Scroll codes over curves of higher genus

Trygve Johnsen [*]

Nils Henry Rasmussen [†]

August 2010

Abstract

We construct linear codes from scrolls over curves of high genus and study the higher support weights d_i of these codes. We embed the scroll into projective space $\mathbb{P}^{k-1}$ and calculate bounds for the d_i by considering the maximal number of $\mathbb{F}_q$-rational points that are contained in a codimension h subspace of $\mathbb{P}^{k-1}$. We find lower bounds of the d_i and for the cases of large i calculate the exact values of the d_i.

This work follows the natural generalisation of Goppa codes to higher-dimensional varieties as studied by S.H. Hansen, C. Lomont and T. Nakashima.

1 Introduction

One way to produce linear q-ary codes with word length n and dimension k is to pick a geometric object T in the projective space $\mathbb{P}^{k-1}$, and let each of the, say n, points of T be represented by an element of $\mathbb{F}_q^k$. Using these k-tuples as the columns of a generator matrix, one defines the code via this generator matrix. The choice of representative for each point, and the ordering of the points, does not change the equivalence class of the code, and hence not the word length and dimension either. For a linear code C, the i'th higher weight d_i is defined as the minimum support weight among all subcodes of C of dimension i. In particular, d_1 is equal to the minimum distance.

Moreover, it is well-known that for $i = 1, \cdots, k$,

$$d_i = n - J_i,$$

where J_i is the the maximal number of $\mathbb{F}_q$- rational points from T on a codimension i linear subspace of $\mathbb{P}^{k-1}$. It is clear that also the d_i are independent of the choice of representative for each point of T.

The aim with this article is to investigate properties of linear error-correcting codes over a finite field $\mathbb{F}_q$, obtained from scrolls that are embeddings of projective bundles of higher rank over curves of higher genus. In [HaJ07], the authors studied properties of linear codes produced from rational normal scrolls, which are naturally embedded projective bundles of type $\mathbb{P}(\mathscr{E})$, where $\mathscr{E} = \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(e_\Delta)$ is a bundle on $\mathbb{P}^1$. In the present work, we will study codes from the projectivised bundles over curves of higher genus in a similar way.

In the present paper, we will let X be a curve of genus g and $\mathscr{E}$ be a semi-stable vector bundle on X, both defined over $\mathbb{F}_q$ and therefore simultaneously over its algebraic closure, and we will embed $T' = \mathbb{P}(\mathscr{E})$ into some projective space $\mathbb{P}^{k-1}$ (over $\mathbb{F}_q$ and over its closure) by the natural line bundle $\mathcal{L}' = \mathcal{O}_{T'}(1)$ such that $k = h^0(X, \mathscr{E}) = h^0(T', \mathcal{L}')$. In this manner, the fibers of the projective bundle are embedded as linear, sub-projective spaces of $\mathbb{P}^{k-1}$.

[*]Dept. of Mathematics, University of Tromsø, N-9037 Tromsø, Norway, Trygve.Johnsen@uit.no

[†]Dept. of Mathematics, University of Bergen, Johs. Bruns gate 12, N-5008 Bergen, Norway, nilshwr@math.uib.no

In other papers, like [Han01], [Lom03] and [Nak06], one also studies projective bundles $T' = \mathbb{P}(\mathscr{E})$ like this for the purpose of producing codes, and one even varies the complete linear system line bundle $a\mathcal{L}' + f_1 + \cdots + f_b$ by which one embeds $\mathbb{P}(\mathscr{E})$ into projective space, where $\mathcal{L}'$ is as described, and the f_j are fibres of $\mathbb{P}(\mathscr{E})$ over points $P_1, \cdots, P_b$ on X. There one gives estimates for the minimum distance d_1 for the codes thus defined, in other words for (the number of points minus) the maximal number of $\mathbb{F}_q$-rational points in a codimension one space in the embedding space. In the present paper, it is not our main purpose to improve the estimates for d_1, but rather to say as much as possible about the d_i for higher $i \leq k$ for our particular linear system $\mathcal{L}'$. We will combine the insight of the mentioned articles about projective bundles in positive characteristic and the techniques of [HaJ07] for rational normal scrolls. To determine the d_i for large i (close to k) an important tool will be Riemann-Roch's theorem for vector bundles on curves, both defined over a finite field.

For somewhat smaller i a main tool to give lower bounds for the weights d_i will be Brill–Noether theory for vector bundles of higher ranks. Especially the non-existence results as in [BGN97], [Bal98], [Re98] and [Mer02] will be useful. We believe that the demonstration of how this kind of mathematics can be applied in a code-theoretic setting is a main point of the article.

We thank Gian Pietro Pirola for helpful remarks during our work on Example 5.2. The second author wishes to thank Anita Buckley for help during his visit to the University of Ljubljana in February 2009.

2 Constructions and presentation of the problem

A linear code C is a linear subspace of $(\mathbb{F}_q)^n$ for some $n \in \mathbb{N}$. We usually denote the dimension of the code by k, and it is defined as $k = \log_q(\#(C))$. For $h = 1, 2, \ldots, k$, let D_h be the set of all linear subspaces of the code C generated by h linearly independent elements in C, and let

$$d_h = \min\{\#(\mathrm{Supp}(E)) \mid E \in D_h\}.$$

We call d_1 the *minimum distance* of the code C. One aim in coding theory is given q, n and k, to maximise d_1. In processes of trellis decoding, or in cryptology, using the generator matrix of C instead as a starting point in connection with the so-called wire-tap channel of type II, it can in some cases be interesting to maximise d_h for higher values of h.

Let X be a non-singular, projective curve of genus g defined over $\mathbb{F}_q$ (see [Ste99, Chapter 5] for definitions), and let $\mathscr{E}$ be a locally free sheaf of rank r on X, where r is some positive integer. Let *$\mathscr{E}$ be defined over* $\mathbb{F}_q$ if there exists an open covering with transition functions consisting of elements of the function field over $\mathbb{F}_q$.

The following proposition is the Riemann–Roch theorem for vector bundles on curves defined over finite fields, and is used repeatedly by other authors, like in [Han01] and [Nak06].

Proposition 2.1. *Over any field k, if X is a curve defined over k and $\mathscr{E}$ is a locally free sheaf of rank r on X, r any positive integer, then*

$$\chi(\mathscr{E}) = \deg(\mathscr{E}) + r(1-g).$$

We will from now on suppose the following: X will denote a non-singular, projective curve of genus $g \geq 0$ defined over the finite field $\mathbb{F}_q$, and $\mathscr{E}$ will denote a locally free, semistable sheaf of rank $r \geq 2$ (and some high degree) defined over $\mathbb{F}_q$ and where $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is very ample.

Let $T' = \mathbb{P}(\mathscr{E})$, and denote $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ by $\mathcal{L}'$. Use $\mathcal{L}'$ to embed T' into projective space $\mathbb{P}^{k-1}$, where $k = h^0(T', \mathcal{L}')$, and denote the isomorphic image by T. Let $\mathcal{L}$ be the line bundle on T corresponding to $\mathcal{L}'$ on T'. Then T will be a scroll in the sense that the fibres of T' over the points of X will be mapped into $\mathbb{P}^{k-1}$ as linear projective (sub)spaces. For each $\mathbb{F}_q$-rational point P on T, choose a set of coordinates $(x_1, \ldots, x_k)$ such that $x_1, \ldots, x_k \in \mathbb{F}_q$. We then define a matrix G where each column is of the form $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k$ are the chosen coordinates of a point P on T. We define C to be the linear code with generator matrix G. The choice of generators of $H^0(\mathcal{L})$ and the ordering of the columns will not affect the equivalence class of the code, and thus not the parameters $n, k, d_1, \dots, d_k$ either. It is for example clear that

$$n = m(q^{r-1} + \cdots + q + 1),$$

where n simultaneously denotes the word length of the code and the number of $\mathbb{F}_q$-rational points on T, and m denotes the number of $\mathbb{F}_q$-rational points on X. We define:

$\mu(\mathscr{E}) := \deg(\mathscr{E})/r$. If $m > \mu(\mathscr{E})$, then the dimension of C is

$$k = h^0(T, \mathcal{L}) = h^0(X, \mathscr{E}).$$

This is true since $m > \mu(\mathscr{E})$ implies that $\deg(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_m)) = \deg(\mathscr{E}) - rm = r(\mu(\mathscr{E}) - m) < 0$, and hence, $h^0(T, \mathcal{L} \otimes \mathscr{O}(-f_1 - \cdots - f_m)) = h^0(X, \pi_*(\mathcal{L} - \mathscr{O}(f_1 + \cdots + f_m))) = h^0(X, \mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_m)) = 0$ since $\mathscr{E}$ and therefore also $\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_m)$ is semi-stable. Here f_i denotes the fibre of T over P_i, for $i = 1, \dots, n$. Hence, the $\mathbb{F}_q$-rational points of T span all of $\mathbb{P}^{k-1}$.

We see that $\mathcal{L}' = \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is very ample on T' if it embeds each fibre of T' as a projective $(r-1)$-subspace of $\mathbb{P}^{k-1}$, and if each pair of two such fibres are mapped onto disjoint $(r-1)$-subspaces, which together impose $2r$ conditions on the hyperplanes in $\mathbb{P}^{k-1}$. A sufficient condition for this to happen, if $\mathscr{E}$ is semistable, is $\deg(\mathscr{E}) > 2gr$, since then

$$h^0(T', \mathcal{L}') - h^0(T', \mathcal{L}' - \mathscr{O}(f_1 + f_2)) = h^0(X, \pi_*\mathcal{L}) - h^0(X, \pi_*(\mathcal{L} - \mathscr{O}(f_1 + f_2))) =$$

$$h^0(X, \mathscr{E}) - h^0(X, \mathscr{E} \otimes \mathscr{O}(-P_1 - P_2)) = (\deg(\mathscr{E}) + r(1-g)) - (\deg(\mathscr{E}) - 2r + r(1-g)) = 2r,$$

since $\deg(\mathscr{E} \otimes \mathscr{O}(-P_1 - P_2)) > r(2g-2)$, and in both cases there is no h^1-term in Riemann–Roch's formula (we have for example: $H^1(X, \mathscr{E} \otimes \mathscr{O}(-P_1 - P_2) = H^0(X, K_X \otimes \mathscr{O}(P_1 + P_2) \otimes \mathscr{E}^\vee) = 0$, since the bundle in the last parenthesis has negative degree and is semi-stable since $\mathscr{E}$ is).

Summing up, we obtain:

Remark 2.2. If $\mathscr{E}$ is semi-stable with $m > \mu(\mathscr{E}) > 2g$, where m is the number of $\mathbb{F}_q$-rational points on X, then $\mathcal{L}'$ is very ample. It follows that T is the isomorphic image of T' and C is an $[n, k]$-code, where $n = m(q^{r-1} + \cdots + q + 1)$ and $k = h^0(T, \mathcal{L}) = h^0(X, \mathscr{E}) = \deg(\mathscr{E}) + r(1-g)$.

Basic Assumption 2.3. *In the rest of the paper (except in Example 5.2) we will assume that C is a code produced from a scroll T as in Remark 2.2, including the assumptions that $\mathscr{E}$ is semi-stable and $m > \mu(\mathscr{E}) > 2g$.*

Our aim is to find a lower bound for $d_1, \dots, d_k$. The number d_k is easily seen to be n, since otherwise there would be a point on T with all coordinates equal to zero, which is impossible.

Notation 2.4. *We denote the maximal number of $\mathbb{F}_q$-rational points on T contained in a codimension h subspace by J_h.*

It is well-known that

$$d_h = n - J_h. \tag{1}$$

In the rest of the article we will determine the J_h for as many h as possible and give good upper bounds for the J_h (lower bounds for the corresponding d_h) for the remaining h.

The following definition makes sense and will be useful:

Definition 2.5. *Let $S_{h,0}$ be the maximal number of fibres of (T over X) contained in a codimension h subspace.*

We then have the following obvious bound:

Remark 2.6. $J_h \leq (q^{r-1} + \cdots + q + 1) \cdot S_{h,0} + (q^{r-2} + \cdots + q + 1) \cdot (m - S_{h,0})$.

Using Equation (1), we obtain

Proposition 2.7. $d_h \geq q^{r-1}(m - S_{h,0})$.

It is desirable to get a better upper bound by determining how a codimension h subspace L containing $S_{h,0}$ fibres intersects other fibres. The fact that the fibres of T over X are linear spaces reduces this to an issue of which dimension $f \cap L$ has for the other fibres f. It is also a priori possible that a codimension h subspace L containing less than $S_{h,0}$ fibres contains a maximal number of $\mathbb{F}_q$-rational points.

The following fact is obvious, but will be used so much throughout that we include it here anyway.

Observation 2.8. *Let $f_1, \dots, f_S$ be S fibres for some integer S. The fibres are contained in a codimension h subspace L if and only if*

$$h^0(T, \mathcal{L} \otimes \mathscr{O}(-f_1 - \cdots - f_S)) \geq h.$$

We have the following preliminary result:

Proposition 2.9. *Let $g \geq 0$ and $h \in \{1, \dots, k\}$. Then*

$$S_{h,0} \leq \mu(\mathscr{E}) - \lfloor (h-1)/r \rfloor .$$

Proof. For $h = 1$, we observe that $h^0(X, \mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{1,0}})) \geq 1$, where $P_1, \cdots, P_{S_{1,0}}$ are points on X corresponding to $S_{1,0}$ fibres in a hyperplane. This implies that $\deg(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{1,0}})) \geq 0$, and hence $S_{1,0} \leq \deg(\mathscr{E})/r = \mu(\mathscr{E})$.

For $h \geq 1$, we show that $S_{h+r,0} \leq S_{h,0} - 1$, i.e., that a codimension $h + r$ subspace L' contains at most $S_{h,0} - 1$ fibres. For arbitrary j, and where C_0 denotes a hyperplane section, the Riemann–Roch theorem gives us

$$\begin{aligned} h^0(T, C_0 - (f_1 + \cdots + f_{j-1})) &= \deg(\mathscr{E}) - rj + (1-g)r \\ &\qquad + h^1(T, C_0 - (f_1 + \cdots + f_{j-1})) + r \\ &\leq \deg(\mathscr{E}) - rj + (1-g)r \\ &\qquad + h^1(T, C_0 - (f_1 + \cdots + f_{j-1} + f_j)) + r \\ &= h^0(T, C_0 - (f_1 + \cdots + f_{j-1} + f_j)) + r. \end{aligned}$$

Set $j = S_{h,0} + 1$. We get $h^0(T, C_0 - (f_1 + \cdots + f_{S_{h,0}})) \leq h^0(T, C_0 - (f_1 + \cdots + f_{S_{h,0}} + f_{S_{h,0}+1})) + r < h + r$, since $S_{h,0}$ is the greatest integer satisfying $h^0(T, C_0 - (f_1 + \cdots + f_{S_{h,0}})) \geq h$. So L' cannot contain $S_{h,0}$ fibres.

Hence, $S_{h+r,0} \leq S_{h,0} - 1$, and

$$S_{h,0} \leq \frac{\deg(\mathscr{E})}{r} - \lfloor (h-1)/r \rfloor .$$

□

The following result follows immediately from Remark 2.6 and Proposition 2.9 and is similar to results in [Han01], [Lom03], and [Nak06].

Corollary 2.10. *$d(C) \geq q^{r-1}(m - \mu(\mathscr{E}))$. In general, $d_h \geq q^{r-1}(m - \mu(\mathscr{E})) + \lfloor (h-r)/r \rfloor)$.*

In Corollary 2.14, Lemma 3.2 and Proposition 4.4, we will improve the preliminary bound in Proposition 2.9 for h in certain (broad) ranges.

The following definitions will be instrumental for many h:

Definition 2.11. *For each non-negative integer d, let $f(d)$ be the maximal value of $h^0(\mathscr{E})$ for all semi-stable vector bundles $\mathscr{E}$ of degree at most d on X, defined over the closure of $\mathbb{F}_q$.*

Moreover, for each positive integer h, let $\phi(h) = \min\{d \mid h^0(\mathscr{E}) \geq h$ for some semi-stable vector bundle $\mathscr{E}$ of degree at most $d\} = \min\{d \mid f(d) \geq h\}$.

Let h be a positive integer. We now have:

Proposition 2.12.
$$S_{h,0} \leq \frac{d}{r} - \frac{\phi(h)}{r} = \mu(\mathscr{E}) - \frac{\phi(h)}{r}.$$

Proof. Assume a codimension h subspace contains S fibres, corresponding to the points $P_1, \dots, P_S$ on C. Then $h^0(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_S)) \geq h$. This immediately implies $\deg(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_S)) \geq \phi(h)$. So we have $\deg(\mathscr{E}) - rS \geq \phi(h)$, which gives us

$$S \leq \frac{d}{r} - \frac{\phi(h)}{r} = \mu(\mathscr{E}) - \frac{\phi(h)}{r}.$$

□

We now for simplicity assume $g \geq 2$. We then have:

Proposition 2.13.

- *For* $0 \leq d \leq r(2g-2)$, *we have* $f(d) \leq r + \frac{d}{2}$.
- *For* $r(2g-2) \leq d \leq r(2g-1)$, *we have* $f(d) \leq rg$.
- *For* $d \geq r(2g-1)$, *we have* $f(d) \leq d + r(1-g)$, *and* $f(d) = d + r(1-g)$ *if semistable bundles of degree* d *exist.*

Proof. The first statement is the Clifford bound given in Theorem 1.1 of [Bal98]. The second and third statements follow from the first statement and Riemann–Roch, which gives $h^0(\mathscr{E}) = d + r(1-g) + h^0(K_X \otimes \mathscr{E}^\vee) = d + r(1-g)$, since $K_X \otimes \mathscr{E}^\vee$ has negative degree and is semi-stable since $\mathscr{E}$ is. □

Corollary 2.14.

- *For* $r \leq h \leq gr$, *we have* $\phi(h) \geq 2(h-r)$ *and* $S_{h,0} \leq \mu(\mathscr{E}) - \frac{2h}{r} + 2$.
- *For* $h \geq gr+1$, *we have* $\phi(h) \geq h + r(g-1)$ *and* $S_{h,0} \leq \mu(\mathscr{E}) - \frac{h}{r} + (1-g)$.

Proof. The lower bounds for $\phi(h)$ follow immediately from Proposition 2.13. The upper bounds for $S_{h,0}$ follow immediately from Proposition 2.12 and the lower bounds for $\phi(h)$. □

To obtain better upper bounds on J_h than the ones we get using Remark 2.6 and the upper bounds on $S_{h,0}$, we have the following helpful result:

Proposition 2.15. *Let* $0 \leq i \leq r-1$, *and let* L *be a codimension* h *subspace that intersects* $\geq s_j$ *fibres in a* $\mathbb{P}^{r-j-1}$ *for* $j = 0, \dots, i$. *Then*

$$s_0 + s_1 + \cdots + s_i \leq \mu(\mathscr{E}) - \frac{\phi(h - s_1 - 2s_2 - \cdots is_i)}{r}.$$

Proof. For $i = 0$, this is only Proposition 2.12. Let $i \geq 1$. We have $\geq s_0$ fibres contained in L, and in addition, L intersects $\geq s_i$ fibres in a $\mathbb{P}^{r-i-1}$ for each $1 \leq i \leq r-1$. For each of these i, choose s_i fibres that intersect L in a $\mathbb{P}^{r-1-i}$ and denote the set of these fibres by F_i. For each fibre in F_i, choose i points such that these points and the intersection of L with the fibre together span the fibre. Let L' be the linear span of L and the $s_1 + 2s_2 + \cdots + is_i$ points we just chose. The codimension of L' is then at least $h - s_1 - 2s_2 - \cdots - is_i$, and L' contains $\geq s_0 + s_1 + \cdots + s_i$ fibres. The proof of Proposition 2.12 then gives the conclusion. □

To improve the (effective) bounds for $f(d)$ and $\phi(h)$ in the range $0 < d < r(2g-2)$ and corresponding range $r < h < gr$, at least in some special cases under further assumptions on X and the bundle $\mathscr{E}$, is a matter of great interest and is essentially the socalled "non-existence" problem in Brill–Noether theory for bundles of higher rank, as addressed in [BGN97], [Re98], [Bal98] and [Mer02]. We will return to this issue in Section 4. For $h \geq gr+1$, there is not much room for improvement, as we will see in the beginning of the next section.

3 Particular bounds in the range $h \geq gr + 1$

We start this section by fixing the following notation.

Notation 3.1. *As before, let $S_{h,0}$ be the maximal number of fibres contained in a codimension h subspace L, where the maximum is taken over all codimension h subspaces L in $\mathbb{P}^{k-1}$. Denote the set of all codimension h subspaces that contain $S_{h,0}$ fibres by $A_{h,0}$. For $1 \leq i \leq r$, denote by $S_{h,i}$ the maximal number of fibres that intersect a codimension h subspace $L \in A_{h,0}$ in a $\mathbb{P}^{r-i-1}$.*

In this section we will now give some bounds for the $S_{h,i}$ for h large enough. In particular, we have the following lower bound for $S_{h,0}$:

Lemma 3.2. *For $h \geq rg + 1$, we have*

$$S_{h,0} \geq \mu(\mathscr{E}) - \frac{h}{r} - g + 1 - \frac{r-1}{r}.$$

It follows that

$$S_{h,0} = \left\lfloor \frac{\deg(\mathscr{E}) - h}{r} \right\rfloor - g + 1 = \frac{\deg(\mathscr{E}) - h'}{r} - g + 1,$$

where $h' = h^0(X, \mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}}))$ and $P_1, \ldots, P_{S_{h,0}}$ is any collection of points corresponding to fibres contained in a codimension h subspace that contains $S_{h,0}$ fibres of T.

Proof. Let $P_1, \ldots, P_{S_{h,0}}$ be points corresponding to fibres as described, and let $h^0(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}})) = h'$. Then $h' \geq h \geq rg + 1$, and the Riemann–Roch theorem gives

$$S_{h,0} = \mu(\mathscr{E}) - \frac{h'}{r} + 1 - g,$$

since we have from Proposition 2.14 that $\deg(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}})) = \deg(\mathscr{E}) - rS_{h,0} \geq \phi(h') \geq (2g-1)r + 1 > (2g-2)r$. By the assumption that $\mathscr{E}$ is semi-stable, there is no h^1-term, and hence this equality follows.

Since $S_{h,0}$ is the largest integer such that there exist points P_i such that $h^0(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}})) \geq h$, we have $h' - h \leq r - 1$ because of the following argument: We just showed that $\deg(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}})) = d - rS_{h,0} \geq (2g-1)r + 1$. It follows that also $h^1(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}} - P_{S_{h,0}+1})) = 0$, and so $h^0(\mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}} - P_{S_{h,0}+1})) = h' - r$, which must be $< h$ because of the definition of $S_{h,0}$. It follows that $h' - h \leq r - 1$, and that the first inequality of the lemma holds.

The equalities at the end of the lemma now follow from Proposition 2.14, stating that $S_{h,0} \leq \mu(\mathscr{E}) - h/r - g + 1$, and from the fact that there is exactly one integer in the interval $[\mu(\mathscr{E}) - h/r - g + 1 - (r-1)/r,\ \mu(\mathscr{E}) - h/r - g + 1]$. □

Corollary 3.3. *For $h \geq rg + 1$, we have $d_h \geq q^{r-1}(m - \left\lfloor \frac{\deg(\mathscr{E}) - h}{r} \right\rfloor + g - 1)$.*

We have the following result for $S_{h,i}$ with $i \geq 1$:

Corollary 3.4. *Let $0 \leq i \leq r - 1$, and let L be a codimension h subspace that intersects $\geq s_j$ fibres in a $\mathbb{P}^{r-j-1}$ for $j = 0, \ldots, i$, where $h \geq rg + s_1 + 2s_2 + \cdots + is_i + 1$. Then*

$$\left(s_0 - \mu(\mathscr{E}) + \frac{h}{r} + g - 1\right) + \left(\frac{r-1}{r}\right) s_1 + \left(\frac{r-2}{r}\right) s_2 + \cdots + \left(\frac{r-i}{r}\right) s_i \leq 0.$$

In particular, if $L \in A_{h,0}$, we have:

$$\left(\frac{r-1}{r}\right) s_1 + \left(\frac{r-2}{r}\right) s_2 + \cdots + \left(\frac{r-i}{r}\right) s_i \leq \frac{h' - h}{r} \leq \frac{r-1}{r},$$

where $h' = h^0(X, \mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}}))$ and the points $P_1, \ldots, P_{S_{h,0}}$ correspond to fibres contained in a codimension h subspace contained in $A_{h,0}$ (see Notation 3.1).

Proof. Since $h \geq rg + s_1 + 2s_2 + \cdots + is_i + 1$, we have $h - s_1 - 2s_2 - \cdots - is_i \geq rg + 1$, so $\phi(h - s_1 - 2s_2 - \cdots - is_i) \geq h - s_1 - 2s_2 - \cdots - is_i + r(g-1)$ by Corollary 2.14. Hence, Proposition 2.15 gives

$$s_0 + s_1 + \cdots + s_i \leq \mu(\mathscr{E}) - \frac{h - s_1 - 2s_2 - \cdots - is_i + r(g-1)}{r}.$$

Rearranging terms, we obtain the first part of the corollary.

The second part of the corollary follows since $h' = \deg(\mathscr{E}) + r(1-g) - rS_0$, and $h' - h \leq r - 1$, as demonstrated in the proof of Lemma 3.2. □

Definition 3.5. *Let $t = h' - h$, where h' was described in Corollary 3.4. Note also that Lemma 3.2 and its proof give the explicit formula: $t = h' - h = \deg(\mathscr{E}) - r\left\lfloor \frac{\deg(\mathscr{E}) - h}{r} \right\rfloor - h$.*

Remark 3.6. One might think of $k - h = k - h' + t = S_{h,0}r + t$ as the dimension of the affine cone in $(\mathbb{F}_q)^k$ of a linear space L in $\mathbb{P}^{k-1}$ containing $S_{h,0}$ fibres and t independent points in another fibre. This only makes sense if the fibres and points impose independent conditions on hyperplanes. We will show that if h and q are big enough, this is indeed the case.

We now make a few essential observations: The last part of Corollary 3.4 reads:

$$(r-1)s_1 + (r-2)s_2 + \cdots + (r-i)s_i \leq t$$

if $h \geq rg + s_1 + 2s_2 + \cdots + is_i + 1$ and $s_0 = S_{h,0}$.

Assume $t = 0$ and $h \geq rg + r$, and that L is a codimension h subspace that contains $S_{h,0}$ fibres and intersects $s_i = 1$ other fibre in a $\mathbb{P}^{r-i-1}$ for some $i \leq r - 1$. Then $h \geq rg + (r-1)s_i + 1 \geq rg + is_i + 1$. But then we obtain (with $s_j = 0$ for all $j \neq i$) that $(r - i)s_i \leq 0$, that is, $s_i = 0$. Hence, $S_{h,i} = 0$ for $i \geq 1$ if $h \geq r(g+1)$ and $t = 0$.

Assume t is any integer satisfying $1 \leq t \leq r - 2$ and $h \geq rg + r - t$, and that L is a codimension h subspace that contains $S_{h,0}$ fibres and intersects $s_i = 1$ other fibre in a $\mathbb{P}^{r-i-1}$ for some $1 \leq i \leq r - (t+1)$. Then $h \geq rg + (r - t - 1)s_i + 1 \geq rg + is_i + 1$. But then we obtain (with $s_j = 0$ for all $j \neq i$) that $(r-i)s_i \leq t$, that is, $s_i = 0$, since $r - i \geq t + 1$. Hence, $S_{h,i} = 0$ for $1 \leq i \leq r - (t+1)$ if $h \geq rg + r - t$.

If t is any integer satisfying $1 \leq t \leq r - 1$ and h also satisfies $h \geq rg + 2(r-t) + 1$, then we conclude in an analogous way that $s_{r-t} \leq 1$. Moreover, it is clear that if $S_{h,0}$ fibres span a codimension $h' = h + t$ plane, then we may just add t independent points in another fibre and thereby span a codimension h plane containing $S_{h,0}$ fibres and intersecting another one in a $\mathbb{P}^{t-1} = \mathbb{P}^{(r-1)-(r-t)}$. Hence, $S_{h,r-t} = 1$.

Moreover, it is then clear that if $h \geq rg + (r-t) + (r-1) + 1 = rg + 2r - t$ and that L is a codimension h subspace that contains $S_{h,0}$ fibres and intersects another one in a $\mathbb{P}^{t-1} = \mathbb{P}^{(r-1)-(r-t)}$, and $r - 1 \geq i \geq 1$, $i \neq r - t$, then the equation $ts_{r-t} + is_i \leq t$ obtained from setting $s_j = 0$ for $j \neq i, r - t$, gives $s_i = 0$.

We sum this up as:

Proposition 3.7. *We have the following:*

(a) If $h \geq rg + r - t$ and $0 \leq t \leq r - 2$, then $S_{h,i} = 0$, for $i = 1, 2, \cdots, r - t - 1$.

(b) If $h \geq rg + 2(r-t) + 1$ and $1 \leq t \leq r - 1$, then $S_{h,r-t} = 1$. If moreover the stronger condition $h \geq rg + 2r - t$ holds, then any element in $A_{h,0}$ intersecting an additional fibre in a $\mathbb{P}^{t-1}$ intersects all other fibers empty.

We then obtain:

Corollary 3.8. *We have the following:*

(a) If $h \geq r(g+1)$ and $t = 0$, then the maximum number of intersection points between an element in $A_{h,0}$ and T is $S_{h,0}\frac{q^r - 1}{q - 1}$.

(b) *If $h \geq rg + (t+1)(r-1)+1$ and $1 \leq t \leq r-1$, and q is big enough, e.g. $q \geq (r-1)(r-2)$, then the maximum number of intersection points between an element in $A_{h,0}$ and T is $S_{h,0}\frac{q^r-1}{q-1} + \frac{q^t-1}{q-1}$.*

Proof. Part (a) follows directly from the case $t = 0$ in part (a) of Proposition 3.7.

Because of Proposition 3.7 (b), part (b) of our corollary follows if we can prove that the number of points in a $\mathbb{P}^{t-1}$ is at least as large as the number of "additional" intersection points of any element in $A_{h,0}$ and T (meaning in addition to the points of the $S_{h,0}$ fibres that are contained in this intersection by definition). By Proposition 3.7 (a), we may restrict ourselves to looking at codimension h spaces that intersect the "additional" fibres of T only in m-spaces where $m < t$.

So assume L is such a codimension h space in $A_{h,0}$, and assume L intersects s_j fibres in a $\mathbb{P}^{r-j-1}$, where $r-1-j \leq t-2$. Then the first part of Corollary 3.4, with $s_0 = S_{h,0}$ and $s_l = 0$ for $l \neq 0, j$, gives $s_j \leq \frac{t}{r-j} \leq t$ if $h \geq rg + js_j + 1$. It will then be enough to assume $h \geq rg + (t+1)(r-1) + 1$ ($\geq rg + (t+1)j + 1$) to conclude $s_j \leq \frac{t}{r-j} \leq t$. (Pick a fibres such that the codimension h subspace contains $S_{h,0}$ fibres and intersects these a fibres in codimension h, where a is an integer with $\frac{t}{r-j} < a \leq t+1$. Then $h \geq rg + (t+1)(r-1)+1 \geq rg + aj + 1$, and we conclude $a \leq \frac{t}{r-j}$ from Corollary 3.4, a contradiction that falsifies the possibility $\frac{t}{r-j} < a$, and we conclude $s_j \leq \frac{t}{r-j}$.) Then it will suffice to find conditions on q such that:

$$\frac{q^t-1}{q-1} \geq \sum_{i=1}^{t-1} \frac{t}{i} \cdot \frac{q^i-1}{q-1}. \tag{2}$$

By expanding both sides as polynomials in q, one sees that it suffices (but is far from necessary) that $q \geq \frac{t}{t-1} + \frac{t}{t-2} + \cdots + \frac{t}{2} + \frac{t}{1}$. It clearly suffices that $q \geq (r-1)(r-2) \geq t(t-1)$. □

3.1 A comparison between elements of $A_{h,0}$ and other codimension h planes

We observe from Corollary 3.8 above, using the identity $d_h = n - J_h$, that as long as J_h is computed by elements of $A_{h,0}$, then d_h is easy to compute as long as $h \geq rg + (t+1)(r-1)+1$ and $q \geq (r-1)(r-2)$. To make sure that d_h and J_h really are computed by elements of $A_{h,0}$, we will have to impose further restrictions on h and q. Here is an analysis:

First we discuss how many fibres s_i that can intersect L in a $\mathbb{P}^{r-i-1}$, $i = 1, 2, \ldots, r-1$, when L contains $s_0 < S_{h,0}$ fibres.

Lemma 3.9. *Let L be a codimension h subspace that contains $S_{h,0} - i$ fibres and intersects s_1 fibres in a $\mathbb{P}^{r-2}$, where $h \geq rg + ir/(r-1) + 3$ and $i \in \{0, 1, \ldots, S_{h,0}\}$. Then*

$$s_1 \leq \left\lfloor \frac{ir}{r-1} \right\rfloor + 1.$$

Proof. We use the first part of Proposition 3.4, with $s_0 = S_{h,0} - i$, and where we use the expression from the proof of Lemma 3.2 for $S_{h,0}$. We then get

$$\frac{h-h'}{r} - i + \left(\frac{r-1}{r}\right) s_1 \leq 0,$$

where $h' = h^0(X, \mathscr{E} \otimes \mathscr{O}(-P_1 - \cdots - P_{S_{h,0}}))$, where the P_i correspond to $S_{h,0}$ fibres contained in a codimension h-space L' which computes $S_{h,0}$. Using that $h' - h \leq r-1$, we get $s_1 \leq ir/(r-1)+1$, if $h \geq rg + s_1 + 1$. But since we assume $h \geq rg + ir/(r-1)+3$, the assumption $h \geq rg + s_1 + 1$ holds as long as $s_1 \leq ir/(r-1)+2$. So in order for s_1 to be $> ir/(r-1)+1$, we must have $s_1 > ir/(r-1)+2$. But then we can't choose a subset of s_1' fibres for $ir/(r-1)+1 < s_1' \leq ir/(r-1)+2$, which is absurd, since this interval contains an integer. □

This enables us to conclude:

Proposition 3.10. *Assume q big enough, for example $q \geq 2g+4$, and $h \geq (r-2)g + \mu(\mathscr{E}) + 1 + \frac{3(r-1)}{r}$ and $h \geq rg+1$. Then there exists a codimension h space $L \in A_{h,0}$ that contains a maximal number of $\mathbb{F}_q$-rational points from T.*

Proof. If $h \geq rg + \frac{rS_{h,0}}{r-1} + 3$, then Lemma 3.9 is applicable for all $i = (0), 1, \cdots, S_{h,0}$, and we conclude that L intersects at most $\frac{ir}{r-1} + 1$ fibres of T in a $\mathbb{P}^{r-2}$ if it contains $S_{h,0} - i$ fibres.

Since $h \geq gr + 1$, we have $S_{h,0} \leq \mu(\mathscr{E}) - \frac{h}{r} - g + 1$ by Lemma 3.2. If we insert the bigger value $\mu(\mathscr{E}) - \frac{h}{r} - g + 1$ for $S_{h,0}$ in the inequality $h \geq rg + \frac{rS_{h,0}}{r-1} + 3$, and the inequality thus obtained holds, then the original inequality also holds. But the condition $h \geq (r-2)g + \mu(\mathscr{E}) + 1 + \frac{3(r-1)}{r}$ in the proposition is precisely the inequality we obtain by inserting this value for $S_{h,0}$.

If $L \in A_{h,0}$, then L contains at least $S_{h,0}\frac{q^r-1}{q-1}$ points on T. If L' is not in $A_{h,0}$, then L' contains $S_{h,0} - i$ fibres, where $i \geq 1$. Then L' contains at most the following number of points:

$$(S_{h,0} - i)\frac{q^r-1}{q-1} + (\frac{ir}{r-1} + 1)\frac{q^{r-1}-1}{q-1} + (m - S_{h,0} + i - \frac{ir}{r-1} - 1)\frac{q^{r-2}-1}{q-1}.$$

It is then enough to prove that

$$i\frac{q^r-1}{q-1} \geq (\frac{ir}{r-1} + 1)\frac{q^{r-1}-1}{q-1} + (m - S_{h,0} + i - \frac{ir}{r-1} - 1)\frac{q^{r-2}-1}{q-1},$$

for $i = 1, \cdots, S_{h,0}$. Writing everything as polynomials in q, we see that it is enough to prove

$$iq^2 \geq (\frac{ir}{r-1} + 1)q + (m - S_{h,0} + i - \frac{ir}{r-1} - 1).$$

This holds for all i if and only if it holds for $i = 1$, and reduces to

$$q^2 \geq \frac{2r-1}{r-1}q + (m - S_{h,0} - \frac{r}{r-1}). \tag{3}$$

Using the Hasse–Weil bound, we see that $m \leq q + 1 + 2g\sqrt{q} \leq (2g+1)q + 1$. Hence, the inequality holds if

$$q^2 - (2g + \frac{3r-2}{r-1})q + (S_{h,0} - 1 + \frac{r}{r-1}) \geq 0.$$

In particular, it holds if $q \geq 2g + 4$. □

We observe that it is possible to modify the proof above to give alternative statements, possibly with harder restrictions on q and milder ones on h, for example like this:

Proposition 3.11. *Assume q big enough, for example $q \geq \max\{2g+4, \frac{4g^2}{i^2} + \frac{2}{i}\}$, and $h \geq rg + \frac{ir}{r-1} + 3$, for some $i \in \{1, \cdots S_{h,0}\}$. Then there exists a codimension h space $L \in A_{h,0}$ that contains a maximal number of $\mathbb{F}_q$-rational points from T.*

Proof. The assumptions on h enable us to apply Lemma 3.9 in the cases where a codimension h plane contains $S_{h,0} - j$ fibres for $j \leq i$. The assumptions on q and the proof of Proposition 3.10 then give that elements of $A_{h,0}$ intersect T in more points than codimension h planes that contain $S_{h,0} - j$ fibres, for $j \leq i$. To prove that elements of $A_{h,0}$ intersect T in more points than codimension h planes that contain $S_{h,0} - j$ fibres for $j \geq i+1$, it suffices to prove that

$$(i+1)\frac{q^r-1}{q-1} \geq m(\frac{q^{r-1}-1}{q-1}).$$

Using the Hasse–Weil bound, we see that this holds if $iq \geq 2gq^{\frac{1}{2}} + 1$, and in particular if $q \geq \frac{4g^2}{i^2} + \frac{2}{i}$. □

3.2 The main result for large h and q

Recall that $t = h' - h = \deg(\mathscr{E}) - r\left\lfloor \frac{\deg(\mathscr{E})-h}{r} \right\rfloor - h$.

Corollary 3.12.

(a) If equations (2) and (3) hold, in particular if $q \geq \max\{(r-1)(r-2), 2g+4\}$, and if in addition $h \geq \max\{(r-2)g + \mu(\mathscr{E}) + 1 + \frac{3(r-1)}{r}, rg + (t+1)(r-1)+1\}$, then

$$J_h = S_{h,0}(\frac{q^r - 1}{q-1}) + \frac{q^t - 1}{q-1}.$$

(b) If equations (2) and (3) hold, in particular if $q \geq \max\{(r-1)(r-2), 2g+4\}$, and if in addition $q \geq \frac{4g^2}{i^2} + \frac{2}{i}$ and $h \geq rg + \frac{ir}{r-1} + 3$, for some $i \in \{1, 2, \cdots, S_{h,0}\}$, and if $h \geq rg + (t+1)(r-1)+1$, then

$$J_h = S_{h,0}(\frac{q^r - 1}{q-1}) + \frac{q^t - 1}{q-1}.$$

Proof. This follows directly from Corollary 3.8 and Propositions 3.10 and 3.11. □

Theorem 3.13. *Under the assumptions of Corollary 3.12, we have:*

$$d_h = (m - S_{h,0})(q^{r-1} + \cdots + q + 1) - (q^{t-1} + \cdots + q + 1) =$$

$$(m - \left\lfloor \frac{\deg(\mathscr{E}) - h}{r} \right\rfloor + g - 1)(q^{r-1} + \cdots + q + 1) - (q^{t-1} + \cdots + q + 1).$$

Proof. This follows from Lemma 3.2 and Corollary 3.12. □

Remark 3.14. *From Corollary 3.12 and the text preceeding Proposition 3.7, it follows that to find a codimension h space that computes J_h for the h in question, you may take the linear space in $\mathbb{P}^{k-1}$ spanned by any choice of $S_{h,0}$ fibres and any choice of $t = h' - h$ linearly independent points in any additional single fibre.*

The appearance of the term $\mu(\mathscr{E})$ in the condition on h in part (a) of Corollary 3.12 implies that it holds for at most $(\frac{r-1}{r}) \cdot k$ of the numbers h between 0 and k (the biggest ones). In reality, since r and g also "count", we can only use (a) of Corollary 3.12 for a somewhat smaller fraction of the h's.

4 Bounds for low h

4.1 Bound for $h = 1$

The integer $S_{1,0}$ is the maximal number of fibres contained in a hyperplane, and is thus equal to the maximal number of points on the curve X that are zero in a global section of $\mathscr{E}$. If we let m be the number of $\mathbb{F}_q$-rational points on X, it is then clear that

$$J_1 = S_{1,0}(q^{r-1} + \cdots + q + 1) + (m - S_{1,0})(q^{r-2} + \cdots + q + 1),$$

since all fibres not contained in a hyperplane H must intersect H in a $\mathbb{P}^{r-2}$. Hence,

$$d_1 = n - J_1 = q^{r-1}(m - S_{1,0}).$$

Remark 4.1. Proposition 2.9 states that $S_{h,0} \leq \mu(\mathscr{E})$ for $h \leq r$. This is in a certain sense a sharp bound: We may construct curves with semi-stable bundles $\mathscr{E}$ of any rank with $S_{h,0} = \mu(\mathscr{E})$ for the corresponding scroll in the following way:

Let X be a curve in projective space such that there exists a hyperplane H that is zero in $\deg(X) > 2g$ distinct $\mathbb{F}_q$-rational points, and let $\mathscr{E} = \mathscr{O}_X(1) \oplus \cdots \oplus \mathscr{O}_X(1)$. Then $\mathscr{E}$ is obviously semistable and has $\mu(\mathscr{E}) = \deg(X)$. (It is easy to check that the tautological line bundle is very ample.) Since $\mathscr{O}_X(1)$ by assumption has a global section s which is zero in $\deg(X)$ distinct $\mathbb{F}_q$-rational points, then so do the global sections $(0, \ldots, 0, s, 0, \ldots, 0)$ of $\mathscr{E}$, and so $S_{h,0} = \mu(\mathscr{E})$ for $h \leq r$, as desired.

4.2 Bound for $h = 2$

For codimensions h, with $2 \leq h \leq r-1$, it is difficult to say much about the J_h and the $S_{h,i}$. We do, however, have the following small result:

Proposition 4.2. *Suppose $S_{1,0} = S_{2,0}$. Then*

$$J_2 = S_{2,0}\left(\frac{q^r-1}{q-1}\right) + S_{2,1}\left(\frac{q^{r-1}-1}{q-1}\right) + (m - S_{2,0} - S_{2,1})\left(\frac{q^{r-2}-1}{q-1}\right),$$

where m is the number of $\mathbb{F}_q$-rational points on the curve X.

Proof. We show that a codimension 2 plane intersecting a maximal number of points must contain a maximal number of fibres. The rest of the statement then follows naturally.

Suppose we have a codimension 2 plane L' containing $S_{2,0}$ fibres, let the plane be defined by two hyperplane sections z_1' and z_2', and let each z_i' contain s_i' fibres. Then L' intersects T in

$$\begin{aligned} J_2' &= S_{2,0}(q^{r-1} + \cdots + q + 1) + (s_1' + s_2' - 2S_{2,0})(q^{r-2} + \cdots + q + 1) \\ &\quad + (m - s_1' - s_2' + S_{h,0})(q^{r-3} + \cdots + q + 1) \\ &= S_{2,0}(q^{r-1} - q^{r-2}) + (s_1' + s_2')q^{r-2} + m(q^{r-3} + \cdots + q + 1) \end{aligned}$$

$\mathbb{F}_q$-rational points, where m is the number of $\mathbb{F}_q$-rational points on X.

Now suppose there is a codimension 2 plane L'' defined by hyperplane sections z_1'' and z_2'', each z_i'' containing s_i'' fibres, and such that L'' contains $S_{2,0} - j$ fibres for some $j \geq 1$. Then L'' intersects T in J_2'' points such that

$$J_2'' - J_2' = -j(q^{r-1} - q^{r-2}) + (s_1'' + s_2'' - s_1' - s_2')q^{r-2}. \tag{4}$$

Now, we assumed that $S_{1,0} = S_{2,0}$, which means that since $z_1' \cap z_2'$ is zero in $S_{2,0}$ fibres, then z_1' and z_2' must each be zero along a maximal number of fibres, and so (4) must be negative, and J_2' must be maximal. □

4.3 Bounds for $r + 1 \leq h \leq gr$

We now study the range $r + 1 \leq h \leq gr$ and look for possible improvements of Corollary 2.14, corresponding to possible improvements of the Clifford bound in Proposition 2.13. Recall the function $f(d)$ introduced in Definition 2.11. The most ambitious conjecture relating to the Clifford bound seems to be the following one, given in [Mer02]. Mercat only states the conjecture for the two first intervals. The last one follows by duality from the first one.

Conjecture 4.3. *Let X be a smooth curve of genus at least 4 and Clifford index γ. Let $\mathscr{E}$ be a semi-stable bundle of rank r, degree d and slope $\frac{d}{r}$. Then:*

- *If $1 \leq \frac{d}{r} \leq \gamma + 2$, then $h^0(\mathscr{E}) \leq f(d) \leq \frac{1}{\gamma+1}(d-r) + r$.*
- *If $\gamma + 2 \leq \frac{d}{r} \leq 2g - 4 - \gamma$, then $h^0(\mathscr{E}) \leq f(d) \leq \frac{d - r\gamma}{2} + r$.*
- *If $2g - 4 - \gamma \leq \frac{d}{r} \leq 2g - 3$, then $h^0(\mathscr{E}) \leq f(d) \leq r(2 - g + \frac{2g-3}{\gamma+1}) + d(\frac{\gamma}{\gamma+1})$.*

We will investigate the consequences of this conjecture:

First, we observe that $d = r$ implies $h^0(\mathscr{E}) \leq r$, and $d = (\gamma + 2)r$ implies $h^0(\mathscr{E}) \leq 2r$, and $d = (2g-4-\gamma)r$ implies $h^0(\mathscr{E}) \leq (g-1-\gamma)r$, and $d = (2g-3)r$ implies $h^0(\mathscr{E}) \leq (g-1)r$, and that the bound for $h^0(\mathscr{E})$ is piecewise linear beteen these values of d. If in addition we set $h^0(\mathscr{E}) \leq r$ also for $0 \leq d \leq r - 1$, and $h^0(\mathscr{E}) \leq d + (2-g)r$ for $(2g-3)r \leq d \leq (2g-2)r$ (the "outer ends" of the Clifford bound), we have an increasing piecewise linear continuous upper bound $L(d)$ for $h^0(\mathscr{E})$ as a function of $d = \deg(\mathscr{E})$ for $0 \leq d \leq (2g-2)r$ (strictly increasing for $r \leq d \leq (2g-2)r$, and which can be slightly improved for the two outer pieces, see Fig. 1 of [Mer02]). We then obtain that $f(d)$ is dominated by this upper bound $L(d)$, and that $\phi(h)$ is at least the inverse of L. We then apply Proposition 2.12:

$$S_{h,0} \leq \mu(\mathscr{E}) - \frac{\phi(h)}{r}.$$

For the interval "in the middle", i.e., $2r \leq h \leq (g-1-\gamma)r$, corresponding to $(\gamma+2)r \leq d \leq (2g-4-\gamma)r$ and $\phi(h) \geq 2h + r(\gamma-2)$, this gives:

$$S_{h,0} \leq \mu(\mathscr{E}) - \frac{2h}{r} + 2 - \gamma. \tag{5}$$

This gives an improvement of γ for the upper bound on $S_{h,0}$ compared with the bound in Corollary 2.14.

For the interval with the smallest h's, i.e., $r \leq h \leq 2r$, corresponding to $r \leq d \leq r(\gamma+2)$ and $\phi(h) \geq (\gamma+1)h - r\gamma$, this gives

$$S_{h,0} \leq \mu(\mathscr{E}) - \frac{(\gamma+1)h}{r} + \gamma.$$

As an example, if $h = 2r$, this gives an improvement of γ for the upper bound on $S_{h,0}$ compared with the bound in Corollary 2.14.

Luckily, there are other, although weaker, results that are theorems, not merely conjectures. In [Mer02], the following theorem is also stated. Mercat only presents the result for the first two intervals, but the last one follows by duality from the first one.

Proposition 4.4. *If $\mathscr{E}$ is a semi-stable rank r bundle of degree d and X is a smooth curve with Clifford index at least 2, then the following holds:*

- *If $1 \leq \mu(\mathscr{E}) \leq 2 + \frac{2}{g-4}$, then $h^0(\mathscr{E}) \leq f(d) \leq \frac{d-r}{g-2} + r$.*
- *If $2 + \frac{2}{g-4} \leq \mu(\mathscr{E}) \leq 2g - 4 - \frac{2}{g-4}$, then $h^0(\mathscr{E}) \leq f(d) \leq \frac{d}{2}$.*
- *If $2g - 4 - \frac{2}{g-4} \leq \mu(\mathscr{E}) \leq 2g - 3$, then $h^0(\mathscr{E}) \leq f(d) \leq d + (3-g)r - \frac{d+r}{g-2}$.*

For the interval "in the middle", i.e., $(1 + \frac{1}{g-4})r \leq h \leq (g - 2 - \frac{1}{g-4})r$, corresponding to $(2 + \frac{2}{g-4})r \leq d \leq (2g - 4 - \frac{2}{g-4})r$ and $\phi(h) \geq 2h$, this gives

$$S_{h,0} \leq \mu(\mathscr{E}) - \frac{2h}{r}$$

and is an improvement of 2 for the upper bound for $S_{h,0}$, compared with the bound in Corollary 2.14.

5 Examples

Example 5.1. We consider the Hermitian curve $x^{j+1} + y^{j+1} + z^{j+1} = 0$ over $\mathbb{F}_q$, where $q = j^2$. We have $g = \frac{j^2-j}{2}$ and $m = j^3 + 1$, and we see that Equation (3) holds if $j \geq 3$, i.e., $q \geq 9$, or $g \geq 3$. It also holds for $j = 2$, $q = 4$, $g = 1$ for h with $S_{h,0}$ big enough.

(a) In the case $g = 1$, $j = 2$, $q = 4$, we have an elliptic curve, and according to an unpublished PhD thesis by Agnes Tillmann, whose proof is recalled in [AEB92] (see also the proof of Corollary 3.1 of [Nak06]), there exists a canonical semistable vector bundle $\mathscr{E}_{d,r}$ defined over $\mathbb{F}_q$ of degree d and rank r for all integers $d \in \mathbb{Z}$ and $r \geq 1$, and hence in particular for all integers d and r that we are interested in.

We observe that Equation (2) holds for all t in question for a lot of r, e.g. $r \leq 7$. Putting $r = 7$, and $\mu(\mathscr{E}) = 8$, we have $m > \mu(\mathscr{E}) > 2g$ as in Remark 2.2, and we obtain a code C with

$k = h^0(\mathscr{E}) = r\mu(\mathscr{E}) = 56$ in addition to $n = 9\cdot(4^6+\cdots+4+1) = 49149$. We may then use Corollary 3.12 (a) and conclude as in the corollary for $h \geq \max\{(r-2)g+\mu(\mathscr{E})+1+\frac{3(r-1)}{r}, rg+(t+1)(r-1)+1\} = \max\{17, rg+(t+1)(r-1)+1\}$, provided that Equation (3) holds. We can then determine d_h, using Theorem 3.13, for all $h \geq 44$, and for $h = 20, 21, 26, 27, 28, 32, 33, 34, 35, 38, 39, 40, 41, 42$. One can even determine d_{14} in the same manner, using Corollary 3.12 (b) for $i = 3$. For $h \geq 49$, the conclusion of Corollary 3.12 obviously holds (for all q), because then there are codimension h planes entirely contained in fibres of T and therefore in T).

We only need $S_{h,0} \geq 1$ to make Equation (3) hold for these h, and this corresponds to $h \leq k - r = 49$. Hence the conclusion of Theorem 3.13 holds for all h listed, including all $h \geq 44$.

(b) We consider the case $j = 4$, $g = 6$, $q = 16$, $m = j^3 + 1 = 65$, a plane quintic curve. We assume that we have a semi-stable bundle $\mathscr{E}'$ of rank 5 and degree -10 (see Example 5.2 below for a candidate). We now tensor the bundle with $\mathcal{O}_{\mathbb{P}^2}(s)$, to obtain a bundle $\mathscr{E}$ of rank 5 and degree $25s - 10$ and slope $\mu = 5s - 2$. To satisfy $\mu(\mathscr{E}) < m = 65$ (See Observation 2.8), we must have $s \leq 13$. So, for simplicity, we set $s = 13$. We observe that $q = 16 \geq (r-1)(r-2) = 12$, and $q \geq 2g + 4 = 16$, so part (a) of Corollary 3.12 can be applied for $h \geq \max\{85, 35 + 4t\} = 85$ (since $t \leq r - 1 = 4$). In this case, the dimension of the code is $k = \deg(\mathscr{E}) + r(1 - g) = 315 - 25 = 290$. The code length is much bigger: $n = 4543825$. But we may also use part (b) of Corollary 3.12 in the case $i = 4$, since we observe that $q = 16 \geq \frac{4g^2}{i^2} + \frac{2}{i} = \frac{19}{2}$, for $i = 4$. This part of the corollary can be applied for $h \geq \max\{38, 51\} = 51$ (putting $t = r - 1$ in the condition).

To use $i = 4$, we must have $S_{h,0} \geq 4$, and this happens if $h \leq k - 4r = 270$. Hence, the J_h and therefore the d_h are all determined by part (b) of Corollary 3.12 for $51 \leq h \leq 270$ in this case. But since part (a) covers the cases $271 \leq h \leq 290$, then all values of d_h for $h \geq 51$ are determined. Considering individual values of t in part (b) of the corollary furthermore gives us d_h for $h = 39, 40, 43, 44, 45, 47, 48, 49, 50$, hence for all $h \geq 47$.

For $h = 39$, we then get $d_h = 1048574$.

For $h \geq 31$, Corollary 3.3 gives $d \geq 16^4 \cdot (70 - \lfloor \frac{315-h}{5} \rfloor \cdot 5)$. For $h = 39$, this is $16^4 \cdot 15 = 983040$.

For the range $6 \leq h \leq 30$, we combine Proposition 2.7 and Corollary 2.14 and obtain $d_h \geq 65536 \cdot \frac{2h}{5}$. For $h = 15$, this gives $d_h \geq 393216$. In this case, the Clifford index γ of the Hermitian curve X is 1, and if we can trust Conjecture 4.3 and Equation (5), we may improve this by 65536 in the range $10 \leq h \leq 20$, for example to $d_{15} \geq 458752$. Unfortunately, Proposition 4.4 cannot be applied here because of the assumption $\gamma \geq 2$. Corollary 2.10 also gives $d(C) \geq 131072$.

For $1 \leq h \leq 5$, we have $d_h \geq 16^4 \cdot 2 = 131072$, by Proposition 2.9 and Remark 4.1.

(c) In the case $g = 21$, $j = 7$, $q = 49$, $m = j^3 + 1 = 344$, we have a plane octic curve, and we assume that there exists a semistable bundle $\mathscr{E}$ on X of degree -24 and rank 9. A candidate could be the kernel of the surjective bundle map $H^0(\mathcal{O}_X(3)) \otimes \mathcal{O}_X \to \mathcal{O}_X(3)$. We tensor this bundle with $\mathcal{O}_{\mathbb{P}^2}(s)$, to obtain a bundle $\mathscr{E}$ of rank 9 and degree $72s - 24$ and slope $8s - \frac{24}{9}$. To satisfy $\mu(\mathscr{E}) < m = 344$ (See Observation 2.8), we must have $s \leq 43$. We choose $s = 40$. So $\deg(\mathscr{E}) = 2856$, and $\mu(\mathscr{E}) = 317.33 > 2g = 42$, so $\mathcal{L}'$ is very ample. We observe that equation (3) holds, and that $q = 49 \geq \frac{t}{t-1} + \frac{t}{t-2} + \cdots + \frac{t}{2} + t$, for $t = 1, 2, \cdots, r - 1 = 8$, so equation (2) also holds. Hence, part (a) of Corollary 3.12 can be applied for $h \geq \max\{468, 198 + 8t\} = 468$ ($t \leq r - 1$). In this case, the dimension of the code is $k = \deg(\mathscr{E}) + r(1 - g) = 2676$. The code length is much bigger: $n =$ approximately $1.17 \cdot 10^{16}$.

But we may also use part (b) of Corollary 3.12 in the case $i = 7$, since we observe that $q = 49 \geq \frac{4g^2}{i^2} + \frac{2}{i} = 36.29$, for $i = 7$. This part of the corollary can be applied for $h \geq \max\{200, 198 + 8t\}$, which gives us d_h for all $h \geq 257$, and several other values for h between 201 and 256. To use $i = 7$, we must have $S_{h,0} \geq 7$, and this happens if $h \leq k - 7r = 2613$. Hence, the J_h and therefore the d_h are all determined by part (b) of Corollary 3.12 for $257 \leq h \leq 2613$ in this case. But since part (a) covers the cases $2614 \leq h \leq 2676$, then all values of d_h for $h \geq 257$ are determined.

For $190 \leq h \leq 256$, Corollary 3.3 can be applied to give a lower bound for d_h.

For the range $10 \leq h \leq 189$, we combine Proposition 2.7 and Corollary 2.14 and obtain $d_h \geq 49^8(\frac{80}{3} + \frac{2h}{9})$. In this case, the Clifford index γ of the Hermitian curve X is 4, and if we can trust Conjecture 4.3 and Equation (5), we may improve this lower bound for d_h by 4×49^8 compared to this bound from Proposition 2.14 in the range $18 \leq h \leq 144$. Proposition 4.4 gives an improvement of 2×49^8 for $10 \leq h \leq 170$ compared to the bound from Proposition 2.14.

Example 5.2. Consider (as in Example 5.1 (b)) the (plane) Hermitian curve X given by $x^5 + y^5 + z^5 = 0$ over $\mathbb{F}_{16}$, and look at the vector bundle $\mathscr{E}$ given as follows: We let $\mathscr{E}'$ be the kernel of the surjective bundle map $H^0(X, \mathscr{O}(2)) \otimes \mathscr{O}_X \to \mathscr{O}(2)$, so that $\mathrm{rank}(\mathscr{E}') = 5$ and $\deg(\mathscr{E}') = -10$. We then let $\mathscr{E} = \mathscr{E}' \otimes \mathscr{O}(2)$, so that $\deg(\mathscr{E}) = 40$.

The vector bundle $\mathscr{E}$ has the following generators:

$$x^2e_1, \quad x^2e_2, \quad x^2e_3, \quad x^2e_4, \quad x^2e_5, \quad \text{for} \quad x \neq 0,$$

$$x^2e_3, \quad xye_3 - y^2e_1, \quad xze_3 - y^2e_2, \quad yze_3 - y^2e_4, \quad z^2e_3 - y^2e_5, \quad \text{for} \quad y \neq 0.$$

We have $h^0(X, \mathscr{E}) = 21$ (this has to be shown using the definition of $\mathscr{E}$, since we don't have $\mu(\mathscr{E}) > 2g - 2$), and so the tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ has 21 global sections. We can't see from the degree that this line bundle is very ample, since we don't have $\mu(\mathscr{E}) > 2g$, but this can be shown directly by regarding the global sections. We can choose 21 generators for the global sections of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$, and the polynomials corresponding to the zero sets of these can then act as the coordinates of $\mathbb{P}^{20}$, which we embed $\mathbb{P}(\mathscr{E})$ into. The global sections of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ correspond to the zero sets of the following polynomials on $\mathbb{P}(\mathscr{E})$:

$$\begin{array}{lll} t_0 = x^2e_1 & t_3 = x^2e_2 & t_6 = x(xe_3 - ye_1) \\ t_1 = xye_1 & t_4 = xye_2 & t_7 = y(xe_3 - ye_1) \\ t_2 = xze_1 & t_5 = xze_2 & t_8 = z(xe_3 - ye_1) \end{array}$$

$$\begin{array}{ll} t_9 = x(ze_1 - xe_4) & t_{12} = x(xe_5 - ze_2) \\ t_{10} = y(ze_1 - xe_4) & t_{13} = y(xe_5 - ze_2) \\ t_{11} = z(ze_1 - xe_4) & t_{14} = z(xe_5 - ze_2) \end{array}$$

$$\begin{array}{lll} t_{15} = y(xe_4 - ye_2) & t_{17} = y(ze_4 - ye_5) & t_{19} = y(ze_3 - ye_4) \\ t_{16} = z(xe_4 - ye_2) & t_{18} = z(ze_4 - ye_5) & t_{20} = z(ze_3 - ye_4) \end{array}$$

Since the curve X is given by $x^5 + y^5 + z^5 = 0$, the embedded scroll then has the equations

$$\begin{array}{l} t_0^5 + t_1^5 + t_2^5 = 0 \\ t_3^5 + t_4^5 + t_5^5 = 0 \\ t_6^5 + t_7^5 + t_8^5 = 0 \\ t_9^5 + t_{10}^5 + t_{11}^5 = 0 \\ t_{12}^5 + t_{13}^5 + t_{14}^5 = 0, \end{array}$$

in addition to the zero set of the 2×2 minors of the matrices

$$\begin{pmatrix} t_0 & t_3 & t_6 & t_9 & t_{12} \\ t_1 & t_4 & t_7 & t_{10} & t_{13} \\ t_2 & t_5 & t_8 & t_{11} & t_{14} \end{pmatrix}$$

and

$$\begin{pmatrix} t_1 & t_4 & t_7 & t_{10} & t_{13} & t_{15} & t_{17} & t_{19} \\ t_2 & t_5 & t_8 & t_{11} & t_{14} & t_{16} & t_{18} & t_{20} \end{pmatrix}.$$

We now try to say something about the minimum distance: Consider the hyperplane $t_{10} = 0$. This contains all fibres over the points on X where $y = 0$. It can be checked that $y = 0$ for 5 distinct $\mathbb{F}_q$-rational points on X. We see that $S_{h,0} \geq 5$ for $h = 1, \ldots, 8$, since there are eight linearly independent hyperplanes that contain the fibres over the points corresponding to $y = 0$ on X, namely $t_1, t_4, t_7, t_{10}, t_{13}, t_{15}, t_{17}, t_{19}$. So there exists a $\mathbb{P}^{12}$ that contains 5 fibres. If we add a sixth fibre, which is a $\mathbb{P}^4$, then these altogether 6 fibres must be contained in a $\mathbb{P}^{17}$. It follows that $S_{3,0} \geq 6$, and therefore also $S_{1,0} \geq 6$, and $d_1 \leq 16^4 \cdot (65 - 6) = 3866624$. We also have

$S_{1,0} \leq \mu(\mathscr{E})$. Hence, $6 \leq S_{1,0} \leq 8 = \mu(\mathscr{E})$, and $3735552 \leq d_1 \leq 3866624$. The length of the scroll code is 4543825.

Since a fibre of T is a $\mathbb{P}^4$, it is clear that $S_{h,0} = 0$ for $h \geq 17$, and $S_{16,0} = 1$. The bounds using Conjecture 4.3 give $S_{h,0} \leq 9 - \frac{2h}{5}$ for $5 \leq h \leq 20$. This gives $S_{h,0} \leq 2, 2, 1, 1, 1$ for $h = 16, 17, 18, 19, 20$, respectively, so it is not always a sharp bound. It does however give $S_{8,0} \leq 5$, so if Conjecture 4.3 holds, we may conclude that $S_{8,0} = 5$.

References

[AEB92] M. Arason, J.K. and Elman, R. and Bill, J., *On indecomposable vector bundles*, Communications in Algebra, 20 (5), pp. 1323–1351, (1992).

[Bal98] E. Ballico, *Brill–Noether theory for vector bundles on projective curves*, Math. Proc. Camb. Phil. Soc. (1998).

[BGN97] L. Brambila-Paz, I. Grzegorczyk, P. E. Newstead, *Geography of Brill–Noether loci for small slopes*, J. Alg. Geom., 6 , pp. 645–669 (1997).

[Han01] S. H. Hansen, *Error-Correcting Codes from Higher-Dimensional Varieties*, Finite Fields Appl. **7**, 530–552 (2001).

[HaJ07] G.M. Hana, T. Johnsen, *Scroll codes*, Designs, Codes and Cryptograhy, 45 (3), pp. 365–377 (2007).

[Lom03] C. C. Lomont, *Error-correcting Codes on Algebraic Surfaces*, math.NT/0309123, (2003).

[Mer02] V. Mercat, *Clifford's theorem and higher rank bundles*, International Journal of Mathematics, 13 (7), pp. 785–796 (2002).

[Nak06] T. Nakashima, *Error-Correcting Codes on Projective Bundles*, Finite Fields Appl. **12**, 222–231 (2006).

[Re98] R. Re, *Multiplication of sections and Clifford bounds for stable vector bundles on curves*, Comm. in Algebra, 26(6) , pp. 1931–1944 (1998).

[Ste99] S. Stepanov, *Codes on Algebraic Curves*, Kluwer Academic/Plenum Publishers, New York, 36 (1999).