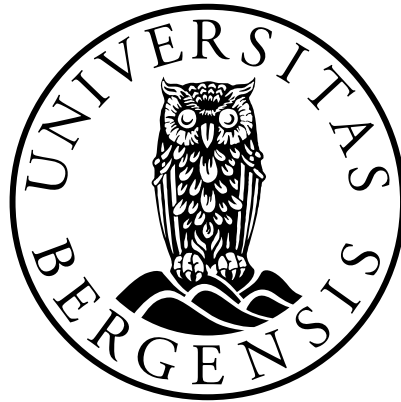


Modulus method and its application to the theory of univalent functions

Belyaeva Elena

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Introduction

Modulus is a characteristic of the family of curves sweeping out the domain on a Riemann surface. The modulus method, called also the method of extremal metric, is one of the powerful tools used in geometric function theory. It has a wide range of applications in different parts of the theory such as extremal problems for conformal and quasiconformal maps, boundary behavior of conformal maps, value distribution theory, potential theory, and many others. In particular, it helped to solve the Denjoy conjecture on integral functions (or Heins' hypothesis), to progress in the problem of Pólya and Szegő on isoperimetric inequalities and to prove the n -segment theorem.

Grötzsch [Grö28] was the first who used it in the form of the strip method in conformal mapping theory. Later, Ahlfors and Beurling [LA50] introduced the notion of extremal length (a reciprocal value of the modulus); this initiated an active development of the method. Now this method is one of the principal methods in the theory of quasiconformal mappings.

Quadratic differentials, another concept accompanying the modulus method, are defined on Riemann surfaces and serve as an important tool both in the theory of differential equations on such surfaces and as a topological characteristic of the surface itself. They naturally appeared in Goluzin and Schiffer's works on the variational method for univalent functions. Later there was shown that the quadratic differentials can be applied to a large variety of extremal problems in the theory of univalent functions.

Teichmüller [Tei40] revealed a close connection between the modulus and quadratic differentials associated with solutions of some extremal problems. This idea, being a heuristic principle, reflects a philosophical underlying basis to a certain class of problems in complex analysis.

In this thesis we consider some known definitions of the modulus and discuss their equivalence. We present an account of results on relations between the modulus method and the problem of the extremal partition of the Riemann surface. Our own results obtained in this thesis are as follows. We study the trajectory structure of some quadratic differential on a Riemann surface depending on a parameter. These trajectories form a dense structure on the Riemann surface except the countable number of cases. We also consider an extremal problem in the theory of univalent functions related to the Löwner lemma on distortion of arcs under conformal maps and solve it using the modulus method.

Chapter 1

Modulus of families of curves

Throughout this work we will use the following notations:

$\mathbb{C} = \{z = x + iy : |z| < \infty\}$ the finite complex plane,

$\mathbb{H}^+ = \{z = x + iy \in \mathbb{C} : y > 0\}$ the upper half-plane of the finite complex plane,

$\mathbb{H}^- = \{z = x + iy \in \mathbb{C} : y < 0\}$ the lower half-plane of the finite complex plane,

$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the compactified complex plane or the Riemann sphere,

$U(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$, $a \in \mathbb{C}$, $r > 0$ the disk of center a and radius r ,

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the unit disk,

$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the boundary of the unit disk,

$S(\phi) = \{z \in \mathbb{D} : 0 < \arg z < \phi\}$ the sector of the unit disk of angle ϕ ,

if $D \subset \hat{\mathbb{C}}$ is a domain, then ∂D is the usual boundary operator of D , \hat{D} is the closure of D ,

$M(\Gamma)$ the modulus of family of curves Γ ,

$M(D, a)$ the reduced modulus of the domain D with respect to the point $a \in D$,

$M(D, a, b)$ the reduced modulus of the digon (D, a, b) with respect to the points $a, b \in \partial D$,

$M_{\Delta}(D, a)$ the reduced modulus of the traingle (D, a) with respect to the point $a \in \partial D$.

1.1 Various definitions of the modulus

The method of extremal metrics is one of the fundamental concepts in geometric function theory closely connected with differential geometry and topology. It is a finest expression for so-called length-area principle where both the length of curves belonging to a specific homotopy classes and the area of a domains swept out by these curves are taken into account and result in one and the same quantity. The length of curves and the area are calculated with respect to a special choice of metric corresponding to an extremal problem one solves. A heuristic Teichmüller principle stands behind this choice by relating the metric and a quadratic differential on a Riemann surface.

The modulus method has various forms. The original one goes back to H.Grötzsch[Grö28] and is called the strip method. It is an essential refinement of the arguments that relate length and area. A strip method operates with the characteristic conformal invariants of a doubly-connected domains and quadrilaterals. This way Grötzsch became the first who

addressed the geometric definition of a quasiconformal mappings. According to this definition, the modulus of family of curves is a geometric quantity which is quasi-invariant under quasiconformal mappings, and therefore define the bounds on the maximal distortion for an appropriate choice of a curve family.

An important step in the development of a modulus method was proposed by L.V. Ahlfors and A. Beurling (1950) [LA50] who introduced the concept of the extremal length of a family of curves. The modulus appeared as a reciprocal quantity later and turned out to be technically more convenient in many problems.

We collect here several definitions of the modulus flashing specific advantages of one or another definition in terms of further applications.

First let us introduce some important terms.

Definition 1.1. *The curve in $\hat{\mathbb{C}}$ is a continuous mapping $\gamma : [a, b] \rightarrow \hat{\mathbb{C}}$. The supremum of sums*

$$\sum_{i=1}^k |\gamma(t_i) - \gamma(t_{i-1})|$$

over all subdivisions $a = t_0 \leq t_1 \leq \dots \leq t_k = b$ of $[a, b]$ is called the length of γ . If the length of γ is finite then γ is rectifiable, otherwise non-rectifiable.

Definition 1.2. *The Baire functions of class n on an interval $[a, b]$ for any ordinal number n are as follows.*

- *The Baire class 0 contains of continuous functions,*
- *The Baire class 1 contains of those functions which are the pointwise limit of a sequence of Baire class 0 functions, but are not of Baire class 0,*
- *In general, the Baire class n contains all functions which are the pointwise limit of a sequence of functions each of which is of Baire class less than n , but do not themselves appears in any lower-numbered class.*

Now we can list the definitions of the modulus along the historical line.

L. Ahlfors, A. Beurling *Conformal Invariants and Function-Theoretic Null-Sets*, Acta Math **83** (1950), 101-129.

Let Ω denote a domain in the extended z -plane, and let Γ denote a family of rectifiable curves in Ω . Consider the class of non-negative functions (weights) $\rho(z)$ in Ω for which the quantities

$$L_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \int_\gamma \rho |dz|,$$

$$A_\rho(\Omega) = \iint_\Omega \rho^2 dx dy, \quad z = x + iy,$$

are defined and not simultaneously 0 or ∞ . The least upper bound

$$\lambda(\Gamma) = \sup_{\rho} \frac{L_{\rho}(\Gamma)^2}{A_{\rho}(\Omega)}$$

with respect to this class is called the *extremal length* of the family Γ . The inverse of this value is called the modulus of the family Γ .

B. Fuglede, *Extremal length and functional completion*, Acta Math **98** (1957), 171-219.

Let Γ denote an arbitrary system of plane curves in \mathbb{R}^2 . Then the modulus $M(\Gamma)$ may be defined by

$$M(\Gamma) = \inf_{f \wedge \Gamma} \iint_{\mathbb{R}^2} f^2 dx dy,$$

where the symbol $f \wedge \Gamma$ (f is associated with the system Γ) means that f is a non-negative Baire function, defined in \mathbb{R}^2 , such that

$$\int_{\gamma} f ds \geq 1, \text{ where } s \text{ is a natural parameter for } \gamma,$$

for every curve $\gamma \in \Gamma$.

J. Jenkins, *Univalent functions and conformal mapping*, Springer, Berlin (1958), 13-14.

Let \mathfrak{R} be a Riemann surface¹. By a conformally invariant metric $\rho(z)|dz|$ defined on \mathfrak{R} we mean an entity which associates to every local uniformizing parameter z of \mathfrak{R} a real-valued non-negative measurable function $\rho(z)$ satisfying the following conditions

- (i) if γ is a rectifiable curve in the parametric neighborhood of z , then $\int_{\gamma} \rho(z)|dz|$ exists (as a Lebesgue-Stieltjes integral), possibly having the value $+\infty$;
- (ii) if $\rho^*(z^*)$ and $\rho(z)$ represent one and the same metric ρ in terms of local uniformizing parameters z^* and z respectively in the local neighbourhoods U_{z^*} and U_z of these parameters, then the relation

$$\rho^*(z^*) = \rho(z) \left| \frac{dz}{dz^*} \right|$$

holds in the points of intersection $U_{z^*} \cap U_z$.

¹Some of the basic facts about Riemann surfaces are presented in Section 2.1.

A curve on a Riemann surface \mathfrak{R} is called *locally rectifiable* if for every compact subcurve lying in the neighborhood of a local uniformizing parameter z on \mathfrak{R} , its image curve in the z -plane is rectifiable.

Let Γ be a family of locally rectifiable curves given on the Riemann surface \mathfrak{R} . We say that the *modulus problem* is defined for Γ if there is a non-void class P of conformally invariant metrics $\rho(z)|dz|$ on \mathfrak{R} , for which $\rho(z)$ is of integrable square so that

$$A_\rho(\mathfrak{R}) = \iint_{\mathfrak{R}} \rho^2 dx dy,$$

is defined and such that $A_\rho(\mathfrak{R})$ and

$$L_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \int_\gamma \rho |dz|,$$

are not simultaneously 0 and ∞ . Then we designate the quantity

$$M(\Gamma) = \inf_{\rho \in P} \frac{A_\rho(\mathfrak{R})}{L_\rho(\Gamma)^2}$$

to the modulus of Γ . This quantity may admit the values 0 and ∞ . The reciprocal of $M(\Gamma)$ is called the extremal length $\lambda(\Gamma)$ of the family Γ as in definition [LA50].

P. M. Tamrazov, *A theorem on line integrals for extremal length*, Dopovidi. Akad. Nauk Ukraïn. RSR, **1** (1966), 51-54

Let Ω be a domain in \mathbb{C} and let a function $\rho(z)$ be real-valued, measurable, almost everywhere non-negative, and from $L^2(\Omega)$. Let this function define a conformal invariant differential metric ρ on Ω by $\rho = \rho(z)|dz|$. If $\tilde{\Omega} = f(\Omega)$, where $w = f(z)$ is a conformal map, then $\rho(f^{-1}(w))|f'(w)|^{-1}|dw| \equiv \tilde{\rho}(w)|dw|$ is the metric $\tilde{\rho}(w)|dw|$ that represents ρ in $\tilde{\Omega}$. Thus, we construct a metric, which is defined on the complete collection of conformally equivalent domains. Let γ be a curve in Ω . The lower Darboux integral

$$\int_\gamma \rho(z)|dz| = l_\rho(\gamma)$$

is said to be the ρ -length of γ . If $\rho(z) \equiv 1$ almost everywhere in Ω , then the 1-length of any rectifiable $\gamma \subset \Omega$ coincides with its euclidean length. In the non-rectifiable case, nevertheless, the integral exists (in the sense of Darboux) and the definition is valid. The integral

$$\iint_{\Omega} \rho^2(z) dx dy = A_{\rho}(\Omega),$$

is called the ρ -area of Ω . Let Γ be a family of curves in Ω . Denote by

$$L_{\rho}(\Gamma) = \inf_{\gamma \in \Gamma} l_{\rho}(\gamma)$$

the ρ -length of the family Γ . Then, the quantity

$$M(\Gamma) = \inf_{\rho} \frac{A_{\rho}(\Omega)}{L_{\rho}^2(\Gamma)}$$

is said to be the *modulus* of the family Γ in Ω where the infimum is taken over all above defined metrics ρ in Ω .

If there exists a metric ρ^* such that the infimum in the definitions of the modulus is attained, then this metric is called *extremal*.

We see that in each of the above definitions the set of curves in the family Γ , admissible metrics and integrals are defined differently. We observe that all these definitions bring us to the same value of modulus in the case of rectifiable curves.

We start with Fuglede's definition. It is given in an essentially different form than the rest of definitions. This form is called *the L-definition of the modulus*, or *L-form*. Let us see that all another definitions can be given in this form, that is

$$M(\Gamma) = \inf_{\rho \wedge \Gamma} A_{\rho}(\Omega)$$

in Fuglede's notation.

Let us consider the definition of Ahlfors and Beurling.

Let ρ be an admissible metric. Consider

$$\tilde{\rho} = \frac{\rho}{L_{\rho}(\Gamma)}.$$

We have $\int_{\gamma} \tilde{\rho} |dz| \geq 1$, so $\tilde{\rho} \wedge \Gamma$. Further,

$$A_{\tilde{\rho}}(\Omega) = \frac{A_{\rho}(\Omega)}{L_{\rho}(\Gamma)^2},$$

which leads to the L-form.

The rest of definitions can be reduced to the L-form in a similar way.

Notice that if $L_{\rho}(\Gamma) = 0$ or $L_{\rho}(\Gamma) = \infty$ then $M(\Gamma) = \infty$ or $M(\Gamma) = 0$ correspondingly.

Let us show that

$$\inf_{\rho \wedge_{\Gamma} \Omega} \iint \rho^2 dx dy = \inf_{\rho \wedge_{\Gamma} \Omega} \iint \rho^2 dx dy$$

ρ is the measurable non-negative function ρ is the Baire non-negative function

It was proved in [Wei74][p. 175] that the Baire functions on \mathbb{R}^k coincide with the Borel measurable functions. For this reason they are all Lebesgue-measurable. According to [Fug57], the Lebesgue measurable function $\rho \geq 0$ may be replaced by a Baire function $\tilde{\rho} \geq \rho$ which equals ρ almost everywhere. We have shown that the definition of Fuglede coincides with all other definitions.

For definition of Tamrazov, observe that for locally rectifiable curves, the Lebesgue integral and the lower Darboux integral coincide. For non-rectifiable curves the Lebesgue integral is not defined, but the modulus of the family of non-rectifiable curves with respect to the lower Darboux integral is equal to 0 ([Väi71][p.20]).

Summing up the above considerations, one can see that all definitions of modulus are equivalent.

We say that the domain Ω and the family of curves Γ on this domain define the *configuration* (Ω, Γ) . In this work we always consider the domain Ω on a Riemann surface together with the family of curves Γ defined on it. When this family is tacitly understood, we just speak about the domain Ω meaning the configuration (Ω, Γ) .

1.2 Simple properties of the modulus

Let us state a simple but important properties of the modulus.

Theorem 1.3. (*conformal invariance*). *Let Γ be a family of curves in a domain $\Omega \in \hat{\mathbb{C}}$, and let $w = f(z)$ be a conformal map of Ω onto $\tilde{\Omega} \in \hat{\mathbb{C}}$. If $\tilde{\Gamma} = f(\Gamma)$, then*

$$M(\Gamma) = M(\tilde{\Gamma}).$$

In other words, the modulus is well-defined in the whole class of conformally invariant domains.

Theorem 1.4. (*uniqueness of the extremal metric*).

(i) *Let ρ_1 and ρ_2 be two extremal metrics for the modulus $M(\Gamma)$. Then, $\rho^* = \rho_1 = \rho_2$ almost everywhere.*

(ii) *Moreover, $L_{\rho^*}(\Gamma) = 1$.*

Theorem 1.5. (*monotonicity*). *If $\Gamma_1 \subset \Gamma_2$ in D , then $M(\Gamma_1) \leq M(\Gamma_2)$.*

The conformal invariance property enables modulus to be an essential tool in quasi-conformal theory, and monotonicity property enables it to be used in various extremal problems.

1.3 Examples (moduli of rectangle and annulus)

The essence of the modulus characteristic can be expressed as follows. Given the domain and the family of curves defined in this domain find an extremal metric for which the length of curves will be minimal, and then find the area of the domain with respect to this metric.

The following simple examples [Jen58] express this general principle.

Theorem 1.6. (*Modulus of a rectangle*)

Let D be the rectangle $\{z = x + iy : 0 < x < l, 0 < y < 1\}$, and let Γ be the family of curves in D that connect the opposite horizontal sides of D . Then, $M(\Gamma) = l$.

Here the extremal metric is $|dz|$, that is the euclidean one.

Theorem 1.7. (*Modulus of an annulus*)

- Let D be the annulus $\{z = re^{i\theta} : 1 < r < R, 0 < \theta < 2\pi\}$, and let Γ be the family of rectifiable curves in D that separate the opposite boundary components of D . Then, $M(\Gamma) = \frac{1}{2\pi} \log R$.
- Let D be the annulus $\{z = re^{i\theta} : 1 < r < R, 0 < \theta < 2\pi\}$, and let Γ^* be the family of rectifiable curves in D that connect the opposite boundary components of D . Then, $M(\Gamma^*) = \frac{2\pi}{\log R}$.

Here in the first case the extremal metric is $\frac{|dz|}{2\pi|z|}$, in the second case $\frac{|dz|}{|z| \log R}$.

The important property following from these examples, is that if Γ^* is the family of curves that connect the pair of complementary sides of a rectangle or Γ^* is defined as in the example of an annulus, then $M(\Gamma)M(\Gamma^*) = 1$.

1.4 Grötzsch lemmas

Theorem 1.8. (*1-st Grötzsch lemma*). Let D be the annulus $D = \{z : r < |z| < R\}$, and let D_1, \dots, D_n be non-overlapping doubly connected domains that separate the boundary components of D . If $M(D)$ stands for the modulus of D with respect to the family of curves that separate its boundary components, then

$$M(D) \geq \sum_{j=1}^n M(D_j).$$

The equality occurs only in the case when $\bigcup_{j=1}^n \hat{D}_j = D$ and ∂D_j are the concentric circles for any j .

Theorem 1.9. (*2-nd Grötzsch lemma*). Let D be a rectangle $\{z = x + iy : 0 < x < l, 0 < y < 1\}$ and D_1, \dots, D_n be non-overlapping quadrilaterals in D with horizontal opposite

sides on those of D . If $M(D)$ is the modulus of D with respect of the family of curves that connect its horizontal sides, then

$$M(D) \geq \sum_{j=1}^n M(D_j).$$

The equality occurs only when $\bigcup_{j=1}^n \hat{D}_j = D$ and when D_j are rectangles.

1.5 Reduced modulus

1.5.1 Circular domains

A simply connected hyperbolic domain $D \subset \hat{\mathbb{C}}$ with a marked point $a \in D$ is called *circular* with respect to the point a . We construct the doubly connected domain $D_\varepsilon = D \setminus U(a, \varepsilon)$ for sufficiently small ε . The quantity

$$M(D, a) = \lim_{\varepsilon \rightarrow 0} \left(M(\Gamma_\varepsilon) + \frac{1}{2\pi} \log \varepsilon \right)$$

is said to be the reduced modulus of the circular domain D with respect to the point a , where Γ_ε is the family of curves that separate the opposite boundary components of D_ε .

By the Riemann mapping theorem there is a unique conformal map $w = f(z)$ of D onto a disk $|w| < R < \infty$, such that $f(a) = 0$, $f'(a) = 1$. The number R is called the conformal radius of D with respect to the point a . We denote it by $R(D, a)$.

In the case $a = \infty$ the function f has the expansion about ∞ as $f(z) = z + a_0 + \frac{a_1}{z} + \dots$ and maps D onto the exterior part of the disk $|w| > R = R(D, \infty)$.

In order to calculate the reduced modulus of a circular domain one can use the following result:

Theorem 1.10. *Let a simply connected hyperbolic domain D have the conformal radius $R(D, a)$ with respect to a fixed point $a \in D$. Then, the quantity $M(D, a)$ exists, finite, and is equal to $\frac{1}{2\pi} \log R(D, a)$.*

Corollary 1.11. *Let D be a simply connected hyperbolic domain, $a \in D$, $|a| < \infty$. If $f(z)$ is a conformal map of D such that $|f(a)| < \infty$, then $M(f(D), f(a)) = M(D, a) + \frac{1}{2\pi} \log |f'(a)|$.*

1.5.2 Digons and triangles

Here we consider two types of domains called triangles and digons, and define reduced moduli for these domains. First, let us introduce some important notions.

The *Stolz angle* at a point $\zeta \in \mathbb{T}$ is the set

$$\Delta_\zeta(\theta, \eta) = \{z \in \mathbb{D} : |\arg(1 - \bar{\zeta}z)| < \theta, |z - \zeta| < \eta\},$$

where $\theta \in (0, \frac{\pi}{2}), \eta \in (0, 2 \cos \theta)$.

Let f be a function from \mathbb{D} to $\hat{\mathbb{C}}$. We say that f has the *angular limit* $a \in \bar{\mathbb{C}}$ at $\zeta \in \mathbb{T}$ if

$$f(z) \rightarrow a \quad \text{as} \quad z \rightarrow \zeta, \quad z \in \Delta_\zeta(\theta, \eta)$$

for each Stolz angle at the point ζ , and denote this limit as $f(\zeta)$.

We say that f has the *angular derivative* $f'(\zeta)$ at $\zeta \in \mathbb{T}$ if the finite angular limit $f(\zeta)$ exists and if

$$\lim_{z \rightarrow \zeta, z \in \Delta_\zeta(\theta, \eta)} \frac{f(z) - f(\zeta)}{z - \zeta} = f'(\zeta).$$

Further, let $D \in \hat{\mathbb{C}}$ be a hyperbolic simply connected domain, $\sigma \in \partial D$, $\zeta \in \mathbb{T}$ and f be a function from \mathbb{D} to D that have the angular limit $f(\zeta) = \sigma$. We define *the inner angle* of a domain D at the point σ as the number

$$\phi = 2 \sup_{f(\Delta_\zeta(\theta, \eta)) \subset D} \theta,$$

for some ν . If there exists a map f_D from the Stolz angle $\Delta_\zeta(\phi, \eta)$ to D that have an angular limit $f_D(\zeta) = \sigma$, and there exist a finite non-zero angular derivative $f'_D(\zeta)$, then the domain D is called *conformal* at the point σ .

Now let us turn to definitions.

- *The digon* (D, a, b) is the hyperbolic simply connected domain $D \subset \mathbb{C}$ with two fixed points a, b on its piecewise smooth boundary. Denote by D_ε the domain $D \setminus \{U(a, \varepsilon_1) \cup U(b, \varepsilon_2)\}$ for $\varepsilon_1, \varepsilon_2$ small enough such that arcs $\partial U(a, \varepsilon_1) \cap D$ and $\partial U(b, \varepsilon_2) \cap D$ are connected and disjoint. Denote by Γ_ε the family of curves joining these arcs in the domain D . If the limit

$$m(D, a, b) = \lim_{\varepsilon_1, \varepsilon_2 \rightarrow 0} \left(\frac{1}{M(\Gamma_\varepsilon)} + \frac{1}{\phi_a} \log \varepsilon_1 + \frac{1}{\phi_b} \log \varepsilon_2 \right)$$

exists, where ϕ_a and ϕ_b are the inner angles of digon (D, a, b) at a and b respectively, then it is called the reduced modulus of the digon (D, a, b) . If the domain D is conformal at the points a and b , then the limit $m(D, a, b)$ exists.

The following theorem describes how the reduced modulus of a digon changes under a conformal mapping.

Theorem 1.12. *Let the digon (D, a, b) be such that the limit $m(D, a, b)$ exists and the inner angles at the vertices a and b of $f(D, a, b)$ are equal to ϕ_a and ϕ_b correspondingly. Suppose that there is a conformal map $f(z)$ of the digon (D, a, b) (which is conformal at a, b) onto a digon $(f(D), f(a), f(b))$, so that there exist the angular limits $f(a), f(b)$ with the inner angles ψ_a and ψ_b at the vertex $f(a)$ and $f(b)$ correspondingly. If the function f has the finite non-zero angular derivatives $f'(a)$ and*

$f'(b)$, then $\phi_a = \psi_a$, $\phi_b = \psi_b$, and the reduced modulus of $(f(D), f(a), f(b))$ exists and changes according to the rule

$$m(f(D), f(a), f(b)) = m(D, a, b) + \frac{1}{\psi_a} \log |f'(a)| + \frac{1}{\psi_b} \log |f'(b)|.$$

- The *triangle* (D, a, z_1, z_2) is the hyperbolic simply connected domain in $D \subset \mathbb{C}$ with three finite fixed boundary points z_1, z_2 , and a on its piecewise smooth boundary. A point a is called a *vertex* of the triangle and an arc between z_1, z_2 which lies on the boundary of D and does not contain a is called a *leg* of the triangle. Denote by D_ε the domain $D \setminus U(a, \varepsilon)$ for ε so small that the arc $\partial U(a, \varepsilon) \cap D$ is connected. Let Γ_ε be the family of curves in D_ε joining the arc $\partial U(a, \varepsilon) \cap D$ with the leg of the triangle. If the limit

$$m_\Delta(D, a, z_1, z_2) = \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{M(\Gamma_\varepsilon)} + \frac{1}{\phi_a} \log \varepsilon \right)$$

exists, where ϕ_a is the inner angle at the point a , then it is called *the reduced modulus of the triangle* (D, a, z_1, z_2) . The sufficient condition for the existence of the above limit is that the domain D is conformal at the point a .

The following theorem describes how the reduced modulus of a triangle changes under a conformal mapping.

Theorem 1.13. *Let a triangle (D, a, z_1, z_2) be such that the limit $m_\Delta(D, a, z_1, z_2)$ exists and the inner angle is equal to ϕ_a . Suppose that there is a conformal map $f(z)$ of the triangle (D, a, z_1, z_2) (which is conformal at a) onto the triangle $f(D, a, z_1, z_2)$, such that there exists the angular limit $f(a)$ with the inner angle ψ_a at the vertex $f(a)$ of $f(D, a, z_1, z_2)$. If the function f has a finite non-zero angular derivative $f'(a)$, then $\phi_a = \psi_a$, and the reduced modulus of $f(D, a, z_1, z_2)$ exists and changes according to the rule*

$$m_\Delta(f(D, a, z_1, z_2)) = m_\Delta(D, a, z_1, z_2) + \frac{1}{\psi_a} \log |f'(a)|.$$

Moreover, if f has the expansion about a

$$f(z) = w_1 + (z - a)^{\frac{\psi_a}{\phi_a}} (c_1 + c_2(z - a) + \dots),$$

then the reduced modulus of $D(a, z_1, z_2)$ is changed by the rule

$$m_\Delta(f(D, a, z_1, z_2)) = M_\Delta(D, a, z_1, z_2) + \frac{1}{\psi_a} \log |c_1|.$$

The reduced modulus of a triangle possesses a monotonicity property which is expressed by the following lemma [Sol99]:

Lemma 1.14. *Assume we have two triangles (D^1, a, z_1^1, z_2^1) and (D^2, a, z_1^2, z_2^2) , having equal inner angles at the vertex a , and for each triangle the reduced modulus is defined. If $D_1 \subseteq D_2$, $z_1^1 z_2^1 \subseteq z_1^2 z_2^2$, and there is a strict inclusion at least in one case, then*

$$m_\Delta(D_1, a, z_1^1, z_2^1) > m_\Delta(D_2, a, z_1^2, z_2^2).$$

Chapter 2

Quadratic differentials

In this chapter we learn some basics of the theory of quadratic differentials on a Riemann surface. For the details about Riemann surfaces and quadratic differentials, see [Spr81] and [Jen58].

2.1 Definitions

Definition 2.1. *The Riemann surface $\mathfrak{R} = (\mathfrak{R}, \{U_\alpha, h_\alpha\}_{\alpha \in I})$ is a connected Hausdorff space \mathfrak{R} together with an open covering $\{U_\alpha\}_{\alpha \in I}$ and a system of homeomorphisms h_α of a sets U_α onto an open sets $V_\alpha = h_\alpha(U_\alpha)$ in the upper half-plane \mathbb{H} with conformal neighbor relations. The latter means, whenever $U_\alpha \cap U_\beta \neq \emptyset$, the composition $h_\alpha \circ h_\beta^{-1}$ is an orientation preserving conformal homeomorphism of the open set $h_\beta(U_\alpha \cap U_\beta)$ onto $h_\alpha(U_\alpha \cap U_\beta)$.*

If, in addition, there exists homeomorphisms h_α of the sets U_α from the open covering in $\hat{\mathbb{H}}$ with $h_\alpha(U_\alpha) \cap \mathbb{R} \neq \emptyset$, then there exists a boundary component of \mathfrak{R} , which is a set $\partial\mathfrak{R} = \{h_\alpha^{-1}(\mathbb{R}) : \alpha \in I\}$.

Let p be a point on \mathfrak{R} . For $p \in U_\sigma$ the function h_σ is called a local parameter about p ; if $p \in \partial\mathfrak{R}$ then the local parameter about p is called boundary..

Every Riemann surface \mathfrak{R} , whether bordered or not, has *the mirror image* (symmetric image). It is defined as a surface

$$\mathfrak{R}^* = (\mathfrak{R}, \{U_\alpha, \bar{h}_\alpha\}),$$

where $\bar{h}_\alpha = s \circ h_\alpha$ is a superposition of the homeomorphism h_α and the reflection s of the complex plane on the real axis. \mathfrak{R}^* is a Riemann surface, because the composition $\bar{h}_\alpha \circ \bar{h}_\beta^{-1}$ is an orientation preserving conformal homeomorphism whenever it has a meaning.

The mirror image \mathfrak{R}^* of a bordered surface can be glued along $\partial\mathfrak{R}$ to the original surface \mathfrak{R} to form a new surface $\tilde{\mathfrak{R}}$, which is called *the double* of \mathfrak{R} .

Remark 2.2. One can always construct a metric on a Riemann surface. Then, we can speak about the distance between points with respect to this metric.

Let us consider the fundamental group $\pi(\mathfrak{R})$ of a loops on \mathfrak{R} . For any subgroup F of this group one can construct a new Riemann surface $\tilde{\mathfrak{R}}$ which is non-branched, unbounded over \mathfrak{R} , and $\pi(\tilde{\mathfrak{R}}) = F$. This surface is called *the covering surface* of \mathfrak{R} .

For the trivial subgroup $F = [1]$ of $\pi(\mathfrak{R})$ consisting of all curves contractible to a point, the covering surface $\tilde{\mathfrak{R}}$ is said to be *the universal cover* of \mathfrak{R} . The surface $\tilde{\mathfrak{R}}$ is simply connected and, by the uniformization theorem of Koebe and Poincare, there is a conformal homeomorphism h of $\tilde{\mathfrak{R}}$ onto either \mathbb{D}, \mathbb{C} , or else $\hat{\mathbb{C}}$, which are called *canonical domains*. The universal cover is to be identified with its canonical conformal image under h . In general, every Riemann surface admits one of the canonical domains as its universal covering surface.

Now we introduce the concept of quadratic differential.

Definition 2.3. *A meromorphic quadratic differential Q defined on a Riemann surface $\mathfrak{R} = (\mathfrak{R}, \{U_\alpha, h_\alpha\}_{\alpha \in I})$ is a set of meromorphic function elements $\{Q_\alpha\}_{\alpha \in I}$; every Q_α is defined in $h_\alpha(U_\alpha)$, and the following identity holds*

$$Q_\alpha(h_\alpha \circ h_\beta^{-1}(z_\beta)) \left(\frac{dz_\alpha}{dz_\beta} \right)^2 = Q_\beta(z_\beta)$$

whenever $U_\beta \cap U_\alpha \neq \emptyset$. The function element Q_β is called a representation of Q in terms of the local coordinate z_β .

Remark. When we speak about quadratic differential on a Riemann surface in general, we write Q , but when we consider it with respect to a local parameter z , we write $Q(z)dz^2$.

Let Q be a meromorphic quadratic differential on a Riemann surface \mathfrak{R} , and let p be a point in \mathfrak{R} . Assume that for some $p \in U_\alpha$ we have $Q_\alpha(h_\alpha(p)) = 0$. Then it is true in every local parameter about p . Indeed, for every U_β in a conformal structure of \mathfrak{R} with $p \in U_\beta$ we have $Q_\beta(h_\beta(p)) = Q_\alpha(h_\alpha \circ h_\beta^{-1}(h_\beta(p))) \left(\frac{dz_\alpha}{dz_\beta} \right)^2$, hence $Q_\beta(h_\beta(p)) = 0$. Therefore, it make sense to speak of zero of the quadratic differential Q . In a similar way one can prove the same for the poles of a quadratic differential.

Definition 2.4. *The critical points of a meromorphic quadratic differential Q are its zeros and poles. All other points of \mathfrak{R} are called regular points of Q . A holomorphic point is either a regular point or a zero. Poles of the first order and zeros will be called finite type critical points, poles of order greater or equal than two will be called infinite type critical points. Denote the set of finite type critical points as C , and the set of infinite type critical points as H .*

It follows from the above definition that the property of being regular points does not depend on the local parameter.

Let p be a regular point, and let z be a local parameter in a neighborhood of p mapping it to the origin. Since $Q(0) \neq 0$, there is a simply connected domain containing the origin in which the two branches of $z \rightarrow \sqrt{Q(z)}$ are single-valued. For a fixed branch of \sqrt{Q} , an integral function

$$N(z) = \int \sqrt{Q(z)} dz$$

is then also single-valued in the neighborhood about the origin and is uniquely determined up to an additive constant. From the invariance of $\sqrt{Q(z)}dz$ under change of the parameter it follows that every N is a well-defined function on \mathfrak{R} about p .

Since $N'(0) = \sqrt{Q(0)} \neq 0$, we conclude that there is a disk about the origin which is mapped injectively to the complex plane under $z \rightarrow N(z)$. It follows that $w = N(z)$ is a local parameter about p . Then

$$dw^2 = Q(z)dz^2$$

and we see that the representation of Q is equal to 1 with respect to w . We say that w is *the natural parameter*.

Let us consider the conformally invariant metric $\sqrt{|Q(z)|}|dz|$ associated with the quadratic differential Q . The element of length in this metric is $|dw| = \sqrt{|Q(z)|}|dz|$. The maximal regular curve on \mathfrak{R} on which the inequality $Q(z)dz^2 > 0$ holds is called *the trajectory* (horizontal arc) of Q . The *orthogonal trajectory* (vertical arc) is the one that satisfies the reverse inequality $Q(z)dz^2 < 0$. Trajectories that have a finite critical points in their closure are called *critical*, other trajectories are *regular*. All trajectories have the property of being geodesics in Q -metric (see [Str67], p. 24). Thus, the trajectory structure of quadratic differential does not depend on its representation in a local parameter.

2.2 Local trajectory structure

In this section we consider the behavior of the trajectories in the neighborhood of points of the following type : regular, zeros and poles.

Let p be a regular point for the quadratic differential Q , and let z be a local parameter about this point. In terms of the natural parameter we have $dw^2 = Q(z)dz^2 > 0$ on the trajectories in a neighborhood about p . Since $dw^2 = du^2 - dv^2 + 2idudv$, on the trajectories we have $v = \text{const}$, therefore the images of trajectories about p are horizontal lines in terms of the natural parameter.

Let $p \in C$ be a critical point of order $n, n \geq -1$. Take a local parameter z about p such that $z(p) = 0$. In a sufficiently small neighbourhood of p one can select a single valued branch of the square root of Q and represent

$$\sqrt{Q(z)} = z^{n/2}(a_0 + a_1z + \dots) \quad (2.1)$$

locally outside $z = 0$.

Mapping via the natural parameter $w = z^{\frac{n+2}{2}}(b_0 + b_1z + \dots)$ transforms the local neighbourhood of the parameter z onto a branched element over w -plane. Introduce a new parameter $\zeta : \zeta^{\frac{n+2}{2}} = w$. We obtain a conformal map $z \rightarrow \zeta$. This map satisfies the equation

$$Q(z)dz^2 = \left(\frac{n+2}{2}\right)^2 \zeta^n d\zeta^2.$$

The trajectory structure of $Q(z)dz^2$ in a neighbourhood of zero is the same as for the differential $\zeta^n d\zeta^2$ up to shift, rotation, or the factor $e^{\frac{2k\pi}{n+2}}, k = 0, \dots, n+1$. One can easily

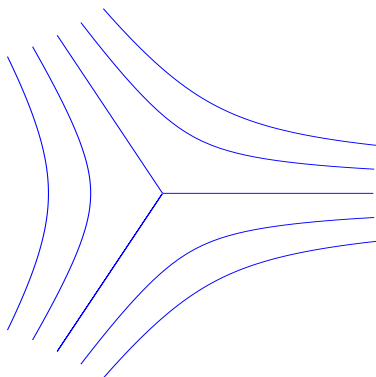


Figure 2.1: Simple zero

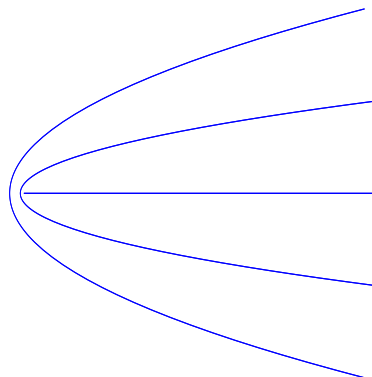
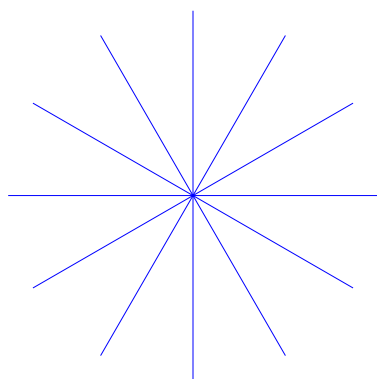
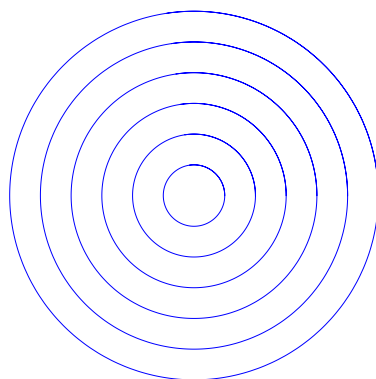
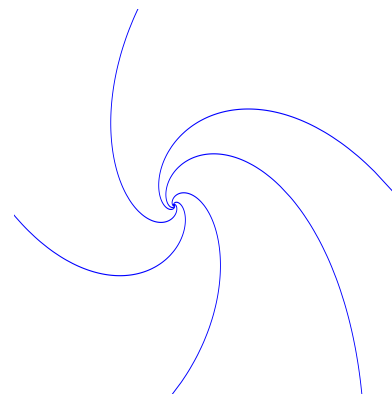


Figure 2.2: Simple pole

Figure 2.3: $a_{-2} > 0$ Figure 2.4: $a_{-2} < 0$ Figure 2.5: $\text{Im } a_{-2} \neq 0$

find the equation for the critical trajectories of this differential and get $\zeta = \rho e^{i\frac{2\pi k}{n+2}}$, $k = 0, \dots, n+1$, $\rho > 0$. Thus, the critical trajectories passing through p are mapped onto $n+2$ rays, emanating from the point 0 with angle $\frac{2\pi}{n+2}$ between neighboring rays. The local trajectory structure about simple zero and simple pole is shown on Figure 2.1 and 2.2 correspondingly.

Let $p \in H$ be a pole of order 2. In this case function (2.1) after integration gives the logarithmic singularity, and in terms of the natural parameter we will have $w = \log z(b_0 + b_1 z + \dots)$. Further exponentiating leads to the conformal map $z \rightarrow \zeta$, that satisfies the equation

$$Q(z)dz^2 = \frac{a_{-2}}{z^2}dz^2.$$

According to the coefficient a_{-2} , the trajectories have the radial form ($a_{-2} > 0$), the circular form ($a_{-2} < 0$), or the spiral form ($\text{Im } a_{-2} \neq 0$) (see Figures 2.3, 2.4 and 2.5).

Let $p \in H$ be a pole of order n greater than 2. As in the previous case, function (2.1) after integration also produces a logarithmic singularity of the form $a \log z$, where $a = a_{|\frac{n}{2}-1}$. This leads to the following representation of quadratic differential:

$$Q(z)dz^2 = \left(\frac{n+2}{2}\zeta^{\frac{n}{2}} + \frac{a}{\zeta}\right)^2 d\zeta^2. \quad (2.2)$$

In case of a pole of odd order $n \geq 3$ or even order with vanishing logarithmic term the quadratic differential has the representation

$$Q(z)dz^2 = \left(\frac{n+2}{2}\right)^2 \zeta^n d\zeta^2. \quad (2.3)$$

In the monograph of Strebel ([Str67], pp.33-34) there was proved that the local trajectory structure in the cases (2.2) and (2.3) is approximately the same. Namely, the critical trajectories in ζ -plane are $|n| - 2$ rays converging to the point 0 with angle $\frac{2\pi}{|n|-2}$ between neighboring rays. As an example, the local trajectory structure about a pole of order 5 is shown on Figure 2.6.

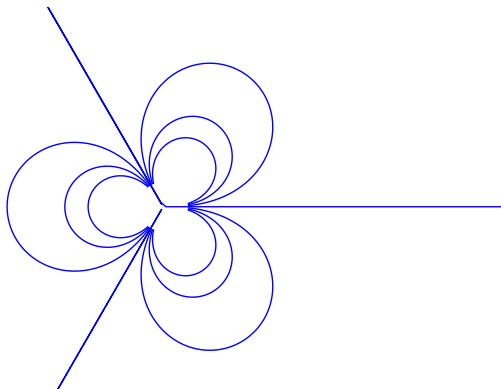


Figure 2.6: Pole of order 5

The local behavior of the trajectories of a quadratic differential was described without proof by Teichmüller at the first time. Later, Shaeffer and Spencer gave the first detailed justification under the stated but inessential condition that the quadratic differential is hyperelliptic. Then, Jenkins [Jen58] simplified this proof and distributed it for the quadratic differential without additional conditions.

2.3 Global trajectory structure

In this section we consider the trajectory structure in large. Following [Jen58], let us confine our attention to a positive quadratic differentials on a finite oriented Riemann surface.

Definition 2.5. *By a positive quadratic differential on a finite oriented Riemann surface \mathfrak{R} we mean a quadratic differential $Q(z)dz^2$ which satisfies the following condition. In terms of a boundary local parameter z , the function $Q(z)$ is regular out of possible simple poles and real-valued on the segment of the real axis corresponding to the boundary points of \mathfrak{R} .*

By the Schwarz's reflection principle, any positive quadratic differential defined on a Riemann surface \mathfrak{R} can be extended onto a mirror image of \mathfrak{R} . Therefore, we agree to regard only positive quadratic differentials.

We denote by \mathfrak{R} a finite oriented Riemann surface, and by $Q(z)dz^2$ a quadratic differential on \mathfrak{R} in the following definitions.

Definition 2.6. *A set K on \mathfrak{R} is called an F -set if any trajectory of $Q(z)dz^2$, which meets K , lies entirely in K .*

Definition 2.7. *An ending domain \mathfrak{E} (relative to $Q(z)dz^2$) is the maximal connected open F -set on \mathfrak{R} with the properties*

- (i) \mathfrak{E} contains no critical point of $Q(z)dz^2$,
- (ii) \mathfrak{E} is swept out by the trajectories of $Q(z)dz^2$, each of which has a limiting end-point in both directions at a given point p in H ,
- (iii) \mathfrak{E} is mapped conformally by $\zeta = \int Q(z)^{\frac{1}{2}}dz$ onto an upper or lower half-plane in the ζ -plane (depending on the direction of integration).

It is seen that p must be a pole of $Q(z)dz^2$ of order at least three.

Definition 2.8. *A strip domain \mathfrak{S} (relative to $Q(z)dz^2$) is the maximal connected open F -set on \mathfrak{R} with the properties*

- (i) \mathfrak{S} contains no critical point of $Q(z)dz^2$,
- (ii) \mathfrak{S} is swept out by the trajectories of $Q(z)dz^2$, each of which has at one direction a limiting end-point $p \in H$, and at another direction a limiting end-point $q \in H$, p and q may coincide,
- (iii) \mathfrak{S} is mapped by $\zeta = \int Q(z)^{\frac{1}{2}}dz$ conformally onto a strip $a < \text{Im } \zeta < b$ (a, b are finite real numbers, $a < b$).

Definition 2.9. *A circle domain \mathfrak{C} (relative to $Q(z)dz^2$) is a maximal connected open F -set on \mathfrak{R} with the properties*

- (i) \mathfrak{C} contains a single double pole p of $Q(z)dz^2$,
- (ii) $\mathfrak{C} - p$ is swept out by the trajectories of $Q(z)dz^2$, each of which is a Jordan curve separating A from the boundary of \mathfrak{C} ,
- (iii) for a suitably chosen purely imaginary constant c the function

$$w = e^{c \int (Q(z))^{\frac{1}{2}} dz}$$

having the value zero at p , maps \mathfrak{C} conformally onto a circle $|w| < R$.

Definition 2.10. A ring domain \mathfrak{D} (relative to $Q(z)dz^2$) is the maximal connected open F -set on \mathfrak{R} with the properties

- (i) \mathfrak{D} contains no critical point of $Q(z)dz^2$,
- (ii) \mathfrak{D} is swept out by the trajectories of $Q(z)dz^2$, each of which is a closed Jordan curve,
- (iii) for a suitably chosen purely imaginary constant c , the function

$$w = e^{c \int (Q(z))^{1/2} dz}$$

maps \mathfrak{D} conformally onto a circular ring

$$r_1 < |w| < r_2, \quad 0 < r_1 < r_2.$$

Definition 2.11. A dense domain \mathfrak{F} (relative to $Q(z)dz^2$) is the maximal connected F -set on \mathfrak{R} with the properties

- (i) \mathfrak{F} does not contain any points of H ,
- (ii) $\mathfrak{F} \setminus C$ is filled with the trajectories of $Q(z)dz^2$, each of which is everywhere dense in \mathfrak{F} .

Now we are ready to state the principal result ([Jen58], p. 37) concerning the global trajectory structure of a quadratic differential.

Theorem 2.12. (Basic Structure Theorem). Let \mathfrak{R} be a finite oriented Riemann surface and let $Q(z)dz^2$ be a positive quadratic differential on \mathfrak{R} where we exclude the following possibilities and all configurations obtained from them by conformal equivalence:

- I. \mathfrak{R} is the z -sphere, $Q(z)dz^2 = dz^2$,
- II. \mathfrak{R} is the z -sphere, $Q(z)dz^2 = \frac{Ke^{i\alpha}}{z^2} dz^2$, where α real and K positive,
- III. \mathfrak{R} is a torus, $Q(z)dz^2$ is regular on \mathfrak{R} .

Let Φ_Q denote the union of all trajectories which have a limiting end-point at a point of \mathbb{C} . Then

- (i) $\mathfrak{R} \setminus \Phi_Q$ consists of a finite number of ending, strip, circle, ring and dense domains,
- (ii) each such domain is bounded by a finite number of trajectories together with the points at which the above trajectories meet; every boundary component of such a domain contains a point of C except that a boundary component of a circle or ring domain may coincide with a boundary component of \mathfrak{R} ; for a strip domain the two boundary elements arising from points of H divide the boundary into two parts on each of which is a point of C ,

- (iii) every pole of $Q(z)dz^2$ of order n greater than two has a neighborhood covered by the inner closure of $n - 2$ end domains and a finite number (possibly zero) of strip domains,
- (iv) every pole of $Q(z)dz^2$ of order two has a neighborhood covered by the inner closure of a finite number of strip domains or has a neighborhood contained in a circle domain,
- (v) the interior of a closure \hat{Q} of Q is an F -set consisting of a finite number of domains on \mathfrak{R} each with a finite number (possibly zero) of boundary components,
- (vi) each boundary component of such a domain is a piecewise analytic curve composed of trajectories and their limiting end-points in C .

Remark. Boundaries of domains of every type for a given quadratic differential on a Riemann surface are defined in the unique way. For example, if a quadratic differential has only closed trajectories, and thereafter, only circle and ring domains, then the boundaries of these domains are defined in the unique way.

2.4 Dense structures depending on a parameter

Consider the quadratic differential

$$\phi(z)dz^2 = \frac{dz^2}{(z - z_1)(z - z_2)(z - z_3)} \quad (2.4)$$

on \hat{C} , where $z_1, z_2, z_3 \in \mathbb{R}$. These points and ∞ are the first order poles of $\phi(z)$. It is readily seen that on a set $(z_1, z_2) \cup (z_3, \infty)$ the quadratic differential is positive. Therefore, the intervals (z_1, z_2) and (z_3, ∞) are a critical trajectories of ϕ . The trajectory structure of ϕ in z -plane is pictured on Figure 2.7.

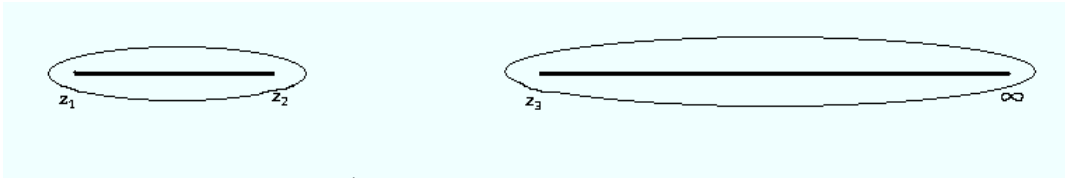


Figure 2.7: The trajectory structure of the differential ϕ on the z -plane. Critical trajectories are in bold

It is obvious that there is one ring domain on \hat{C} .

Let us map the points (z_1, z_2, z_3, ∞) to $(-\frac{1}{k}, -1, 1, \frac{1}{k})$ by a Möbius automorphism

$$\zeta(z) = \frac{z(1 + 2\lambda - k) - z_1(1 - k) - 2\lambda z_3}{z(1 - 2k\lambda - k) + 2k\lambda z_3 - (1 - k)z_1},$$

where $\lambda = \frac{z_1 - z_2}{z_2 - z_3}$, $\frac{1}{k} = 1 + 2\lambda + 2\sqrt{\lambda^2 + \lambda}$.

In the parameter ζ differential (2.4) takes the form:

$$\phi(\zeta)d\zeta^2 = \frac{\sigma^2 d\zeta^2}{(\zeta^2 - (\frac{1}{k})^2)(\zeta^2 - 1)},$$

where $\sigma = \sqrt{\frac{2\lambda(1-(\frac{1}{k})^2)}{z_1 - z_2}}$.

In what follows we need some facts from elliptic function theory. Let us denote a complete and associated complete elliptic integrals of the first kind by $\mathbf{K} = \mathbf{K}(k)$ and $\mathbf{K}' = \mathbf{K}(k')$ respectively [Akh70].

The mapping $w(\zeta) = \sigma \int_0^\zeta \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}$ is obviously a natural parameter for differential (2.4). The function $w(\zeta)$ is an elliptic integral which maps the quadrilateral $\mathbb{H}^+(-\frac{1}{k}, -1, 1, \frac{1}{k})$ onto the rectangle $Q(-\sigma\mathbf{K} + i\sigma\mathbf{K}', -\sigma\mathbf{K}, \sigma\mathbf{K}, \sigma\mathbf{K} + i\sigma\mathbf{K}')$. According to reflection principle, a function w admits an analytic continuation through any side λ of the quadrilateral \mathbb{H}^+ to the lower half-plane, and moreover, this analytic continuation is a conformal mapping of the lower half-plane on the rectangle which is symmetric to Q with respect to the image of λ under w . Thus, the global analytic function W obtained by various analytic continuations of w , maps z -plane onto a lattice of rectangles $\Theta(\sigma\mathbf{K}', \sigma\mathbf{K})$ in the W -plane where each rectangle of the lattice is an image of a half-plane (upper or lower). An edges of the lattice $\Theta(\sigma\mathbf{K}', \sigma\mathbf{K})$ are an images of the real axis in ζ -plane. Notice that the inverse function w^{-1} maps the rectangle with a sides of horizontal and vertical length $2\sigma\mathbf{K}$ and $2\sigma\mathbf{K}'$ respectively onto the whole ζ -plane.

It can easily be checked that the images of points z_1, z_2, z_3 and ∞ under the mapping $w \circ \zeta$ are the vertex of a rectangle Q . Therefore a nodes of the lattice Θ are the first order poles of the differential ϕ in w -plane. Hence, a critical trajectories of ϕ in the w -plane will go through a vertex of the lattice $\Theta(\sigma\mathbf{K}', \sigma\mathbf{K})$ parallel to the real axis. The trajectory structure of differential (2.4) for the natural parameter w is drawing in Figure 2.8.

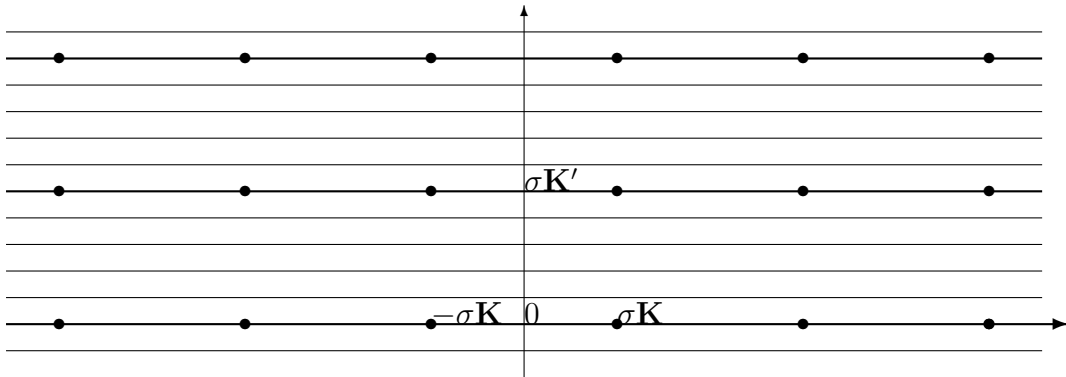


Figure 2.8: The trajectory structure for ϕ for natural parameter w

Now consider the quadratic differential

$$\psi(z)dz^2 = \frac{e^{i\alpha}dz^2}{(z-z_1)(z-z_2)(z-z_3)}, \quad \alpha \neq 0 \quad (2.5)$$

Let us see how the trajectory structure of ψ looks like.

In terms of a parameter $w(= u + iv)$ defined above this differential is represented as $e^{i\alpha}dw^2$. It follows that in w -plane the trajectories of ψ are the lines

$$\begin{cases} v = \tan \frac{\alpha}{2}u + C & \text{for } 0 < \alpha < \pi, \\ u = C & \text{for } \alpha = \pi, \\ v = -\cot \frac{\alpha}{2}u + C & \text{for } \pi < \alpha < 2\pi, \end{cases} \quad (2.6)$$

where $C = \text{const}$.

Let us consider the trajectory structure of differential (2.5) in w - and ζ -plane in dependence of an angle α .

From equations (2.6) one can see that after rotating all the trajectories of the differential (2.4) in w -plane by the angle $\frac{\alpha}{2}$, we obtain the trajectory structure of differential (2.5) in this parameter. Notice that the property of being a critical trajectory under this operation in general is not conserved. That is, those trajectories of differential (2.4) which after rotating cross at most one node of the lattice $\Theta(\sigma\mathbf{K}, \sigma\mathbf{K}')$ became the critical trajectories for differential (2.5).

It is easily shown that if we have

$$\tan \frac{\alpha}{2} = \frac{p\mathbf{K}'}{q\mathbf{K}} \quad \text{for some } p, q \in \mathbb{Z}, \quad (2.7)$$

then the critical trajectories of differential (2.5) in w -plane cross the nodes of a lattice $\Theta(p\sigma\mathbf{K}', q\sigma\mathbf{K}) \subset \Theta(\sigma\mathbf{K}', \sigma\mathbf{K})$. Then it can easily be checked that the trajectory structure of (2.5) is periodic on a rectangles with the sides of horizontal and vertical length $2q\sigma\mathbf{K}$ and $2p\sigma\mathbf{K}'$ respectively. Notice that the inverse function w^{-1} maps each of these rectangles onto the whole ζ -plane.

Hence, in this case each of two critical trajectories of differential (2.5) in ζ -plane have two end-points from the set $(-\frac{1}{k}, -1, 1, \frac{1}{k})$ and cross the upper and lower half-plane of ζ finitely many times.

If an angle α is such that condition (2.7) is not satisfied, then the critical trajectories in w -plane will never cross the nodes of a lattice $\Theta(\sigma\mathbf{K}', \sigma\mathbf{K})$. Hence the trajectory structure of differential (2.5) in w -plane will not be periodic, and in ζ -plane the critical trajectories cross an upper and lower half-plane infinitely many times. In this case we will have a *dense trajectory structure* on ζ -plane.

To conclude with, we have the global trajectory structure for differential (2.5) consisting of one ring domain for countably many α satisfying condition (2.7).

On Figure 2.9 are presented the critical trajectory structures for different angles α in ζ -plane.

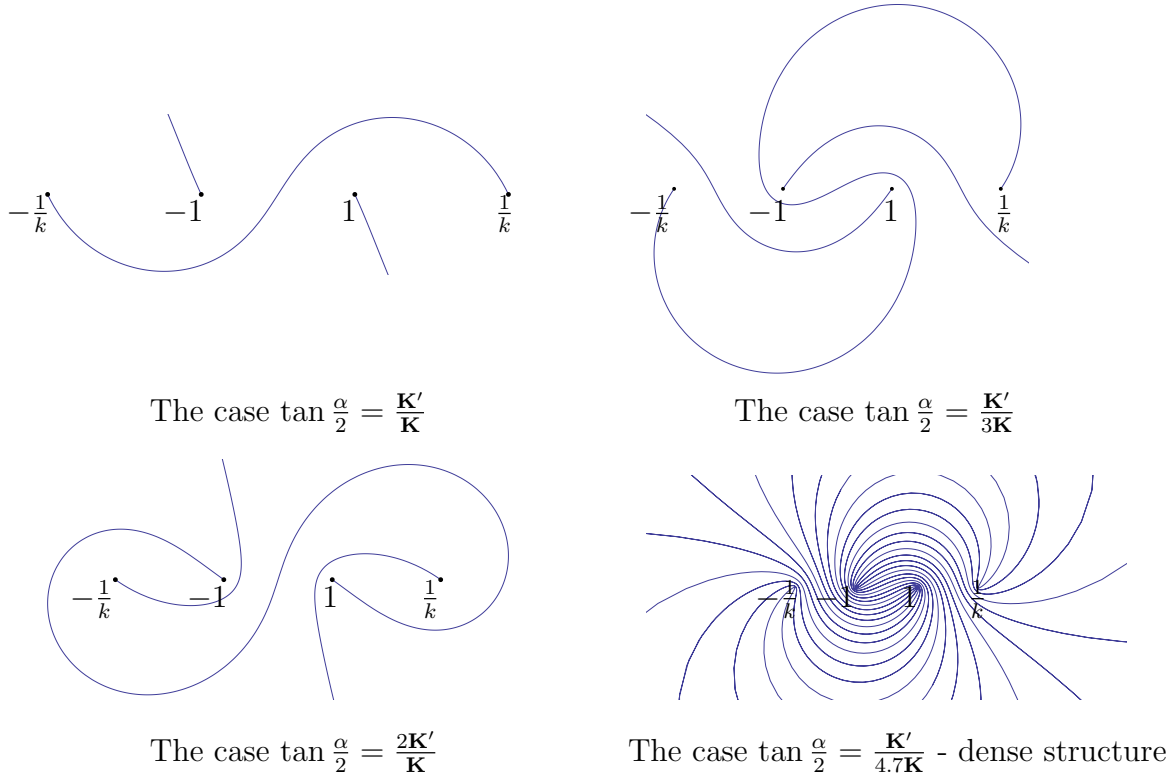


Figure 2.9: The trajectory structure for various angles α in ζ -plane

Let us find the analytic expression for a critical trajectories of differential (2.5) in ζ -plane. Each of the following equations

$$\int_0^{\zeta(t)} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \begin{cases} t + i(\tan \frac{\alpha}{2}t + n\sigma\mathbf{K}') & \text{for } 0 < \alpha < \pi, \\ it + n\sigma\mathbf{K} & \text{for } \alpha = \pi, \\ t + i(-\cot \frac{\alpha}{2}t + n\sigma\mathbf{K}') & \text{for } \pi < \alpha < 2\pi, \end{cases} \quad (2.8)$$

where $n \in \mathbb{Z}$, defines the critical trajectories $\zeta(t)$ of differential (2.5). We get one trajectory for odd n and another for even n correspondingly.

Equations (2.8) are equivalent to

$$\zeta(t) = \begin{cases} sn(t + i(\tan \frac{\alpha}{2}t + n\sigma\mathbf{K}')) & \text{for } 0 < \alpha < \pi, \\ sn(it + n\sigma\mathbf{K}) & \text{for } \alpha = \pi, \\ sn(t + i(-\cot \frac{\alpha}{2}t + n\sigma\mathbf{K}')) & \text{for } \pi < \alpha < 2\pi, \end{cases}$$

where $sn(x) = sn(x, k)$ is the Jacobi's sine elliptic function [Akh70].

Chapter 3

Moduli and extremal partition

3.1 Introduction

In this chapter we introduce a generalized modulus problem for the Riemann surface \mathfrak{R} and show its connection to some particular quadratic differentials defined on \mathfrak{R} . This connection is one of the examples for Teichmüller's heuristic principle which states that the solution of a certain type of extremal problems in geometric function theory is, in general, associated with quadratic differentials.

In our case the functional treated in the problem is a weighted sum of moduli or reduced moduli of non-overlapping domains. These domains form a partition on \mathfrak{R} which is defined by a set of homotopy classes of curves given on \mathfrak{R} .

Problems of such type was first considered by Jenkins [Jen57]. He considered the finite type Riemann surfaces without boundary and the ring type domains. Later, Kuzmina and Emel'yanov ([Kuz86], [Eme86]) expanded this problem. Namely, they considered surfaces with boundary, added domains of other types and defined their reduced moduli. In this chapter we give the most general formulation of the problem following [Sol99].

3.2 General conditions

Let us describe a general conditions of the problems which we will work with.

Let three system of points: $A = \{a_i\}_{i=1}^L$, $B = \{b_i\}_{i=1}^M$, $C = \{c_i\}_{i=1}^N$ be marked on a Riemann surface \mathfrak{R} and let other three systems of points $A' = \{a'_i\}_{i=1}^{L'}$, $B' = \{b'_i\}_{i=1}^{M'}$, $C' = \{c'_i\}_{i=1}^{N'}$ be marked on the (possible) boundary of \mathfrak{R} (any of the systems above may be empty).

By \mathfrak{R}' , $\hat{\mathfrak{R}}$ and $(\partial\mathfrak{R})'$ we denote punctured in the above point systems the Riemann surface \mathfrak{R} , its closure $\hat{\mathfrak{R}}$ and its boundary $\partial\mathfrak{R}$ correspondingly. Let $(\mathfrak{R}, \varepsilon)$ denote a surface without closed neighbourhoods of points from $B \cup C \cup B' \cup C'$ with radius ε .

Let us consider a homotopy classes on \mathfrak{R}' of the following types:

- (1) Jordan curves which are not null-homotopic,

- (2) Jordan curves which are homotopic to some point of the set B ,
- (3) Jordan arcs having end-points from the set $C \cup C'$ which are not homotopic to 0,
- (4) Jordan arcs having end-points on $(\partial\mathfrak{R})'$ which are not homotopic to 0,
- (5) Jordan arcs having end-points on $(\partial\mathfrak{R})'$ which are homotopic to some point from the set B' ,
- (6) Jordan arcs having one end-point on the set $C \cup C'$ and another end-point on $(\partial\mathfrak{R})'$ which are not homotopic to 0.

Choose systems A, A', B, B', C, C' such that there exists at least one non-empty homotopy class of some type from the above list.

Definition 3.1. *A free family of homotopy classes on \mathfrak{R}' is a finite set of homotopy classes $\mathcal{H} = \{H_i^j, j = 1, \dots, n; 1 \leq i \leq 6\}$, where H_i^j is a j -th homotopy class of type i , including all possible homotopy classes of second and fifth type for all points from the set $B \cup B'$, and at most one homotopy class of third or sixth type for every point from the set $C \cup C'$, with the following conditions:*

- (i) all classes H_i^j are distinct;
- (ii) there exists curves γ_m^j in H_i^j , such that $\gamma_m^j \cap \gamma_p^q = \emptyset$, if $(m, p) \neq (n, q)$.

In what follows we construct a partition of \mathfrak{R} associated with the free family of homotopy classes $\{H_i^j\}$ in a sense that we describe below.

First let us set the relation between a domains and a homotopy classes.

The configuration (D_i^j, Γ) is associated to the homotopy class H_i^j (of one of the above six types) if $\Gamma \subset H_i^j$. In this case Γ is defined uniquely by the domain D_i^j and the homotopy class H_i^j . We say that the domain D_i^j is associated to the homotopy class H_i^j .

Definition 3.2. *A system of domains $\mathcal{D} = \{D_i^j, j = 1, \dots, n, 1 \leq i \leq 6\}$ is said to be associated with the free family of homotopy classes \mathcal{H} if each domain D_i^j is associated to the homotopy class H_i^j .*

We include in consideration the cases where some of the domains D_1^j and D_4^j can be degenerate.

To each domain D_i^j from \mathcal{D} we assign a non-negative number α^j , which is called *the weight* of this domain.

Denote by $I(c)$ the set of indexes j such that the curves in a homotopy class H_3^j or H_6^j have one end-point at $c \in C \cup C'$. If the curves of a homotopy class H_3^j have both end-points at c , then the index j occurs in a set $I(c)$ twice. Then, define numbers

$$\beta_c = \sum_{j \in I(c)} h^j.$$

For $j \in I(c)$ denote by ϕ_c^j the inner angle (for the definition of inner angle see Section 1.5.2) of a domain D_i^j at the boundary point c .

Definition 3.3. A system of domains \mathcal{D} satisfies the weight-angle conditions if for every digon D_3^j and triangle D_6^j the following identity holds:

$$\phi_c^j \beta_c = 2\pi \alpha^j,$$

where α^j is the weight of a corresponding digon or triangle.

A system \mathcal{D} is called *admissible* with respect to weights $\{\alpha^j\}$ if it satisfies the weight-angle conditions.

3.3 Modulus problem

3.3.1 Statement

Given the sets of points A, B, C, A', B', C' and the free family of homotopy classes \mathcal{H} . Define a set of nonnegative weights $\{\alpha^j\}_{j=1}^n$. Let \mathcal{D} be a system of domains associated with \mathcal{H} which is admissible with respect to $\{\alpha^j\}$. By $\text{Adm}P^*(\mathcal{D})$ we denote the class of a conformal invariant metrics $\rho|dz|$ on \mathfrak{R}' such that

- for each locally rectifiable curve $\gamma \in H_i^j$, $i \in \{1, 2, 4, 5\}$ we have the inequality

$$\int_{\gamma} \rho(z)|dz| \geq \alpha^j,$$

- for each locally rectifiable curve $\gamma \in H_3^j$ we have the inequality

$$\lim_{\varepsilon_1, \varepsilon_2 \rightarrow 0} \left\{ \int_{\gamma(\varepsilon_1, \varepsilon_2)} \rho(z)|dz| + \frac{\beta_{c_p}}{2\pi} \log \varepsilon_1 + \frac{\beta_{c_q}}{2\pi} \log \varepsilon_2 \right\} \geq \lambda_j;$$

where the curves in H_3^j connect points $c_p, c_q \in C \cup C'$, and $\lambda_j = \frac{\alpha_j}{m(D_{3^j}^j, c_p, c_q)}$

- for each locally rectifiable curve $\gamma \in H_6^j$ we have the inequality

$$\lim_{\varepsilon \rightarrow 0} \left\{ \int_{\gamma(\varepsilon)} \rho(z)|dz| + \frac{\beta_{c_p}}{2\pi} \log \varepsilon \right\} \geq \lambda_j,$$

where the curves in H_6^j have the end-point at $c_p \in C \cup C'$, and $\lambda_j = \frac{\alpha_j}{m_{\Delta}(D_{6^j}^j, c_p)}$;

- there exists the finite limit

$$\begin{aligned} \mathcal{M}(\{\alpha^j\}_{j=1}^n, \mathcal{D}) &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_{(\mathfrak{R}', \varepsilon)} \rho^2(z) dx dy + \frac{1}{2\pi} \sum_{H_2^j} \alpha_j^2 \log \varepsilon + \right. \\ &\quad \left. \frac{1}{\pi} \sum_{H_5^j} \alpha_j^2 \log \varepsilon + \frac{1}{2\pi} \sum_{H_3^j} \alpha_j^2 \log \varepsilon + \frac{1}{\pi} \sum_{H_6^j} \alpha_j^2 \log \varepsilon \right\}. \end{aligned}$$

3.3.2 Goal

The modulus problem \mathcal{M} consists of finding the infimum

$$\mathcal{M} = \inf_{\mathcal{D}, \rho(\mathcal{D})} \mathcal{M}(\{\alpha^j\}_{j=1}^n, \mathcal{D})$$

where \mathcal{D} is an admissible system associated with the free homotopy class \mathcal{H} and $\rho \in \text{Adm}P^*$ is an admissible metric.

The problem can be split in the following steps:

1. Find an infimum

$$\inf_{\rho \in \text{Adm}P^*(\mathcal{D})} \mathcal{M}(\{\alpha^j\}_{j=1}^n, \mathcal{D})$$

for each admissible domain system \mathcal{D} .

2. Find an infimum

$$\mathcal{M} = \inf_{\mathcal{D}} \inf_{\rho \in \text{Adm}P^*(\mathcal{D})} \mathcal{M}(\{\alpha^j\}_{j=1}^n, \mathcal{D})$$

over all admissible domain systems \mathcal{D} .

This formulation of the modulus problem is equivalent to the one given in [Kuz86].

3.4 Extremal partition problem

3.4.1 Statement

Given the sets of points A, B, C, A', B', C' , the free family of homotopy classes \mathcal{H} , and a set of nonnegative weights $\{\alpha^j\}_{j=1}^n$. Let \mathcal{D} be a system of domains associated with \mathcal{H} , which is admissible with respect to $\{\alpha^j\}$. Define a functional

$$\mathcal{P}(\{\alpha^j\}_{j=1}^n, \mathcal{D}) = \sum_{H_i^j, i \in \{1,2,4,5\}} \alpha^j M(D_i^j) - \sum_{H_i^j, i \in \{3,6\}} \alpha^j M(D_i^j).$$

3.4.2 Goal

The extremal partition problem \mathcal{P} consist of finding the supremum

$$\mathcal{P} = \sup_{\mathcal{D} \text{ is admissible}} \mathcal{P}(\{\alpha^j\}_{j=1}^n, \mathcal{D})$$

3.5 Equivalence of problems

In the case $B = B' = C = C' = \emptyset$ the solution of problems \mathcal{M} and \mathcal{P} and their equivalence are the fundamental results of Jenkins [Jen57]. The case $B, B' \neq \emptyset, C = C' = \emptyset$ was treated by Strebel [Str67]. The most general case was treated by Kuzmina [Kuz86] and Emel'yanov [Eme86]; their results were completed by Solynin [Sol99].

Here we will formulate the main theorem regarding the case when none of the sets of marked points A, B, C, A', B', C' might be empty.

Theorem 3.4. *Given the Riemann surface \mathfrak{R} with marked sets of points A, B, C on \mathfrak{R} and A', B', C' on $\partial\mathfrak{R}'$. Let us define the free family of homotopy classes of curves $\mathcal{H} = \{H_i^j\}$ on $\mathfrak{R}' \cup \partial\mathfrak{R}'$ together with a system of non-negative weights $\alpha = \{\alpha_i^j\}$.*

There exists a unique quadratic differential $Q(z)dz^2$, real-valued on $(\partial\mathfrak{R})'$ and possessing the following properties:

- (i) $Q(z)dz^2$ has no dense domains on \mathfrak{R} (that is, the complement to a union of all critical trajectories of Q , denoted by Φ_Q , is empty);
- (ii) Its system of domains $\mathfrak{R} \setminus \Phi_Q$ is a solution of the extremal partition problem on the Riemann surface \mathfrak{R} ;
- (iii) The metric $(Q(z))^{\frac{1}{2}}|dz|$ is extremal in the corresponding modulus problem on \mathfrak{R} ;
- (iv) $Q(z)dz^2$ has a simple poles at the points of a set $A \cup A'$, a poles of second order at the points of a set $B \cup B'$ and critical points of another type at the points of a set $C \cup C'$ (that is, zeros or poles of order greater than two);
- (v) The trajectories in a domains $\tilde{D}_i^j, i \in \{1, 2, 4, 5\}$ have length equal to α^j in the Q -metric, and orthogonal trajectories in the domains $\tilde{D}_i^j, i \in \{3, 6\}$ have length equal to α^j in Q -metric.

The proof of Theorem 3.6 can be found in [Sol99].

Chapter 4

Löwner lemma and its generalizations

In this chapter we consider an extremal problem for univalent functions $f(z)$ that map the unit disk \mathbb{D} into itself. As the preliminary part we give some facts about the non-euclidean metric in the unit disk.

4.1 Preliminaries

In what follows let us denote the group of conformal maps of \mathbb{D} onto \mathbb{D} by $\text{Möb}(\mathbb{D})$, that is

$$\text{Möb}(\mathbb{D}) = \left\{ \tau(z) = e^{i\alpha} \frac{z - z_0}{1 - \bar{z}_0 z}, \quad |z_0| < 1 \right\}.$$

Let A be an arc on \mathbb{T} . One can construct the function

$$\omega(z, A) = \frac{1}{2\pi} \int_A \frac{1 - |z|^2}{|\zeta - z|^2} |d\zeta|$$

defined by the Poisson integral (see [Pom92], p. 42) of the characteristic function of a set A . It is not hard to prove the following properties:

- $\omega(z, A)$ is a harmonic function satisfying the condition $0 \leq \omega(z, A) \leq 1$ for $z \in \mathbb{D}$,
- $\omega(z, A) = \begin{cases} 1 & \text{for } z \in A, \\ 0 & \text{for } z \in \mathbb{T} \setminus A, \end{cases}$
- $\omega(0, A) = \frac{\Lambda(A)}{2\pi}$, where $\Lambda(A)$ is the length of A in the euclidean metric,
- For $\tau \in \text{Möb}(\mathbb{D})$, we have $\omega(\tau(z), \tau(A)) = \omega(z, A)$.

A function $\omega(z, A)$ is called *the harmonic measure* of the arc A at the point z with respect to the unit disk \mathbb{D} .

Another property of the harmonic measure is expressed in Löwner's lemma [Pom92]

Lemma 4.1. (*Löwner's lemma.*) Let ϕ be analytic function in \mathbb{D} , and let $A, B \subset \mathbb{T}$ be measurable. If the angular limit $\phi(\zeta)$ exists for all $\zeta \in A$, and if

$$\phi(\mathbb{D}) \subset \mathbb{D}, \quad \phi(A) \subset B \subset \mathbb{T}$$

then

$$\omega(\phi(z), B) \geq \omega(z, A) \quad \text{for } z \in \mathbb{D}.$$

In particular, if $\phi(0) = 0$ then $\Lambda(B) \geq \Lambda(A)$.

4.2 Extremal problem

In the paper [CP02] there was obtained the following result:

Lemma 4.2. For a conformal map $f : \mathbb{D} \rightarrow \mathbb{D}$, $f(0) = 0$, which has the finite angular limits $f(e^{i\theta}) = e^{i\sigma}$, $\sigma \in [\sigma_1, \sigma_2]$, $\theta \in [\theta_1, \theta_2]$, $0 < \theta_2 - \theta_1 < 2\pi$, $\sigma_2 - \sigma_1 < 2\pi$, there is an estimate of its derivative at the origin:

$$\frac{1 - \cos \frac{\theta_2 - \theta_1}{2}}{1 - \cos \frac{\sigma_2 - \sigma_1}{2}} \leq |f'(0)| \leq 1.$$

The left-hand side inequality is sharp with the Pick extremal function $g_\alpha^\zeta(z)$, $\zeta = \frac{\theta_1 + \theta_2}{2}$ and $\sigma_1 = \arg g_\alpha^\zeta(e^{i\theta_1})$, $\sigma_2 = \arg g_\alpha^\zeta(e^{i\theta_2})$. The right-hand side inequality is obvious due to Schwarz lemma.

Let us replace here the condition $f(0) = 0$ by $f(z_0) = z_0$, where $z_0 \in \hat{\mathbb{D}}$ and try to obtain the similar estimations for $|f'(z_0)|$. Consider the following cases:

- $z_0 \in \mathbb{D}$. Take $\tau(z) = \frac{z - z_0}{1 - \bar{z}_0 z} \in \text{Möb}(\mathbb{D})$ mapping z_0 to 0. Function $g = \tau \circ f \circ \tau^{-1}$ is a conformal map $g : \mathbb{D} \rightarrow \mathbb{D}$ with $g(0) = 0$ which has the finite angular limits $g(e^{i\theta}) = e^{i\sigma}$, $\theta \in [\theta_1 + 2 \arg(1 - z_0 e^{i\theta_1}), \theta_2 + 2 \arg(1 - z_0 e^{i\theta_2})]$, $\sigma \in [\sigma_1 + 2 \arg(1 - z_0 e^{i\sigma_1}), \sigma_2 + 2 \arg(1 - z_0 e^{i\sigma_2})]$. The derivative of g at the origin is equal to

$$g'(0) = f'(z_0),$$

and we obtain the same estimate for $|f'(z_0)|$ as in Lemma 4.2.

- $z_0 \in \mathbb{T}$, $z_0 = e^{i\zeta}$.

Let us consider the triangle $f(D, z_0, \theta_1, \theta_2)^1$ that belongs to the family of all triangles in \mathbb{D} with the vertex at the point z_0 , with the inner angle π at z_0 , and the opposite leg lying on \mathbb{T} and connecting the points $e^{i\sigma_1}$, $e^{i\sigma_2}$. Since for the conformal map f from Lemma 4.2 we have $f(\mathbb{D}) \subseteq \mathbb{D}$, it follows from Lemma 1.14 that

¹In what follows, we denote the triangle $(D, z_0, e^{i\theta_1}, e^{i\theta_2})$ that have the domain $D = \mathbb{D} \setminus (-e^{\frac{\theta_1 + \theta_2}{2}}, z_0]$ by $(D, z_0, \theta_1, \theta_2)$.

$m_\Delta(D, z_0, \sigma_1, \sigma_2) = \inf_f m_\Delta(f(D, z_0, \theta_1, \theta_2))$, where extremal map is from Möb(\mathbb{D}) that maps points $(z_0, e^{i\theta_1}, e^{i\theta_2})$ to $(z_0, e^{i\sigma_1}, e^{i\sigma_2})$ correspondingly.

According to the rule of change of the reduced modulus under a conformal map $f(z)$ (see Theorem 1.13) we have

$$m_\Delta(D, z_0, \theta_1, \theta_2) + \frac{1}{\pi} \log |f'(z_0)| \geq m_\Delta(D, z_0, \sigma_1, \sigma_2). \quad (4.1)$$

In order to estimate the derivative of f at the point z_0 , we have only to find the number $m_\Delta(D, z_0, \alpha_1, \alpha_2)$ for any α_1, α_2 with $0 < \alpha_2 - \alpha_1 < 2\pi$.

First, let us compute $m_\Delta(D, -1, -\alpha, \alpha)$. We transfer this triangle onto the triangle $(D, 0, -\pi, \pi)$ by the map $w(z)$ which is the solution of the equation

$$\frac{3(z + \frac{1}{z}) - 6 \cos \alpha - 2 + (z + \frac{1}{z}) \cos \alpha}{(1 - \cos \alpha)(\frac{1}{2}(z + \frac{1}{z}) + 1)} = w + \frac{1}{w}.$$

The function w has the expansion about -1 as

$$\begin{aligned} w(z) = & \frac{-16(1 + \cos \alpha)}{1 - \cos \alpha} (z + 1)^{-2} - \frac{16}{(1 - \cos \alpha)^2} (z + 1)^{-1} + \frac{-2(\cos^2 \alpha + 2 \cos \alpha - 1)}{(1 - \cos \alpha)^2} + \\ & + \frac{1 - \cos \alpha}{16(1 + \cos \alpha)} (z + 1)^2 + O((z + 1)^3). \end{aligned}$$

By the rule of change of the reduced modulus we have

$$m_\Delta(D, 0, -\pi, \pi) = m_\Delta(D, -1, -\alpha, \alpha) + \frac{1}{2\pi} \log \frac{(1 - \cos \alpha)}{16(1 + \cos \alpha)}.$$

One can easily check that $m_\Delta(D, 0, -\pi, \pi) = 0$. Hence, we have

$$m_\Delta(D, -1, -\alpha, \alpha) = \frac{1}{2\pi} \log \frac{16(1 + \cos \alpha)}{(1 - \cos \alpha)}. \quad (4.2)$$

Using the equality (4.2) and the rule of change of the reduced modulus we obtain

$$m_\Delta(D, z_0, \alpha_1, \alpha_2) = \frac{1}{\pi} \log \frac{4}{|w'_\alpha(z_0)|}, \quad (4.3)$$

where $w_\alpha(z)$ is the Möbius map that transforms points $(z_0, e^{i\alpha_1}, e^{i\alpha_2})$ to $(-1, -i, i)$.

Denote by w_σ and w_θ the Möbius transformations from \mathbb{D} to \mathbb{D} that maps points $(z_0, e^{i\sigma_1}, e^{i\sigma_2})$ and $(z_0, e^{i\theta_1}, e^{i\theta_2})$ correspondingly to $(-1, -i, i)$.

From (4.1) and (4.3) it follows that

$$\frac{|w'_\theta(z_0)|}{|w'_\sigma(z_0)|} \leq |f'(z_0)|,$$

where

$$\frac{|w'_\theta(z_0)|}{|w'_\sigma(z_0)|} = \frac{\sin \frac{\theta_1 - \theta_2}{2} \sin \frac{\zeta - \theta_2}{2} \sin \frac{\zeta - \theta_1}{2}}{\sin \frac{\sigma_1 - \sigma_2}{2} \sin \frac{\zeta - \sigma_2}{2} \sin \frac{\zeta - \sigma_1}{2}}.$$

Finally, let us formulate a generalization of Lemma 4.2 which collects all the obtained results:

Lemma 4.3. *For a conformal map $f : \mathbb{D} \rightarrow \mathbb{D}$, which has the finite angular limits $f(e^{i\theta}) = e^{i\sigma}$, $\sigma \in [\sigma_1, \sigma_2]$, $\theta \in [\theta_1, \theta_2]$, $0 < \theta_2 - \theta_1 < 2\pi$, $\sigma_2 - \sigma_1 < 2\pi$ and fixes a point in $\bar{\mathbb{D}}$ the following estimates of the derivative of f holds:*

- *If f fixes point $z_0 \in \mathbb{D}$, then*

$$\frac{1 - \cos \frac{\theta_2 - \theta_1}{2}}{1 - \cos \frac{\sigma_2 - \sigma_1}{2}} \leq |f'(z_0)| \leq 1;$$

The left-hand side inequality is sharp with the Pick extremal function $g_\alpha^\zeta(z)$, $\zeta = \frac{\theta_1 + \theta_2}{2}$ and $\sigma_1 = \arg g_\alpha^\zeta(e^{i\theta_1})$, $\sigma_2 = \arg g_\alpha^\zeta(e^{i\theta_2})$. The right-hand side inequality is obvious due to Schwarz lemma.

- *If f fixes point $z_0 = e^{i\zeta} \in \mathbb{T}$, then*

$$\frac{\sin \frac{\theta_1 - \theta_2}{2} \sin \frac{\zeta - \theta_2}{2} \sin \frac{\zeta - \theta_1}{2}}{\sin \frac{\sigma_1 - \sigma_2}{2} \sin \frac{\zeta - \sigma_2}{2} \sin \frac{\zeta - \sigma_1}{2}} \leq |f'(z_0)|.$$

The left-hand side inequality is sharp with the Möbius function $w(z) \in \text{Möb}(\mathbb{D})$ that maps points $(z_0, e^{i\theta_1}, e^{i\theta_2})$ to $(z_0, e^{i\sigma_1}, e^{i\sigma_2})$ correspondingly.

Bibliography

- [Akh70] N. I. Akhiezer, *Elements of the theory of elliptic functions*, Nauka, Moscow, 1970.
- [CP02] A. Vasiliev C. Pommerenke, *Angular derivatives of bounded univalent functions and extremal partitions of the unit disk*, Pacific J. Math. **206** (2002), no. 2, 425–450.
- [Eme86] E. G. Emel'yanov, *Problems of extremal decomposition*, Zap. Nauchn. Sem. St.-Petersb. Otdel. Mat. Inst. Steklov **154** (1986), 76–89.
- [Fug57] B. Fuglede, *Extremal length and functional completion*, Acta Math. **98** (1957), 171–219.
- [Grö28] H. Grötzsch, *über einige extremalprobleme der konformen abbildungen. i-ii*, Ber. Verh. - Sächs. Akad. Wiss. Leipzig, Math. - Phys. Kl. **80** (1928), 367–376.
- [Jen57] J. Jenkins, *On the existence of certain general extremal metric*, Annals of Math. **66** (1957), no. 3, 440–453.
- [Jen58] ———, *Univalent functions and conformal mapping*, Springer-Verlag, Berlin, 1958.
- [Kuz86] G. Kuzmina, *Extremal properties of quadratic differentials with strip-shaped domains in the structure of the trajectories*, Zap. Nauchn. Sem. St.-Petersb. Otdel. Mat. Inst. Steklov **154** (1986), no. 4, 110–129.
- [LA50] A. Beurling L. Ahlfors, *Conformal invariants and function-theoretic null sets*, Acta Math. **83** (1950), 101–129.
- [Pom92] Ch. Pommerenke, *Boundary behavior of conformal maps*, Springer-Verlag, Germany, 1992.
- [Sol99] A. Yu. Solynin, *Moduli and extremal metric problems*, Algebra i analiz **11** (1999), no. 1, 3–86, transl. in St. Petersburg Math. J., **11**, (2000), no. 1, 1-65.
- [Spr81] G. Springer, *Introduction to riemann surfaces*, Chelsea Publ., New-York, 1981.
- [Str67] K. Strebel, *On quadratic differentials with closed trajectories and second order poles*, J. Anal. Math. **19** (1967), 373–382.

- [Tei40] O. Teichmüller, *Extremale quasikonforme abbildungen and quadratische differentiale*, Abhandl. Preuss. Akad. Wiss., Math. - Naturwiss. Kl. **22** (1940), 3–197.
- [Väi71] J. Väisälä, *Lectures on n -dimension quasiconformal mappings*, Springer-Verlag, Germany, 1971.
- [Wei74] A. Weir, *General integration and measure, volume 2*, Cambridge University Press, Great Britain, 1974.