Some Regularity Results for Certain Weakly Quasiregular Mappings on the Heisenberg Group and Elliptic Equations

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Dedicated to my parents and my friends

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Introduction

We begin with historical remarks. The research in quasiconformal maps goes back to Lavrentiev's work in 1930s. The first monograph on quasiconformal maps was made in 1966 by Ahlfors [1]. Reshetnyak [58] in 1967 introduced space mappings with bounded distortion in the Euclidean space $\mathbb{R}^n$ of higher dimension $n \geq 3$. Mappings of this kind are not necessary homeomorphic and represent a nice generalization of the classical notion of analytic functions. Systematic study in this area was initiated by Reshetnyak, we refer the reader to monographs [59, 60] and references therein. Simultaneously, the authors of [51] introduced the definition of quasiregular mappings. Later it was proved that the analytical definition of the mapping with bounded distortion and the geometrical notion of quasiregular mappings leads to the same mappings in Euclidean space $\mathbb{R}^n$.

The Reshetnyak theorem, see, for example [59, Chapter 3 section 5], which exhibited a remarkable relation between quasiregular maps and elliptic equations, allowed to make an essential progress in nonlinear potential theory, in degenerate elliptic equations and later in subelliptic equations, see, for instance [9, 10, 29]. Let us recall the Reshetnyak theorem. Suppose that Ω and G are open subsets of $\mathbb{R}^n$, and $f\colon \Omega \to G$ is a quasiregular mapping. Assume that a function $v \in C^1(G)$ is a weak solution in G of the equation $\operatorname{div}\left(|\nabla v|^{n-2}\nabla v\right) = 0$. Then the function $u = v \circ f$ is a weak solution in the domain Ω of the equation

$$\operatorname{div}\left(\langle \theta(x,f)\nabla u(x), \nabla u(x)\rangle^{(n-2)/2}\theta(x,f)\nabla u(x)\right) = 0,$$

where $\theta(x,f) = J_f(x)^{2/n}[Df(x)]^{-1}[D^T f(x)]^{-1}$ and $J_f(x)$ is the Jacobi determinant of f. A simple example of this theorem can be constructed in the case of $n = 2$ in the context of complex function theory. Consider a function $u(x,y)$ in the complex plane $\mathbb{C}$, fix a point a in the unit disk $\mathbb{D}$ and set the Möbius transform

$$w = \frac{z-a}{1-\bar{a}z}. \tag{1}$$

It is not hard to verify that the equality

$$(1-|w|^2)^2 \frac{\partial^2}{\partial w \partial \bar{w}} u = (1-|z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}} u \tag{2}$$

holds and therefore the Möbius transform w preserves the Laplace equation $(\partial_x^2 + \partial_y^2)u = 0$ in the unit disk $\mathbb{D}$. However, the conformal maps in $\mathbb{R}^n$ possess more complicated properties. For example, let us fix a point $a \in B = \{x \in \mathbb{R}^n : |x| < 1\}$ and consider a space mapping

$$y = \varphi_a(x) := j_0 \circ \psi \circ j_a(x) \tag{3}$$

where $j_0(x) := x/|x|^2$, $\quad j_a(x) := (x-a)/(|x-a|^2) + a$ and $\psi(x) := (1-|a|^2)(x-a) - a$. Observe that the space mapping (3) is a generalized version of the Möbius transform (1) in $\mathbb{R}^n$. It is easy to prove that φ_a is a conformal mapping that maps the open set B into itself. Moreover, it maps three points $a, \frac{a}{|a|^2}, \frac{a}{|a|}$ to three points $0, \infty, 1$, respectively (see for example [35]). It was shown by Hua [30] that

$$(1-|y|^2)^n \sum_{i=1}^{n} \partial_{y_i} \left[(1-|y|^2)^{2-n} \partial_{y_i} u\right] = (1-|x|^2)^n \sum_{i=1}^{n} \partial_{x_i} \left[(1-|x|^2)^{2-n} \partial_{x_i} u\right]. \tag{4}$$

The equation (4) generalizes (2) for $n \geq 2$. It is not hard to show that the degenerate elliptic equation $\operatorname{div}(\omega(x)\nabla u) = 0$ is invariant in the unit ball B under the transform (3), and the weight $\omega(x) = (1-|x|^2)^{2-n}$ is not even an admissible weight for odd n, $n \geq 3$.

The example constructed above illustrates that there are some common features in quasiregular maps and elliptic equations. The progress in the study of quasiconformal and quasiregular maps always provides new methods for the theory of elliptic equations. Gehring [17] proved that the Jacobian of quasiconformal maps has a higher integrability property. Shortly thereafter, Meyers and Elcrat [52] obtained the higher integrability result for elliptic systems by making use of Gehring's technique. A well-known result from harmonic analysis [25, theorem 9.33] states that a function $\omega(x)$ is locally higher integrable if and only if $\omega(x)$ is an A_∞-weight and thus it is open-ended. Such property is usually called the self-improving property and the proof is essentially reduced to the use of the reverse Hölder inequality and harmonic analysis techniques. Another characterization of A_∞-weight in terms of Gurov-Reshetnyak condition [26] was obtained by Korenovskyy, Lerner and Stokolos [42]. We shall discuss this topic in different geometrical settings, such as the Euclidean space, the Heisenberg group and the Carnot-Carathéodory space.

The self-improving integrability of quasiregular maps in the planar case is well-understood. Consider solutions of the Beltrami equation $\partial_z f - \mu \partial_{\bar{z}} f = 0$ in the plane, where μ is a bounded measurable function, $\|\mu\|_\infty = k < 1$. The famous problem proposed by Gehring and Reich [18] asks to determine the minimal requirement of the type $f \in W^{1,q}_{\text{loc}}$ which guarantees continuity of any solution of the Beltrami equation. A deep result of Astala [2] says that $f \in W^{1,1+k+\epsilon}$ implies $f \in W^{1+1/k}$ and thus f is a quasiregular map. On the other hand, Iwaniec showed in [32] that $q < 1 + k$ is not sufficient for the continuity. Petermichl and Volberg showed in [57] that the solution is always continuous for the borderline case $q = 1 + k$. There are no good estimates for these thresholds for Euclidean spaces of higher dimensions or for the Carnot-Carathéodory space. Unfortunately, the results of the thesis do not provide much progress in this respect.

The thesis is organized as follows. In chapter 1, we set up a higher integrability result for the horizontal part of certain weakly quasiregular maps on the Heisenberg group. Unlike the Euclidean case, the exponential of the integrability is not near the homogeneous dimension Q that is not analogous to the Euclidean setting. Chapter 2 is devoted to the study of self-improving regularity for certain subelliptic equations. The difficulty of this problem in the Carnot group is that the Whitney extension theorem and the main result in the Carnot group can be obtained only for fourth-order homogeneous subelliptic systems from the arguments in [44]. Since the p-sub-Laplace equation is a very special case of the nonlinear subelliptic equations we can establish a better result in this case via the arguments from [12]. Chapter 3 provides a discussion of self-improving regularity for the degenerate elliptic equations in the Euclidean space. The main result of Chapter 3 extends a result of Lewis from [44] to the degenerate elliptic systems. The proof relies on the weighted pointwise Sobolev inequality for higher order derivatives which is a useful tool in study of higher order degenerate elliptic systems.

Chapter 1

Higher Integrability for Certain Weakly Quasiregular Maps on the Heisenberg Group

This chapter studies quasiregular mappings or, in another terminology, mappings with bounded distortion on the Heisenberg group. We remind the definition of a quasiregular mapping on $\mathbb{R}^n$. We set $W^{1,n}_{loc}(\mathbb{R}^n)$ to be the first order Sobolev space in $\mathbb{R}^n$ and Df be the differential of a map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Definition 1.1 ([59])**.** *Let Ω be an open subset of $\mathbb{R}^n$. If $f : \Omega \longrightarrow \mathbb{R}^n$ is a continuous map and $f \in W^{1,n}_{loc}(\Omega)$, then there exists a constant $K > 0$, such that*

$$\max_{|\xi|=1} |Df(x) \cdot \xi|^n \leq K \det Df, \tag{1.1}$$

$$\det Df \leq K \min_{|\xi|=1} |Df(x) \cdot \xi|^n \tag{1.2}$$

then f is called the **quasiregular map**. *Moreover, if f is a homeomorphism, then f is called the* **quasiconformal map**.

The study of higher integrability property for quasiconformal mappings in $\mathbb{R}^n$ traces back to the work of Gehring [17]. He proved that if f is a quasiconformal mapping, then the differential Df has higher integrability property: $Df \in L^{n+\epsilon}$. Iwaniec and Martin [31, 33, 34] showed that if f satisfies (1.1) and (1.2), then there exits $\delta > 0$ such that $Df \in L^{n-\delta}$ implies $Df \in L^n$. The higher integrability result can be used to study the removability property of quasiregular mappings. It is a question of interest to know whether it is possible to establish the same result for the Heisenberg group. The work [33] can give a clue in this topic. In fact, the use of the Beurling operator is the basic tool in the study of quasiconformal and quasiregular mappings on the even dimensional Euclidean space. The Beurling operator plays a crucial role in the Hodge decomposition. Since the tangent space of the even dimensional Euclidean space has an even dimensional basis $\{\partial_1, \cdots \partial_{2l}\}$, the conclusions in [33] yield that for each element ∂_k, $1 \leq k \leq l$ of the basis, we can find a conjugate vector field $\overline{\partial}_k = \partial_{k+l}$ in the same basis and a bounded Beurling operator S such that $\partial_k = S \circ \overline{\partial}_k$.

The study of the boundedness of the Beurling operator is an interface between harmonic analysis and quasiconformal mappings. The development of both theory and technique of the harmonic analysis has had a big influence on the study of quasiconformal and quasiregular mappings in recent years. We refer the reader, for example, to [43] for the most recent excellent progress in Astala's conjecture regarding the distortion of the Hausdorff dimension under quasiconformal mappings. The novelty in this paper is the proof of the "conformal outside" part, and that estimate relies on the boundedness of the Beurling operator on a non-doubling measure space. The main idea of the proof of the boundedness of the Beurling operator follows the spirit of [23], where the author uses stopping time arguments and construction of exceptional sets. These technique can be traced back to the Fefferman and the Carleson works concerning the convergence of Fourier series, see [6, 13]. For the systematic study of this topic, see also [3].

Unfortunately, this approach does not work even for the lowest dimensional Heisenberg group. This happens due to the absence of a bounded Beurling type operator for the Heisenberg group. To illustrate this we consider the Heisenberg group $(x, y, t) \in \mathbb{H}^1$ with its left invariant vector fields $X = \partial/\partial x + 2y\partial/\partial t$ and $Y = \partial/\partial y - 2x\partial/\partial t$. We know that the operator $\overline{Z} = \frac{1}{2}(X + iY)$ is exactly Lewy's example regarding the unsolvable partial differential operator, see [46], meanwhile the construction of the Beurling operator on the complex plane requires the solvability of the operator $\overline{\partial} = \partial_x + i\partial_y$. Therefore, unlike the even dimensional Euclidean space, it is not wisely to look for a singular integral operator S such that $Z = S \circ \overline{Z}$, where $Z = \frac{1}{2}(X - iY)$.

The authors of [15], provide a simple way to tackle this problem. But the new difficulty arises: the integral zero condition for the Jacobian of the differential of a quasiconformal map on the Heisenberg group does not hold. This is even false for case of the Jacobian of horizontal differential. Indeed, if we consider a map $F = (f, g, h) : \mathbb{H}^1 \longrightarrow \mathbb{H}^1$ with $f, g \in C_0^\infty(\Omega)$, where Ω is an open subset of $\mathbb{H}^1$, then we get for the horizontal differential D_H

$$\int_\Omega \det D_H F dx = \int_\Omega \det \begin{pmatrix} Xf, & Yf \\ Xg, & Yg \end{pmatrix} dx = \int_\Omega XfYg - YfXg dx = -\int_\Omega f[X, Y]g dx.$$

Observe that the integral may not vanish for all possible choices of functions $f, g \in C_0^\infty(\Omega)$. Anyway, this observation shed some light on the study of quasiregular maps on the Heisenberg group. Unlike the Euclidean case, we shall impose some additional conditions on quasiconformal mappings on the Heisenberg group and investigate the higher integrability property in this special case.

We start from the definition of the Heisenberg group. The Heisenberg group $\mathbb{H}^n$ is the set of points $\mathbf{x} = (x', t) \in \mathbb{C}^n \times \mathbb{R}$, $x' \in \mathbb{C}^n$, $x' = (x_1, \ldots, x_n) + i(x_{n+1}, \ldots, x_{2n})$, endowed with the group multiplication defined by $\mathbf{x} \cdot \mathbf{y} = (x' + y', t + s + 2\mathrm{Im} x'\overline{y'})$. We denote by $Q = 2n + 2$ the homogenous dimension of the Heisenberg group $\mathbb{H}^n$. The quasinorm is defined by $|(x', t)|_H = (|x'|^4 + |t|^2)^{1/4}$. There exists the Carnot-Carathéodory metric $d_C(x, y)$ on the Heisenberg group, see [11]. It is defined as the minimum over all lengths of rectifiable curves connecting points x and y. The metric $d_C(x, y)$ is equivalent to the quasinorm $|x^{-1}y|_H$, see [14]. The left invariant vector fields are defined as follows:

$$X_k = \partial/\partial x_k + 2x_{k+n}\partial/\partial t, \quad X_{k+n} = \partial/\partial x_{k+n} - 2x_k\partial/\partial t, \quad T = \partial/\partial t, \quad k = 1, \ldots, n.$$

The vector fields X_k, $k = 1, ..., 2n$ are called horizontal derivatives of the Heisenberg group. We let $dx_1, \ldots, dx_{2n}$, and $\tau = 2\sum_j (x_j dy_j - y_j dx_j) + dt$ be the left invariant 1-forms dual to the basis $X_k, T, k = 1, \ldots, 2n$.

Let Ω be an open set in $\mathbb{H}^n$ and $1 < p < \infty$. We denote by $HW^{1,p}(\Omega)$ the horizontal Sobolev space on the Heisenberg group:

$$HW^{1,p}(\Omega) = \overline{\{f \in C_0^\infty(\Omega) : X_k f \in L^p \ \ k = 1, \ldots, 2n\}}.$$

We set $W^{1,p}(\Omega)$ to be the Sobolev space for all derivatives. The subbundle $HT = \text{span}\,\{X_1, \ldots, X_{2n}\}$ is called the horizontal subbundle of the tangent bundle. We say that one form ω is contact if $\omega(v) = 0$ for any $v \in HT$. A transform $f = (f_1, \ldots, f_{2n}, f_{2n+1}) : \Omega \to \mathbb{H}^n$ is called contact if its differential defines a contact form. It is indicated in [28] that the differential of f can be written as follows:

$$Df := \begin{pmatrix} D_H f & * \\ 0 & \lambda \end{pmatrix}, \quad \text{where } D_H f := \begin{pmatrix} X_1 f_1 & \cdots & X_{2n} f_1 \\ & \cdots & \\ X_1 f_{2n} & \cdots & X_{2n} f_{2n} \end{pmatrix}.$$

The horizontal differential $D_H f(x)$ is the linear map $HT_x \to HT_x$, $x \in \mathbb{H}^n$.

1.1 A Variant Bounded Distortion

We follow the definition of the quasiregular mapping given in [28] and we introduce the weakly quasiregular map on the Heisenberg group in spirit of [31] and [33].

Definition 1.2. *If $f : \Omega \to \mathbb{H}^n$ is a contact continuous map such that $f \in W^{1,q}_{loc}(\Omega)$ and there exists a constant $K > 0$ with*

$$\max_{|\xi|=1} |D_H f(x) \cdot \xi|^Q \le K \det Df, \tag{1.3}$$

$$\det Df \le K \min_{|\xi|=1} |D_H f(x) \cdot \xi|^Q, \tag{1.4}$$

then f is called the quasiregular map in the case $q = Q$. While if $q < Q$ we call the map f the **weakly quasiregular map**.

The inequalities (1.3) and (1.4) imply

$$\max_{\xi \in HT,\ |\xi|=1} |D_H f(x) \cdot \xi| \le K^{2/Q} \min_{\xi \in HT,\ |\xi|=1} |D_H f(x) \cdot \xi|. \tag{1.5}$$

Consider the collection of multiindices

$$\mathfrak{I} = \{(i_1, \ldots, i_n) : |i_k - i_l| \ne n \quad \text{for all integers} \quad 1 \le k, l \le n\}.$$

We fix an index $I \in \mathfrak{I}$, and denote the n-form $\omega = dx^I := dx^{i_1}_{i_1} \wedge \ldots \wedge dx^{i_n}_{i_n}$. The definition of pull-back $\Gamma_\# \colon \ \bigwedge^n \longrightarrow \bigwedge^n$ of a linear transform Γ can be found in [33, page 39] and we follow the notations therein.

We denote by L_k, $1 \leq k \leq n$, the vector fields either X_k or X_{k+n}. Let $J = (j_1, ..., j_n)$ be any n-tuple and let $f_{j_k} \in C_0^\infty(\Omega)$, $k = 1, ..., n$, be smooth functions. We introduce a quantity

$$L_J(x, f) = \det \begin{pmatrix} L_1 f_{j_1} & \cdots & L_n f_{j_1} \\ & \cdots & \\ L_1 f_{j_n} & \ldots & L_n f_{j_n} \end{pmatrix}.$$

Since L_k are skew symmetric and $[L_k, L_j] = 0$ for all $1 \leq k, j \leq n$, we have $\int_\Omega L_J(x, f)dx = 0$, where the observation is true due to [22, page 606]. Next, we write $D_H f \cdot (D_H f)^T = O \cdot \Gamma^2 \cdot O^T$, where O is an orthogonal matrix and

$$\Gamma = \begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_{2n} \end{pmatrix}.$$

The matrix Γ is diagonal with nonnegative diagonal terms γ_k for every $1 \leq k \leq 2n$. We follow the method developed in [33] and obtain

$$|D_H f|^n = \max_{\xi \in HT,\ |\xi|=1} |D_H f \cdot \xi|^n = \max_{1 \leq k \leq 2n} \gamma_k^n.$$

Moreover, we get

$$\max_{1 \leq k \leq 2n} \gamma_k^n \leq K^{2n/Q} \min_{1 \leq k \leq 2n} \gamma_k^n \leq K^{2n/Q} |\Gamma_\sharp \omega| = K^{2n/Q} |(D_H f)_\sharp \omega|$$

from (1.5). Arguments in [33, page 39] imply

$$(D_H f)_\#(\omega) = (D_H f)_\#(dx^I) = \sum_{|J|=n} L_{I,J}(x, f) dx^J,$$

where $L_{I,J}(x, f)$ denotes the determinant of the $(n \times n)$-minor obtained by fixing all j-th rows with $j \in J$ and all i-th columns with $i \in I$. If we impose the condition

$$L_{I,J}(x, f) \geq 0 \text{ or } \leq 0 \text{ for all multiindeces } |J| = n \text{ and for } I \in \mathfrak{I} \tag{1.6}$$

then we arrive to the following estimates that give a more suitable form of bounded distortion.

$$|D_H f|^n \leq K^{2n} \left(\sum_J L_{I,J}(x, f)^2 \right)^{\frac{1}{2}} \leq K^{2n} \sum_J L_{I,J}(x, f). \tag{1.7}$$

The advantage of (1.7) is the vanishing of $\int_\Omega L_{I,J}(x, f)dx$ for all $f \in C_0^\infty(\Omega)$ and for all multiindeces $|J| = n$. If we use the notation $L(x, f) = \sum_J L_{I,J}(x, f)$, then we get

$$\int_{F_\lambda} L(x, f)dx = - \int_{\Omega - F_\lambda} L(x, f)dx$$

for some subset $F(\lambda) \subset \Omega$.

We use the tensor product of $f_H = (f_1, \cdots, f_{2n})$ by $D_H\phi$ with $\phi \in C_0^\infty(\Omega)$

$$f_H \otimes D_H\phi = \begin{pmatrix} f_1X_1\phi & \cdots & f_1X_{2n}\phi \\ & \cdots & \\ f_{2n}X_1\phi & \cdots & f_{2n}X_{2n}\phi \end{pmatrix}$$

in order to prove the following result in analogy with discussions in [15].

Theorem 1.3. *There exist a number $q(n,K) < n$, $n \geq 2$ such that for every $q \in (q(n,K), n)$ and every contact mapping $f \in HW^{1,q}_{loc}(\Omega, \mathbb{H}^n)$ satisfying (1.3)-(1.7), the Caccioppoli type inequality*

$$\|\phi D_H f\|_{L^q} \leq C(q,n,K)\|f_H \otimes D_H\phi\|_{L^q}, \tag{1.8}$$

holds for any $\phi \in C_0^\infty(\Omega)$ and the horizontal part $f_H = (f_1, ..., f_{2n})$ of the mapping f.

If we assume for the moment, that Theorem 1.3 is true, then the higher integrability of $D_H f$ can be proved by making use of the Poincaré inequality, what shows the following theorem.

Theorem 1.4. *There exists a number $1 < q(n,K) < n$, $n \geq 2$, such that for every $q \in (q(n,K), n)$, and for every contact mapping $f \in HW^{1,q}_{loc}(\Omega, \mathbb{H}^n)$ satisfying (1.3)-(1.7) we have*

$$X_k f_j \in L^{q(Q-1)/(Q-q)}_{loc}(\Omega)$$

for any k, j, $1 \leq k, j \leq 2n$.

Proof. Indeed, if we take a ball $B = B(x_0, 2r) \subset \Omega$ in the Carnot-Carathéodory metric and a function $\phi \in C_0^\infty(2B)$, $\phi \equiv 1$ on B such that $|X_k\phi| \leq C/r$ for all $k = 1, \cdots, 2n$, then we get

$$\begin{aligned}\left(\fint_B |D_H f|^{q(Q-1)/(Q-q)} dx\right)^{(Q-q)/(Q-1)q} &\leq C(q,n,K)\frac{1}{r}\left(\fint_{2B} |f_k(x) - (f_k)_{2B}|^{q(Q-1)/(Q-q)} dx\right)^{(Q-q)/(Q-1)q} \\ &\leq C(q,n,K)\frac{1}{r}\left(\fint_{2B} |f_k(x) - (f_k)_{2B}|^{qQ/(Q-q)} dx\right)^{(Q-q)/Qq},\end{aligned}$$

from (1.8) since the volume of the ball $B(x_0, r)$ satisfies the relation $|B(x_0, r)| \approx r^Q$. Applying the Hölder inequality and the sharp form of the Poincaré inequality [48], we obtain

$$\left(\fint_B |D_H f|^{q(Q-1)/(Q-q)} dx\right)^{(Q-q)/(Q-1)q} \leq C(q,n,K)\left(\fint_{CB} |D_H f|^q dx\right)^{1/q}.$$

This gives the higher integrability of $D_H f \in L^{q(Q-1)/(Q-q)}_{loc}$. Therefore we have proven Theorem 1.4. □

Remark 1.5. The inequalities (1.3) and (1.4) show that we actually proved

$$\left(\fint_B (\det Df)^{q(Q-1)/Q(Q-q)} dx\right)^{(Q-q)/(Q-1)q} \leq C(q,n,K)\left(\fint_{\widehat{C}B} (\det Df)^{q/Q} dx\right)^{1/q}$$

for some constants $C(q,n,K) > 0$ and $\widehat{C} > 0$. This implies that $(\det Df)^{q/Q}$ is A_∞-weight.

1.2 Proof of Theorem 1.3

In the following arguments the neighborhood of $\mathbf{x} = (x', s)$ is defined by

$$B(\mathbf{x}, r) := \left\{ \mathbf{y} = (y', t) \in \mathbb{H}^n : \Big(\sum_{i=1}^{2n} |x_i - y_i|^2 \Big)^{1/2} \le r \quad \text{and} \quad |t - s - \mathrm{Im} x' \overline{y'}| \le r^2 \right\}. \tag{1.9}$$

and the metric d is defined by

$$d(\mathbf{x}, \mathbf{y}) = \max \left\{ |x' - y'|,\ |t - s - \mathrm{Im} x' \overline{y'}|^{1/2} \right\} \quad \mathbf{x}, \mathbf{y} \in \mathbb{H}^n. \tag{1.10}$$

It is easy to see that $d(\mathbf{x}, \mathbf{y})$ is a quasi-metric and equivalent to the quasimetric $|\mathbf{x}^{-1}\mathbf{y}|_H$ or the Carnot-Carathéodory metric $d_C(\mathbf{x}, \mathbf{y})$. The neighborhoods defined in (1.9) also form a topology basis.

1.2.1 Auxiliary Lemmas

We need to establish a geometric lemma which is obvious in the Euclidean case. We will use the projection map $\pi_{\mathbb{C}^n} \mathbf{x} = x'$, where $\mathbf{x} = (x', t) \in \mathbb{H}^n$.

Lemma 1.6. *We set* $d_{\mathbf{x}_1} = 2\mathrm{dist}(\mathbf{x}_1, \mathbb{H}^n - B(\mathbf{x}_0, r))$ *for a fixed* $\mathbf{x}_1 \in B(\mathbf{x}_0, r)$. *Then we have*

$$|B(\mathbf{x}_1, Cd_{\mathbf{x}_1}) \cap (\mathbb{H}^n - B(\mathbf{x}_0, r/2))| \ge C(n) |B(\mathbf{x}_1, d_{\mathbf{x}_1})|$$

for some constant $C > 0$ *that only depends on* n.

Proof. Let $\mathbf{x}_1 = (\mathbf{x}_1', s)$ and $\mathbf{x}_0 = (x_0', s_0)$. Since the closure $\overline{B\mathbf{x}_0, r)}$ is a compact set, there exists a point $\mathbf{y}_1 \in \partial B(x_0, r)$ such that $d(\mathbf{x}_1, \mathbf{y}_1) = \mathrm{dist}(\mathbf{x}_1, \mathbb{H}^n - B(\mathbf{x}_0, r))$. If $|x_1' - y_1'| = d(\mathbf{x}_1, \mathbf{y}_1)$, then there exists $\mathbf{x}_2 = (x_2', s) \in B(\mathbf{x}_1, d_{\mathbf{x}_1})$ with $x_2' \in \pi_{\mathbb{C}^n}(\mathbb{H}^n - B(\mathbf{x}_0, r))$ such that

$$U := \{x' \in \mathbb{C}^n : |x' - x_2'| \le (1/100) d_{\mathbf{x}_1}\} \subset \{x' \in \mathbb{C}^n : |x' - x_1'| \le d_{\mathbf{x}_1}\} \cap \pi_{\mathbb{C}^n}(\mathbb{H}^n - B(\mathbf{x}_0, r)),$$

where $\pi_{\mathbb{C}^n}$ denotes the projection operator from the Heisenberg group $\mathbb{H}^n$ to $\mathbb{C}^n$.

Next we aim to show that if $|t - s - \mathrm{Im} x' \overline{x_2'}| < (d_{\mathbf{x}_1}/100)^2$ for all $x' \in U$, then $|t - s - \mathrm{Im} x' \overline{x_1'}| < Cd_{\mathbf{x}_1}^2$. In fact, we have

$$\begin{aligned} |t - s - \mathrm{Im} x' \overline{x_1'}| &= |2(t-s) - (t-s) - \mathrm{Im} x_2' \overline{x_1'} - \mathrm{Im} x' \overline{x_2'} + \mathrm{Im} x_2' \overline{x_2'} - \mathrm{Im}(x' - x_2')\overline{(x_1' - x_2')}| \\ &\le |t - s - \mathrm{Im} x_2' \overline{x_1'}| + |t - s - \mathrm{Im} x' \overline{x_2'}| + |t - s - \mathrm{Im} x_2' \overline{x_2'}| + |(x' - x_2')(x_1' - x_2')| \\ &\le 4 d_{\mathbf{x}_1}^2. \end{aligned}$$

Therefore, we have proved the inclusion $B(\mathbf{x}_2, d_{\mathbf{x}_1}/100) \subset B(\mathbf{x}_1, Cd_{\mathbf{x}_1}) \cap (\mathbb{H}^n - B(\mathbf{x}_0, r))$. We estimate

$$|B(\mathbf{x}_1, Cd_{\mathbf{x}_1}) \cap (\mathbb{H}^n - B(\mathbf{x}_0, r))| \ge |B(\mathbf{x}_2, d_{\mathbf{x}_1}/100)| \ge C(n) |B(\mathbf{x}_1, d_{\mathbf{x}_1})|.$$

On the other hand, if the minimum is attained at the ”bottom” or ”top”, that means for $\mathbf{y}_1 = (y_1', s_1) \in \partial B(\mathbf{x}_0, r)$, we have $|x_1' - y_1'| < |s - s_1 - \mathrm{Im} x_1' \overline{y_1'}|^{1/2} = d(\mathbf{x}_1, \mathbf{y}_1) = d_{\mathbf{x}_1}/2$. We assert that the boundary point $\mathbf{y}_1$ must have the property $|y_1' - x_0'| \le |s_1 - s_0 - \mathrm{Im} y_1' \overline{x_0'}|^{1/2}$. If this were not true, the point $(y_1', s_1 \pm \epsilon)$ would also lie on the boundary. But this could change the value of distance from $\mathbf{x}_1$ to the boundary. By this argument we must have $|s_1 - s_0 - \mathrm{Im} y_1' \overline{x_0'}| = r^2$ and for the fixed y_1', x_0', s_1 is unique on one side.

We only consider the case when $(y_1', s_1 - \epsilon)$ lies outside of the domain $B(\mathbf{x}_0, r)$, since arguments for the case when $(y_1', s_1 + \epsilon)$ is outside of the domain are similar. We take a point $\mathbf{x}_3 = (y_1', s_1 - d_{\mathbf{x}_1}^2)$. We can show that $B(\mathbf{x}_3, d_{\mathbf{x}_1}/100) \subset \mathbb{H}^n - B(\mathbf{x}_0, r/2)$. In fact, we have

$$\begin{aligned} |t - s - \mathrm{Im} y' \overline{x_0'}| &= |t - (s_1 - d_{\mathbf{x}_1}^2) - \mathrm{Im} y' \overline{y_1'} + (s_1 - d_{\mathbf{x}_1}^2) - s_0 - \mathrm{Im} y_1' \overline{x_0'} - \mathrm{Im}\left((y' - y_1')(x_0' - y_1')\right)| \\ &\ge r^2 + d_{\mathbf{x}_1}^2 - (d_{\mathbf{x}_1}/100)^2 - r d_{\mathbf{x}_1}/100 \ge (r/2)^2 \end{aligned}$$

for any $(y', t) \in B(\mathbf{x}_3, d_{x_1}/100)$. The last step is followed from the fact $d_{\mathbf{x}_1} \le r$. We also need to show $B(\mathbf{x}_3, d_{\mathbf{x}_1}/100) \subset B(\mathbf{x}_1, C d_{\mathbf{x}_1})$ for some constant $C > 0$. This is easy case, since

$$d((y', t), x_1) \le C\left[d((y', t), x_3) + d(x_3, y_1) + d(y_1, x_1)\right] \le C d_{x_1}$$

for any $(y', t) \in B(\mathbf{x}_3, d_{\mathbf{x}_1}/100)$. So we have $B(\mathbf{x}_3, d_{\mathbf{x}_1}/100) \subset B(\mathbf{x}_1, C d_{\mathbf{x}_1}) \cap (\mathbb{H}^n - B(\mathbf{x}_0, r/2))$, that completes the proof of Lemma 1.6. □

The following Lemma was proved in [27] for vector fields satisfying the Hörmander condition. Here we provide a simpler proof for the homogeneous group.

Lemma 1.7. *Let Ω be an arbitrary domain in $\mathbb{H}^n$. If f is a Lipschitz function on $\Omega \subset \mathbb{H}^n$, then $f \in HW^{1,\infty}(\Omega)$.*

Proof. If we write $y = \exp(tX_k)$ and $y_0 = \exp(X_k)$ for horizontal vector field X_k then $y = t y_0$. Since $|f(xy) - f(x)| \le C|y|_H$ and

$$X_k f = \frac{d}{dt} f(x \exp(tX_k))\bigg|_{t=0} = \lim_{t \to 0} \frac{f(x \exp(tX_k)) - f(x)}{t}, \tag{1.11}$$

we get

$$\left| f(x \exp(tX_k)) - f(x) \right| \le C |t y_0|_H.$$

Therefore, we have

$$\left| [f(x \exp(tX_k)) - f(x)]/t \right| \le C |y_0|_H < \infty$$

by (1.11) for all $|t| < \delta$, where δ is a small enough positive number. There exists a function $g \in L^\infty(\Omega)$ such that there is a sequence $t_j \to 0$ with

$$[f(x \exp(t_j X_k)) - f(x)]/t_j \rightharpoonup g \quad \text{weakly in} \quad L^2_{loc}(\Omega).$$

On the other hand,

$$\begin{aligned} \int_\Omega \phi [f(x \exp(t_j X_k)) - f(x)]/t_j dx &= \int_\Omega f(x)[\phi(x \exp(-t_j X_k)) - \phi(x)]/t_j dx \\ &\to -\int_\Omega (X_k \phi) f(x) dx \end{aligned}$$

for any test function $\phi \in C_0^\infty(\Omega)$. This implies $\int_\Omega \phi g dx = -\int_\Omega (X_k \phi) f dx$ and $X_k f \in L^\infty(\Omega)$. □

Lemma 1.8 (Pointwise Sobolev inequality [36, 47]). *Let $u \in W^{1,p}(\mathbb{H}^n)$, $1 < p < \infty$, and let $x \in B = B(x_0, r)$. Then there exist constants $c > 0$ and $C > 0$ such that*

$$|u(x) - u_B| \leq crM(|X_k u|\chi_{CB})(x_0))$$

$$|u(x) - u(y)| \leq cd_C(x,y)[M(|X_k u|)(x) + M(|X_k u|)(y)],$$

where $M(f)$ denotes the Hardy-Littlewood maximal function defined on the Heisenberg group, X_k is any horizontal derivative and χ_G is the characteristic function of the set G.

1.2.2 Whitney Extensions for Horizontal Components f_k, $k = 1, \cdots, 2n$

We assume that $\phi \in C_0^\infty(\Omega)$ and supp $\phi \subset B_0 := \{x \in \mathbb{H}^n : \ d(x,0) < r/2\}$. Let $g = |\phi D_H f| + |f_H \otimes D_H \phi|$ and let

$$F_\lambda = \{x \in B_0 : M(g) \leq \lambda\} \quad \text{for} \quad \lambda > 0,$$

where $M(g)$ is the maximal function of g on the Heisenberg group. We aim to show that $u_k = f_k \phi$, $1 \leq k \leq 2n$, are the Lipschitz functions on the closed set $E(\lambda) = F(\lambda) \cup (\mathbb{H}^n - B)$, where $B = \{x \in \mathbb{H}^n : \ d(x,0) < r\}$. We will consider three cases.

Supposing $x, \ y \in F(\lambda)$, the Lemma 1.8 implies

$$\begin{aligned} |u_k(x) - u_k(y)| &\leq cd_C(x,y)[M(|X_i u|)(x) + M(|X_i u|)(y)] \\ &\leq cd_C(x,y)[M(|g|)(x) + M(|g|)(y)] \\ &\leq c\lambda d_C(x,y). \end{aligned}$$

If $x, \ y \in \mathbb{H}^n - B$, then $u_k(x) = u_k(y) = 0$. We set $B_1 := \{z \in B(x, d_x) : u_k(z) = 0\}$. Lemma 1.6 implies

$$|B_1| \geq |B(x, Cd_x) \cap (\mathbb{H}^n - B(x_0, r/2))| \geq C(n)|B(x, d_x)| \tag{1.12}$$

for the case $x \in F(\lambda)$ and $y \in \mathbb{H}^n - B$. Basing on (1.12) and the Poincaré inequality, we get

$$\begin{aligned} \left| \fint_{B(x,d_x)} u_k(y)dy \right| &\leq C(n)\frac{|B_1|}{|B(x,d_x)|}\left| \fint_{B(x,d_x)} u_k(y)dy \right| \\ &\leq C(n)\left(\frac{|B_1|}{|B(x,d_x)|}\left| \fint_{B(x,d_x)} u_k(y)dy \right| + \frac{1}{|B(x,d_x)|}\int_{B(x,d_x)-B_1} |u_k - (u_k)_{B(x,d_x)}|dy \right) \\ &\leq C(n)\fint_{B(x,d_x)} |u_k - (u_k)_{B(x,d_x)}|dy \\ &\leq C(n)d_x\left(\fint_{B(x,d_x)} |X_i u|^{Q/(Q+1)}dy \right)^{Q+1/Q}. \end{aligned}$$

Therefore, by the Hölder inequality we have

$$(u_k)_{B(x,d_x)} \leq C(n)d_x \fint_{B(x,d_x)} |X_i u| \leq C(n)d_x M(g)(x) \leq C(n)\lambda d_C(x,y).$$

This yields

$$\begin{aligned}|u_k(x) - u_k(y)| &\le |u_k(x) - (u_k)_{B(x,d_x)}| + |(u_k)_{B(x,d_x)}| \\ &\le C(n)d_x \fint_{B(x,d_x)} |X_i u| + C(n)\lambda d_C(x,y) \\ &\le C(n)\lambda d_C(x,y).\end{aligned}$$

We have proven that u_k is a Lipschitz function on $E(\lambda)$ and Lipschitz constant is $C(n)\lambda$. Since $(\mathbb{H}^n, d_C)$ is a metric space, we can use the McShane extension theorem [53]. We extend u_k to the Lipschitz extension function u_k^λ defined on $\mathbb{H}^n$, which can be constructed as follows

$$u_k^\lambda(x) = \sup_{x_1 \in E(\lambda)} [u_k(x_1) - C(n)\lambda d_C(x_1, x)].$$

1.2.3 Stopping Time Arguments

Consider any cofactor $L_{I,J}(x, \phi f)$ that was defined in Subsection 1.1. We denote by $(j_1, \cdots j_n)$ a multiindex from J and let $u_{j_1}^\lambda$ be a Lipschitz extension of ϕf_1. Lemma 1.7 shows that $u_{j_1}^\lambda \in HW^{1,\infty}(\Omega)$. Let $f_\lambda = (u_{j_1}^\lambda, \phi f_{j_2}, \cdots \phi f_{j_n})$. By approximation arguments and the Hölder inequality we get

$$\int_\Omega L_{I,J}(x, f_\lambda) dx = 0$$

for all n-tuples from I and J.

Since $|f_{j_1} D_H \phi| \le C(n)|f_H \otimes D_H \phi|$ and $|X_i(\phi f_{j_1})| \le C(n) g$ for all $i \in I$, we have

$$\int_{F(\lambda)} \phi^n L_{I,J}(x, f) \le C(n) \left(\lambda \int_{\Omega - F(\lambda)} g^{n-1} + \int_{F(\lambda)} |f_H \otimes D_H \phi| g^{n-1} \right)$$

from the above estimates. We multiply by $\lambda^{-1-\epsilon}$ both sides and interchange the integrations to obtain

$$\int_\Omega \phi^n L_{I,J}(x, f) M(g)(x)^{-\epsilon} \le C(n) \left(\frac{\epsilon}{1-\epsilon} \int_\Omega g^{n-1} M(g)(x)^{1-\epsilon} dx + \int_\Omega |f_H \otimes D_H \phi| g^{n-1} M(g)(x)^{-\epsilon} dx \right)$$

for all n-tuples from I and J. The definition of the variant bounded distortion of quasiregular maps on the Heisenberg group (1.7) essentially implies

$$\int_\Omega \phi^n |D_H f|^n M(g)(x)^{-\epsilon} \le \frac{C(n) K^2 \epsilon}{1-\epsilon} \int_\Omega g^{n-1} M(g)(x)^{1-\epsilon} dx + C(n) K^2 \int_\Omega |f_H \otimes D_H \phi| g^{n-1} M(g)(x)^{-\epsilon} dx. \tag{1.13}$$

Next, by the Hölder inequality, and the Hardy-Littlewood maximal theorem we have

$$\begin{aligned}\int_\Omega |\phi D_H f|^{n-\epsilon} dx &\le \left(\int_\Omega |\phi D_H f|^n M(g)^{-\epsilon} dx \right)^{(n-\epsilon)/n} \left(\int_\Omega M(g)^{n-\epsilon} dx \right)^{\epsilon/n} \\ &\le C(n) \left(\int_\Omega |\phi D_H f|^n M(g)^{-\epsilon} dx \right)^{(n-\epsilon)/n} \left(\int_\Omega g^{n-\epsilon} dx \right)^{\epsilon/n}\end{aligned}$$

for a small enough $\epsilon > 0$ and an integer $n \geq 2$. If the following estimate is true

$$\int_\Omega g^{n-\epsilon} dx \geq 2^{n-\epsilon} \int_\Omega |\phi D_H f|^{n-\epsilon} dx,$$

then Theorem 1.3 has been proven. In the case

$$\int_\Omega g^{n-\epsilon} dx \leq 2^{n-\epsilon} \int_\Omega |\phi D_H f|^{n-\epsilon} dx,$$

making use of the above estimates, the Hölder inequality, the Hardy-Littlewood maximal theorem, and the fact $g \leq M(g)$ one gets

$$\begin{aligned} \int_\Omega g^{n-\epsilon} dx &\leq C(n) \int_\Omega |\phi D_H f|^n M(g)^{-\epsilon} dx \\ &\leq \frac{C(n)K^2\epsilon}{1-\epsilon} \int_\Omega g^{n-\epsilon} dx + C(n)K^2 \left(\int_\Omega |f_H \otimes D_H \phi|^{n-\epsilon} dx \right)^{1/(n-\epsilon)} \left(\int_\Omega g^{n-\epsilon} dx \right)^{(n-\epsilon-1)/(n-\epsilon)} . \end{aligned}$$

This leads to the estimate

$$\int_\Omega g^{n-\epsilon} dx \leq C(n) \int_\Omega |f_H \otimes D_H \phi|^{n-\epsilon} dx.$$

The proof of Theorem 1.3 is completed.

Chapter 2

Self-Improving Regularity for the Very Weak Solutions of Subelliptic Equations

We begin with the definition of elliptic systems in $\mathbb{R}^n$. Let m be an integer number greater than or equal to 1. Introduce the notations $P = \prod_{0\le|I|\le m}\mathbb{R}^N$ and $D^m u = (u, \partial_x u, \cdots, \partial_x^{\sigma} u)$ for all $|\sigma| = m$. In order to simplify the notation, we denote by $\partial^m u$ the summation $\sum_{|\sigma|=m}\partial_x^{\sigma} u$. We assume in this chapter that Ω is a bounded domain in $\mathbb{R}^n$. Let $A = (A_\sigma)$: $\Omega \times P \to \mathbb{R}^N$ be a function such that $A_\sigma(\cdot, D^m u(x))$, $x \in \Omega$, is a measurable function on Ω, satisfying the following conditions:

$$\sum_{|\sigma|=m} A_\sigma(x, D^m u(x)) \cdot \partial_x^{\sigma} u(x) \ge \gamma |\partial^m u|^p - a(x) \qquad a.e. \text{ in } \Omega \tag{2.1}$$

and

$$|A_\sigma(x, D^m u(x))| \le |\partial_x^{\sigma} u(x)|^{p-1} + b_\sigma(x) \qquad a.e. \text{ in } \Omega, \tag{2.2}$$

where $|\sigma| \le m$ and $a(x)$, $b_\sigma(x)$ are nonnegative integrable functions. We say that a function $u \in W^{m,p}(\Omega)$ is a weak solution of

$$\sum_{|\sigma|=0}^{m} (-1)^{|\sigma|} \partial_x^{\sigma} A_\sigma(x, D^m u(x)) = 0$$

on an open set Ω if

$$\sum_{|\sigma|=0}^{m} \int_\Omega A_\sigma(x, D^m u(x)) \partial_x^{\sigma} \phi dx = 0 \tag{2.3}$$

for any test function $\phi = (\phi_1, \cdots, \phi_N) \in C_0^\infty(\Omega)$.

In [52], authors extended Gehring's lemma, regarding the higher integrability [17] in a more general form, and proved that the weak solution of elliptic system (2.1)-(2.3) has the higher integrability property. That is to say, there exists an $\epsilon > 0$ such that the weak solution $u \in W^{m,p+\epsilon}(\Omega)$. There are several monographs studying and generalizing this question (see, for example [4, 8, 19]). Motivated by the Iwaniec and Martin work on the integrability of weakly quasiregular maps [31, 33, 34], Lewis [44] introduced a very weak solutions for elliptic systems;

that is, if u satisfies (2.3) and $u \in W^{1,q}(\Omega)$ for $q < p$ then u is a very weak solution of elliptic system (2.1)-(2.3). Lewis [44] obtained a higher integrability property for the very weak solutions of the elliptic systems that is the extension of the result in [52]. Also, a number of authors have given regularity results concerning parabolic equations. Kinnunen and Lewis [40, 41] proved the result for the first order parabolic equations and Bögelein [5] generalized these results for the higher order parabolic systems. It is worth mentioning that there are another ways to study the higher integrability problem for elliptic equations, see [29] for the second order degenerate elliptic equations and [9] for the second order subelliptic equations.

It is also interesting to know whether the Lewis celebrated result [44] can be extended to the Carnot-Caratheodory space. However up to now, it is only known that this kind of extension works for the particular case for the second order system. Zatorska-Goldstein [16] got a higher integrability result for the second order subelliptic equations by combining the technique in [44] with the Young inequality. This kind of technique was shown to be extremely useful in the study of the first order parabolic equations, see [40, 41]. The difficulty in this problem is that the Whitney extension seems to be more complicated in the Carnot group than in the Euclidean space. We shall discuss a certain specific subelliptic equations in this chapter.

In the first section, we prove the self-improving regularity result for weak solutions of the fourth order homogeneous subelliptic system. In Euclidean space, a typical fourth order homogeneous elliptic equation is the biharmonic equation $\Delta^2 u = \Delta \circ \Delta u = 0$, $\Delta = \partial^2_{x_1} + \cdots + \partial^2_{x_n}$, which appears in the linear elasticity theory and the Stokes flows. We will consider a generalized form of this kind of equation that is given in more general geometric setting, namely, on the Carnot group $\mathbb{G}$.

In the second section, we consider the p-sub-Laplace equation. In this particular case, there are some nice properties of weak and very weak solutions. We follow the approach of [12], where it is claimed that a special kind of function Λ, defined by the p-sub-Laplace equation, belongs to the Hardy space $H^1(\mathbb{R}^n)$ if we have the decomposition $\Lambda = \overrightarrow{E} \cdot \overrightarrow{B}$, where $\overrightarrow{E} \in (L^p(\mathbb{R}^n))^n$ is the "electric field"; that is $\operatorname{div}\overrightarrow{E} = 0$, $\overrightarrow{B} \in (L^{p'}(\mathbb{R}^n))^n$ is the "magnetic field" defined by $\operatorname{curl}\overrightarrow{B} = 0$ with $\frac{1}{p'} + \frac{1}{p} = 1$. Unlike the approach in [54], this technique provides another point of view on studying higher integrability of determinants. Making use of this technique is useful for study of determinants of vector fields satisfying the Hörmander hypoellipticity condition (see [21, 22]).

2.1 Self-Improving Regularity for the Weak Solutions of Fourth order Homogeneous Subelliptic Systems

The Carnot group is a connected simply connected Lie group, whose Lie algebra $\mathfrak{g}$ is nilpotent and graded:

$$\mathfrak{g} = V_1 \oplus ... \oplus V_n, \qquad [V_1, V_j] = V_{j+1}, \quad j < n, \qquad [V_1, V_n] = 0.$$

Let $Y_1, Y_2, ..., Y_M$ be left invariant vector fields on $\mathbb{G}$ that form a basis of Lie algebra $\mathfrak{g}$, here $M = \dim \mathbb{G}$. We say that a vector field Y_i has a degree d_i if $Y_i \in V_{d_i}$. The vector fields $X_1, X_2, \ldots, X_{n_1}$, that form the basis of V_1, are called the horizontal derivatives on $\mathbb{G}$. If $I = (i_1, ..., i_M)$ is a multiindex, then we denote by $Y^I = Y_{i_1}^{i_1} Y_{i_2}^{i_2} \cdots Y_{i_M}^{i_M}$ the differential operator of order $|I| = i_1 + \ldots + i_M$, and by $d(I) = d_1 i_1 + \ldots + d_M i_M$ the homogeneous degree of the multiindex. The exponential mapping $x = \exp \sum_{i=1}^M x_i Y_i$ is a diffeomorphism of $\mathfrak{g}$ onto $\mathbb{G}$ and we use it to introduce the normal coordinates. Let $\eta_i(x) = x_i$, $i = 1, \cdots, M$ be the coordinate functions. We denote by $Q = \sum_{i=1}^M d_i$ the homogeneous dimension of the Carnot group.

Let us define the homogeneous polynomial on the Carnot group. We denote by $\eta^I = \eta_1^{i_1} \cdots \eta_M^{i_M}$ the monomial of homogeneous degree $d(I) = d_1 i_1 + \ldots + d_M i_M$. A homogeneous polynomial of homogeneous degree d is a linear combination of monomials of the same homogeneous degree d. We say that a polynomial has homogeneous degree d if it is a linear combination of monomials with the homogeneous degree at most d. We let $|\cdot|_G$ be a quasinorm in the Carnot group. We shall use the notation $|B|$ to denote the Haar measure of a set B.

Let $X^I = X_1^{i_1} X_2^{i_2} \cdots X_{n_1}^{i_{n_1}}$ be the m-th order horizontal derivative, Ω be an open bounded domain in the Carnot group. Define the functional space $L_m^p(\Omega)$ of functions $u : \Omega \to \mathbb{R}^N$ as follows:

$$L_m^p(\Omega) = \left\{ u = (u_1, ..., u_N) : \ u_k \in L^p(\Omega), \ \|X^I u_k\|_{L^p(\Omega)} < \infty, |I| \leq m, k = 1, ..., N \right\}.$$

We consider the horizontal Sobolev space on the Carnot group defined by

$$HW^{m,p}(\Omega) = \overline{C^\infty(\Omega) \cap L_m^p(\Omega)}.$$

To define the fourth order homogeneous subelliptic systems on the Carnot group G we follow the definition of the higher order elliptic systems in $\mathbb{R}^n$. Denote $P = \prod_{0 \leq |I| \leq 2} \mathbb{R}^N$ and $D^2 u = (u, X^1 u, X^\sigma u)$ for all $|\sigma| = 2$. Let $A = (A_\sigma)$: $\Omega \times P \to \mathbb{R}^N$ be a function such that $A_\sigma(\cdot, D^2 u(x))$, $x \in \Omega$ is measurable in Ω and satisfies the following conditions:

$$A_\sigma(x, D^2 u(x)) \cdot X^\sigma u(x) \geq \gamma |X^\sigma u(x)|^p \tag{2.4}$$

almost everywhere in Ω and

$$|A_\sigma(x, D^2 u(x))| \leq |X^\sigma u(x)|^{p-1} + b_\sigma(x) \tag{2.5}$$

almost everywhere in Ω, where $|\sigma| = 2$. We say that $u(x) \in HW^{2,q}(\Omega)$ is the weak solution of fourth order homogeneous elliptic system

$$\sum_{|\sigma|=2} X^\sigma A_\sigma(x, D^2 u(x)) = 0,$$

if u satisfies the following identity

$$\sum_{|\sigma|=2} \int_\Omega A_\sigma(x, D^2 u(x)) X^\sigma \phi(x) dx = 0 \tag{2.6}$$

for any $\phi = (\phi_1, ..., \phi_N) \in C_0^\infty(\Omega)$ and $q = p$. We say that u is the very weak solution of (2.6) if $q < p$. In order to simplify the notation, we denote by $X^2 u$ the summation with respect to the indices i and j, that is $\sum_{i,j=1}^{n_1} X_i X_j u$.

In this chapter we follow the approach, developed in [44], to obtain a self-improving integrability result for subelliptic equations. Our principal result states as follows

Theorem 2.1. *Let Ω be a bounded domain on the Carnot group $\mathbb{G}$, $u \in HW_{loc}^{2,r}(\Omega)$ and A satisfies* (2.4)-(2.6). *Then for $p > 1$ there exists a $\delta = \delta(Q, N, \gamma, p) > 0$ such that if $r = p - \delta$, then $u \in HW_{loc}^{2,p+\delta}(\Omega)$.*

2.1.1 Preliminary Lemmas

First of all, we need to establish the pointwise Sobolev inequality for higher order derivatives on the Carnot group $\mathbb{G}$, which is the content of Lemma 2.2. For a locally integrable function f on $\mathbb{G}$, and $B(x,r) = \{y \in \mathbb{G} : d_C(x,y) < r\}$, where d_C is the Carnot- Caratheodory metric on the Carnot group we let

$$f_B = \frac{1}{|B|}\int_B f dx = \fint_B f dx.$$

We define the center maximal function

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|dy,$$

and a localized operator with respect to any fixed subset $B \subset \mathbb{G}$,

$$\mathfrak{M}_B(f)(x) = M(f\chi_B)(x),$$

where χ_B is the characteristic function of B. We let $\mathfrak{M}_B^k(f)(x)$ to be the k-times composition operator of $\mathfrak{M}_B(f)$, that is if $\mathfrak{M}_B^{k-1}(f)(x)$ is defined then

$$\mathfrak{M}_B^k(f)(x) = \sup_{r>0} \fint_{B(x,r)} \mathfrak{M}_B^{k-1}(f)(y)\chi_B(y)dy$$

is defined inductively for $k \geq 2$. We remind that the Carnot-Caratheodory metrc $d_C(x,y)$ on the Carnot group is defined as infimum over lengths of all absolutely continuous curves $\gamma : [0,1] \to \mathbb{G}$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\dot{\gamma} \in \operatorname{span}\{X_1, \cdots, X_{n_1}\}$.

Lemma 2.2. *Let l be a positive integer, $1 < q < \infty$, $r > 0$, $x_0 \in \Omega$, and $B_0 = B(x_0, r)$. If $u \in HW^{l,q}(\Omega)$ in a bounded open subset $\Omega \subset \mathbb{G}$ and $\int_{B_0} X^\alpha u = 0$ for $0 \leq |\alpha| \leq l-1$, then there exist constants $C_1(n_1, Q, l, q)$ and $C > 0$ such that*

$$|u(x)| \leq C_1 r^l \mathfrak{M}_B^l\Big(\sum_{|\sigma|=l} |X^\sigma u|\Big)(x), \quad x \in B = B(x_0, Cr). \tag{2.7}$$

Moreover, if $1 < s < q$ and $ls < Q$, then there exists $C_2(n_1, Q, l, q) > 0$ such that

$$|u(x)| \leq C_2 r^l \left(\fint_B \mathfrak{M}_B^l\Big(\sum_{|\sigma|=l} |X^\sigma u|\Big)^s(x)dx\right)^{\frac{l}{Q}} \mathfrak{M}_B^l\Big(\sum_{|\sigma|=l} |X^\sigma u|\Big)^{\frac{s}{s^*}}(x), \tag{2.8}$$

$$\left(\fint_B |u(x)|^{s^*} dx\right)^{\frac{1}{s^*}} \leq C_2 r^l \left(\fint_B \mathfrak{M}_B^l\Big(\sum_{|\sigma|=l} |X^\sigma u|\Big)^s(x)dx\right)^{\frac{1}{s}}, \tag{2.9}$$

where $s^* = \frac{Qs}{Q-ls}$. Furthermore, if $ls > Q$, then

$$|u(x)| \leq \widehat{C_2} r^l \left(\fint_B \mathfrak{M}_B^l\Big(\sum_{|\sigma|=l} |X^\sigma u|\Big)(x)^s dx\right)^{\frac{1}{s}} \tag{2.10}$$

where $\widehat{C_2} = \widehat{C_2}(Q, l, s)$.

Proof. We start our proof from the first order case, and then continue to show the higher order result by induction. We know from papers [36, 47], that the Morrey type inequality holds for the Carnot group. Since $u_B = 0$ then

$$\begin{aligned}|u(x)| = |u - u_{B_0}| &\le C \int_B \sum_{k=1}^{n_1} |X_k u(y)| d_C(x,y)^{-Q+1} dy \\ &= C \int_{\{y\in B: d_C(x,y)<\eta\}} + \int_{\{y\in B: d_C(x,y)\ge\eta\}} \sum_{k=1}^{n_1} |X_k u(y)| d_C(x,y)^{-Q+1} dy := I + II\end{aligned}$$

for some $\eta > 0$ that will be chosen later. To estimate I we write I in the form

$$\begin{aligned}I &= C \sum_{k=0}^{\infty} \int_{\{y\in B: 2^{-k-1}\eta \le d_C(x,y) < 2^{-k}\eta\}} \sum_{k=1}^{n_1} |X_k u(y)| d_C(x,y)^{-Q+1} dy \\ &\le C \sum_{k=0}^{\infty} 2^{-k}\eta \frac{1}{(2^{-k}\eta)^Q} \int_{\{y\in B: d_C(x,y) < 2^{-k}\eta\}} \sum_{k=1}^{n_1} |X_k u(y)| dy.\end{aligned}$$

Due to the relation $|B(x, 2^{-k}\eta)| \approx (2^{-k}\eta)^Q$, we get

$$I \le C\eta M(\sum_{k=1}^{n_1} |X_k u| \chi_B)(x).$$

If $\eta > Cr$ then $II = 0$ and the above estimate shows that (2.7) holds for $l = 1$. In the case $\eta < Cr$ we apply the Hölder inequality with the exponent s, $1 < s < Q$ for the second term and deduce

$$II \le C \left(\int_{\{y\in B: d_C(x,y)\ge\eta\}} (\sum_{k=1}^{n_1} |X_k u(y)|)^s dy \right)^{\frac{1}{s}} \left(\int_{\{y\in B: d_C(x,y)\ge\eta\}} d_C(x,y)^{(-Q+1)s'} dy \right)^{\frac{1}{s'}}.$$

Since $d_C \approx |x^{-1}y|_G$, we can estimate

$$\int_{\{y\in B: d_C(x,y)\ge\eta\}} d_C(x,y)^{(-Q+1)s'} dy \le C_2 \int_{\{y\in B: |x^{-1}y|_G \ge C\eta\}} |x^{-1}y|_G^{(-Q+1)s'} dy = \int_{\mathbb{G}} H(x^{-1}y) dy,$$

where $H(z) = C_2 |z|_G^{(-Q+1)s'} \chi_{\{z\in\mathbb{G}: |z|_G \ge C\eta\}}$. The biinvarians of the Haar measure on the Carnot group and [14, corollary 1.16] imply

$$\int_{\{y\in B: d_C(x,y)\ge\eta\}} d_C(x,y)^{(-Q+1)s'} dy \le C(s,Q)\eta^{Q-(Q-1)s'}.$$

Therefore, we obtain

$$II \le C\eta^{-\frac{1}{s}Q+1} \left(\int_{\{y\in B: d_C(x,y)\ge\eta\}} (\sum_{k=1}^{n_1} |X_k u(y)|)^s dy \right)^{\frac{1}{s}}.$$

We choose

$$\eta = \left(\int_{\{y\in B: d_C(x,y)\geq \eta\}} (\sum_{k=1}^{n_1} |X_k u(y)|)^s dy \right)^{\frac{1}{Q}} M(\sum_{k=1}^{n_1} |X_k u|\chi_B)^{-\frac{s}{Q}}(x)$$

in order to get $I = II$. Thus we obtain the chain of estimates

$$\begin{aligned}
|u(x)| &\leq Cr\left(\fint_B (\sum_{k=1}^{n_1} |X_k u(y)|)^s dy \right)^{\frac{1}{Q}} M(\sum_{k=1}^{n_1} |X_k u|\chi_B)^{\frac{s}{s^*}}(x) \\
&\leq Cr\left(\fint_B M(\sum_{k=1}^{n_1} |X_k u|\chi_B)^s(y) dy \right)^{\frac{1}{Q}} M(\sum_{k=1}^{n_1} |X_k u|\chi_B)^{\frac{s}{s^*}}(x) \\
&= Cr\left(\fint_B \mathfrak{M}_B(\sum_{k=1}^{n_1} |X_k u|)^s(y) dy \right)^{\frac{1}{Q}} \mathfrak{M}_B(\sum_{k=1}^{n_1} |X_k u|)^{\frac{s}{s^*}}(x).
\end{aligned}$$

This finishes the proof for the case $l = 1$.

For the general case, we assume that the theorem already holds for $l-1$, and we proceed to show the theorem for $l > 1$. We repeat the above arguments for $x \in B$, applying the induction hypothesis to every $X_k u$, and get

$$\begin{aligned}
I &\leq C\eta M(\sum_{k=1}^{n_1} |X_k u|\chi_B)(x) \\
&\leq C(n_1)C_2\eta r^{l-1}\left(\fint_B \mathfrak{M}_B^{l-1}(\sum_{|\sigma|=l} |X^\sigma u|)^s(x)dx \right)^{\frac{l-1}{Q}} M\left(\mathfrak{M}_B^{l-1}(\sum_{|\sigma|=l} |X^\sigma u|)(\cdot)\chi_B(\cdot) \right)^{\frac{s}{s^*}}(x) \\
&\leq C(n_1)C_2\eta r^{l-1}\left(\fint_B \mathfrak{M}_B^{l-1}(\sum_{|\sigma|=l} |X^\sigma u|)^s(x)dx \right)^{\frac{l-1}{Q}} \left(\mathfrak{M}_B^{l}(\sum_{|\sigma|=l} |X^\sigma u|(\cdot)\chi_B(\cdot))(x) \right)^{\frac{s}{s^*}}
\end{aligned}$$

for $ls < Q$ and $s^* = \frac{Qs}{Q-ls+s}$. The last step followed from the Hölder inequality and the definition of the localized operator. If $\eta < Cr$, then (2.7) holds for l. If not, we need to estimate second term once again. Applying the Hölder inequality with the exponent s^* for II, the similar arguments show

$$\begin{aligned}
II &\leq C\eta^{-\frac{1}{s^*}Q+1}|B|^{\frac{1}{s^*}}\left(\sum_{k=1}^{n_1} \fint_B |X_k u(y)|^{s^*} dy \right)^{\frac{1}{s^*}} \\
&\leq C\eta^{-\frac{1}{s^*}Q+1}|B|^{\frac{1}{s^*}} r^{l-1}\left(\fint_B \mathfrak{M}_B^{l-1}(\sum_{|\sigma|=l} |X^\sigma u|)^s(x)dx \right)^{\frac{1}{s}}.
\end{aligned}$$

We set

$$\eta^{\frac{Q}{s^*}} = |B|^{\frac{1}{s^*}}\left(\fint_B \mathfrak{M}_B^{l-1}(\sum_{|\sigma|=l} |X^\sigma u|)^s(x)dx \right)^{\frac{1}{s}-\frac{l-1}{Q}} \left(\mathfrak{M}_B^{l}(\sum_{|\sigma|=l} |X^\sigma u|(\cdot))(x) \right)^{-\frac{s}{s^*}}.$$

After simplifying and making use of the fact $|B| \approx Cr^Q$, we get

$$|u(x)| \leq C_2 r^l \left(\fint_B \mathfrak{M}_B^l (\sum_{|\sigma|=l} |X^\sigma u|)^s(x) dx \right)^{\frac{l}{Q}} \mathfrak{M}_B^l (\sum_{|\sigma|=l} |X^\sigma u|)^{\frac{Q-ls}{Q}}(x),$$

which is the required estimate. The inequality (2.9) follows immediately. To prove (2.10), observe that $s' < Q/(Q-1)$, we have

$$\int_{\{y \in B: d_C(x,y) \leq r\}} d_C(x,y)^{(-Q+1)s'} dy \leq C(s,Q) r^{Q-(Q-1)s'}.$$

Therefore,

$$\begin{aligned} |u(x)| &\leq \int_{\{y \in B: d_C(x,y) \leq r\}} \sum_{k=1}^{n_1} |X_k u|(y) d_C(x,y)^{-Q+1} dy \\ &\leq C(s,Q) \left(\int_{\{y \in B: d_C(x,y) \leq r\}} \left[\sum_{k=1}^{n_1} |X_k u|(y) \right]^s dy \right)^{\frac{1}{s}} r^{Q/s' - (Q-1)} \\ &\leq C(s,Q) r \left(\fint_{\{y \in B: d_C(x,y) \leq r\}} \left[\sum_{k=1}^{n_1} |X_k u|(y) \right]^s dy \right)^{\frac{1}{s}}. \end{aligned}$$

This proves (2.10) for $l = 1$, and it is easy to prove the general case by induction. This completes the proof of Lemma 2.2. □

In the Euclidean space it is very useful to consider a special type of polynomial $P(x, B)$ of degree m, which satisfies $\int_B (\partial^\alpha u(x) - \partial^\alpha P(x,B))\, dx = 0$ for all $|\alpha| \leq m-1$. Such a polynomial is called the *fitting polynomial* that received its named after P. W. Jones's celebrated paper [37] concerning the extension problem of the Sobolev space on (ϵ, δ)-domains. In the present paper we should prove that such kind of polynomials also exist on the Carnot group. Our proof is based on Nhieu's ideas [56]. In fact, the proof can be rather straightforward extended to an arbitrary Carnot group. For this purpose we state the following lemma.

Lemma 2.3. *(Existence of fitting polynomial on Carnot groups) For any fixed measurable subset $B \subset \bar{B} \subset \Omega$, $0 < |B| < \infty$ and any $u \in W^{m,p}(B)$, m and p are any fixed positive integer, there exists a polynomial $P(x, u, B)$ (may not unique) such that*

$$X^I P = 0 \text{ for all } |I| = m, \text{ and } \int_B X^I(u - P) dx = 0 \text{ for all } |I| \leq m-1.$$

Proof. We divide the proof into 3 steps.

Step 1: Some properties of homogeneous polynomials on the Carnot group.

Remind from [14] that the left invariant vector fields can be written as

$$X_k = \partial/\partial\eta_k + \sum_{d_i > d_k} P_{ik} \partial/\partial\eta_i,$$

where P_{ik} is a homogeneous polynomial of homogeneous degree $d_i - d_k$. The higher derivatives can be written

$$X^I = \sum_{d(K)\geq d(I),\ |K|\leq|I|} P_{IK}(\partial/\partial\eta)^K,$$

where P_{IK} is a homogeneous polynomial of homogeneous degree $d(K) - d(I)$. Therefore, it is easy to check that if P is a homogeneous polynomial of the homogeneous degree m, then the homogeneous degree of $X^I P$ equals

$$\deg_{\mathbb{G}}(X^I P) = d(K) - d(I) + m - d(K) = m - d(I).$$

We conclude that if $d(I) > m$, then $X^I P = 0$ and if $d(I) = m$, then $X^I P$ is a constant. Since we only take into account horizontal derivatives, we get $d(I) = |I|$. Therefore we have $\deg_{\mathbb{G}}(X^I P) = m-|I|$.

Step 2. The proof of Lemma 2.3 will be completed by proving the following statement: for any fixed integer l, $0 \leq l \leq m-1$ there exists a homogeneous polynomial P_l of degree $\deg_{\mathbb{G}}(P_l) = l$ such that $\int_B X^I(u - P_l)dx = 0$ for all $|I| = l$.

If this statement is true, we first find a homogeneous polynomial P_{m-1} such that $\int_B X^I(u - P_{m-1})dx = 0$ for all multiindeces $|I| = m-1$. Next, we let $g = f - P_{m-1}$ and find a homogeneous polynomial P_{m-2} such that $\int_B X^I(g-P_{m-2})dx = 0$ for all $|I| = m-2$. Continue to repeat this process until we find a 0-degree homogeneous polynomial P_0. Then we assert that $P = P_0 + P_1 + \cdots + P_{m-1}$ is the desired polynomial.

Observe that if multiindex I satisfies $|I| = m$, then $X^I P = \sum_{k=0}^{m-1} X^I P_k = 0$.

For any fixed multiindex I with $|I| = l \leq m-1$, we have showed that homogeneous polynomials, constructed above, satisfy $\int_B X^I(u - P_{m-1} - \cdots - P_l)dx = 0$. Since $X^I(\sum_{k=0}^{l-1} P_k) = 0$ we get

$$\int_B X^I(u-P)dx = \int_B \left(X^I(u - P_{m-1} - \cdots - P_l) - X^I(\sum_{k=0}^{l-1} P_k) \right) dx = 0.$$

This finishes the proof of Step 2.

Since $X^l P_l$ is a constant, in order to find the homogeneous polynomial it suffices to determine its coefficients, that is to find a solution of the overdetermined linear system:

$$X^I P_l = \fint_B X^I u, \quad |I| = l. \tag{2.11}$$

To complete the proof of Lemma 2.3, it is enough to show that the linear system (2.11) has solutions. It reduces to the linear algebra problem to indicate whether $\{X^I P_l : |I| = l\}$ and $\{X^I u : |I| = l\}$ has the same linear dependent relation. This will be proved in the third step.

Step 3. We prove that the equality

$$\sum_{|I|=l} c_I X^I P_l = 0,$$

implies

$$\sum_{|I|=l} c_I X^I u = 0 \tag{2.12}$$

for any choice of real numbers c_I.

Indeed, first, we assert that for the same set $\{c_I : |I| = l\}$, the equality

$$\sum_{|I|=l} c_I X^I P = 0 \tag{2.13}$$

holds for any polynomial P.

Indeed, if the homogeneous degree of P satisfies $k = \deg_{\mathbb{G}} P \leq l$, then (2.13) holds automatically. If $k = \deg_{\mathbb{G}} P > l$, we write this polynomial in the form $P = \sum_{j=0}^{k} P_j$ and assert that $X^I P(0) = 0$. In fact, $X^I P = \sum_{j \geq l+1} X^I P_j$, since $X^I P_j$ is the homogeneous polynomial of the homogeneous degree $j - d(I) = j - |I| = j - l \geq 1$. Therefore $X^I P_j(0) = 0$ for all $j \geq l+1$. Thus the assertion is true.

We denote $\tau_x(y) = x \cdot y$ the left translation on the Carnot group $\mathbb{G}$ which is a diffeomorphism of $\mathbb{G}$. Composition $P \circ \tau_x$ is a polynomial if P is a polynomial. Since X^I are invariant under the left translation, we have

$$\sum_{|I|=l} c_I X^I P(x) = \sum_{|I|=l} c_I (X^I P) \circ \tau_x(0) = \sum_{|I|=l} c_I (X^I P \circ \tau_x)(0) = 0.$$

We prove the equality (2.12) for $u \in C_0^\infty(\Omega)$. In [14, page 34-35], one can find the Taylor series on the Carnot group. We apply this formula to u for any fixed x_0. Let

$$P(x, x_0) = \sum_{d(I) \leq k} a_I(x_0) \frac{\eta^I(x_0^{-1} x)}{I!}$$

be the Taylor polynomial of u with homogeneous degree $k > l$. Then we can conclude that

$$\sum_{|I|=l} c_I X^I P(x, x_0) = 0.$$

Since $X^I P(x, x_0)|_{x=x_0} = X^I u(x_0)$, we get

$$\sum_{|I|=l} c_I X^I u(x_0) = 0$$

for any fixed $x_0 \in \Omega$.

The last step is to show that the equality (2.12) holds for $u \in HW^{m,p}(B)$. This follows immediately from the definition of the Sobolev space on Carnot groups. Applying approximation arguments to $u \in C_0^\infty(B)$, we get that the conclusion holds for the Sobolev space. This completes the proof of Lemma 2.3. □

Another approach to prove the existence of fitting polynomial can be found in [49].

Lemma 2.4. *Let Ω be a domain on the Carnot group $\mathbb{G}$ and $x_0 \in \Omega$. Assume that $\lambda > 0$, $r > 0$, and $1 < q < \infty$. For $u = (u_1, ..., u_N) \in HW^{2,q}(\Omega)$ with* supp $u \subset \overline{B(x_0, r)} \subset \Omega$*, we denote $B = \overline{B(x_0, Cr)}$ for some $C > 1$ and*

$$F(\lambda) = \{x \in \Omega \colon \mathfrak{M}_B^2(|X^2 u|)(x) \leq \lambda\} \cap B \neq \emptyset.$$

Then $u|_{F(\lambda)}$ has the Whitney extension v to $\mathbb{G}$ satisfying
(1) $v = u$ on $F(\lambda)$,
(2) $X_k v = X_k u$ on $F(\lambda)$,
(3) $|X^\sigma v| \le c\lambda$ a.e. on $\mathbb{G}$ for $|\sigma| = 2$.

Proof. We need to verify that u satisfies the conditions of the Whitney extension theorem shown in [63]. We know by approximation arguments that $\int_{2B} X_k u dx = 0$. Applying Lemma 2.2, we get

$$|X_k u| \le Cr\mathfrak{M}^2_B(|X^2u|)(x) \le C\lambda r \quad \text{for any} \quad x \in F(\lambda)$$

and for $k = 1, ..., n_1$. Moreover, we have $|u - u_{2B}| \le C\lambda r^2$. If $x_1 \in \left(\frac{3}{2}B\right)^c$, then $u(x_1) = 0$ and

$$\begin{aligned} |u_{2B}| &= |u(x_1) - u_{2B}| \le Cr^2\mathfrak{M}^2_B(|X^2u|)(x_1) \\ &\le Cr^{2-Q}\int_{2B}\mathfrak{M}_B(|X^2u|)(y)dy \le C\lambda r^2. \end{aligned}$$

Therefore $|u(x)| \le C\lambda r^2$.

The first order Taylor polynomial for the function u can be written in the form

$$P(x, x_0) = u(x_0) + \sum_{k=1}^{n_1} X_k u(x_0)\, \eta_k(x_0^{-1}x).$$

We can choose a polynomial $Q(x, y_0, s)$ of homogenous degree 1 satisfying

$$\int_{B(y_0,s)}(u - Q)dx = 0 \quad \text{and} \quad \int_{B(y_0,s)} X_k(u - Q)dx = 0 \ \text{ for } \ k = 1, ..., n_1$$

for any fixed $y_0 \in F(\lambda)$ and $s > 0$ by Lemma 2.3. Using Lemma 2.2, we get

$$|\, u - Q \,| \le C\lambda r^2 \ \text{ and } \ |\, X_k(u - Q) \,| \le C\lambda r \ \text{ on } \ F(\lambda) \cap B(y_0, s). \tag{2.14}$$

Since Q is the polynomial of the homogenous degree 1, it contains only η_k for index k varying only from 1 to n_1. Therefore, Q can be written as

$$Q(x) = Q(x_0) + \sum_{k=1}^{n_1}(X_k Q)(x_0)\, \eta_k(x_0^{-1}x)$$

and from (2.14) we can immediately deduce

$$|Q(x_0) - u(x_0)| \le C\lambda r^2, \ \ |X_k Q(x_0) - X_k u(x_0)| \le C\lambda r.$$

Since P and Q are polynomials of homogenous degree 1, their first order horizontal derivatives are identically constant, therefore $X_k Q(x) \equiv X_k Q(x_0)$ and $X_k P(x_0, x) \equiv X_k P(x_0, x)|_{x=x_0} = X_k u(x_0)$. This implies

$$|P - Q| \le C\lambda r^2 \quad \text{and} \quad |X_k(P - Q)| \le C\lambda r.$$

So we get

$$|P - u| \le C\lambda r^2 \quad \text{and} \quad |X_k(P - u)| \le C\lambda r$$

and this shows, that u satisfies the condition of [63, Theorem 2]. Thus, we conclude that requirements (1) and (2) of the statement of Lemma 2.4 hold and the observation from [63, page 611] implies that $|X^\sigma v| \le c\lambda$ on $F(\lambda)^c$. This completes the proof of Lemma 2.4. □

In order to prove the main result of the present chapter, we recall the following Gehring's lemma on metric measure spaces X, d, μ, where d is a distance and μ is a doubling measure.

Lemma 2.5. *[16] Let $q \in [q_0, 2Q]$, where $q_0 > 1$. Assume that functions f, g, defined on a metric measure space $(X,\ d,\ \mu)$, are nonnegative and $g \in L^q_{\text{loc}}(X,\mu)$, $f \in L^{r_0}_{\text{loc}}(X,\mu)$ for some $r_0 > q$. If there exist nonnegative constants $b > 1$ and θ such that for every ball $B \subset \beta B \subset X$, $\beta > 1$, the following estimate holds*

$$\fint_B g^q d\mu \leq b\left[\left(\fint_{\beta B} g d\mu\right)^q + \fint_{\beta B} f^q d\mu\right] + \theta \fint_{\beta B} g^q d\mu,$$

then there exist nonnegative constants θ_0 and ϵ_0, $\theta_0 = \theta_0(q_0, Q, C_d, \beta)$ and $\epsilon_0 = \epsilon_0(b, q_0, Q, C_d, \beta)$ such that if $0 < \theta < \theta_0$ then $g \in L^p_{\text{loc}}(X,\mu)$ for $p \in [q, q+\epsilon_0)$ and moreover

$$\left(\fint_B g^p d\mu\right)^{1/p} \leq C\left[\left(\fint_{\beta B} g^q d\mu\right)^q + \left(\fint_{\beta B} f^p d\mu\right)^{1/p}\right]$$

for $C = C(b, q_0, Q, C_d, \beta)$.

2.1.2 Proof of Theorem 2.1

Our approach is due to [44] and we divide the proof into several steps.

The distributional set of maximal function on the Carnot group is an open set.

For any fixed locally integrable function f, we prove that the set $\{x \in \mathbb{G} :\ M(f) > t\}$ is an open set for all $t > 0$. This assertion is equivalent to saying that the maximal function $Mf(x)$ is lower semi-continuous. Obviously, any average of the function f

$$x \to \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy$$

is continuous. Therefore

$$\begin{aligned}\{x \in \mathbb{G} :\ M(f) > t\} &= \left\{x \in \mathbb{G} :\ \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy > t\right\}\\ &= \bigcup_{r>0}\left\{x \in \mathbb{G} :\ \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy > t\right\}.\end{aligned}$$

is an open set. This proves the assertion.

Stopping time arguments.

Let u, Ω be as in Theorem 2.1. Suppose $B(z_0, R)$ be the any Carnot-Caratheodory ball in Ω. We fix a point $x_0 \in B(z_0, R/2)$. Let $r = R/4C$, $C > 1$, and denote by B_r the ball $B(x_0, r)$. There exists $\varphi \in C_0^\infty(B)$, $B = \overline{B(x_0, Cr)}$, such that $\varphi \equiv 1$ on $B(x_0, r)$, supp $\varphi \subset B$ and $|X^\sigma \varphi(x)| \leq C_1(\varphi) r^{-|\sigma|}$, $|\sigma| \leq 2$, see [14]. By Lemma 2.3, there exists a polynomial P such that $\int_{B(x_0,R)} X^I(u - P)dx = 0$ for any $|I| \leq 1$. Denote $u_0 = (u-P)\varphi(x)$ and let

$$E(\lambda) = \left\{x : \mathfrak{M}^2_B(|X^2 u_0|)(x) \leq \lambda\right\} \ \text{ and } \ F(\lambda) = E(\lambda) \cap B.$$

So that $F(\lambda)$ is a closed set. Applying Lemma 2.4, we extend $u_0|_{F(\lambda)}$ to v on $\mathbb{G}$. Next, denote by θ a test function $\theta \in C_0^\infty(B(x_0, R))$, which $\theta(x)|_{B(x_0,3Cr/2)} \equiv 1$. Define $\bar{v}(x) = v(x)\theta(x)$. We can show that $|X^\eta \bar{v}| \le C_1 \sum_{|\sigma|\le 2} |X^\sigma v| \le C_1(1 + \lambda + \lambda^2)$, and thus $X^\eta \bar{v} \in L^\infty(\Omega)$ for all $|\eta| \le 2$. Since Ω is bounded, we can deduce $X^\eta \bar{v} \in L^q(\Omega)$ for $q \ge p - \delta$. We assert that (2.6) holds if we substitute ϕ by $\bar{v}$, that is

$$\sum_{|\sigma|=2} \int_\Omega A_\sigma(x, D^2u(x)) X^\sigma \bar{v}(x) dx = 0.$$

In fact, we choose a sequence $\phi_i \in C_0^\infty(\Omega)$ such that $\|\phi_i - \bar{v}\|_{W^{2,q}(\Omega)} \to 0$. Therefore

$$\begin{aligned}
\left| \sum_{|\sigma|=2} \int_\Omega A_\sigma(x, D^2u(x)) X^\sigma \bar{v}(x) dx \right| &= \left| \sum_{|\sigma|=2} \int_\Omega A_\sigma(x, D^2u(x)) \left(X^\sigma \bar{v}(x) - X^\sigma \phi_i(x)\right) dx \right| \\
&\le \left| \sum_{|\sigma|=2} \int_\Omega \left(|X^\sigma u(x)|^{p-1} + b_\sigma(x) \right) \left(X^\sigma \bar{v}(x) - X^\sigma \phi_i(x)\right) dx \right| \\
&\le C_1 \sum_{|\sigma|=2} \left\| |X^\sigma u(x)|^{p-\delta} + b_\sigma(x) dx \right\|_{L^{(p-\delta)/(1-\delta)}(\Omega)} \\
&\times \|X^\sigma \bar{v}(x) - X^\sigma \phi_i(x)\|_{L^q(\Omega)} \to 0.
\end{aligned}$$

The last step is followed from the Hölder inequality.

We split Ω into two sets $\Omega = F(\lambda) \cup (\Omega - F(\lambda))$ and obtain

$$\begin{aligned}
\sum_{|\sigma|=2} \int_{F(\lambda)} A_\sigma(x, D^2u(x)) X^\sigma u_0(x) dx &= \sum_{|\sigma|=2} \int_{\Omega - F(\lambda)} A_\sigma(x, D^2u(x)) X^\sigma \bar{v}(x) dx \\
&\le C_1 \lambda \sum_{|\sigma|=2} \int_{B(x_0,R) - F(\lambda)} \left(|X^\sigma u(x)|^{p-1} + b_\sigma(x) \right) dx.
\end{aligned} \tag{2.15}$$

The last inequality is followed from (2.5). We assert that there exists $\lambda_0 > 0$ such that $E(\lambda) = F(\lambda)$ for any $\lambda > \lambda_0$. To prove this assertion notice that we have

$$\begin{aligned}
\mathfrak{M}_B^2(|X^2 u_0|)(x) &\le c r^{-Q} \int_{B(x_0,4Cr)} \mathfrak{M}_B(|X^2 u_0|)(x) dx \\
&\le C \min_{B(x_0,8Cr)} \mathfrak{M}_B^2(|X^2 u_0|)(x)
\end{aligned} \tag{2.16}$$

for any $x \in \Omega - B(x_0, 3Cr)$. Setting

$$\lambda_0 = c r^{-Q} \int_{B(x_0,4Cr)} \mathfrak{M}_B(|X^2 u_0|)(x) dx,$$

we finish to prove the assertion. Multiply both sides of (2.15) by $\lambda^{-1-\delta}$ and integrate on (λ_0, ∞).

Interchange the integration on both of sides. We get for the right hand side

$$
\begin{aligned}
K &= \int_{\lambda_0}^{\infty} \lambda^{-1-\delta} \sum_{|\sigma|=2} \int_{F(\lambda)} A_\sigma(x, D^2u(x)) X^\sigma u_0(x) dx d\lambda \\
&= C_1 \int_{\lambda_0}^{\infty} \lambda^{-\delta} \sum_{|\sigma|=2} \int_{B(x_0,R)-F(\lambda)} \left(|X^\sigma u(x)|^{p-1} + b_\sigma(x)\right) dx d\lambda \\
&\le C_1 \sum_{|\sigma|=2} \int_0^{\infty} \lambda^{-\delta} \int_{B(x_0,R)-F(\lambda)} \left(|X^\sigma u(x)|^{p-1} + b_\sigma(x)\right) dx d\lambda \\
&= C_1 \frac{1}{1-\delta} \sum_{|\sigma|=2} \int_{B(x_0,R)} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{1-\delta} \left(|X^\sigma u(x)|^{p-1} + b_\sigma(x)\right) dx \\
&\le C_1 \frac{1}{1-\delta} \int_{B(x_0,R)} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{p-\delta} + \mathfrak{M}_B^2(|X^2 u_0|)(x)^{1-\delta} b_\sigma(x) dx \\
&\le C_1 \frac{1}{1-\delta} \int_{B(x_0,R)} |X^2 u_0|^{p-\delta} dx + \int_{B(x_0,R)} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{1-\delta} b_\sigma(x) dx,
\end{aligned}
$$

where the last step follows from the Hardy-Littlewood maximal theorem. Let

$$
F_1^{p-\delta} = \mathfrak{M}_B^2(|X^2 u_0|)(x)^{1-\delta} b_\sigma(x).
$$

Then

$$
K \le \int_{B(x_0,2Cr)} F_1^{p-\delta} dx + c \int_{B(x_0,2Cr)} |X^2 u_0|^{p-\delta} dx \tag{2.17}
$$

and we have

$$
F_1 \in L^{p+\alpha}(B(x_0, 2Cr)) \tag{2.18}
$$

for certain $\alpha > 0$. It remains to estimate the lower bounds for K. In fact, we can write

$$
\begin{aligned}
K &= \sum_{|\sigma|=2} \int_{\lambda_0}^{\infty} \lambda^{-1-\delta} \left(\int_{F(\lambda_0)} + \int_{F(\lambda)-F(\lambda_0)} A_\sigma(x, D^2u(x)) X^\sigma u_0(x) \right) dx d\lambda \\
&= \frac{1}{\delta} \lambda_0^{-\delta} \int_{F(\lambda_0)} A_\sigma(x, D^2u(x)) X^\sigma u_0(x) dx + \\
&\frac{1}{\delta} \int_{\Omega-E(\lambda_0)} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{-\delta} A_\sigma(x, D^2u(x)) X^\sigma u_0(x) dx \\
&\ge \frac{1}{\delta} \int_{B(x_0,Cr)} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{-\delta} A_\sigma(x, D^2u(x)) X^\sigma u_0(x) dx - \\
&\frac{1}{\delta} \int_{E(\lambda_0)} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{-\delta} A_\sigma(x, D^2u(x)) X^\sigma u_0(x) dx := L_2 - L_1.
\end{aligned} \tag{2.19}
$$

The estimate of L_1.

First of all, we need to prove the estimate

$$
|X^I u_0(x)| \le C_1 \mathfrak{M}_B^2(|X^2 u|)(x). \tag{2.20}
$$

We assume for all multiindeces I that $|I| = 2$ and this assumption will play a crucial role in the proof of Theorem 2.1. Thus we can write

$$\begin{aligned} X^I u_0 &= \sum_{\alpha+\beta=I} X^\alpha (u-P)(x) X^\beta \varphi(x) \\ &= \varphi X^I (u-P) + \sum_{\alpha+\beta=I,\ |\alpha|<2} X^\alpha (u-P)(x) X^\beta \varphi(x) \\ &= \varphi X^I u + \sum_{\alpha+\beta=I,\ |\alpha|<2} X^\alpha (u-P)(x) X^\beta \varphi(x). \end{aligned}$$

The last step is followed from the equality $X^I P = 0$. We have

$$|X^\alpha (u-P)(x)| \le C_1 r^{2-|\alpha|} \mathfrak{M}_B^2(|X^2 u|)(x)$$

from Lemma 2.2. Combining this result with the fact $|X^\beta \varphi(x)| \le C_1 r^{-|\beta|}$, we get

$$|X^\alpha (u-P)(x) X^\beta \varphi(x)| \le C_1 \mathfrak{M}_B^2(|X^2 u|)(x)$$

for all multiindices α and β, $0 \le |\alpha| < 2$, $0 < |\beta| \le 2$. Since φ is bounded it trivially follows that

$$|X^I u(x)| \le C_1 \mathfrak{M}_B^2(|X^2 u|)(x).$$

This implies the inequality (2.20). We can conclude that $u(x) = u_0(x)$ on $B(x_0, r)$ and we have the inequality

$$|X^I u_0 - \varphi X^I u| \le C_1 r^{|\alpha|-2} \sum_{0\le|\alpha|<2} X^\alpha (u-P)(x) \tag{2.21}$$

for all multiindices I, $|I| = 2$.

Suppose $0 < \eta \le \frac{1}{2}$ and split $E(\lambda_0) = E_1(\lambda_0) \cup (E(\lambda_0) - E_1(\lambda_0))$, where

$$E_1(\lambda_0) = \{x \in E(\lambda_0) :\ |X^2 u(x)| \ge \eta^{-1}\lambda_0\}.$$

We have the chain of inequalities

$$\begin{aligned} L_1 &\le \frac{1}{\delta} \int_{E(\lambda_0)} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{-\delta} |X^2 u_0(x)| \left(|X^2 u(x)|^{p-1} + b_\sigma(x)\right) dx \\ &\le \frac{1}{\delta} \int_{E(\lambda_0)} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{1-\delta} \left(|X^2 u(x)|^{p-1} + b_\sigma(x)\right) dx \\ &\le \frac{1}{\delta} \int_{E(\lambda_0)} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{1-\delta} |X^2 u(x)|^{p-1} dx + \frac{1}{\delta} \int_{E(\lambda_0)} F_1^{p-\delta} dx. \end{aligned}$$

by (2.5). We also know

$$\mathfrak{M}_B^2(|X^2 u_0|)(x) \le \lambda_0 \le \eta\, |X^2 u(x)|$$

for any $x \in E_1(\lambda_0)$. We continue for $|X^2 u(x)| \ge \eta^{-1}\lambda_0$:

$$L_1 \le \frac{1}{\delta} \eta^{1-\delta} \int_{E(\lambda_0)} |X^2 u(x)|^{p-\delta} dx + \frac{1}{\delta} \int_{E(\lambda_0)} F_1^{p-\delta} dx.$$

On the other hand, $|X^2u(x)| < \eta^{-1}\lambda_0$ for any $x \in E(\lambda_0) - E_1(\lambda_0)$. Applying (2.16) and (2.19), we get the pointwise estimate for a fixed $t > 0$

$$\begin{aligned}\mathfrak{M}_B^2(|X^2u_0|)(x)^{-\delta}|X^2u_0(x)|\left(|X^2u(x)|^{p-1} + b_\sigma(x)\right)\\ &\le \lambda_0^{2-\delta}\eta^{\delta-2} + \lambda_0^{-\delta}|X^2u_0(x)|\, b_\sigma(x)\\ &\le \eta^{2-p}\left(\fint_B |X^2u|^t dx\right)^{1/t} + \lambda_0^{-\delta}|X^2u_0(x)|\, b_\sigma(x)\end{aligned}$$

by the Hardy-Littlewood maximal theorem. Recalling the definition of λ_0 and making use of the notation

$$F_2^{p-\delta} = \lambda_0^{-\delta}|X^2u_0(x)|\, b_\sigma(x),$$

we get the estimate

$$L_1 \le C_1\frac{\eta^{1-p}}{\delta}r^Q\left(\fint_{B(x_0,2Cr)}|X^2u|^t dx\right)^{(p-\delta)/t} + \frac{1}{\delta}\int_{B(x_0,2Cr)} F_2^{p-\delta}dx,$$

where we have used the fact $|B(x_0, 2Cr)| \approx r^Q$. Therefore, we have proved the estimate for L_1:

$$\begin{aligned}L_1 \le \frac{1}{\delta}\eta^{1-\delta}\int_{B(x_0,2Cr)}|X^2u(x)|^{p-\delta}dx + \frac{1}{\delta}\int_{B(x_0,2Cr)} F_3^{p-\delta}dx\\ + \frac{\eta^{1-p}}{\delta}r^Q\left(\fint_{B(x_0,2Cr)}|X^2u(x)|^t dx\right)^{(p-\delta)/t},\end{aligned} \tag{2.22}$$

where F_3 is the integrable function defined by

$$F_3^{p-\delta} = F_1^{p-\delta} + F_2^{p-\delta} \text{ and } F_3 \in L^{p+\alpha}(B(x_0, 2R)).$$

Decomposition of L_2

In order to estimate L_2, we need to decompose L_2 in a more suitable way. Denote by D_1 the set

$$D_1 = \{x \in B(x_0, Cr) - B(x_0, r) : \mathfrak{M}_B^2(|X^2u_0|)(x) \le \delta\, \mathfrak{M}_B^2(|X^2u|)(x)\}$$

and set $D_2 = B(x_0, Cr) - (D_1 \cup B(x_0, r))$. We get

$$\begin{aligned}\delta L_2 &= \left(\int_{B-D_1} + \int_{D_1}\right)\mathfrak{M}_B^2(|X^2u_0|)(x)^{-\delta}A_\sigma(x, D^2u(x))X^\sigma u_0(x)dx\\ &\ge \int_{B-D_1}\mathfrak{M}_B^2(|X^2u_0|)(x)^{-\delta}A_\sigma(x, D^2u(x))X^\sigma u_0(x)dx-\\ &\int_{D_1}\mathfrak{M}_B^2(|X^2u_0|)(x)^{-\delta}|X^2u_0(x)|\left(|X^2u(x)|^{p-1} + b_\sigma(x)\right)dx\end{aligned}$$

Denote the second term by H_3 and decompose the first term into two parts as follows

$$\begin{aligned}\delta L_2 \ge &\int_{B-D_1}\mathfrak{M}_B^2(|X^2u_0|)(x)^{-\delta}A_\sigma(x, D^2u(x)) \cdot \varphi X^\sigma u(x)dx-\\ &\int_{B-D_1}\mathfrak{M}_B^2(|X^2u_0|)(x)^{-\delta}A_\sigma(x, D^2u(x)) \cdot (\varphi X^\sigma u(x) - X^\sigma u_0(x))dx\\ &- H_3 := H_1 - H_2 - H_3.\end{aligned}$$

Consider H_1. Since $B = D_1 \cup D_2 \cup B(x_0, r)$, we have the following estimate by (2.4)

$$\begin{aligned} H_1 &= \int_{B(x_0,r)\cup D_2} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{-\delta} A_\sigma(x, D^2u(x)) \cdot \varphi X^\sigma u(x) dx \\ &\geq \int_{B(x_0,r)} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{-\delta} A_\sigma(x, D^2u(x)) \cdot X^\sigma u(x) dx \\ &\geq \gamma \int_{B(x_0,r)} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{-\delta} |X^\sigma u(x)|^p dx := J_1. \end{aligned}$$

To estimate H_2 we will use the estimate (2.20) and the equality $u_0(x) = u(x)$ on $B(x_0, r)$. We rewrite H_2 as follows

$$\begin{aligned} H_2 &= \int_{B(x_0,r)\cup D_2} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{-\delta} A_\sigma(x, D^2u(x)) \cdot (\varphi X^\sigma u(x) - X^\sigma u_0(x)) dx \\ &\leq \int_{D_2} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{-\delta} \left(|X^2 u(x)|^{p-1} + b_\sigma(x)\right) \cdot |\varphi X^\sigma u(x) - X^\sigma u_0(x)| dx \\ &\leq \sum_{0\leq|\alpha|<2} r^{-2+|\alpha|} \int_{D_2} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{-\delta} \left(|X^2 u(x)|^{p-1} + b_\sigma(x)\right) \cdot |X^\alpha(u(x) - P(x))| dx. \end{aligned}$$

Denote by J_2 the last term of the above inequality. We continue and get for H_3

$$\begin{aligned} H_3 &= \int_{D_1} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{-\delta} |X^2 u_0(x)| \left(|X^2 u(x)|^{p-1} + b_\sigma(x)\right) dx \\ &\leq \int_{D_1} \mathfrak{M}_B^2(|X^2 u_0|)(x)^{1-\delta} \left(|X^2 u(x)|^{p-1} + b_\sigma(x)\right) dx := J_3. \end{aligned}$$

Then we arrive at the following relation

$$\delta\, L_2 \geq J_1 - J_2 - J_3. \tag{2.23}$$

The estimate of J_1

We can write

$$\begin{aligned} \mathfrak{M}_B(|X^2 u_0|)(x) &= M(|X^2 u_0| \chi_B)(x) \\ &\leq M(|X^2 u_0| \chi_{B_r})(x) + M(|X^2 u_0| \chi_{B-B_r})(x) \end{aligned}$$

for any $x \in B(x_0, \frac{r}{2})$. For the second term we deduce

$$M(|X^2 u_0| \chi_{B-B_r})(x) \leq \fint_{B(x_0,Cr)} |X^2 u_0| dx.$$

Therefore, we obtain

$$\mathfrak{M}_B(|X^2 u_0|)(x) \leq M(|X^2 u_0| \chi_{B_r})(x) + \fint_{B(x_0,Cr)} |X^2 u_0| dx.$$

Moreover, by the same arguments, we have

$$\begin{aligned}
\mathfrak{M}_B^2(|X^2u_0|)(x) &\le M\left(M(|X^2u_0|\chi_{B_r})(\cdot)\chi_B(\cdot)\right)(x) + \fint_{B(x_0,Cr)} |X^2u_0|dx \\
&\le M\left(M(|X^2u_0|\chi_{B_r})(\cdot)\chi_{B_r}(\cdot)\right)(x) \\
&+ M\left(M(|X^2u_0|\chi_{B_r})(\cdot)\chi_{B-B_r}(\cdot)\right)(x) + \fint_{B(x_0,Cr)} |X^2u_0|dx \\
&\le \mathfrak{M}_{B_r}^2(|X^2u_0|)(x) + \fint_{B(x_0,Cr)} M(|X^2u_0|)(x)\chi_B dx + \fint_{B(x_0,Cr)} |X^2u_0|dx \\
&\le \mathfrak{M}_{B_r}^2(|X^2u_0|)(x) + 2\fint_{B(x_0,Cr)} \mathfrak{M}_B^2(|X^2u_0|)(x)dx.
\end{aligned}$$

Since $X^\sigma u_0(x) = X^\sigma u(x)$ on $B(x_0,r)$ for all multiindices σ, $|\sigma| = 2$ we produce the estimate

$$\mathfrak{M}_B^2(|X^2u_0|)(x) \le \mathfrak{M}_{B_r}^2(|X^2u|)(x) + C_1 \fint_{B(x_0,Cr)} \mathfrak{M}_B^4(|X^2u|)(x)dx \tag{2.24}$$

by (2.19). Having established (2.24), we construct the set D as

$$D = \left\{x \in B(x_0, r/2):\ \mathfrak{M}_{B_r}^2(|X^2u|)(x) \ge C_1 \fint_{B(x_0,Cr)} \mathfrak{M}_B^4(|X^2u|)(x)dx\right\}.$$

We immediately obtain that if $x \in D$, then we have

$$\mathfrak{M}_B^2(|X^2u_0|)(x) \le C_1^2\mathfrak{M}_{B_r}^2(|X^2u|)(x) \tag{2.25}$$

by (2.24). It is known that the quantity $\left(\mathfrak{M}_B^2(|X^2u_0|)(x)\right)^{-\delta}$ is A_p-weight if $2\delta \le p-1$ by [16, Lemma 4.1]. This leads to the lower bound for J_1

$$\begin{aligned}
J_1 &= \gamma \sum_{|\sigma|=2} \int_{B(x_0,r)} \mathfrak{M}_B^2(|X^2u_0|)(x)^{-\delta}|X^\sigma u(x)|^p dx \\
&\ge C_1 \int_{B(x_0,r)} \mathfrak{M}_B^2(|X^2u_0|)(x)^{-\delta}\mathfrak{M}_B^2(|X^2u|)(x)^p dx \\
&\ge C_1 \int_D \mathfrak{M}_{B_r}^2(|X^2u|)(x)^{p-\delta}dx \\
&\ge C_1 \int_{B_{r/2}} \mathfrak{M}_{B_r}^2(|X^2u|)(x)^{p-\delta}dx - C_1 \int_{B_{r/2}-D} \mathfrak{M}_{B_r}^2(|X^2u|)(x)^{p-\delta}dx \\
&\ge C_1 \int_{B_{r/2}} |X^2u|^{p-\delta}dx - C_1 r^Q\left(\fint_{B(x_0,2Cr)} |X^2u|^t dx\right)^{(p-\delta)/t}.
\end{aligned} \tag{2.26}$$

The estimate of J_2

To estimate J_2, we set $t = \frac{p+1}{2}$ and $\hat{t} = \max\left\{t, p - t\left(\frac{2-|\alpha|}{Q}\right) - \delta, p-\delta-1\right\}$ and we consider three cases.

If $t(2-|\alpha|) < Q$, we apply Lemma 2.2 (2.8) to α derivative of $u-P$, and get

$$|X^\alpha(u-P)(x)| \le C_2 r^{2-|\alpha|}\left(\fint_B \mathfrak{M}_B^2(|X^2u|)(x)^t dx\right)^{\frac{2-|\alpha|}{Q}} \mathfrak{M}_B^2(|X^2u|)(x)^{1-t\frac{2-|\alpha|}{Q}},$$

We get for $x \in D_2$,

$$\begin{aligned} & r^{-2+|\alpha|}\mathfrak{M}_B^2(|X^2u_0|)(x)^{-\delta}|X^2u|^{p-1}|X^\alpha(u-P)|(x) \\ & \le C(\delta)\left(\fint_B \mathfrak{M}_B^2(|X^2u|)(x)^t dx\right)^{\frac{2-|\alpha|}{Q}} \mathfrak{M}_B^2(|X^2u|)(x)^{p-\delta-t\frac{2-|\alpha|}{Q}} \\ & \le C(\delta)\left(\fint_B \mathfrak{M}_B^2(|X^2u|)(x)^{\hat{t}} dx\right)^{\frac{2-|\alpha|}{Q}\frac{t}{\hat{t}}} \mathfrak{M}_B^2(|X^2u|)(x)^{p-\delta-t\frac{2-|\alpha|}{Q}}. \end{aligned}$$

Therefore, we have the estimate for J_2

$$\begin{aligned} J_2 & \le C(\delta)r^Q\left(\fint_B \mathfrak{M}_B^2(|X^2u|)(x)^{\hat{t}} dx\right)^{\frac{2-|\alpha|}{Q}\frac{t}{\hat{t}}} \fint_B \mathfrak{M}_B^2(|X^2u|)(x)^{p-\delta-t\frac{2-|\alpha|}{Q}} dx \\ & \le C(\delta)r^Q\left(\fint_B |X^2u|(x)^{\hat{t}} dx\right)^{(p-\delta)/\hat{t}}. \end{aligned} \tag{2.27}$$

In case $t(2-|\alpha|) = Q$, we may take $1 < \bar{t} < t$ and define $\hat{\bar{t}}$ similarly. We can get (2.27) once again. If $t(2-|\alpha|) > Q$, we apply Lemma 2.2 (2.10) to α derivative of $u-P$. We get in this case

$$|X^\alpha(u-P)(x)| \le C_2 r^{2-|\alpha|}\left(\fint_B \mathfrak{M}_B^2(|X^2u|)(x)^t dx\right)^{\frac{1}{t}} \le C_2 r^{2-|\alpha|}\left(\fint_B \mathfrak{M}_B^2(|X^2u|)(x)^{\hat{t}} dx\right)^{\frac{1}{\hat{t}}}.$$

Therefore, from Hardy-Littlewood maximal theorem we have

$$\begin{aligned} J_2 & \le C(\delta)r^Q\left(\fint_B \mathfrak{M}_B^2(|X^2u|)(x)^{\hat{t}} dx\right)^{\frac{1}{\hat{t}}} \fint_B \mathfrak{M}_B^2(|X^2u|)(x)^{p-\delta-1} dx \\ & \le C(\delta)r^Q\left(\fint_B |X^2u|(x)^{\hat{t}} dx\right)^{(p-\delta)/\hat{t}}. \end{aligned} \tag{2.28}$$

Summing over α, we conclude that

$$J_2 \le C(\delta)r^Q\left(\fint_B |X^2u|(x)^{\hat{t}} dx\right)^{(p-\delta)/\hat{t}} \tag{2.29}$$

by (2.27) and (2.28).

The estimate of J_3

By the definition of the set D and by the Hardy-Littlewood maximal theorem we have

$$\begin{aligned} J_3 &= \int_{D_1} \mathfrak{M}_B^2(|X^2u_0|)(x)^{1-\delta}\left(|X^2u(x)|^{p-1}+b_\sigma(x)\right)dx \\ &\le \int_{D_1} \mathfrak{M}_B^2(|X^2u|)(x)^{1-\delta}\left(|X^2u(x)|^{p-1}+b_\sigma(x)\right)dx \\ &\le \delta^{1-\delta}\int_{B(x_0,2Cr)} |X^2u_0|^{p-\delta}dx + \int_{B(x_0,2Cr)} F_1^{p-\delta}dx. \end{aligned} \tag{2.30}$$

Final estimations

We get

$$\begin{aligned} \delta\, L_2 \ge{}& C_2\int_{B_{r/2}} |X^2u|^{p-\delta}dx - C_2 r^Q\left(\fint_{B(x_0,2Cr)}|X^2u|^t dx\right)^{(p-\delta)/t} \\ &- C_2\int_{B(x_0,2Cr)} F_1^{p-\delta}dx - C(\delta)r^Q\left(\fint_B \mathfrak{M}_B^2(|X^2u|)(x)^{\hat{t}}dx\right)^{(p-\delta)/\hat{t}} - C_2\delta^{1-\delta}\int_{B(x_0,2Cr)}|X^2u_0|^{p-\delta}dx \end{aligned}$$

from (2.22), (2.25), (2.28) and (2.30). Therefore, since $B(x_0,2Cr)\approx r^Q$ we can deduce

$$\begin{aligned} \delta r^{-Q}L_2 \ge{}& C_2\fint_{B_{r/2}}|X^2u|^{p-\delta}dx - C_2\left(\fint_{B(x_0,2Cr)}|X^2u|^t dx\right)^{(p-\delta)/t} \\ &- C_2\fint_{B(x_0,2Cr)} F_1^{p-\delta}dx - C_2\delta^{1-\delta}\fint_{B(x_0,2Cr)}|X^2u_0|^{p-\delta}dx. \end{aligned}$$

Combining the above estimates, we have

$$\begin{aligned} \delta r^{-Q}K \ge{}& C_2\fint_{B_{r/2}}|X^2u|^{p-\delta}dx - C_2(\eta^{1-\delta}+\delta^{1-\delta})\fint_{B(x_0,2Cr)}|X^2u(x)|^{p-\delta}dx \\ &- \sum_{k=1}^{3}\fint_{B(x_0,2Cr)}F_k^{p-\delta}dx - C_2(\eta^{1-p}+1)\left(\fint_{B(x_0,2Cr)}|X^2u|^t dx\right)^{(p-\delta)/t} \end{aligned}$$

by (2.18) and (2.21). Applying (2.17) which is the upper bounds of K, we obtain

$$\begin{aligned} \fint_{B_{r/2}}|X^2u|^{p-\delta}dx \le{}& C_2\sum_{k=1}^{3}\fint_{B(x_0,2Cr)}F_k^{p-\delta}dx \\ &+ C_2(\eta^{1-\delta}+\delta^{1-\delta}+\delta)\fint_{B(x_0,2Cr)}|X^2u_0|^{p-\delta}dx \\ &+ C_2(\eta^{1-p}+1)\left(\fint_{B(x_0,2Cr)}|X^2u|^t dx\right)^{(p-\delta)/t}. \end{aligned}$$

Now we take t, $0<t<p-\delta$ and denote $f^{1/t}=c\sum_{i=1}^{3}F_i$, $q=(p-\delta)/t$. Applying Lemma 3.7 to $|X^2u|^t$, we have proved that the highest horizontal derivatives have the higher integrability.

If $X^\sigma u(x) \in L^q(B(x_0, 2R)$, for some $q > p$ and the multiindex σ, $|\sigma| = 2$, then the Poincaré inequality enables us to prove that both $X_k u$ and u have the higher integrability. Since $u \in HW^{2,p-\delta}$ function u and its derivatives $X_k u$ are locally integrable. We have

$$\begin{aligned}||X_k u||_{L^q(B)} &\leq C(B,u) + ||X_k u - (X_k u)_B||_{L^q(B)}\\ &\leq C(B,u) + C_2 \sum_{j=1}^{n_1} ||X^2 u||_{L^q(B)} < \infty\end{aligned}$$

for any ball $B \subset \Omega$ by the Poincaré inequality [36]. Moreover

$$\begin{aligned}||u||_{L^q(B)} &\leq C_1(B,u) + ||u - u_B||_{L^q(B)}\\ &\leq C_1(B,u) + C_2 \sum_{j=1}^{n_1} ||X_j u||_{L^q(B)} < \infty.\end{aligned}$$

This completes the proof of Theorem 2.1.

2.2 Hardy Space Estimate for Weak Solutions of Sub-Laplace Equations

Let $\{X_1, \ldots, X_m\}$ be a set of real C^∞-smooth vector fields on an open bounded domain $\Omega \subset \mathbb{R}^n$. This set satisfies the Hörmander condition if there exists an integer s such that the family of commutators of the vector fields up to the length s, i.e. the family of vector fields

$$X_1, \ldots, X_m,\ [X_{j_1}, X_{j_1}], \ldots,\ [X_{j_1}, [X_{j_2}, [\ldots, X_{j_s}]] \ldots], \quad j_k = 1, \ldots, s,$$

spans the tangent space $T_x\mathbb{R}^n$ at every point $x \in \mathbb{R}^n$.

In [55], the authors define a (quasi)metric ρ on Ω. We say that an absolutely continuous curve $\gamma : [a,b] \to \mathbb{R}^n$ is *admissible*, if there exist functions $\alpha_j : [a,b] \to \mathbb{R}$, $j = 1, \ldots, k$, such that

$$\dot{\gamma} = \sum_{j=1}^{m} \alpha_j(t) X_j(\gamma(t)) \text{ and } \sum_{j=1}^{m} \alpha_j(t)^2 \leq 1.$$

The distance $\rho(x,y)$ between points x and y is defined as the infimum of those $T > 0$ for which there exists an admissible curve $\gamma : [a,b] \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma(T) = y$. For $x \in \Omega$ and $\delta > 0$, let $B(x,\delta) = \{y \in \Omega : \rho(x,y) < \delta\}$ be the ball centered at x with radius δ with respect to the metric ρ. In general it does not need to be a metric. When the family of vector fields $X_1, \ldots, X_k$ satisfies the Hörmander condition, then ρ is a metric and we say that $(\mathbb{R}^n, \rho)$ is a Carnot-Carathéodory space. The set $\{X_1, \ldots, X_m\}$ is often called the *horizontal* vector fields in Ω.

It was shown in [55] that these balls satisfy doubling property for Lebesgue measure. More precisely, for any compact subset K of Ω, there exist positive constants C_K and $\delta'(K)$ such that for all δ, $0 < \delta < \delta'(K)$ and all x in K

$$|B(x, 2\delta)| \leq C_K |B(x,\delta)|, \tag{2.31}$$

where $|\cdot|$ denotes the Lebesgue measure. We denote by C_d the best constant of the estimate (2.31), then the Carnot-Carathéodory space $(\mathbb{R}^n, \rho)$ with a Lebesgue measure has the homogeneous dimension $Q = \log_2 C_d$. We define the local Hardy-Littlewood maximal function

$$M_\Omega(f)(x) = \sup_{0<\delta<\delta'} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y)|dy.$$

It follows from (2.31) that the Hardy-Littlewood maximal function maps $L^p(\Omega)$ to $L^p(K)$ for any K compact subset of Ω and $1 < p < \infty$.

Next, we are going to define the Hardy space $H^1(\Omega)$. Fix a smooth bump function Φ in the unit ball of the Carnot-Carathéodory space and let $\Phi_\delta(y) = \delta^{-n}\Phi_\delta(\delta^{-1}y)$. For any x_0 in Ω and $\delta > 0$ small enough, the push-forward of Φ_δ by any of the coordinate maps constructed in [55] gives a smooth bump $\Phi_\delta^{x_0}$ supported in the ball $B(x_0, \delta)$. One can check that for any compact subset K of Ω and for arbitrary $x \in K$

$$\left|\Phi_\delta^{x_0}(x)\right| \le C_K |B(x_0,\delta)|^{-1} \quad \text{and} \quad \left|X_k(\Phi_\delta^{x_0})(x)\right| \le C_K \delta^{-1} |B(x_0,\delta)|^{-1}, \;\; k = 1, \ldots, m, \tag{2.32}$$

when $0 < \delta < \delta''(K)$, where δ'' is a small constant such that $B(x_0, \delta'') \subset \Omega$.

For a function f on Ω and $\delta > 0$, let

$$\tilde{M}_\delta(f)(x_0) = \sup_{0<\sigma<\delta} \left| \int_\Omega f(y)\Phi_\sigma^{x_0}(y)dy \right|. \tag{2.33}$$

We say that f lies in $H^1(\Omega)$ if for any compact subset K of Ω, there exists a $\delta_0(K) > 0$ such that $\tilde{M}_{\delta_0(K)}(f)(x)$ is in $L^1(K)$. We define the Hardy space norm of f on K by setting

$$\|f\|_{H^1(\Omega)} = \sup_{K\subset\Omega} \left\|\tilde{M}_{\delta_0(K)}(f)(x)\right\|_{L^1(K)}. \tag{2.34}$$

Given a first-order differential operator $X = (X_1, \ldots, X_m)$, we define the Sobolev space $W^{1,p}(\Omega)$ in the following way:

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : X_j u \in L^p(\Omega), \quad j = 1, \ldots, m \right\},$$

where $X_j u$ is the distributional derivative. The $W^{1,p}$-norm $\|\cdot\|_{1,p}$ is defined by

$$\|u\|_{1,p} = \|u\|_{L^p} + \|Xu\|_{L^p},$$

where $Xu = (X_1 u, \ldots, X_m u)$ is the horizontal gradient and its length is given by

$$|Xu(x)| = \left(\sum_{k=1}^m |X_k u(x)|^2 \right)^{\frac{1}{2}}.$$

We denote by X_j^* the C^∞-smooth vector field which is the formal adjoint to X_j in L^2, i.e.

$$\int_\Omega f X_j^* g dx = -\int_\Omega g X_j f dx \;\text{ for functions } f, g \in C_0^\infty(\Omega).$$

We say that $u(x) \in W^{1,p}(\Omega)$ is the weak solution of the p-Sub-Laplace equation

$$\sum_{i=1}^{m} X_i^* \left(|Xu|^{p-2} X_i u \right) = 0,$$

if u satisfies the following identity

$$\sum_{i=1}^{m} \int_{\Omega} |Xu|^{p-2} X_i u X_i \phi dx = 0 \tag{2.35}$$

for any $\phi \in C_0^{\infty}(\Omega)$.

Our result states as follows.

Theorem 2.6. *Let u be the weak solution of the p-Sub-Laplace equation (2.35) and* $\frac{Q+1}{Q} < p < Q+1$.

(a) If $u \in W^{1,q}(\Omega)$, $\frac{Q}{Q+1} p < q \leq p$, *then we have*

$$\| |Xu|^p \|_{H^\gamma} \leq C \|u\|_{1,q}^p$$

where $\gamma = \frac{q}{p}$, $\frac{Q}{Q+1} < \gamma \leq 1$.

(b) If $u \in W^{1,q}(\Omega)$, *where* $\frac{Q}{Q+1} p \leq q < p$ *then* $u \in W^{1,p}(\Omega)$.

Proof. Let $t > 0$ and $x \in \Omega$, we set $B_x = B(x,t)$ and $u_{B_x} = |B_x|^{-1} \int_{B_x} u(y) dy$. We find a smooth function Φ_t^x such that $\Phi_t^x \geq 0$ and $\Phi_t^x \equiv 1$ on $B(x, t/2)$. We first write

$$\Phi_t^x(y) |Xu(y)|^p = \sum_{i=1}^{m} X_i (u - u_{B_x}) \overline{X_i u} \cdot |Xu|^{p-2} \Phi_t^x(y). \tag{2.36}$$

Since

$$\begin{aligned} \sum_{i=1}^{m} X_i \left[(u - u_{B_x}) \Phi_t^x(y) \right] \overline{X_i u} \cdot |Xu|^{p-2} &= \sum_{i=1}^{m} (u - u_{B_x}) \overline{X_i u} \cdot |Xu|^{p-2} X_i \Phi_t^x(y) \\ &\quad + \sum_{i=1}^{m} X_i (u - u_{B_x}) \overline{X_i u} \cdot |Xu|^{p-2} \Phi_t^x(y), \end{aligned}$$

by differentiating of product and from (2.36), we get

$$\begin{aligned} \Phi_t^x(y) |Xu(y)|^p &= \sum_{i=1}^{m} X_i \left[(u - u_{B_x}) \Phi_t^x(y) \right] \overline{X_i u} \cdot |Xu|^{p-2} \\ &\quad - \sum_{i=1}^{m} (u - u_{B_x}) \overline{X_i u} \cdot |Xu|^{p-2} X_i \Phi_t^x(y). \end{aligned}$$

It is easy to see that $\tilde{u} = (u - u_{B_x})\Phi_t^x \in W^{1,p}(\Omega)$. Thus there exists a sequence $\phi_N \in C_0^\infty(\Omega)$ such that $\|\phi_N - \tilde{u}\|_{1,q'} \to 0$ where $q < q' < p$. We have

$$\begin{aligned}\sum_{i=1}^m \int_\Omega X_i\left[(u-u_{B_x})\Phi_t^x(y)\right]\overline{X_i u}\cdot|Xu|^{p-2}dy &= \sum_{i=1}^m \int_\Omega X_i\left(\tilde{u}-\phi_N\right)\overline{X_i u}\cdot|Xu|^{p-2}dy \\ &\le \|Xu\|_{L^q}^{p-1}\left\|X(\phi_N-\tilde{u})\right\|_{q'} \to 0.\end{aligned}$$

Therefore, we obtain

$$\int_\Omega \Phi_t^x(y)|Xu(y)|^p dy = -\sum_{i=1}^m \int_\Omega (u-u_{B_x})\overline{X_i u}\cdot|Xu|^{p-2}X_i\Phi_t^x(y)dy.$$

From (2.32), we get

$$\left|\int_\Omega \Phi_t^x(y)|Xu(y)|^p dy\right| \le \sum_{i=1}^m \fint_{B_x} t^{-1}|u-u_{B_x}||X_i u|\cdot|Xu|^{p-2}dy.$$

Remind the Poincaré inequality on the Carnot-Carathéodory space [48]:

$$\left(\fint_{B_x}|u(x)-\fint_{B_x} u dy|^s dx\right)^{\frac{1}{s}} \le Ct\sum_{i=1}^m\left(\fint_{2B_x}|X_i u|^r dx\right)^{\frac{1}{r}}$$

for $\frac{1}{Q} < \frac{1}{r} < 1$, $\frac{1}{r} - \frac{1}{Q} \le \frac{1}{s} \le 1$. It follows that

$$\left|\int_\Omega \Phi_t^x(y)|Xu(y)|^p dy\right| \le C\sum_{i=1}^m\sum_{j=1}^m\left(\fint_{2B_x}|X_j u|^r dx\right)^{\frac{1}{r}}\left(\fint_{B_x}\left(|X_i u|\cdot|Xu|^{p-2}\right)^{s'}dx\right)^{\frac{1}{s'}}$$

where $\frac{1}{r} = \frac{1}{Q} + \frac{1}{s}$.

Since $|X_i u| \le |Xu|$ for $r = \frac{Qp}{Q+1}$ we have

$$\int_{B(x,t/2)}|Xu(y)|^p dy \le \left|\int_\Omega \Phi_t^x(y)|Xu(y)|^p dy\right| \le C(n,\Phi)\left(\fint_{B_x}|Xu|^{\frac{Qp}{Q+1}}dx\right)^{\frac{Q+1}{Q}}.$$

This proves part (b), and moreover,

$$\sup_{t>0}\left|\int_\Omega \Phi_t^x(y)|Xu(y)|^p dy\right| \le C(n,\Phi)\left[M_\Omega|Xu|^{\frac{Qp}{Q+1}}(x)\right]^{\frac{Q+1}{Q}}. \tag{2.37}$$

Therefore, we can get the Hardy space estimate as follows:

$$\begin{aligned}\||Xu|^p\|_{H^\gamma} &= C(n,\Phi)\left\|M_\Omega|Xu|^{\frac{Qp}{Q+1}}(x)\right\|_{L^{\gamma\frac{Q+1}{Q}}}^{\frac{Q+1}{Q}} \\ &\le C(n,\Phi)\left\||Xu|^{\frac{Qp}{Q+1}}(x)\right\|_{L^{\gamma\frac{Q+1}{Q}}}^{\frac{Q+1}{Q}} \\ &= C(n,\Phi)\|Xu(x)\|_{L^{\gamma p}}^p \le C(n,\Phi)\|u\|_{1,q}^p,\end{aligned}$$

by boundedness of the Hardy-Littlewood maximal function. This finishes the proof of (a). □

Remark 2.7. Notice that if $u \in W^{1,p}(\Omega)$ then for any compact set $K \subset \Omega$ we have

$$\int_K |Xu|^p \log^+ |Xu| dx < +\infty.$$

This means that if the horizontal gradient Xu of a solution u belongs to $L^p(\Omega)$, then it has locally higher integrability $L^p \log^+ L(\Omega)$. In fact, the author in [16, Theorem 1.2] actually proved a stronger result that if $u \in W^{1,p}(\Omega)$ then $u \in W^{1,p+\epsilon}(\Omega)$ for some $\epsilon > 0$. Combining this result with Theorem 2.6 (b), we come to the following proposition:

Proposition 2.8. *If u is a weak solution of (2.35) and $u \in W^{1,q}(\Omega)$, where $\frac{Q}{Q+1}p \le q < p$, then $u \in W^{1,p+\epsilon}(\Omega)$.*

Remark 2.9. From the estimate (2.37), we have

$$\begin{aligned} \left| \left\{ x \in \mathbb{R}^n : \sup_{t>0} \left| \int_\Omega \Phi_t^x(y) |Xu(y)|^p dy \right| > \lambda \right\} \right| &\le \left| \left\{ x \in \mathbb{R}^n : \left[M_\Omega |Xu|^{\frac{Qp}{Q+1}}(x) \right]^{\frac{Q+1}{Q}} > C(n,\Phi)\lambda \right\} \right| \\ &\le C(n,\Phi)\lambda^{-\frac{Q}{Q+1}} \|Xu(x)\|_{L^q}^q \end{aligned}$$

for the end point case with $\gamma = \frac{Q}{Q+1}$. We conclude that $|Xu|^p$ belongs to the weak type Hardy space $H^{\frac{Q}{Q+1}}$, which was introduced by Grafakos [20] and

$$\| |Xu|^p \|_{\text{weak}-H^{\frac{Q}{Q+1}}} \le C \|u\|_{1,q}^p.$$

Remark 2.10. If we restrict the Carnot-Carathéodory space to the Heisenberg group, then from the arguments in [22], we state without proof the following compactness result:

Proposition 2.11. *Let u_k be solutions of (2.35) and $\|u_k\|_{1,p} \le C$ for $k \ge 1$. Then there exists some subsequence u_{k_j} of u_k such that $|Xu_{k_j}|^p$ converges $*$-weakly in H^1. Moreover if $Xu_k \to Xu$ a.e. for some u, then $|Xu_k|^p \to |Xu|^p$ converges $*$-weakly in H^1.*

Remark 2.12. Let u_k be solutions of (2.35). We set

$$\mathfrak{P} = \{q > 0 : \ q < p \text{ and } u \in W^{1,q}(\Omega) \text{ implies } u \in W^{1,p}(\Omega)\}.$$

It is of interest to know whether $q = \frac{Qp}{Q+1}$ is the best lower bounds of the set $\mathfrak{P}$? In other words, is it possible that the set $\mathfrak{P}$ is an open set? If the answer is affirmative, the integrability property of the solution of (2.35) is open-ended. Keith and Zhong [38] proved that the $(1,r)$-type Poincaré inequality is open-ended. The proof of part (b) of Theorem 2.6 shows the investigation of open-ended property for the Poincaré inequality in general case is rather important. For the definition of open-ended property see Subsection 3.4.

Chapter 3

Self-Improving Regularity for Very Weak Solutions of Degenerate Elliptic Systems

In this chapter, we consider degenerate elliptic systems in Euclidean space $\mathbb{R}^n$. We aim to find a result analogous to a result from [44] for degenerate elliptic equations. In comparison with [29, Lemma 3.38, Theorem 3.58] our result reveals some new aspects of a measure μ defined by A_p-weight.

Let $(\mathbb{R}^n, \mu)$ be a measure space, where $d\mu = \omega(x)dx$ and $\omega(x)$ is A_p-weight for some $p \geq 1$. i.e. $\frac{1}{|Q|}\int_Q \omega \left(\frac{1}{|Q|}\int_Q \omega^{1-p'}\right)^{p-1} \leq c$, where Q is an arbitrary cube in $\mathbb{R}^n$. Let m be an integer number such that $m \geq 1$. Denote $P = \prod_{0\leq|I|\leq m} \mathbb{R}^N$ and $D^m u = (u, \partial_x u, \cdots, \partial_x^\sigma u)$ for all $|\sigma| = m$. Let Ω be a bounded domain in $\mathbb{R}^n$, $A = (A_\sigma)$: $\Omega \times P \to \mathbb{R}^N$ be a function such that $A_\sigma(\cdot, D^m u(x))$, $x \in \Omega$, is a measurable function on Ω satisfying the following conditions:

$$\sum_{|\sigma|=m} A_\sigma(x, D^m u(x)) \cdot \partial_x^\sigma u(x) \geq \gamma\omega(x)|\partial^m u|^p \quad \text{a.e. in} \quad \Omega \tag{3.1}$$

and

$$|A_\sigma(x, D^m u(x))| \leq \omega(x)|\partial_x^\sigma u(x)|^{p-1} \quad \text{a.e. in} \quad \Omega, \tag{3.2}$$

where $|\sigma| \leq m$. We set $H^{1,p}(\Omega, \mu)$ to be the weighted Sobolev space defined in [29]. Similarly, we introduce the definition of the higher order weighted Sobolev spaces.

Definition 3.1. *For a function $\varphi \in C^\infty(\Omega)$ we let*

$$||\varphi||_{m,p} = \sum_{|\sigma|=0}^{m} \left(\int_\Omega |\partial^\sigma \varphi(x)|^p \omega(x)dx\right)^{1/p}.$$

The weighted Sobolev space $H^{m,p}(\Omega, \mu)$ is defined to be the completion of

$$\left\{\varphi \in C^\infty(\Omega) : ||\varphi||_{m,p} < \infty\right\}.$$

In other words, a function $u \in H^{m,p}(\Omega, \mu)$ if and only if $u \in L^p(\Omega, \mu)$ and there are functions v_σ, such that for some sequence $\varphi_i \in C^\infty(\Omega)$ we have convergence

$$\int_\Omega |\varphi_i - u|^p \omega(x)dx \to 0$$

and

$$\int_{\Omega} |\partial_x^{\sigma} \varphi_i - v_{\sigma}|^p \omega(x) dx \to 0$$

as $i \to 0$. *The function* v_{σ} *is called* σ*-th derivative of* u *in* $H^{m,p}(\Omega)$.

We say that a function $u \in H^{m,q}(\Omega, \mu)$ is a weak solution of

$$\sum_{|\sigma|=0}^{m} (-1)^{|\sigma|} \partial_x^{\sigma} A_{\sigma}(x, D^m u(x)) = 0$$

on an open set Ω if $q = p$ and

$$\sum_{|\sigma|=0}^{m} \int_{\Omega} A_{\sigma}(x, D^m u(x)) \partial_x^{\sigma} \phi dx = 0 \tag{3.3}$$

for any test function $\phi = (\phi_1, \cdots, \phi_N) \in C_0^{\infty}(\Omega)$. We say that $u \in H^{m,q}(\Omega, \mu)$ is the very weak solution if (3.3) holds and $q < p$.

Definition 3.2. *For the locally integrable function* u *denote by* $\widetilde{\partial_x^{\sigma}} u$ *the* σ*-th weak derivatives; that is*

$$\int_{\Omega} \phi \widetilde{\partial_x^{\sigma}} u dx = (-1)^{|\sigma|} \int_{\Omega} u \partial_x^{\sigma} \phi dx$$

for any $\phi \in C_0^{\infty}(\Omega)$. *We say that* $u \in W^{m,p}(\Omega, \mu)$ *if* $\widetilde{\partial_x^{\sigma}} u \in L^p(\Omega, \mu)$ *for all* $|\sigma| \leq m$.

Observe that we have the identity $\widetilde{\partial_x} \circ \widetilde{\partial_x} = \widetilde{\partial_x^2}$ due to

$$\int_{\Omega} \phi \widetilde{\partial_x} (\widetilde{\partial_x} u) dx = - \int_{\Omega} \widetilde{\partial_x} u \partial_x \phi dx = \int_{\Omega} \phi \widetilde{\partial_x^2} u dx,$$

and moreover, we have the semigroup property

$$\widetilde{\partial_x^{\sigma}} \circ \widetilde{\partial_x^{\gamma}} = \widetilde{\partial_x^{\sigma+\gamma}} \tag{3.4}$$

for any multiindeces σ and γ.

In this chapter we follow the approach, developed in [44], to obtain a self-improving integrability result for degenerate systems in the weighted space. Our principal result states as follows.

Theorem 3.3. *Let* Ω *be a bounded domain in* Ω, $p > 1$, *and* $\omega(x)$ *is* A_p*-weight. Assume that* $u \in H_{loc}^{m,r}(\Omega, \mu)$ *and* A_{σ} *satisfies* (3.56)-(3.3). *Then there exists* $\delta = \delta(n, N, \gamma, p, [\omega]_{A_p}) > 0$ *such that if* $r = p - \delta$, *then* $u \in H_{loc}^{m,p+\delta}(\Omega, \mu)$.

We postpone the proof of Theorem 3.3 to Section 3.4 and start from some auxiliary results.

3.1 Preliminary Lemmas

For $\omega(x) \in A_p$, there exists a $p_0 < p$ such that $\omega(x) \in A_{p_0}$ (see for example [29, 25]). We set $q_0 = q_0([\omega]_{A_p}) = \inf\{p_0 : \omega \in A_{p_0},\ 1 < p_0 < p\}$ and fix a positive number $q_0 < q < p$.

It is important to introduce another weighted Sobolev space $W^{m,p}(\Omega, \omega)$ as follows.

Definition 3.4. *For the locally integrable function u denote by* $\widetilde{\partial}_x^\sigma u$ *the* σ*-th weak derivatives; that is* $\int_\Omega \phi \widetilde{\partial}_x^\sigma u dx = (-1)^{|\sigma|} \int_\Omega u \partial_x^\sigma \phi dx$ *for any* $\phi \in C_0^\infty(\Omega)$*. We say that* $u \in W^{m,p}(\Omega, \omega)$ *if* $\widetilde{\partial}_x^\sigma u \in L^p(\Omega, \omega)$ *for all* $|\sigma| \le m$*.*

The author of [39] shows that $H^{m,p}(\Omega, \omega) = W^{m,p}(\Omega, \omega)$ for $m = 1$. His arguments also can be successfully applied for $m \ge 2$. We sketch the proof of this lemma for the completeness.

Lemma 3.5. *[39]* $H^{m,p}(\Omega, \omega) = W^{m,p}(\Omega, \omega)$.

Proof. We divide the proof into three steps.

Step 1. Let $\eta(x) \in C_0^\infty(\Omega)$, $\eta(x) = \eta_1(|x|)$ and $\int_{\mathbb{R}^n} \eta(x) = 1$, where $\eta_1(r)$ is a decreasing function. If ω is A_p-weight and $f \in L^p(\mathbb{R}^n, \omega)$, then $\eta_j * f \to f$ in $L^p(\mathbb{R}^n, \omega)$.

Step 2. We show that $H^{m,p}(\Omega, \omega) \subset W^{m,p}(\Omega, \omega)$. If $D \subset \mathbb{R}^n$ is a bounded domain recalling the embedding property [39]

$$L^p(D, \omega) \subset L^{p/q_0}(D). \tag{3.5}$$

So, for any $u \in H^{m,p}(\mathbb{R}^n, \omega)$, we have $u \in H^{m,p/q_0}(D, dx)$ whenever D is a bounded open subset of Ω. Moreover, any σ-th derivative of u that is in $H^{m,p}(\Omega, \omega)$ is also the σ-th weak derivative.

Since this assertion is true for $m = 1$, we prove the assertion for higher order derivatives by induction. If we have $H^{m-1,p}(\Omega, \omega) \subset W^{m-1,p}(\Omega, \omega)$ and $m \ge 2$, since $u \in H^{m,p}(\Omega, \omega)$, there exists $\varphi_j \in C_0^\infty(\Omega)$, and $\partial_x^\sigma \varphi_j \to \widetilde{\partial}_x^\sigma u$ in $L^p(\Omega, \omega)$ with $0 \le |\sigma| \le m - 1$ and $\partial_x^\tau \varphi_j \to v_\tau$ where $|\tau| = m$. We set e_k be the multiindex with k-th component 1 and 0 elsewhere. For any multiindex $|\sigma| = m - 1$, from Definition 3.4 we can verify that $\widetilde{\partial}_x^{\sigma+e_k} u(x) = \widetilde{\partial}_{x_k}(\widetilde{\partial}_x^\sigma u)(x)$. We take $\psi \in C_0^\infty(\Omega)$ and from (3.5),

$$\left| \int_\Omega \widetilde{\partial}_x^\sigma u \partial_{x_k} \psi - (-1) v_{\sigma+e_k} \psi dx \right| \le \max |D\psi| \left(\int_\Omega |\widetilde{\partial}_x^\sigma u - \partial_x^\sigma \varphi_j| dx + \int_\Omega |v_{\sigma+e_k} - \partial_x^{\sigma+e_k} \varphi_j| dx \right)$$
$$\to 0 \quad \text{as} \quad j \to \infty.$$

It follows that $\widetilde{\partial}_x^{\sigma+e_k} u(x) = v_{\sigma+e_k} \in L^p(\Omega, \omega)$. This completes the proof of Step 2.

Step 3. We aim to show the inclusion $W^{m,p}(\Omega, \omega) \subset H^{m,p}(\Omega, \omega)$. Let $u \in W^{m,p}(\Omega, \omega)$ and D be a bounded domain in Ω. It suffices to prove $u \in H^{m,p}(D, \omega)$, see [29, 1.15]. Since $u_j = \eta_j * u \in C^\infty(\mathbb{R}^n)$ and $\partial_x^\sigma u_j = \eta_j * \widetilde{\partial}_x^\sigma u(x)$ [64, Lemma 2.13], we have $\|u_j - u\|_{m,p,\omega} \to 0$. Therefore, $u \in H^{m,p}(\Omega, \omega)$. □

Let (R^n, d, μ) be the metric space with doubling measure. Let f be a locally integrable function on this measure space. We introduce arbitrary maximal function with respect to the measure μ which is defined by

$$\widetilde{M}_\mu(f)(x) = \sup_{x \in Q} \frac{1}{\mu(Q)} \int_Q |f| d\mu,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x. We define the centered maximal function

$$M_\mu(f)(x) = \sup_{\delta>0} \frac{1}{\mu(B(x,\delta))} \int_{B(x,\delta)} |f| d\mu$$

where $B(x,\delta) = \{y \in \mathbb{R}^n : d(x,y) < \delta\}$. We start from the observation that, the doubling condition of the measure μ implies the inequality $M_\mu(f)(x) \leq C\widetilde{M}_\mu(f)(x)$ for some constant $C > 0$. We choose an arbitrary cube Q, containing x. Let $r = 2\sqrt{n}l(Q)$ and $Q' = 100\sqrt{n}Q$, where $l(Q)$ is the length of the edge of the cube Q. Then $Q \subset B(x,r) \subset Q'$ and therefore $\mu(B(x,r)) \leq \mu(Q') \leq C\mu(Q)$. So we have

$$\frac{1}{\mu(Q)} \int_Q |f| d\mu \leq C \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| d\mu \leq C M_\mu(f)(x)$$

and, consequently, we know that these two maximal functions are pointwise comparable

$$C' M_\mu(f)(x) \leq \widetilde{M}_\mu(f)(x) \leq C M_\mu(f)(x). \tag{3.6}$$

From Vitali lemma and Marcinkiewicz interpolation theorem, we have the following lemma

Lemma 3.6. *[61] (1) If $f \in L^1(\mathbb{R}^n,\mu)$, then for every $\lambda > 0$,*

$$\mu\{x : M_\mu(f) > \lambda\} \leq \frac{c}{\lambda} \int_{R^n} |f(y)| d\mu(y)$$

(2) If $f \in L^p(\mathbb{R}^n,\mu)$, $1 < p \leq \infty$, then $M_\mu(f) \in L^p(\mathbb{R}^n,\mu)$ and

$$\left\| M_\mu(f) \right\|_{L^p(\mathbb{R}^n,\mu)} \leq C_p \|f\|_{L^p(\mathbb{R}^n,\mu)}$$

where the bound C_p depends only on c, n and p.

More specifically, for $d\mu = \omega(x)dx$ and $d(x,y)$ is the Euclidean metric $|x-y|$, we define a localized maximal operator with respect to any fixed subset $B \subset \mathbb{R}^n$, $M_B(f)(x) = M_\mu(f\chi_B)(x)$, where χ_B is the characteristic function of B. We let $M_B^k(f)(x)$ to be the k times composition operator of M_B on f. That is if $M_B^{k-1}(f)(x)$ is defined, then

$$M_B^k(f)(x) = \sup_{r>0} \frac{1}{\omega(B(x,r))} \int_{B(x,r)} M_B^{k-1}(f)(y)\chi_B(y)\omega(y)dy$$

is defined inductively for $k \geq 2$. We write

$$\mathrm{M}_B(f) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f|\chi_B dy$$

be the classical Hardy-Littlewood maximal function. Since $\omega(x) \in A_q$, we get

$$\left(\int_B \omega(x)^{1/(1-q)} dx \right)^{q-1} \leq C \frac{|B|^q}{\int_B \omega(x)dx} \approx C \frac{r^{nq}}{\omega(B)}. \tag{3.7}$$

We can deduce the following estimate

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}|f|\chi_B dy \leq \frac{1}{|B(x,r)|}\left(\int_{B(x,r)}|f|^q\chi_B\omega(y)dy\right)^{1/q}\left(\int_B \omega(y)^{-q/(q-1)}\omega(y)dy\right)^{1-1/q} \leq M_B(|f|^q)(x)^{1/q}. \tag{3.8}$$

Note that from (3.8) we actually proved

$$\mathrm{M}_B^k(f)(x) \leq M_B^k(|f|^q)(x)^{1/q} \tag{3.9}$$

for $k \geq 1$ and any locally integrable function f.

To simplify the notation, we write $⨍_B f(x)\omega(x)dx$ to be $\frac{1}{\omega(B)}\int_B f(x)\omega(x)dx$ and $\partial^\sigma u$ to be the σ-th derivative of u when $u \in H^{m,p}(\Omega,\omega)$. We shall use a theorem of Gehring in doubling measure space [16]. It is also worth to mention that Martin and Milman [50] extended the Gehring's lemma for non-doubling measures.

Lemma 3.7. *[50, 16] Let $q \in [q_0, 2n]$, where $q_0 > 1$. Assume that functions f, g, defined on $(\mathbb{R}^n,\omega)$, are nonnegative and $g \in L^q_{\mathrm{loc}}(\mathbb{R}^n,\omega)$, $f \in L^{r_0}_{\mathrm{loc}}(\mathbb{R}^n,\omega)$ for some $r_0 > q$. If there exist nonnegative constants $b > 1$ and θ such that for every ball $B \subset \beta B \subset \mathbb{R}^n$, $\beta > 1$, the following estimate holds*

$$⨍_B g^q\omega(x)dx \leq b\left[\left(⨍_{\beta B} g\omega(x)dx\right)^q + ⨍_{\beta B} f^q\omega(x)dx\right] + \theta ⨍_{\beta B} g^q\omega(x)dx,$$

then there exist nonnegative constants θ_0 and ϵ_0, $\theta_0 = \theta_0(q_0, Q, C_d, \beta)$ and $\epsilon_0 = \epsilon_0(b, q_0, Q, C_d, \beta)$ such that if $0 < \theta < \theta_0$ then $g \in L^p_{\mathrm{loc}}(\mathbb{R}^n,\omega)$ for $p \in [q, q+\epsilon_0)$ and moreover

$$\left(⨍_B g^p\omega(x)dx\right)^{1/p} \leq C\left[\left(⨍_{\beta B} g^q\omega(x)dx\right)^q + \left(⨍_{\beta B} f^p\omega(x)dx\right)^{1/p}\right]$$

for $C = C(b, q_0, Q, C_d, \beta)$.

3.2 Weighted Pointwise Sobolev Inequality

The following lemma extends [44, Lemma 2.1] to the case of the weighted Sobolev space.

Lemma 3.8. *Let $l > 0$ be an integer, $r > 0$, $x_0 \in \Omega$, and $B = B(x_0, r) \subset \Omega$. If $u \in W^{l,p}(B,\omega)$, $\int_B \partial^\alpha u\, dx = 0$ for $0 \leq |\alpha| \leq l-1$, and $x \in B$, then there exists $C_1(n,l,p) > 0$ such that*

$$|u(x)| \leq C_1 r^l M_B^l(|\partial^l u|^q)(x)^{\frac{1}{q}} \quad \text{a. e. in } B \tag{3.10}$$

Moreover, if $q \leq s \leq p$, and $ls < nq$, then there exists $C_2(n,l,p) > 0$ such that

$$|u(x)| \leq C_2 r^l \left(⨍_B M_B^l(|\partial^l u|^q)(x)^{\frac{s}{q}}\omega(x)dx\right)^{\frac{l}{nq}} M_B^l(|\partial^l u|^q)(x)^{\frac{s}{qs^*}} \quad \text{a. e. in } B \tag{3.11}$$

$$\left(\fint_B |u(x)|^{s^*}\omega(x)dx\right)^{\frac{1}{s^*}} \le C_2 r^l \left(\fint_B M_B^l(|\partial^l u|^q)(x)^{\frac{s}{q}}\omega(x)dx\right)^{\frac{1}{s}} \tag{3.12}$$

where $s^* = \frac{nqs}{nq-ls}$. *Furthermore, if* $ls > nq$, *then*

$$|u(x)| \le \widehat{C_2} r^l \left(\fint_B M_B^l(|\partial^l u|^q)^{s/q}\omega(x)dx\right)^{1/s} \quad \text{a. e. in } B \tag{3.13}$$

where $\widehat{C_2} = \widehat{C_2}(n, l, s)$.

Proof. Recalling the embedding property (3.5), we conclude that

$$\begin{aligned}\left|u(x) - \frac{1}{|B|}\int_B u(y)dy\right| &\le C(n)\int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}}dy \\ &\le Cr\mathrm{M}_B(|\nabla u|)(x) \le CrM_B(|\nabla u|^q)(x)^{1/q} \quad \text{a. e. in } B\end{aligned}$$

where $\nabla u = (\partial_{x_1}u, \cdots, \partial_{x_n}u)$ is the distributional derivatives of u. From an iteration argument, we have

$$|u(x)| \le C_1 r^l M_B^l(|\partial^l u|^q)(x)^{1/q} \quad \text{a. e. in } B.$$

Next, making use of arguments from [29, page 306] for $q < s < nq$, we can write

$$|u(x)| \le C_1\eta \mathrm{M}_B(|\nabla u|)(x) + C_2\eta^{(s-nq)/s}\left(\int_B |\nabla u|^s\omega(x)dx\right)^{1/s}\left(\int_B \omega(x)^{1/(1-q)}dx\right)^{(q-1)/s} = I + II. \tag{3.14}$$

Thus we have

$$|u(x)| \le C\mathrm{M}_B(|\nabla u|)(x)^{(nq-s)/nq}\left(\int_B |\nabla u|^s\omega(x)dx\right)^{1/nq}\left(\int_B \omega(x)^{1/(1-q)}dx\right)^{(q-1)/nq} \tag{3.15}$$

by taking

$$\eta = \left(\frac{\left(\int_B |\nabla u|^p\omega(x)dx\right)^{1/s}\left(\int_B \omega(x)^{1/(1-q)}dx\right)^{(q-1)/s}}{\mathrm{M}_B(|\nabla u|)(x)}\right)^{s/nq}.$$

Combining (3.7) with (3.15), we obtain

$$\begin{aligned}|u(x)| &\le Cr\mathrm{M}_B(|\nabla u|)(x)^{(nq-s)/nq}\left(\fint_B |\nabla u|^s\omega(x)dx\right)^{1/nq} \\ &\le CrM_B(|\nabla u|^q)(x)^{(nq-s)/nq^2}\left(\fint_B M_B(|\nabla u|^q)^{s/q}\omega(x)dx\right)^{1/nq}.\end{aligned}$$

The last step is followed by the fact $|\nabla u| \le M_B(|\nabla u|^q)(x)^{1/q}$, $\omega-$a. e. in B. As a result, we have proved (3.11) for the case $l = 1$. Now, we need only to show (3.11) since the inequality (3.12) is an immediate consequence of (3.11).

Applying the induction argument we show that if (3.11) already holds for $l-1$, then it also holds for l. We repeat the arguments above, for $x \in B$ and $(l-1)s < ls < nq$, observing that $\bar{s} = \frac{nqs}{nq-ls+s} > s$ and $\bar{s} < nq$. Replace $\bar{s}$ by s in (3.14) and get the estimate of I

$$\begin{aligned}
I &= C_1 \eta \mathrm{M}_B(|\nabla u|)(x) \\
&\le C\eta r^{l-1} \left(\fint_B M_B^{l-1}(|\partial^l u|^q)(x)^{s/q} \omega(x) dx \right)^{(l-1)/nq} \mathrm{M}_B(M_B^{l-1}(|\partial^l u|^q)(\cdot)^{s/q\bar{s}})(x) \\
&\le C\eta r^{l-1} \left(\fint_B M_B^{l-1}(|\partial^l u|^q)(x)^{s/q} \omega(x) dx \right)^{(l-1)/nq} M_B^l(|\partial^l u|^q)(x)^{s/q\bar{s}}
\end{aligned}$$

by (3.11), and the induction hypothesis where the last step is follows from (3.9) and Hölder inequality. On the other hand, we estimate II as follows:

$$\begin{aligned}
II &= C_2 \eta^{(\bar{s}-nq)/\bar{s}} \left(\int_B |\nabla u|^{\bar{s}} \omega(x) dx \right)^{1/\bar{s}} \left(\int_B \omega(x)^{1/(1-q)} dx \right)^{(q-1)/\bar{s}} \\
&\le C_2 \omega(B)^{1/\bar{s}} \eta^{(\bar{s}-nq)/\bar{s}} r^{l-1} \left(\fint_B M_B^{l-1}(|\partial^l u|^q)(x)^{s/q} \omega(x) dx \right)^{1/s} \left(\int_B \omega(x)^{1/(1-q)} dx \right)^{(q-1)/\bar{s}} \\
&\le C_2 r^{nq/\bar{s}} \eta^{(\bar{s}-nq)/\bar{s}} r^{l-1} \left(\fint_B M_B^{l-1}(|\partial^l u|^q)(x)^{s/q} \omega(x) dx \right)^{1/s}.
\end{aligned}$$

by (3.7) and induction procedure. Choosing

$$\eta = r \left(\fint_B M_B^{l-1}(|\partial^l u|^q)(x)^{s/q} \omega(x) dx \right)^{1/nq} M_B^l(|\partial^l u|^q)(x)^{-s/nq^2},$$

we get $I = II$ in this case and we obtain

$$|u(x)| \le C_2 r^l \left(\fint_B M_B^l(|\partial^l u|^q)(x)^{s/q} \omega(x) dx \right)^{l/nq} M_B^l(|\partial^l u|^q)(x)^{s/qs^*},$$

that proves (3.11) for the general case. To prove (3.13), observe that $(s/q)' < n/(n-1)$, we have $\int_{\{y \in B: |x-y| \le r\}} |x-y|^{(-n+1)(s/q)'} dy \le C(s,n) r^{n-(n-1)(s/q)'}$. Therefore,

$$\begin{aligned}
|u(x)| &\le C(n) \int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy \\
&\le C(s,n) \left(\int_{\{y \in B: |x-y| \le r\}} |\nabla u(y)|^{s/q} dy \right)^{q/s} r^{n/(s/q)'-(n-1)} \\
&\le C(s,n) r \left(\fint_B |\nabla u(y)|^s \omega(x) dy \right)^{1/s}.
\end{aligned}$$

This proves (3.13) for $l = 1$, and it is easy to prove the general case by induction. This completes the proof of Lemma 3.8. □

As an application of Lemma 3.8, we have the following lemma which will be used later. We fix a $q_1 > 1$ such that $q < q_1 < p$

Lemma 3.9. *Let $\lambda > 0$. If $h \in W^{m,q_1}(\Omega, \omega)$,* supp$h$ $\subset \frac{1}{2}B = \{x \in \Omega : |x - x_0| < r\}$*, and*

$$F(\lambda) = E(\lambda) \cap \bar{B} \neq \emptyset \;\text{ where }\; E(\lambda) = \left\{x \in \mathbb{R}^n : M_B^m(|\partial^m h|_1^q)(x)^{1/q_1} \leq \lambda\right\},$$

then $h\,|_{F(\lambda)}$ has an extension H to $\mathbb{R}^n$ satisfying

(i) $H = h$ on $F(\lambda)$ and supp $H \subset B(x_0, 4r)$*,*

(ii) $H \in W^{m,\infty}(\mathbb{R}^n)$ with $\|\partial^\sigma H\|_{L^\infty(\mathbb{R}^n)} \leq c\lambda r^{m-|\sigma|}$, $0 \leq |\sigma| \leq m$,

(iii) $|\partial^\sigma (H - h)(x)| \leq c\lambda d(x)^{m-|\sigma|}$ a. e. *for $0 \leq |\sigma| \leq m - 1$, where $d(x)$ denotes the distance from x to $F(\lambda)$.*

Proof. From (3.5), we know that $h \in W^{m,q_1/q_0}(\Omega)$. Observe that

$$\int_B \partial^\sigma h dx = 0 \quad \text{when} \quad 0 < |\sigma| \leq m,$$

and, by Lemma 3.8, we have $|\partial^\sigma h(x)| \leq c\lambda r^{m-|\sigma|}$ for any $x \in F(\lambda)$ and $0 < |\sigma| \leq m$. On the other hand,

$$\left|h(x) - \frac{1}{|B|}\int_B h\right| \leq cr^m M_B^m(|\partial^m h|^{q_1})(x)^{1/q_1} \quad \text{a. e. in } B. \tag{3.16}$$

Therefore we can find a point $x_1 \in \{\frac{4r}{3} < |x - x_0| < \frac{3r}{2}\}$ such that 3.16 holds for x_1. Since $h(x_1) = 0$, we get

$$\left|\frac{1}{|B|}\int_B h\right| \leq cr^m M_B^m(|\partial^m h|^{q_1})(x_1)^{1/q_1} \leq cr^m \left(\frac{1}{\omega(B(x_1, \frac{3r}{4}))}\int_{B(x_0,r)} |h|^{q_1}\omega(x)dx\right)^{1/q_1}.$$

Since $\omega(B(x_0, r)) \leq \omega(B(x_1, 5r)) \leq C\omega(B(x_1, \frac{3r}{4}))$, we get

$$\left|\frac{1}{|B|}\int_B h\right| \leq c' r^m \min_{x\in B} \widetilde{M}_B^m(|\partial^m h|^{q_1})(x_1)^{1/q_1} \leq c' r^m \min_{x\in B} M_B^m(|\partial^m h|^{q_1})(x_1)^{1/q_1}$$

where $\widetilde{M}_B$ is the localized arbitrary maximal function and the last step is followed by (3.6). This implies $|\partial^\sigma h(x)| \leq c\lambda r^{m-|\sigma|}$ for any $x \in F(\lambda)$ and $0 \leq |\sigma| \leq m$. From a similar argument of [44, Lemma 2.2] and Lemma 3.8, for any $z_0 \in F(\lambda)$ and $s > 0$ we have

$$|\partial^\sigma (u - Q_{m-1})(y)| \leq c\lambda s^{m-|\sigma|} \quad y \in F(\lambda), \;\; 0 \leq |\alpha| \leq m$$

where Q_{m-1} is the Taylor polynomial of degree $m-1$. So $h\,|_{F(\lambda)}$ satisfies the conditions of Whitney extension theorem. □

3.3 The A_p weight on metric space with doubling measures

Consideration is given to A_p weight the metric space $(\mathbb{R}^n, d, \mu)$ with doubling measures. More specifically, the measure μ satisfies a so-called doubling D_b condition which states as follows.

Definition 3.10. *[62] We say that $\mu \in D_b$, if there are a constant $k > 0$ and a number $b > 0$ with the property that for all $x \in \mathbb{R}^n$, $t \geq 1$ and $r > 0$ we have*

$$\mu(B(x,tr)) \leq kt^b \mu(B(x,r)).$$

We follow the definition of A_p weight on the metric space $(\mathbb{R}^n, d, \mu)$ which was introduced in [62, page 4].

Definition 3.11. *[62] The $A_p(\mu)$ condition for ω with respect to the measure μ, that is*

$$\frac{1}{\mu(B)}\int_B \omega d\mu \left(\frac{1}{\mu(B)}\int_B \omega^{-1/(p-1)} d\mu\right)^{p-1} < c$$

for all the ball $B \subset \mathbb{R}^n$.

We will also need the following lemma.

Lemma 3.12. *[62] Let $p > 1$. The estimate $\|M_\mu(f)\|_{L^p(v)} \leq C_p \|f\|_{L^p(v)}$ holds for every $f \in L^p(v)$ if and only if v is a weighted measure with respect to μ and the weight $\omega \in A_p(\mu)$.*

We are going to prove the following lemma which in analogue with [25, Theorem 9.2.7].

Lemma 3.13. *Let f be a locally integrable function on $(\mathbb{R}^n, \mu)$, $1 \leq q < p$ and $0 < \delta < p - q$. Then $M_\mu(|f|^q)(x)^{-\delta/q}$ is the $A_{p/q}(\mu)$ weight.*

Proof. The proof is divided into several parts.

Step 1. Recalling the Kolmogorov theorem on the measure space [61, page 43]. Let $S : L^1(\mathbb{R}^n, \mu) \to L^{1,\infty}(\mathbb{R}^n, \mu)$ be a weak-$(1,1)$ type operator and A is a finite measurable set. Then

$$\int_A |S(f)(x)|^{\epsilon'} d\mu \leq (1-\epsilon')^{-1} \|S\|^{\epsilon'}_{L^1 \to L^{1,\infty}} \mu(A)^{1-\epsilon'} \left(\int_{\mathbb{R}^n} |f| d\mu\right)^{\epsilon'}. \tag{3.17}$$

for all $0 < \epsilon' < 1$. From Lemma 3.6, we have (3.18) is true when S replaced by M_μ.

Step 2. Let $0 < \epsilon < q$. We prove that $M_\mu(|f|^q)(x)^{\epsilon/q}$ is an $A_1(\mu)$ weight. The assertion is reduced to showing that

$$\frac{1}{\mu(Q)}\int_Q M_\mu(|f|^q)(y)^{\epsilon/q} d\mu(y) \leq C(n,\epsilon,q) M_\mu(|f|^q)(x)^{\epsilon/q} \quad \mu \ \ a.e. \tag{3.18}$$

for arbitrary cube $Q \subset \mathbb{R}^n$ containing x.

We split the function $f(x) = f(x)\chi_{3\sqrt{n}Q} + f(x)\chi_{(3\sqrt{n}Q)^c} := f_1 + f_2$. On the one hand, we have

$$\begin{aligned}\frac{1}{\mu(Q)}\int_Q M_\mu(|f_1|^q)(y)^{\epsilon/q}d\mu(y) &\le C(n,\epsilon)\left(\frac{1}{\mu(Q)}\int_{\mathbb{R}^n}|f_1|^q d\mu\right)^{\epsilon/q}\\ &\le C(n,\epsilon)\left(\frac{1}{\mu(Q)}\int_{3\sqrt{n}Q}|f|^q d\mu\right)^{\epsilon/q}\\ &\le C(n,\epsilon)M_\mu(|f|^q)(x)^{\epsilon/q}\end{aligned}$$

by (3.17). On the other hand, observe that $M_\mu(|f_2|^q)(y) \le C\widetilde{M}_\mu(|f_2|^q)(x)$ for any $x, y \in Q$. Combining this estimate and (3.6), we have $M_\mu(|f_2|^q)(y) \le CM_\mu(|f_2|^q)(x) \le CM_\mu(|f|^q)(x)$. Therefore,

$$\frac{1}{\omega(Q)}\int_Q M_\mu(|f_2|^q)(y)^{\epsilon/q}\omega(y)dy \le C(n)M_\mu(|f|^q)(x)^{\epsilon/q}.$$

This proves (3.18).

Final Step. We show that $\xi(x) = M_\mu(|f|^q)(x)^{-\delta/q}$ is an $A_{p/q}(\mu)$ weight..

Denote by B an arbitrary ball in $\mathbb{R}^n$ and obtain

$$\begin{aligned}\frac{1}{\mu(B)}\int_B \xi(x)^{-\frac{q}{p-q}}d\mu &= \frac{1}{\mu(B)}\int_B M_\mu(|f|^q)(x)^{\delta/(p-q)}d\mu\\ &\le C(n,p,q)M_\mu(|f|^q)(x)^{\delta/(p-q)}\end{aligned}$$

from (3.18). Then we get

$$\xi(x)\left(\frac{1}{\mu(B)}\int_B \xi(x)^{-q/(p-q)}d\mu\right)^{(p-q)/q} \le C(n,p,q)$$

and thus

$$\left(\frac{1}{\mu(B)}\int_B \xi(x)d\mu\right)\left(\frac{1}{\mu(B)}\int_B \xi(x)^{-q/(p-q)}d\mu\right)^{(p-q)/q} \le C(n,p,q).$$

This completes the proof. □

Since the A_p weight $\omega(x)$ satisfies the D_b condition (see, for example [25, page 284]), from Lemma 3.12 and Lemma 3.13 we get the estimate

$$\int_\Omega M_B(|h|^q)^{p/q}M_B(|f|^q)(x)^{-\delta/q}\omega(x)dx \le C\int_\Omega |h|^p M_B(|f|^q)(x)^{-\delta/q}\omega(x)dx, \tag{3.19}$$

for any function $h \in L^p(\mathbb{R}^n, \eta(x)dx)$ where $\eta(x) = M_B(|f|^q)(x)^{-\delta/q}\omega(x)$.

3.4 Proof of Theorem 3.3

Let u, Ω be as in Theorem 3.3. Suppose $B(z_0, R)$ is any ball in Ω. We fix a point $x_0 \in B(z_0, R/2)$. Let $0 < r < R/32$ and $B_r = B(x_0, r)$. There exists a smooth function $\varphi \in C_0^\infty(B)$, where $B = \overline{B(x_0, 2r)}$, such that $\varphi \equiv 1$ on $B(x_0, r)$, supp $\varphi \subset B$ and $|\partial^\sigma \varphi(x)| \le Cr^{-|\sigma|}$, $|\sigma| \le m$. There exists a polynomial P such that $\int_{B(x_0,8r)} \partial^I(u-P)dx = 0$ for any $|I| \le m-1$. Denote $u_0 = (u-P)\varphi(x)$ and $\bar{v}$ be the Whitney extension of $u_0 \mid_{F(\lambda)}$. We assert that (3.3) holds if we substitute ϕ by $\bar{v}$; that is

$$\sum_{|\sigma|=0}^{m} \int_\Omega A_\sigma(x, D^m u(x)\partial_x^\sigma \bar{v}(x)dx = 0.$$

In fact, we choose a sequence $\phi_i \in C_0^\infty(\Omega)$ such that $\|\phi_i - \bar{v}\|_{H^{m,(p-\delta)/(1-\delta)}(\Omega,\, \mu)} \to 0$ as $i \to \infty$. Therefore

$$\begin{aligned}\left|\sum_{|\sigma|=0}^{m} \int_\Omega A_\sigma(x, D^m u(x))\partial_x^\sigma \bar{v}(x)dx\right| &= \left|\sum_{|\sigma|=0}^{m} \int_\Omega A_\sigma(x, D^m u(x)) \left(\partial_x^\sigma \bar{v}(x) - \partial_x^\sigma \phi_i(x)\right) dx\right| \\ &\le \sum_{|\sigma|=0}^{m} \int_\Omega |\partial_x^\sigma u(x)|^{p-1} \left|\partial_x^\sigma \bar{v}(x) - \partial_x^\sigma \phi_i(x)\right| \omega(x)dx \\ &\le \sum_{|\sigma|=0}^{m} \left\|\partial_x^\sigma u\right\|_{L^{p-\delta}(\Omega,\, \mu)}^{p-1} \left\|\partial_x^\sigma \bar{v} - \partial_x^\sigma \phi_i\right\|_{L^{(p-\delta)/(1-\delta)}(\Omega,\, \mu)} \to 0.\end{aligned}$$

The last step follows from the Hölder inequality.

We split Ω into two sets $\Omega = F(\lambda) \cup (\Omega - F(\lambda))$ and obtain

$$\begin{aligned}\sum_{|\sigma|=0}^{m} \int_{F(\lambda)} A_\sigma(x, D^m u(x))\partial_x^\sigma u_0(x)dx &= -\sum_{|\sigma|=0}^{m} \int_{\Omega - F(\lambda)} A_\sigma(x, D^m u(x))\partial_x^\sigma v(x)dx \\ &\le \sum_{|\sigma|=0}^{m-1} \int_{B(x_0,8r)-F(\lambda)} A_\sigma(x, D^m u(x))\partial_x^\sigma (u_0 - v)(x)dx \\ &- \sum_{|\sigma|=0}^{m-1} \int_{B(x_0,8r)-F(\lambda)} A_\sigma(x, D^m u(x))\partial_x^\sigma u_0(x)dx \\ &+ c\lambda \int_{B(x_0,8r)-F(\lambda)} |\partial_x^m u(x)|^{p-1}\omega(x)dx \\ &= J_1 + J_2 + J_3.\end{aligned} \tag{3.20}$$

We need to verify that $F(\lambda)$ is a nonempty set. We assert that there exists $\lambda_0 > 0$ such that $E(\lambda) = F(\lambda)$ for any $\lambda > \lambda_0$. Notice that if $x \in \mathbb{R}^n - B(x_0, 3r)$, then

$$\omega(B(x_0, 2r)) \le \omega(B(x, 7r)) \le C\omega(B(x, r)).$$

This implies

$$\begin{aligned} M_B^m(|\partial_x^m u_0|^{q_1})(x) &\le c\frac{1}{\omega(B(x_0,4r))}\int_{B(x_0,4r)} M_B^{m-1}(|\partial_x^m u_0|^{q_1})(x)\omega(x)dx \\ &\le c\min_{B(x_0,8r)} M_B^m(|\partial_x^m u_0|^{q_1})(x). \end{aligned} \tag{3.21}$$

We denote

$$\lambda_0 = c\left(⨍_{B(x_0,4r)} M_B^{m-1}(|\partial_x^m u_0|^{q_1})(x)\omega(x)dx\right)^{1/q_1},$$

We find that the assertion is true and $F(\lambda) \neq \emptyset$ for $\lambda > \lambda_0$. Multiplying both sides of (3.20) by $\lambda^{-1-\delta}$, integrating on (λ_0, ∞), and interchanging the order of integration on both sides, we get

$$\begin{aligned} &\delta^{-1}K \\ &= \delta^{-1}\sum_{|\sigma|=0}^{m}\int_{\mathbb{R}^n-F(\lambda_0)} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1}A_\sigma(x,D^m u(x))\partial_x^\sigma u_0(x)dx \\ &+ \delta^{-1}\lambda_0^{-\delta}\sum_{|\sigma|=0}^{m}\int_{F(\lambda_0)} A_\sigma(x,D^m u(x))\partial_x^\sigma u_0(x)dx \\ &\le \sum_{i=1}^{3}\int_{\lambda_0}^{\infty}\lambda^{-(1+\delta)}J_i d\lambda = \sum_{i=1}^{3}K_i. \end{aligned} \tag{3.22}$$

If we set

$$\lambda_0' = c\left(⨍_{B(x_0,4r)} |\partial_x^m u_0|^{q_1}(x)\omega(x)dx\right)^{1/q_1},$$

we conclude that $\lambda_0 \ge \lambda_0'$.

The estimate of K_1

Let $0 \le l \le m-1$. We define the following quantities:

$$\begin{aligned} \alpha &= \frac{(p-1)\epsilon}{100n(p+\epsilon)^2}, \\ \tilde{p} &= 1-\frac{(m-l)(1-\alpha)p}{nq_1}, \\ \gamma_1 &= \left[\left(\int_{B(z_0,\frac{3R}{4})} M_B^{2m}(|\partial^m u|^{q_1})^{\frac{n(1-\alpha)}{m-l}}\omega(x)dx\right)^{\frac{1}{q_1}} \frac{|B(z_0,\frac{3R}{4})|}{\omega(B(z_0,\frac{3R}{4}))^{\frac{1}{q_1}}}\right]^{\frac{m-l}{n}}, \\ \gamma_1' &= \left[\frac{1}{\omega(B(x_0,4r))}\int_{B(x_0,4r)} |\partial_x^m u_0|^{p-\delta}(x)\omega(x)dx\right]^{(1-\alpha)/(p-\delta)}, \end{aligned} \tag{3.23}$$

$$\gamma_2 = \left[\left(\int_{B(z_0, \frac{3R}{4})} M_B^{2m} (|\partial^m u|^{q_1})^{\frac{p(1-\alpha)}{q_1}} \omega(x) dx \right)^{\frac{1}{q_1}} \frac{|B(z_0, \frac{3R}{4})|}{\omega(B(z_0, \frac{3R}{4}))^{\frac{1}{q_1}}} \right]^{\frac{m-l}{n}},$$

$$\tau_1 = \left[\left(\int_{B(z_0, \frac{3R}{4})} M_B^{m} (|\partial^m u_0|^{q_1})^{\frac{n(1-\alpha)}{m-l}} \omega(x) dx \right)^{\frac{1}{q_1}} \frac{|B(z_0, \frac{3R}{4})|}{\omega(B(z_0, \frac{3R}{4}))^{\frac{1}{q_1}}} \right]^{\frac{m-l}{n}},$$

$$\tau_1' = \left[\frac{1}{\omega(B(x_0, 4r))} \int_{B(x_0,4r)} |\partial_x^m u|^{p-\delta}(x) \omega(x) dx \right]^{(1-\alpha)/(p-\delta)},$$

$$\tau_2 = \left[\left(\int_{B(z_0, \frac{3R}{4})} M_B^{m} (|\partial^m u_0|^{q_1})^{\frac{p(1-\alpha)}{q_1}} \omega(x) dx \right)^{\frac{1}{q_1}} \frac{|B(z_0, \frac{3R}{4})|}{\omega(B(z_0, \frac{3R}{4}))^{\frac{1}{q_1}}} \right]^{\frac{m-l}{n}}.$$

We can write

$$\begin{aligned}
\partial^m u_0 &= \sum_{\alpha+\beta=m} \partial^\alpha (u-P)(x) \partial^\beta \varphi(x) \\
&= \varphi(x) \partial^m (u-P)(x) + \sum_{\alpha+\beta=m,\ |\alpha|<m} \partial^\alpha (u-P)(x) \partial^\beta \varphi(x) \\
&= \varphi(x) \partial^m u(x) + \sum_{\alpha+\beta=m,\ |\alpha|<m} \partial^\alpha (u-P)(x) \partial^\beta \varphi(x).
\end{aligned}$$

We have

$$|\partial^\alpha (u-P)(x)| \leq C r^{m-|\alpha|} M_B^{m-|\alpha|} (|\partial^m u|^q)(x)^{\frac{1}{q}} \leq C r^{m-|\alpha|} M_B^{m-|\alpha|} (|\partial^m u|^{q_1})(x)^{\frac{1}{q_1}}$$

from (3.10) and by $\partial^m P = 0$. Combining this estimate with the fact that $|\partial^\beta \varphi(x)| \leq C r^{-|\beta|}$, we get

$$|\partial^\alpha (u-P)(x) \partial^\beta \varphi(x)| \leq C M_B^m (|\partial^m u|^{q_1})(x)^{\frac{1}{q_1}}$$

for all multiindices α and β, $0 \leq |\alpha| < m$, $0 < |\beta| \leq m$. Since φ is a bounded function, we know that

$$|\partial^m u(x)| \leq C M_B^m (|\partial^m u|^{q_1})(x)^{\frac{1}{q_1}}.\ \ \mu - \text{a. e},$$

see, for example [61]. Therefore,

$$|\partial^m u_0| \leq C M_B^m (|\partial^m u|^{q_1})(x)^{\frac{1}{q_1}} \quad \mu - \text{a. e.} \tag{3.24}$$

and

$$|J_1| \leq c\lambda \sum_{l=0}^{m-1} \int_{\mathbb{R}^n - E(\lambda)} |\partial^m u|^{p-1} d^{m-l}(x) \omega(x) dx$$

from (3.57). Observe that $d^{m-l}(x) \leq C|B(x_0,8r)-E(\lambda)|^{\frac{m-l}{n}}$. We fix $0<t<(p-\delta)/q_1$ and deduce

$$\begin{aligned}
\lambda d^{m-l}(x) &\leq C\lambda^{1-\frac{m-l}{n}t}\left[\int_{B(z_0,\frac{3R}{4})} M_B^m(|\partial^m u_0|^{q_1})^{\frac{t}{q_1}}\,dx\right]^{\frac{m-l}{n}} \\
&\leq C\lambda^{1-\frac{m-l}{n}t}\left[\left(\int_{B(z_0,\frac{3R}{4})} M_B^m(|\partial^m u_0|^{q_1})^t\omega(x)dx\right)^{\frac{1}{q_1}}\left(\int_{B(z_0,\frac{3R}{4})}\omega(x)^{-\frac{1}{q_1-1}}\right)^{\frac{q_1-1}{q_1}}\right]^{\frac{m-l}{n}} \\
&\leq C\lambda^{1-\frac{m-l}{n}t}\left[\left(\int_{B(z_0,\frac{3R}{4})} M_B^m(|\partial^m u_0|^{q_1})^t\omega(x)dx\right)^{\frac{1}{q_1}}\frac{|B(z_0,\frac{3R}{4})|}{\omega(B(z_0,\frac{3R}{4}))^{\frac{1}{q_1}}}\right]^{\frac{m-l}{n}} \\
&\leq C\min\{\lambda^{\alpha}\tau_1,\lambda^{\tilde{p}}\tau_2\} \leq C\min\{\lambda^{\alpha}\gamma_1,\lambda^{\tilde{p}}\gamma_2\},
\end{aligned} \tag{3.25}$$

where the last step is followed from (3.24). Using (3.25), we get

$$|J_1| \leq c\sum_{l=0}^{m-1}\int_{\mathbb{R}^n-E(\lambda)}|\partial^m u|^{p-1}\min\{\lambda^{\alpha}\gamma_1,\lambda^{\tilde{p}}\gamma_2\}\omega(x)dx.$$

If $p \geq \frac{q_1 n}{m-l}$, then we choose $\lambda^{\alpha}\gamma_1$ that provides the minimum in (3.25). In this case,

$$\begin{aligned}
K_1 &\leq C\sum_{l=0}^{m-1}\gamma_1\int_{B(x_0,8r)-E(\lambda_0)}\int_0^{M_B^m(|\partial^m u_0|^{q_1})^{\frac{1}{q_1}}}\lambda^{\alpha-\delta-1}d\lambda|\partial^m u|^{p-1}\omega(x)dx \\
&\leq C\sum_{l=0}^{m-1}\gamma_1\int_{B(x_0,8r)} M_B^m(|\partial^m u_0|^{q_1})^{\frac{\alpha-\delta}{q_1}}|\partial^m u|^{p-1}\omega(x)dx \\
&\leq C\sum_{l=0}^{m-1}\gamma_1\int_{B(x_0,8r)} M_B^{2m}(|\partial^m u|^{q_1})^{\frac{\alpha-\delta}{q_1}}|\partial^m u|^{p-1}\omega(x)dx,
\end{aligned}$$

where the last step follows from (3.24). While in the case $p < \frac{q_1 n}{m-l}$, the quantity $\lambda^{\tilde{p}}\gamma_2$ should be taken into account, and thus,

$$\begin{aligned}
K_1 &\leq C\sum_{l=0}^{m-1}\gamma_2\int_{B(x_0,8r)-E(\lambda_0)}\int_0^{M_B^m(|\partial^m u_0|^{q_1})^{\frac{1}{q_1}}}\lambda^{\tilde{p}-\delta-1}d\lambda|\partial^m u|^{p-1}\omega(x)dx \\
&\leq C\sum_{l=0}^{m-1}\gamma_2\int_{B(x_0,8r)} M_B^m(|\partial^m u_0|^{q_1})^{\frac{\tilde{p}-\delta}{q_1}}|\partial^m u|^{p-1}\omega(x)dx \\
&\leq C\sum_{l=0}^{m-1}\gamma_2\int_{B(x_0,8r)} M_B^{2m}(|\partial^m u|^{q_1})^{\frac{\tilde{p}-\delta}{q_1}}|\partial^m u|^{p-1}\omega(x)dx.
\end{aligned}$$

We use arguments similar to [44], and conclude that

$$K_1 \leq \int_{B(x_0,8r)} F_1^{p-\delta}\omega(x)dx, \quad \text{with} \quad F_1 \in L^{p+\alpha'}(B(x_0,8r),\mu), \tag{3.26}$$

provided $(p+\alpha')(p-\min\{\alpha,\tilde{p}\}-\delta)/(p-\delta) < p-\delta$ and thus

$$\delta < \min\{\alpha,\tilde{p}\} - \alpha' + \alpha' \min\{\alpha,\tilde{p}\}/p. \tag{3.27}$$

The estimate of K_2

We write

$$|J_2| \le \sum_{l=0}^{m-1} \int_{B(x_0,8r)-F(\lambda)} |\partial^m u|^{p-1} \partial_x^l u_0(x)\omega(x)dx$$

and distinguish several cases. First of all, we assert that

$$|\partial^l u_0|(x) \le C(m,l,s,\Omega)\lambda^\alpha M_B^m(|\partial^m u_0|^{q_1})(x)^{(1-\alpha)/q_1}. \tag{3.28}$$

for all the $0 \le l \le m-1$. If $(m-l)q_1 > nq$, we use inequality (3.11) with $s = q_1$, and from the fact that $q_1 > q > 1$, Hardy-Littlewood maximal theorem and Hölder inequality, we obtain

$$\begin{aligned}
|\partial^l u_0|(x) &\le C_2 r^{m-l} \left(\fint_B M_B^m(|\partial^m u_0|^q)(x)^{\frac{q_1}{q}} \omega(x)dx \right)^{(m-l)/nq} M_B^m(|\partial^m u_0|^q)(x)^{(1/q)(1-q_1(m-l)/nq)} \\
&\le C_2 r^{m-l} \left(\fint_B |\partial^m u_0|(x)^{q_1} \omega(x)dx \right)^{(m-l)/nq} M_B^m(|\partial^m u_0|^q)(x)^{(1/q)(1-q_1(m-l)/nq)} \\
&\le C_2 r^{m-l} {\lambda'_0}^{(q_1/q)(m-l)/n} M_B^m(|\partial^m u_0|^{q_1})(x)^{(1/q_1)-(m-l)/nq}
\end{aligned}$$

We can take a small $0 < \alpha < 1 < (q_1/q)(m-l)/n$, from the fact $\lambda'_0 < \lambda_0 < M_B^m(|\partial^m u_0|^{q_1})(x)^{1/q_1}$ and $\lambda_0 \le \lambda$, we conclude that

$$\begin{aligned}
|\partial^l u_0|(x) &\le C_2 r^{m-l} \lambda^\alpha M_B^m(|\partial^m u_0|^{q_1})(x)^{(1-\alpha)/q_1} \\
&\le C_2(m,l,s,\Omega)\lambda^\alpha M_B^m(|\partial^m u_0|^{q_1})(x)^{(1-\alpha)/q_1}.
\end{aligned} \tag{3.29}$$

If $(m-l)q_1 = nq$, we use Hölder inequality and apply the argument with q_1 replaced by $\bar{q}_1$ where $q < \bar{q}_1 < q_1$. If $(m-l)q_1 > nq$, we apply (3.13) with $s = q_1$ to conclude that

$$|\partial^l u_0|(x) \le C r^{m-l} \lambda'_0 \le \widehat{C_2}(m,l,s,\Omega)\lambda^\alpha M_B^m(|\partial^m u_0|^{q_1})(x)^{(1-\alpha)/q_1}. \tag{3.30}$$

This proves the assertion (3.28).

If $p - \delta > \frac{nq}{m-l}$, we assert that

$$|\partial^l u_0|(x) \le C'(m,l,s,\Omega)\tau'_1(l) M_B^m(|\partial^m u_0|^{q_1})(x)^{\alpha/q_1}. \tag{3.31}$$

From (3.4), we get

$$\begin{aligned}
|\partial^l u_0|(x) &\le C r^{m-l} \lambda'_0 \\
&\le \widehat{C_2}(m,l,s,\Omega) \left(\fint_B |\partial^m u_0|(x)^{q_1} \omega(x)dx \right)^{(1-\alpha)/q_1} M_B^m(|\partial^m u_0|^{q_1})(x)^{\alpha/q_1} \\
&\le \widehat{C_2}(m,l,s,\Omega)\gamma'_1(l) M_B^m(|\partial^m u_0|^{q_1})(x)^{\alpha/q_1} \\
&\le \widehat{C_2}(m,l,s,\Omega)\tau'_1(l) M_B^m(|\partial^m u_0|^{q_1})(x)^{\alpha/q_1},
\end{aligned}$$

which implies that (3.31) is true for the case $(m-l)q_1 > nq$. In case $(m-l)q_1 \le nq$, we can choose $q < s < nq/(m-l) < p-\delta$ so that $(m-l)s/nq = 1-\alpha$. We apply Lemma 3.8 (3.11) to the l derivative of u_0, from Hardy-Littlewood maximal theorem, we have

$$\begin{aligned}|\partial^l u_0|(x) &\le C_2 r^{m-l}\left(\fint_B M_B^m(|\partial^m u_0|^q)(x)^{\frac{s}{q}}\omega(x)dx\right)^{(m-l)/nq} M_B^m(|\partial^m u_0|^q)(x)^{(1/q)(1-s(m-l)/nq)}\\ &\le C_2 r^{m-l}\left(\fint_B |\partial^m u_0|(x)^s\omega(x)dx\right)^{(m-l)/nq} M_B^m(|\partial^m u_0|^q)(x)^{\alpha/q}\\ &\le C_2(m,l,s,\Omega)\gamma_1' M_B^m(|\partial^m u_0|^q)(x)^{\alpha/q} \le C_2(m,l,s,\Omega)\tau_1' M_B^m(|\partial^m u_0|^q)(x)^{\alpha/q}.\end{aligned}$$

This proves the assertion (3.31). Therefore,

$$|\partial^l u_0| = |\partial^l u_0|^\alpha |\partial^l u_0|^{1-\alpha} \le C(l)\lambda^\alpha \tau_1' M_B^m(|\partial^m u_0|^q)^{\frac{\alpha-\alpha^2}{q}} \tag{3.32}$$

from inequalities (3.28) and (3.31) evaluated in the power α and $1-\alpha$, respectively. We proceed to estimate K_2:

$$\begin{aligned}|K_2| &\le \sum_{l=0}^{m-1} C(l)\tau_1' \int_{B(x_0,8r)} |\partial^m u|^{p-1}\left(\int_0^{M_B^m(|\partial^m u_0|^{q_1})^{\frac{1}{q_1}}} \lambda^{\beta_0\alpha-1-\delta}d\lambda\right) M_B^m(|\partial^m u_0|^{q_1})^{\frac{\alpha-\alpha^2}{q_1}}\omega(x)dx\\ &\le \sum_{l=0}^{m-1} C(l)\tau_1' \int_{B(x_0,8r)} |\partial^m u|^{p-1} M_B^m(|\partial^m u_0|^{q_1})^{\frac{\alpha-\alpha^2-\delta}{q_1}}\omega(x)dx\\ &\le \sum_{l=0}^{m-1} C(l)\tau_1' \int_{B(x_0,8r)} |\partial^m u|^{p-1} M_B^{2m}(|\partial^m u|^{q_1})^{\frac{\alpha-\alpha^2-\delta}{q_1}}\omega(x)dx.\end{aligned} \tag{3.33}$$

In the case $p-\delta \le \frac{nq}{m-l}$, using (3.11) once again, we obtain

$$\begin{aligned}|\partial^l u_0| &\le C r^{m-l}\gamma_2 |B(z_0,3R/4)|^{-\frac{m-l}{n}} M_B^m(|\partial^m u_0|^{q_1})^{\frac{1}{q_1}(1-\frac{p(m-l)(1-\alpha)}{q_1 n})}\\ &\le C\gamma_2 M_B^m(|\partial^m u_0|^{q_1})^{\frac{1}{q_1}(1-\frac{p(m-l)(1-\alpha)}{qn})} = C\gamma_2 M_B^m(|\partial^m u_0|^{q_1})^{\frac{\tilde{p}}{q_1}}.\end{aligned} \tag{3.34}$$

We write $|\partial^l u_0| = |\partial^l u_0|^\alpha \times |\partial^l u_0|^{1-\alpha}$ and get

$$|\partial^l u_0| \le C(n,R)\gamma_2^{1-\alpha}\lambda^{\beta_0\alpha} M_B^m(|\partial^m u_0|^{q_1})^{\frac{(1-\beta_0/q_1)\alpha+(1-\alpha)\tilde{p}}{q_1}} \tag{3.35}$$

from (3.29) and (3.34). We finish to estimate K_2 in this case:

$$\begin{aligned}|K_2| &\le C\sum_{l=0}^{m-1}\gamma_2(l)^{1-\alpha}\int_{B(x_0,8r)}|\partial^m u|^{p-1}\left(\int_0^{M_B^m(|\partial^m u_0|^{q_1})^{\frac{1}{q_1}}}\lambda^{\beta_0\alpha-1-\delta}d\lambda\right)M_B^m(|\partial^m u_0|^{q_1})^{\frac{(1-\beta_0/q_1)\alpha+(1-\alpha)\tilde{p}}{q_1}}\omega(x)dx\\ &\le C\sum_{l=0}^{m-1}\gamma_2(l)^{1-\alpha}\int_{B(x_0,8r)}|\partial^m u|^{p-1}M_B^m(|\partial^m u_0|^{q_1})^{\frac{c_2(\beta_0)\alpha-\delta+(1-\alpha)\tilde{p}}{q_1}}\omega(x)dx\\ &\le C\sum_{l=0}^{m-1}\gamma_2(l)^{1-\alpha}\int_{B(x_0,8r)}|\partial^m u|^{p-1}M_B^{2m}(|\partial^m u|^{q_1})^{\frac{c_2(\beta_0)\alpha-\delta+(1-\alpha)\tilde{p}}{q_1}}\omega(x)dx.\end{aligned} \tag{3.36}$$

Combining (3.36) with (3.33) we deduce

$$K_2 \le \int_{B(x_0,8r)} F_2^{p-\delta}\omega(x)dx, \quad \text{with} \quad F_2 \in L^{p+\alpha'}(B(x_0,8r),\mu). \tag{3.37}$$

by the Muckenhaupt theorem, provided

$$(p+\alpha')(p-\min\{c_2(\beta_0)\alpha+(1-\alpha)\tilde{p}, c_1(\beta_0)\alpha-\alpha^2\}-\delta)/(p-\delta) < p-\delta$$

and thus

$$\delta < \min\{c_2(\beta_0)\alpha+(1-\alpha)\tilde{p}, c_1(\beta_0)\alpha-\alpha^2\}-\alpha'+\alpha' \min\{c_2(\beta_0)\alpha+(1-\alpha)\tilde{p}, c_1(\beta_0)\alpha-\alpha^2\}/p. \tag{3.38}$$

The upper bound of K

Since we have

$$\begin{aligned} K_3 &\le C\int_{B(x_0,8r)} |\partial^m u|^{p-1} M_B^m(|\partial^m u_0|^{q_1})^{\frac{1-\delta}{q_1}}\omega(x)dx \\ &\le C\int_{B(x_0,8r)} |\partial^m u|^{p-1} M_B^{2m}(|\partial^m u|^{q_1})^{\frac{1-\delta}{q_1}}\omega(x)dx \\ &\le C\int_{B(x_0,8r)} M_B^{2m}(|\partial^m u|^{q_1})^{\frac{p-\delta}{q_1}}\omega(x)dx \le C\int_{B(x_0,8r)} |\partial^m u|^{p-\delta}\omega(x)dx, \end{aligned}$$

where the last step follows from the Muckenhaupt theorem under the condition $q_1 < p-\delta$. We conclude that

$$\begin{aligned} K \le \delta \int_{B(x_0,8r)} F_3^{p-\delta}\omega(x)dx + \delta\int_{B(x_0,8r)} |\partial^m u|^{p-\delta}\omega(x)dx, \\ \text{with} \quad F_3 \in L^{p+\alpha'}(B(x_0,8r),\mu). \end{aligned} \tag{3.39}$$

The lower bounds of K

Since supp $u_0 \subset B(x_0,2r)$, we can write

$$\begin{aligned} K &= \sum_{|\sigma|=0}^{m} \int_{\mathbb{R}^n - F(\lambda_0)} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} A_\sigma(x, D^m u(x))\partial_x^\sigma u_0(x)dx \\ &+ \lambda_0^{-\delta}\sum_{|\sigma|=0}^{m}\int_{F(\lambda_0)} A_\sigma(x, D^m u(x))\partial_x^\sigma u_0(x)dx \\ &\ge \sum_{|\sigma|=m}\int_{B(x_0,2r)} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} A_\sigma(x, D^m u(x))\partial_x^\sigma u_0(x)dx \\ &- c\sum_{|\sigma|=0}^{m-1}\int_{B(x_0,2r)} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} |\partial^m u|^{p-1}|\partial_x^\sigma u_0(x)|\omega(x)dx \\ &- c\int_{E(\lambda_0)} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1}|\partial^m u|^{p-1}|\partial^m u_0(x)|\omega(x)dx \\ &= L_1 - L_2 - L_3. \end{aligned} \tag{3.40}$$

We start from estimates of L_2 and L_3.

The estimate of L_2

Following the same method we used above, we obtain that

$$L_2 \leq \int_{B(x_0,8r)} F_1^{p-\delta}\omega(x)dx, \quad \text{with} \quad F_1 \in L^{p+\alpha'}(B(x_0,8r),\mu), \tag{3.41}$$

from (3.4) and (3.34).

The estimate of L_3

Suppose $0 < \eta \leq \frac{1}{2}$ and split $E(\lambda_0) = E_1(\lambda_0) \cup (E(\lambda_0) - E_1(\lambda_0))$, where

$$E_1(\lambda_0) = \{x \in E(\lambda_0): \ |\partial^m u| \geq \eta^{-1}\lambda_0\}.$$

We have the estimate

$$L_3 \leq \int_{E(\lambda_0)} M_B^m(|\partial^m u_0|^{q_1})(x)^{(1-\delta)/q_1}|\partial^m u|^{p-1}\omega(x)dx.$$

We also know that

$$M_B^m(|\partial^m u_0|^{q_1})(x)^{1/q_1} \leq \lambda_0 \leq \eta\, |\partial^m u|$$

for any $x \in E_1(\lambda_0)$. We continue and for $|\partial^m u| \geq \eta^{-1}\lambda_0$ we get

$$L_3 \leq \eta^{1-\delta}\int_{E_1(\lambda_0)} |\partial^m u|^{p-\delta}\omega(x)dx. \tag{3.42}$$

On the other hand, $|\partial^m u| < \eta^{-1}\lambda_0$ for any $x \in E(\lambda_0) - E_1(\lambda_0)$. Applying (3.21) and (3.24), we get the pointwise estimate

$$\begin{aligned} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1}|\partial^m u_0(x)||\partial^m u(x)|^{p-1} &\leq M_B^m(|\partial^m u_0|^{q_1})(x)^{(1-\delta)/q_1}\eta^{1-p}\lambda_0^{p-1} \\ &= \eta^{\delta-p}\lambda_0^{p-\delta} \leq \eta^{\delta-p}\left[\min_{B(x_0,8r)} M_B^m(|\partial_x^m u_0|^{q_1})(x)^{1/q_1}\right]^{p-\delta}. \end{aligned}$$

Since for a fixed t_0 such that $t_0 > q$, the following inequalities

$$\begin{aligned} \min_{B(x_0,8r)} M_B^m(|\partial_x^m u_0|^{q_1})(x)^{1/q_1} &\leq \left(\fint_{B(x_0,8r)} M_B^m(|\partial_x^m u_0|^{q_1})(x)^{t_0/q_1}\omega(x)dx\right)^{1/t_0} \\ &\leq \left(\fint_{B(x_0,8r)} M_B^{2m}(|\partial_x^m u|^{q_1}\chi_{B(x_0,8r)})(x)^{t_0/q_1}\omega(x)dx\right)^{1/t_0} \\ &\leq \left(\fint_{B(x_0,8r)} |\partial_x^m u|^{t_0}\omega(x)dx\right)^{1/t_0}, \end{aligned}$$

hold by the Muckenhaupt theorem, we obtain the estimate

$$L_3 \leq \eta^{1-\delta}\int_{B(x_0,8r)} |\partial^m u|^{p-\delta}\omega(x)dx + \eta^{\delta-p}\omega(x_0,8r)\left(\fint_{B(x_0,8r)} |\partial_x^m u|^{t_0}\omega(x)dx\right)^{1/t_0} \tag{3.43}$$

from (3.42).

Decomposition of L_1

In order to estimate L_1, we need to decompose L_1 in a more suitable way. Denote by D_1 the set

$$D_1 = \{x \in B(x_0, 2r) - B(x_0, r) : \ M_B^m(|\partial^m u_0|^{q_1})(x)^{1/q_1} \leq \delta\, M_B^m(|\partial^m u|^{q})(x)^{1/q_1}\}$$

and set $D_2 = B(x_0, 2r) - (D_1 \cup B(x_0, r))$. We get

$$\begin{aligned} L_1 &= \sum_{|\sigma|=m} \left(\int_{B(x_0,2r)-D_1} + \int_{D_1} \right) M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} A_\sigma(x, D^m u(x)) \partial^\sigma u_0(x) dx \\ &\geq \sum_{|\sigma|=m} \int_{B(x_0,2r)-D_1} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} A_\sigma(x, D^m u(x)) \partial^\sigma u_0(x) dx \\ &\quad - \sum_{|\sigma|=m} \int_{D_1} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} |\partial^m u_0(x)| |\partial^m u(x)|^{p-1} \omega(x) dx. \end{aligned}$$

Denote the second term by H_3 and decompose the first term into two parts,

$$\begin{aligned} L_1 &\geq \sum_{|\sigma|=m} \int_{B(x_0,2r)-D_1} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} A_\sigma(x, D^m u(x)) \cdot \varphi \partial^\sigma u(x) dx \\ &\quad - \sum_{|\sigma|=m} \int_{B(x_0,2r)-D_1} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} A_\sigma(x, D^m u(x)) \cdot (\varphi \partial^\sigma u(x) - \partial^\sigma u_0(x)) dx - H_3 \\ &= H_1 - H_2 - H_3. \end{aligned}$$

Consider H_1 first. Since $B(x_0, 2r) = D_1 \cup D_2 \cup B(x_0, r)$, we have the following estimate

$$\begin{aligned} H_1 &= \sum_{|\sigma|=m} \int_{B(x_0,r)\cup D_2} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} A_\sigma(x, D^m u(x)) \cdot \varphi \partial^\sigma u(x) dx \\ &\geq \sum_{|\sigma|=m} \int_{B(x_0,r)} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} A_\sigma(x, D^m u(x)) \cdot \partial^\sigma u(x) dx \\ &\geq \gamma \int_{B(x_0,r)} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} |\partial^m u(x)|^p \omega(x) dx := J_1 \end{aligned}$$

To estimate H_2 we use the estimate (3.24) and the equality $u_0(x) = u(x)$ on $B(x_0, r)$. We rewrite H_2 as follows

$$\begin{aligned} H_2 &= \sum_{|\sigma|=m} \int_{B(x_0,r)\cup D_2} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} A_\sigma(x, D^m u(x)) \cdot (\varphi \partial^\sigma u(x) - \partial^\sigma u_0(x)) dx \\ &\leq \sum_{0\leq|\alpha|<m} r^{-m+|\alpha|} \int_{D_2} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} |\partial^m u(x)|^{p-1} |\partial^\alpha (u(x) - P(x))| dx. \end{aligned}$$

Denote by J_2 the last term of the above inequality. We continue and estimate H_3:

$$\begin{aligned} H_3 &= \int_{D_1} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} |\partial^m u_0(x)| |\partial^m u(x)|^{p-1} \omega(x) dx \\ &\leq \int_{D_1} M_B^m(|\partial^m u_0|^{q_1})(x)^{(1-\delta)/q_1} |\partial^m u(x)|^{p-1} \omega(x) dx := J_3. \end{aligned}$$

Then we arrive at the following relation

$$L_1 \geq J_1 - J_2 - J_3. \tag{3.44}$$

The estimate of J_1

Let $B_r = B(x_0, r)$. We can write

$$\begin{aligned} M_B(|\partial^m u_0|^{q_1})(x)^{1/q_1} &= M(|\partial^m u_0 \chi_B|^{q_1})(x)^{1/q_1} \\ &\leq M(|\partial^m u_0 \chi_{B_r}|^{q_1})(x)^{1/q_1} + M(|\partial^m u_0 \chi_{B-B_r}|^{q_1})(x)^{1/q_1} \\ &\leq M_{B_r}(|\partial^m u_0|^{q_1})(x)^{1/q_1} + \left(\fint_{B(x_0,8r)} |\partial^m u_0|^{q_1} \omega(x) dx \right)^{1/q_1} \end{aligned}$$

for any $x \in B(x_0, \frac{r}{2})$. Moreover, by the same arguments as (2.24), we have

$$\begin{aligned} M_B^2(|\partial^m u_0|^{q_1})(x)^{1/q_1} &\leq M\left(M_{B_r}(|\partial^m u_0|^{q_1})(x)\chi_B\right)^{1/q_1} + \left(\fint_{B(x_0,8r)} |\partial^m u_0|^{q_1} \omega(x) dx \right)^{1/q_1} \\ &\leq M_{B_r}^2(|\partial^m u_0|^{q_1})(x)^{1/q_1} + \left(\fint_{B(x_0,8r)} |M_{B_r}(\partial^m u_0)|^{q_1} \omega(x) dx \right)^{1/q_1} + \left(\fint_{B(x_0,8r)} |\partial^m u_0|^{q_1} \omega(x) dx \right)^{1/q_1} \\ &\leq M_{B_r}^2(|\partial^m u_0|^{q_1})(x)^{1/q_1} + c\left(\fint_{B(x_0,8r)} |M_B(\partial^m u_0)|^{q_1} \omega(x) dx \right)^{1/q_1}. \end{aligned}$$

Since $\partial^m u_0(x) = \partial^m u(x)$ on $B(x_0, r)$ we get the estimate

$$M_B^m(|\partial^m u_0|^{q_1})(x)^{1/q_1} \leq M_{B_r}^m(|\partial^m u|^{q_1})(x)^{1/q_1} + c\left(\fint_{B(x_0,8r)} M_B^{2m}(|\partial^m u|)^{q_1} \omega(x) dx \right)^{1/q_1} \tag{3.45}$$

by induction. Next, we construct the set G:

$$G = \left\{ x \in B(x_0, \frac{r}{2}) : \ M_{B_r}^m(|\partial^m u|^{q_1})(x)^{1/q_1} \geq C_1 \left(\fint_{B(x_0,8r)} M_B^{2m}(|\partial^m u|^{q_1}) \omega(x) dx \right)^{1/q_1} \right\}.$$

We see at once that if $x \in G$, then

$$M_B^m(|\partial^m u|^{q_1})(x)^{-\delta/q_1} \leq cC_1 M_{B_r}^m(|\partial^m u|^{q_1})(x)^{1/q_1}. \tag{3.46}$$

It is known from Lemma 3.13 that the quantity $M_{B_r}^m(|\partial^m u_0|^{q_1})(x)^{1/q_1}$ is A_{p/q_1}-weight in the measure space $(\mathbb{R}^n,\mu)$ if $\delta < p-q_1$. Since $t_0 > q_1$, we can find the lower bound for J_1:

$$
\begin{aligned}
J_1 &= \gamma \int_{B(x_0,r)} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} |\partial^m u(x)|^p \omega(x)dx \\
&\geq C\gamma \int_{B(x_0,r)} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} M_{B_r}^m(|\partial^m u|^{q_1}\chi_{B_r})(x)^{p/q_1}\omega(x)dx \\
&\geq C \int_G M_{B_r}^m(|\partial^m u|^{q_1})(x)^{(p-\delta)/q_1}\omega(x)dx \\
&= C \int_{B_{r/2}} M_{B_r}^m(|\partial^m u|^{q_1})(x)^{(p-\delta)/q_1}\omega(x)dx - C_1 \int_{B_{r/2}-G} M_{B_r}^m(|\partial^m u|^{q})(x)^{(p-\delta)/q_1}\omega(x)dx \\
&\geq C \int_{B_{r/2}} |\partial^m u|^{p-\delta}\omega(x)dx - C_1\omega(B(x_0,8r))\left(\fint_{B(x_0,8r)} M_{B_r}^m(|\partial^m u|^{q_1})(x)\omega(x)dx\right)^{(p-\delta)/q_1} \\
&\geq C \int_{B_{r/2}} |\partial^m u|^{p-\delta}\omega(x)dx - C_1\omega(B(x_0,8r))\left(\fint_{B(x_0,8r)} |\partial^m u|^{t_0}\omega(x)dx\right)^{(p-\delta)/t_0}.
\end{aligned}
\tag{3.47}
$$

The estimate of J_2

To estimate J_2, we set $\hat{t} = \max\left\{q_1, p - q_1\left(\frac{m-l}{nq}\right) - \delta, p-\delta-1\right\}$ and we consider three cases.

If $q_1(m-l) < nq$, we apply Lemma 3.8 (3.11) to the l derivative of $u-P$ with $s = q_1$,

$$
|\partial^l(u-P)(x)| \leq C_2 r^{m-l}\left(\fint_B M_B^m(|\partial^m u|^q)(x)^{q_1/q}\omega(x)dx\right)^{\frac{m-l}{nq}} M_B^m(|\partial^m u|^q)(x)^{(1/q)(1-(m-l)q_1/nq)},
$$

From Hölder inequality and $x \in D_2$,

$$
\begin{aligned}
M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} &\leq C(\delta) M_B^m(|\partial^m u|^{q_1})(x)^{-\delta/q_1} \\
&\leq C(\delta) M_B^m(|\partial^m u|^{q})(x)^{-\delta/q}.
\end{aligned}
$$

From Hardy-Littlewood maximal theorem, we get

$$
\begin{aligned}
& r^{l-m} M_B^m(|\partial^m u|^{q_1})(x)^{-\delta/q_1}|\partial^m u|^{p-1}|\partial^l(u-P)|(x) \\
&\leq C(\delta)\left(\fint_B M_B^m(|\partial^m u|^q)(x)^{q_1/q}\omega(x)dx\right)^{\frac{m-l}{nq}} M_B^m(|\partial^m u|^q)(x)^{(1/q)[p-\delta-(m-l)q_1/nq]} \\
&\leq C(\delta)\left(\fint_B M_B^m(|\partial^m u|^q)(x)^{\hat{t}/q}\omega(x)dx\right)^{q_1\frac{m-l}{nq\hat{t}}} M_B^m(|\partial^m u|^q)(x)^{(1/q)[p-\delta-(m-l)q_1/nq]}.
\end{aligned}
$$

Therefore, by Hardy-Littlewood maximal theorem, we have the estimate for J_2

$$
\begin{aligned}
J_2 &\leq C(\delta)\omega(B)\left(\fint_B M_B^m(|\partial^m u|^q)(x)^{\hat{t}/q}\omega(x)dx\right)^{(p-\delta)/\hat{t}} \\
&\leq C(\delta)\omega(B)\left(\fint_B |\partial^m u|(x)^{\hat{t}}\omega(x)dx\right)^{(p-\delta)/\hat{t}}.
\end{aligned}
\tag{3.48}
$$

In case $q_1(m-l) = nq$, we may take $1 < \bar{t} < t$ and define $\hat{\hat{t}}$ similarly. We can get (2.27) once again. If $q_1(m-l) > nq$, we apply Lemma 3.8 (3.13) to l derivative of $u-P$. We get in this case

$$|\partial^l(u-P)(x)| \le C_2 r^{m-l}\left(\fint_B M_B^m(|\partial^m u|^q)(x)^{q_1/q}\omega(x)dx\right)^{\frac{1}{q_1}} \le C_2 r^{m-l}\left(\fint_B M_B^m(|\partial^m u|^q)(x)^{\hat{t}/q}\omega(x)dx\right)^{\frac{1}{\hat{t}}}.$$

Therefore we have

$$\begin{aligned} J_2 &\le C(\delta)\omega(B)\left(\fint_B M_B^m(|\partial^m u|^q)(x)^{\hat{t}/q}\omega(x)dx\right)^{\frac{1}{\hat{t}}}\fint_B M_B^m(|\partial^m u|^q)(x)^{(p-\delta-1)/q}\omega(x)dx \\ &\le C(\delta)\omega(B)\left(\fint_B |\partial^m u|(x)^{\hat{t}}\omega(x)dx\right)^{(p-\delta)/\hat{t}}. \end{aligned} \tag{3.49}$$

Summing over l, we conclude that

$$J_2 \le C(\delta)\omega(B)\left(\fint_B |\partial^m u|(x)^{\hat{t}}\omega(x)dx\right)^{(p-\delta)/\hat{t}} \tag{3.50}$$

by (3.48) and (3.49).

The estimate of J_3

By the definition of the set D_1 and by the Hardy-Littlewood maximal theorem we write

$$\begin{aligned} J_3 &= \int_{D_1} M_B^m(|\partial^m u_0|^q)(x)^{(1-\delta)/q}|\partial^m u(x)|^{p-1}\omega(x)dx \\ &\le \delta^{1-\delta}\int_{D_1} M_B^m(|\partial^m u|^q)(x)^{(1-\delta)/q}|\partial^m u(x)|^{p-1}\omega(x)dx \\ &\le \delta^{1-\delta}\int_{B(x_0,8r)} |\partial^m u(x)|^{p-\delta}\omega(x)dx. \end{aligned} \tag{3.51}$$

Final Estimations

The estimations (3.47)-(3.51) imply

$$\begin{aligned} \fint_{B(x_0,r/2)} |\partial^m u|^{p-\delta}\omega(x)dx &\le C\omega(B(x_0,8r))^{-1}L_2 + \fint_{B(x_0,8r)} F_4^{p-\delta}\omega(x)dx \\ &+ c\delta^{1-\delta}\fint_{B(x_0,8r)} |\partial^m u|^{p-\delta}\omega(x)dx + c\left(\fint_{B(x_0,8r)} |\partial^m u|^{t_0}\omega(x)dx\right)^{(p-\delta)/t_0} \\ &+ C(\delta)\left(\fint_B |\partial^m u|(x)^{\hat{t}}dx\right)^{(p-\delta)/\hat{t}}. \end{aligned} \tag{3.52}$$

On the other hand, since $K = L_1 - L_2 - L_3$, it follows from (3.51), (3.42) and (3.43) that

$$\begin{aligned} \fint_{B(x_0,r/2)} |\partial^m u|^{p-\delta}\omega(x)dx &\le c\omega(B(x_0,8r))^{-1}K + \fint_{B(x_0,8r)} (F_1^{p-\delta} + F_4^{p-\delta})\omega(x)dx \\ &+ c(\delta^{1-\delta} + \eta^{1-\delta})\fint_{B(x_0,8r)} |\partial^m u|^{p-\delta}\omega(x)dx \\ &+ c\eta^{\delta-p}\left(\fint_{B(x_0,8r)} |\partial^m u|^{t_0}\omega(x)dx\right)^{(p-\delta)/t_0}. \end{aligned} \tag{3.53}$$

Combining (3.53) with (3.39), we get

$$\begin{aligned}\fint_{B(x_0,r/2)} |\partial^m u|^{p-\delta}\omega(x)dx \le \fint_{B(x_0,8r)} & F^{p-\delta}\omega(x)dx \\ &+ c(\delta^{1-\delta}+\eta^{1-\delta})\fint_{B(x_0,8r)} |\partial^m u|^{p-\delta}\omega(x)dx \\ &+ c\eta^{\delta-p}\left(\fint_{B(x_0,8r)} |\partial^m u|^{t_0}\omega(x)dx\right)^{(p-\delta)/t_0},\end{aligned} \tag{3.54}$$

where $F^{p-\delta} = \sum F_i^{p-\delta}$. We take the quantity $c(\delta^{1-\delta}+\eta^{1-\delta})$ in (3.54) to be sufficiently small, for instance, less than $1/2$. This leads to

$$\begin{aligned}&\fint_{B(x_0,r/2)} |\partial^m u|^{p-\delta}\omega(x)dx \le \fint_{B(x_0,8r)} F^{p-\delta}\omega(x)dx \\ &+ \frac{1}{2}\fint_{B(x_0,8r)} |\partial^m u|^{p-\delta}\omega(x)dx + C_0\left(\fint_{B(x_0,8r)} |\partial^m u|^{t_0}\omega(x)dx\right)^{(p-\delta)/t_0}\end{aligned} \tag{3.55}$$

for some large $C_0 > 0$.

In view of (3.55), we can use Lemma 3.7 by letting $g = |\partial^m u|^{t_0}$, $f = F^{t_0}$ and $\theta = \frac{1}{2}$. We apply Theorem 3.3 for the value of δ satisfying (3.27), (3.38) and $\delta < p - q$.

Remark 3.14. In the Euclidean case, [44] is the classical paper regarding the very weak solutions of elliptic systems. But there is a mistake in that paper. From Sobolev inequality, we should restrict $s \ge 1$, but in the page 1528 of that paper, s may be less than 1 when we set $s = \frac{\alpha n}{m-l}$. The Lemma 2.2 (2.10), Lemma 3.8 (3.13), estimates of J_2 in chapter 2, estimates of K_2 and J_2 in chapter 3 were suggested by Lewis [45], these modifications are very efficient to overcome these difficulties in the estimates of the lower order terms. In fact, the $E(\lambda)$ in [44] should be defined by

$$E(\lambda) = \left\{x \in \mathbb{R}^n : M^m(|\partial^m u|^t)(x)^{1/t} \le \lambda\right\}$$

where t is fixed and we should take $1 < t < p$.

If we modified the techniques of chapter 3, it is not hard to establish the same result for the elliptic systems in the following more general form:

$$\sum_{|\sigma|=m} A_\sigma(x, D^m u(x)) \cdot \partial_x^\sigma u(x) \ge \gamma\omega(x)|\partial^m u|^p - a(x) \quad \text{a.e. in} \quad \Omega \tag{3.56}$$

and

$$|A_\sigma(x, D^m u(x))| \le \omega(x)|\partial_x^\sigma u(x)|^{p-1} + b_\sigma(x) \quad \text{a.e. in} \quad \Omega, \tag{3.57}$$

where $a(x) \in L^r(\mathbb{R}^n, \omega)$ and $b_\sigma(x) \in L^{p_\sigma}(\mathbb{R}^n, \omega)$. But, as indicated by Lewis [45], we shall write the lower bounds of K differently:

$$\begin{aligned}
K &= \int_{B(y_0,2r)} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} < A(x, D^m u(x)) - \hat{A}(x, D^m u(x)), D^m u_0(x) > dx \\
&\geq c \int_{B(y_0,r)} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} |\partial u(x)|^p \omega(x) dx - c \int_{D_1} M_B^m(|\partial^m u_0|^{q_1})(x)^{(1-\delta)/q_1} |\partial u(x)|^{p-1} \omega(x) dx \\
&- c \sum_{l=0}^{m-1} r^{l-m} \int_{D_2} M_B^m(|\partial^m u_0|^{q_1})(x)^{-\delta/q_1} |\partial u(x)|^{p-1} |\partial^l (u-P)|(x) \omega(x) dx \\
&- \int_{B(y_0,2r)} a(x)\omega(x) dx - c \int_{B(y_0,2r)} M_B^m(|\partial^m u_0|^{q_1})(x)^{(1-\delta)/q_1} b_\sigma(x) \omega(x) dx \\
&= L_1 - L_2 - L_3 - L_4 - L_5.
\end{aligned}$$

Remark 3.15. In the situation when $d\mu = \omega(x)dx$ is defined by a quasiconformal map, namely $\omega(x) = (\det Df)^{1-p/n}$, where f is a quasiconformal map, one needs to find a different approach to this problem.

Remark 3.16. We hope that this kind of techniques could shed some light on the study of self-improving regularity problem of the degenerate elliptic equation with double weight, namely let $A_\sigma(x, D^m u(x))$ be a measurable function satisfies

$$\sum_{|\sigma|=m} A_\sigma(x, D^m u(x)) \cdot \partial_x^\sigma u(x) \geq \gamma \omega_1(x) |\partial_x^m u(x)|^p \qquad a.e. \text{ in } \Omega \tag{3.58}$$

and

$$|A_\sigma(x, D^m u(x))| \leq \omega_2(x) |\partial_x^\sigma u(x)|^{p-1} \qquad a.e. \text{ in } \Omega, \tag{3.59}$$

where $|\sigma| \leq m$ and $\omega_1(x)$, $\omega_2(x)$ are two weights. For example, we can assume that (ω_1, ω_2) satisfies the following two-weight Muckenhaupt condition:

$$\frac{1}{|Q|} \int_Q \omega_1(x) dx \left(\frac{1}{|Q|} \int_Q \omega_2(x)^{1-p'} \right)^{p-1} dx \leq c.$$

This kind of degenerate elliptic equations was initiated by Chanillo and Wheeden [7].

Remark 3.17. It is an interesting problem to understand whether the main results in [5, 40, 41] can be extended to the degenerate parabolic systems

$$\partial_t u - \omega_3(x) \mathrm{div} A(x, \nabla u) = 0,$$

where $\omega_3(x)$ is an admissible weight and $A(x, \nabla u)$ satisfies (3.58) and (3.59) for $m = 1$.

Bibliography

[1] L. Ahlfors, *Lectures on quasiconformal mappings,* van Nostrand. (1966).

[2] K. Astala, *Area distortion of quasiconformal mappings,* Acta Math., **173** (1994), 37-60.

[3] K. Astala, T. Iwaniec and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane,* Princeton University Press, (2009).

[4] A. Bensoussan and J. Frehse, *Regularity results for nonlinear elliptic systems and applications,* Springer-Verlag, New York, (2002).

[5] V. Bögelein, *Regularity results for weak and very weak solutions of higher order parabolic systems.* Ph.D. Thesis in Universität Erlangen-Nürnberg, (2007).

[6] L. Carleson, *On convergence and growth of partial sums of Fourier series,* Acta Math., **116** (1966) 135-157.

[7] S. Chanillo and R. L. Wheeden, *Harnack's inequality and mean-value inequalities for solutions of degenerate elliptic equations,* Comm. Part. Diff. Equ., **11** (1986), 1111-1134.

[8] Y. Chen and L. Wu *Second order elliptic equations and elliptic Systems,* Translations of Mathematical Monographs, Amer. Math. Soc. (1998).

[9] V. M. Chernikov and S. K. Vodop'yanov, *Sobolev spaces and hypoelliptic equations. I,* Siberian Advances in Mathematics, Siberian Adv. Math., **6** (1996), 27-67.

[10] V. M. Chernikov and S. K. Vodop'yanov, *Sobolev spaces and hypoelliptic equations. II,* Siberian Advances in Mathematics, Siberian Adv. Math., **6** (1996), 64-96.

[11] W. L. Chow, Ü*ber systeme von linearen partiellen differentialgleichungen erster ordung,* Math. Ann., **117** (1939), 98-105.

[12] R. R. Coifman, P. L. Lions, Y. Meyer and S. Semmes, *Compensated compactness and Hardy spaces,* J. Math. Pures Appl., **72** (1993), 247-286.

[13] C. Fefferman, *Pointwise convergence of Fourier series,* Ann. of Math., **98** (1973), 551-571.

[14] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups,* Mathematical Notes, Princeton Univ. Press, Princeton, (1982).

[15] D. Faraco and X. Zhong, *A short proof of the self-improving regularity of quasiregular mappings,* Proc. Amer. Math. Soc., **134** (2006), 193-206.

[16] A. Zatorska-Goldstein, *Very weak solutions of nonlinear subelliptic equations,* Annales Acad. Sci. Fenn. Math., **30** (2005), 407-436.

[17] F. W. Gehring, *The L^p integrability of the partial derivatives of a quasiconformal mapping,* Acta Math., **130**, (1973), 265-277.

[18] F. W. Gehring and E. Reich, *Area distortion under quasiconformal mappings,* Ann. Acad. Sci. Fenn. Ser. Math. **388** (1966), 1-15.

[19] M. Giaquinta, *Multiple integrals in the calculus of variation and non-linear elliptic systems,* Annals of Math. Studies, **105**, Princeton University Press, Princeton (1983).

[20] L. Grafakos, *Hardy space estimates for multilinear operators. II.* Rev. Mat. Iberoam., **8** (1992), 69-92.

[21] L. Grafakos, *H^1 boundedness of determinants of vector fields,* Proc. Amer. Math. Soc., **125** (1997), 3279-3288.

[22] L. Grafakos and R. Rochberg, *Compensated compactness and the Heisenberg group,* Math. Ann., **301** (1995), 601-611.

[23] L. Grafakos, T. Tao and E. Terwilleger, *L^p bounds for a maximal dyadic sum operator,* Math. Zeit., **246** (2004), 321-337.

[24] L. Grafakos, *Classical Fourier Analysis,* GTM 249, Springer-Verlag, New York, (2008).

[25] L. Grafakos, *Modern Fourier Analysis,* GTM 250, Springer-Verlag, New York, (2008).

[26] L. G. Gurov and Yu. G. Reshetnyak, *A certain analogue of the concept of a function with bounded mean oscillation,* (Russian) Sibirsk. Mat. Z. **17** (1976), 540-546.

[27] P. Hajlasz and P. Koskela, *Sobolev met Poincaré,* Memoirs Amer. Math. Soc., **688** (2000), 1-101.

[28] J. Heinonen and I. Holopainen, *Quasiregular maps on Carnot groups,* J. Geom. Anal., **71** (1997), 109-148.

[29] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear potential theory of degenerate elliptic equations,* Oxford Mathematical Monographs, Clarendon Press, Oxford, (1993).

[30] L.-K. Hua, *Starting with the unit circle - Background to higher analysis,* Springer-Verlag, New York, (1981)

[31] T. Iwaniec, *p-harmonic tensors and quasiregular mappings,* Ann. of Math., **136** (1992), 589-624.

[32] T. Iwaniec, *"L^p theory of quasiregular mappings" in Quasiconformal Space Mappings,* Lecture Notes in Math. **1508**, Springer, Berlin, 1992.

[33] T. Iwaniec and G. Martin, *Quasiregular mappings in even dimensions,* Acta Math., **170** (1993), 29-81.

[34] T. Iwaniec and G. Martin, *Geometric function theory and non-linear analysis,* Oxford Mathematical Monographs. Clarendon Press, Oxford University Press, New York, (2001).

[35] T. Iwaniec and J. Onninen, *p-harmonic energy of deformations between punctured balls,* Adv. Calc. Var. **2** (2009), 93-107.

[36] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition,* Duke Math. J., **53** (1986), 503-523.

[37] P. W. Jones, *Quasiconformal mappings and extendability of functions in Sobolev spaces,* Acta Math., **147** (1981), 71-88.

[38] S. Keith and X. Zhong, *The Poincaré inequality is an open ended condition,* Ann. of Math. **167** (2008), 575-599.

[39] T. Kilpeläinen, *Weighted Sobolev spaces and capacity,* Ann. Acad. Sci. Fenn. Math., **19** (1994), 95-113.

[40] J. Kinnunen and J. L. Lewis, *Higher integrability for parabolic systems of p-Laplacian type.* Duke Math. J., **102** (2000), 253-271.

[41] J. Kinnunen and J. L. Lewis, *Very weak solutions of parabolic systems of p-Laplacian type,* Ark. Mat., **40** (2002), 105-132.

[42] A. A. Korenovskyy, A. K. Lerner and A. M. Stokolos, *A note on the Gurov-Reshetnyak condition,* Math. Res. Lett. **9** (2002), 579-583.

[43] M. Lacey, E. Sawyer, and I. Uriarte-Tuero, *Astala's conjecture on distortion of Hausdorff measure under quasiconformal maps,* Acta Math., **204** (2010), 273-292.

[44] J. L. Lewis, *On the very weak solutions of certain elliptic systems,* Comm. Part. Diff. Equ., **18** (1993), 1515-1537.

[45] J. L. Lewis, *Personal communications,* 2011.

[46] H. Lewy, *An example of a smooth linear partial differential equation without solution,* Ann. of Math., **66** (1957), 155-158.

[47] G. Lu, *Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications,* Rev. Mat. Iberoamericana, **8** (1992), 367-439.

[48] G. Lu, *The sharp Poincaré inequality for free vector fields: An endpoint result,* Rev. Mat. Iberoamericana, **10** (1994), 453-466.

[49] G. Lu, *Local and global interpolation inequalities on the Folland-Stein Sobolev spaces and polynomials on stratified groups,* Math. Res. Lett., **4** (1997) 777-790.

[50] J. Martin and M. Milman, *Gehring's lemma for non doubling measures,* Michigan Math. J., **47** (2000) 559-573.

[51] O. Martio, S. Rickman, and J. Väisälä, *Definitions for quasiregular mappings,* Ann. Acad. Sci. Fenn. Ser. A I **448**, (1969), 1-40.

[52] N. G. Meyers and A. Elcrat, *Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions,* Duke Math. J., **42** (1975), 121-136.

[53] E. J. McShane, *Extension of range of functions,* Bull. Amer. Math. Soc., **12** (1934), 837-842.

[54] S. Müller, *Higher integrability of Determinants and weak convergence in L^1,* J. Reine. Angew. Math., **412** (1990), 20-34.

[55] A. Nagel, E. M. Stein and S. Wainger, *Balls and metrics defined by vector fields I, Basic Properties,* Acta Math., **155** (1985), 103-147.

[56] D. M. Nhieu, *Extension theorems for Sobolev spaces in the Heisenberg group,* Ph.D. Thesis in Pudure University, (1995).

[57] S. Petermichl and A. Volberg, *Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular,* Duke Math. J. **112** (2002), 281-305.

[58] Yu. G. Reshetnyak, *Space mappings with bounded distortion,* Sibirsk. Mat. Zh., **8** (1967), 629-658.

[59] Yu. G. Reshetnyak, *Space mappings with bounded distortion,* Translations of Mathematical Monographs, Amer. Math. Soc. (1989).

[60] S. Rickman, *Quasiregular mappings,* Springer-Verlag, Berlin, (1993).

[61] E. M. Stein, *Harmonic analysis: Real variable methods, orthogonality, and oscillatory integrals,* Princeton University Press, (1993).

[62] J. O. Strömberg and A. Torchinsky. *Weighted Hardy spaces,* Lecture Notes in Mathematics, **1381**, Springer-Verlag, (1989).

[63] S. K. Vodop'yanov and I. M. Pupyshev, *Whitney-type theorems on extension of functions on Carnot group,* Sib. Math. Zh., **47** (2006), 601-620.

[64] W. P. Ziemer, *Weakly differentiable functions-Sobolev spaces and functions of bounded variation,* GTM 120, Springer-Verlag, New York, (1989).