Injective Braids, Braided Operads and Double Loop Spaces

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Introduction

In topological spaces, a double loop space is the space of based maps from the two-sphere into a space with basepoint. If we identify all the points in the equator of a two-sphere we get a wedge of two two-spheres. We can use this to define a multiplication on a double loop space, making it a commutative monoid in the homotopy category. Given a double loop space, we would like to find a commutative monoid that is weakly equivalent to the double loop space. This is not always possible.

For infinite loop spaces the situation is similar. They are homotopy commutative monoids, but not necessarily weakly equivalent to a commutative monoid. In [SS11], Sagave and Schlichtkrull construct the symmetric monoidal diagram category $\mathcal{S}^{\mathcal{I}}$ of $\mathcal{I}$-spaces, which is Quillen equivalent to topological spaces. In this category every connected E_∞-space, that is, a connected space with structure corresponding to that of an infinite loop space, is weakly equivalent to a commutative monoid. There is also a chain of Quillen equivalences from the category of commutative monoids in $\mathcal{S}^{\mathcal{I}}$ to the category of infinite loop spaces in topological spaces. This means that every infinite loop space can be represented by a commutative monoid in $\mathcal{S}^{\mathcal{I}}$.

In this thesis we construct a braided monoidal diagram category $\mathcal{S}^{\mathfrak{B}}$ which is Quillen equivalent to the category of simplicial sets, $\mathcal{S}$. The induced equivalence on homotopy categories maps a commutative monoid in $\mathcal{S}^{\mathfrak{B}}$ to a space that is weakly equivalent to a double loop space, if it is connected.

This thesis is organized as follows.

In Chapter 1 we recall some definitions and results about braid groups and braided monoidal categories.

In Chapter 2 we define the category $\mathfrak{B}$ of injective braids. We give a topological definition of the injective braids inspired by the topological definition of the braid groups. Finally we define a braided strict monoidal structure on $\mathfrak{B}$.

We begin Chapter 3 with some examples of $\mathfrak{B}$-spaces, i.e. objects in $\mathcal{S}^{\mathfrak{B}}$. Then we show in general that if $\mathcal{A}$ is a small braided monoidal category, the diagram category $\mathcal{S}^{\mathcal{A}}$ inherits a braided monoidal structure. The monoidal product of two objects is defined as a left Kan extension. We use the functoriality of the left Kan extension, proved in the appendix, to define the rest of the structure and prove that this is in fact a braided monoidal structure. In the last section of Chapter 3 we give $\mathcal{S}^{\mathfrak{B}}$ a projective model structure. Here a morphism is a weak equivalence if the induced map on homotopy colimits is a weak equivalence. With this model structure the adjunction, $\mathrm{colim}_{\mathfrak{B}}\colon \mathcal{S}^{\mathfrak{B}} \rightleftarrows \mathcal{S} : \mathrm{const}$, is a Quillen equivalence.

In Chapter 4 we follow [Fie] in defining the concept of braided operads. A braided operad is a sequence of spaces with the same structure as an operad, except that there is an action of the braid groups instead of the symmetric groups. A B_∞ operad is a braided operad where each space is contractible and the action of the braid groups are free. We construct a braided operad such that the associated monad is the same as the monad associated to the little rectangles operad. This implies that a connected B_∞-space is weakly equivalent to a double loop space. We also construct a braided analog of the Barratt-Eccles operad and show that it acts on the nerve of a braided strict monoidal category. All of this can be found in [Fie], but in this reference there are few details in the definitions and constructions and no details in the proofs of the results. Therefore we construct and prove everything carefully here. Then we show that the braided analog of the Barratt-Eccles operad acts on the homotopy colimit of a commutative monoid.

In Chapter 5 we show that the two $\mathfrak{B}$-spaces $X^\bullet$ and $\mathcal{N}\mathcal{A}_\bullet$, constructed in Chapter 3, are in fact commutative monoids. Then we show that taking homotopy colimits yields spaces that are weakly equivalent to $\Omega^2\Sigma^2 X$ and $\mathcal{N}\mathcal{A}$ respectively. In the latter case we construct a chain of weak equivalences where each map is a morphism of $\mathcal{N}\mathcal{B}$-spaces. In conclusion we have represented the B_∞-spaces $\Omega^2\Sigma^2 X$ and $\mathcal{N}\mathcal{A}$ as commutative $\mathfrak{B}$-space monoids.

The Appendix A is devoted to studying the functoriality of the left Kan extension.

Chapter 1

Preliminaries

1.1 Braid groups

In this section we define the braid groups, and review some results about these from Section 1 in [Bir74].

Definition 1.1.1 (Page 5 in [Bir74])**.** Let

$$F_{0,n}\mathbb{R}^2 = \{(z_1, \dots z_n) \,|\, z_i \in \mathbb{R}^2 \text{ and } z_i \neq z_j \text{ if } i \neq j \text{ for } i = 1, \dots, n\},$$

we call the fundamental group $\pi_1(F_{0,n}, ((1,0), \dots, (n,0)))$ the pure braid group on n strings, denoted by $\mathcal{P}_n$.

Let

$$B_{0,n}\mathbb{R}^2 = F_{0,n}\mathbb{R}^2/(z_1, \dots z_n) \sim (z_{\sigma^{-1}(1)}, \dots z_{\sigma^{-1}(n)})$$

for any permutation σ of the set $\{1, \dots, n\}$. We define the braid group on n strings, $\mathcal{B}_n$ to be the fundamental group $\pi_1(B_{0,n}, [(1,0), \dots, (n,0)])$.

Remark 1.1.2 (Geometric interpretation of braids, page 5-6 in [Bir74]). Any element in $\pi_1(B_{0,n}, [(1,0), \dots, (n,0)])$ is represented by a loop

$$f\colon ([0,1], 0, 1) \to (B_{0,n}, [(1,0), \dots, (n,0)])$$

which lifts uniquely to a path

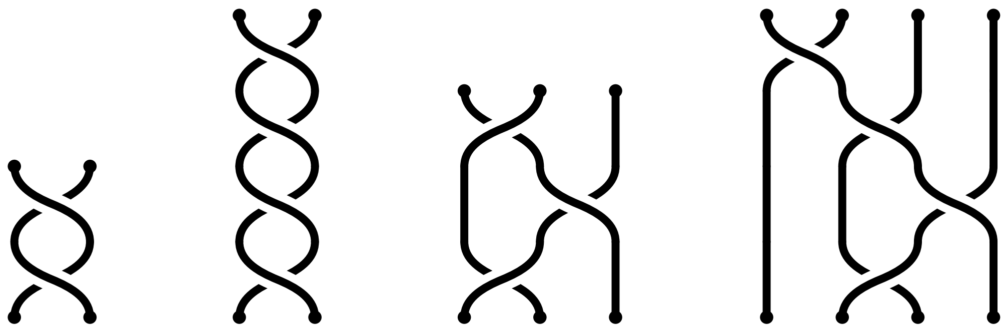
$$f\colon ([0,1], 0, 1) \to (F_{0,n}, [(1,0), \dots, (n,0)]).$$

If $f(t) = (f_1(t), \dots, f_n(t))$ for $0 \leq t \leq 1$, then each of the coordinate functions f_i defines an arc

$$a_i = \{(f_i(t), t) \,|\, 0 \leq t \leq 1\}$$

in $\mathbb{R}^2 \times [0,1]$. Since $f(t)$ is in $F_{0,n}\mathbb{R}^2$, $f_i(t) \neq f_j(t)$ if $i \neq j$, and so the arcs $a_1, \dots a_n$ are disjoint. Their union $a = a_1 \cup \dots \cup a_n$ is called a geometric braid. The arc a_i is called the ith string in the braid.

From this geometric interpretation we can illustrate braids by picture like these:



Proposition 1.1.3 (Proposition 1.1 in [Bir74])**.** *The natural projection*

$$F_{0,n}\mathbb{R}^2 \to B_{0,n}\mathbb{R}^2$$

is a regular covering map. The group of covering transformations is the symmetric group Σ_n on n letters. Therefore we get a short exact sequence

$$\mathcal{P}_n \to \mathcal{B}_n \xrightarrow{\Phi_n} \Sigma_n.$$

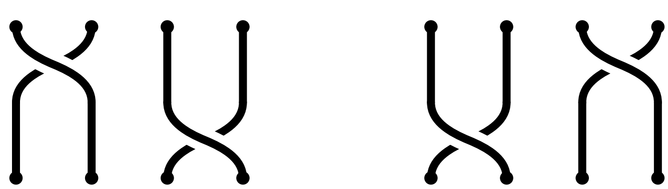
Proposition 1.1.4 (Theorem 1.8 in [Bir74])**.** *The braid group $\mathcal{B}_n$ admits a presentation with generators $\zeta_1, \ldots, \zeta_{n-1}$ and defining relations*

$$\begin{aligned} \zeta_i\zeta_j &= \zeta_j\zeta_i \qquad \text{if } |i-j| \geq 2,\ 1 \leq i,j \leq n-1 \\ \zeta_i\zeta_{i+1}\zeta_i &= \zeta_{i+1}\zeta_i\zeta_{i+1} \qquad 1 \leq i \leq n-2. \end{aligned}$$

In the geometric interpretation, Remark 1.1.2, the generators yield pictures like these:

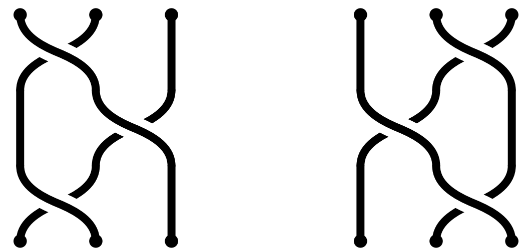
$\zeta_{\mathbf{i}}$ $\qquad$ $\zeta_{\mathbf{i}}^{-\mathbf{1}}$

We say that ζ_i braids the ith string over string $i+1$, and ζ_i^{-1} braids the ith string under string $i+1$. An illustration of the first type of relation in Propostion 1.1.4:



$\zeta_{\mathbf{3}}\zeta_{\mathbf{1}}$ $\qquad$ $\zeta_{\mathbf{1}}\zeta_{\mathbf{3}}$

An illustration of the second type of relation in Propostion 1.1.4:



$\zeta_1\zeta_2\zeta_1$ $\zeta_2\zeta_1\zeta_2$

1.2 Braided monoidal categories

Here we state the definition of a braided monoidal category and other related definitons for easy reference.

Definition 1.2.1 (From Section 1 in [JS93])**.** A monoidal category, $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r})$, consists of a category $\mathcal{A}$ together with a functor $\oplus\colon\ \mathcal{A}\times\mathcal{A}\to\mathcal{A}$, called the monoidal product, an object I, called the unit object, and natural isomorphisms

$$\mathsf{a} = \mathsf{a}_{a,b,c}\colon\ (a\oplus b)\oplus c \to a\oplus(b\oplus c),$$

$$\mathsf{l} = \mathsf{l}_a\colon\ I\oplus a\to a, \qquad \mathsf{r} = \mathsf{r}_a\colon\ a\oplus I\to a$$

called the associativity, left unit and right unit constraints respectively, such that, for all objects a, b, c, d in $\mathcal{A}$, the following two diagrams commute:

$$\begin{array}{ccccc} & & (a\oplus(b\oplus c))\oplus d & \xrightarrow{\mathsf{a}} & a\oplus((b\oplus c)\oplus d) \\ & \nearrow^{\mathsf{a}\oplus\mathrm{id}} & & & \downarrow{\scriptstyle \mathrm{id}\oplus\mathsf{a}} \\ ((a\oplus b)\oplus c)\oplus d & & & & \\ & \searrow_{\mathsf{a}} & & & \\ & & (a\oplus b)\oplus(c\oplus d) & \xrightarrow{\mathsf{a}} & a\oplus(b\oplus(c\oplus d)) \end{array} \tag{1.1}$$

$$\begin{array}{ccc} (a\oplus I)\oplus b & \xrightarrow{\mathsf{a}} & a\oplus(I\oplus b) \\ & {}_{\mathsf{r}\oplus\mathrm{id}}\searrow \quad \swarrow_{\mathrm{id}\oplus\mathsf{l}} & \\ & a\oplus b & . \end{array} \tag{1.2}$$

A monoidal category is called strict when all the constraints a, l and r are idetity morphisms.

We sometimes write: "Let $\mathcal{A}$ be a monoidal category", then the rest of the structure is implicit.

Definition 1.2.2 (From Section 1 in [JS93])**.** Let $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r})$ and $(\mathcal{A}', \oplus', I', \mathsf{a}', \mathsf{l}', \mathsf{r}')$ be monoidal categories. A lax/strong/strict monoidal functor (F, φ, ϑ) from the former monoidal category to the latter consists of a functor $F\colon \mathcal{A} \to \mathcal{A}'$, a family of natural morphisms/isomorphisms/identity morphisms:

$$\varphi_{a,b}\colon\ F(a) \oplus' F(b) \to F(a \oplus b),$$

and a morphism/isomorphism/identity morphism $\vartheta\colon\ I' \to F(I)$, such that the following diagrams commute:

$$\begin{tikzcd}[column sep=small]
 & F(a) \oplus' (F(b) \oplus' F(c)) \arrow[dr, "\mathrm{id}\oplus'\varphi"] & \\
(F(a) \oplus' F(b)) \oplus' F(c) \arrow[ur, "\mathsf{a}'"] \arrow[d, "\varphi\oplus'\mathrm{id}"'] & & F(a) \oplus' F(b \oplus c) \arrow[d, "\varphi"] \\
F(a \oplus b) \oplus' F(c) \arrow[dr, "\varphi"'] & & F(a \oplus (b \oplus c)) \\
 & F((a \oplus b) \oplus c) \arrow[ur, "F(\mathsf{a})"] &
\end{tikzcd} \tag{1.3}$$

$$\begin{tikzcd}
F(a) \oplus' I' \arrow[r, "\mathsf{r}'"] \arrow[d, "\mathrm{id}\oplus'\vartheta"'] & F(a) \\
F(a) \oplus' F(I) \arrow[r, "\varphi"] & F(a \oplus I) \arrow[u, "F(\mathsf{r})"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I' \oplus' F(a) \arrow[r, "\mathsf{l}'"] \arrow[d, "\vartheta\oplus'\mathrm{id}"'] & F(a) \\
F(I) \oplus' F(a) \arrow[r, "\varphi"] & F(I \oplus a). \arrow[u, "F(\mathsf{l})"']
\end{tikzcd} \tag{1.4}$$

Definition 1.2.3 (Definitions 2.1 and 2.2 in [JS93])**.** A braided monoidal category, $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \mathsf{b})$, consists of a monoidal category $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r})$ and a family of natural isomorphisms

$$\mathsf{b} = \mathsf{b}_{a,b}\colon\ a \oplus b \to b \oplus a$$

in $\mathcal{A}$, called the braiding, such that the two following diagrams commute:

$$\begin{tikzcd}
 & (b \oplus a) \oplus c \arrow[r, "\mathsf{a}"] & b \oplus (a \oplus c) \arrow[dr, "\mathrm{id}\oplus\mathsf{b}"] & \\
(a \oplus b) \oplus c \arrow[ur, "\mathsf{b}\oplus\mathrm{id}"] \arrow[dr, "\mathsf{a}"'] & & & b \oplus (c \oplus a) \\
 & a \oplus (b \oplus c) \arrow[r, "\mathsf{b}"'] & (b \oplus c) \oplus a \arrow[ur, "\mathsf{a}"'] &
\end{tikzcd} \tag{1.5}$$

$$\begin{tikzcd}
 & a \oplus (c \oplus b) \arrow[r, "\mathsf{a}^{-1}"] & (a \oplus c) \oplus b \arrow[dr, "\mathsf{b}\oplus 1"] & \\
a \oplus (b \oplus c) \arrow[ur, "\mathrm{id}\oplus\mathsf{b}"] \arrow[dr, "\mathsf{a}^{-1}"'] & & & (c \oplus a) \oplus b \\
 & (a \oplus b) \oplus c \arrow[r, "\mathsf{b}"'] & c \oplus (a \oplus b) \arrow[ur, "\mathsf{a}^{-1}"'] &
\end{tikzcd} \tag{1.6}$$

A braided strict monoidal category is a braided monoidal category such that the moinidal structure is strict.

Definition 1.2.4 (Definitions 2.1 and 2.2 in [JS93])**.** A symmetric monoidal category is a braided monoidal category $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \mathsf{b})$ such that

$$\mathsf{b}_{b,a} \circ \mathsf{b}_{a,b} = \mathrm{id}$$

for all $a, b \in \mathcal{A}$.

Definition 1.2.5 (Definition 2.3 in [JS93])**.** A lax/strong/strict monoidal functor

$$(F, \varphi, \vartheta)\colon\ (\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \mathsf{b}) \to (\mathcal{A}', \oplus', I', \mathsf{a}', \mathsf{l}', \mathsf{r}', \mathsf{b}')$$

between braided monoidal categories is called braided when the following diagram commutes:

$$\begin{array}{ccc} F(a) \oplus' F(b) & \xrightarrow{\varphi} & F(a \oplus b) \\ {\scriptstyle \mathsf{b}'}\big\downarrow & & \big\downarrow{\scriptstyle F(\mathsf{b})} \\ F(b) \oplus' F(a) & \xrightarrow{\varphi} & F(b \oplus a) \end{array} \tag{1.7}$$

for all a, b in $\mathcal{A}$.

Proposition 1.2.6 (Braided coherence)**.** *Let $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \mathsf{b})$ be a braided monoidal category. Let X be an n-fold monoidal product of objects $X_1, \ldots, X_n$ in $\mathcal{A}$. Let X' be an n-fold monoidal product of a permutation of the objects $X_1, \ldots, X_n$. Suppose we have two morphisms $f_1, f_2\colon\ X \to X'$ where*

$$f_i = (\mathrm{id} \oplus g^i_{j_i} \oplus \mathrm{id}) \circ \cdots \circ (\mathrm{id} \oplus g^i_1 \oplus \mathrm{id})$$

for $i = 1, 2$ and some j_1 and j_2, where each g^i_k is either a or b. Ignoring the associativity morphisms, f_i induces a braid ξ_i on n-strings. If $\xi_1 = \xi_2$ then $f_1 = f_2$.

Proof. This is a direct consequence of Theorem 2 in Section XI.5 [ML98]. □

Lemma 1.2.7. *Let $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \mathsf{b})$ be a braided monoidal category. For every object a in $\mathcal{A}$ the equations*

$$\mathsf{b}_{I,a} \circ \mathsf{b}_{a,I} = \mathrm{id} \qquad \mathsf{b}_{a,I} \circ \mathsf{b}_{I,a} = \mathrm{id}$$

hold.

Proof. It is proved in Proposition 2.1 in [JS93] that $\mathsf{l} \circ \mathsf{b}_{a,I} = \mathsf{r}$ and $\mathsf{r} \circ \mathsf{b}_{I,a} = \mathsf{l}$. Combining them yields $\mathsf{r} \circ \mathsf{b}_{I,a} \circ \mathsf{b}_{a,I} = \mathsf{r}$ and since r is an isomorphism, the first equation follows. The seqond equation follws from the first. □

Lemma 1.2.8. *Let $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r})$ be a small braided strict monoidal category. Let the braids $\xi_1, \ldots, \xi_{m-1}$ be the generators for the braid group $\mathcal{B}_m$, see Proposition 1.1.4. Given an n-tuple $(a_1, \ldots, a_m)$ of objects in $\mathcal{A}$, let ξ_i also denote the morphism*

$$\mathrm{id} \oplus \mathsf{b}_{a_i, a_{i+1}} \oplus \mathrm{id} \colon\ a_1 \oplus \cdots \oplus a_i \oplus a_{i+1} \oplus \cdots \oplus a_m \to a_1 \oplus \cdots \oplus a_{i+1} \oplus a_i \oplus \cdots \oplus a_m.$$

Similarly let ξ_i^{-1} denote the morphism

$$\mathrm{id} \oplus \mathsf{b}^{-1}_{a_{i+1}, a_i} \oplus \mathrm{id} \colon\ a_1 \oplus \cdots \oplus a_i \oplus a_{i+1} \oplus \cdots \oplus a_m \to a_1 \oplus \cdots \oplus a_{i+1} \oplus a_i \oplus \cdots \oplus a_m.$$

Let ξ be a braid on m strings, write ξ as a product $\zeta_n \cdots \zeta_1$ where each ζ_i is either a generator or the inverse of a generator. This induces a morphism

$$\zeta_n \cdots \zeta_1 \colon\ a_1 \oplus \cdots\cdots \oplus a_m \to a_{\Phi(\xi)^{-1}(1)} \oplus \cdots \oplus a_{\Phi(\xi)^{-1}(m)}.$$

This morphism only depends on ξ and not on how ξ is factored.

Proof. It is only the last statement that requires a proof, but this is a consequence of braided coherence, see Lemma 1.2.6. □

Definition 1.2.9 (From Section VII.3 in [ML98]). A monoid in a monoidal category $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r})$ is an object a together with two arrows $\mu\colon\ a \oplus a \to a$ and $\eta\colon\ I \to a$ such that the diagrams

$$\begin{array}{ccccc} a \oplus (a \oplus a) & \xrightarrow{\mathsf{a}} & (a \oplus a) \oplus a & \xrightarrow{\mu \otimes \mathrm{id}} & a \oplus a \\ \downarrow{\scriptstyle \mathrm{id} \oplus \mu} & & & & \downarrow{\scriptstyle \mu} \\ a \oplus a & & \xrightarrow{\quad\quad \mu \quad\quad} & & a \end{array} \tag{1.8}$$

$$\begin{array}{ccccc} I \oplus a & \xrightarrow{\eta \oplus \mathrm{id}} & a \oplus a & \xleftarrow{\mathrm{id} \oplus \eta} & a \oplus I \\ & {\scriptstyle \mathsf{l}} \searrow & \downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle \mathsf{r}} & \\ & & a & & \end{array} \tag{1.9}$$

are commutative.

A morphism of monoids $f\colon\ (a, \mu, \eta) \to (a', \mu', \eta')$ is a morphism $f\colon\ a \to a'$ in $\mathcal{A}$ such that

$$f \circ \mu = \mu' \circ (f \oplus f)\colon\ a \oplus a \to a' \qquad f \circ \eta = \eta'\colon\ I \to a'.$$

Definition 1.2.10 (Commutative monoid). A commutative monoid in a braided commutative category $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \mathsf{b})$ is a monoid (a, μ, η) such that

$$\mu \circ \mathsf{b}_{a,a} = \mu\colon\ a \oplus a \to a.$$

We finish this section with two categorical definitions.

Definition 1.2.11 (From Section II.6 [ML98])**.** Given functors

$$T\colon\ \mathcal{E} \to \mathcal{C} \leftarrow \mathcal{D} : S$$

the comma category $(T \downarrow S)$ has as objects all triples (e, d, f) with e an object in $\mathcal{E}$, d an object in $\mathcal{D}$ and $f\colon\ T(e) \to S(d)$ a morphism in $\mathcal{C}$. The comma category has as morphism $(e, d, f) \to (e', d', f')$ all pairs (k, h) of arrows $k\colon\ e \to e'$, $h\colon\ d \to d'$ such that the diagram

$$\begin{array}{ccc} T(e) & \xrightarrow{T(k)} & T(e') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S(d) & \xrightarrow{S(h)} & S(d') \end{array}$$

commutes. The composite of two composable morphisms (k, h) and (k', h') is $(k' \circ k, h' \circ h)$.

For an object c in $\mathcal{C}$, we can view c as a functor from a category with a unique object and only the identity morphism. Objects in $(c \downarrow S)$ are then pairs (d, f), $d \in \mathcal{D}$ and $f\colon\ c \to S(d)$. A morphims from (d, f) to (d', f') is a morphism $h\colon\ d \to d'$ such that $S(h) \circ f = f'$. Similarly we get the comma category $(T \downarrow c)$.

Definition 1.2.12 (Example 1.4 in Chapter I [GJ99])**.** Let $\mathcal{E}$ be a small category. The nerve, $\mathcal{N}\mathcal{E}$, of $\mathcal{E}$ is the simplicial set with

$$(\mathcal{N}\mathcal{E})_k = \{e_o \xrightarrow{f_1} e_1 \xrightarrow{f_2} \ldots \xrightarrow{f_k} e_k \,|\, \text{for composible morphisms } f_i \text{ in } \mathcal{E}\}.$$

The simplicial structure maps are defined by

$$d_i(e_0 \to \ldots \to e_k) = (e_0 \to \ldots \to e_{i-1} \to e_{i+1} \to \ldots \to e_k) \text{ for } 0 < i < k,$$

$$d_0(e_0 \to \ldots \to e_k) = (e_1 \to \ldots \to e_k),$$
$$d_k(e_0 \to \ldots \to e_k) = (e_0 \to \ldots \to e_{k-1})$$

and

$$s_i(e_0 \to \ldots \to e_k) = (e_0 \to \ldots \to e_i = e_i \to \ldots \to e_k) \text{ for } 0 \le i \le k.$$

A functor $T\colon\ \mathcal{E} \to \mathcal{D}$ induce a simplicial set map on the nerves of the categories by

$$\mathcal{N}(T)_k(e_o \xrightarrow{f_1} e_1 \xrightarrow{f_2} \ldots \xrightarrow{f_k} e_k) = (T(e_o) \xrightarrow{T(f_1)} T(e_1) \xrightarrow{T(f_2)} \ldots \xrightarrow{T(f_k)} T(e_k)).$$

It is easy to check that the nerve is a functor from the category of small categories to simplicial sets.

Chapter 2

The category of injective braids $\mathfrak{B}$

2.1 Injective braids

In this section we will define the concept of injective braids, and a category $\mathfrak{B}$ where the morphisms are the injective braids.

Very informally we can say that injective braids relates to injective functions the same way as braids relates to permutations. Where braids can be illustrated by pictures like in Figure 2.1, the pictures in Figure 2.2 illustrate injective braids.

Figure 2.1: Two braids with underlying permutation $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 3$.

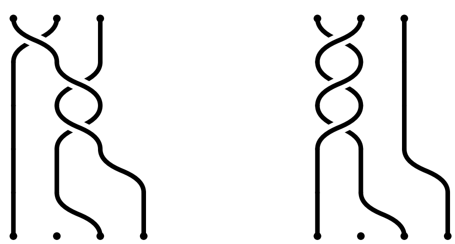


Figure 2.2: Two injective braids with underlying injective map $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 4$.

We can generalise the topological definition of the braid groups 1.1.1 to a definition of injective braids. In this definition a braid on n strings is a homotopy class of an n-tuple of paths in $\mathbb{R}^2$ such that the endpoints of the paths is a permutation of the starting points, and sucht that at any given time none of the paths intersect.

Definition 2.1.1. Let m and n be integers greater than or equal to 0. For $m > 0$, an injective braid α on m strings into n points is a homotopy class of an m-tuple $(\alpha_1, \dots, \alpha_m)$ of paths

$$\alpha_i\colon\ I = [0,1] \to \mathbb{R}^2$$

in $\mathbb{R}^2$, satisfying the following: Each α_i starts in $(i,0)$ and ends in one of the points $\{(1,0), ..., (n,0)\}$. We also require that $\alpha_i(t)$ is not equal to $\alpha_j(t)$ when $i \neq j$ for all t in I.

Two m-tuples $(\alpha_1, \dots, \alpha_m)$ and $(\beta_1, \dots, \beta_m)$ are homotopic if there exists an m-tuple of homotopies

$$H_i\colon\ I \times I \to \mathbb{R}^2$$

from α_i to β_i, fixing endpoints, such that $H_i(s,t) \neq H_j(s,t)$ for $i \neq j$ and all (s,t) in $I \times I$. It is easily seen that this is an equivalence relation.

For $m = 0$ we say that there is one, and only one, injective braid on 0 strings into n points for each $n \geq 0$.

Note that the condition that $\alpha_i(t) \neq \alpha_j(t)$ when $i \neq j$ implies that the α_i's have different endpoints, so n is nessesarily greater than or equal to m for an injective braid on m strings into n points. When $m = n$, the definition of an injective braid coincides with the topological definition of a braid.

Definition 2.1.2. An injective braid α on m strings into n points defines an injective map $\Phi(\alpha)\colon\ \{1, \dots, m\} \to \{1, \dots, n\}$ by choosing a representative $(\alpha_1, \dots, \alpha_m)$ and letting $\Phi(\alpha)(i) = \mathrm{pr}_1 \alpha_i(1)$. This is well defined since the homotopies fix endpoints. When m is 0, the map $\Phi(\alpha)$ is the inclusion of the empty set.

Definition 2.1.3. Let $\mathfrak{B}$ be the category with objects the sets $\mathbf{n} = \{1, \dots, n\}$ for each natural number n and $\mathbf{0} = \emptyset$, and with morphisms from $\mathbf{m}$ to $\mathbf{n}$ the set of injective braids on m strings into n points.

We compose a morphism $\alpha\colon\ \mathbf{k} \to \mathbf{m}$ with a morphism $\beta\colon\ \mathbf{m} \to \mathbf{n}$ by choosing representatives $(\alpha_1, \dots, \alpha_k)$ and $(\beta_1, \dots, \beta_m)$, and define the composite $\beta \circ \alpha$ to be the homotopy class of

$$(\beta_{\Phi(\alpha)(1)} \cdot \alpha_1, ..., \beta_{\Phi(\alpha)(k)} \cdot \alpha_k).$$

The product path $\beta_{\Phi(\alpha)(i)} \cdot \alpha_i$ is defined by

$$t \mapsto \begin{cases} \alpha_i(2t) & t \in [0, \frac{1}{2}] \\ \beta_{\Phi(\alpha)(i)}(2t-1) & t \in [\frac{1}{2}, 1] \end{cases} .$$

It is easy to check that this is well defined. When $k = 0$, the composite $\beta \circ \alpha$ is the unique injective braid on 0 strings into n points.

There is a unique morphism from 0 to each object in $\mathfrak{B}$, so $\mathbf{0}$ is an initial object in the category. For n less than m there are no morphisms from $\mathbf{m}$ to $\mathbf{n}$, and when $n = m$, the set of endomorphisms of $\mathbf{n}$ is the braid group on n strings.

Definition 2.1.4. We define categories $\mathcal{B}$, Σ, $\mathcal{I}$ and $\mathcal{M}$ with the same objects as $\mathfrak{B}$ and where

- $\star$ $\mathcal{B}$ has braids as morphisms,
- $\star$ Σ has permutations as morphisms,
- $\star$ $\mathcal{I}$ has injective functions as morphisms and
- $\star$ $\mathcal{M}$ has injective order preserving functions as morphisms.

We define a functor $\Phi\colon \mathfrak{B} \to \mathcal{I}$ by $\Phi(\mathbf{m}) = \mathbf{m}$ and $\Phi(\alpha)$ the injective function defined in Definition 2.1.2. This restricts to a functor $\mathcal{B} \to \Sigma$ which we also denote by Φ.

For an injective order preserving function $\mu\colon \mathbf{m} \to \mathbf{n}$ let $(\mu_1, \dots, \mu_m)$ be defined by $\mu_i(t) = (i, 0)(1-t) + (\mu(i), 0)t$ for $0 \le t \le 1$. Since μ is order preserving, $\mu_i(t) \ne \mu_j(t)$ for $i \ne j$ and for all t in I. Therefore $(\mu_1, \dots, \mu_m)$ represents an injective braid which we denote $\Upsilon(\mu)$. If ν is an order preserving function from $\mathbf{n}$ to $\mathbf{p}$ it is easy to see that $\Upsilon(\nu) \circ \Upsilon(\mu) = \Upsilon(\nu \circ \mu)$. This together with $\Upsilon(\mathbf{m}) = \mathbf{m}$ determines a functor $\Upsilon\colon \mathcal{M} \to \mathfrak{B}$.

Lemma 2.1.5. *The funtors from the previous definition fit into the commutative diagram:*

$$\begin{array}{ccccc} \mathcal{B} & \subseteq & \mathfrak{B} & \overset{\Upsilon}{\nwarrow} & \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} & & \\ \Sigma & \subseteq & \mathcal{I} & \supseteq & \mathcal{M}. \end{array} \tag{2.1}$$

Proof. Follows directly from the definitions. □

Lemma 2.1.6. *Every morphism α in $\hom_{\mathfrak{B}}(\mathbf{m}, \mathbf{n})$ has a unique decompostion as a pair (μ, ζ) where $\mu \in \hom_{\mathcal{M}}(\mathbf{m}, \mathbf{n})$ and $\zeta \in \mathcal{B}_m$ such that $\alpha = \Upsilon(\mu) \circ \zeta$.*

Proof. Let $j_1, \dots, j_m$ be integers between 1 and m such that $\Phi(\alpha)(j_1) < \dots < \Phi(\alpha)(j_m)$. It follows that $\mu(i) = \Phi(\alpha)(j_i)$, for $i = 1, \dots, m$, determines an injective order preserving function from $\mathbf{m}$ to $\mathbf{n}$.

We define paths $\bar{\mu}_i$ for $i = 1, \dots, m$ by $\bar{\mu}_i(t) = (\mu(i), 0)(1-t) + (i, 0)t$ for $t \in I$. Choose a representative $(\alpha_1, \dots, \alpha_m)$ for α. Since the path $\bar{\mu}_{j_i}$ starts in $(\mu(j_i), 0) = \alpha_i(1)$ and ends in $(j_i, 0)$ the homotopy class of $(\bar{\mu}_{j_1} \cdot \alpha_1, \dots, \bar{\mu}_{j_m} \cdot \alpha_m)$ is a braid on m strings and we define this to be ζ.

Each path $\bar{\mu}_{j_i}$ is the reverse path of μ_{j_i} in the definition of $\Upsilon(\mu)$. Hence it is clear that $\Upsilon(\mu) \circ \zeta$, which is represented by $(\mu_{j_1} \cdot \bar{\mu}_{j_1} \cdot \alpha_1, \dots, \mu_{j_m} \cdot \bar{\mu}_{j_m} \cdot \alpha_m)$, is equal to α.

The morphism μ is uniquely determined by the numbers $\Phi(\alpha)(j_i)$ and we see from the construction that ζ is then also uniquely determined. □

Lemma 2.1.7. *Given μ in $\hom_{\mathcal{M}}(\mathbf{m}, \mathbf{n})$ and ξ in $\mathcal{B}_n$, let $\mu^*(\xi)$ in $\mathcal{B}_m$ and $\xi_*(\mu)$ in $\hom_{\mathcal{M}}(\mathbf{m}, \mathbf{n})$ be the uniquely determined morphisms such that the diagram*

$$\begin{array}{ccc} \mathbf{m} & \xrightarrow{\Upsilon(\mu)} & \mathbf{n} \\ {\scriptstyle\mu^*(\xi)}\downarrow & & \downarrow{\scriptstyle\xi} \\ \mathbf{m} & \xrightarrow{\Upsilon(\xi_*(\mu))} & \mathbf{n} \end{array}$$

is commutative.

The morphism set $\hom_{\mathfrak{B}}(\mathbf{m},\mathbf{n})$ *can be identified with* $\hom_{\mathcal{M}}(\mathbf{m},\mathbf{n}) \times \mathcal{B}_m$*, and under this identification composition is given by*

$$(\nu,\xi)\circ(\mu,\zeta) = (\nu\circ\xi_*(\mu), \mu^*(\xi)\circ\zeta).$$

Proof. The identification $\hom_{\mathfrak{B}}(\mathbf{m},\mathbf{n}) \cong \hom_{\mathcal{M}}(\mathbf{m},\mathbf{n}) \times \mathcal{B}_m$ follows from Lemma 2.1.6. The equation

$$(\nu,\xi)\circ(\mu,\zeta) \cong \Upsilon(\nu)\circ\xi\circ\Upsilon(\mu)\circ\zeta = \Upsilon(\nu)\circ\Upsilon(\xi_*(\mu))\circ\mu^*(\xi)\circ\zeta = \\ \Upsilon(\nu\circ\xi_*(\mu))\circ\mu^*(\xi)\circ\zeta \cong (\nu\circ\xi_*(\mu), \mu^*(\xi)\circ\zeta)$$

yields the composition rule. □

2.2 A braided monoidal structure on $\mathfrak{B}$

The purpose of this section is to define a braided strict monoidal structure on $\mathfrak{B}$. By restricting this structure, the category of braids $\mathcal{B}$ is also braided strict monoidal.

We start with a monoidal structure.

Definition 2.2.1 (A strict monoidal structure on $\mathfrak{B}$)**.** We define the monoidal product $\mathbf{m}\oplus\mathbf{n}$ of two objects $\mathbf{m}$ and $\mathbf{n}$ to be the set $\{1,\ldots,m+n\}$.

Before we proceed with the definition, we observe that for a morphism α from $\mathbf{m}$ to $\mathbf{m}'$ there is always a representative $(\alpha_1,\ldots,\alpha_m)$ with

$$\tfrac{1}{2} < \mathrm{pr}_1\alpha_i(t) < m(1-t)+m't+\tfrac{1}{2}$$

for all i and t, for an illustration see Firgure 2.3. We call such a representative a good representative for α.

Given two morphisms $\alpha\colon \mathbf{m}\to\mathbf{m}'$ and $\beta\colon \mathbf{n}\to\mathbf{n}'$ we choose good representatives $(\alpha_1,\ldots,\alpha_m)$ and $(\beta_1,\ldots,\beta_n)$ for α and β respectively. Let $\alpha\oplus\beta\colon \mathbf{m}\oplus\mathbf{n}\to\mathbf{m}'\oplus\mathbf{n}'$ be the homotopy class of $(\alpha_1,...,\alpha_m,\beta_1',...,\beta_n')$ where

$$\beta_j'(t) = (m,0)(1-t)+(m',0)t+\beta_j(t)$$

for $1\le j\le n$.

We have to check that this tuple satisfies the conditions in Definition 2.1.1. Each of the paths β_j' starts in $(m+j,0)$ and ends in $(m',0)+\beta_j(1)$ so the requirements for the endpoints are fulfilled. Since $(\alpha_1,\ldots,\alpha_m)$ and $(\beta_1,\ldots,\beta_n)$ represents injective braids we know that

$$\beta_i'(t) = (m,0)(1-t)+(m',0)t+\beta_i(t) \neq (m,0)(1-t)+(m',0)t+\beta_j(t) = \beta_j'(t)$$

for $1\le i<j\le n$ and for all t, similarly for the paths α_i. What is left is to check that $\alpha_i(t)\neq\beta_j'(t)$ for $1\le i\le m$, $1\le j\le n$ and for all t, but this follows from the inequality

$$\mathrm{pr}_1\alpha_i(t) < (1-t)m+tm'+\tfrac{1}{2} < (1-t)m+tm'+\mathrm{pr}_1\beta_j(t) = \mathrm{pr}_1\beta_j'(t).$$

Hence $(\alpha_1, ..., \alpha_m, \beta'_1, ..., \beta'_n)$ represents an injective braid.

If we have two good representatives $(\alpha_1^1, \dots, \alpha_m^1)$ and $(\alpha_1^2, \dots, \alpha_m^2)$ of a morphism α, then it is possible to choose a good homotopy between them. That is a tuple of homotopies such that

$$\tfrac{1}{2} < \mathrm{pr}_1 H_i(s,t) < m(1-t) + m't + \tfrac{1}{2}$$

for each of the homoptoies H_i. Suppose we have good homotopies $H\colon (\alpha_1^1, \dots, \alpha_m^1) \to (\alpha_1^2, \dots, \alpha_m^2)$ and $G\colon (\beta_1^1, \dots, \beta_m^1) \to (\beta_1^2, \dots, \beta_m^2)$ between good representatives for α and β respectively. Then we can let $H \oplus G$ be the tuple $(H_1, \dots, H_m, G'_1, \dots, G'_n)$ where G'_j is defined similarly to β'_j. This will give a homotopy from $(\alpha_1^1, ..., \alpha_m^1, \beta_1^{1\prime}, ..., \beta_n^{1\prime})$ to $(\alpha_1^2, ..., \alpha_m^2, \beta_1^{2\prime}, ..., \beta_n^{2\prime})$. This shows that $\alpha \oplus \beta$ is well defined.

It is clear that if we have morphisms $\alpha_i\colon \mathbf{m_i} \to \mathbf{m_{i+1}}$ and $\beta_i\colon \mathbf{n_i} \to \mathbf{n_{i+1}}$ for $i = 1, 2$ then

$$(\alpha_1 \oplus \beta_1) \circ (\alpha_2 \oplus \beta_2) = (\alpha_1 \circ \alpha_2) \oplus (\beta_1 \circ \beta_2),$$

so we get a functor $\oplus\colon \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ which is the monoidal product.

The monoidal product is strictly associative,

$$\mathbf{k} \oplus (\mathbf{m} \oplus \mathbf{n}) = \{1, \dots, k+m+n\} = (\mathbf{k} \oplus \mathbf{m}) \oplus \mathbf{n}$$

and this is natural in $\mathbf{k}$, $\mathbf{m}$ and $\mathbf{n}$. The object $\mathbf{0}$ is clearly a strict unit,

$$\mathbf{0} \oplus \mathbf{m} = \{1, \dots, m\} = \mathbf{m},$$

similarly $\mathbf{m} \oplus \mathbf{0} = \mathbf{m}$, and this is natural in $\mathbf{m}$.

When we have strict associativity and a strict unit, the associativity pentagon (1.1) and the triangle for unit (1.2) automatically commute.

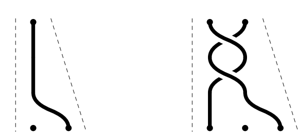


Figure 2.3: Good representatives.

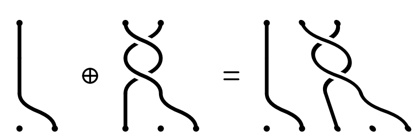


Figure 2.4: Illustration of the monoidal product of two morphisms.

Definition 2.2.2 (A strict monoidal structure on $\mathcal{I}$)**.** We define the monoidal product $\mathbf{m} \oplus \mathbf{n}$ of two objects $\mathbf{m}$ and $\mathbf{n}$ to be the set $\{1, \dots, m+n\}$.

Given two morphisms $\phi\colon \mathbf{m} \to \mathbf{m}'$ and $\psi\colon \mathbf{n} \to \mathbf{n}'$ define $\phi \oplus \psi$ by

$$\phi \oplus \psi(i) = \begin{cases} \phi(i) & 1 \le i \le m \\ m' + \psi(i-m) & m+1 \le i \le m+n. \end{cases}$$

As in Definition 2.2.1 we have strict associativity and $\mathbf{0}$ is a strict unit.

Lemma 2.2.3. *The strict monoidal structure on $\mathfrak{B}$ from Definition 2.2.1 restricts to $\mathcal{B}$. The strict monoidal structure on $\mathcal{I}$ from Definition 2.2.2 restricts to Σ and $\mathcal{M}$.*

The functors in Diagram 2.1 are strict monoidal functors.

Proof. Everything follows easily from the definitions. □

Lemma 2.2.4. *Let $\alpha = (\mu, \zeta)$ and $\beta = (\nu, \xi)$ be morphisms in $\mathfrak{B}$, then*

$$\alpha \oplus \beta = (\mu, \zeta) \oplus (\nu, \xi) = (\mu \oplus \nu, \zeta \oplus \xi)$$

under the identification $\hom_{\mathfrak{B}}(\mathbf{m}, \mathbf{n}) \cong \hom_{\mathcal{M}}(\mathbf{m}, \mathbf{n}) \times \mathcal{B}_m$.

Proof. The functoriality of the monoidal product implies that

$$\alpha \oplus \beta = (\Upsilon(\mu) \circ \zeta) \oplus (\Upsilon(\nu) \circ \xi) = (\Upsilon(\mu) \oplus \Upsilon(\nu)) \circ (\zeta \oplus \xi).$$

This is equal to $(\Upsilon(\mu \oplus \nu)) \circ (\zeta \oplus \xi)$ since Υ is a strict monoidal functor, see the previous lemma. □

We now turn to the braiding of the monoidal product on $\mathfrak{B}$. Intuitively there are two ways to define a braiding. For a product $\mathbf{m} \oplus \mathbf{n}$ we can move the m strings over the n strings while all the way keeping the order internally among the m and n strings respectively. Or we could move the m strings under the n strings. We choose the first option, but the second would work just as well.

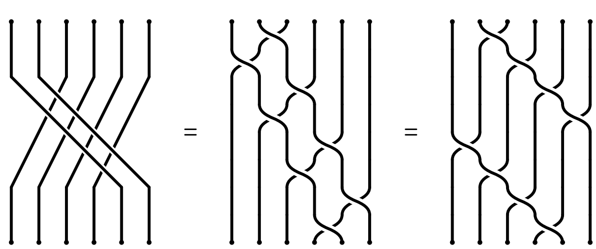


Figure 2.5: Different representatives for the braiding $\mathbf{2} \oplus \mathbf{4} \to \mathbf{4} \oplus \mathbf{2}$.

The following definition only defines the isomorphisms for the braiding. The naturality and the commutativity of the hexagon diagrams will be proven in subsequent results.

Definition 2.2.5. For objects $\mathbf{m}$ and $\mathbf{n}$ in $\mathfrak{B}$ we define the braiding $b_{\mathbf{m},\mathbf{n}}\colon \mathbf{m}\oplus\mathbf{n}\to\mathbf{n}\oplus\mathbf{m}$ to be the homotopy class of

$$(b_1^{\mathbf{m},\mathbf{n}},\dots b_m^{\mathbf{m},\mathbf{n}}, b_{m+1}^{\mathbf{m},\mathbf{n}},\dots b_{m+n}^{\mathbf{m},\mathbf{n}}),$$

where the paths in the tuple are defined like this:

$$b_i^{\mathbf{m},\mathbf{n}}(t) = \begin{cases} (i,-t) & t\in[0,\frac14] \\ (i+2nt-\frac{n}{2},-\frac14) & t\in[\frac14,\frac34] \\ (i+n,t-1) & t\in[\frac34,1] \end{cases} \quad \text{for } 1\le i\le m$$

and

$$b_{m+j}^{\mathbf{m},\mathbf{n}}(t) = \begin{cases} (m+j,t) & t\in[0,\frac14] \\ (\frac{3m}{2}-2m+j,\frac14) & t\in[\frac14,\frac34] \\ (j,1-t) & t\in[\frac34,1] \end{cases} \quad \text{for } 1\le j\le n.$$

It is easy to check that this tuple represents an injective braid.

Lemma 2.2.6. *The braiding $b_{\mathbf{m},\mathbf{n}}$ is a braid on $m+n$ strings, so we can write it as a product of the generators of $\mathcal{B}_{m+n}$. Two easy ways of doing this are:*

$$b_{\mathbf{m},\mathbf{n}} = (\zeta_n\cdots\zeta_{m+n-2}\zeta_{m+n-1})\cdots(\zeta_2\cdots\zeta_m\zeta_{m+1})(\zeta_1\cdots\zeta_{m-1}\zeta_m) \tag{2.2}$$

$$b_{\mathbf{m},\mathbf{n}} = (\zeta_n\cdots\zeta_2\zeta_1)\cdots(\zeta_{m+n-2}\cdots\zeta_m\zeta_{m-1})(\zeta_{m+n-1}\cdots\zeta_{m+1}\zeta_m) \tag{2.3}$$

Proof. The first one moves each of the m strings one step to the right, starting with the rightmost one, and then starts over agian until all the m strings has been moved over the n strings. The second one moves string number m all the way to the right, and then moves string $m-1$ and so on. See Figure 2.5.

It is intuitively clear that any two ways of moving the m strings over the n string while all the way keeping the order internally among the m strings and the n strings are homotopic. □

Lemma 2.2.7. *The braiding is natural in* $\mathbf{m}$ *and* $\mathbf{n}$.

Proof. Given maps $\alpha\colon \mathbf{m}\to\mathbf{m}'$ and $\beta\colon \mathbf{n}\to\mathbf{n}'$ we have to show that

$$(\beta\oplus\alpha)\circ b_{\mathbf{m},\mathbf{n}} = b_{\mathbf{m}',\mathbf{n}'}\circ(\alpha\oplus\beta).$$

Let $(\alpha_1,\dots,\alpha_m)$ and $(\beta_1,\dots,\beta_n)$ be good representatives for α and β respectively, as in Definition 2.2.1, that in addition satiesfies

$$-\tfrac14 < \mathrm{pr}_2\alpha_i(t), \mathrm{pr}_2\beta_j(t) < \tfrac14$$

for $1\le i\le m$ and $1\le j\le n$.

Then we can write up an explicit homotopy from

$$\alpha_i'(t)\cdot b_i^{\mathbf{m},\mathbf{n}}(t) = ((n,0)(1-t)+(n',0)t+\alpha_i(t))\cdot b_i^{\mathbf{m},\mathbf{n}}(t)$$

to $b^{\mathbf{m}',\mathbf{n}'}_{\Phi(\alpha)(i)}(t) \cdot \alpha_i(t)$:

$$H_i(s,t) = \begin{cases} b_i^{\mathbf{m},\mathbf{n}}(2t) & t \in [0, \frac{1-s}{2}] \\ \alpha_i(2t+s-1) + ((i,0) - b_i^{\mathbf{m},\mathbf{n}}(1-s))(2t+s-2) & \\ \quad +(b^{\mathbf{m}',\mathbf{n}'}_{\Phi(\alpha)(i)}(1-s) - \alpha_i(1))(2t+s-1) & t \in [\frac{1-s}{2}, \frac{2-s}{2}] \\ b^{\mathbf{m}',\mathbf{n}'}_{\Phi(\alpha)(i)}(2t-1) & t \in [\frac{2-s}{2}, 1] \end{cases}$$

for each $1 \leq i \leq m$ and an explicit homotopy from $\beta_j(t) \cdot b^{\mathbf{m},\mathbf{n}}_{m+j}(t)$ to

$$b^{\mathbf{m}',\mathbf{n}'}_{m'+\Phi(\beta)(j)}(t) \cdot \beta_j'(t) = b^{\mathbf{m}',\mathbf{n}'}_{m'+\Phi(\beta)(j)}(t) \cdot ((m,0)(1-t) + (m',0)t + \beta_j(t)) :$$

$$H_{m+j}(s,t) = \begin{cases} b^{\mathbf{m},\mathbf{n}}_{m+j}(2t) & t \in [0, \frac{1-s}{2}] \\ \beta_j(2t+s-1) + ((j,0) - b^{\mathbf{m},\mathbf{n}}_{m+j}(1-s))(2t+s-2) & \\ \quad +(b^{\mathbf{m}',\mathbf{n}'}_{m'+\Phi(\beta)(j)}(1-s) - \beta_j(1))(2t+s-1) & t \in [\frac{1-s}{2}, \frac{2-s}{2}] \\ b^{\mathbf{m}',\mathbf{n}'}_{m'+\Phi(\beta)(j)}(2t-1) & t \in [\frac{2-s}{2}, 1] \end{cases}$$

for each $1 \leq j \leq n$.

These homotopies gives a homotopy from the chosen representative of $(\beta \oplus \alpha) \circ b_{\mathbf{m},\mathbf{n}}$ to the chosen representative of $b_{\mathbf{m}',\mathbf{n}'} \circ (\alpha \oplus \beta)$. □

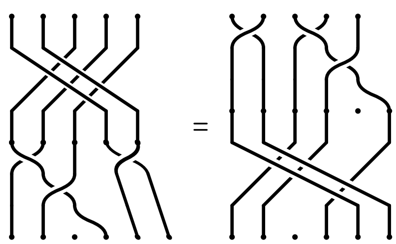


Figure 2.6: An illustration of the naturality of the braiding.

Lemma 2.2.8. *The hexagons,* (1.5) *and* (1.6)*, for the braiding commute.*

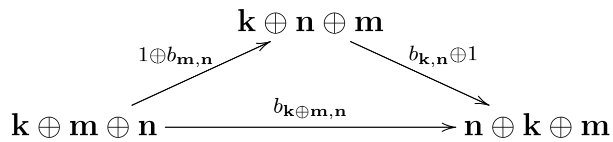
Proof. Since we have a strict monidal product the hexagonal diagrams reduce to triangles. We show the commutativity of the first one

$$\begin{array}{ccccc} & & \mathbf{m} \oplus \mathbf{k} \oplus \mathbf{n} & & \\ & \overset{b_{\mathbf{k},\mathbf{m}} \oplus 1}{\nearrow} & & \overset{1 \oplus b_{\mathbf{k},\mathbf{n}}}{\searrow} & \\ \mathbf{k} \oplus \mathbf{m} \oplus \mathbf{n} & & \xrightarrow{\quad b_{\mathbf{k},\mathbf{m}\oplus\mathbf{n}} \quad} & & \mathbf{m} \oplus \mathbf{n} \oplus \mathbf{k} \end{array} \tag{2.4}$$

by using the equation (2.2):

$$\begin{aligned} 1 \oplus b_{\mathbf{k},\mathbf{n}} \circ b_{\mathbf{k},\mathbf{m}} \oplus 1 = {} & (\zeta_{m+n} \cdots \zeta_{k+m+n-2}\zeta_{k+m+n-1}) \cdots (\zeta_{m+2} \cdots \zeta_{k+m}\zeta_{k+m+1}) \\ & \cdot (\zeta_{m+1} \cdots \zeta_{k+m-1}\zeta_{k+m})(\zeta_m \cdots \zeta_{k+m-2}\zeta_{k+m-1}) \cdots (\zeta_2 \cdots \zeta_k\zeta_{k+1})(\zeta_1 \cdots \zeta_{k-1}\zeta_k) \\ & = b_{\mathbf{k},\mathbf{m}\oplus\mathbf{n}}. \end{aligned}$$

The commutativity of



(2.5)

is just as easy, using the equation (2.3):

$$\begin{aligned} b_{\mathbf{k},\mathbf{n}} \oplus 1 \circ 1 \oplus b_{\mathbf{m},\mathbf{n}} &= (\zeta_n \cdots \zeta_2 \zeta_1) \cdots (\zeta_{k+n-2} \cdots \zeta_k \zeta_{k-1})(\zeta_{k+n-1} \cdots \zeta_{k+1} \zeta_k) \\ &\cdot (\zeta_{k+n} \cdots \zeta_{k+2} \zeta_{k+1}) \cdots (\zeta_{k+m+n-2} \cdots \zeta_{k+m} \zeta_{k+m-1})(\zeta_{k+m+n-1} \cdots \zeta_{k+m+1} \zeta_{k+m}) \\ &= b_{\mathbf{k}\oplus\mathbf{m},\mathbf{n}}. \end{aligned}$$

□

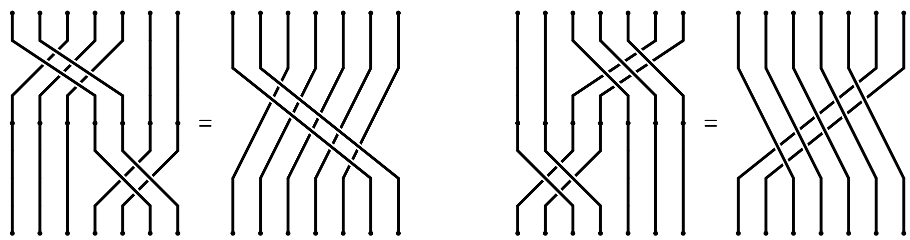


Figure 2.7: An illustration of the two hexagons commuting.

We have now completed the proof of:

Proposition 2.2.9. *$\mathfrak{B}$ is a braided strict monoidal category with the monoidal product from Definition 2.2.1 and the braiding from Definition 2.2.5.*

Proposition 2.2.10. *For objects $\mathbf{m}$ and $\mathbf{n}$ in $\mathcal{I}$ we define the symmetry isomorphism $c_{\mathbf{m},\mathbf{n}} \colon \mathbf{m} \oplus \mathbf{n} \to \mathbf{n} \oplus \mathbf{m}$ by*

$$c_{\mathbf{m},\mathbf{n}}(i) = \begin{cases} n + i & \text{for } 1 \leq i \leq m \\ i - m & \text{for } m + 1 \leq i \leq m + n. \end{cases}$$

This determines a symmetric strict monoidal structure on $\mathcal{I}$.

Proof. Let $\phi \colon \mathbf{m} \to \mathbf{m}'$ and $\psi \colon \mathbf{n} \to \mathbf{n}'$ be morphisms in $\mathcal{I}$. Direct verification shows that $c_{\mathbf{m}',\mathbf{n}'} \circ (\phi \oplus \psi) = (\psi \oplus \phi) \circ c_{\mathbf{m},\mathbf{n}}$. The left side evaluated at i is:

$$\begin{cases} c_{\mathbf{m}',\mathbf{n}'}(\phi(i)) & = & n' + \phi(i) & 1 \leq i \leq m \\ c_{\mathbf{m}',\mathbf{n}'}(m' + \psi(i-m)) & = & m' + \psi(i-m) - m' & m+1 \leq i \leq m+n, \end{cases}$$

which is the same as the right side evaluated at i:

$$\begin{cases} (\psi \oplus \phi)(n+i) & = & n' + \phi(n+i-n) & 1 \leq i \leq m \\ (\psi \oplus \phi)(i-m) & = & \psi(i-m) & m+1 \leq i \leq m+n. \end{cases}$$

Similary direct computation shows that $c_{\mathbf{n},\mathbf{m}} \circ c_{\mathbf{m},\mathbf{n}} = \mathrm{id}_{\mathbf{m}\oplus\mathbf{n}}$ so that c is indeed symmetric. For a symmetric structure it is enough to show that Hexagon 1.5 commutes. Since we have a strict monoidal structure this diagram reduces to a triangle like Diagram 2.4, with the b's replaced by the corresponding c's. The computation

$$(\mathrm{id}_{\mathbf{m}} \oplus c_{\mathbf{k},\mathbf{n}}) \circ (c_{\mathbf{k},\mathbf{m}} \oplus \mathrm{id}_{\mathbf{n}})(i) = \begin{cases} (\mathrm{id}_{\mathbf{m}} \oplus c_{\mathbf{k},\mathbf{n}})(m+i) & 1 \le i \le k \\ (\mathrm{id}_{\mathbf{m}} \oplus c_{\mathbf{k},\mathbf{n}})(i-k) & k+1 \le i \le k+m \\ (\mathrm{id}_{\mathbf{m}} \oplus c_{\mathbf{k},\mathbf{n}})(i) & k+m+1 \le i \le k+m+n \end{cases}$$

$$= \begin{cases} n+m+i & 1 \le i \le k \\ i-k & k+1 \le i \le k+m \\ i-k & k+m+1 \le i \le k+m+n \end{cases} = c_{\mathbf{k},\mathbf{m}\oplus\mathbf{n}}(i).$$

shows that this commutes. □

Recall from the definition of symmetric monoidal category 1.2.1 that a symmetric structure is also a braided structure. So the last statement in the following lemma makes sense.

Lemma 2.2.11. *The braided strict monoidal structure on $\mathfrak{B}$ restricts to $\mathcal{B}$. The symmetric strict monoidal structure on $\mathcal{I}$ restricts to Σ.*

The functors in the square part of Diagram 2.1 are braided strict monoidal functors.

Proof. Everything follows easily from the definitions. □

Note that the symmetric structure on $\mathcal{I}$ does not restrict to $\mathcal{M}$ since the symmetry morphisms are not order preserving.

Chapter 3

Diagram spaces

3.1 Examples of $\mathfrak{B}$-spaces

Definition 3.1.1. A $\mathfrak{B}$-space is a functor from $\mathfrak{B}$ to the category of simplicial sets, which we denote by $\mathcal{S}$.

Example 3.1.2 (Free $\mathfrak{B}$-spaces)**.** For every $\mathbf{m}$ in $\mathfrak{B}$, and every simplicial set K, we can construct a free $\mathfrak{B}$-space $F_{\mathbf{m}}(K)$ as follows:

$$F_{\mathbf{m}}(K)(\mathbf{n}) = \hom_{\mathfrak{B}}(\mathbf{m}, \mathbf{n}) \times \mathrm{K},$$

and for a morphism $\alpha\colon \mathbf{n} \to \mathbf{p}$ we define $F_{\mathbf{m}}(K)(\alpha)$ to be postcomposition by α.

Note that the functor $F_{\mathbf{0}}(K)$ is a constant $\mathfrak{B}$-space in the sense that $F_{\mathbf{0}}(K)(\mathbf{n}) \cong K$ and $F_{\mathbf{0}}(K)(\alpha) \cong \mathrm{id}_{\mathrm{K}}$ for every object $\mathbf{n}$ and every morphism α in $\mathfrak{B}$ respectively.

Recall that the category $\mathcal{I}$ is the category with the same objects as $\mathfrak{B}$ and with injective functions as morphism. An $\mathcal{I}$-space determines a $\mathfrak{B}$-space by precomposition with the functor Φ from $\mathfrak{B}$ to $\mathcal{I}$ defined in Definition 2.1.4. We give a couple of examples of $\mathcal{I}$-spaces.

Example 3.1.3 ($\mathcal{I}$-space)**.** Let X be a based simplicial set, with base point $*$. For a morphism $\phi\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ we define a function X^{ϕ} from $X^{\times m}$ to $X^{\times n}$ by

$$X_s^{\phi}(x_1, \ldots, x_m) = (x_{\phi^{-1}(1)}, \ldots, x_{\phi^{-1}(n)})$$

in degree s, where $x_{\phi^{-1}(i)} = *$ if i is not in the image of ϕ. It is easily seen that $X^{\psi \circ \phi} = X^{\psi} \circ X^{\phi}$ and $X^{\mathrm{id}_{\mathbf{m}}}$ is obviously the identity on $X^{\times m}$, so we get a functor $X^{\bullet}\colon \mathcal{I} \to \mathcal{S}$ with $X^{\mathbf{m}} = X^{\times m}$.

Definition 3.1.4. Given an injective function ϕ from $\mathbf{m}$ to $\mathbf{n}$ we can extend it to a permutation of n points in several ways, but an obvious choice would be to let it be order preserving in the $n - m$ last points. To be precise, let $j_{m+1} < \cdots < j_n$ be the elements in the complement of the image of ϕ. We define a permutation ς_{ϕ} by

$$\varsigma_{\phi}(i) = \begin{cases} \phi(i) & 1 \le i \le m \\ j_i & m+1 \le i \le n. \end{cases}$$

Lemma 3.1.5. *If two permutations σ, σ' in Σ_n have the property that $\sigma(i) = \sigma'(i)$ for $1 \le i \le m$ for some $m \le n$ then*

$$\sigma \circ (\omega \oplus \mathrm{id}_{\mathbf{n-m}}) \circ \sigma^{-1} = \sigma' \circ (\omega \oplus \mathrm{id}_{\mathbf{n-m}}) \circ \sigma'^{-1}$$

for any $\omega \in \Sigma_m$.

Proof. The stated property is equivalent to $\sigma' = \sigma \circ (\mathrm{id}_{\mathbf{m}} \oplus \tau)$ for some $\tau \in \Sigma_{n-m}$. Substituting $\sigma \circ (\mathrm{id}_{\mathbf{m}} \oplus \tau)$ for σ' in the above equation yields the result. □

Example 3.1.6 ($\mathcal{I}$-space)**.** Let Σ_m denote the group of permutaions of the elements in the set $\mathbf{m}$, that is $\Sigma_m = \hom_I(\mathbf{m}, \mathbf{m})$. We define a functor $\mathcal{N}\Sigma_*\colon \mathcal{I} \to \mathcal{S}$ on an object $\mathbf{m}$ to be the nerve, see Definition 1.2.12, of the group Σ_m. An injective function $\phi\colon \mathbf{m} \to \mathbf{n}$ determines a group homomorphism $\Sigma_m \to \Sigma_n$ by

$$\sigma \mapsto \varsigma_\phi \circ (\sigma \oplus \mathrm{id}_{\mathbf{n-m}}) \circ \varsigma_\phi^{-1},$$

where ς_ϕ is the permutation defined in the previous definition. This group homomorphism induces a simplicial set map on the nerves of the groups.

If we have injective functions $\phi\colon \mathbf{m} \to \mathbf{n}$ and $\psi\colon \mathbf{n} \to \mathbf{p}$, then the equation

$$\varsigma_\psi \circ (\varsigma_\phi \oplus \mathrm{id}_{\mathbf{p-n}})(i) = \varsigma_{\psi\circ\phi}(i)$$

holds for $1 \le i \le m$ so by the previous lemma

$$\varsigma_\psi \circ (\varsigma_\phi \oplus \mathrm{id}_{\mathbf{p-n}}) \circ (\sigma \oplus \mathrm{id}_{\mathbf{p-m}}) \circ (\varsigma_\phi^{-1} \oplus \mathrm{id}_{\mathbf{p-n}}) \circ \varsigma_\psi^{-1} = \varsigma_{\psi\circ\phi} \circ (\sigma \oplus \mathrm{id}_{\mathbf{p-m}}) \circ \varsigma_{\psi\circ\phi}^{-1}$$

for any $\sigma \in \Sigma_m$. This proves the functoriality.

Definition 3.1.7. Let $\iota_{\mathbf{n-m}}$ denote the injective braid $\mathbf{0} \to \mathbf{n-m}$. Given an injective braid α from $\mathbf{m}$ to $\mathbf{n}$, let $\tilde{\varsigma}_\alpha$ be a braid on n strings, such that $\Phi(\tilde{\varsigma}_\alpha) = \varsigma_{\Phi(\alpha)}$, and such that $\tilde{\varsigma}_\alpha \circ (\mathrm{id}_{\mathbf{m}} \oplus \iota_{\mathbf{n-m}}) = \alpha$. Informally the latter condition says that if we take away the $n-m$ last strings from $\tilde{\varsigma}_\alpha$ we get α. We say that any such $\tilde{\varsigma}_\alpha$ extends the injective braid α to a braid. See Figure 3.1.

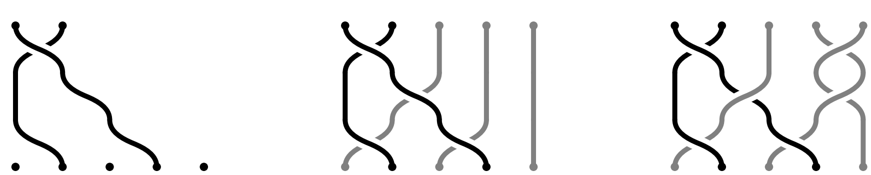


Figure 3.1: Two extensions of an injective braid to a braid.

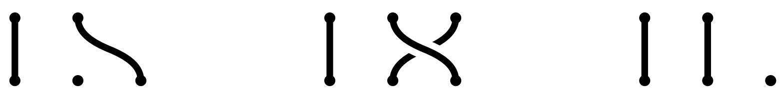
Remark 3.1.8. There is not an analogous of Lemma 3.1.5 for braids. One could still try to get the equation

$$\tilde{\varsigma}_\beta \circ (\tilde{\varsigma}_\alpha \oplus \mathrm{id}_{\mathbf{p}-\mathbf{n}}) \circ (\zeta \oplus \mathrm{id}_{\mathbf{p}-\mathbf{m}}) \circ (\tilde{\varsigma}_\alpha^{-1} \oplus \mathrm{id}_{\mathbf{p}-\mathbf{n}}) \circ \tilde{\varsigma}_\beta^{-1} = \tilde{\varsigma}_{\beta\circ\alpha} \circ (\zeta \oplus \mathrm{id}_{\mathbf{p}-\mathbf{m}}) \circ \tilde{\varsigma}_{\beta\circ\alpha}^{-1}$$

for braids $\tilde{\varsigma}_\alpha$, $\tilde{\varsigma}_\beta$ and $\tilde{\varsigma}_{\beta\circ\alpha}$ extending injective braids α, β and $\beta\circ\alpha$ respectively.

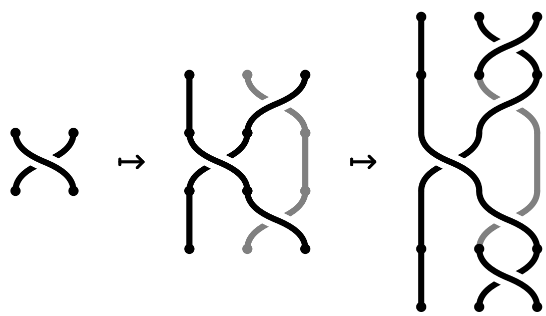
We would then have to braid the extensions in a consistent manner. We could try by letting all the new strings lie under the strings from the injective braid, and let it be order preserving on the new strings, see the middle braid in Figure 3.1.

This does not work. The illustration shows, from left to right, the injective braids $\alpha\colon \mathbf{2} \to \mathbf{3}$, $\beta\colon \mathbf{3} \to \mathbf{3}$ and $\beta\circ\alpha\colon \mathbf{2} \to \mathbf{3}$:

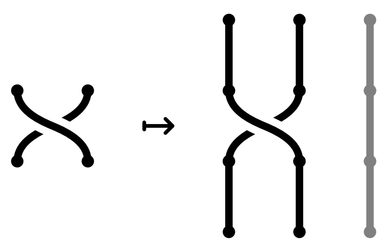


We map a braid $\zeta\colon \mathbf{2} \to \mathbf{2}$ first to $\tilde{\varsigma}_\alpha \circ (\zeta \oplus \mathrm{id}_{\mathbf{3}-\mathbf{2}}) \circ \tilde{\varsigma}_\alpha^{-1}$, and then on to

$$\tilde{\varsigma}_\beta \circ (\tilde{\varsigma}_\alpha \oplus \mathrm{id}_{\mathbf{3}-\mathbf{3}}) \circ (\zeta \oplus \mathrm{id}_{\mathbf{3}-\mathbf{2}}) \circ (\tilde{\varsigma}_\alpha^{-1} \oplus \mathrm{id}_{\mathbf{3}-\mathbf{3}}) \circ \tilde{\varsigma}_\beta^{-1} :$$



Then we map the same braid, ζ, to $\tilde{\varsigma}_{\beta\circ\alpha} \circ (\zeta \oplus \mathrm{id}_{\mathbf{3}-\mathbf{2}}) \circ \tilde{\varsigma}_{\beta\circ\alpha}^{-1}$:



We see that the two results are not equal.

No matter how we extend α we can always find a β such that the results are different. Therefore there is no $\mathfrak{B}$-space analog of Example 3.1.6 with the permutation groups replaced by the braid groups.

Example 3.1.9. Let $\mathcal{A}$ be a small braided strict monoidal category, with unit I. For $\mathbf{n} \in \mathfrak{B}$ we define categories $\mathcal{A}_\mathbf{n}$, where $\mathcal{A}_\mathbf{0}$ is the category with one object and one morphism. For

$n > 0$ we construct $\mathcal{A}_{\mathbf{n}}$ in the following way: The objects are n-tuples of objects in $\mathcal{A}$, and the set of morphisms from $(a_1, \dots, a_n)$ to $(b_1, \dots, b_n)$ is

$$\hom_{\mathcal{A}}(a_1 \oplus \cdots \oplus a_n, b_1 \oplus \cdots \oplus b_n).$$

We want to define a functor $\mathcal{A}_\bullet$ from $\mathfrak{B}$ to the category of small categories that is $\mathcal{A}_{\mathbf{n}}$ on objects. For a morphism α in $\hom_{\mathfrak{B}}(\mathbf{m}, \mathbf{n})$ we have to define a functor $\mathcal{A}_\alpha \colon \mathcal{A}_{\mathbf{m}} \to \mathcal{A}_{\mathbf{n}}$. On an object $(a_1, \dots, a_m)$ we let it be $(a_{\Phi(\alpha)^{-1}(1)}, \dots, a_{\Phi(\alpha)^{-1}(n)}) \in \mathcal{A}_{\mathbf{n}}$, where $a_{\Phi(\alpha)^{-1}(i)} = I$ if i is not in the image of $\Phi(\alpha)$.

Extend α to a braid $\tilde{\varsigma}_\alpha$ as in Definition 3.1.7. We use the braiding in $\mathcal{A}$ to define the map

$$\tilde{\varsigma}_\alpha \colon\ a_1 \oplus \cdots \oplus a_m \oplus I \oplus \cdots \oplus I \to a_{\Phi(\alpha)^{-1}(1)} \oplus \cdots \oplus a_{\Phi(\alpha)^{-1}(n)}$$

braided the same way as $\tilde{\varsigma}_\alpha$, see Lemma 1.2.8.

For a morphism $f\colon\ a_1 \oplus \cdots \oplus a_m \to b_1 \oplus \cdots \oplus b_m$ in $\mathcal{A}_{\mathbf{m}}$ we set $\mathcal{A}_\alpha(f)$ to

$$a_{\Phi(\alpha)^{-1}(1)} \oplus \cdots \oplus a_{\Phi(\alpha)^{-1}(n)} \xrightarrow{\tilde{\varsigma}_\alpha^{-1}} a_1 \oplus \cdots \oplus a_m \oplus I \oplus \cdots \oplus I$$
$$\xrightarrow{f \oplus \mathrm{id}} b_1 \oplus \cdots \oplus b_m \oplus I \oplus \cdots \oplus I \xrightarrow{\tilde{\varsigma}_\alpha} b_{\Phi(\alpha)^{-1}(1)} \oplus \cdots \oplus b_{\Phi(\alpha)^{-1}(n)},$$

for an illustration see Figure 3.2. Since

$$\tilde{\varsigma}_\alpha \circ (g \oplus \mathrm{id}) \circ \tilde{\varsigma}_\alpha^{-1} \circ \tilde{\varsigma}_\alpha \circ (f \oplus \mathrm{id}) \circ \tilde{\varsigma}_\alpha^{-1} = \tilde{\varsigma}_\alpha \circ (gf \oplus \mathrm{id}) \circ \tilde{\varsigma}_\alpha^{-1},$$

for any two composable morphisms f and g in $\mathcal{A}_{\mathbf{m}}$ and identity morphisms are clearly preserved, it follows that $\mathcal{A}_\alpha$ is a functor.

It remains to prove that $\mathcal{A}_\bullet$ is a functor. Suppose we have $\alpha\colon\ \mathbf{m} \to \mathbf{n}$ and $\beta\colon\ \mathbf{n} \to \mathbf{p}$, we must show that $\mathcal{A}_\beta \circ \mathcal{A}_\alpha = \mathcal{A}_{\beta\circ\alpha}$. This is easily seen to hold on objects. Let f be a morphism in $\mathcal{A}_{\mathbf{m}}$, we have

$$\mathcal{A}_\beta(\mathcal{A}_\alpha(f)) = \tilde{\varsigma}_\beta \circ (\tilde{\varsigma}_\alpha \oplus \mathrm{id}_{\mathbf{p}-\mathbf{n}}) \circ (f \oplus \mathrm{id}_{\mathbf{p}-\mathbf{m}}) \circ (\tilde{\varsigma}_\alpha^{-1} \oplus \mathrm{id}_{\mathbf{p}-\mathbf{n}}) \circ \tilde{\varsigma}_\beta^{-1}$$

and

$$\mathcal{A}_{\beta\circ\alpha}(f) = \tilde{\varsigma}_{\beta\circ\alpha} \circ (f \oplus \mathrm{id}_{\mathbf{p}-\mathbf{m}}) \circ \tilde{\varsigma}_{\beta\circ\alpha}^{-1}.$$

Let $\iota_{\mathbf{k}}$ be the injective braid $\mathbf{0} \to \mathbf{k}$ for $\mathbf{k}$ in $\mathfrak{B}$, then

$$\tilde{\varsigma}_\beta \circ (\tilde{\varsigma}_\alpha \oplus \mathrm{id}_{\mathbf{p}-\mathbf{n}}) \circ (\mathrm{id}_{\mathbf{m}} \oplus \iota_{\mathbf{p}-\mathbf{m}}) = \tilde{\varsigma}_\beta \circ (\alpha \oplus \iota_{\mathbf{p}-\mathbf{n}}) = \beta \circ \alpha = \tilde{\varsigma}_{\beta\circ\alpha} \circ (\mathrm{id}_{\mathbf{m}} \oplus \iota_{\mathbf{p}-\mathbf{m}}),$$

we will show in Lemma 3.1.10 that this implies that $\mathcal{A}_\beta \circ \mathcal{A}_\alpha = \mathcal{A}_{\beta\circ\alpha}$. Hence $\mathcal{A}_\bullet$ is a functor.

Finally we get a $\mathfrak{B}$-space, $N\mathcal{A}_\bullet$, by composing $\mathcal{A}_\bullet$ with the nerve functor, see Definition 1.2.12.

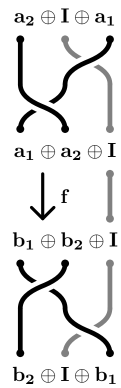


Figure 3.2: $\mathcal{A}_\alpha(f)$

Lemma 3.1.10. *Let $(a_1, \ldots, a_m, c, \ldots, c)$ and $(b_1, \ldots, b_m, c, \ldots, c)$ be two n-tuples of objects in a small braided strict monoidal category $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r})$, such that*

$$\mathsf{b}_{a_i,c} = \mathsf{b}_{c,a_i}^{-1} \text{ and } \mathsf{b}_{b_i,c} = \mathsf{b}_{c,b_i}^{-1}$$

for $1 \le i \le m$. In particular this holds if c is the unit I, see Lemma 1.2.7, or if $\mathcal{A}$ is symmetric, see Definition 1.2.4.

Let $\iota_{\mathbf{n-m}}$ denote the injective braid $\mathbf{0} \to \mathbf{n-m}$. Let ξ and ξ' be two elements in $\mathcal{B}_n$ such that

$$\xi \circ (\mathrm{id}_{\mathbf{m}} \oplus \iota_{\mathbf{n-m}}) = \xi' \circ (\mathrm{id}_{\mathbf{m}} \oplus \iota_{\mathbf{n-m}}).$$

We call this injective braid α. Informally ξ and ξ' becomes equal if we remove the $n-m$ last strings. Then for any map $f\colon a_1 \oplus \cdots \oplus a_m \to b_1 \oplus \cdots \oplus b_m$, the two maps

$$a_{\Phi(\alpha)^{-1}(1)} \oplus \cdots \oplus a_{\Phi(\alpha)^{-1}(n)} \xrightarrow[(\xi')^{-1}]{\xi^{-1}} a_1 \oplus \cdots \oplus a_m \oplus c \oplus \cdots \oplus c \xrightarrow[f\oplus \mathrm{id}]{f\oplus \mathrm{id}}$$
$$b_1 \oplus \cdots \oplus b_m \oplus c \oplus \cdots \oplus c \xrightarrow[\xi']{\xi} b_{\Phi(\alpha)^{-1}(1)} \oplus \cdots \oplus b_{\Phi(\alpha)^{-1}(n)} \quad (3.1)$$

are equal. Here $a_{\Phi(\alpha)^{-1}(i)} = b_{\Phi(\alpha)^{-1}(i)} = c$ if i is not in the image of α.

Proof. Write ξ as a product $\zeta_n \cdots \zeta_1$ where each ζ_i is either a generator or the inverse of a generator. The last map in 3.1, ξ factors as $(\mathrm{id} \oplus \zeta_n \oplus \mathrm{id}) \circ \cdots \circ (\mathrm{id} \oplus \zeta_1 \oplus \mathrm{id})$.

If ζ_i braids c over b_j or if ζ_i braids b_j under c for some j, then replace ζ_i with ζ_i^{-1}. This yields a new braid $\tilde{\xi}$, but the maps

$$\xi, \tilde{\xi}\colon\ b_1 \oplus \cdots \oplus b_m \oplus c \oplus \cdots \oplus c \to b_{\Phi(\alpha)^{-1}(1)} \oplus \cdots \oplus b_{\Phi(\alpha)^{-1}(n)}$$

are equal since the two maps

$$\zeta_i, \zeta_i^{-1}\colon\ c \oplus b_j \to b_j \oplus c$$

are the same by assumption. And the same holds if we replace the b's by a's.

Let ρ be braid on $n-m$ strings such that when we only consider the $n-m$ last strings in $\tilde{\xi} \circ (\mathrm{id}_{\mathbf{m}} \oplus \rho)$ it is orderpreserving with no braiding.

Repeat this procedure with ξ' to obtain $\tilde{\xi}' \circ (\mathrm{id}_{\mathbf{m}} \oplus \rho')$. Now $\tilde{\xi} \circ (\mathrm{id}_{\mathbf{m}} \oplus \rho)$ and $\tilde{\xi}' \circ (\mathrm{id}_{\mathbf{m}} \oplus \rho')$ are equal on the m first strings, on the $n-m$ last strings both are order preserving without braiding. And in both cases the $n-m$ last strings will lie under the m first strings, hence they are the same braid. We have

$$\tilde{\xi} \circ (\mathrm{id}_{\mathbf{m}} \oplus \rho) \circ (f \oplus \mathrm{id}) \circ (\tilde{\xi} \circ (\mathrm{id}_{\mathbf{m}} \oplus \rho))^{-1} = \tilde{\xi} \circ (f \oplus (\rho \circ \rho^{-1})) \circ \tilde{\xi}^{-1} = \xi \circ (f \oplus \mathrm{id}) \circ \xi^{-1}$$

and similarly

$$\tilde{\xi}' \circ (\mathrm{id}_{\mathbf{m}} \oplus \rho') \circ (f \oplus \mathrm{id}) \circ (\tilde{\xi}' \circ (\mathrm{id}_{\mathbf{m}} \oplus \rho'))^{-1} = \xi' \circ (f \oplus \mathrm{id}) \circ \xi'^{-1}.$$

Since $\tilde{\xi} \circ (\mathrm{id}_{\mathbf{m}} \oplus \rho) = \tilde{\xi}' \circ (\mathrm{id}_{\mathbf{m}} \oplus \rho')$ it follows that $\xi \circ (f \oplus \mathrm{id}) \circ \xi^{-1} = \xi' \circ (f \oplus \mathrm{id}) \circ \xi'^{-1}$. $\square$

Example 3.1.11. Let $\mathcal{A}$ be a small braided strict monoidal category, with unit I, and S be a subset of the objects of $\mathcal{A}$, containing I. We can generalise the previous example by constructing a $\mathfrak{B}$-space $\mathcal{N}\mathcal{A}^S_\bullet$ in the same way as we constructed $\mathcal{N}\mathcal{A}_\bullet$, with one difference: The objects in $\mathcal{A}^S_{\mathbf{n}}$ are n-tuples of objects in S.

It does also make sense to define $\mathcal{N}\mathcal{A}^S_\bullet$ for a category $\mathcal{A}$ which is not small, when S is a set of objects in $\mathcal{A}$.

Note that S does not not need to be closed under the monoidal product. If it is, then $\mathcal{N}\bar{S}_\bullet = \mathcal{N}\mathcal{A}^S_\bullet$, where $\bar{S}$ is the full subcategory of $\mathcal{A}$ generated by S.

Example 3.1.12 ($\mathcal{I}$-space)**.** When $\mathcal{C}$ is a small symmetric strict monoidal category, the functor $\mathcal{N}\mathcal{C}_\bullet\colon \mathfrak{B} \to \mathcal{S}$ factors through $\mathcal{I}$.

For the $\mathfrak{B}$-space $\mathcal{N}\mathcal{A}_\bullet$ we used the unit I to define $\mathcal{A}_\alpha(a_1, \ldots, a_m)$ so that Lemma 3.1.10 would give $\mathcal{A}_\beta(\mathcal{A}_\alpha(f)) = \mathcal{A}_{\beta\circ\alpha}(f)$. In the symmetric case this is no longer necessary. For any object $c \in \mathcal{C}$ let ${}_c\mathcal{C}_{\mathbf{m}} = \mathcal{C}_{\mathbf{m}}$ for $\mathbf{m} \in \mathcal{C}$. For an injective function $\phi\colon \mathbf{m} \to \mathbf{n}$ let

$${}_c\mathcal{C}_\phi(c_1, \ldots, c_m) = (c_{\phi^{-1}(1)}, \ldots, c_{\phi^{-1}(n)}),$$

where $c_{\phi^{-1}(i)} = c$ if i is not in the image of ϕ, and let ${}_c\mathcal{C}_\phi(f) = \varsigma_\phi \circ (f \oplus \mathrm{id}_{\mathbf{n-m}}) \circ \varsigma_\phi^{-1}$ for f in ${}_c\mathcal{C}_{\mathbf{m}}$. Now Lemma 3.1.10 implies the functoriality of ${}_c\mathcal{C}_\phi$ by a similar argument to that in Example 3.1.9.

Composing with the nerve functor we get an $\mathcal{I}$-space $\mathcal{N}{}_c\mathcal{C}_\bullet$. If S is a subset of the objects in $\mathcal{C}$, including c, we get a $\mathcal{I}$-space $\mathcal{N}{}_c\mathcal{C}^S_\bullet$ in the same way as we got the $\mathfrak{B}$-space in Example 3.1.11.

Remark 3.1.13. The category ${}_{\mathbf{1}}\mathcal{I}^{\{\mathbf{1}\}}_{\mathbf{m}}$ has one object, the m-tuple $(\mathbf{1}, \ldots, \mathbf{1})$, and morphisms $\hom_{\mathcal{I}}(\mathbf{m}, \mathbf{m}) = \Sigma_m$, so $\mathcal{N}{}_{\mathbf{1}}\mathcal{I}^{\{\mathbf{1}\}}_{\mathbf{m}}$ is isomorpic to $\mathcal{N}\Sigma_{\mathbf{m}}$. This isomorphism is natural in $\mathbf{m}$, so the two $\mathcal{I}$-spaces $\mathcal{N}{}_{\mathbf{1}}\mathcal{I}^{\{\mathbf{1}\}}_\bullet$ and $\mathcal{N}\Sigma_*$ isomorphic.

This highlights that the reason why we can not define a $\mathfrak{B}$-space version of $\mathcal{N}\Sigma_*$ with the braid groups, is the same reason as why there are objects a such that we can not use any object a instead of I to define $\mathcal{A}_\alpha(a_1, \ldots, a_m)$.

Remark 3.1.14. Let $\alpha = (\mu, \zeta)\colon \mathbf{m} \to \mathbf{n}$ be an injective braid. For a morphism f in $\mathcal{A}_{\mathbf{m}}$ the morphism

$$\tilde{\zeta}_\alpha \circ (f \oplus \mathrm{id}_{\mathbf{n}-\mathbf{m}}) \circ \tilde{\zeta}_\alpha^{-1}$$

is equal to

$$\zeta \circ f \circ \zeta^{-1}\colon\ a_{\Phi(\zeta)^{-1}(1)} \oplus \cdots \oplus a_{\Phi(\zeta)^{-1}(m)} \to b_{\Phi(\zeta)^{-1}(1)} \oplus \cdots \oplus b_{\Phi(\zeta)^{-1}(m)},$$

since $\mathcal{A}$ is a strict monoidal category, so we could have defined $\mathcal{A}_\alpha(f)$ in this way. Functoriality of $\mathcal{A}_\bullet$ would then have been easier to show, but this approach relies heavily on the fact that I is the unit for the monoidal product. It would obscure the role I plays in the definition of $\mathcal{N}\mathcal{A}_\bullet$ and the connection to $\mathcal{N}_c\mathcal{C}_\bullet$ would then at best be harder to see.

3.2 A braided monoidal structure on diagram spaces

In this section $\mathcal{A}$ denotes a small monoidal category $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r})$. We denote by $\mathcal{S}$ is the category of simplicial sets. This is a symmetric monoidal category with the categorial product as the monoidal product and the terminal simplicial set $*$ as unit. The goal is to show how we get a braided monoidal structure on the category of $\mathcal{A}$-spaces. We will achieve this by using the functoriality of the left Kan extension, see Appendix A.

Definition 3.2.1. Let $\mathcal{S}^{\mathcal{A}}$ denote the category with objects the functors from $\mathcal{A}$ to $\mathcal{S}$ and with natural transformations as morphisms. The objects in $\mathcal{S}^{\mathcal{A}}$ are also called $\mathcal{A}$-spaces.

For two $\mathcal{A}$-spaces X and Y let

$$X \times Y\colon\ \mathcal{A} \times \mathcal{A} \to \mathcal{S}$$

be the functor defined by $(X \times Y)(a_1, a_2) = X(a_1) \times Y(a_2)$ and $(X \times Y)(f, g) = X(f) \times Y(g)$.

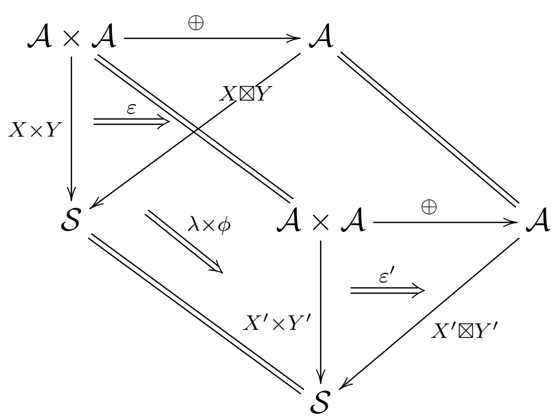
We now define the functor $\boxtimes$ which will be the monoidal product in $\mathcal{S}^{\mathcal{A}}$.

Definition 3.2.2 (Monoidal product)**.** We define the functor $\boxtimes\colon\ \mathcal{S}^{\mathcal{A}} \times \mathcal{S}^{\mathcal{A}} \to \mathcal{S}^{\mathcal{A}}$ on an object (X, Y) as the left Kan extension of $X \times Y$ along $\oplus$, the monoidal product in $\mathcal{A}$. By Corollary 2 in [ML98, Section X.3] this Kan extension always exists because $\mathcal{S}$ has all colimits and $\mathcal{A}$ is small. The same section shows that we can write the left Kan extension as a pointwise colimit, in this case we get:

$$X \boxtimes Y(a) = \operatorname*{colim}_{a_1 \oplus a_2 \to a} X(a_1) \times Y(a_2).$$

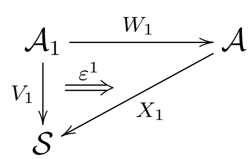
A morphism $(\lambda, \phi)\colon\ (X, Y) \to (X', Y')$ in $\mathcal{S}^{\mathcal{A}} \times \mathcal{S}^{\mathcal{A}}$ consists of two natural transformations $\lambda\colon\ X \to X'$ and $\phi\colon\ Y \to Y'$, so we get a natural transformation $\lambda \times \phi\colon\ X \times Y \to X' \times Y'$. By the functoriality of the left Kan extension, see Theorem A.0.4, we get a natural transformation $\lambda \boxtimes \phi\colon\ X \boxtimes Y \to X' \boxtimes Y'$. See Diagram (3.2).

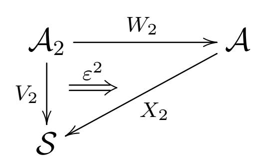
This preserves identity morphisms and composition, again by the functoriality of the left Kan extension, so $\boxtimes$ is indeed a functor.



(3.2)

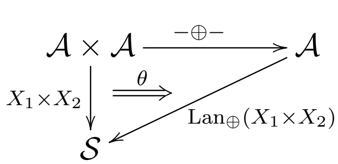
Lemma 3.2.3. *Suppose we have functors $X_1, X_2 \colon \mathcal{A} \to \mathcal{S}$ and that (X_1, ε^1) is the left Kan extension of $V_1 \colon \mathcal{A}_1 \to \mathcal{S}$ along $W_1 \colon \mathcal{A}_1 \to \mathcal{A}$ and that (X_2, ε^2) is the left Kan extension of $V_2 \colon \mathcal{A}_2 \to \mathcal{S}$ along $W_2 \colon \mathcal{A}_2 \to \mathcal{A}$.*

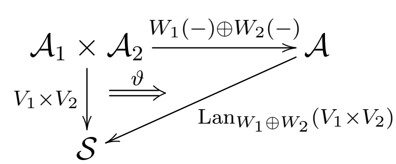




Let $(\mathrm{Lan}_{\oplus}(X_1 \times X_2), \theta)$ *be the left Kan extension of* $X_1 \times X_2$ *along* $\oplus \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. *Let* $(\mathrm{Lan}_{W_1 \oplus W_2}(V_1 \times V_2), \vartheta)$ *be the left Kan extension of* $V_1 \times V_2$ *along*

$$W_1(-) \oplus W_2(-) \colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}.$$





Then $\mathrm{Lan}_{\oplus}(X_1 \times X_2)$ *is isomorphic to* $\mathrm{Lan}_{W_1 \oplus W_2}(V_1 \times V_2)$.

Proof. The functor $\mathrm{Lan}_{W_1 \oplus W_2}(V_1 \times V_2)$ equals $\mathrm{Lan}_{\oplus \circ (W_1 \times W_2)}(V_1 \times V_2)$. By Lemma A.0.6

$$\mathrm{Lan}_{\oplus \circ (W_1 \times W_2)}(V_1 \times V_2) \cong \mathrm{Lan}_{\oplus}(\mathrm{Lan}_{W_1 \times W_2}(V_1 \times V_2))$$

and by Lemma A.0.7

$$\mathrm{Lan}_{\oplus}(\mathrm{Lan}_{W_1 \times W_2}(V_1 \times V_2)) \cong \mathrm{Lan}_{\oplus}(\mathrm{Lan}_{W_1} V_1 \times \mathrm{Lan}_{W_2} V_2),$$

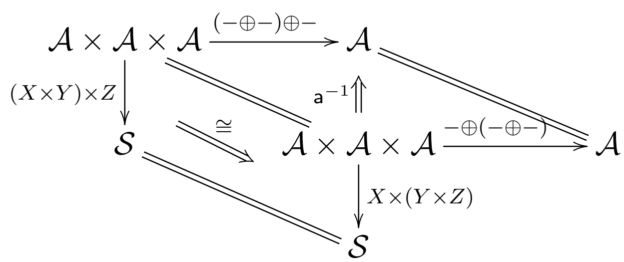
which is equal to $\mathrm{Lan}_{\oplus}(X_1 \times X_2)$. □

Definition 3.2.4 (Associativity constraint)**.** We want to define the associativity contraint for the monoidal structure on $\mathcal{S}^{\mathcal{A}}$. First we note that Lemma 3.2.3 implies that $(X \boxtimes Y) \boxtimes Z$ is naturally isomorphic to the left Kan extension of $(X \times Y) \times Z$ along $(- \oplus -) \oplus -$.

Similarly $X \boxtimes (Y \boxtimes Z)$ is naturally isomorphic to the left Kan extension of $X \times (Y \times Z)$ along $-\oplus(-\oplus-)$. Then by Theorem A.0.4 the natural isomorphisms $(X \times Y) \times Z \cong X \times (Y \times Z)$ and $\mathsf{a}^{-1}\colon\ -\oplus(-\oplus-) \xrightarrow{\cong} (-\oplus-)\oplus-$ induce an isomorphism

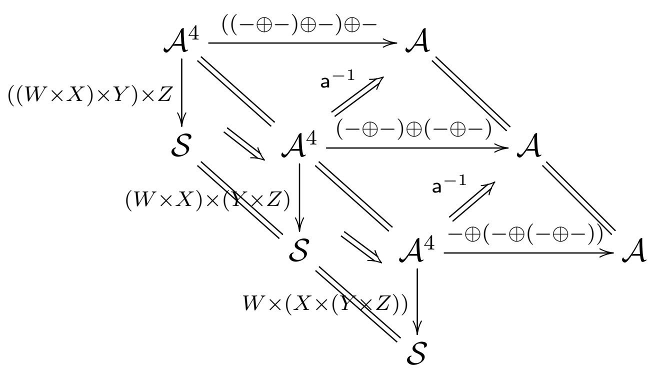
$$\mathfrak{a}\colon\ (X \boxtimes Y) \boxtimes Z \xrightarrow{\cong} X \boxtimes (Y \boxtimes Z)$$

that is natural in X, Y and Z.

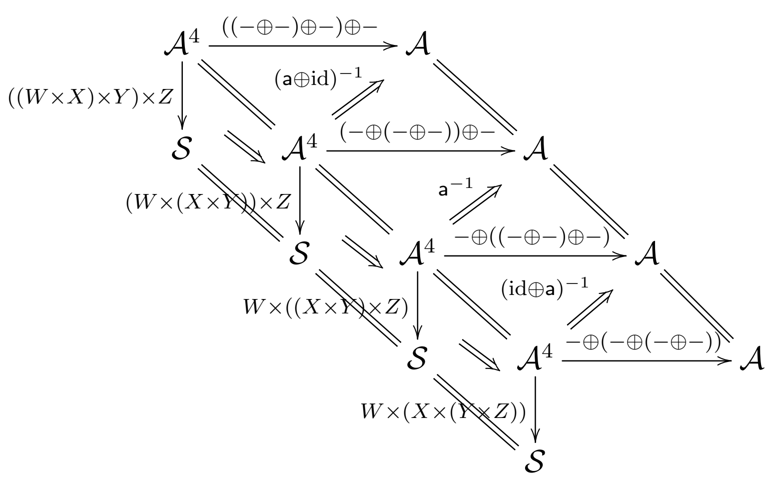


Lemma 3.2.5. *The associativity pentagon* (1.1) *for* $\boxtimes$ *commutes for all* $\mathcal{A}$*-spaces* W, X, Y *and* Z*.*

Proof. The morphism $\mathfrak{a}_{W,X,Y\boxtimes Z}\circ\mathfrak{a}_{W\boxtimes X,Y,Z}$ is the left Kan extension functor of the composite of the morphisms:



in the left Kan category. The morphism $(\mathrm{id}\boxtimes\mathfrak{a}_{X,Y,Z})\circ\mathfrak{a}_{W,X\boxtimes Y,Z}\circ(\mathfrak{a}_{W,X,Y}\boxtimes\mathrm{id})$ is the left Kan extension functor of the composite of the morphisms:



in the left Kan category.

The natural transformations in the parallellograms to the right are the associativity constraints for the monoidal product $\oplus$ in $\mathcal{A}$. Since the associativity pentagon commutes in $\mathcal{A}$, the the parallellograms to the right compose to the same thing in the upper and lower route respectively. Similarly the parallellograms to the left in the two diagrams compose to the same thing in the upper and lower route respectively, since $\mathcal{S}$ is a monidal category. So evaluating the left Kan extension functor on the two diagrams gives the same result and the pentagon commutes. $\square$

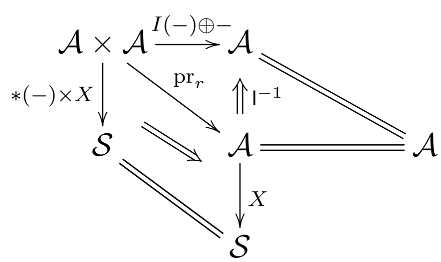
Definition 3.2.6 (Unit)**.** Let U be the functor $\mathcal{A} \to \mathcal{S}$ with $U(a) = \hom_{\mathcal{A}}(I, a) \times *$ and $U(f)$ defined by postcomposition, for objects a and morphisms f in $\mathcal{A}$.

Remark 3.2.7. Let $*(-)$ denote the constant $\mathcal{A}$-space taking all objects to $*$. Let $I(-)$ denote the constant functor $\mathcal{A} \to \mathcal{A}$ sending every object in $\mathcal{A}$ to I. It is then easy to check that U is the left Kan extension of $*\colon \mathcal{A} \to \mathcal{S}$ along $I\colon \mathcal{A} \to \mathcal{A}$.

The $\mathcal{A}$-space U will be the unit for the monoidal structure on $\mathcal{S}^{\mathcal{A}}$.

Definition 3.2.8 (Unit constraints)**.** For a $\mathcal{A}$-space X the product $U \boxtimes X$ is isomorphic to the left Kan extension of $*(-) \times X$ along $I(-) \oplus -$ by Remark 3.2.7 and Lemma 3.2.3.

Using the left unit constraints in $\mathcal{A}$ and $\mathcal{S}$ we get isomorphisms $*(a_1) \times X(a_2) \cong X(a_2)$ and $a_2 \cong I(a_1) \oplus a_2$ that are natural in a_1 and a_2. Theorem A.0.4 then gives an isomorphism $\mathfrak{l}\colon U \boxtimes X \to X$ that is natural in X. We let this be the left unit constraint. We get a right unit constraint $\mathfrak{r}\colon X \boxtimes U \to X$ similarly.



Lemma 3.2.9. *The triangle for unit* (1.2) *commutes for all $\mathcal{A}$-spaces X and Y.*

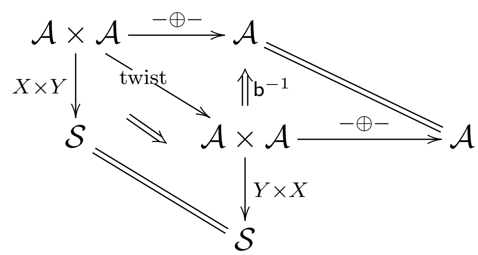
Proof. The proof is similar to the proof of Lemma 3.2.5 using that the triangle for unit commutes in the monoidal categories $\mathcal{A}$ and $\mathcal{S}$. $\square$

We have now completed the proof of the following proposition.

Proposition 3.2.10. *If $\mathcal{A}$ is a small monoidal category, the definitions in this section define a monoidal structure on $\mathcal{S}^{\mathcal{A}}$.*

For the rest of this section let $\mathcal{A}$ be a small braided monoidal category $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \mathsf{b})$.

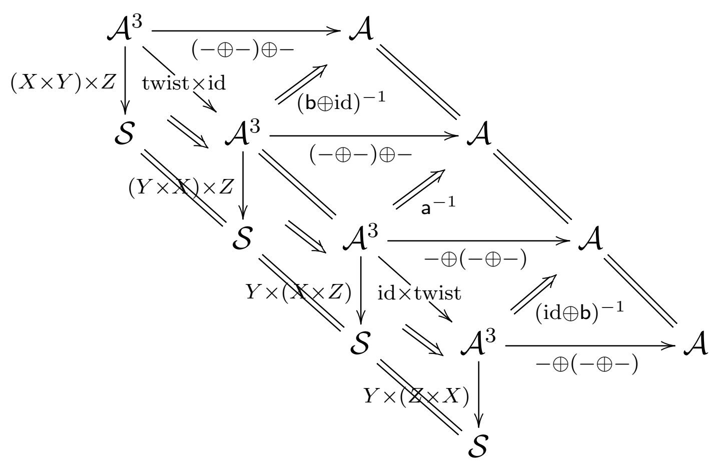
Definition 3.2.11 (Braiding)**.** The braidings in $\mathcal{A}$ and $\mathcal{S}$ give isomorphisms $\mathsf{b}_{a_1,a_2}\colon a_1 \oplus a_2 \xrightarrow{\cong} a_2 \oplus a_1$ and $X(a_1) \times Y(a_2) \cong Y(a_2) \times X(a_1)$ that are natural in a_1 and a_2. Theorem A.0.4 then provides an isomorphism $\mathfrak{b}_{X,Y}\colon X \boxtimes Y \xrightarrow{\cong} Y \boxtimes X$ that is natural in X and Y. We let this be the braiding of the monoidal product.



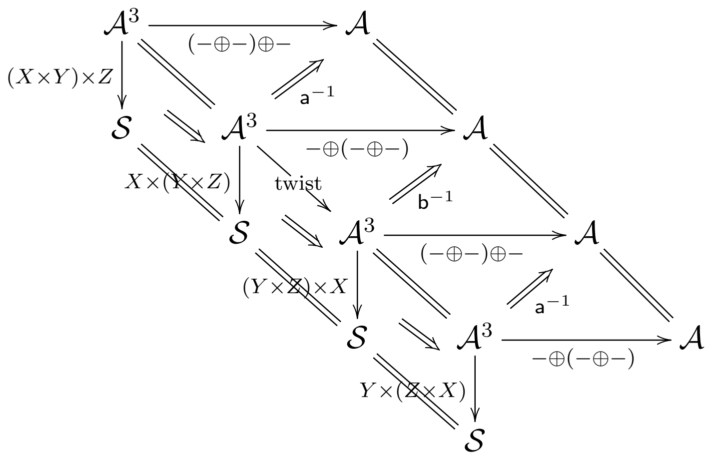
Lemma 3.2.12. *The hexagons for the braiding* (1.5) *and* (1.6) *commute for all $\mathcal{A}$-spaces X, Y and Z.*

Proof. We show that Diagram (1.5) commutes, the commutativity of the other hexagon is proven similarly.

The morphism $(\mathrm{id} \boxtimes \mathfrak{b}_{X,Z}) \circ \mathfrak{a}_{Y,X,Z} \circ (\mathfrak{b}_{X,Y} \boxtimes \mathrm{id})$ is the left Kan extension functor of the composite of the morphisms:



in the left Kan category. The morphism $\mathfrak{a}_{Y,Z,X} \circ \mathfrak{b}_{X,Y \boxtimes Z} \circ \mathfrak{a}_{X,Y,Z}$ is the left Kan extension functor of the composite of the morphisms:



in the left Kan category.

The parallellograms to the right in the two diagrams compose to the same thing in the upper and lower route respectively, since $\mathcal{A}$ is a braided monidal category. The parallellograms to the left in the two diagrams compose to the same thing in the upper and lower route respectively, since $\mathcal{S}$ is a symmetric monidal category. So evaluating the left Kan extension functor on the two diagrams gives the same result and the hexagon commutes. □

Theorem 3.2.13. *Let $\mathcal{A}$ be a small braided monoidal cartegory and let $\mathcal{S}$ be the cartegory of simplicial sets. The definitions in this section defines a braided monoidal structure on the category of $\mathcal{A}$-spaces, $\mathcal{S}^{\mathcal{A}}$. The monoidal product is closed, that is, there is a functor* $\mathrm{Hom}\colon (\mathcal{S}^{\mathcal{A}})^{\mathrm{op}} \times \mathcal{S}^{\mathcal{A}} \to \mathcal{S}^{\mathcal{A}}$ *and a family of natural isomorphisms*

$$\hom_{\mathcal{S}^{\mathcal{A}}}(X \boxtimes Y, Z) \cong \hom_{\mathcal{S}^{\mathcal{A}}}(X, \mathrm{Hom}(Y, Z)).$$

The functor Hom *is defined as the end:*

$$\mathrm{Hom}(Y, Z) = \int_{a\in\mathcal{A}} \mathrm{Map}_{\mathcal{S}}(Y(a), Z(- \oplus a))$$

on $\mathcal{A}$-spaces Y and Z.

Proof. The first claim is proven by the lemmas in this section. In the category of simplicial set we have a famliy of natural isomorphisms:

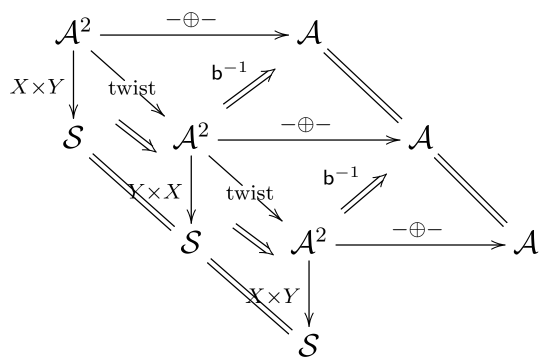
$$\hom_{\mathcal{S}}(X(d) \times Y(a), Z(d \oplus a)) \cong \hom_{\mathcal{S}}(X(d), \mathrm{Map}_{\mathcal{S}}(Y(a), Z(d \oplus a))).$$

Using this the second claim now follows from the universal properties of the Kan extension and the end construction. □

Corollary 3.2.14 (Corollary of the proof)**.** *If $\mathcal{A}$ is symmetric monoidal, the definitions in this section gives a symmetric monoidal structure on $\mathcal{S}^{\mathcal{A}}$.*

Proof. The only thing to check is that $\mathfrak{b}_{Y,X} \circ \mathfrak{b}_{X,Y}$ equals $\mathrm{id}_{X\boxtimes Y}$, so that the braiding for the braided monoidal structure on $\mathcal{S}^{\mathcal{A}}$, Theorem 3.2.13, is in fact a symmetry.

The morphism $\mathfrak{b}_{Y,X} \circ \mathfrak{b}_{X,Y}$ is the left Kan extension functor of the composite of the morphisms:



in the left Kan category. The natural transformations in the parallellograms compose to the identity, since $\mathcal{A}$ is symmetric monoidal. The parallellograms to the left compose to the identity since $\mathcal{S}$ is symmetric monidal. So evaluating the left Kan extension functor on the diagram gives the identity on $X \boxtimes Y$. □

3.3 A model structure on $\mathcal{S}^{\mathfrak{B}}$

The goal of this section is to define a model structure on $\mathfrak{B}$-spaces and establish a Quillen equivalence between this model category and the category of simplicial sets. This will be analogous to parts of what is done in Section 6 [SS11].

Lemma 3.3.1. *$\mathcal{S}^{\mathfrak{B}}$ is a simplicial category with mapping space defined as the end*

$$\mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(X, Y) = \int_{\mathbf{n}\in\mathfrak{B}} \mathrm{Map}_{\mathcal{S}}(X(\mathbf{n}), Y(\mathbf{n})),$$

where $\mathrm{Map}_{\mathcal{S}}$ is the function complex defined in Section I.5 [GJ99]. The tensor is defined by

$$(X \otimes K)(\mathbf{n}) = X(\mathbf{n}) \times K$$

and cotensor defined by

$$X^K(\mathbf{n}) = \mathrm{Map}_{\mathcal{S}}(K, X(\mathbf{n}))$$

for $\mathfrak{B}$-spaces X and Y and a simplicial set K.

This is easy to verify using the simplicial structure on simplicial sets, see [GJ99, page 84] and the universal property of the end construction.

Lemma 3.3.2. *For every object $\mathbf{m}$ in $\mathfrak{B}$ there is an adjuncction*

$$F_{\mathbf{m}} \colon \mathcal{S} \rightleftarrows \mathcal{S}^{\mathfrak{B}} : \mathrm{Ev}_{\mathbf{m}} \tag{3.3}$$

where $\mathrm{Ev}_{\mathbf{m}}$ *evaluates a* $\mathfrak{B}$*-space on* $\mathbf{m}$*, and* $F_{\mathbf{m}}(K)$ *is the free* $\mathfrak{B}$*-space defined by*

$$F_{\mathbf{m}}(K) = \hom_{\mathfrak{B}}(\mathbf{m}, -) \times K \tag{3.4}$$

for a simplicial set K*.*

We want to use Theorem 2.1.19 in [Hov99] to get the model structure. In that theorem there are some smallness requirements, so first we give the definitions necessary to define smallness and then we prove a Lemma 3.3.6 which takes care of all smallness issues.

Definition 3.3.3 (Definition 2.1.1 in [Hov99])**.** Suppose $\mathcal{C}$ is a category with all small colimits, and λ is an ordinal. A λ sequence in $\mathcal{C}$ is a colimit preserving functor $A\colon \lambda \to \mathcal{C}$, commonly written as

$$A_0 \to A_1 \to \ldots \to A_\vartheta \to \ldots.$$

Since A preserves colimits, for all limit ordinals $\theta < \lambda$, the induced map

$$\operatorname*{colim}_{\vartheta<\theta} A_\vartheta \to A_\theta$$

is an isomorphism. We refer to the map $A_0 \to \operatorname{colim}_{\vartheta<\lambda} A_\vartheta$ as the composition of the λ-sequence, though actually the composition is not unique, but only unique up to isomorphism under A, since the colimit is not unique. If $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$ and every map $A_\vartheta \to A_{\vartheta+1}$ for $\vartheta + 1 < \lambda$ is in $\mathcal{D}$, we refer to the composition $A_0 \to \operatorname{colim}_{\vartheta<\lambda} A_\vartheta$ as a transfinite composition of maps of $\mathcal{D}$.

Definition 3.3.4 (Definition 2.1.2 in [Hov99])**.** Let κ be a cardinal. An ordinal λ is κ-filtered if it is a limit ordinal and, if $S \subseteq \lambda$ and $|S| \leq \kappa$, then $\sup(S) < \lambda$.

Definition 3.3.5 (Definition 2.1.3 in [Hov99])**.** Suppose $\mathcal{C}$ is a category with all small colimits, $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$, X is an object of $\mathcal{C}$ and κ is a cardinal. We say that X is κ-small relative to $\mathcal{D}$ if, for all κ-filtered ordinals λ and all λ-sequences

$$A_0 \to A_1 \to \cdots \to A_\vartheta \to \cdots$$

such that each map $A_\vartheta \to A_{\vartheta+1}$ is in $\mathcal{D}$ for $\vartheta + 1 < \lambda$, the map of sets

$$\Theta\colon \operatorname*{colim}_{\vartheta<\lambda} \hom_{\mathcal{C}}(X, A_\vartheta) \to \hom_{\mathcal{C}}(X, \operatorname*{colim}_{\vartheta<\lambda} A_\vartheta) \tag{3.5}$$

is an isomorphism. We say that X is small relative to $\mathcal{D}$ if it is κ-small relative to $\mathcal{C}$ for some κ. We say that X is small if it is small relative to $\mathcal{C}$ itself.

The proof of the next lemma is similar to the proof of Lemma 3.1.1 in [Hov99] which shows that all simplical sets are small.

Lemma 3.3.6. *All objects in* $\mathcal{S}^{\mathfrak{B}}$ *are small.*

Proof. Suppose we are in the situation of Definition 3.3.5 with $\mathcal{C} = \mathcal{D} = \mathcal{S}^{\mathfrak{B}}$ and we want to show that the map (3.5) is a bijection. Since every simplicial set is small, [Hov99, Lemma 3.1.1], we know that $X(\mathbf{m})$ is $\kappa_{\mathbf{m}}$-small. The proof shows that $\kappa_{\mathbf{m}}$ is the cardinality of the set of simplexes in $X(\mathbf{m})$. Let ω be the cardinality of the natural numbers and define κ by

$$\kappa = \sup\{\bigcup_{\mathbf{m}\in\mathfrak{B}} \{\kappa_{\mathbf{m}}\} \cup \{\omega\}\}.$$

We will show that X is κ small.

Injectivity of the map (3.5) is the easiest part. Suppose f and g are different elements in $\operatorname{colim}_{\vartheta<\lambda} \hom_{\mathcal{C}}(X, A_\vartheta)$, then there is an $\mathbf{m}$ such that $f_{\mathbf{m}}$ is not equal to $g_{\mathbf{m}}$ in $\operatorname{colim}_{\vartheta<\lambda} \hom_{\mathcal{C}}(X(\mathbf{m}), A_\vartheta(\mathbf{m}))$. Because the simplicial set $X(\mathbf{m})$ is small, we get

$$\Theta(f)_{\mathbf{m}} = \Theta_{\mathbf{m}}(f_{\mathbf{m}}) \neq \Theta_{\mathbf{m}}(g_{\mathbf{m}}) = \Theta(g)_{\mathbf{m}}$$

so $\Theta(f) \neq \Theta(g)$.

To show surjectivity, let $h\colon X \to \operatorname{colim}_{\vartheta<\lambda} A_\vartheta$ be an element in $\hom_{\mathcal{C}}(X, \operatorname{colim}_{\vartheta<\lambda} A_\vartheta)$. Because simplicial sets are small, there is a $\theta_{\mathbf{m}} < \lambda$ such that h factors through $A_{\theta_{\mathbf{m}}}$, for every $\mathbf{m} \in \mathfrak{B}$. Let

$$\theta = \sup\{\cup_{\mathbf{m}\in\mathfrak{B}}\{\theta_{\mathbf{m}}\}\}.$$

The cardinality of $\{\cup_{\mathbf{m}\in\mathfrak{B}}\{\theta_{\mathbf{m}}\}\}$ is ω which is not greater than κ, so $\theta < \lambda$ since λ is κ-filtered. We get a collection of simplicial set maps $h'_{\mathbf{m}}\colon X(\mathbf{m}) \to A_\theta(\mathbf{m})$ which maps to h by Θ, but they do not necessarily determine a morphism of $\mathfrak{B}$-spaces. We define a set

$$\{(x, \alpha) \mid x \in X(\mathbf{m})_s,\ \alpha \in \hom_{\mathfrak{B}}(\mathbf{m}, \mathbf{n}), \text{ for some } \mathbf{m}, \mathbf{n} \in \mathfrak{B} \text{ and } s \geq 0\}.$$

The cardinality of this set is less than or equal to $\omega \cdot \kappa \cdot \omega \cdot \omega = \kappa$. Since the collection of $h'_{\mathbf{m}}$ map to a morphism of $\mathfrak{B}$-spaces by Θ, they do commute with the structure maps of the $\mathfrak{B}$-spaces in the colimit of the A_ϑ's. So for each pair (x, α) there is a $\vartheta_{(x,\alpha)}$ sucht that $h'_{\mathbf{m}}(X(\alpha)(x))$ is equal to $A_\theta(\alpha)(h'_{\mathbf{n}}(x))$ in $A_{\vartheta_{(x,\alpha)}}$. If we set $\vartheta = \sup\{\cup_{(x,\alpha)}\{\vartheta_{(x,\alpha)}\}\}$, we get $\vartheta < \lambda$ since λ is κ-filtered. This means that we can factor h through A_ϑ, and the map (3.5) is surjective. □

Definition 3.3.7 (Definitions 2.1.7 and 2.1.9 in [Hov99])**.** Let I be a class of maps in a category $\mathcal{C}$.

- ⋆ A map is I-injective if it has the right lifting property with respect to every map in I. The class of I-injective maps is denoted I-inj.
- ⋆ A map is I-projective if it has the left lifting property with respect to every map in I. The class of I-projective maps is denoted I-proj.
- ⋆ A map is an I-cofibration if it has the left lifting property with respect to every I-injective map. The class of I-cofibrations is the class (I-inj)-proj and is denoted I-cof.

- ⋆ A map is an I-fibration if it has the right lifting property with respect to every I-projective map. The class of I-fibrations is the class (I-proj)-inj and is denoted I-fib.
- ⋆ Let I be a set of maps in a category $\mathcal{C}$ containing all small colimits. A relative I-cell complex is a transfinite composition of pushouts of elements of I. That is, if $f\colon X \to Y$ is a relative I-cell complex, then there is an ordinal λ and a λ-sequence $A\colon \lambda \to \mathcal{C}$ such that f is the composition of A and such that, for each ϑ such that $\vartheta + 1 < \lambda$, there is a pushout square

$$\begin{array}{ccc} V_\vartheta & \longrightarrow & A_\vartheta \\ {\scriptstyle g_\vartheta}\downarrow & \ulcorner & \downarrow \\ W_\vartheta & \longrightarrow & A_{\vartheta+1} \end{array}$$

  such that $g_\vartheta \in I$. We denote the collection of relative I-cell complexes by I-cell. We say that $X \in \mathcal{C}$ is an I-cell complex if the map $\emptyset \to X$ is a relative I-cell complex.

Theorem 3.3.8 (Theorem 2.1.19 in [Hov99])**.** *Suppose $\mathcal{C}$ is a category with all small colimits and limits. Suppose $\mathcal{W}$ is a subcategory of $\mathcal{C}$, and $\mathcal{I}$ and $\mathcal{J}$ are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with $\mathcal{I}$ as the set of generating cofibrations, $\mathcal{J}$ as the set of generating acyclic cofibrations, and $\mathcal{W}$ as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

1. *The subcategory $\mathcal{W}$ has the two out of three property and is closed under retracts.*
2. *The domains of $\mathcal{I}$ are small relative to $\mathcal{I}$-cell.*
3. *The domains of $\mathcal{J}$ are small relative to $\mathcal{J}$-cell.*
4. *$\mathcal{J}$-cell $\subseteq \mathcal{W} \cap \mathcal{I}$-cof.*
5. *$\mathcal{I}$-inj $\subseteq \mathcal{W} \cap \mathcal{J}$-inj.*
6. *Either $\mathcal{W} \cap \mathcal{I}$-cof $\subseteq \mathcal{J}$-cof or $\mathcal{W} \cap \mathcal{J}$-inj $\subseteq \mathcal{I}$-inj.*

The next result is analogous to to a special case of Proposition 6.7 in [SS11].

Lemma 3.3.9 (Level model structure)**.** *There is a cofibrantly generated model structure, called the level model structure, on $\mathcal{S}^{\mathfrak{B}}$, where a map $X \to Y$ is a weak equivalence (respectively fibration) if $X(\mathbf{m}) \to Y(\mathbf{m})$ is a weak equivalence (respectively fibration) for every object $\mathbf{m}$ in $\mathfrak{B}$.*

The generating cofibrations are

$$I_l = \{F_{\mathbf{m}}(i) \mid \mathbf{m} \in \mathfrak{B},\ i\colon\ \partial\Delta^k \subseteq \Delta^k,\ k \geq 0\}$$

and the generating acyclic cofibrations are

$$J_l = \{F_{\mathbf{m}}(j) \mid \mathbf{m} \in \mathfrak{B},\ j\colon\ \Lambda^k_l \subseteq \Delta^k,\ 0 \leq l \leq k\}.$$

Proof. We use Theorem 3.3.8 to show that I_l and J_l are the generating cofibrations and acyclic cofibrations, respectively, for a model structure with weak equivalences defined as above. The fibrations in this model structure are the maps with the right lifting property with respect to every map in J_l. Using the adjunction (3.3) we see that these are precisely the levelwise fibrations. In the same way we get that the I_l-injective maps are precisely the levelwise acyclic fibrations. This implies that condition 5 and 6 in Theorem 3.3.8 holds.

Condition 1 is obviously satisfied and the two smallness requirements follow from Lemma 3.3.6.

We have J_l-cof $\subseteq I_l$-cof since I_l-inj $\subseteq J_l$-inj. Combining this with the fact that J_l-cell $\subseteq J_l$-cof [Hov99, 2.1.10], the thing that remains for condition 4 to hold, is that the maps in J_l-cell are levelwise weak equivalencces. That the morphisms in J_l are levelwise acyclic cofibrations follows easily from the definition. A map in J_l-cell is a transfinite composition of pushouts of maps in J_l and the class of acyclic cofibrations in $\mathcal{S}$ is closed under such a composition. Therefore every map in J_l-cell is a levelwise weak equivalence. □

Lemma 3.3.10. *With the level model structure from Lemma 3.3.9 and the simplicial structure defined in Lemma 3.3.1, $\mathcal{S}^{\mathfrak{B}}$ is a simplicial model category.*

Proof. Axiom SM7 for a simplicial model category is equivalent to the requirement that for all cofibrations $i\colon K \to L$ of simplicial sets and all fibrations $q\colon X \to Y$ in $\mathcal{S}^{\mathfrak{B}}$, $X^L \to X^K \times_{Y^K} Y^L$ is a fibration, which is acyclic if i or q is, see Proposition 3.13 in Chapter II in [GJ99]. This morphism evaluated at $\mathbf{m}$ in $\mathfrak{B}$ is

$$\mathrm{Map}_{\mathcal{S}}(L, X(\mathbf{m})) \to \mathrm{Map}_{\mathcal{S}}(K, X(\mathbf{m})) \times_{\mathrm{Map}_{\mathcal{S}}(L,Y(\mathbf{m}))} \mathrm{Map}_{\mathcal{S}}(L, Y(\mathbf{m})).$$

Since both the fibrations and weak equivalences in $\mathcal{S}^{\mathfrak{B}}$ are the levelwise fibrations and levelwise weak equivalences respectively, the axiom is satisfied because $\mathcal{S}$ is a simplicial model category. □

We now recall some general constructions that can be done in any simplicial model category category.

Definition 3.3.11 (The pushout product map)**.** For a map $f\colon X \to Y$ in $\mathcal{S}^{\mathfrak{B}}$ and $i\colon K \to L$ in $\mathcal{S}$ the pushout product map $f\Box i$ is the map

$$(X \otimes L) \amalg_{(X\otimes K)} (Y \otimes K) \to Y \otimes L$$

induced by $\mathrm{id} \otimes i\colon Y \otimes K \to Y \otimes L$ and $f \otimes \mathrm{id}\colon X \otimes L \to Y \otimes L$.

Definition 3.3.12 (Simplicial homotopy)**.** For two maps $f, g\colon X \to Y$ in $\mathcal{S}^{\mathfrak{B}}$ we can define a simplicial homotopy from f to g to be a map $H\colon X \otimes \Delta^1 \to Y$ such that $H \circ (\mathrm{id} \otimes i_0) = f$ and $H \circ (\mathrm{id} \otimes i_1) = g$, where i_0, i_1 are the two inclusions of Δ^0 in Δ^1.

Lemma 3.3.13. *Suppose we have a simplicial homotopy between f and g, then the simplicial set maps*

$$f^*, g^*\colon \mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(Y, Z) \to \mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(X, Z)$$

are homotopic for every object Z in $\mathcal{S}^{\mathfrak{B}}$.

In particular this means that if $f\colon X \to Y$ is a simplicial homotopy equivalence, then $f^*\colon \mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(Y,Z) \to \mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(X,Z)$ is a weak equivalence of simplicial sets for all objects $Z \in \mathcal{S}^{\mathfrak{B}}$.

Proof. A simplicial homotopy H from X to Y induces a map

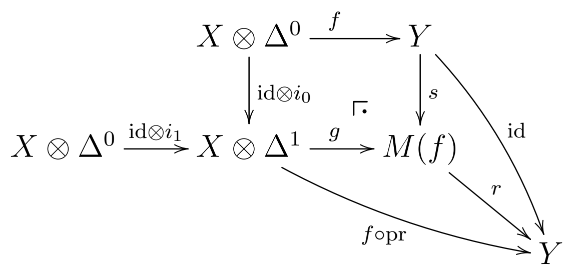
$$\mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(Y,Z) \xrightarrow{H^*} \mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(X \otimes \Delta^1, Z) \cong \mathrm{Map}_{\mathcal{S}}(\Delta^1, \mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(X,Z)),$$

which has an adjoint map

$$\mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(Y,Z) \times \Delta^1 \to \mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(X,Z),$$

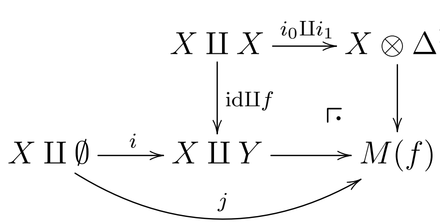
and this is a homotopy from f^* to g^*. □

Definition 3.3.14 (Mapping cylinder)**.** The mapping cylinder $M(f)$ of a map $f\colon X \to Y$ in $\mathcal{S}^{\mathfrak{B}}$ is the pushout in the diagram below:



The $\mathfrak{B}$-space Y is a strong deformation retract of $M(f)$, meaning that $rs = \mathrm{id}_Y$ and there is a simplicial homotopy from sr to the identity on $M(f)$ relative Y. In particular r is a simplicial homotopy equivalence. Let $j\colon X \otimes \Delta^0 \to M(f)$ be the composition of $\mathrm{id} \otimes i_1$ with g. We can then factor f as $r \circ j$.

The mapping cylinder is also the pushout of the following diagram:



The morphism $i_0 \amalg i_1$ is a cofibration, see Lemma 3.5 [GJ99, Chapter II], and if Y is cofibrant, the inclusion i is also a cofibration,hence the pushout of that map is a cofibration. Then j is a cofibration since it is the composition of two cofibrations, and we get a factorisation of f into a cofibration followed by a simplicial homotopy equivalence if Y is cofibrant.

Definition 3.3.15. [GJ99, Section II.8] We say a commutative square of morphisms in a proper closed model category

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle f} \\ W & \longrightarrow & Z \end{array}$$

is homotopy cartesian if for any factorisation of f into an acyclic cofibration i followed by a fibration p, the induced map i_* is a weak equivalence.

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle i_*} & & {\scriptstyle \sim}\downarrow{\scriptstyle i} \\ W \times_Z \tilde{Y} & \longrightarrow & \tilde{Y} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ W & \longrightarrow & Z \end{array}$$

Remark 3.3.16. In fact, it follows from Lemma 8.16 in [GJ99, Chapter II] that it suffices to find one such factorization $f = pi$ such that i_* is a weak equivalence. We will also use Lemma 8.22(1) in in the same section. It says that if the two horizontal maps in a commutative square are weak equivalences, then the square is homotopy cartesian.

Definition 3.3.17. [BK72, Section XII.5.1] For a functor X from a small category $\mathcal{A}$ to simplicial sets, the homotopy colimit is the simplicial set with k-simplices

$$(\operatorname*{hocolim}_{\mathcal{A}} X)_k = \coprod_{a_0 \leftarrow a_1 \leftarrow \cdots \leftarrow a_k} X(a_k)_k \tag{3.6}$$

for morphisms $a_0 \leftarrow a_1, \ldots, a_{k-1} \leftarrow a_k$ in $\mathcal{A}$. The simplicial structure maps are defined by

$$d_i(a_0 \leftarrow \cdots \leftarrow a_k, x) = (a_0 \leftarrow \cdots \leftarrow a_{i-1} \leftarrow a_{i+1} \leftarrow \cdots \leftarrow a_k, d_i(x)) \text{ for } 0 < i < k,$$

$$d_0(a_0 \leftarrow \cdots \leftarrow a_k, x) = (a_1 \leftarrow \cdots \leftarrow a_k, d_0(x)),$$
$$d_k(a_0 \leftarrow \cdots \leftarrow a_k, x) = (a_0 \leftarrow \cdots \leftarrow a_{k-1}, d_k(x))$$

and

$$s_i(a_0 \leftarrow \cdots \leftarrow a_k, x) = (a_0 \leftarrow \cdots \leftarrow a_i = a_i \leftarrow \cdots \leftarrow a_k, s_i(x)) \text{ for } 0 \leq i \leq k.$$

Lemma 3.3.18. *The homotopy colimit preserves colimits, more precisely, given a colimit* $\operatorname{colim}_{d \in \mathcal{D}} X_d$ *in* $\mathcal{S}^{\mathcal{A}}$*, then there is a natural isomorphism*

$$\operatorname*{hocolim}_{\mathcal{A}}(\operatorname*{colim}_{d \in \mathcal{D}} X_d) \cong \operatorname*{colim}_{d \in \mathcal{D}}(\operatorname*{hocolim}_{\mathcal{A}} X_d).$$

The homotopy colimit preserves tensors, that is, given a simplicial set K *and a functor* $X\colon \mathcal{A} \to \mathcal{S}$*, there is a natural isomorphism*

$$\operatorname*{hocolim}_{\mathcal{A}}(X) \times K \cong \operatorname*{hocolim}_{\mathcal{A}}(X \otimes K).$$

Proof. The homotopy colimit is left adjoint to the functor

$$\mathrm{Map}_{\mathcal{S}}(\mathcal{N}(-\downarrow\mathcal{A}), -)\colon \mathcal{S}\to\mathcal{S}^{\mathcal{A}},$$

see Section XII.2.2 [BK72]. This proves the first claim, since left adjoint functors preserve colimits, for the dual result see Theorem 1 Section V.5 [ML98].

The equations below define a simplicial set map that is natural in both X and K:

$$\begin{aligned}(\underset{\mathcal{A}}{\mathrm{hocolim}}(X)\times K)_k &= (\coprod_{a_0\leftarrow a_1\leftarrow\cdots\leftarrow a_k} X(a_k)_k)\times K_k \cong \coprod_{a_0\leftarrow a_1\leftarrow\cdots\leftarrow a_k}(X(a_k)_k\times K_k)\\ &= \coprod_{a_0\leftarrow a_1\leftarrow\cdots\leftarrow a_k}((X\otimes K)(a_k)_k) = \underset{\mathcal{A}}{\mathrm{hocolim}}(X\otimes K)_k,\end{aligned}$$

this proves the second claim. □

We will need the following result from [RSS01] for the proof of the next proposition.

Lemma 3.3.19. *[RSS01, Proposition 4.4] Let $\mathcal{C}$ be a small category, let $X\to Y$ be a map of $\mathcal{C}$-diagrams in $\mathcal{S}$, and let $\alpha\colon \mathbf{k}\to\mathbf{l}$ be a morphism in $\mathcal{C}$. Consider the two squares*

$$\begin{array}{ccc} X(\mathbf{k}) & \longrightarrow & Y(\mathbf{k}) \\ \downarrow{\scriptstyle X(\alpha)} & & \downarrow{\scriptstyle Y(\alpha)} \\ X(\mathbf{l}) & \longrightarrow & Y(\mathbf{l}) \end{array} \qquad\qquad \begin{array}{ccc} X(\mathbf{k}) & \longrightarrow & Y(\mathbf{k}) \\ \downarrow & & \downarrow \\ \mathrm{hocolim}_{\mathcal{C}}X & \longrightarrow & \mathrm{hocolim}_{\mathcal{C}}Y \end{array}$$

If the left hand square is homotopy cartesian for every α, then the right hand square is homotopy cartesian for every object $\mathbf{k}$.

Definition 3.3.20. For a morphism $\alpha\colon \mathbf{m}\to\mathbf{n}$ in $\mathfrak{B}$, we factor $\alpha^*\colon F_{\mathbf{n}}(*)\to F_{\mathbf{m}}(*)$ through the mapping cylinder, see Definition 3.3.14,

$$\alpha^*\colon F_{\mathbf{n}}(*)\xrightarrow{j_\alpha} M(\alpha^*)\xrightarrow{r_\alpha} F_{\mathbf{m}}(*).$$

The map r_α is then by construction a simplicial homotopy equivlaence, see Definition 3.3.14, and therefore also a levelwise weak equivalence. The space $F_{\mathbf{m}}(*)$ is cofibrant because the map $\emptyset = F_{\mathbf{m}}(\emptyset)\to F_{\mathbf{m}}(*)$ is a generating cofibration. By the argument made after the definition of the mapping cylinder, the morphism j_α is a cofibration in the level model structure.

Lemma 3.3.21. *The simplicial set map*

$$\underset{\mathfrak{B}}{\mathrm{hocolim}}(\alpha^*)\colon\ \underset{\mathfrak{B}}{\mathrm{hocolim}}\,F_{\mathbf{n}}(*)\to\underset{\mathfrak{B}}{\mathrm{hocolim}}\,F_{\mathbf{m}}(*)$$

is a weak equivalence for any $\alpha\colon \mathbf{m}\to\mathbf{n}$ in $\mathfrak{B}$.

Proof. From the definition of the homotopy colimit 3.3.17 and the definiton of $F_{\mathbf{n}}(*)$, in Lemma 3.3.2, it is easy to see that $\operatorname{hocolim}_{\mathfrak{B}} F_{\mathbf{n}}(*)$ is isomorphic to the nerve, see Definition 1.2.12, of the comma category $(\mathbf{n} \downarrow \mathfrak{B})$, see Definition 1.2.11. This comma category has an initial object, so $\operatorname{hocolim}_{\mathfrak{B}} F_{\mathbf{n}}(*)$ is contractible for every $\mathbf{n}$ in $\mathfrak{B}$. It follows that $\operatorname{hocolim}_{\mathfrak{B}}(\alpha^*)$ is a weak equivalence. □

The next result is analogous to a special case of Proposition 6.16 in [SS11].

Proposition 3.3.22 (Model structure)**.** *There is a cofibrantly generated model structure on $\mathcal{S}^{\mathfrak{B}}$, where a map $X \to Y$ is*

- *a weak equivalence if the induced map on the homotopy colimits is a weak equivalence of simplicial sets,*
- *a cofibration if it is a cofibration in the level model structure from Lemma 3.3.9,*
- *a fibration if it is a levelwise fibration and for every morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathfrak{B}$, we get a homotopy cartesian square*

$$\begin{array}{ccc} X(\mathbf{m}) & \xrightarrow{X(\alpha)} & X(\mathbf{n}) \\ \downarrow & & \downarrow \\ Y(\mathbf{m}) & \xrightarrow{Y(\alpha)} & Y(\mathbf{n}). \end{array} \tag{3.7}$$

The set of generating cofibrations for this model structure is $I = I_l$, and the generating acyclic cofibrations are

$$J = J_l \cup \bar{J} := J_l \cup \{j_\alpha \square i \mid \alpha\colon \mathbf{m} \to \mathbf{n},\ j_\alpha \text{ defined in 3.3.20},\ i\colon \partial\Delta^k \to \Delta^k,\ k \geq 0\}.$$

Lemma 3.3.23. *The J-injective morphisms are precisely the fibrations defined above.*

Proof. This is the same argument as in the proof of Lemma 3.4.12 in [HSS00] in the case of symmetric spectra. We have J-inj $= J_l$-inj$\cap\bar{J}$-inj and J_l-inj$=$ {levelwise fibrations} as before.

A map $f\colon X \to Y$ in $\mathcal{S}^{\mathfrak{B}}$ has the right lifting property (RLP) with respect to all $j_\alpha \square i$ if and only if every $\operatorname{Map}_\square(j_\alpha, f)$ has the RLP with respect to all i. This is because the diagram

$$\begin{array}{ccc} (M(\alpha^*) \otimes \partial\Delta^k) \coprod_{F_{\mathbf{n}}(*)\otimes\partial\Delta^k} (F_{\mathbf{n}}(*) \otimes \Delta^k) & \longrightarrow & X \\ \downarrow{\scriptstyle j_\alpha \square i} & & \downarrow{\scriptstyle f} \\ M(\alpha^*) \otimes \Delta^k & \longrightarrow & Y \end{array}$$

is adjoint to the diagram

$$\begin{array}{ccc} \partial\Delta^k & \longrightarrow & \operatorname{Map}_{\mathcal{S}^{\mathfrak{B}}}(M(\alpha^*), X) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \operatorname{Map}_\square(j_\alpha, f)} \\ \Delta^k & \longrightarrow & \operatorname{Map}_{\mathcal{S}^{\mathfrak{B}}}(M(\alpha^*), Y) \times_{\operatorname{Map}_{\mathcal{S}^{\mathfrak{B}}}(F_{\mathbf{n}}(*), Y)} \operatorname{Map}_{\mathcal{S}^{\mathfrak{B}}}(F_{\mathbf{n}}(*), X). \end{array}$$

The map $\mathrm{Map}_\square(j_\alpha, f)$ has the RLP with respect to all i if and only if $\mathrm{Map}_\square(j_\alpha, f)$ is an acyclic fibration of simplicial sets. We know that j_α is a cofibration in the level model structure, see Definition 3.3.20, so if f is a levelwise fibration, Proposition 3.3.10 implies that $\mathrm{Map}_\square(j_\alpha, f)$ is a fibration. So f is J-injective if and only if it is a levelwise fibration and $\mathrm{Map}_\square(j_\alpha, f)$ is a weak equivalence for all α.

In the next diagram of simplicial sets, the lower square is a pullback. The map r_α is a simplicial homotopy equivalence, see Definition 3.3.20, so by Lemma 3.3.13, r_α^* is a weak equivalence. When we assume that f is a levelwise fibration, the map p is a fibration which then implies that q is a weak equivalence since $\mathcal{S}$ is proper. Then from the top square we get that $\mathrm{Map}_\square(\alpha^*, f)$ is a weak equivalence if and only if $\mathrm{Map}_\square(j_\alpha, f)$ is.

$$\begin{array}{ccc}
\mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(F_{\mathbf{m}}(*), X) & \xrightarrow[\sim]{r_\alpha^*} & \mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(M(\alpha^*), X) \\
\downarrow{\scriptstyle \mathrm{Map}_\square(\alpha^*, f)} & & \downarrow{\scriptstyle \mathrm{Map}_\square(j_\alpha, f)} \\
\mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(F_{\mathbf{m}}(*), Y) \times_{\mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(F_{\mathbf{n}}(*), Y)} \mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(F_{\mathbf{n}}(*), X) & \xrightarrow{q} & \mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(M(\alpha^*), Y) \times_{\mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(F_{\mathbf{n}}(*), Y)} \mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(F_{\mathbf{n}}(*), X) \\
\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
\mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(F_{\mathbf{m}}(*), Y) & \xrightarrow[\sim]{r_\alpha^*} & \mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(M(\alpha^*), Y)
\end{array}$$

Since we have natural isomorphisms $F_{\mathbf{m}}(K \times L) \cong F_{\mathbf{m}}(K) \otimes L$ for simplicial sets K and L, Lemma 2.9 in Chapter II in [GJ99] says that $\mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(F_{\mathbf{m}}(*), X)$ is naturally isomorphic to $\mathrm{Map}_{\mathcal{S}}(*, X(\mathbf{m})) \cong X(\mathbf{m})$. Therefore $\mathrm{Map}_\square(\alpha^*, f)$ is naturally ismorphic to

$$X(\mathbf{m}) \to Y(\mathbf{m}) \times_{Y(\mathbf{n})} X(\mathbf{n}).$$

The conclusion is that f is J-injective if and only if it is a levelwise fibration and $X(\mathbf{m}) \to Y(\mathbf{m}) \times_{Y(\mathbf{n})} X(\mathbf{n})$ is a weak equivalence for every morphism $\alpha \colon \mathbf{m} \to \mathbf{n}$ in $\mathfrak{B}$, which is the criterion for being a fibration in Proposition 3.3.22. □

Proof of Proposition 3.3.22. Again we use Theorem 3.3.8 to show that this defines a model structure. Condition 1 follows from the two out of three property of weak equivalences in $\mathcal{S}$. Lemma 3.3.6 implies that the two smallness requirements hold.

The I-injective maps are the levelwise acyclic fibrations. For a levelwise weak equivalence the diagram (3.7) is homotopy cartesian for all maps α in $\mathfrak{B}$, see Remark 3.3.16. A levelwise weak equivalence also induces a weak equivalence on the homotopy colimits, this is a consequence of Proposition 1.8 [GJ99, Chapter IV]. So an I-injective map is both a weak equivalence and J-injective, which is condition 5 in Theorem 3.3.8. For the inclusion in the other direction, we apply Lemma 3.3.19 to a map which is both J-injective and a weak equivalence on the homotopy colimits, and get that the map is a levelwise acyclic fibration. Hence condition 6 is satisfied.

By the same argument as in the proof of Lemma 3.3.9 we get that J-cell$\subseteq I$-cof which is the first part of condition 4. The second part is that relative J-cell complexes should be weak equivalences. The strategy for this is to show that the homotopy colimit of a generating acyclic cofibration is an acyclic cofibration. Then the same will hold for the relative J-cell complexes because the homotopy colimit preserves colimits, see 3.3.18.

The free $\mathfrak{B}$-space $F_{\mathbf{m}}(K)$ is isomorphic to $F_{\mathbf{m}}(*) \otimes K$. The homotopy colimit preserves tensors, so $\operatorname{hocolim}_{\mathfrak{B}}(F_{\mathbf{m}}(K)) \cong \operatorname{hocolim}_{\mathfrak{B}}(F_{\mathbf{m}}(*)) \otimes K$. We see that the generating cofibrations are taken to cofibrations in $\mathcal{S}$, and the elements in J_l are taken to acyclic cofibrations. Since the homotopy colimit preserves colimits, it takes all of I-cell to cofibrations of simplicial sets. By Corollorary 2.1.15 in [Hov99] any map in I-cof is a retract of a map in I-cell. So the homotopy colimit of any map in I-cof is a cofibration.

Lemma 3.3.21 shows that $\operatorname{hocolim}_{\mathfrak{B}}(\alpha^*)$ is a weak quivalence. The map r_α is a levelwise weak equivalence, so the induced map on the homotopy colimits is also a weak equivalence. Therefore by the two out of three property $\operatorname{hocolim}_{\mathfrak{B}}(j_\alpha)$ is a weak equivalence. The homotopy colimit preserves colimits and tensors, see Lemma 3.3.18, so $\operatorname{hocolim}_{\mathfrak{B}}(j_\alpha \square i)$ is isomorphic to $\operatorname{hocolim}_{\mathfrak{B}}(j_\alpha) \square i$. The weak equivalence $\operatorname{hocolim}_{\mathfrak{B}}(j_\alpha)$ is also a cofibration, since j_α is an I-cofibration. This and the fact that i is a cofibration implies that $\operatorname{hocolim}_{\mathfrak{B}}(j_\alpha) \square i$ is an acyclic cofibration.

Now we know that all the generating acyclic cofibration are taken to acyclic cofibrations by the homotopy colimit, this also extends to J-cell since the homotopy colimit preserves colimits. In particular the morphisms in J-cell are weak equivalences, and this completes the proof of condition 4 and also the whole proof. □

Proposition 3.3.24. *With the model structure in Proposition 3.3.22 and the simplicial structure in Lemma 3.3.1 the category of $\mathfrak{B}$-spaces is a simplicial model category.*

Proof. SM7 is equivalent to the requirement that for all cofibrations j is $\mathcal{S}^{\mathfrak{B}}$ and all cofibrations i in $\mathcal{S}$, the map $j \square i$ is a cofibration which is acyclic if j or i is. We have the same cofibrations as in the level model structure, and this together with the same simplicial structure is a simplicial model structure, so $j \square i$ is a cofibration which is acyclic if i is. Since $\operatorname{hocolim}_{\mathfrak{B}}(j \square i) = \operatorname{hocolim}_{\mathfrak{B}}(j) \square i$, it is also acyclic if j is acyclic. □

The next result is analogous to Theorem 3.3 in [SS11].

Proposition 3.3.25. *There is a Quillen equivalence between $\mathcal{S}^{\mathfrak{B}}$ with the model structure in Proposition 3.3.22 and simplicial sets with the usual model structure by the adjunction*

$$\operatorname{colim}_{\mathfrak{B}} \colon \mathcal{S}^{\mathfrak{B}} \rightleftarrows \mathcal{S} : \operatorname{const}_{\mathfrak{B}}.$$

Proof. The constant functor preserves fibrations and acyclic fibrations, so the adjunction is a Quillen adjunction [Hov99, Lemma 1.3.4].

Since $\mathbf{0}$ is an initial object in $\mathfrak{B}$, the constant functor is isomorphic to $F_{\mathbf{0}}$, and using the adjunction (3.3) and that the acyclic fibrations are the levelwise acyclic fibrations, we get that $\operatorname{const}(K)$ is cofibrant for every simplicial set K. Proposition 18.9.4 in [Hir03] shows that the map $\operatorname{hocolim}_{\mathfrak{B}} X \to \operatorname{colim}_{\mathfrak{B}} X$ is a weak equivalence for all cofibrant $\mathfrak{B}$-spaces X.

The commutative diagram

$$\begin{array}{ccc} \operatorname{hocolim}_{\mathfrak{B}} X & \longrightarrow & \operatorname{hocolim}_{\mathfrak{B}} \operatorname{const}_{\mathfrak{B}}(K) \\ \downarrow \sim & & \downarrow \sim \\ \operatorname{colim}_{\mathfrak{B}} X & \longrightarrow & \operatorname{colim}_{\mathfrak{B}} \operatorname{const}_{\mathfrak{B}}(K) = K \end{array}$$

shows that if X is cofibrant, then a morphism $X \to \operatorname{const}_{\mathfrak{B}}(K)$ is a weak equivalence if and only if $\operatorname{colim}_{\mathfrak{B}} X \to K$ is a weak equivalence, so the adjunction is a Quillen equivalence. $\square$

Chapter 4

Braided operads

In this chapter we define braided operads, which are similar to operads, see Definition 1.1 in [May72], but with the actions of the symmetric groups Σ_n replaced by actions of the braid groups $\mathcal{B}_n$. We also define a certain class of braided operads called B_∞ operads similar to the E_∞ operads. Then we construct a B_∞ operad which acts on double loop spaces, and show that if a braided operad acts on a connected space, then the space is weakly equivalent to a double loop space. Finally we construct a braided analog of the Barratt-Eccles operad and show that it acts on the nerve of any small braided strict category.

All of this can be found in the article [Fie], but there are few details in the definitions and constructions and no details in the proofs. So we prove everything carefully here.

4.1 Braided operads and monads

In this section we work in the category of compactly generated Hausdorff topological spaces, see Section VII.8 in [ML98], denoted by $\mathcal{U}$. We write spaces for short. Let $\mathcal{T}$ be the corresponding category of pointed spaces.

Definition 4.1.1 (Braided operad)**.** [Fie, Definition 3.2] A braided operad $\mathcal{C}$ consists of spaces $\mathcal{C}(j)$ for $j \geq 0$ with $\mathcal{C}(0)$ a single point $*$, together with the following data:

(a) Continuous functions

$$\gamma\colon \mathcal{C}(k) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) \to \mathcal{C}(j)$$

where $j = \Sigma j_s$, such that the following associativity formula is satisfied for all $c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$ and $e_t \in \mathcal{C}(i_t)$:

$$\gamma(\gamma(c; d_1, \ldots, d_k); e_1, \ldots, e_j) = \gamma(c; f_1, \ldots, f_k),$$

where $f_s = \gamma(d_s; e_{j_1+\cdots+j_{s-1}+1}, \ldots, e_{j_1+\cdots+j_s})$, and $f_s = *$ if $j_s = 0$.

(b) An identity element $1 \in \mathcal{C}(1)$ such that $\gamma(1; d) = d$ for all $d \in \mathcal{C}(j)$ and $\gamma(c; 1^k) = c$ for $c \in \mathcal{C}(k)$, $1^k = (1, \ldots, 1) \in \mathcal{C}(1)^k$.

(c) A right action of the braid group $\mathcal{B}_j$ on $\mathcal{C}(j)$ such that the following equivariance formulas are satisfied for all $c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$, $\xi \in \mathcal{B}_k$, and $\zeta_s \in \mathcal{B}_{j_s}$:

$$\gamma(c\xi; d_1, \ldots, d_k) = \gamma(c; d_{\xi^{-1}(1)}, \ldots, d_{\xi^{-1}(k)})\xi(j_1, \ldots, j_k) \text{ and}$$

$$\gamma(c; d_1\zeta_1, \ldots, d_k\zeta_k) = \gamma(c; d_1, \ldots, d_k)(\zeta_1 \oplus \cdots \oplus \zeta_k),$$

where $\xi \in \mathcal{B}_k$ acts on $\{1, \ldots, k\}$ via the projection $\mathcal{B}_k \to \Sigma_k$, $\xi(j_1, \ldots, j_k)$ is the braid obtained from ξ by replaing the ith string of ξ by j_i parallel strings, and $\zeta_1 \oplus \cdots \oplus \zeta_k$ denotes the monoidal product of the braids $\zeta_1, \ldots, \zeta_k$ in $\mathcal{B}$.

Definition 4.1.2. A morphism of braided operads $\varphi\colon \mathcal{C} \to \mathcal{C}'$ is a sequence of $\mathcal{B}_k$ equivariant continuous functions

$$\varphi_k\colon \mathcal{C}(k) \to \mathcal{C}'(k)$$

such that $\varphi_k(1) = 1$ and the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{C}(k) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) & \xrightarrow{\gamma} & \mathcal{C}(j) \\
{\scriptstyle \varphi_k \times \varphi_{j_1} \times \cdots \times \varphi_{j_k}} \downarrow & & \downarrow {\scriptstyle \varphi_j} \\
\mathcal{C}'(k) \times \mathcal{C}'(j_1) \times \cdots \times \mathcal{C}'(j_k) & \xrightarrow{\gamma'} & \mathcal{C}'(j)
\end{array}$$

where $j = j_1 + \cdots + j_k$.

The next definition is analogous to the definition of an E_∞ operad, see Definition 3.5 and following paragraphs in [May72].

Definition 4.1.3 (B_∞ operad)**.** [Fie, Page 14] A B_∞ operad is a braided operad $\mathcal{C}$ where each space $\mathcal{C}(j)$ is contractible, and the action of $\mathcal{B}_j$ on $\mathcal{C}(j)$ is free for every $j \geq 0$.

The next definition is analogous to the description in [May72, Lemma 1.4] of an action of an operad.

Definition 4.1.4. An action of a braided operad $\mathcal{C}$ on a pointed space X consists of maps

$$\theta_k\colon \mathcal{C}(k) \times X^{\times k} \to X$$

for $k \geq 0$ where θ_0 maps the single point of $\mathcal{C}(0)$ to the basepoint of X, and such that

(a) the following diagrams are commutative, where $j = j_1 + \cdots + j_k$ and u denotes the evident shuffle homeomorphism:

$$\begin{array}{ccccc}
\mathcal{C}(k) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) \times X^{\times j} & \xrightarrow{\gamma \times \mathrm{id}} & \mathcal{C}(j) \times X^{\times j} & & \\
& & & \searrow^{\theta_j} & \\
{\scriptstyle \mathrm{id} \times u} \downarrow & & & & X \\
& & & \nearrow_{\theta_k} & \\
\mathcal{C}(k) \times \mathcal{C}(j_1) \times X^{\times j_1} \times \cdots \times \mathcal{C}(j_k) \times X^{\times j_k} & \xrightarrow{\mathrm{id} \times \theta_{j_1} \times \cdots \times \theta_{j_k}} & \mathcal{C}(k) \times X^{\times k} & &
\end{array}$$

(b) $\theta_1(1; x) = x$ for $x \in X$, and

(c) $\theta_k(c\xi; y) = \theta_k(c; \Phi(\xi)y)$ for c in $\mathcal{C}(k)$, ξ in $\mathcal{B}_k$ and y in $X^{\times k}$.

Definition 4.1.5. Let $\mathcal{C}$ be a braided operad, and $\mathcal{T}$ be the category of pointed spaces. We define a category $\mathcal{C}[\mathcal{T}]$ with objects pointed spaces with an action of $\mathcal{C}$. A morphism from X to X' in this category is a pointed continuous function $f\colon X \to X'$ such that the following diagrams commute for all $k \geq 0$:

$$\begin{array}{ccc} \mathcal{C}(k) \times X^{\times k} & \xrightarrow{\theta_k} & X \\ {\scriptstyle \mathrm{id}\times f^k}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathcal{C}(k) \times (X')^{\times k} & \xrightarrow{\theta'_k} & X' \end{array}$$

Definition 4.1.6 (Monad)**.** [May72, Definition 2.1] A monad $(\mathbb{C}, \mu, \eta)$ in $\mathcal{T}$ consists of a functor $\mathbb{C}\colon \mathcal{T} \to \mathcal{T}$ together with natural transformations

$$\mu\colon \mathbb{C}\mathbb{C} \to \mathbb{C} \text{ and } \eta\colon \mathrm{id}_{\mathcal{T}} \to \mathbb{C}$$

such that the following diagrams are commutative for all X in $\mathcal{T}$:

$$\begin{array}{ccccc} \mathbb{C}X & \xrightarrow{\mathbb{C}\eta(X)} & \mathbb{C}^2X & \xleftarrow{\eta(\mathbb{C}X)} & \mathbb{C}X \\ & \searrow\!\!\!\!= & \big\downarrow{\scriptstyle \mu(X)} & =\!\!\!\!\swarrow & \\ & & \mathbb{C}X & & \end{array} \qquad \begin{array}{ccc} \mathbb{C}^3X & \xrightarrow{\mu(\mathbb{C}X)} & \mathbb{C}^2X \\ {\scriptstyle \mathbb{C}\mu(X)}\big\downarrow & & \big\downarrow{\scriptstyle \mu(X)} \\ \mathbb{C}^2X & \xrightarrow{\mu(X)} & \mathbb{C}X. \end{array} \tag{4.1}$$

A morphism $\vartheta\colon (\mathbb{C}, \mu, \eta) \to (\mathbb{C}', \mu', \eta')$ of monads in $\mathcal{T}$ is a natural transformation of functors $\vartheta\colon \mathbb{C} \to \mathbb{C}'$ such that the following diagrams are commutative for all X in $\mathcal{T}$:

$$\begin{array}{ccc} & X & \\ {\scriptstyle \eta}\swarrow & & \searrow{\scriptstyle \eta'} \\ \mathbb{C}X & \xrightarrow{\vartheta} & \mathbb{C}'X \end{array} \qquad \begin{array}{ccc} \mathbb{C}\mathbb{C}X & \xrightarrow{\vartheta\circ\mathbb{C}\vartheta = \mathbb{C}'\vartheta\circ\vartheta} & \mathbb{C}'\mathbb{C}'X \\ {\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle \mu'} \\ \mathbb{C}X & \xrightarrow{\vartheta} & \mathbb{C}'X. \end{array}$$

Definition 4.1.7 (Algebra over monad)**.** [May72, Definition 2.2] An algebra (X, χ) over a monad $(\mathbb{C}, \mu, \eta)$ in a category $\mathcal{T}$ is an object X in $\mathcal{T}$ together with a map

$$\chi\colon \mathbb{C}X \to X$$

in $\mathcal{T}$ such that the following diagrams are commutative:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \mathbb{C}X \\ & =\!\!\!\!\searrow & \big\downarrow{\scriptstyle \chi} \\ & & X \end{array} \qquad \begin{array}{ccc} \mathbb{C}\mathbb{C}X & \xrightarrow{\mu} & \mathbb{C}X \\ {\scriptstyle \mathbb{C}\chi}\big\downarrow & & \big\downarrow{\scriptstyle \chi} \\ \mathbb{C}X & \xrightarrow{\chi} & X. \end{array}$$

A morphism $f\colon (X,\chi) \to (X',\chi')$ of $\mathbb{C}$-algebras is a map $f\colon X \to X'$ in $\mathcal{T}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C}X & \xrightarrow{\mathbb{C}f} & \mathbb{C}X' \\ \chi\downarrow & & \downarrow\chi' \\ X & \xrightarrow{f} & X'. \end{array}$$

The category of $\mathbb{C}$-algebras and their morphisms will be denoted by $\mathbb{C}[\mathcal{T}]$.

Definition 4.1.8. [Fie, Definition 3.3] Let $\mathcal{T}$ be the category of pointed spaces. The monad $\mathbb{C}$ in $\mathcal{T}$ associated to a braided operad $\mathcal{C}$, is defined as follows:

$$\mathbb{C}X = (\coprod_{n\geq 0} \mathcal{C}(n) \times X^{\times n})/\sim$$

with basepoint $[1,*]$, for a space X with basepoint $*$. The equivalence relation $\sim$ is generated by the following relations:

1. $(c\xi, y) \sim (c, \Phi(\xi)y)$ for c in $\mathcal{C}(n)$, ξ in $\mathcal{B}_n$, and $y \in X^{\times n}$. The functor Φ is defined in Definition 2.1.4 and the permutaion $\Phi(\xi)$ acts on $X^{\times n}$ by permuting the coordinates

2. $(\gamma(c; 1^i, *, 1^{n-1-i}), (x_1, \ldots, x_{n-1})) \sim (c, (x_1, \ldots, x_i, *, x_{i+1}, \ldots, x_{n-1}))$ for $1 \leq i \leq n$, c in $\mathcal{C}(n)$, $x_s \in X$, γ is the structure map of $\mathcal{C}$, 1 is the unit in $\mathcal{C}(1)$, $*$ represents the single point in $\mathcal{C}(0)$ in the left side of the equation and the basepoint in X in the right side of the equation.

One can show that $\mathbb{C}X$ is in $\mathcal{T}$ in the same way as this is shown for the monad associated to an operad in Propostion 2.6 [May72].

The multiplication $\mu\colon \mathbb{C}\mathbb{C} \to \mathbb{C}$ is the natural transformation determined by

$$\mu_X([c; [d_1, y_1], \ldots, [d_k, y_k]]) = [\gamma(c; d_1, \ldots, d_k), y_1, \ldots, y_k]$$

for c in $\mathcal{C}(k)$, d_s in $\mathcal{C}(j_s)$ and y_s in X^{j_s}. That this respects the first generating relation follows from the first equivariance condition for a braided operad. That it respects the second generating relation follows from the associativity of γ together with the condition on the unit. That μ is associative follows again from the associativity of γ.

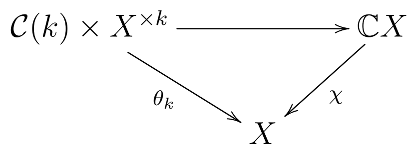
The unit $\eta\colon \mathrm{id}_{\mathcal{T}} \to \mathbb{C}$ is the natural transformation determined by

$$\eta_X(x) = [1, x]$$

for $x \in X$. That Diagram (4.1) commutes follows from the equations $\gamma(1; d) = d$ and $\gamma(c; 1^k) = c$.

The next result is analogous to Proposition 2.8 in [May72] for operads and monads associated to operads. It can be proven in the same way too, so we leave out the proof.

Proposition 4.1.9. *Let $\mathcal{C}$ be a braided operad and let $\mathbb{C}$ be its associated monad. Then for a pointed space X there is a one-to-one correspondence between $\mathcal{C}$-actions θ and $\mathbb{C}$-algebra structure maps $\chi\colon \mathbb{C}X \to X$ defined by letting θ correspond to χ if and only if the following diagrams are commutative for all k:*



where the horisontal morphism is the inclusion into the colimit. Moreover, this correspondence defines an isomorphism between $\mathcal{C}[\mathcal{T}]$, see Definition 4.1.5, and $\mathbb{C}[\mathcal{T}]$, see Definition 4.1.7.

Lemma 4.1.10. *A braided operad $\mathcal{C}$ determines a contravariant functor $\mathcal{C}(\cdot)$ from $\mathfrak{B}$ to $\mathcal{U}$.*

Proof. We use that any morphism from $\mathbf{m}$ to $\mathbf{n}$ in $\mathfrak{B}$ can be decomposed as a pair (μ, ζ) where $\mu \in \hom_{\mathcal{M}}(\mathbf{m}, \mathbf{n})$ and $\zeta \in \mathcal{B}_m$, see Lemma 2.1.6. Compostion is then given by the formula

$$(\nu, \xi) \circ (\mu, \zeta) = (\nu \circ \xi_*(\mu), \mu^*(\xi) \circ \zeta),$$

with $\xi_*(\mu)$ and $\mu^*(\xi)$ defined in Lemma 2.1.7.

We define $\mathcal{C}(\mathbf{m})$ to be $\mathcal{C}(m)$. For $\mu \in \hom_{\mathcal{M}}(\mathbf{m}, \mathbf{n})$ let $\mu'(i) = 1 \in \mathcal{C}(1)$ if i is in the image of μ and $* \in \mathcal{C}(0)$ otherwise. Then we can define the the function $\mathcal{C}(\mu, \zeta)\colon \mathcal{C}(n) \to \mathcal{C}(m)$ as

$$\mathcal{C}(\mu, \zeta)(c) = \gamma(c; \mu'(1), \dots, \mu'(n))\zeta$$

for $(\mu, \zeta)\colon \mathbf{m} \to \mathbf{n}$ in $\mathfrak{B}$ and $c \in \mathcal{C}(n)$. Identity morphisms are preserved since

$$\gamma(c; 1, \dots, 1)\mathrm{id} = c.$$

We check that composition is preserved:

$$\begin{aligned}
&\mathcal{C}(\mu, \zeta)(\mathcal{C}(\nu, \xi)(c)) \\
=&\gamma(\gamma(c; \nu'(1), \dots, \nu'(p))\xi; \mu'(1), \dots, \mu'(n))\zeta \\
=&\gamma(\gamma(c; \nu'(1), \dots, \nu'(p)); \mu'(\xi^{-1}(1)), \dots, \mu'(\xi^{-1}(n)))(\mu^*(\xi) \circ \zeta) \quad \text{by 4.1.1}(c) \text{ \& def of } \mu^*(\xi) \\
=&\gamma(\gamma(c; \nu'(1), \dots, \nu'(p)); (\xi_*(\mu))'(1), \dots, (\xi_*(\mu))'(n))(\mu^*(\xi) \circ \zeta) \quad \text{by def of } \xi_*(\mu)
\end{aligned}$$

By 4.1.1(a) this is equal to $\gamma(c; f_1, \dots, f_p)(\mu^*(\xi) \circ \zeta)$, if i is not in the image of ν

$$f_i = * = (\nu \circ \xi_*(\mu))'(i),$$

otherwise

$$f_i = \gamma(1; \xi_*(\mu)'(\nu^{-1}(i))) = \xi_*(\mu)'(\nu^{-1}(i)) = (\nu \circ \xi_*(\mu))'(i).$$

So we get

$$\begin{aligned}&\mathcal{C}(\mu,\zeta)(\mathcal{C}(\nu,\xi)(c))\\ =&\gamma(c;(\nu\circ\xi_*(\mu))'(1),\dots,(\nu\circ\xi_*(\mu))'(p))(\mu^*(\xi)\circ\zeta)\\ =&\mathcal{C}(\nu\circ\xi_*(\mu),\mu^*(\xi)\circ\zeta)(c)\\ =&\mathcal{C}((\nu,\xi)\circ(\mu,\zeta))(c)\end{aligned}$$

which shows that $\mathcal{C}(\cdot)$ is a contravariant functor. □

Definition 4.1.11. From a pointed space X we can construct a functor $\mathcal{I}$ to $\mathcal{U}$, the category of spaces, in the same way as we do in the simplicial case in Example 3.1.3. Let $X^\bullet$ denote this functor precomposed with $\Phi\colon \mathfrak{B}\to\mathcal{I}$.

Proposition 4.1.12. *Let $\mathcal{C}$ be a braided operad, and let X be a pointed space. The monad $\mathbb{C}$ associated to $\mathcal{C}$ evaluated at X is homeomorphic to the coend*

$$\int^{\mathfrak{B}}\mathcal{C}(\cdot)\times X^\bullet,$$

with basepoint $[1,*]$, where $\mathcal{C}(\cdot)$ is the functor $\mathfrak{B}^{\mathrm{op}}\to\mathcal{U}$ from Lemma 4.1.10 and $X^\bullet\colon \mathfrak{B}\to\mathcal{U}$ is defined in Definition 4.1.11.

Proof. The coend $\int^{\mathfrak{B}}\mathcal{C}(\cdot)\times X^{(\cdot)}$ is the coequalizer of the diagram

$$\coprod_{(\mu,\zeta)\colon\ \mathbf{m}\to\mathbf{n}}\mathcal{C}(n)\times X^{\times m}\rightrightarrows\coprod_{\mathbf{m}\in\mathfrak{B}}\mathcal{C}(m)\times X^{\times m}$$

So $\int^{\mathfrak{B}}\mathcal{C}(\cdot)\times X^\bullet$ is homeomorphic to $(\coprod_{\mathbf{m}\in\mathfrak{B}}\mathcal{C}(m)\times X^{\times m})/\sim'$ where $\sim'$ is the equivalence relation generated by

$$(\mathcal{C}(\mu,\zeta)(c),y)\sim'(c,X^\bullet(\mu,\zeta)(y))$$

for $c\in\mathcal{C}(n)$, $y\in X^{\times m}$ and for all morphisms $(\mu,\zeta)\colon\ \mathbf{m}\to\mathbf{n}$ in $\mathfrak{B}$. More explicitly the generating relations are of the form

$$(\gamma(c;\mu'(1),\dots,\mu'(n))\zeta,y)\sim'(c,X^\bullet(\mathrm{id},\mu)(\Phi(\zeta)y)).\tag{4.2}$$

To see that this space is the same as $\mathbb{C}X=(\coprod_{\mathbf{m}\in\mathfrak{B}}\mathcal{C}(m)\times X^{\times m})/\sim$ from Definition 4.1.8 we must show that the two equivalence relations are the same. The first generating relation $(c\zeta,y)\sim(c,\Phi(\zeta)y)$ from Definition 4.1.8 is of the form (4.2) with $\mu=\mathrm{id}$. The second one:

$$(\gamma(c;1^i,*,1^{n-1-i}),(x_1,\dots,x_{n-1}))\sim(c,(x_1,\dots,x_i,*,x_{i+1},\dots,x_{n-1}))$$

is of the form (4.2) with $\zeta=\mathrm{id}$ and $\mu\colon\ \mathbf{n-1}\to\mathbf{n}$ the injective order preserving function with image $\{1,\dots,i-1,i+1,\dots,n\}$.

The only thing left is to check that

$$[\gamma(c;\mu'(1),\dots,\mu'(n))\zeta,y]=[c,X(\mathrm{id},\mu)\Phi(\zeta)y]$$

in $\mathbb{C}X$. Write μ as a composition $\mu_{n-m} \circ \cdots \circ \mu_1$, where $\mu_i \in \hom_{\mathcal{M}}(\mathbf{m}+\mathbf{i}-\mathbf{1}, \mathbf{m}+\mathbf{i})$. Then we have a chain of generating relations in $\mathbb{C}X$:

$$\begin{aligned}
&(\gamma(c;\mu'(1),\ldots,\mu'(n))\zeta, y) \\
\sim&(\gamma(c;\mu'(1),\ldots,\mu'(n)), \Phi(\zeta)y) && \text{by 4.1.8(1)} \\
=&(\mathcal{C}(\mathrm{id},\mu_1)(c), \Phi(\zeta)y) \\
=&(\mathcal{C}(\mathrm{id},\mu_1)\circ\cdots\circ\mathcal{C}(\mathrm{id},\mu_{n-m})(c), \Phi(\zeta)y) \\
\sim&(\mathcal{C}(\mathrm{id},\mu_2)\circ\cdots\circ\mathcal{C}(\mathrm{id},\mu_{n-m})(c), X^\bullet(\mathrm{id},\mu_1)(\Phi(\zeta)y)) && \text{by 4.1.8(2)} \\
\sim&\cdots\sim(c, X^\bullet(\mathrm{id},\mu)(\Phi(\zeta)y)) && \text{by 4.1.8(2)}
\end{aligned}$$

so the two equivalence relations are equal. The baspoint in both $\mathbb{C}X$ and $\int^{\mathfrak{B}} \mathcal{C}(\cdot) \times X^\bullet$ is $[1, *]$ so they are the same pointed space. □

4.2 Braided operads and double loop spaces

In this section we construct a B_∞ operad and show that the monad associated to this braided operad is the same as the monad associated to the little rectangles operad, see Definition 4.1 [May72]. Then we use this to show that if any B_infty operad acts on a connected space, then the space is weakly equivalent to a double loop space.

Here we work in the category of compactly generated Hausdorff topological spaces, see Section VII.8 in [ML98], denoted by $\mathcal{U}$. We write spaces for short. Let $\mathcal{T}$ be the corresponding category of pointed spaces.

Example 4.2.1 (Little cubes operad)**.** [May72, Definition 4.1] Let I^n denote the unit n-cube and let J^n denote its interior. An little n-cube is a linear embedding f of J^n in J^n, with paralell axes; thus

$$f = f_1 \times \cdots \times f_n$$

where $f_i \colon J \to J$ is a linear function $f_i(t) = (y_i - x_i)t + x_i$, with $0 \le x_i < y_i \le 1$.

Define $\mathcal{C}_n^\Sigma(j)$ to be the set of those j-tuples $< c_1, \ldots, c_j >$ of little n-cubes such that the images of the c_r are pairwise disjoint. Let ${}^jJ^n$ denote the disjoint union of j copies of J^n. Regard $< c_1, \ldots, c_n >$ as a map ${}^jJ^n \to J^n$, and topologize $\mathcal{C}_n^\Sigma(j)$ as a subspace of the space of all continuous functions ${}^jJ^n \to J^n$. Write $\mathcal{C}_n^\Sigma(0) =<>$, and regard $<>$ as the unique "embedding" of the empty set in J^n.

The requisite data are defined by

(a) $\gamma(c; d_1, \ldots, d_k) = c \circ (d_1 \amalg \cdots \amalg d_k) \colon {}^{j_1}J^n \amalg \cdots \amalg {}^{j_k}J^n \to J^n$, for $c \in \mathcal{C}_n^\Sigma(k)$ and $d_s \in \mathcal{C}_n^\Sigma(j_s)$.

(b) $1 \in \mathcal{C}_n^\Sigma(1)$ is the identity function; and

(c) $< c_1, \ldots, c_j > \sigma =< c_{\sigma(1)}, \ldots, c_{\sigma(j)} >$ for $\sigma \in \Sigma_j$.

The associativity, unitary, and equivariance formulas requires of an operad are trivial to verify, and the action of Σ_j on $\mathcal{C}_n^{\Sigma}(j)$ is free in view of the requirement that the component little cubes of a point of $\mathcal{C}_n^{\Sigma}(j)$ have disjoint images.

Definition 4.2.2. [May72, Page 31] Define a morphism of operads

$$\mathcal{C}_n^{\Sigma} \to \mathcal{C}_{n+1}^{\Sigma}$$

by $< c_1, \dots, c_j > \mapsto < c_1 \times 1, \dots, c_j \times 1 >$, where 1 is the identity on J.

Lemma 4.2.3. *Let*

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{f} & Y \end{array}$$

be a commutative diagram of continuous functions with p a covering map, and

$$f_*(\pi_1(B, i(a))) \subseteq p_*(\pi_1(X, \bar{f}(a)))$$

for some $a \in A$. If in addition A is path connected and $fi(A)$ is evenly covered, there exists a unique lift in the diagram.

Proof. We can lift f to a unique map $\tilde{f}$ such that $\tilde{f}i(a) = \bar{f}(a)$ for some $a \in A$, see for instance [Hat02, Proposition 1.33]. So we need only show that $\tilde{f}i = \bar{f}$. Since A is connected and $fi(A)$ is evenly covered, the image of A under $\tilde{f}i$ is contained in one of the sheets. This sheet is homeomorphic to $fi(A)$, so now a diagram chase will show that $\tilde{f}i = \bar{f}$. □

Construction 4.2.4. [Fie, Example 3.3] We will construct a B_∞ operad $\tilde{\mathcal{C}}_2^{\mathcal{B}}$ from the little rectangles operad $\mathcal{C}_2^{\Sigma}$ from Example 4.2.1. Let $\tilde{\mathcal{C}}_2^{\mathcal{B}}(j)$ be the universal covering space of $\mathcal{C}_2^{\Sigma}(j)$ for each $j \geq 0$. For convenience we denote the covering map by p regardless of j.

Let $\mathcal{C}_1(j)$ denote the path component of the little intervals operad $\mathcal{C}_1^{\Sigma}(j)$ where $f_1 < \cdots < f_j$. The identity element $1 \in \mathcal{C}_1^{\Sigma}(1)$ is in $\mathcal{C}_1(1)$ and γ restricts to the $\mathcal{C}_1(j)$'s. But there is of course no action of Σ_j on $\mathcal{C}_1(j)$. This type of structure, that is a sequence of spaces together with the data of (a) and (b) in Definition 4.1.1 is called a non-Σ operad. Each of the spaces $\mathcal{C}_1(j)$ is contractible. We identify $\mathcal{C}_1$ with its image under the operad morphism $\mathcal{C}_1^{\Sigma} \to \mathcal{C}_2^{\Sigma}$, Definition 4.2.2. The morphism takes an interval to a column with the same width and horisontal position.

Since $\mathcal{C}_1(j)$ is contractible the inverse image under the covering map $p\colon \tilde{\mathcal{C}}_2^{\mathcal{B}}(j) \to \mathcal{C}_2^{\Sigma}(j)$ is homemorphic to a disjoint union of copies of $\mathcal{C}_1(j)$. Choose one of the copies and denote it by $\tilde{\mathcal{C}}_1(j)$. Let $h\colon \mathcal{C}_1(j) \to \tilde{\mathcal{C}}_1(j)$ denote the homeomorphism that inverse to p.

We will use the homeomorphism $h\colon \mathcal{C}_1(j) \to \tilde{\mathcal{C}}_1(j)$ to specify what the structure maps should be on $\tilde{\mathcal{C}}_1(j)$, and then lift the structure maps in $\mathcal{C}_2^{\Sigma}$ to structure maps on $\tilde{\mathcal{C}}_2^{\mathcal{B}}(j)$ that respect this.

As the identity element $\tilde{1}$ in $\tilde{\mathcal{C}}_2^{\mathcal{B}}(1)$ we choose the element $h(1) \in \tilde{\mathcal{C}}_1(1)$, where $1 \in \mathcal{C}_1(1)$.

By Lemma 4.2.3 there exits a unique lift in the diagram below, and we define $\tilde{\gamma}$ as this lift.

$$\begin{array}{ccccccccc}
\tilde{\mathcal{C}}_1(k)\times\tilde{\mathcal{C}}_1(j_1)\times\cdots\times\tilde{\mathcal{C}}_1(j_k) & \xrightarrow[\cong]{h^{-1}} & \mathcal{C}_1(k)\times\mathcal{C}_1(j_1)\times\cdots\times\mathcal{C}_1(j_k) & \xrightarrow{\gamma} & \mathcal{C}_1(j) & \xrightarrow[\cong]{h} & \tilde{\mathcal{C}}_1(j) & \xrightarrow{i} & \tilde{\mathcal{C}}_2^{\mathcal{B}}(j) \\
\downarrow{\scriptstyle i} & & & \tilde{\gamma} \nearrow & & & & & \downarrow{\scriptstyle p} \\
\tilde{\mathcal{C}}_2^{\mathcal{B}}(k)\times\tilde{\mathcal{C}}_2^{\mathcal{B}}(j_1)\times\cdots\times\tilde{\mathcal{C}}_2^{\mathcal{B}}(j_k) & \xrightarrow{p} & \mathcal{C}_2^{\Sigma}(k)\times\mathcal{C}_2^{\Sigma}(j_1)\times\cdots\times\mathcal{C}_2^{\Sigma}(j_k) & & & \xrightarrow{\gamma} & & & \mathcal{C}_2^{\Sigma}(j)
\end{array}$$

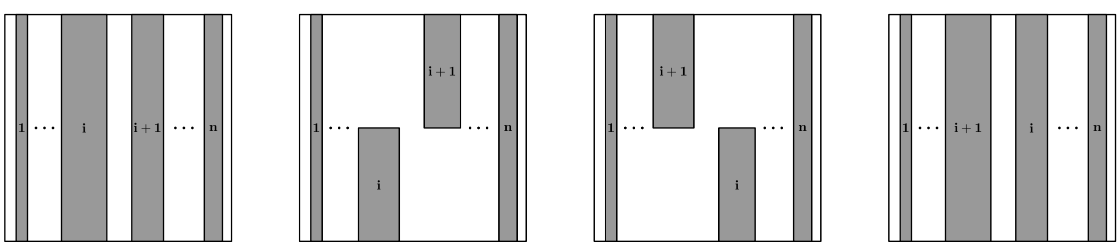
Recall the definition of $F_{0,n}\mathbb{R}^2$ from Defintion 1.1.1. The pure braid group, $\mathcal{P}_n$, on n strings is defined as the fundamental group of $F_{0,n}\mathbb{R}^2$ with basepoint $((1,0),\ldots,(n,0))$. The space $F_{0,n}\mathbb{R}^2$ is path connected so if we change basepoint we get an isomorphic group. For convenience we can exchange $\mathbb{R}^2$ with J^2, see Example 4.2.1, and the fundamental group of $F_{0,n}J^2$ with any basepoint is still isomorphic to $\mathcal{P}_n$. The fundamental group of $(F_{0,n}J^2)/\Sigma_n$ with any basepoint is isomorpic to the braid group, $\mathcal{B}_n$.

There is a homotopy equivalence $\Theta\colon \mathcal{C}_2^{\Sigma}(n) \to F_{0,n}J^2$, see Theorem 4.8 in [May72]. Informally Θ takes each little rectangle to the midpoint of the rectangle. Let P denote the composite

$$\mathcal{C}_2^{\Sigma}(n) \xrightarrow{\Theta} F_{0,n}J^2 \to (F_{0,n}J^2)/\Sigma_n,$$

where the last map is the projection. We will use P to define the action of $\mathcal{B}_n$ on $\tilde{\mathcal{C}}_2^{\mathcal{B}}(n)$.

Let ξ be an element in $\mathcal{B}_n$ and σ be the projection of it in Σ_n. Choose any point $\tilde{x}$ in $\tilde{\mathcal{C}}_1(n)$, and let $\alpha_{\tilde{x},\xi}$ be any path in $\mathcal{C}_2^{\Sigma}(n)$ from $x = p\tilde{x}$ to $x\sigma$ such that $P(\alpha_{\tilde{x},\xi})$ represents ξ in $\pi_1(F_{0,n}J^2/\Sigma_n, P(x))$. The next illustration shows one possible choice of such a path for a generator ζ_i.



By writing ξ as a product of generators we can therefore always find such an $\alpha_{\tilde{x},\xi}$.

Lift $\alpha_{\tilde{x},\xi}$ to a path $\tilde{\alpha}_{\tilde{x},\xi}$ in $\tilde{\mathcal{C}}_2^{\mathcal{B}}(n)$ starting in $\tilde{x}$, and define $\tilde{x}\xi$ to be the endpoint of $\tilde{\alpha}_{\tilde{x},\xi}$. This is well defined, because if $[P\alpha_{\tilde{x},\xi}] = [P\alpha'_{\tilde{x},\xi}]$ in $\pi_1(F_{0,n}J^2/\Sigma_n, P(x))$, the two paths $\alpha_{\tilde{x},\xi}$ and $\alpha'_{\tilde{x},\xi}$ are path homotopic in $\mathcal{C}_2^{\Sigma}(n)$ via a homotopy that fix the endpoints, so they lift to paths that have the same endpoint.

We define the action of ξ on $\tilde{\mathcal{C}}_1(n)$ to be the lift in the diagram below.

$$\begin{array}{ccccc}
\{\tilde{x}\} & \longrightarrow & \{\tilde{x}\xi\} & \longrightarrow & \tilde{\mathcal{C}}_2^{\mathcal{B}}(n) \\
\downarrow & & \nearrow & & \downarrow{\scriptstyle p} \\
\tilde{\mathcal{C}}_1(n) & \xrightarrow{p} & \mathcal{C}_1(n) & \xrightarrow{\sigma} & \mathcal{C}_2^{\Sigma}(n)
\end{array}$$

We want to know that this definition is independent of the point $\tilde{x}$. Let $\tilde{y}$ be any other point in $\tilde{\mathcal{C}}_1(n)$, and $y = p(\tilde{y})$. From the proof of the lifting propety, [Hat02, Proposition 1.33], we learn how ξ acts on $\tilde{y}$. Take any path ω from x to y in $\mathcal{C}_1(n)$, and let $\bar{\omega}$ denote the reversed path. Lift the product path $\omega \cdot \alpha_{\tilde{x},\xi} \cdot \bar{\omega}$ to a path in $\tilde{\mathcal{C}}_2^{\mathcal{B}}(n)$ starting at $\tilde{y}$. Then $\tilde{y}\xi$ is by definition the endpoint of this path. The path $P(\omega)$ determines an isomorphism

$$\pi_1(F_{0,n}J^2/\Sigma_n, P(x)) \to \pi_1(F_{0,n}J^2/\Sigma_n, P(y)).$$

Since $\tilde{\mathcal{C}}_1(n)$ is contractible this isomorphism is canonical. Therefore the result would have been the same if we had started with ξ in $\pi_1(F_{0,n}J^2/\Sigma_n, P(y))$. So th action of ξ on $\tilde{\mathcal{C}}_1(n)$ is independent of $\tilde{x}$.

Now we can define the map $\xi\colon \tilde{\mathcal{C}}_2^{\mathcal{B}}(n) \to \tilde{\mathcal{C}}_2^{\mathcal{B}}(n)$ as the lift in the diagram below.

$$\begin{tikzcd}
\tilde{\mathcal{C}}_1(n) \arrow[rr, "\xi"] \arrow[d, "i"] & & \tilde{\mathcal{C}}_2^{\mathcal{B}}(n) \arrow[d, "p"] \\
\tilde{\mathcal{C}}_2^{\mathcal{B}}(n) \arrow[r, "p"] \arrow[urr, dotted] & \mathcal{C}_2^{\Sigma}(n) \arrow[r, "\sigma"] & \mathcal{C}_2^{\Sigma}(n)
\end{tikzcd}$$

To see that this determines an action of the braid group, we have to check that $\xi \circ \zeta = \xi\zeta$. It is enough to check this for an element $\tilde{x} \in \tilde{\mathcal{C}}_1(n)$ since two lifts of the same map are the same if they agree on a point. We know how $\tilde{x}(\xi\zeta)$ is defined, but $\tilde{x}\xi$ is not in $\tilde{\mathcal{C}}_1(n)$, so we go the construction of the lifting in [Hat02, Proposition 1.33] to see how $(\tilde{x}\xi)\zeta$ is defined. We take the path $\tilde{\alpha}_{\tilde{x},\xi}$ from $\tilde{x}$ to $\tilde{x}\xi$, map it with $p(\zeta)p$ to $\alpha_{\tilde{x},\xi}p(\zeta)$ in $\mathcal{C}_2^{\Sigma}(n)$. We lift the path $(\alpha_{\tilde{x},\xi}p(\zeta)) \cdot \alpha_{\tilde{x},\zeta}$ to a path starting in $\tilde{x}$, then $(\tilde{x}\xi)\zeta$ is the endpoint of this lift. So since

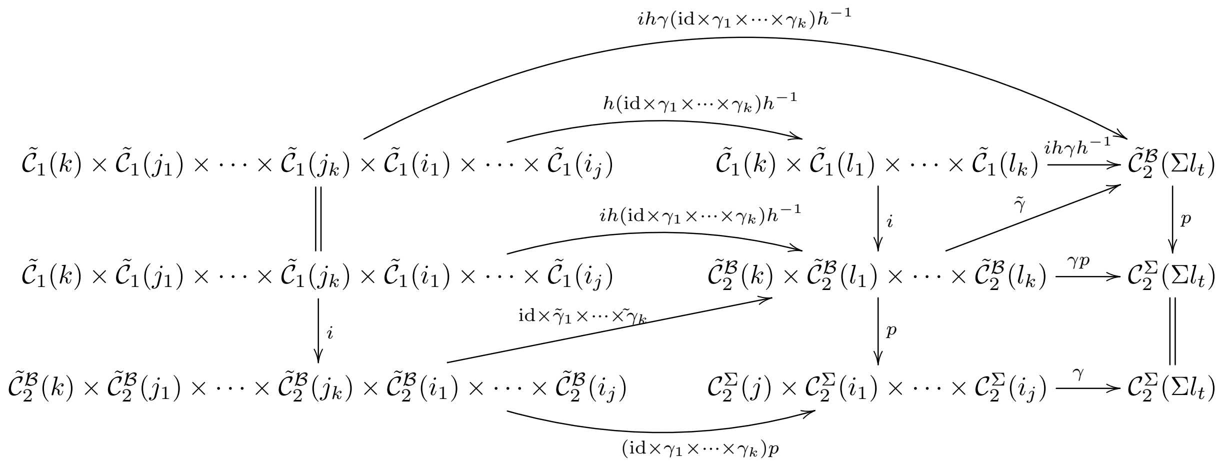
$$[P((\alpha_{\tilde{x},\xi}p(\zeta)) \cdot \alpha_{\tilde{x},\zeta})] = [P(\alpha_{\tilde{x},\xi}p(\zeta))][P\alpha_{\tilde{x},\zeta}] = \xi \circ \zeta = [P\alpha_{\tilde{x},\xi\zeta}]$$

in $\pi_1(F_{0,n}J^2/\Sigma_n, P(p(\tilde{x})))$ we are done.

Proposition 4.2.5. *The construction in 4.2.4 is a B_∞ operad.*

Proof. We first check the associativity condition. Writing out the definition of $\tilde{\gamma}(\tilde{\gamma} \times \mathrm{id})$ and $\tilde{\gamma}(\mathrm{id} \times \tilde{\gamma}_1 \times \cdots \times \tilde{\gamma}_k)$, we find that they are lifts in the next two outer diagrams respectively.

$$\begin{tikzcd}
\tilde{\mathcal{C}}_1(k) \times \tilde{\mathcal{C}}_1(j_1) \times \cdots \times \tilde{\mathcal{C}}_1(j_k) \times \tilde{\mathcal{C}}_1(i_1) \times \cdots \times \tilde{\mathcal{C}}_1(i_j) \arrow[r, "h(\gamma\times\mathrm{id})h^{-1}"] \arrow[d, equal] \arrow[rr, bend left, "ih\gamma(\gamma\times\mathrm{id})h^{-1}"] & \tilde{\mathcal{C}}_1(j) \times \tilde{\mathcal{C}}_1(i_1) \times \cdots \times \tilde{\mathcal{C}}_1(i_j) \arrow[r, "ih\gamma h^{-1}"] \arrow[d, "i"] & \tilde{\mathcal{C}}_2^{\mathcal{B}}(\Sigma i_s) \arrow[d, "p"] \\
\tilde{\mathcal{C}}_1(k) \times \tilde{\mathcal{C}}_1(j_1) \times \cdots \times \tilde{\mathcal{C}}_1(j_k) \times \tilde{\mathcal{C}}_1(i_1) \times \cdots \times \tilde{\mathcal{C}}_1(i_j) \arrow[r, "ih(\gamma\times\mathrm{id})h^{-1}"] \arrow[d, "i"] & \tilde{\mathcal{C}}_2^{\mathcal{B}}(j) \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(i_1) \times \cdots \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(i_j) \arrow[r, "\gamma p"] \arrow[d, "p"] \arrow[ur, "\tilde{\gamma}"] & \mathcal{C}_2^{\Sigma}(\Sigma i_s) \arrow[d, equal] \\
\tilde{\mathcal{C}}_2^{\mathcal{B}}(k) \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(j_1) \times \cdots \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(j_k) \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(i_1) \times \cdots \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(i_j) \arrow[r, "(\gamma\times\mathrm{id})p"] \arrow[ur, "\tilde{\gamma}\times\mathrm{id}"] & \mathcal{C}_2^{\Sigma}(j) \times \mathcal{C}_2^{\Sigma}(i_1) \times \cdots \times \mathcal{C}_2^{\Sigma}(i_j) \arrow[r, "\gamma"] & \mathcal{C}_2^{\Sigma}(\Sigma i_s)
\end{tikzcd}$$

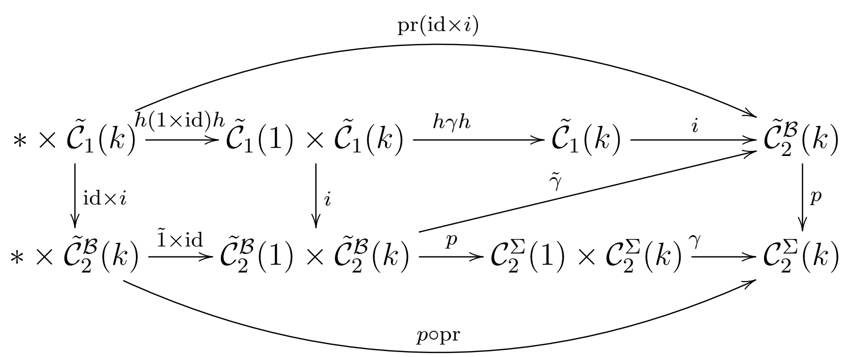


The two outer diagrams are equal since γ is associative in $\mathcal{C}_1$ and $\mathcal{C}_2^{\Sigma}$, so $\tilde{\gamma}$ is associative as well.

If we write $\tilde{1}\colon * \to \tilde{\mathcal{C}}_2^{\mathcal{B}}(1)$ for the map taking the single point to $\tilde{1}$ in $\tilde{\mathcal{C}}_2^{\mathcal{B}}(1)$, the first condition in Definition 4.1.1(b) is that the map

$$* \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(k) \xrightarrow{\tilde{1}\times\mathrm{id}} \tilde{\mathcal{C}}_2^{\mathcal{B}}(1) \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(k) \xrightarrow{\tilde{\gamma}} \tilde{\mathcal{C}}_2^{\mathcal{B}}(k)$$

is equal to the projection $\mathrm{pr}\colon\ * \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(k) \to \tilde{\mathcal{C}}_2^{\mathcal{B}}(k)$. The following diagram shows how $\tilde{\gamma} \circ (\tilde{1} \times \mathrm{id})$ is defined.



The projection onto $\tilde{\mathcal{C}}_2^{\mathcal{B}}(k)$ is also a lift in the outer diagram, so $\mathrm{pr} = \tilde{\gamma}(\tilde{1} \times \mathrm{id})$.

The second condition in 4.1.1(b) is that the map

$$\tilde{\mathcal{C}}_2^{\mathcal{B}}(k) \times * \times \cdots \times * \xrightarrow{\mathrm{id}\times\tilde{1}\times\cdots\times\tilde{1}} \tilde{\mathcal{C}}_2^{\mathcal{B}}(k) \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(1) \times \cdots \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(1) \xrightarrow{\tilde{\gamma}} \tilde{\mathcal{C}}_2^{\mathcal{B}}(k)$$

is equal to the projection $\mathrm{pr}\colon \tilde{\mathcal{C}}_2^{\mathcal{B}}(k) \times * \times \cdots \times * \to \tilde{\mathcal{C}}_2^{\mathcal{B}}(k)$. The following diagram shows how $\tilde{\gamma} \circ (\mathrm{id} \times \tilde{1} \times \cdots \times \tilde{1})$ is defined.

$$\begin{tikzcd}
\tilde{\mathcal{C}}_1(k) \times * \times \cdots \times * \arrow[r, "h(\mathrm{id}\times 1\times\cdots\times 1)h"] \arrow[d, "i\times\mathrm{id}\times\cdots\times\mathrm{id}"] \arrow[rrr, bend left, "\mathrm{pr}(i\times\mathrm{id}\times\cdots\times\mathrm{id})"] & \tilde{\mathcal{C}}_1(k) \times \tilde{\mathcal{C}}_1(1) \times \cdots \tilde{\mathcal{C}}_1(1) \arrow[r, "h\gamma h"] \arrow[d, "i"] & \tilde{\mathcal{C}}_1(k) \arrow[r, "i"] & \tilde{\mathcal{C}}_2^{\mathcal{B}}(k) \arrow[d, "p"] \\
\tilde{\mathcal{C}}_2^{\mathcal{B}}(k) \times * \times \cdots \times * \arrow[r, "\mathrm{id}\times\tilde{1}\times\cdots\times\tilde{1}"] \arrow[rrr, bend right, "p\circ\mathrm{pr}"'] & \tilde{\mathcal{C}}_2^{\mathcal{B}}(k) \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(1) \times \cdots \tilde{\mathcal{C}}_2^{\mathcal{B}}(1) \arrow[r, "p"] \arrow[rru, "\tilde{\gamma}"] & \mathcal{C}_2^{\Sigma}(k) \times \mathcal{C}_2^{\Sigma}(1) \times \cdots \mathcal{C}_2^{\Sigma}(1) \arrow[r, "\gamma"] & \mathcal{C}_2^{\Sigma}(k)
\end{tikzcd}$$

The projection onto $\tilde{\mathcal{C}}_2^{\mathcal{B}}(k)$ is also a lift in the outer diagram, so $\mathrm{pr} = \tilde{\gamma}(\mathrm{id} \times \tilde{1} \times \cdots \times \tilde{1})$.

There are two equivariance conditions to check. Let ξ be an element in $\mathcal{B}_k$, then the map

$$\tilde{\gamma}(\xi \times \mathrm{id} \times \cdots \times \mathrm{id})\colon\ \tilde{\mathcal{C}}_2^{\mathcal{B}}(k) \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(j_1) \times \cdots \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(j_k) \to \tilde{\mathcal{C}}_2^{\mathcal{B}}(j)$$

is a lift of $\gamma(\xi \times \mathrm{id} \times \cdots \times \mathrm{id})p$, and

$$\tilde{\mathcal{C}}_2^{\mathcal{B}}(k)\times\tilde{\mathcal{C}}_2^{\mathcal{B}}(j_1)\times\cdots\times\tilde{\mathcal{C}}_2^{\mathcal{B}}(j_k) \cong \tilde{\mathcal{C}}_2^{\mathcal{B}}(k)\times\tilde{\mathcal{C}}_2^{\mathcal{B}}(j_{\xi^{-1}(1)})\times\cdots\times\tilde{\mathcal{C}}_2^{\mathcal{B}}(j_{\xi^{-1}(k)}) \xrightarrow{\tilde{\gamma}} \tilde{\mathcal{C}}_2^{\mathcal{B}}(j) \xrightarrow{\xi(j_1,\ldots,j_k)} \tilde{\mathcal{C}}_2^{\mathcal{B}}(j)$$

is a lift of

$$\mathcal{C}_2^{\Sigma}(k)\times\mathcal{C}_2^{\Sigma}(j_1)\times\cdots\times\mathcal{C}_2^{\Sigma}(j_k) \cong \mathcal{C}_2^{\Sigma}(k)\times\mathcal{C}_2^{\Sigma}(j_{\xi^{-1}(1)})\times\cdots\times\mathcal{C}_2^{\Sigma}(j_{\xi^{-1}(k)}) \xrightarrow{\gamma} \mathcal{C}_2^{\Sigma}(j) \xrightarrow{\xi(j_1,\ldots,j_k)} \mathcal{C}_2^{\Sigma}(j)$$

Since $\mathcal{C}_2^{\Sigma}$ is an operad, $\gamma(\xi \times \mathrm{id} \times \cdots \times \mathrm{id})p$ is equal to $(\gamma\xi(j_1, \ldots, j_k)) \circ p$. Therefore the two maps $\tilde{\gamma}(\xi \times \mathrm{id} \times \cdots \times \mathrm{id})$ and $\tilde{\gamma}\xi(j_1, \ldots, j_k)$ are lifts of the same map, so if we can show that they agree on one point, they agree on every point.

Let c_x denote the path that is constant x. Take any point $(\tilde{x}, \tilde{x}_1, \ldots, \tilde{x}_k)$ in $\tilde{\mathcal{C}}_1(k) \times \tilde{\mathcal{C}}_1(j_1) \times \cdots \times \tilde{\mathcal{C}}_1(j_k)$, and lift the path

$$\gamma(\alpha_{\tilde{x},\xi} \times c_{x_1} \times \cdots \times c_{x_k})$$

to a path in $\tilde{\mathcal{C}}_2^{\mathcal{B}}(j)$ starting in $\tilde{\gamma}(\tilde{x}, \tilde{x}_1, \ldots, \tilde{x}_k)$. Then $\tilde{\gamma}(\tilde{x}\xi, \tilde{x}_1, \cdots, \tilde{x}_k)$ is the endpoint of this path. If we map this path to $F(\mathbb{R}^2; j)/\Sigma_j$ it is not in general a loop. So let ω be any path in $\mathcal{C}_1(j)$ from $\gamma(x, x_{\xi^{-1}(1)}, \ldots, x_{\xi^{-1}(k)})$ to $\gamma(x, x_1, \ldots, x_k)$, then the image of

$$\gamma(\alpha_{\tilde{x},\xi} \times c_{x_1} \times \cdots \times c_{x_k})\cdot \omega \tag{4.3}$$

in $F(\mathbb{R}^2; j)/\Sigma_j$ is a loop that represents $\xi(j_1, \ldots, j_k)$. Since ω is a path entirely in $\mathcal{C}_1(j)$, we know that if we lift ω to a path $\tilde{\omega}$ starting in $\tilde{\gamma}(\tilde{x}, \tilde{x}_{\xi^{-1}(1)}, \ldots, \tilde{x}_{\xi^{-1}(k)})$ it will end in $\tilde{\gamma}(\tilde{x}, \tilde{x}_1, \ldots, \tilde{x}_k)$. That means that $\tilde{\gamma}(\tilde{x}\xi, \tilde{x}_1, \cdots, \tilde{x}_k)$ is also the endpoint of the lift of the path (4.3), but this is by definition $\tilde{\gamma}(\tilde{x}, \tilde{x}_{\xi^{-1}(1)}, \ldots, \tilde{x}_{\xi^{-1}(k)})\xi(j_1, \ldots, j_k)$, so they must be equal.

The maps

$$(\xi_1 \oplus \cdots \oplus \xi_k)\tilde{\gamma}\colon\ \tilde{\mathcal{C}}_2^{\mathcal{B}}(k) \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(j_1) \times \cdots \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(j_k) \to \tilde{\mathcal{C}}_2^{\mathcal{B}}(j) \text{ and}$$

$$\tilde{\gamma}(\mathrm{id} \times \xi_1 \times \cdots \times \xi_k) \colon \tilde{\mathcal{C}}_2^{\mathcal{B}}(k) \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(j_1) \times \cdots \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(j_k) \to \tilde{\mathcal{C}}_2^{\mathcal{B}}(j)$$

are lifts of $(\xi_1 \oplus \cdots \oplus \xi_k)\gamma p$ and $\gamma(\mathrm{id} \times \xi_1 \times \cdots \times \xi_k)p$ respectively. The last two maps are equal since $\mathcal{C}_2^{\Sigma}$ is an operad. So if we can show that the two lifts agree on one point, they must agree on all points. Let $(\tilde{x}, \tilde{x}_1, \ldots, \tilde{x}_k)$ be any point in $\tilde{\mathcal{C}}_1(k) \times \tilde{\mathcal{C}}_1(j_1) \times \cdots \times \tilde{\mathcal{C}}_1(j_k)$. Lift the path $\gamma(c_x, \alpha_{\tilde{x}_1,\tilde{\xi}_1}, \ldots, \alpha_{\tilde{x}_k,\tilde{\xi}_k})$ to a path in $\tilde{\mathcal{C}}_2^{\mathcal{B}}(j)$ starting in $\tilde{\gamma}(\tilde{x}, \tilde{x}_1, \ldots, \tilde{x}_k)$, the endpoint of this is $\tilde{\gamma}(\tilde{x}, \tilde{x}_1\tilde{\xi}_1, \ldots, \tilde{x}_k\tilde{\xi}_k)$. The endpoint is also $\tilde{\gamma}(\tilde{x}, \tilde{x}_1, \ldots, \tilde{x}_k)(\xi_1 \oplus \cdots \oplus \xi_k)$, since the image of $\gamma(c_x, \alpha_{\tilde{x}_1,\tilde{\xi}_1}, \ldots, \alpha_{\tilde{x}_k,\tilde{\xi}_k})$ in $F(\mathbb{R}^2; j)/\Sigma_j$ is a loop that represents $\xi_1 \oplus \cdots \oplus \xi_k$. □

Proposition 4.2.6. *[Fie, Page 17] The monad $\tilde{\mathbb{C}}_2^{\mathcal{B}}$ associated to the braided operad $\tilde{\mathcal{C}}_2^{\mathcal{B}}$ from Construction 4.2.4 is the same as the monad $\mathbb{C}_2^{\Sigma}$, associated to the operad $\mathcal{C}_2^{\Sigma}$ from Example 4.2.1.*

Proof. Recall from Proposition 1.1.3 that the pure braid group on m strings, $\mathcal{P}_m$, is the kernel of $\Phi \colon \mathcal{B}_m \to \Sigma_m$. The space $\mathcal{C}_2^{\Sigma}(m)$ has fundamental group $\mathcal{P}_m$ and universal covering space $\tilde{\mathcal{C}}_2^{\mathcal{B}}(m)$. Hence $\tilde{\mathcal{C}}_2^{\mathcal{B}}(m)/\mathcal{P}_m$ is homeomorphis to $\mathcal{C}_2^{\Sigma}(m)$.

For a pointed space X,

$$\tilde{\mathbb{C}}_2^{\mathcal{B}} X = \Big(\coprod_{\mathbf{m} \in \mathfrak{B}} \tilde{\mathcal{C}}_2^{\mathcal{B}}(m) \times X^{\times m}\Big)/\sim_{\mathfrak{B}},$$

where $\sim_{\mathfrak{B}}$ is the equivalence relation generated by the two relations in Definition 4.1.8. The action of any pure braid ρ on $\tilde{\mathcal{C}}_2^{\mathcal{B}}(m)$ defines a map $\tilde{\mathcal{C}}_2^{\mathcal{B}}(m) \to \tilde{\mathcal{C}}_2^{\mathcal{B}}(m)$ which is the same as the deck transformation corresponding to ρ. The first relation in Definition 4.1.8 implies that $[c\rho, y] = [c, y]$ for any pure braid ρ. Hence

$$(\tilde{\mathcal{C}}_2^{\mathcal{B}}(m) \times X^{\times m})/(\tilde{c}\xi, y) \sim (\tilde{c}, \Phi(\xi)y)$$

for $\xi \in \mathcal{B}_m$, is homeomorphic to

$$(\tilde{\mathcal{C}}_2^{\mathcal{B}}(m)/\mathcal{P}_m \times X^{\times m})/([\tilde{c}]\sigma, y) \sim ([\tilde{c}], \sigma y)$$

for $\sigma \in \Sigma_m$, which again is homeomorphic to

$$(\mathcal{C}_2^{\Sigma}(m) \times X^{\times m})/(p(\tilde{c})\sigma, y) \sim (p(\tilde{c}), \sigma y)$$

for $\sigma \in \Sigma_m$, and where p is the covering map $p \colon \tilde{\mathcal{C}}_2^{\mathcal{B}}(m) \to \mathcal{C}_2^{\Sigma}(m)$. Under this homeomorphism the generating relation

$$(\tilde{\gamma}(\tilde{c}; \tilde{1}^i, *, \tilde{1}^{n-1-i}), (x_1, \ldots, x_{n-1})) \sim (\tilde{c}, (x_1, \ldots, x_i, *, x_{i+1}, \ldots, x_{n-1}))$$

in $\tilde{C}_2^{\mathcal{B}} X$ corresponds to the generating relation

$$(\gamma(p(\tilde{c}); 1^i, *, 1^{n-1-i}), (x_1, \ldots, x_{n-1})) \sim (p(\tilde{c}), (x_1, \ldots, x_i, *, x_{i+1}, \ldots, x_{n-1}))$$

in $\mathbb{C}_2^{\Sigma} X$, this follows easily from the definition of the braided operad structure on $\tilde{\mathcal{C}}_2^{\mathcal{B}}$, see Construction 4.2.4.

The structure maps of the monads are defined similarly, so it is easy to see that the homeomorphim respects them, hence the two monads are homeomorphic. □

Definition 4.2.7. A space X in $\mathcal{T}$ is a double loop space, if it equals the space of based maps from the two-sphere into Y, for some Y in $\mathcal{T}$.

Proposition 4.2.8 (Proposition 3.4 in [Fie])**.** *If a B_∞ operad acts on a connected pointed space X, then X is weakly equivalent to a double loop space.*

Proof. Let $\mathcal{D}$ be any B_∞ operad, and let $\tilde{\mathcal{C}}_2^{\mathcal{B}}$ be the B_∞ operad from Construction 4.2.4. We form the braided operad $\mathcal{D} \times \tilde{\mathcal{C}}_2^{\mathcal{B}}$ in the obvious way by $(\mathcal{D} \times \tilde{\mathcal{C}}_2^{\mathcal{B}})(j) = \mathcal{D}(j) \times \tilde{\mathcal{C}}_2^{\mathcal{B}}(j)$,

$$(\gamma \times \tilde{\gamma})(c \times \tilde{c}; d_1 \times \tilde{d}_1, \ldots, d_k \times \tilde{d}_k) = \gamma(c; d_1, \ldots, d_k) \times \tilde{\gamma}(\tilde{c}; \tilde{d}_1, \ldots, \tilde{d}_k),$$

$1 = 1 \times \tilde{1}$ and $(c \times \tilde{c})\xi = c\xi \times \tilde{c}\xi$. Denote the monad associated to $\mathcal{D} \times \tilde{\mathcal{C}}_2^{\mathcal{B}}$ by $\mathbb{D} \times \tilde{\mathbb{C}}_2^{\mathcal{B}}$ (this is not necessarily the product of $\mathbb{D}$ and $\tilde{\mathbb{C}}_2^{\mathcal{B}}$ in the category of monads.

The braided operad $\mathcal{D} \times \tilde{\mathcal{C}}_2^{\mathcal{B}}$ is again a B_∞ operad, and the projection $\pi\colon (\mathcal{D} \times \tilde{\mathcal{C}}_2^{\mathcal{B}})(j) \to \tilde{\mathcal{C}}_2^{\mathcal{B}}(j)$ is a homotopy equivalence for each j. Similary to in Proposition 3.4 in [May72] one can show that this implies that the associated map

$$\pi\colon (\mathbb{D} \times \tilde{\mathbb{C}}_2^{\mathcal{B}})X \to \tilde{\mathbb{C}}_2^{\mathcal{B}}X \tag{4.4}$$

is a weak homotopy equivalence.

If $\mathcal{D}$ acts on X we can use the projection π to get an action of $\mathcal{D} \times \tilde{\mathcal{C}}_2^{\mathcal{B}}$ on X. Let $\chi\colon (\mathbb{D} \times \tilde{\mathbb{C}}_2^{\mathcal{B}})X \to X$ be the associated algebra morphism, see Proposition 4.1.9. There is a weak equivalence of $(\mathbb{D} \times \tilde{\mathbb{C}}_2^{\mathcal{B}})$-algebras

$$X \xleftarrow{\varepsilon(\chi)} B(\mathbb{D} \times \tilde{\mathbb{C}}_2^{\mathcal{B}}, \mathbb{D} \times \tilde{\mathbb{C}}_2^{\mathcal{B}}, X) \tag{4.5}$$

see Theorem 9.10 and Corollary 11.10 in [May72]. Here B denotes the realization of the two sided bar construction defined in Section 9 [May72].

The monad $\tilde{\mathbb{C}}_2^{\mathcal{B}}$ is the same as $\mathbb{C}_2^{\Sigma}$, see Proposition 4.2.6. In Theorem 5.2 [May72] a morphism of monads $\alpha_2\colon \mathbb{C}_2^{\Sigma} \to \Omega^2\Sigma^2$, where Σ denotes suspension.

Since the map (4.4) is a weak homotopy equivalence the same arguments as in the proof of Theorem 13.1 shows that there is a $\mathbb{D} \times \tilde{\mathbb{C}}_2^{\mathcal{B}}$-algebra morphism

$$\begin{aligned} B(\mathbb{D} \times \tilde{\mathbb{C}}_2^{\mathcal{B}}, \mathbb{D} \times \tilde{\mathbb{C}}_2^{\mathcal{B}}, X) &\xrightarrow{B(\pi,\mathrm{id},\mathrm{id})} B(\tilde{\mathbb{C}}_2^{\mathcal{B}}, \mathbb{D} \times \tilde{\mathbb{C}}_2^{\mathcal{B}}, X) \\ &\xrightarrow{B(\alpha_2,\mathrm{id},\mathrm{id})} B(\Omega^2\Sigma^2, \mathbb{D} \times \tilde{\mathbb{C}}_2^{\mathcal{B}}, X) \xrightarrow{\gamma^2} \Omega^2 B(\Sigma^2, \mathbb{D} \times \tilde{\mathbb{C}}_2^{\mathcal{B}}, X) \end{aligned}$$

which is a weak equivalence. Together with (4.5) this gives a chain of weak equivalences from X to a double loop space. □

4.3 The braided analog of the Barratt-Eccles operad

In this section we go back to working in the category of simplicial sets. By braided operad, B_∞ operad etc. we will mean simplplicial versions of the structures defined in the topological setting in Section 4.1. We start by constructing a braided operad analogous to the Barratt-Eccles operad defined in [BE74]. Then we show that this acts on the nerve of a small braided strict monoidal category and on the homotopy colimit of a commutative monoid in a certain diagram space.

First we translate Proposition 4.2.8 into the simplicial setting.

Proposition 4.3.1. *If a simplicial B_∞ operad acts on a pointed simplicial set X, then X is weakly equivalent to a double loop space.*

Proof. By Proposition 4.2.8 the realization $|X|$ is weakly equivalent to $\Omega^2 T$ for a pointed space T. From the proof of Theorem 11.4 Chapter I in [GJ99] we get that $X \to S|X|$ is a weak equivalence where S denotes the singular functor from spaces to simplicial sets. The singular functor preserves weak equivalences, see page 63 [GJ99], so $S|X|$ is weakly equivalent to $S\Omega^2 T$. This is by definition

$$\hom_{\mathcal{U}}(|\Delta^-|, \Omega^2 T) \cong \hom_{\mathcal{T}}(|\Delta^-|_+, \Omega^2 T),$$

where $+$ denotes a disjoin basepoint. The functor Ω^2 is right adjoint to $S^2 \wedge -$, see Section VII.9 [ML98] hence we get an isomorphism

$$\hom_{\mathcal{T}}(|\Delta^-|_+, \Omega^2 T) \cong \hom_{\mathcal{T}}(S^2 \wedge |\Delta^-|_+, T).$$

The realization preserves products so

$$\hom_{\mathcal{T}}(S^2 \wedge |\Delta^-|_+, T) \cong \hom_{\mathcal{T}}(|(\Delta^2/\partial\Delta^2) \wedge \Delta^-_+|, T),$$

and finally, from the adjunction $|-|\colon \mathcal{S}_* \rightleftarrows \mathcal{T} : S$ we get

$$\begin{aligned}\hom_{\mathcal{T}}(|(\Delta^2/\partial\Delta^2) \wedge \Delta^-_+|, T) &\cong \hom_{\mathcal{S}_*}((\Delta^2/\partial\Delta^2) \wedge \Delta^-_+, S(T)) \\ &= \mathrm{Map}_{\mathcal{S}_*}(\Delta^2/\partial\Delta^2, S(T)) = \Omega^2 S(T).\end{aligned}$$

So there is a chain of weak equivalences connecting X and $\Omega^2 S(T)$. □

Proposition 4.3.2. *[Fie, Example 3.2] There is a B_∞ operad, $\mathcal{NB}$, constructed from the braid groups similar to how the Barratt-Eccles operad is constructed from the symmetric groups.*

Proof. We construct the braided operad by first defining categories $\mathcal{B}(k)$ for $k \geq 0$ and a "braided operad structure" on the category level, such that when we apply the nerve functor, see Definition 1.2.12, we get a braided operad.

Let $\mathcal{B}(k)$ be the comma category $(\mathbf{k} \downarrow \mathcal{B}_k)$, see Definition 1.2.11, where we view $\mathcal{B}_k$ as the full subcategory of $\mathcal{B}$ with one object $\mathbf{k}$. The objects of $\mathcal{B}(k)$ are braids on k strings and between any two objects ξ and ζ there is a unique morphism $\zeta\xi^{-1}\colon \xi \to \zeta$.

We define the functor

$$\gamma\colon \mathcal{B}(k) \times \mathcal{B}(j_1) \times \cdots \times \mathcal{B}(j_k) \to \mathcal{B}(j), \text{ for } j = j_1 + \cdots + j_k$$

on an object $(\xi, \zeta_1, \dots, \zeta_k)$ as

$$\gamma(\xi, \zeta_1, \dots, \zeta_k) = \xi(j_1, \dots, j_k)(\zeta_1 \oplus \dots \oplus \zeta_k)$$

where $\xi(j_1, \dots, j_k)$ is the braid obtained from ξ by replaing the ith string of ξ by j_i parallel strings, and $\zeta_1 \oplus \dots \oplus \zeta_k$ denotes the monoidal product of the braids $\zeta_1, \dots, \zeta_k$ in $\mathcal{B}$.

Since there is a unique morphism between any two objects we do not have to specify γ on morphisms. But when we want to define an action of the braided operad on a space, we have to know what it is. So we will say what the sturcture maps are on the morphisms, but we will only check associativity and equivariance on objects. On the morphism

$$(\xi'\xi^{-1}\colon\ \xi \to \xi') \times (\zeta_1'\zeta_1^{-1}\colon\ \zeta_1 \to \zeta_1') \times \dots \times (\zeta_k'\zeta_k^{-1}\colon\ \zeta_k \to \zeta_k')$$

γ has to be

$$\begin{aligned}(\xi'\xi^{-1})(j_{\xi^{-1}(1)}, \dots, j_{\xi^{-1}(k)})(\zeta'_{\xi^{-1}(1)}\zeta^{-1}_{\xi^{-1}(1)} \oplus \dots \oplus \zeta'_{\xi^{-1}(1)}\zeta^{-1}_{\xi^{-1}(k)})\colon \\ \xi(j_1, \dots, j_k)(\zeta_1 \oplus \dots \oplus \zeta_k) \to \xi'(j_1, \dots, j_k)(\zeta_1' \oplus \dots \oplus \zeta_k').\end{aligned}$$

The associativity condition for γ holds:

$$\begin{aligned}&\gamma(\xi; \gamma(\xi_1; \zeta_1, \dots, \zeta_{j_1}), \dots, \gamma(\xi_k; \zeta_{j_1+\dots+j_{k-1}+1}, \dots, \zeta_j)) \\ =&\gamma(\xi; \xi_1(i_1, \dots, i_{j_1})(\zeta_1 \oplus \dots \oplus \zeta_{j_1}), \dots, \xi_k(i_{j_1+\dots+j_{k-1}+1}, \dots, i_j)(\zeta_{j_1+\dots+j_{k-1}+1} \oplus \dots \oplus \zeta_j)) \\ =&\xi((i_1 + \dots + i_{j_1}), \dots, (i_{j_1+\dots+j_{k-1}+1} + \dots + i_j)) \\ &\quad(\xi_1(i_1, \dots, i_{j_1})(\zeta_1 \oplus \dots \oplus \zeta_{j_1}) \oplus \dots \oplus \xi_k(i_{j_1+\dots+j_{k-1}+1}, \dots, i_j)(\zeta_{j_1+\dots+j_{k-1}+1} \oplus \dots \oplus \zeta_j)) \\ =&\xi((i_1 + \dots + i_{j_1}), \dots, (i_{j_1+\dots+j_{k-1}+1} + \dots + i_j)) \\ &\quad(\xi_1(i_1, \dots, i_{j_1}) \oplus \dots \oplus \xi_k(i_{j_1+\dots+j_{k-1}+1}, \dots, i_j))(\zeta_1 \oplus \dots \oplus \zeta_j) \\ =&(\xi(j_1, \dots, j_k)(\xi_1 \oplus \dots \oplus \xi_k))(i_1, \dots, j_1)(\zeta_1 \oplus \dots \oplus \zeta_j) \\ =&\gamma(\gamma(\xi; \xi_1, \dots, \xi_k); \zeta_1, \dots, \zeta_j).\end{aligned}$$

We denote the unique object in $\mathcal{B}(1)$ by $\mathrm{id}_{\mathcal{B}_1}$. Then

$$\gamma(\mathrm{id}_{\mathcal{B}_1}; \xi) = \mathrm{id}_{\mathcal{B}_1}(k)\xi = \mathrm{id}_{\mathcal{B}_k}\xi = \xi \text{ and}$$

$$\gamma(\xi; \mathrm{id}_{\mathcal{B}_1}, \dots, \mathrm{id}_{\mathcal{B}_1}) = \xi(1, \dots, 1)(\mathrm{id}_{\mathcal{B}_1} \oplus \dots \oplus \mathrm{id}_{\mathcal{B}_1}) = \xi,$$

so $\mathrm{id}_{\mathcal{B}_1}$ satiesfies the conditions for the unit in Definition 4.1.1.

There is a right action of $\mathcal{B}_k$ on $\mathcal{B}(k)$ by right multiplication,

$$\xi \mapsto \xi\varrho$$

for an object ξ in $\mathcal{B}(k)$ and a braid ϱ in $\mathcal{B}_k$. The action on a morphism $\xi'\xi^{-1}\colon\ \xi \to \xi'$ in $\mathcal{B}(k)$ is

$$(\xi'\xi^{-1}\colon\ \xi \to \xi') \mapsto (\xi'\xi^{-1}\colon\ \xi\varrho \to \xi'\varrho),$$

for $\varrho \in \mathcal{B}_k$. Direct computation shows that the equivariance conditions for γ, Definition 4.1.1, are satiesfied. We check the first one:

$$\begin{aligned}&\gamma(\xi\varrho, \zeta_1, \dots, \zeta_k)\\ =&(\xi\varrho)(j_1, \dots, j_k)(\zeta_1 \oplus \cdots \oplus \zeta_k)\\ =&\xi(j_{\varrho^{-1}(1)}, \dots, j_{\varrho^{-1}(k)})\varrho(j_1, \dots, j_k)(\zeta_1 \oplus \cdots \oplus \zeta_k)\\ =&\xi(j_{\varrho^{-1}(1)}, \dots, j_{\varrho^{-1}(k)})(\zeta_{\varrho^{-1}(1)} \oplus \cdots \oplus \zeta_{\varrho^{-1}(k)})\varrho(j_1, \dots, j_k)\\ =&\gamma(\xi; \zeta_{\varrho^{-1}(1)}, \dots, \zeta_{\varrho^{-1}(k)})\varrho(j_1, \dots, j_k).\end{aligned}$$

Then the second one:

$$\begin{aligned}&\gamma(\xi; \zeta_1\varrho_1, \dots, \zeta_k\varrho_k)\\ =&(\xi)(j_1, \dots, j_k)(\zeta_1\varrho_1 \oplus \cdots \oplus \zeta_k\varrho_k)\\ =&(\xi)(j_1, \dots, j_k)(\zeta_1 \oplus \cdots \oplus \zeta_k)(\varrho_1 \oplus \cdots \oplus \varrho_k)\\ =&\gamma(\xi; \zeta_1, \dots, \zeta_k)(\varrho_1 \oplus \cdots \oplus \varrho_k).\end{aligned}$$

Applying the nerve functor, we get a braided operad $\mathcal{NB}$. The category $(\mathbf{k} \downarrow \mathcal{B}_k)$ has an initial object $\mathrm{id}_{\mathcal{B}_k}$ so $\mathcal{NB}(k)$ is contractible for each k, and the action of the braid group is free for each k, so $\mathcal{NB}$ is a B_∞ operad. □

Proposition 4.3.3. *[Fie, Page 18] The B_∞ operad $\mathcal{NB}$ acts on the nerve of any small braided strict monoidal category $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \mathsf{b})$. Hence it follows that if $\mathcal{NA}$ is connected, then it is weakly equivalent to a double loop space, see Proposition 4.3.1.*

Proof. We define all the structure on the category level, when we then apply the nerve functor we get an action of $\mathcal{NB}$ on the nerve.

Now we define a functor $\theta_k \colon \mathcal{B}(k) \times \mathcal{A}^{\times k} \to \mathcal{A}$ by

$$\begin{aligned}(\zeta, a_1, \dots, a_k) &\mapsto a_{\zeta^{-1}(1)} \oplus \cdots \oplus a_{\zeta^{-1}(k)}\\ (\xi\colon \zeta \to \xi\zeta, f_1, \dots, f_k) &\mapsto \xi \circ (f_{\zeta^{-1}(1)} \oplus \cdots \oplus f_{\zeta^{-1}(k)}),\end{aligned}$$

for $k = 0$ let $\theta_0(\mathrm{id}_{\mathcal{B}_1}) = I$, where on the right hand side ξ is the morphism that the braid ξ induces, see Lemma 1.2.8. Composition is preserved:

$$\begin{aligned}&\theta_k(\xi'\colon \xi\zeta \to \xi'\xi\zeta; f'_1, \dots, f'_k) \circ \theta_k(\xi\colon \zeta \to \xi\zeta; f_1, \dots, f_k)\\ =&(\xi' \circ (f'_{(\xi\zeta)^{-1}(1)} \oplus \cdots \oplus f'_{(\xi\zeta)^{-1}(k)})) \circ (\xi \circ (f_{\zeta^{-1}(1)} \oplus \cdots \oplus f_{\zeta^{-1}(k)}))\\ =&\xi'\xi \circ (f'_{(\zeta)^{-1}(1)} \oplus \cdots \oplus f'_{(\zeta)^{-1}(k)}) \circ (f_{\zeta^{-1}(1)} \oplus \cdots \oplus f_{\zeta^{-1}(k)})\\ =&\xi'\xi \circ (f'_{(\zeta)^{-1}(1)}f_{\zeta^{-1}(1)} \oplus \cdots \oplus f'_{(\zeta)^{-1}(k)}f_{\zeta^{-1}(k)})\\ =&\theta_k(\xi'\xi\colon \zeta \to \xi'\xi\zeta; f'_1f_1, \dots, f'_kf_k),\end{aligned}$$

the second equality follows from the naturality of the braiding in $\mathcal{A}$, the third is the functoriality of the monoidal product, the others are by definition.

There are three things to check for this to be an action of the braided operad, see Definition 4.1.4. Condition 4.1.4(b), $\theta_1(\mathrm{id}_{\mathcal{B}_1}; a) = a$, is immediate from the definition. Let $\varrho \in \mathcal{B}_k$, the next equations show condition 4.1.4(c) for objects:

$$\theta_k(\zeta\varrho; a_1, \dots, a_k) = a_{(\zeta\varrho)^{-1}(1)} \oplus \cdots \oplus a_{(\zeta\varrho)^{-1}(k)} \\ = a_{\zeta^{-1}(\varrho^{-1}(1))} \oplus \cdots \oplus a_{\zeta^{-1}(\varrho^{-1}(k))} = \theta_k(\zeta; \varrho(a_1, \dots, a_k)).$$

Similarly it also holds for morphisms, since $(\xi\colon\ \zeta \to \xi\zeta)\varrho = (\xi\colon\ \zeta\varrho \to \xi\zeta\varrho)$. The upper route in the diagram in condition 4.1.4(a) for objects gives:

$$\begin{aligned} &\theta_j(\gamma(\zeta; \varrho_1, \dots, \varrho_k); a_1, \dots, a_j) \\ =&\theta_j(\zeta(j_1, \dots, j_k)(\varrho_1 \oplus \cdots \oplus \varrho_k); a_1, \dots, a_j) \\ =&a_{(\zeta(j_1,\dots,j_k)(\varrho_1\oplus\cdots\oplus\varrho_k))^{-1}(1)}, \dots, a_{(\zeta(j_1,\dots,j_k)(\varrho_1\oplus\cdots\oplus\varrho_k))^{-1}(j)}, \end{aligned}$$

the lower route in the same diagram gives:

$$\begin{aligned} &\theta_k(\zeta; \theta_{j_1}(\varrho_1; a_1, \dots, a_{j_1}), \dots, \theta_{j_k}(\varrho_k; a_{j_1+\cdots+j_{k-1}+1}, \dots, a_j)) \\ =&\theta_k(\zeta; (a_{\varrho_1^{-1}(1)} \oplus \cdots \oplus a_{\varrho_1^{-1}(j_1)}), \dots, (a_{\varrho_k^{-1}(j_1+\cdots+j_{k-1}+1)} \oplus \cdots \oplus a_{\varrho_k^{-1}(j)})) \\ =&(a_{\varrho^{-1}_{\zeta^{-1}(1)}(1)} \oplus \cdots \oplus a_{\varrho^{-1}_{\zeta^{-1}(1)}(j_1)}) \oplus \cdots \oplus (a_{\varrho^{-1}_{\zeta^{-1}(k)}(j_1+\cdots+j_{k-1}+1)} \oplus \cdots \oplus a_{\varrho^{-1}_{\zeta^{-1}(k)}(j)}) \\ =&a_{(\zeta(j_1,\dots,j_k)(\varrho_1\oplus\cdots\oplus\varrho_k))^{-1}(1)} \oplus \cdots \oplus a_{(\zeta(j_1,\dots,j_k)(\varrho_1\oplus\cdots\oplus\varrho_k))^{-1}(j)} \end{aligned}$$

we see that the two composites are equal. Now we check the same for morphisms, first the upper route:

$$\begin{aligned} &\theta_j(\gamma(\xi\colon\ \zeta \to \xi\zeta; \xi_1\colon\ \varrho_1 \to \xi_1\varrho_1, \dots, \xi_k\colon\ \varrho_k \to \xi_k\varrho_k), f_1, \dots, f_j) \\ =&\theta_j(\xi(j_{\zeta^{-1}(1)}, \dots, j_{\zeta^{-1}(k)})(\xi_{\zeta^{-1}(1)} \oplus \cdots \oplus \xi_{\zeta^{-1}(k)})\colon\ \zeta(j_1, \dots, j_k)(\varrho_1 \oplus \cdots \oplus \varrho_k) \\ &\quad \to (\xi\zeta)(j_1, \dots, j_k)(\xi_1\varrho_1 \oplus \cdots \oplus \xi_k\varrho_k), f_1, \dots, f_j) \\ =&\xi(j_{\zeta^{-1}(1)}, \dots, j_{\zeta^{-1}(k)})(\xi_{\zeta^{-1}(1)} \oplus \cdots \oplus \xi_{\zeta^{-1}(k)}) \\ &\quad \circ (f_{(\zeta(j_1,\dots,j_k)(\varrho_1\oplus\cdots\oplus\varrho_k))^{-1}(1)} \oplus \cdots \oplus f_{(\zeta(j_1,\dots,j_k)(\varrho_1\oplus\cdots\oplus\varrho_k))^{-1}(j)}) \end{aligned}$$

then the lower route:

$$\begin{aligned} &\theta_k(\xi\colon\ \zeta \to \xi\zeta, \theta_{j_1}(\xi_1\colon\ \varrho_1 \to \xi_1\varrho_1, f_1, \dots, f_{j_1}), \dots, \theta_{j_k}(\xi_k\colon\ \varrho_k \to \xi_k\varrho_k, f_{j_1+\cdots+j_{k-1}+1}, \dots, f_j)) \\ =&\theta_k(\xi\colon\ \zeta \to \xi\zeta, \xi_1(f_{\varrho_1^{-1}(1)} \oplus \cdots \oplus f_{\varrho_1^{-1}(j_1)}), \dots, \xi_k(f_{\varrho_k^{-1}(j_1+\cdots+j_{k-1}+1)} \oplus \dots \oplus f_{\varrho_k^{-1}(j)})) \\ =&\xi \circ ((\xi_{\zeta^{-1}(1)}(f_{\varrho^{-1}_{\zeta^{-1}(1)}(1)} \oplus \cdots \oplus f_{\varrho^{-1}_{\zeta^{-1}(1)}(j_1)})) \oplus \cdots \oplus \\ &\quad (\xi_{\zeta^{-1}(k)}(f_{\varrho^{-1}_{\zeta^{-1}(k)}(j_1+\cdots+j_{k-1}+1)} \oplus \cdots \oplus f_{\varrho^{-1}_{\zeta^{-1}(k)}(j)}))) \\ =&\xi(j_{\zeta^{-1}(1)}, \dots, j_{\zeta^{-1}(k)})(\xi_{\zeta^{-1}(1)} \oplus \cdots \oplus \xi_{\zeta^{-1}(k)}) \\ &\quad \circ (f_{\varrho^{-1}_{\zeta^{-1}(1)}(1)} \oplus \cdots \oplus f_{\varrho^{-1}_{\zeta^{-1}(1)}(j_1)} \oplus \cdots \oplus f_{\varrho^{-1}_{\zeta^{-1}(k)}(j_1+\cdots+j_{k-1}+1)} \oplus \dots \oplus f_{\varrho^{-1}_{\zeta^{-1}(k)}(j)}) \\ =&\xi(j_{\zeta^{-1}(1)}, \dots, j_{\zeta^{-1}(k)})(\xi_{\zeta^{-1}(1)} \oplus \cdots \oplus \xi_{\zeta^{-1}(k)}) \\ &\quad \circ (f_{(\zeta(j_1,\dots,j_k)(\varrho_1\oplus\cdots\oplus\varrho_k))^{-1}(1)} \oplus \cdots \oplus f_{(\zeta(j_1,\dots,j_k)(\varrho_1\oplus\cdots\oplus\varrho_k))^{-1}(j)}) \end{aligned}$$

and they are the same. □

Proposition 4.3.4. *Let $\mathcal{A}$ be a small category and $\mathcal{S}$ the category of simplicial sets. Let $*$ be a one point set, and denote the nerve functor by $\mathcal{N}$. The homotopy colimit of a functor $X\colon \mathcal{A} \to \mathcal{S}$ is the simplicial set with s-simplices*

$$\operatorname*{hocolim}_{\mathcal{A}}(X)_s = \coprod_{a_0 \leftarrow \cdots \leftarrow a_s} X(a_s)_s,$$

see Definition 3.3.17 for the definition of the simplicial structure maps. The simplicial set $\operatorname{hocolim}_{\mathcal{A}}(X)$ is isomorphic to the diagonal of the bisimplicial set

$$t \mapsto \mathcal{N}(* \downarrow X(\cdot)_t).$$

Proof. The comma category $(* \downarrow X(\cdot)_t)$ has as objects pairs (a, x), where $a \in \mathcal{A}$ and x is an element in $X(a)_t$. A morphism from (a, x) to (a', x') is a morphism $f\colon\ a \to a'$ in $\mathcal{A}$ such that $X(f)_t(x) = x'$. The set of morphisms in $(* \downarrow X(\cdot)_t)$ is isomorphic to

$$\{(f, x) | x \in X(a)_t\,,\ f\colon\ a \to a'\}.$$

The set of pairs of composable morphisms is isomorphic to

$$\{(g, f, x) | x \in X(a)_t\,,\ f\colon\ a \to a'\,,\ g\colon\ a' \to a''\}.$$

Continuing like this, we see that

$$\mathcal{N}(* \downarrow X(\cdot)_t)_s = \coprod_{a_0 \leftarrow \cdots \leftarrow a_s} X(a_s)_t.$$

If we use the simplicial structure of X to make a bisimplicial set $t \mapsto \mathcal{N}(* \downarrow X(\cdot)_t)$, then the diagonal of this is a simplicial set with the set of simplices isomorphic to set of simplicies of $\operatorname{hocolim}_{\mathcal{A}}(X)$ and corresponding simplicial structure. □

Lemma 4.3.5. *Let $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, (\mathsf{b}))$ be a small (braided) monoidal category and $\mathcal{S}$ the category of simplicial sets. The category $\mathcal{S}^{\mathcal{A}}$ has an induced (braided) monoidal structure, $(\mathcal{S}^{\mathcal{A}}, \oplus, U, \mathfrak{a}, \mathfrak{l}, \mathfrak{r}, (\mathfrak{b}))$, see Proposition 3.2.10. Let (X, μ, η) be a (commutative) monoid in $\mathcal{S}^{\mathcal{A}}$, see Definitions 1.2.9 and 1.2.10. The comma category*

$$(* \downarrow X(\cdot)_t)$$

is (braided) monoidal for all $t \geq 0$. In both cases the monoidal structure is strict if the monoidal structure on $\mathcal{A}$ is.

Proof. The monoidal product $Y \boxtimes Z$ in for objects $Y, Z \in \mathcal{S}^{\mathcal{A}}$ is defined via the left Kan extension, this construction also comes with a natural transformation

$$\varepsilon^{Y,Z}\colon\ Y \times Z \to (Y \boxtimes Z)(- \oplus -).$$

If we have two natural transformations $\theta\colon Y \to Y'$ and $\vartheta\colon Z \to Z'$, the product $\theta \boxtimes \vartheta$ is the natural transformation $Y \boxtimes Z \to Y' \boxtimes Z'$ such that

$$\varepsilon^{Y',Z'}(\theta \times \vartheta) = (\theta \boxtimes \vartheta)\varepsilon^{Y,Z}. \tag{4.6}$$

We define a monoidal product on $(* \downarrow X(\cdot)_t)$ in the following way:

$$(a_1, x_1) \oplus (a_2, x_2) = (a_1 \oplus a_2, \mu_{a_1 \oplus a_2}\varepsilon^{X,X}_{a_1,a_2}(x_1, x_2))$$

for objects and

$$\begin{aligned}(f_1\colon (a_1, x_1) \to (a'_1, x'_1)) \oplus (f_2\colon (a_2, x_2) \to (a'_2, x'_2))\\ = f_1 \oplus f_2\colon (a_1 \oplus a_2, \mu_{a_1\oplus a_2}\varepsilon^{X,X}_{a_1,a_2}(x_1, x_2)) \to (a'_1 \oplus a'_2, \mu_{a'_1\oplus a'_2}\varepsilon^{X,X}_{a'_1,a'_2}(x'_1, x'_2))\end{aligned}$$

for morphisms. That $f_1 \oplus f_2$ is really a morphism in the comma category follows from the naturality of μ and $\varepsilon^{X,X}$.

We want to use $\mathsf{a}\colon (a_1 \oplus a_2) \oplus a_3 \to a_1 \oplus (a_2 \oplus a_3)$ as the associativity isomorphism, but first we have to show that it is a morphism in the comma category. From the definitoin of the associativity isomorphism in $\mathcal{S}^{\mathcal{A}}$ via the left Kan extension, we can extract that $\mathfrak{a}\colon (X \boxtimes X) \boxtimes X \to X \boxtimes (X \boxtimes X)$ is the unique natural transformation such that

$$((X\boxtimes X)\boxtimes X)(\mathsf{a}_{(a_1\oplus a_2)\oplus a_3})\varepsilon_{(a_1\oplus a_2),a_3}(\varepsilon_{a_1,a_2}\times \mathrm{id}_{a_3}) = \mathfrak{a}^{-1}_{(a_1\oplus a_2)\oplus a_3}\varepsilon_{a_1,(a_2\oplus a_3)}(\mathrm{id}_{a_1}\times\varepsilon_{a_2,a_3}). \tag{4.7}$$

The isomorphism $\mathsf{a}_{(a_1\oplus a_2)\oplus a_3}$ is a morphism in the comma category if

$$\begin{aligned}X_t(\mathsf{a}_{(a_1\oplus a_2)\oplus a_3})\mu_{(a_1\oplus a_2)\oplus a_3}\varepsilon_{(a_1\oplus a_2),a_3}((\mu_{(a_1\oplus a_2)} \times \mathrm{id}_{a_3})\varepsilon_{(a_1\oplus a_2)}(x_1, x_2), x_3)\\ = \mu_{a_1\oplus(a_2\oplus a_3)}\varepsilon_{a_1,(a_2\oplus a_3)}(x_1, (\mu_{(a_1\oplus a_2)} \times \mathrm{id}_{a_3})\varepsilon_{(a_1\oplus a_2)}(x_2, x_3)).\end{aligned}$$

Using the equations above this will follow from the fact that μ is associative.

$$\begin{aligned}
&X_t(\mathsf{a}_{(a_1\oplus a_2)\oplus a_3})\mu_{(a_1\oplus a_2)\oplus a_3}\varepsilon_{(a_1\oplus a_2),a_3}(\mu_{(a_1\oplus a_2)} \times \mathrm{id}_{a_3})(\varepsilon_{(a_1\oplus a_2)} \times \mathrm{id}_{a_3}) \\
=&X_t(\mathsf{a}_{(a_1\oplus a_2)\oplus a_3})\mu_{(a_1\oplus a_2)\oplus a_3}(\mu\boxtimes \mathrm{id})_{(a_1\oplus a_2)\oplus a_3}\varepsilon_{(a_1\oplus a_2),a_3}(\varepsilon_{(a_1\oplus a_2)} \times \mathrm{id}_{a_3}) && \text{by (4.6)}\\
=&\mu_{(a_1\oplus a_2)\oplus a_3}(\mu\boxtimes \mathrm{id})_{(a_1\oplus a_2)\oplus a_3}((X\boxtimes X)\boxtimes X)_t(\mathsf{a}_{(a_1\oplus a_2)\oplus a_3})\\
&\qquad\varepsilon_{(a_1\oplus a_2),a_3}(\varepsilon_{(a_1\oplus a_2)} \times \mathrm{id}_{a_3}) && \text{by naturality}\\
=&\mu_{(a_1\oplus a_2)\oplus a_3}(\mu\boxtimes \mathrm{id})_{(a_1\oplus a_2)\oplus a_3}\mathfrak{a}^{-1}_{(a_1\oplus a_2)\oplus a_3}\varepsilon_{a_1,(a_2\oplus a_3)}(\mathrm{id}_{a_3}\times\varepsilon_{a_2,a_3}) && \text{by (4.7)}\\
=&\mu_{a_1\oplus(a_2\oplus a_3)}(\mathrm{id}\boxtimes\mu)_{a_1\oplus(a_2\oplus a_3)}\varepsilon_{a_1,(a_2\oplus a_3)}(\mathrm{id}_{a_3}\times\varepsilon_{a_2,a_3}) && \text{by ass. of } \mu\\
=&\mu_{a_1\oplus(a_2\oplus a_3)}\varepsilon_{a_1,(a_2\oplus a_3)}(\mathrm{id}_{a_1}\times\mu_{(a_2\oplus a_3)})(\mathrm{id}_{a_3}\times\varepsilon_{a_2,a_3}) && \text{by (4.6)}
\end{aligned}$$

The unit for the monoidal product in $\mathcal{S}^{\mathcal{A}}$ is U where $U(a)_t = \hom_{\mathcal{A}}(I, a)$. The left unit constraint $\mathfrak{l}\colon U \boxtimes X \to X$ is the natural transformation such that

$$\mathfrak{l}\circ\varepsilon^{U,X} = X(\mathsf{l}^{-1})\circ \mathrm{pr}_X. \tag{4.8}$$

As unit in the comma category we take $(I, \eta_I(\mathrm{id}_I))$. We check that we can use the left uint constraint $\mathsf{l}_a\colon I \oplus a \to a$ in $(* \downarrow X(\cdot)_t)$ as well.

$$
\begin{aligned}
& X_t(\mathsf{l})\mu_{I\oplus a}\varepsilon^{X,X}_{I,a}(\eta_I(\mathrm{id}_I), x) & \\
=& X_t(\mathsf{l})\mu_{I\oplus a}\varepsilon^{X,X}_{I,a}(\eta_I \times \mathrm{id}_a)(\mathrm{id}_I, x) & \\
=& X_t(\mathsf{l})\mu_{I\oplus a}(\eta \boxtimes \mathrm{id})_{I\oplus a}\varepsilon^{U,X}_{I,a}(\mathrm{id}_I, x) & \text{by (4.6)}\\
=& X_t(\mathsf{l})\mathfrak{l}_{I\oplus a}\,\varepsilon^{U,X}_{I,a}(\mathrm{id}_I, x) & \text{since } X \text{ is a monoid}\\
=& \mathrm{pr}_X(\mathrm{id}_I, x) = x & \text{by (4.8)}
\end{aligned}
$$

The argument for the right unit constraint is similar.

Since we used the associativity and unit isomorphisms from $\mathcal{A}$ we know that they are natural and that the associativity pentagon (1.1) and triangle for the unit (1.2) are commutative. This shows that $(* \downarrow X(\cdot)_t)$ is a monoidal category. It is obious that this is a strict monoidal structure if $\mathcal{A}$ is a strict monoidal category.

If $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \mathsf{b})$ is a braided monoidal category, and (X, μ, η) a commutative moniod, we show that we can define a braiding

$$(a_1, x_1) \oplus (a_2, x_2) \to (a_2, x_2) \oplus (a_1, x_1)$$

by $\mathsf{b}\colon a_1 \oplus a_2 \to a_2 \oplus a_1$. The braiding $\mathfrak{b}\colon X \boxtimes X \to X \boxtimes X$ is the unique natural transformation such that

$$\varepsilon^{X,X}_{a_1,a_2} \circ \mathfrak{b}_{a_1,a_2} = (X \boxtimes X)_t(\mathsf{b}^{-1}) \circ \varepsilon^{X,X}_{a_2,a_1} \circ \mathrm{twist}. \tag{4.9}$$

When we show below that b is a morphism in the comma category we really need X to be a commutative monoid.

$$
\begin{aligned}
& X_t(\mathsf{b})\mu_{a_1\oplus a_2}\varepsilon^{X,X}_{a_1,a_2}(x_1, x_2) & \\
=& \mu_{a_2\oplus a_1}(X \boxtimes X)_t(\mathsf{b})\varepsilon^{X,X}_{a_1,a_2}(x_1, x_2) & \text{by naturality}\\
=& \mu_{a_2\oplus a_1}\mathfrak{b}^{-1}_{a_2,a_1}\varepsilon^{X,X}_{a_2,a_1}(x_2, x_1) & \text{by (4.9)}\\
=& \mu_{a_2\oplus a_1}\varepsilon^{X,X}_{a_2,a_1}(x_2, x_1) & \text{since } X \text{ is commutative}
\end{aligned}
$$

The hexagons (1.5) and (1.6) for the braiding commute and the braiding is natural since we used the braiding from $\mathcal{A}$, so this defines a braided monoidal structure on $(* \downarrow X(\cdot)_t)$. □

Proposition 4.3.6. *Let $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \mathsf{b})$ be a small braided strict monoidal category and (X, μ, η) a commutative monoid in $\mathcal{S}^{\mathcal{A}}$. Then the braided operad $\mathcal{NB}$ acts on*

$$\operatorname*{hocolim}_{\mathcal{A}}(X) = |t \mapsto \mathcal{N}(* \downarrow X(\cdot)_t)|.$$

Hence it follows that if $\mathrm{hocolim}_{\mathcal{A}}(X)$ is connected, then it is weakly equivalent to a double loop space, see Proposition 4.3.1.

Proof. The previous lemma combined with Proposition 4.3.3 shows that $\mathcal{NB}$ acts on the space $\mathcal{N}(* \downarrow X(\cdot)_t)$ for every t. If we can show that the actions are compatible with the simplicial structure maps in the t direction, $\mathcal{NB}$ will also act on the homotopy colimit. The action of $\mathcal{NB}(k)$ in degree (s,t) is

$$\begin{aligned}(\xi_1, \ldots, \xi_s, \zeta) \times (f_1^1, \ldots f_s^1, x^1 \in X(a_s^1)_t) \times \cdots \times (f_1^k, \ldots f_s^k, x^k \in X(a_s^k)_t) \\ \mapsto (\xi_1(f_1^{\zeta^{-1}(1)} \oplus \cdots \oplus f_1^{\zeta^{-1}(k)}), \ldots, \xi_s(f_s^{(\xi_{s-1}\cdots\xi_1\zeta)^{-1}(1)} \oplus \cdots \oplus f_s^{(\xi_{s-1}\cdots\xi_1\zeta)^{-1}(k)}), \\ (\mu_{(a_s^{\zeta^{-1}(1)} \oplus \cdots \oplus a_s^{\zeta^{-1}(k)})} \circ \varepsilon_{(a_s^{\zeta^{-1}(1)}, \ldots, a_s^{\zeta^{-1}(k)})})(x^{\zeta^{-1}(1)}, \ldots, x^{\zeta^{-1}(k)})).\end{aligned}$$

A simplicial structure map in the t direction will only affect the x^i on the left side, and $\mu\varepsilon(x^{\zeta^{-1}(1)}, \ldots, x^{\zeta^{-1}(k)})$ on the right. Since μ and ε are natural transformations of simplicial diagrams, they commute with the simplicial structure maps. □

Chapter 5

Commutative monoids in $\mathcal{S}^{\mathfrak{B}}$

In this chapter we fix a small braided strict monoidal category $(\mathcal{A}, \oplus, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \mathsf{b})$, and $\mathcal{NA}_\bullet$ denotes the $\mathfrak{B}$-space constructed in Example 3.1.9. In the first section we prove that this $\mathfrak{B}$ space is a commutative monoid in $\mathcal{S}^{\mathfrak{B}}$. We also show that $X^\bullet$ from Example 3.1.3 is a commutative $\mathfrak{B}$-space monoid.

In the next section we show that the homotopy colimit of this commutative monoid is weakly equivalent to the nerve of $\mathcal{A}$.

5.1 Examples

Example 5.1.1. Let X be a pointed simplicial set, with basepoint $*$. Then the $\mathfrak{B}$-space $X^\bullet$ from Example 3.1.3 is a commutative monoid in the braided monoidal category $\mathcal{S}^{\mathfrak{B}}$.

Proof. The natural isomorphism $X^{\times m} \times X^{\times n} \cong X^{\times m+n}$ induce a $\mathfrak{B}$-space morphism

$$X^\bullet \boxtimes X^\bullet \to X^\bullet$$

which we take to be the product μ. For $\mathbf{m}$ let $U(\mathbf{m}) \to X^{\times m}$ be determined by sending the one point in $U(\mathbf{m})$ to $(*, \ldots, *)$. We let the resulting functor $\eta \colon U \to X^\bullet$ be the unit. It is easy to verify that this is a monoid.

From the definition of the braiding of $\mathcal{S}^{\mathfrak{B}}$ and the functoriality of the Kan extension we see that the composite

$$X^\bullet \boxtimes X^\bullet \xrightarrow{\mathfrak{b}} X^\bullet \boxtimes X^\bullet \xrightarrow{\mu} X^\bullet$$

is induced by

$$X^{\times m} \times X^{\times n} \cong X^{\times n} \times X^{\times m} \cong X^{\times n+m} \xrightarrow{X^{b^{-1}_{\mathbf{m},\mathbf{n}}}} X^{\times m+n}.$$

This is obviously equal to the map that induces μ, so the monoid is commutative. $\square$

Proposition 5.1.2. *The $\mathfrak{B}$-space, $\mathcal{NA}_\bullet$, constructed in Example 3.1.9 is a commutative monoid in the braided monoidal category $\mathcal{S}^{\mathfrak{B}}$.*

Proof. First we have to define the product $\mu\colon \mathcal{N}\mathcal{A}_\bullet \boxtimes \mathcal{N}\mathcal{A}_\bullet \to \mathcal{N}\mathcal{A}_\bullet$. By the universal property of the Kan extension, we can do this by specifying a natural transformation from $\mathcal{N}\mathcal{A}_\bullet \times \mathcal{N}\mathcal{A}_\bullet$ to $\mathcal{N}\mathcal{A}_\bullet(- \oplus -)$. Because of the natural isomorphism $\mathcal{N}\mathcal{A}_{\mathbf{m}} \times \mathcal{N}\mathcal{A}_{\mathbf{n}} \cong \mathcal{N}(\mathcal{A}_{\mathbf{m}} \times \mathcal{A}_{\mathbf{n}})$ it is enough to give a natural transformation $\mu'\colon \mathcal{A}_\bullet \times \mathcal{A}_\bullet \to \mathcal{A}_\bullet(- \oplus -)$.

We define the functor $\mu'_{\mathbf{m},\mathbf{n}}\colon \mathcal{A}_{\mathbf{m}} \times \mathcal{A}_{\mathbf{n}} \to \mathcal{A}_{\mathbf{m}\oplus\mathbf{n}}$ by

$$\mu'_{\mathbf{m},\mathbf{n}}((a_1,\dots,a_m)\times(b_1,\dots,b_n)) = (a_1,\dots a_m, b_1,\dots,b_n)$$

on the objects and by

$$\mu'_{\mathbf{m},\mathbf{n}}(f\times g) = f\oplus g$$

on the morphisms.

To check that this is natural in $\mathbf{m}$ and $\mathbf{n}$, let $\alpha\colon \mathbf{m}\to\mathbf{m}'$ and $\beta\colon \mathbf{n}\to\mathbf{n}'$ be morphisms in $\mathfrak{B}$. To shorten notation, we write $\alpha^{-1}(i)$ for $\Phi(\alpha)^{-1}(i)$ for any injective braid α. First we check on the objects:

$$\begin{aligned}\mathcal{A}_{\alpha\oplus\beta}(\mu'_{\mathbf{m},\mathbf{n}}((a_1,\dots a_m)\times(b_1,\dots,b_n))) &= \mathcal{A}_{\alpha\oplus\beta}(a_1,\dots,a_m,b_1,\dots,b_n)\\ &= (a_{(\alpha\oplus\beta)^{-1}(1)},\dots,a_{(\alpha\oplus\beta)^{-1}(m')},b_{(\alpha\oplus\beta)^{-1}(m'+1)-m},\dots,b_{(\alpha\oplus\beta)^{-1}(n')-m})\end{aligned}$$

it follows from the definition of the monoidal product in $\mathfrak{B}$ that this is the same as

$$\mu'_{\mathbf{m}',\mathbf{n}'}(\mathcal{A}_\alpha\times\mathcal{A}_\beta)((a_1,\dots,a_m)\times(b_1,\dots,b_n)) = (a_{\alpha^{-1}(1)},\dots a_{\alpha^{-1}(m')},b_{\beta^{-1}(1)},\dots,b_{\beta^{-1}(n')}).$$

Then for the morphisms, we have

$$\mathcal{A}_{\alpha\oplus\beta}(\mu'_{\mathbf{m},\mathbf{n}}(f\times g))) = \tilde{\varsigma}_{\alpha\oplus\beta}\circ(f\oplus g\oplus \mathrm{id}_{\mathbf{m}'-\mathbf{m}+\mathbf{n}'-\mathbf{n}})\circ\tilde{\varsigma}^{-1}_{\alpha\oplus\beta} \tag{5.1}$$

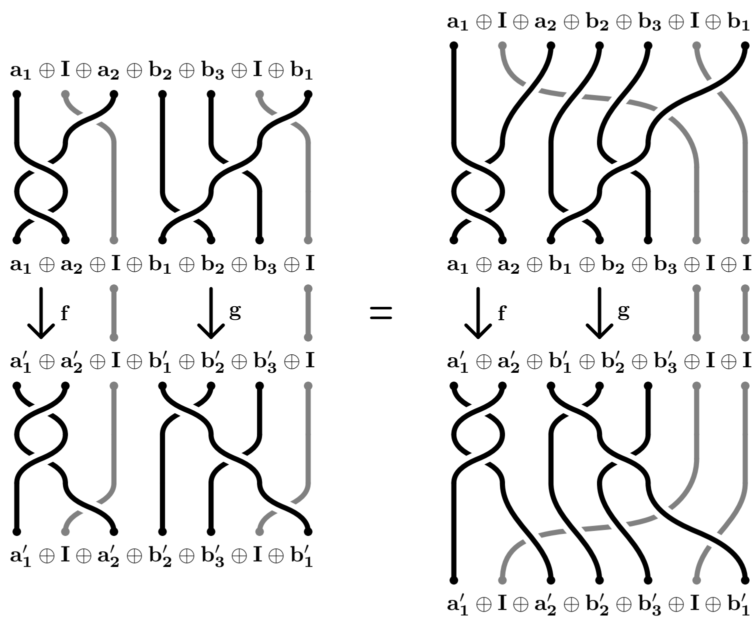
and

$$\begin{aligned}&\mu'_{\mathbf{m}',\mathbf{n}'}((\mathcal{A}_\alpha\times\mathcal{A}_\beta)(f\times g))\\ =&(\tilde{\varsigma}_\alpha\circ(f\oplus\mathrm{id}_{\mathbf{m}'-\mathbf{m}})\circ\tilde{\varsigma}_\alpha^{-1})\oplus(\tilde{\varsigma}_\beta\circ(g\oplus\mathrm{id}_{\mathbf{n}'-\mathbf{n}})\circ\tilde{\varsigma}_\beta^{-1})\\ =&(\tilde{\varsigma}_\alpha\oplus\tilde{\varsigma}_\beta)\circ(f\oplus\mathrm{id}_{\mathbf{m}'-\mathbf{m}}\oplus g\oplus\mathrm{id}_{\mathbf{n}'-\mathbf{n}})\circ(\tilde{\varsigma}_\alpha^{-1}\oplus\tilde{\varsigma}_\beta^{-1})\\ =&(\tilde{\varsigma}_\alpha\oplus\tilde{\varsigma}_\beta)\circ(\mathrm{id}_{\mathbf{m}}\oplus(\mathsf{b}_{\mathbf{n},\mathbf{m}'-\mathbf{m}}\circ\mathsf{b}^{-1}_{\mathbf{n},\mathbf{m}'-\mathbf{m}})\oplus\mathrm{id}_{\mathbf{n}'-\mathbf{n}})\\ &\qquad\circ(f\oplus\mathrm{id}_{\mathbf{m}'-\mathbf{m}}\oplus g\oplus\mathrm{id}_{\mathbf{n}'-\mathbf{n}})\circ(\tilde{\varsigma}_\alpha^{-1}\oplus\tilde{\varsigma}_\beta^{-1})\\ =&(\tilde{\varsigma}_\alpha\oplus\tilde{\varsigma}_\beta)\circ(\mathrm{id}_{\mathbf{m}}\oplus\mathsf{b}_{\mathbf{n},\mathbf{m}'-\mathbf{m}}\oplus\mathrm{id}_{\mathbf{n}'-\mathbf{n}})\circ(f\oplus g\oplus\mathrm{id}_{\mathbf{m}'-\mathbf{m}}\oplus\mathrm{id}_{\mathbf{n}'-\mathbf{n}})\\ &\qquad\circ(\mathrm{id}_{\mathbf{m}}\oplus\mathsf{b}^{-1}_{\mathbf{n},\mathbf{m}'-\mathbf{m}}\oplus\mathrm{id}_{\mathbf{n}'-\mathbf{n}})\circ(\tilde{\varsigma}_\alpha^{-1}\oplus\tilde{\varsigma}_\beta^{-1}).\end{aligned}$$

The last equality is by naturality of the braiding, the others are just by definition. Using Lemma 3.1.10 the equations

$$\begin{aligned}&(\tilde{\varsigma}_\alpha\oplus\tilde{\varsigma}_\beta)\circ(\mathrm{id}_{\mathbf{m}}\oplus\mathsf{b}_{\mathbf{n},\mathbf{m}'-\mathbf{m}}\oplus\mathrm{id}_{\mathbf{n}'-\mathbf{n}})\circ(\mathrm{id}_{\mathbf{m}}\oplus\mathrm{id}_{\mathbf{n}}\oplus\iota_{\mathbf{m}'-\mathbf{m}}\oplus\iota_{\mathbf{n}'-\mathbf{n}})\\ =&(\tilde{\varsigma}_\alpha\oplus\tilde{\varsigma}_\beta)\circ(\mathrm{id}_{\mathbf{m}}\oplus(\mathsf{b}_{\mathbf{n},\mathbf{m}'-\mathbf{m}}\circ(\mathrm{id}_{\mathbf{n}}\oplus\iota_{\mathbf{m}'-\mathbf{m}}))\oplus\iota_{\mathbf{n}'-\mathbf{n}})\\ =&(\tilde{\varsigma}_\alpha\oplus\tilde{\varsigma}_\beta)\circ(\mathrm{id}_{\mathbf{m}}\oplus((\iota_{\mathbf{m}'-\mathbf{m}}\oplus\mathrm{id}_{\mathbf{n}})\circ\mathsf{b}_{\mathbf{n},\mathbf{0}})\oplus\iota_{\mathbf{n}'-\mathbf{n}}) &&\text{by nat. of } \mathsf{b}\\ =&(\tilde{\varsigma}_\alpha\oplus\tilde{\varsigma}_\beta)\circ(\mathrm{id}_{\mathbf{m}}\oplus\iota_{\mathbf{m}'-\mathbf{m}}\oplus\mathrm{id}_{\mathbf{n}}\oplus\iota_{\mathbf{n}'-\mathbf{n}})\\ =&\alpha\oplus\beta &&\text{by Def. 3.1.7}\\ =&\tilde{\varsigma}_{\alpha\oplus\beta}\circ(\mathrm{id}_{\mathbf{m}}\oplus\mathrm{id}_{\mathbf{n}}\oplus\iota_{\mathbf{m}'-\mathbf{m}}\oplus\iota_{\mathbf{n}'-\mathbf{n}}) &&\text{by Def. 3.1.7}\end{aligned}$$

show that the two functions are the same. The next illustration is of the naturality for the morphisms.



Let $T\colon \mathfrak{B} \to \mathbf{Cat}$ be the constant functor that takes every object to the terminal category. For every $\mathbf{n}$ there is a functor that sends the only object in the terminal category to the object $(I, \ldots, I)$ in $\mathcal{A}_{\mathbf{n}}$. These functors induce a natural transformation

$$\eta\colon \mathcal{N}T \to \mathcal{N}\mathcal{A}_{\bullet}.$$

It requires a little work, but it is straightforward to check that the axioms for a monoid holds.

Finally we want to show that this monoid is commutative. From the definition of the braiding of $\mathcal{S}^{\mathfrak{B}}$ and the functoriality of the Kan extension we see that the composite

$$\mathcal{N}\mathcal{A}_{\bullet} \boxtimes \mathcal{N}\mathcal{A}_{\bullet} \xrightarrow{\mathfrak{b}} \mathcal{N}\mathcal{A}_{\bullet} \boxtimes \mathcal{N}\mathcal{A}_{\bullet} \xrightarrow{\mu} \mathcal{N}\mathcal{A}_{\bullet}$$

is induced by

$$\mathcal{N}(\mathcal{A}_{\mathbf{m}} \times \mathcal{A}_{\mathbf{n}} \cong \mathcal{A}_{\mathbf{n}} \times \mathcal{A}_{\mathbf{m}} \xrightarrow{\mu'_{\mathbf{n},\mathbf{m}}} \mathcal{A}_{\mathbf{n}\oplus\mathbf{m}} \xrightarrow{\mathcal{A}_{\mathsf{b}^{-1}_{\mathbf{m},\mathbf{n}}}} \mathcal{A}_{\mathbf{m}\oplus\mathbf{n}}). \tag{5.2}$$

Before applying the nerve functor, this functor acts on objects in this way:

$$\begin{aligned}(a_1, \ldots, a_m) \times (b_1, \ldots, b_n) &\mapsto (b_1, \ldots, b_n) \times (a_1, \ldots, a_m)\\ &\mapsto (b_1, \ldots, b_n, a_1, \ldots, a_m) \mapsto (a_1, \ldots, a_m, b_1, \ldots, b_n)\end{aligned}$$

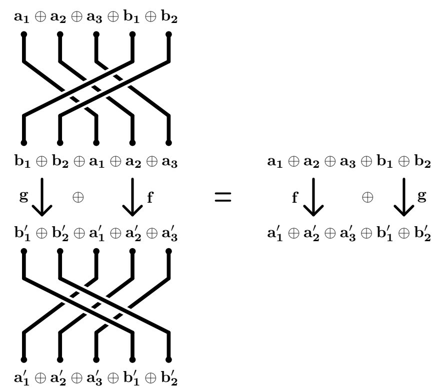
and on morphisms in this way:

$$f \times g \mapsto g \times f \mapsto g \oplus f \mapsto \mathsf{b}^{-1}_{\mathbf{m},\mathbf{n}} \circ (g \oplus f) \circ \mathsf{b}_{\mathbf{m},\mathbf{n}}.$$

By naturality of the braiding, the map $\mathsf{b}_{\mathbf{m},\mathbf{n}}^{-1} \circ (g \oplus f) \circ \mathsf{b}_{\mathbf{m},\mathbf{n}}$ equals $f \oplus g$, see also the next figure for an illustration of this. This shows that the functor (5.2) is equal to

$$\mathcal{N}(\mathcal{A}_{\mathbf{m}} \times \mathcal{A}_{\mathbf{n}} \xrightarrow{\mu'_{\mathbf{m},\mathbf{n}}} \mathcal{A}_{\mathbf{m}\oplus\mathbf{n}}),$$

which induces the multiplication, μ, so the monoid is commutative.



□

5.2 Commutative monoids and double loop spaces

We proved in Proposition 4.3.6 that the homotopy colimit of a commutative $\mathfrak{B}$-space monoid is an $\mathcal{NB}$-space. The natural question now is: Given a simplicial set K with an $\mathcal{NB}$-action can we find a commutative $\mathfrak{B}$-space monoid Z such that $\operatorname{hocolim}_{\mathfrak{B}} Z \simeq K$ via a chain of $\mathcal{NB}$-space morphisms? In this section we show that if $K = \mathcal{NA}$, then $Z = \mathcal{NA}_\bullet$ will do.

But first we show that the double loop space $\Omega\Sigma^2 X$ is weakly equivalent to the homotopy colimit of the commutative monoid, $X^\bullet$, from Example 5.1.1, if X is a connected pointed simplicial set. This result is analogous to Proposition 4.5 in [Sch07].

Theorem 5.2.1. *If X is a connected pointed simplicial set then there is a chain of weak equivalences from $\Omega\Sigma^2 X$ to $\operatorname{hocolim}_{\mathfrak{B}}(X^\bullet)$, where $X^\bullet$ is the commutative monoid from Example 5.1.1.*

Sketch of proof. The double loop space $\Omega\Sigma^2 X$ is weakly equivalent to $\mathbb{C}_2^\Sigma X$ by Theorem 2.7 in [May72]. From Proposition 4.2.6 we get $\mathbb{C}_2^\Sigma X = \tilde{\mathbb{C}}_2^{\mathcal{B}} X$. As in the proof of Proposition 4.2.8 we can construct the product of the braided operads $\mathcal{NB}$ and $\tilde{\mathcal{C}}_2^{\mathcal{B}}$ and get weak equivalences

$$\tilde{\mathbb{C}}_2^{\mathcal{B}} X \leftarrow (\tilde{\mathbb{C}}_2^{\mathcal{B}} \times \mathbb{NB})X \to \mathbb{NB}X.$$

Let $\bar{X}_t$ be the set of t simplices of X except the basepoint. The relations of type 2 in Definition 4.1.8 implies that

$$(\mathbb{NB}X)_t \cong \coprod_{n \geq 0} (\mathcal{NB}(n)_t \times_{\mathcal{B}_n} \bar{X}_t^{\times n}).$$

Now we can use the same teqnique as in Section 4.1 [Sch07], to get a weak equivalence

$$\mathcal{N}(* \downarrow X_t^\bullet) \to (\mathbb{NB}X)_t.$$

Viewing $\mathbb{NB}X$ as a bisimplicial set, constant in the vertical direction we get a morphism of bisimplicial sets $\{t \mapsto \mathcal{N}(* \downarrow X_t^\bullet)\} \to \mathbb{NB}X$ that is a weak equivalence in the horisontal degrees. Taking diagonals yields a weak equivalence

$$\operatorname*{hocolim}_{\mathfrak{B}}(X^\bullet) \to \mathbb{NB}X.$$

So we have a chain of weak equivalences from $\Omega\Sigma^2 X$ to $\operatorname{hocolim}_{\mathfrak{B}}(X^\bullet)$. □

For the other theorem we will need some definitions and auxiliary results.

Definition 5.2.2. Let $\mathfrak{B}_+$ denote the full subcategory of $\mathfrak{B}$ with all objects except $\mathbf{0}$.

Lemma 5.2.3. *For each $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{B}_+$ the simplicial set function*

$$\mathcal{NA}_\alpha\colon \mathcal{NA}_{\mathbf{m}} \to \mathcal{NA}_{\mathbf{n}}$$

is a weak equivalence, where $\mathcal{NA}_\bullet$ is the $\mathfrak{B}$-space constructed in Example 3.1.9.

Proof. For any $\mathbf{k}$ we can define a functor $\Gamma_k\colon \mathcal{A}_{\mathbf{k}} \to \mathcal{A}_{\mathbf{1}}$ by $\Gamma_k(a_1, \dots, a_k) = a_1 \otimes \cdots \otimes a_k$ and $\Gamma_k(f) = f$. Let β be any injective braid from $\mathbf{1}$ to $\mathbf{k}$. Is easy to see that there is a natural transformation from $\Gamma_k \circ \mathcal{A}_\beta$ to the identity on $\mathcal{A}_{\mathbf{1}}$, and one from $\mathcal{A}_\beta \circ \Gamma_k$ to the identity on $\mathcal{A}_{\mathbf{k}}$. Therefore $\mathcal{NA}_\beta$ is a weak homotopy equivalence.

Now let β be any injective braid from $\mathbf{1}$ to $\mathbf{m}$. Then the following diagram commutes

$$\begin{array}{ccccc} & & \mathcal{NA}_{\mathbf{1}} & & \\ & \swarrow^{\mathcal{NA}_\beta} & & \searrow^{\mathcal{NA}_{\alpha\circ\beta}} & \\ \mathcal{NA}_{\mathbf{m}} & & \xrightarrow{\mathcal{NA}_\alpha} & & \mathcal{NA}_{\mathbf{n}} \end{array}$$

and by the two out of three property $\mathcal{NA}_\alpha$ is a weak equivalence. □

Definition 5.2.4 (From Definition 19.6.1 in [Hir03])**.** Let $\mathcal{D}$ be a subcategory of a small category $\mathcal{C}$. The subcategory $\mathcal{D}$ is a homotopy right cofinal in $\mathcal{C}$ if for every object c in $\mathcal{C}$ the space $\mathcal{N}(c \downarrow \mathcal{D})$ is contractible.

Lemma 5.2.5 (From Theorem 19.6.13 in [Hir03])**.** *Let $\mathcal{D}$ be a subcategory of a small category $\mathcal{C}$. If $\mathcal{D}$ is homotopy right cofinal in $\mathcal{C}$ then for every functor $X\colon \mathcal{C} \to \mathcal{S}$, the canonical map*

$$\operatorname*{hocolim}_{\mathcal{D}} X \to \operatorname*{hocolim}_{\mathcal{C}} X$$

is a weak eqiuvalence of simplicial sets.

Lemma 5.2.6. *The subcategory $\mathfrak{B}_+$ is homotopy right cofinal in $\mathfrak{B}$.*

Proof. Fix $\mathbf{k}$ in $\mathfrak{B}$. The comma category $(\mathbf{k} \downarrow \mathfrak{B})$ has an initial object, so $\mathcal{N}(\mathbf{k} \downarrow \mathfrak{B})$ is contractible. We have a functor

$$i\colon (\mathbf{k} \downarrow \mathfrak{B}_+) \to (\mathbf{k} \downarrow \mathfrak{B})$$

induced by the inclusion of $\mathfrak{B}_+$ in $\mathfrak{B}$. Let $\iota_{\mathbf{1}}$ denote the map $\mathbf{0} \to \mathbf{1}$ in $\mathfrak{B}$, it induces a functor

$$p\colon (\mathbf{k} \downarrow \mathfrak{B}) \to (\mathbf{k} \downarrow \mathfrak{B}_+)$$

by sending an object $\alpha\colon \mathbf{k} \to \mathbf{m}$ to $\alpha \oplus \iota_{\mathbf{1}}\colon \mathbf{k} \to \mathbf{m}+\mathbf{1}$ and a morphism β to $\beta \oplus \mathrm{id}_{\mathbf{1}}$. The map $\iota_{\mathbf{1}}$ induces natural transformations $\mathrm{id}_{(\mathbf{k}\downarrow\mathfrak{B}_+)} \to pi$ and $\mathrm{id}_{(\mathbf{k}\downarrow\mathfrak{B})} \to ip$. A natural transformation from a functor F to a functor G induces a homotopy from the nerve of F to the nerve of G. Therefore

$$\mathcal{N}(\mathbf{k} \downarrow \mathfrak{B}_+) \simeq \mathcal{N}(\mathbf{k} \downarrow \mathfrak{B}) \simeq *,$$

so $\mathcal{N}(\mathbf{k} \downarrow \mathfrak{B}_+)$ is contractible for every $\mathbf{k}$ in $\mathfrak{B}$. □

Definition 5.2.7. Let $\mathcal{N}\tilde{\mathcal{A}}_\bullet$ be the $\mathfrak{B}$-space defined by

$$\mathcal{N}\tilde{\mathcal{A}}_{\mathbf{m}} = \mathcal{N}\mathcal{A}_{\mathbf{1}\oplus\mathbf{m}} \text{ and } \mathcal{N}\tilde{\mathcal{A}}_\alpha = \mathcal{N}\mathcal{A}_{\mathrm{id}_{\mathbf{1}}\oplus\alpha}$$

for objects $\mathbf{m}$ and morphisms α in $\mathfrak{B}$. This is obviously functorial.

For $\mathbf{m}$ in $\mathfrak{B}$ we can define a functor

$$\mathcal{A}_{\mathbf{m}} \to \mathcal{A}_{\mathbf{1}\oplus\mathbf{m}}$$

by $(a_1,\ldots,a_m) \mapsto (I, a_1, \ldots, a_m)$ and $g \mapsto \mathrm{id}_I \oplus g$, and a functor

$$\mathcal{A}_{\mathbf{1}} \to \mathcal{A}_{\mathbf{1}\oplus\mathbf{m}}$$

by $a \mapsto (a, I, \ldots, I)$ and $f \mapsto f \oplus \mathrm{id}_I \oplus \cdots \oplus \mathrm{id}_I$. Both functors are natural in $\mathbf{m}$ and we get morphisms of $\mathfrak{B}$-spaces

$$\mathrm{const}(\mathcal{N}\mathcal{A}) \to \mathcal{N}\tilde{\mathcal{A}}_\bullet \leftarrow \mathcal{N}\mathcal{A}_\bullet. \tag{5.3}$$

Lemma 5.2.8. *The morphisms in* (5.3) *are morphisms of* $\mathcal{N}\mathcal{B}$*-spaces.*

Sketch of proof. First we have to show that $\mathcal{N}\mathcal{B}$ acts on the $\mathfrak{B}$-spaces. The action of $\mathcal{N}\mathcal{B}$ on $\mathcal{N}\mathcal{A}$ from Propositoin 4.3.3 gives an action on $\mathrm{const}(\mathcal{N}\mathcal{A})$.

We can define a functor

$$\mathcal{B}(k) \times \mathcal{A}_{\mathbf{m_1}} \times \cdots \times \mathcal{A}_{\mathbf{m_k}} \to \mathcal{A}_{\mathbf{m_1}\oplus\cdots\oplus\mathbf{m_k}}$$

by

$$(\zeta, (a_1^1, \ldots, a_{m_1}^1), \ldots, (a_1^k, \ldots, a_{m_k}^k)) \mapsto (a_1^{\zeta^{-1}(1)}, \ldots, a_{m_{\zeta^{-1}(1)}}^{\zeta^{-1}(1)}, \ldots, a_1^{\zeta^{-1}(k)}, \ldots, a_{m_{\zeta^{-1}(k)}}^{\zeta^{-1}(k)})$$
$$(\xi\colon \zeta \to \xi\zeta, f^1, \ldots, f^k) \mapsto \xi(m_{\zeta^{-1}(1)}, \ldots, m_{\zeta^{-1}(1)}) \circ (f^{\zeta^{-1}(1)} \oplus \cdots \oplus f^{\zeta^{-1}(k)}),$$

the functoriality is a consequence of the naturality of the braiding. The same type of argument used in Proposition 5.1.2 to show that $\mu'_{\mathbf{m},\mathbf{n}}$ is nautral shows that the functor here is natural in $\mathbf{m_1}, \dots, \mathbf{m_k}$. The family of such functors satisfy diagrams like those in Definition 4.1.4, by similar arguments as in the proof of Propositoin 4.3.3. By the universal property of the left kan extension this determines an action of $\mathcal{NB}$ on $\mathcal{NA}_\bullet$.

We now combine the action of $\mathcal{NB}$ on const($\mathcal{NA}$) and $\mathcal{NA}_\bullet$ to get an action of $\mathcal{NB}$ on $\mathcal{N}\tilde{\mathcal{A}}_\bullet$ We define a functor

$$\mathcal{B}(k) \times \mathcal{A}_{\mathbf{1}\oplus\mathbf{m_1}} \times \cdots \times \mathcal{A}_{\mathbf{1}\oplus\mathbf{m_k}} \to \mathcal{A}_{\mathbf{1}\oplus\mathbf{m_1}\oplus\cdots\oplus\mathbf{m_k}}$$

on objects by

$$(\zeta, (a_0^1, a_1^1, \dots, a_{m_1}^1), \dots, (a_0^k, a_1^k, \dots, a_{m_k}^k)) \mapsto (a_0^{\zeta^{-1}(1)} \oplus \cdots \oplus a_0^{\zeta^{-1}(k)}, a_1^{\zeta^{-1}(1)}, \dots, a_{m_{\zeta^{-1}(1)}}^{\zeta^{-1}(1)}, \dots, a_1^{\zeta^{-1}(k)}, \dots, a_{m_{\zeta^{-1}(k)}}^{\zeta^{-1}(k)}).$$

Send a morphism

$$(\xi\colon\ \zeta \to \xi\zeta, f^1, \dots, f^k)\colon\ (\zeta, (a_0^1, a_1^1, \dots, a_{m_1}^1), \dots, (a_0^k, a_1^k, \dots, a_{m_k}^k)) \to (\xi\zeta, (b_0^1, b_1^1, \dots, b_{m_1}^1), \dots, (b_0^k, b_1^k, \dots, b_{m_k}^k))$$

to

$$\mathrm{braid} \circ \xi \circ (f^{\zeta^{-1}(1)} \oplus \cdots \oplus f^{\zeta^{-1}(k)}) \circ \mathrm{braid}^{-1},$$

where

$$a_0^{\zeta^{-1}(1)} \oplus \cdots \oplus a_0^{\zeta^{-1}(k)} \oplus a_1^{\zeta^{-1}(1)} \oplus \cdots \oplus a_{m_{\zeta^{-1}(1)}}^{\zeta^{-1}(1)} \oplus \cdots \oplus a_1^{\zeta^{-1}(k)} \oplus \cdots \oplus a_{m_{\zeta^{-1}(k)}}^{\zeta^{-1}(k)}$$
$$\xrightarrow{\mathrm{braid}^{-1}} a_0^{\zeta^{-1}(1)} \oplus a_1^{\zeta^{-1}(1)} \oplus \cdots \oplus a_{m_{\zeta^{-1}(1)}}^{\zeta^{-1}(1)} \oplus \cdots \oplus a_0^{\zeta^{-1}(k)} \oplus a_1^{\zeta^{-1}(k)} \oplus \cdots \oplus a_{m_{\zeta^{-1}(k)}}^{\zeta^{-1}(k)}$$

braids the a_0's under the a_i's and all the time keeping the order between the a_i's for $i > 0$ and similarly for

$$b_0^{(\xi\zeta)^{-1}(1)} \oplus b_1^{(\xi\zeta)^{-1}(1)} \oplus \cdots \oplus b_{m_{(\xi\zeta)^{-1}(1)}}^{(\xi\zeta)^{-1}(1)} \oplus \cdots \oplus b_0^{(\xi\zeta)^{-1}(k)} \oplus b_1^{(\xi\zeta)^{-1}(k)} \oplus \cdots \oplus b_{m_{(\xi\zeta)^{-1}(k)}}^{(\xi\zeta)^{-1}(k)} \xrightarrow{\mathrm{braid}}$$
$$b_0^{(\xi\zeta)^{-1}(1)} \oplus \cdots \oplus b_0^{(\xi\zeta)^{-1}(k)} \oplus b_1^{(\xi\zeta)^{-1}(1)} \oplus \cdots \oplus b_{m_{(\xi\zeta)^{-1}(1)}}^{(\xi\zeta)^{-1}(1)} \oplus \cdots \oplus b_1^{(\xi\zeta)^{-1}(k)} \oplus \cdots \oplus b_{m_{(\xi\zeta)^{-1}(k)}}^{(\xi\zeta)^{-1}(k)}.$$

That this gives an action of $\mathcal{NB}$ on $\mathcal{N}\tilde{\mathcal{A}}_\bullet$ can be proved by carefully combining the corresponding arguments in the cases of $\mathcal{NA}$ and $\mathcal{NA}_\bullet$.

The morphisms preserve the action of $\mathcal{NB}$ by construction. □

Theorem 5.2.9. *There is a chain of weak equivalences*

$$\mathcal{NA} \leftarrow \operatorname*{hocolim}_{\mathfrak{B}}(\mathrm{const}(\mathcal{NA})) \to \operatorname*{hocolim}_{\mathfrak{B}}(\mathcal{N}\tilde{\mathcal{A}}_\bullet) \leftarrow \operatorname*{hocolim}_{\mathfrak{B}}(\mathcal{NA}_\bullet) \tag{5.4}$$

connecting $\mathcal{NA}$ *and* $\mathrm{hocolim}_{\mathfrak{B}}(\mathcal{NA}_\bullet)$. *Each of the maps is a morphism of* $\mathcal{NB}$*-spaces.*

Proof. The simplicial set $\mathrm{hocolim}_{\mathfrak{B}}(\mathrm{const}(\mathcal{N}\mathcal{A}))$ is isomorphich to $\mathcal{N}\mathfrak{B} \times \mathcal{N}\mathcal{A}$ and the left map is the projection. The action of $\mathcal{N}\mathcal{B}$ on $\mathcal{N}\mathfrak{B} \times \mathcal{N}\mathcal{A}$ is diagonal, so the projection is is an $\mathcal{N}\mathcal{B}$-morphism. Since $\mathcal{N}\mathfrak{B}$ is contractible the projection is a homotopy equivalence.

The other two maps are $\mathrm{hocolim}_{\mathfrak{B}}$ of the maps in (5.3). In Proposition 6.5 in [Sch09] it is proven that $\mathrm{hocolim}_{\mathcal{I}}$ determines a functor

$$\mathcal{S}^{\mathcal{I}}[\mathcal{E}] \to \mathcal{S}[\mathcal{E} \times \mathcal{E}],$$

where $\mathcal{I}$ is defined in Definition 2.1.4 and $\mathcal{E}$ is the Barratt-Eccles operad. Then we can precompse with the diagonal $\mathcal{E} \to \mathcal{E} \times \mathcal{E}$ to get to $\mathcal{S}[\mathcal{E}]$. The same theqnique can be used to show that $\mathrm{hocolim}_{\mathfrak{B}}$ determines a functor

$$\mathcal{S}^{\mathfrak{B}}[\mathcal{N}\mathcal{B}] \to \mathcal{S}[\mathcal{N}\mathcal{B}].$$

So what is left is to show that the two rightmost maps are weak equivalences. By Lemma 5.2.3, the $\mathfrak{B}$-space morphism $\mathrm{const}(\mathcal{N}\mathcal{A}) \to \mathcal{N}\tilde{\mathcal{A}}_\bullet$ is a levelwise weak equivalence, hence the induced map on homotopy comlimits is a weak equivalence of simplicial sets. Likewise $\mathcal{N}\mathcal{A}_\bullet \to \mathcal{N}\tilde{\mathcal{A}}_\bullet$ is a weak equivalence in all levels except for $\mathbf{0}$ and the simplicial set map $\mathrm{hocolim}_{\mathfrak{B}_+}(\mathcal{N}\mathcal{A}_\bullet) \to \mathrm{hocolim}_{\mathfrak{B}_+}(\mathcal{N}\tilde{\mathcal{A}}_\bullet)$ is a weak equivalence. In the diagram

$$\begin{array}{ccc} \mathrm{hocolim}_{\mathfrak{B}_+}(\mathcal{N}\mathcal{A}_\bullet) & \xrightarrow{\sim} & \mathrm{hocolim}_{\mathfrak{B}_+}(\mathcal{N}\tilde{\mathcal{A}}_\bullet) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{hocolim}_{\mathfrak{B}}(\mathcal{N}\mathcal{A}_\bullet) & \longrightarrow & \mathrm{hocolim}_{\mathfrak{B}}(\mathcal{N}\tilde{\mathcal{A}}_\bullet) \end{array}$$

the vertical maps are weak equivelences by Lemma 5.2.6 and Lemma 5.2.5, and so the result follows from the two out of three property for weak equivalences. □

Remark 5.2.10. The action of $\mathcal{N}\mathcal{B}$ on $\mathrm{hocolim}_{\mathfrak{B}}(\mathcal{N}\mathcal{A}_\bullet)$ from Theorem 5.2.9 is the same as the one from Proposition 4.3.6.

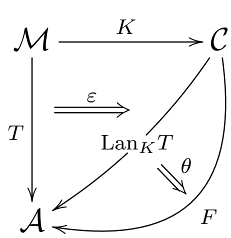
Appendix A

Functoriality of the left Kan extension

The purpose of this appendix is to show that the left Kan extension is functorial. This property is used extensively when we in Section 3.2 use left Kan extensions to define a braided monidal structure on a diagram category.

We start by fixing some notation. We will denote by $\mathcal{S}$ the category of simplicial sets. Suppose we have two functors $G, H\colon \mathcal{C} \to \mathcal{D}$, a natural transformation $\theta\colon G \to H$ and a functor $F\colon \mathcal{B} \to \mathcal{C}$. There is an induced natural transformation from $G \circ F$ to $H \circ F$ which we denote by θ_F, it is determined by $(\theta_F)_b = \theta_{F(b)}$. In diagrams we will use a double arrow to indicate a natural transformation.

Definition A.0.1 (Dual to the definition of a right Kan extension in Section X.3 [ML98])**.** Given functors $K\colon \mathcal{M} \to \mathcal{C}$ and $T\colon \mathcal{M} \to \mathcal{A}$, a left Kan extension of T along K is a pair $(\mathrm{Lan}_K T, \varepsilon)$ consisting of a functor $\mathrm{Lan}_K T\colon \mathcal{C} \to \mathcal{A}$ and a natural transformation $\varepsilon\colon T \to \mathrm{Lan}_K T \circ K$ with the following universal property:
For every pair $(F\colon \mathcal{C} \to \mathcal{A}, \vartheta\colon T \to F \circ K)$ there is a unique natural transformation $\theta\colon \mathrm{Lan}_K T \to F$ such that $\theta_K \circ \varepsilon = \vartheta$.



We often refer to the left Kan extension $\mathrm{Lan}_K T$, the natural transformation ε is then implicit. From the univiversal property it follows that if a left Kan extension of T along K exists, the functor $\mathrm{Lan}_K T$ is unique up to natural isomorphism. This justifies writing *the* left Kan extension.

When the left Kan extension $\mathrm{Lan}_K T$ existsion exists we can write it as a pointwise colimit

$$\mathrm{Lan}_K T(c) = \operatorname*{colim}_{K(m)\to c} T(m), \tag{A.1}$$

where the colimit is taken over the comma category $(K \downarrow c)$. Furthermore, the map $\mathrm{colim}_{K(m)\to c} T(m) \to \mathrm{colim}_{K(m)\to c'} T(m)$, induced by a morphism $f\colon c \to c'$, agrees with $\mathrm{Lan}_K T(f\colon c \to c')$. This is straightforward to check from the definitions. The converse is also true: The dual of Theorem 1 in Section X.3 [ML98] implies that if the colimit in (A.1) exists for each c in $\mathcal{C}$, then the left Kan extension of T along K exists.

As corollary we get that if the category $\mathcal{M}$ is small and $\mathcal{A}$ has all small colimits, then the left Kan extension of any functor $T\colon \mathcal{M} \to \mathcal{A}$ along any $K\colon \mathcal{M} \to \mathcal{C}$ exists.

Definition A.0.2. We define the Left Kan category $\mathcal{L}$ where an objects is a pair

$$\langle T\colon \mathcal{M} \to \mathcal{S}, K\colon \mathcal{M} \to \mathcal{C}\rangle$$

with T is a functor from a small category $\mathcal{M}$ to the category of simplicial sets $\mathcal{S}$ and K is a functor from $\mathcal{M}$ to a category $\mathcal{C}$. A morphism

$$F\colon \langle T\colon \mathcal{M} \to \mathcal{S}, K\colon \mathcal{M} \to \mathcal{C}\rangle \to \langle T'\colon \mathcal{M}' \to \mathcal{S}, K'\colon \mathcal{M}' \to \mathcal{C}'\rangle$$

consists of two functors $F_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{M}'$ and $F_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}'$ and two natural transformations $\theta\colon T \to T' \circ F_{\mathcal{M}}$ and $\kappa\colon K' \circ F_{\mathcal{M}} \to F_{\mathcal{C}} \circ K$.

The composite $\tilde{F} = F' \circ F$ is defined by the equations

$$\tilde{F}_{\mathcal{M}} = F'_{\mathcal{M}'} \circ F_{\mathcal{M}} \text{ and } \tilde{F}_{\mathcal{C}} = F'_{\mathcal{C}'} \circ F_{\mathcal{C}}$$

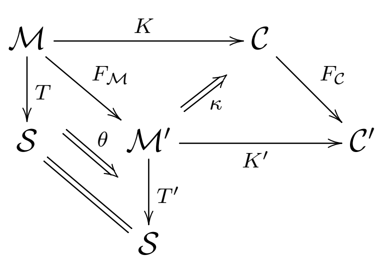
for the functors, and the morphisms

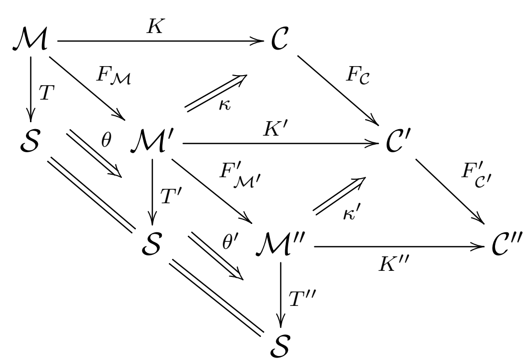
$$\tilde{\theta}_m\colon T(m) \xrightarrow{\theta_m} T' \circ F_{\mathcal{M}}(m) \xrightarrow{\theta'_{F_{\mathcal{M}}(m)}} T'' \circ F'_{\mathcal{M}'} \circ F_{\mathcal{M}}(m) \text{ and}$$

$$\tilde{\kappa}_m\colon K'' \circ F'_{\mathcal{M}'} \circ F_{\mathcal{M}}(m) \xrightarrow{\kappa'_{F_{\mathcal{M}}(m)}} F'_{\mathcal{C}'} \circ K' \circ F_{\mathcal{M}}(m) \xrightarrow{F'_{\mathcal{C}'}(\kappa_m)} F'_{\mathcal{C}'} \circ F_{\mathcal{C}} \circ K(m)$$

for the natural transformations.

The diagram on the left illustrates a morphism, and the one on the right illustrates two composable morphisms.





Definition A.0.3. We define the Lax Twisted Arrow category $\mathcal{T}$ with functors to simplicial sets as objects. A morphism G from a functor $L\colon \mathcal{C} \to \mathcal{S}$ to a functor $L'\colon \mathcal{C}' \to \mathcal{S}$ consists of a functor

$$G_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}'$$

and a natural transformation

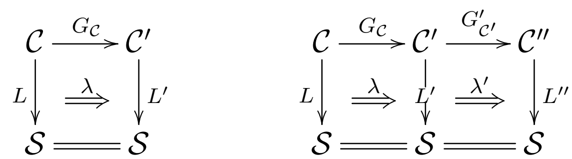
$$\lambda\colon L \to L' \circ G_{\mathcal{C}}.$$

The composite $\tilde{G} = G' \circ G$ is determined by the equation $\tilde{G}_{\mathcal{C}} = G'_{\mathcal{C}'} \circ G_{\mathcal{C}}$ for the functor, and the morphism

$$\tilde{\lambda}_c\colon L(c) \xrightarrow{\lambda_c} L' \circ G_{\mathcal{C}}(c) \xrightarrow{\lambda'_{G_{\mathcal{C}}(c)}} L'' \circ G'_{\mathcal{C}'} \circ G_{\mathcal{C}}(c)$$

for the natural transformation.

The diagram on the left illustrates a morphism, and the one on the right illustrates two composable morphisms.



Theorem A.0.4 (Functoriality of the left Kan extension. Together with Martin Stolz)**.** *There is a functor*

$$\mathcal{L} \to \mathcal{T}$$

defined on objects by

$$\langle T\colon \mathcal{M} \to \mathcal{S}, K\colon \mathcal{M} \to \mathcal{C}\rangle \mapsto \langle \mathrm{Lan}_K T\colon \mathcal{C} \to \mathcal{S}\rangle,$$

where $(\mathrm{Lan}_K T, \varepsilon)$ *is the left Kan extension of* T *along* K*.*

A morphism

$$F\colon \langle T\colon \mathcal{M} \to \mathcal{S}, K\colon \mathcal{M} \to \mathcal{C}\rangle \to \langle T'\colon \mathcal{M}' \to \mathcal{S}, K'\colon \mathcal{M}' \to \mathcal{C}'\rangle$$

is sent to the morphism $\mathrm{Lan}_K T \to \mathrm{Lan}_{K'} T'$ *consisting of* $F_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}'$ *and the natural transformasjon* $\lambda\colon \mathrm{Lan}_K T \to \mathrm{Lan}_{K'} T' \circ F_{\mathcal{C}}$ *defined below.*

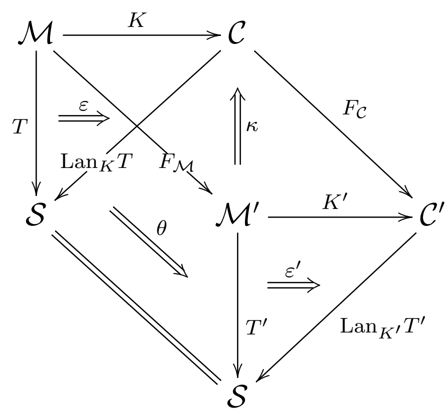
We can define a natural transformation

$$\varphi\colon T \to \mathrm{Lan}_{K'} T' \circ F_{\mathcal{C}} \circ K$$

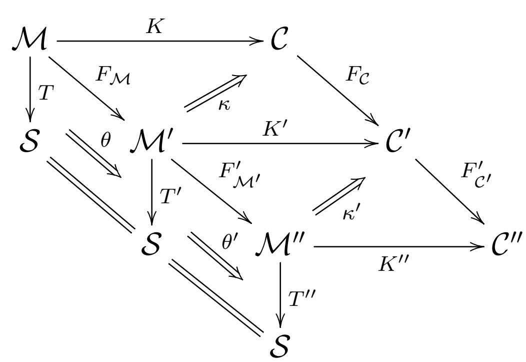
by

$$T(m) \xrightarrow{\theta_m} T' \circ F_{\mathcal{M}}(m) \xrightarrow{\varepsilon'_{F_{\mathcal{M}}(m)}} \mathrm{Lan}_{K'} T' \circ K' \circ F_{\mathcal{M}}(m) \xrightarrow{\mathrm{Lan}_{K'} T'(\kappa_m)} \mathrm{Lan}_{K'} T' \circ F_{\mathcal{C}} \circ K(m).$$

We let λ *be the unique natural transformation* $\lambda\colon \mathrm{Lan}_K T \to \mathrm{Lan}_{K'} T' \circ F_{\mathcal{C}}$*, with* $\varphi = \lambda_K \circ \varepsilon$*, that the universal property of the Kan extension provides.*



Proof. This clearly preserves identity morphisms, so the only thing to check is that composition is preserved.



Let $F\colon \langle T, K\rangle \to \langle T', K'\rangle$ and $F'\colon \langle T', K'\rangle \to \langle T'', K''\rangle$ be two composable morphisms in $\mathcal{L}$, and let $(\mathrm{Lan}_K T, \varepsilon)$ be the left Kan extensions of T along K, and similarly for $(\mathrm{Lan}_{K'} T', \varepsilon')$ and $(\mathrm{Lan}_{K''} T'', \varepsilon'')$.

The comoposition of F and F' in $\mathcal{L}$ is determined by $F'_{\mathcal{M}'} \circ F_{\mathcal{M}},\ F'_{\mathcal{C}'} \circ F_{\mathcal{C}}$,

$$\tilde{\theta}_m = \theta'_{F_{\mathcal{M}}(m)} \circ \theta_m \text{ and} \tag{A.2}$$

$$\tilde{\kappa}_m = F'_{\mathcal{C}'}(\kappa_m) \circ \kappa'_{F_{\mathcal{M}}(m)}. \tag{A.3}$$

This is then sent to

$$F'_{\mathcal{C}'} \circ F_{\mathcal{C}}\colon\ \mathcal{C} \to \mathcal{C}''$$

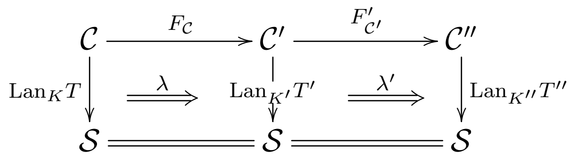
and the unique natural transformation $\bar{\lambda}\colon\ \mathrm{Lan}_K T \to \mathrm{Lan}_{K''} T'' \circ F'_{\mathcal{C}'} \circ F_{\mathcal{C}}$ such that

$$\bar{\varphi} = \bar{\lambda}_K \circ \varepsilon \tag{A.4}$$

for $\bar{\varphi}$ defined by

$$\bar{\varphi}_m = \mathrm{Lan}_{K''} T''(\tilde{\kappa}_m) \circ \varepsilon''_{F'_{\mathcal{M}'} F_{\mathcal{M}}(m)} \circ \tilde{\theta}_m. \tag{A.5}$$

If we instead apply the functor to each of the morphisms we get the two morphisms picutred in the diagram:



The natural transformations λ and λ' are defined using the universal property of the left Kan extension. We define $\lambda\colon \mathrm{Lan}_K T \to F_{\mathcal{A}} \circ \mathrm{Lan}_{K'} T' \circ F_{\mathcal{C}}$ as the unique natural transformation such that

$$\varphi = \lambda_K \circ \varepsilon, \tag{A.6}$$

for φ induced by

$$\varphi_m = \mathrm{Lan}_{K'} T'(\kappa_m) \circ F_{\mathcal{A}}(\varepsilon'_{F_{\mathcal{M}}(m)}) \circ \theta_m. \tag{A.7}$$

Similarly the natural transformation $\lambda'\colon \mathrm{Lan}_{K'} T' \to F'_{\mathcal{A}'} \mathrm{Lan}_{K''} T'' F'_{\mathcal{C}'}$ is the unique one with

$$\varphi' = \lambda'_{K'} \circ \varepsilon', \tag{A.8}$$

for φ' induced by

$$\varphi'_{m'} = \mathrm{Lan}_{K''} T''(\kappa'_{m'}) \circ \varepsilon''_{F'_{\mathcal{M}'}(m')} \circ \theta'_{m'}. \tag{A.9}$$

We then compose in $\mathcal{T}$ and obtain the morphism consiting of the functor

$$F'_{\mathcal{C}'} \circ F_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}''$$

and the natural transformation

$$\lambda'_{F_{\mathcal{C}}} \circ \lambda. \tag{A.10}$$

For the two resulting morphisms, we see that the functors are equal, so what is left to show is that the natural transformations $\bar{\lambda}$ and $\lambda'_{F_{\mathcal{C}}} \circ \lambda$ are the same. Since $\bar{\lambda}$ is determined by (A.4), we will achieve this by showing

$$\bar{\varphi}_m = \lambda'_{F_{\mathcal{C}}(K(m))} \circ \lambda_{K(m)} \circ \varepsilon_m$$

for each m in $\mathcal{M}$. We expand the right side of the equation:

$$\begin{aligned}
&\lambda'_{F_{\mathcal{C}}K(m)} \circ \lambda_{K(m)} \circ \varepsilon_m \\
&= \lambda'_{F_{\mathcal{C}}K(m)} \circ \varphi_m && \text{by (A.6)} \\
&= \lambda'_{F_{\mathcal{C}}K(m)} \circ \mathrm{Lan}_{K'} T'(\kappa_m) \circ \varepsilon'_{F_{\mathcal{M}}(m)} \circ \theta_m && \text{by (A.7)} \\
&= \mathrm{Lan}_{K''} T'' F'_{\mathcal{C}'}(\kappa_m) \circ \lambda'_{K' F_{\mathcal{M}}(m)} \circ \varepsilon'_{F_{\mathcal{M}}(m)} \circ \theta_m && \text{naturality of } \lambda' \\
&= \mathrm{Lan}_{K''} T'' F'_{\mathcal{C}'}(\kappa_m) \circ \varphi'_{F_{\mathcal{M}}(m)} \circ \theta_m && \text{by (A.8)} \\
&= \mathrm{Lan}_{K''} T'' F'_{\mathcal{C}'}(\kappa_m) \circ \mathrm{Lan}_{K''} T''(\kappa'_{F_{\mathcal{M}}(m)}) \circ \varepsilon''_{F'_{\mathcal{M}'} F_{\mathcal{M}}(m)} \circ \theta'_{F_{\mathcal{M}}(m)} \circ \theta_m && \text{by (A.9)} \\
&= \mathrm{Lan}_{K''} T''(F'_{\mathcal{C}'}(\kappa_m) \circ \kappa'_{F_{\mathcal{M}}(m)}) \circ \varepsilon''_{F'_{\mathcal{M}'} F_{\mathcal{M}}(m)} \circ \theta'_{F_{\mathcal{M}}(m)} \circ \theta_m
\end{aligned}$$

And then the left side:

$$\begin{aligned}
&\bar{\varphi}_m \\
&= \mathrm{Lan}_{K''}T''(\tilde{\kappa}_m) \circ \varepsilon''_{F'_{\mathcal{M}'}F_{\mathcal{M}}(m)} \circ \tilde{\theta}_m && \text{by (A.5)} \\
&= \mathrm{Lan}_{K''}T''(F'_{\mathcal{C}'}(\kappa_m) \circ \kappa'_{F_{\mathcal{M}}(m)}) \circ \varepsilon''_{F'_{\mathcal{M}'}F_{\mathcal{M}}(m)} \circ \theta'_{F_{\mathcal{M}}(m)} \circ \theta_m && \text{by (A.3) and (A.2)}
\end{aligned}$$

We see that the equation $F_{\mathcal{A}}(\lambda'_{F_{\mathcal{C}}(K(m))}) \circ \lambda_{K(m)} \circ \varepsilon_m = \bar{\varphi}_m$ holds for all objects m in $\mathcal{M}$, which completes the proof. □

Remark A.0.5. This can be done more generally by considering functors $T\colon \mathcal{M} \to \mathcal{A}$ and $K\colon \mathcal{M} \to \mathcal{C}$ such that the left Kan extension of T along K exists. The functoriality in $\mathcal{A}$ is contravariant. For our purpose we only need the case where $\mathcal{A} = \mathcal{S}$ and $F_{\mathcal{S}} = \mathrm{id}_{\mathcal{S}}$. This simplifies the notation in this appendix considerably, we have therefore chosen not to write the result in full generality.

We conclude this appendix with a couple of other useful results about left Kan extensions.

Lemma A.0.6. *Suppose we have a functors $X\colon \mathcal{A} \to \mathcal{S}$, $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$. Let*

$$(\mathrm{Lan}_F X, \varepsilon^F)$$

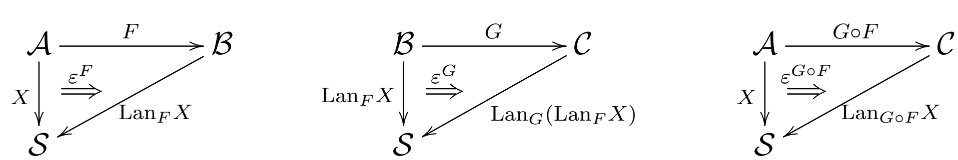
be the left Kan extension of $X\colon \mathcal{A} \to \mathcal{S}$ along $F\colon \mathcal{A} \to \mathcal{B}$,

$$(\mathrm{Lan}_G(\mathrm{Lan}_F X), \varepsilon^G)$$

the left Kan extension of $\mathrm{Lan}_F X\colon \mathcal{B} \to \mathcal{S}$ *along $G\colon \mathcal{B} \to \mathcal{C}$, and let*

$$(\mathrm{Lan}_{G\circ F} X, \varepsilon^{G\circ F})$$

be the left Kan extension of $X\colon \mathcal{A} \to \mathcal{S}$ along $G \circ F\colon \mathcal{A} \to \mathcal{C}$. Then $\mathrm{Lan}_G(\mathrm{Lan}_F X)$ *is isomorphich to* $\mathrm{Lan}_{G\circ F} X$.



Proof. By the universal property of the left Kan extension, the natural transformation $\varepsilon^{G\circ F}\colon X \to \mathrm{Lan}_{G\circ F}X \circ G \circ F$ induces a unique morphism $\theta\colon \mathrm{Lan}_F X \to \mathrm{Lan}_{G\circ F}X \circ G$ such that

$$\theta_F \circ \varepsilon^F = \varepsilon^{G\circ F}. \tag{A.11}$$

This again induces a unique morphism $\kappa\colon \mathrm{Lan}_G(\mathrm{Lan}_F X) \to \mathrm{Lan}_{G\circ F}X$ such that

$$\kappa_G \circ \varepsilon^G = \theta. \tag{A.12}$$

The natural transformation $\varepsilon^G_F \circ \varepsilon^F\colon\ X \to \mathrm{Lan}_G(\mathrm{Lan}_F X)\circ G \circ F$ induces a unique morphism $\lambda\colon\ \mathrm{Lan}_{G\circ F} X \to \mathrm{Lan}_G(\mathrm{Lan}_F X)$ such that

$$\lambda_{G\circ F} \circ \varepsilon^{G\circ F} = \varepsilon^G_F \circ \varepsilon^F. \tag{A.13}$$

If we can show that κ and λ are inverses of each other we are done. By the univeral property of the Kan extension, $\kappa \circ \lambda$ is the identity if

$$(\kappa \circ \lambda)_{G\circ F} \circ \varepsilon^{G\circ F} = \varepsilon^{G\circ F}.$$

This follows from equations (A.13), (A.12) and (A.11). Similarly $\lambda \circ \kappa$ is the identity if

$$(\lambda \circ \kappa)_G \circ \varepsilon^G = \varepsilon^G\colon\ \mathrm{Lan}_F X \to \mathrm{Lan}_{G\circ F} X \circ G$$

using the universality of the Kan extension once again, this equality holds if

$$(\lambda \circ \kappa)_{G\circ F} \circ \varepsilon^G_F \circ \varepsilon^F = \varepsilon^G \circ \varepsilon^F.$$

This follows from equations (A.12), (A.11) and (A.13). □

For two functors $V_1\colon\ \mathcal{A}_1 \to \mathcal{S}$ and $V_2\colon\ \mathcal{A}_2 \to \mathcal{S}$ let $V_1 \times V_2\colon\ \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{S}$ be the functor defined by

$$(V_1 \times V_2)(a_1, a_2) = V_1(a_1) \times V_2(a_2)$$

on objects $a_i \in \mathcal{A}_i$ and

$$(V_1 \times V_2)(f_1, f_2) = V_1(f_1) \times V_2(f_2)$$

on morphisms $a_i \in \mathcal{A}_i$, $i = 1, 2$.

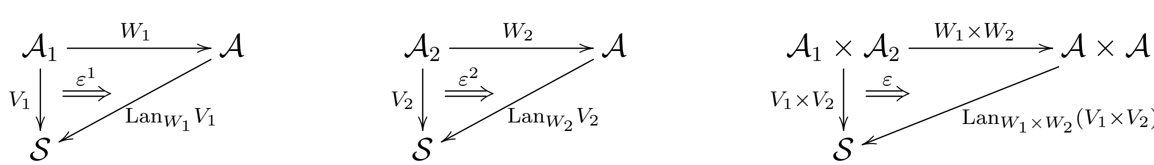
Lemma A.0.7. *Suppose we have functors* $V_i\colon\ \mathcal{A}_i \to \mathcal{S}$ *and* $W_i\colon\ \mathcal{A}_i \to \mathcal{A}$ *for* $i = 1, 2$. *Let*

$$(\mathrm{Lan}_{W_i} V_i, \varepsilon^i)$$

denote the left Kan extension of $V_i\colon\ \mathcal{A}_i \to \mathcal{S}$ *along* $W_i\colon\ \mathcal{A}_i \to \mathcal{A}$ *for* $i = 1, 2$, *and let*

$$(\mathrm{Lan}_{W_1\times W_2}(V_1 \times V_2), \varepsilon)$$

be the left Kan extension of $V_1 \times V_2\colon\ \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{S}$ *along* $W_1 \times W_2\colon\ \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A} \times \mathcal{A}$. *Then* $\mathrm{Lan}_{W_1} V_1 \times \mathrm{Lan}_{W_2} V_2$ *is isomorphic to* $\mathrm{Lan}_{W_1\times W_2}(V_1 \times V_2)$.



Proof. By the universal property of the left Kan extension, the natural transformation

$\varepsilon^1 \times \varepsilon^2 \colon V_1 \times V_2 \to (\mathrm{Lan}_{W_1} V_1 \circ W_1) \times (\mathrm{Lan}_{W_2} V_2 \circ W_2) = (\mathrm{Lan}_{W_1} V_1 \times \mathrm{Lan}_{W_2} V_2)(W_1 \times W_2)$

induces a unique morphism $\kappa \colon \mathrm{Lan}_{W_1 \times W_2}(V_1 \times V_2) \to \mathrm{Lan}_{W_1} V_1 \times \mathrm{Lan}_{W_2} V_2$ such that

$$\kappa_{W_1 \times W_2} \circ \varepsilon = \varepsilon^1 \times \varepsilon^2. \tag{A.14}$$

The natural transformation

$$\varepsilon \colon V_1 \times V_2 \to \mathrm{Lan}_{W_1 \times W_2}(V_1 \times V_2)(W_1 \times W_2)$$

is a family of simplicial set functions

$$V_1(a_1) \times V_2(a_2) \to \mathrm{Lan}_{W_1 \times W_2}(V_1 \times V_2)(W_1(a_1) \times W_2(a_2))$$

that is natural in $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$. Under the bijection

$$\hom_{\mathcal{S}}(K, \mathrm{Map}_{\mathcal{S}}(A, B)) \to \hom_{\mathcal{S}}(A \times K, B),$$

which is natural in the simplicial sets A,B and K, see [GJ99, prop 5.1], κ corresponds to a family of simplicical set functions

$$V_1(a_1) \to \mathrm{Map}_{\mathcal{S}}(V_2(a_2), \mathrm{Lan}_{W_1 \times W_2}(V_1 \times V_2)(W_1(a_1) \times W_2(a_2)))$$

that is natural in $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$. The universal property of the Kan extension now gives a family

$$\mathrm{Lan}_{W_1} V_1(a) \to \mathrm{Map}_{\mathcal{S}}(V_2(a_2), \mathrm{Lan}_{W_1 \times W_2}(V_1 \times V_2)(a \times W_2(a_2)))$$

that is natural in $a \in \mathcal{A}$ and $a_2 \in \mathcal{A}_2$. Using the bijection from [GJ99, prop 5.1] the other way, we get a family

$$\mathrm{Lan}_{W_1} V_1(a) \times V_2(a_2) \to \mathrm{Lan}_{W_1 \times W_2}(V_1 \times V_2)(a \times W_2(a_2))$$

that is natural in $a \in \mathcal{A}$ and $a_2 \in \mathcal{A}_2$. Doing the same for V_2 we get a natural transformation

$$\lambda \colon \mathrm{Lan}_{W_1} V_1 \times \mathrm{Lan}_{W_2} V_2 \to \mathrm{Lan}_{W_1 \times W_2}(V_1 \times V_2),$$

that is the unique one such that

$$\lambda_{W_1 \times W_2} \circ (\varepsilon^1 \times \varepsilon^2) = \varepsilon. \tag{A.15}$$

What is left is to check that κ and λ are inverses of each other. That $\lambda \circ \kappa$ is the identity follows from equations (A.14) and (A.15) similarly as in the previous proof. Going back and forth with the bijection from [GJ99, prop 5.1] as above we can show that $\kappa \circ \lambda$ is the identity if

$$(\kappa \circ \lambda)_{W_1 \times W_2} \circ (\varepsilon^1 \times \varepsilon^2) = \varepsilon^1 \times \varepsilon^2,$$

this follows from equations (A.15) and (A.14). □

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