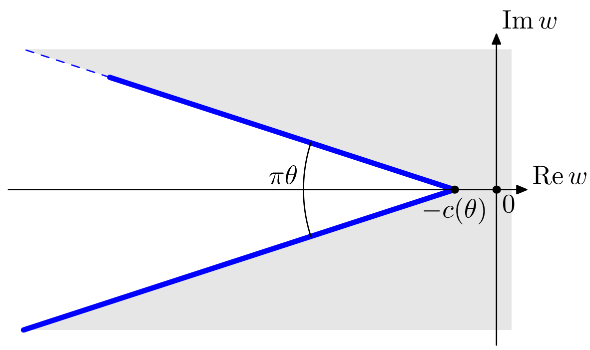

Figure 4.2: The subordination chain in the second example

The initial domain D_0 is the component of $\mathbb{C} \setminus \Gamma$ containing 0, and all the other domains are of the form $D_t = \mathbb{C} \setminus \Gamma_t$. The conformal map $F_0 : \mathbb{D} \to D_0$ can be represented as the composition of the map $g(z)$ of $\mathbb{D}$ onto $\mathbb{H}$

$$g(z) = \frac{w - \overline{w} z e^{-i\alpha}}{1 - z e^{-i\alpha}}, \quad w \in \mathbb{H},$$

and the map $f(z)$ of $\mathbb{H}$ onto D_0 given by

$$f(z) = -e^{i\frac{\pi\theta}{2}} z^{2-\theta} - c(\theta).$$

In order to satisfy the condition $F_0(0) = 0$, the value of the parameter w in $g(z)$ must be $w = f^{-1}(0) = ic(\theta)^{\frac{1}{2-\theta}}$.

The condition $F_0'(0) = 1$ (i. e., $\operatorname{rad}(D_0) = 1$) allows us to determine the value of $c(\theta)$. Since

$$F_0'(0) = f'(ic(\theta)^{\frac{1}{2-\theta}}) \cdot g'(0) = 2(2-\theta)c(\theta)e^{-i\alpha},$$

we conclude that $\alpha = 0$ and

$$c(\theta) = \frac{1}{2(2-\theta)}.$$

4.3.2 The Time t_0 of Reaching the Corner

After the tip of the slit reaches the point $-c(\theta)$, the structure of the maps in the subordination chain $F(z,t)$ changes significantly. In order to calculate the time t_0 when this happens, we should find the conformal radius of the corresponding domain D_{t_0}.

The upper half-plane can be mapped onto this domain by the function

$$f(z) = z^2 e^{i\left(\pi + \frac{\pi\theta}{2}\right)} - \frac{1}{2(2-\theta)}.$$

The inverse image of zero is $z_0 = \frac{e^{i\frac{\pi}{2}\left(1-\frac{\theta}{2}\right)}}{\sqrt{2(2-\theta)}}$, the derivative is $f'(z) = 2e^{i\left(\pi + \frac{\pi\theta}{2}\right)}$, $f'(z_0) = -\frac{2e^{i\left(\frac{\pi}{2} + \frac{\pi\theta}{4}\right)}}{\sqrt{2(2-\theta)}}$.

The unit disc can be mapped onto the upper half-plane by a Möbius transformation sending 0 to z_0:

$$g(z) = \frac{z_0 - \bar{z}_0 z e^{-i\alpha}}{1 - z e^{-i\alpha}},$$

$$g'(0) = 2e^{i\left(\frac{\pi}{2}-\alpha\right)} \operatorname{Im} z_0 = 2e^{i\left(\frac{\pi}{2}-\alpha\right)} \frac{\cos\frac{\pi\theta}{4}}{\sqrt{2(2-\theta)}}.$$

The derivative of the composite map is thus

$$f'(z_0)g'(0) = \frac{2\cos\frac{\pi\theta}{4}}{2-\theta} e^{i\left(\frac{\pi\theta}{4}-\alpha\right)}.$$

It should be real-valued, hence $\alpha = \frac{\pi\theta}{4}$. Now we can write $g(z)$ explicitly:

$$g(z) = -\frac{ie^{-i\frac{\pi\theta}{4}}}{\sqrt{2(2-\theta)}} \frac{z - e^{i\left(\pi - \frac{\pi\theta}{4}\right)}}{z - e^{i\frac{\pi\theta}{4}}}.$$

The conformal radius is thus $\frac{2\cos\frac{\pi\theta}{4}}{2-\theta}$ and the point $-c(\theta)$ is reached at the time

$$t_0 = \ln \frac{2\cos\frac{\pi\theta}{4}}{2-\theta}.$$

We can also see that

$$u(t_0) = \alpha + 2\arg z_0 = \pi - \frac{\pi\theta}{4}.$$

4.3.3 Mapping Functions $F(z,t)$ for $0 < t < t_0$

The function

$$(w-a)^{1-\frac{\theta}{2}} w^{\frac{\theta}{2}}, \quad a > 0$$

maps the upper half-plane $\mathbb{H}$ onto $\mathbb{H}$ with a slit set at the angle $\frac{\pi\theta}{2}$ to the negative real line. In order to get the evolution domain D_t for some $0 < t < t_0 = \ln \frac{\cos\frac{\pi\theta}{4}}{1-\frac{\theta}{2}}$, we square this function, then rotate the obtained domain by $\pi + \frac{\pi\theta}{2}$ and, finally, shift it to the left. That is, the upper half-plane can be mapped onto the evolution domain D_t by

$$f(w,t) = -e^{i\frac{\pi\theta}{2}} (w - a(t))^{2-\theta} w^{\theta} - c(\theta).$$

The derivative of the function $f(w)$ is

$$f'(w) = -e^{i\frac{\pi\theta}{2}} \left(\frac{(2-\theta)(w-a)^{2-\theta} w^{\theta}}{(w-a)} + \frac{\theta (w-a)^{2-\theta} w^{\theta}}{w} \right)$$
$$= -e^{i\frac{\pi\theta}{2}} \frac{(w-a)^{2-\theta} w^{\theta}}{(w-a)w} (2w - a\theta).$$

Denote by $w_0 = w_0(t)$ the inverse image of zero under $f(w,t)$, i. e., the solution of the equation

$$-e^{i\frac{\pi\theta}{2}}(w_0 - a(t))^{2-\theta} w_0^\theta = c(\theta), \tag{4.11}$$

lying in the upper half-plane.

In order to satisfy the requirement $F(0,t) = 0$, the function $F(z,t)$ must be of the form $F(z,t) = f(g(z,t),t)$, where $g(z,t)$ is the function which maps the unit disc onto the upper half-plane and sends 0 to w_0:

$$g(z,t) = \frac{w_0(t) - \overline{w_0(t)} z e^{-i\alpha}}{1 - z e^{-i\alpha}},$$

$$g'(0,t) = 2e^{i\left(\frac{\pi}{2}-\alpha\right)} \operatorname{Im} w_0.$$

The condition $F'(0,t) = e^t$ gives an additional equation for determining $\alpha(t)$, $w_0(t)$ and $a(t)$:

$$-e^{i\frac{\pi\theta}{2}} \frac{(w_0 - a)^{2-\theta} w_0^\theta}{(w_0 - a) w_0} (2w_0 - a\theta)\, 2e^{i\left(\frac{\pi}{2}-\alpha\right)} \operatorname{Im} w_0 = e^t. \tag{4.12}$$

The critical points of $f(w)$ are 0, $\frac{a\theta}{2}$ and a. The point $\frac{a\theta}{2}$ is, in fact, the inverse image of the slit's tip in the upper half-plane, and it follows that

$$e^{iu(t)} = e^{i\alpha} \frac{w_0(t) - \frac{a(t)\theta}{2}}{\overline{w_0(t)} - \frac{a(t)\theta}{2}} = e^{i\alpha} \frac{(2w_0(t) - a(t)\theta)^2}{|2w_0(t) - a(t)\theta|^2}. \tag{4.13}$$

4.3.4 Mapping Functions $F(z,t)$ for $t > t_0$

The upper half-plane $\mathbb{H}$ can be mapped onto D_t, $t > t_0$ by the function

$$f(z,t) = (z^2 + \tau) e^{\pi i\left(1+\frac{\theta}{2}\right)} - c(\theta), \quad \tau = \tau(t) > 0,$$

$$f'(z,t) = 2e^{\pi i\left(1+\frac{\theta}{2}\right)} z.$$

The inverse image of the tip of the slit, i. e., $f^{-1}(\Gamma_t, t)$ is zero. The inverse image of zero under $f(z,t)$ is $w_0(t) = f^{-1}(0,t) = \sqrt{c(\theta) e^{-\pi i\left(1+\frac{\theta}{2}\right)} - \tau}$. By the law of cosines we find that

$$|w_0|^2 = |c(\theta) e^{-\pi i\left(1+\frac{\theta}{2}\right)} - \tau| = \tau^2 + c(\theta)^2 + 2c(\theta)\tau \cos\frac{\pi\theta}{2}.$$

By the law of sines,

$$\arg w_0^2 = \arg\left(c(\theta) e^{-\pi i\left(1+\frac{\theta}{2}\right)} - \tau\right) = \arcsin \frac{c(\theta)\sin\frac{\theta}{2}}{\tau^2 + c(\theta)^2 + 2c(\theta)\tau\cos\frac{\pi\theta}{2}}.$$

The imaginary part of w_0 is

$$\operatorname{Im} w_0 = \sqrt{\tau^2 + c(\theta)^2 + 2c(\theta)\tau\cos\frac{\pi\theta}{2}}\, \sin\frac{1}{2}\arcsin\frac{c(\theta)\sin\frac{\theta}{2}}{\tau^2 + c(\theta)^2 + 2c(\theta)\tau\cos\frac{\pi\theta}{2}}.$$

The derivative of $f(z,t)$ at w_0 is

$$f'(w_0,t) = 2e^{\pi i\left(1+\frac{\theta}{2}\right)}\sqrt{c(\theta)e^{-\pi i\left(1+\frac{\theta}{2}\right)} - \tau}.$$

Its absolute value and argument are

$$|f'(w_0,t)| = 2\sqrt{\tau^2 + c(\theta)^2 + 2c(\theta)\tau\cos\frac{\pi\theta}{2}},$$

$$\arg f'(w_0,t) = \pi\left(1+\frac{\theta}{2}\right) + \frac{1}{2}\arcsin\frac{c(\theta)\sin\frac{\theta}{2}}{\tau^2 + c(\theta)^2 + 2c(\theta)\tau\cos\frac{\pi\theta}{2}}.$$

The suitable map of the unit disc $\mathbb{D}$ onto the upper half-plane $\mathbb{H}$ is of the form

$$g(z,t) = \frac{w_0(t) - \overline{w_0(t)}ze^{-i\alpha}}{1 - ze^{-i\alpha}}, \quad \alpha = \alpha(t) > 0,$$

$$g'(0,t) = 2e^{i\left(\frac{\pi}{2}-\alpha\right)}\sqrt{\tau^2 + c(\theta)^2 + 2c(\theta)\tau\cos\frac{\pi\theta}{2}}\,\sin\frac{1}{2}\arcsin\frac{c(\theta)\sin\frac{\theta}{2}}{\tau^2 + c(\theta)^2 + 2c(\theta)\tau\cos\frac{\pi\theta}{2}}.$$

The functions $F(z,t)$, $t > t_0$ can be represented as the composition $F(z,t) = f(g(z,t),t)$. The condition $|F'(0,t)| = e^t$ gives us an equation for determining $\tau(t)$

$$4\left(\tau^2 + c(\theta)^2 + 2c(\theta)\tau\cos\frac{\pi\theta}{2}\right)\sin\frac{1}{2}\arcsin\frac{c(\theta)\sin\frac{\theta}{2}}{\tau^2 + c(\theta)^2 + 2c(\theta)\tau\cos\frac{\pi\theta}{2}} = e^t.$$

Thus, $\tau(t)$ is a differentiable function of t. The condition $\arg F'(0,t) = 0$ makes it possible to determine $\alpha(t)$:

$$\alpha(t) = \frac{3\pi}{2} + \frac{\theta}{2} + \frac{1}{2}\arcsin\frac{c(\theta)\sin\frac{\theta}{2}}{\tau(t)^2 + c(\theta)^2 + 2c(\theta)\tau(t)\cos\frac{\pi\theta}{2}}.$$

The driving term $u(t)$ is defined by

$$e^{iu(t)} = e^{i\alpha}\frac{w_0 - 0}{\overline{w_0} - 0},$$

i. e., $u(t) = \alpha(t) + 2\arg w_0$, or,

$$u(t) = \frac{3\pi}{2} + \frac{\theta}{2} + \frac{3}{2}\arcsin\frac{c(\theta)\sin\frac{\theta}{2}}{\tau^2 + c(\theta)^2 + 2c(\theta)\tau\cos\frac{\pi\theta}{2}}.$$

This is a differentiable function of τ, and, consequently, a differentiable function of t. That means, that the interval (t_0,∞) does not contribute to the norm $\|u\|_{\frac{1}{2}loc}$.

Another way to see this is to use the truncation property. Since the slit $\Gamma([t_0,+\infty))$ is C^∞-smooth, the corresponding driving term is also C^∞-smooth.

4.3.5 Determining the Parameters for $0 < t < t_0$ and Calculating $\|u\|_{\frac{1}{2}loc}$

We can express a from (4.11):

$$a = w_0 - c(\theta)^{\frac{1}{2-\theta}} e^{\frac{\pi i}{2}} w_0^{-\frac{\theta}{2-\theta}}. \tag{4.14}$$

This number must be real positive, i. e.

$$\operatorname{Im} w_0 = \operatorname{Im}\left(c(\theta)^{\frac{1}{2-\theta}} e^{\frac{\pi i}{2}} w_0^{-\frac{\theta}{2-\theta}} \right).$$

If we denote $\phi = \arg w_0$, we can rewrite this equation in the form

$$|w_0| \sin\phi = c(\theta)^{\frac{1}{2-\theta}} |w_0|^{-\frac{\theta}{2-\theta}} \sin\left(\frac{\pi}{2} - \frac{\theta\phi}{2-\theta}\right),$$

$$|w_0|^{\frac{2}{2-\theta}} = c(\theta)^{\frac{1}{2-\theta}} \frac{\cos\frac{\theta\phi}{2-\theta}}{\sin\phi}, \tag{4.15}$$

$$|w_0| = \frac{1}{\sqrt{2(2-\theta)}} \left(\frac{\cos\frac{\theta\phi}{2-\theta}}{\sin\phi} \right)^{1-\frac{\theta}{2}}.$$

Thus, we have obtained an expression for the absolute value of w_0 in terms of its argument.

The following estimate is important for choosing the branch of the argument. From (4.11) we have

$$(2-\theta)\arg(w_0 - a) + \theta \arg w_0 = \pi - \frac{\pi\theta}{2}.$$

Since $a > 0$,

$$2 \arg w_0 \le \pi - \frac{\pi\theta}{2},$$

$$\arg w_0 \le \frac{\pi}{2} - \frac{\pi\theta}{4}.$$

Moreover, if $t \to t_0$, then $a \to 0$ and $\phi = \arg w_0 \to \frac{\pi}{2} - \frac{\pi\theta}{4}$.

We can rewrite (4.12) as

$$\frac{c(\theta)}{w_0(w_0 - a)} (2w_0 - a\theta)\, 2e^{i\left(\frac{\pi}{2} - \alpha\right)} \operatorname{Im} w_0 = e^t.$$

Using (4.14) we can get

$$w_0 - a = c(\theta)^{\frac{1}{2-\theta}} e^{\frac{\pi i}{2}} w_0^{-\frac{\theta}{2-\theta}},$$

and

$$2w_0 - a\theta = (2-\theta)w_0 + \theta c(\theta)^{\frac{1}{2-\theta}} e^{\frac{\pi i}{2}} w_0^{-\frac{\theta}{2-\theta}} = (2-\theta) w_0^{-\frac{\theta}{2-\theta}} \left(w_0^{\frac{2}{2-\theta}} + \frac{\theta}{2-\theta} c(\theta)^{\frac{1}{2-\theta}} e^{\frac{\pi i}{2}} \right).$$

So, we can rewrite the condition 4.12 as

$$\left(w_0^{\frac{2}{2-\theta}} + \frac{\theta}{2-\theta}c(\theta)^{\frac{1}{2-\theta}}e^{\frac{\pi i}{2}}\right)c(\theta)^{-\frac{1}{2-\theta}}\frac{\operatorname{Im} w_0}{w_0}e^{-i\alpha} = e^t. \tag{4.16}$$

We can rewrite the expression in parentheses in terms of $\phi = \arg w_0$.

$$\begin{aligned}
w_0^{\frac{2}{2-\theta}} + \frac{\theta}{2-\theta}c(\theta)^{\frac{1}{2-\theta}}e^{\frac{\pi i}{2}} &= \left(|w_0|^{\frac{2}{2-\theta}}\left(\cos\frac{2\phi}{2-\theta} + i\sin\frac{2\phi}{2-\theta}\right) + i\frac{\theta}{2-\theta}c(\theta)^{\frac{1}{2-\theta}}\right)\\
&= \left(c(\theta)^{\frac{1}{2-\theta}}\frac{\cos\frac{\theta\phi}{2-\theta}}{\sin\phi}\left(\cos\frac{2\phi}{2-\theta} + i\sin\frac{2\phi}{2-\theta}\right) + i\frac{\theta}{2-\theta}c(\theta)^{\frac{1}{2-\theta}}\right)\\
&= \frac{c(\theta)^{\frac{1}{2-\theta}}}{\sin\phi}\left(\cos\frac{\theta\phi}{2-\theta}\cos\frac{2\phi}{2-\theta} + i\left(\cos\frac{\theta\phi}{2-\theta}\sin\frac{2\phi}{2-\theta} + \frac{\theta}{2-\theta}\sin\phi\right)\right).
\end{aligned}$$

The absolute value of this expression is

$$\begin{aligned}
&\left|w_0^{\frac{2}{2-\theta}} + \frac{\theta}{2-\theta}c(\theta)^{\frac{1}{2-\theta}}e^{\frac{\pi i}{2}}\right|\\
&\quad= \frac{c(\theta)^{\frac{1}{2-\theta}}}{\sin\phi}\sqrt{\cos^2\frac{\theta\phi}{2-\theta}\cos^2\frac{2\phi}{2-\theta} + \left(\cos\frac{\theta\phi}{2-\theta}\sin\frac{2\phi}{2-\theta} + \frac{\theta}{2-\theta}\sin\phi\right)^2}\\
&\quad= \frac{c(\theta)^{\frac{1}{2-\theta}}}{\sin\phi}\sqrt{\cos^2\frac{\theta\phi}{2-\theta} + \left(\frac{\theta}{2-\theta}\right)^2\sin^2\phi + \frac{2\theta}{2-\theta}\sin\phi\cos\frac{\theta\phi}{2-\theta}\sin\frac{2\phi}{2-\theta}}.
\end{aligned}$$

Hence, if we take absolute values in 4.16, we get

$$\sqrt{\cos^2\frac{\theta\phi}{2-\theta} + \left(\frac{\theta}{2-\theta}\right)^2\sin^2\phi + \frac{2\theta}{2-\theta}\sin\phi\cos\frac{\theta\phi}{2-\theta}\sin\frac{2\phi}{2-\theta}} = e^t,$$

or

$$t = \frac{1}{2}\ln\left(\cos^2\frac{\theta\phi}{2-\theta} + \left(\frac{\theta}{2-\theta}\right)^2\sin^2\phi + \frac{2\theta}{2-\theta}\sin\phi\cos\frac{\theta\phi}{2-\theta}\sin\frac{2\phi}{2-\theta}\right).$$

We can write first terms in the Taylor expansion of the expression on the right-hand side when $\phi \to 0$ and $\phi \to \frac{\pi}{2} - \frac{\pi\theta}{4}$.

$$t = \frac{2\theta}{(2-\theta)^2}\phi^2 - \frac{2\theta}{3}\frac{\theta^2+5\theta+4}{(2-\theta)^4}\phi^4 + O(\phi^6), \quad \phi \to 0.$$

$$t = \ln\frac{2\cos\frac{\pi\theta}{4}}{2-\theta} - \frac{\theta}{2-\theta}\left(\phi - \frac{\pi}{2} + \frac{\pi\theta}{4}\right)^2 + O\left(\left(\phi - \frac{\pi}{2} + \frac{\pi\theta}{4}\right)^3\right), \quad \phi \to \frac{\pi}{2} - \frac{\pi\theta}{4}.$$

That means

$$\phi = \frac{2-\theta}{\sqrt{2\theta}}\sqrt{t} + O(t), \quad t \to 0, \tag{4.17}$$

$$\phi = \frac{\pi}{2} - \frac{\pi\theta}{4} - \sqrt{\frac{2-\theta}{\theta}}\sqrt{t_0 - t} + O(t_0 - t), \quad t \to t_0.$$

We can express $e^{i\alpha}$ from (4.16)

$$e^{i\alpha} = \left(w_0^{\frac{2}{2-\theta}} + \frac{\theta}{2-\theta}c(\theta)^{\frac{1}{2-\theta}}e^{\frac{\pi i}{2}}\right)c(\theta)^{-\frac{1}{2-\theta}}\frac{\operatorname{Im} w_0}{w_0}e^{-t}.$$

Using (4.13) we can express $e^{iu(t)}$

$$e^{iu(t)} = e^{i\alpha}\frac{(2w_0 - a\theta)^2}{|2\bar{w}_0 - a\theta|^2} = e^{i\alpha}\frac{(2-\theta)^2 w_0^{-\frac{2\theta}{2-\theta}}\left(w_0^{\frac{2}{2-\theta}} + \frac{\theta}{2-\theta}c(\theta)^{\frac{1}{2-\theta}}e^{\frac{\pi i}{2}}\right)^2}{(2-\theta)^2|w_0|^{-\frac{2\theta}{2-\theta}}\left|w_0^{\frac{2}{2-\theta}} + \frac{\theta}{2-\theta}c(\theta)^{\frac{1}{2-\theta}}e^{\frac{\pi i}{2}}\right|^2}$$

$$= \frac{w_0^{-\frac{2\theta}{2-\theta}}\left(w_0^{\frac{2}{2-\theta}} + i\frac{\theta}{2-\theta}c(\theta)^{\frac{1}{2-\theta}}\right)^3}{|w_0|^{-\frac{2\theta}{2-\theta}}\left|w_0^{\frac{2}{2-\theta}} + i\frac{\theta}{2-\theta}c(\theta)^{\frac{1}{2-\theta}}\right|^2}c(\theta)^{-\frac{1}{2-\theta}}\frac{\operatorname{Im} w_0}{w_0}e^{-t}$$

$$= w_0^{-\frac{2+\theta}{2-\theta}}\left(w_0^{\frac{2}{2-\theta}} + i\frac{\theta}{2-\theta}c(\theta)^{\frac{1}{2-\theta}}\right)^3\frac{c(\theta)^{-\frac{1}{2-\theta}}\operatorname{Im} w_0 e^{-t}}{\left|w_0^{\frac{2}{2-\theta}} + i\frac{\theta}{2-\theta}c(\theta)^{\frac{1}{2-\theta}}\right|^2}.$$

Thus,

$$u(t) = -\frac{2+\theta}{2-\theta}\arg w_0 + 3\arg\left(w_0^{\frac{2}{2-\theta}} + i\frac{\theta}{2-\theta}c(\theta)^{\frac{1}{2-\theta}}\right).$$

This can be rewritten in terms of ϕ:

$$u(t) = -\frac{2+\theta}{2-\theta}\phi + 3\arg\left(\cos\frac{\theta\phi}{2-\theta}\cos\frac{2\phi}{2-\theta} + i\left(\cos\frac{\theta\phi}{2-\theta}\sin\frac{2\phi}{2-\theta} + \frac{\theta}{2-\theta}\sin\phi\right)\right)$$

$$= -\frac{2+\theta}{2-\theta}\phi + 3\arctan\frac{\cos\frac{\theta\phi}{2-\theta}\sin\frac{2\phi}{2-\theta} + \frac{\theta}{2-\theta}\sin\phi}{\cos\frac{\theta\phi}{2-\theta}\cos\frac{2\phi}{2-\theta}}.$$

The first terms in the Taylor expansion are

$$u(t) = 2\frac{2+\theta}{2-\theta}\phi - 4\theta\frac{2+\theta}{(2-\theta)^3}\phi^3 + O(\phi^5), \quad \phi \to 0,$$

$$u(t) = u(t_0) + 4\frac{1-\theta}{2-\theta}\left(\phi - \frac{\pi}{2} + \frac{\pi\theta}{4}\right) + O\left(\phi - \frac{\pi}{2} + \frac{\pi\theta}{4}\right)^2, \quad \phi \to \frac{\pi}{2} - \frac{\pi\theta}{4}.$$

Using the expression (4.17) for ϕ we finally get

$$u(t) = (2+\theta)\sqrt{\frac{2}{\theta}}\sqrt{t} + O(t), \quad t \to 0,$$

$$u(t) = u(t_0) - 4\frac{1-\theta}{\sqrt{2-\theta}\sqrt{\theta}}\sqrt{t_0 - t} + O(t_0 - t), \quad t \to t_0.$$

$u(t)$ is differentiable on $(0, t_0)$ and (t_0, ∞). The norm of the driving term is thus given by

$$\|u(t)\|_{\frac{1}{2}loc} = \max\left((2+\theta)\sqrt{\frac{2}{\theta}}, 4\frac{|1-\theta|}{\sqrt{2-\theta}\sqrt{\theta}}\right).$$

The function $(2+\theta)\sqrt{\frac{2}{\theta}}$ decreases from $+\infty$ to 0 as θ varies between 0 and 2. The function $4\frac{|1-\theta|}{\sqrt{2-\theta}\sqrt{\theta}}$ first decreases from $+\infty$ to 0 as θ increases from 1 to 2, and then increases to $+\infty$ as $\theta \to 2$.

$$(2+\theta)\sqrt{\frac{2}{\theta}} = 4\frac{|1-\theta|}{\sqrt{2-\theta}\sqrt{\theta}} \quad \text{for } \theta = \theta_0 = 3\sqrt{5} - 5.$$

For $0 < \theta < \theta_0$,

$$(2+\theta)\sqrt{\frac{2}{\theta}} > 4\frac{|1-\theta|}{\sqrt{2-\theta}\sqrt{\theta}}.$$

For $\theta > \theta_0$,

$$(2+\theta)\sqrt{\frac{2}{\theta}} < 4\frac{|1-\theta|}{\sqrt{2-\theta}\sqrt{\theta}}.$$

Thus,

$$\|u(t)\|_{\frac{1}{2}loc} = \begin{cases} (2+\theta)\sqrt{\frac{2}{\theta}}, & 0 < \theta < \theta_0, \\ 4\frac{|1-\theta|}{\sqrt{2-\theta}\sqrt{\theta}}, & \theta > \theta_0. \end{cases}$$

The minimal value is obtained when $\theta = \theta_0$. In this case, $\|u\|_{\frac{1}{2}loc} = 3\sqrt{2}\,\frac{\sqrt{5}-1}{\sqrt{3\sqrt{5}-5}} \approx 4.0125$, which is very close to Lind's bound.

4.3.6 Non-Slit Disks

The geometry of the images of the unit disc under the function $f(z,t)$ changes significantly as t increases.

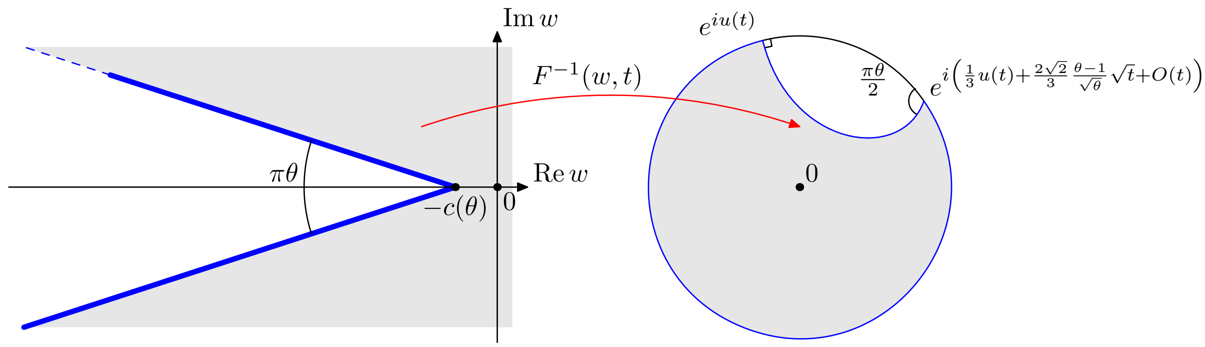

Figure 4.3: $f(\mathbb{D},t)$ for $0<t<t_0$.

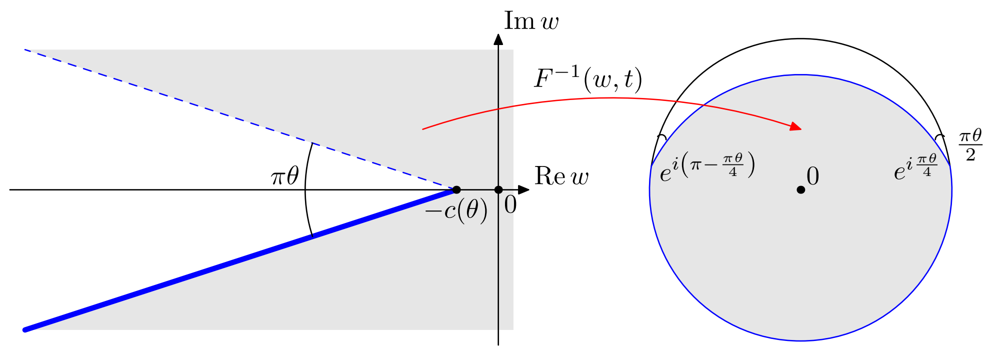

Figure 4.4: $f(\mathbb{D},t_0)$.

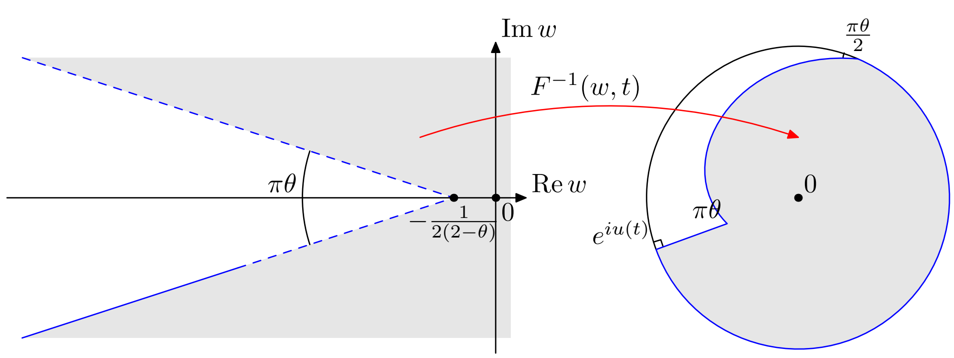

Figure 4.5: $f(\mathbb{D},t)$ for $t>t_0$.

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