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A UNIFIED MULTILEVEL FRAMEWORK OF UPSCALING AND DOMAIN DECOMPOSITION

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Summary. We consider multiscale preconditioners for a class of mass-conservative domain-decomposition (MCDD) methods. For the application of reservoir simulation, we need to solve large linear systems, arising from finite-volume discretisations of elliptic PDEs with highly variable coefficients. We introduce an algebraic framework, based on probing, for constructing mass-conservative operators on a multiple of coarse scales. These operators may further be applied as coarse spaces for additive Schwarz preconditioners. By applying different local approximations to the Schur complement system based on a careful choice of probing vectors, we show how the MCDD preconditioners can be both efficient preconditioners for iterative methods or accurate upscaling techniques for the heterogeneous elliptic problem. Our results show that the probing technique yield better approximation properties compared with the reduced boundary condition commonly applied with multiscale methods.

1 INTRODUCTION

Challenges within flow in porous media include complex geological structures with spatial variability on multiple scales. Reservoir simulations (i.e. groundwater flow, oil recovery, CO_2 storage) often involve large spatial scales, where we need to solve large linear systems repeatedly in time. The potential u within the reservoir is governed by an elliptic PDE, with highly variable tensor coefficients $k(\mathbf{x})$,

$$-\nabla\cdot(\mathbf{K}(\mathbf{x})\nabla u(\mathbf{x})) = q \qquad \mathbf{x}\in\Omega. \tag{1}$$

Here Ω is a two dimensional domain, $\mathbf{K}$ is the permeability and q represents the source terms. Standard two-level domain-decomposition methods, using e.g. piecewise linear

basis functions for the coarse space, where the oscillating coefficients are assumed to be resolved at the coarse scale, in general perform poorly for these problems[5], where the condition number will have a dependence on the largest ratio in coefficients k. Multiscale methods[3] is introduced as an upscaling technique for constructing robust coarse spaces, with harmonic basis functions. The multiscale problem is solved directly on the coarse scale, and resolved on the fine scale as a linear combination of the basis functions. This is equivalent with one fine-scale iteration, using the multiscale method as a two-level additive Schwarz preconditioner for domain decomposition[8].

In some cases the solution may be too expensive to compute on the fine scale at each time step, and we are forced to do upscaling. However, the coarse-scale operator may produce non-physical oscillations in the solution[6], which can only be reduced by iterating on the fine-scale residual. We will introduce an adaptive framework for constructing coarse spaces for the class of mass-conservative domain-decomposition (MCDD) methods introduced by Nordbotten and Bjørstad[8], which can act as either an accurate upscaling method, or an efficient preconditioner. The framework is based on algebraic approximations to the Schur complement, by using the interface probing technique[2]. Most multiscale methods are based on a geometric upscaling of fine-scale information, however, this does not naturally generalize from two- to multi-level methods, or arbitrary geometries and dimensions. The probing technique on the other hand is only based on neighbour element relations, which is independent upon the underlying geometry, and we show in Section 3 how this approach can be extended to construct multilevel preconditioners. For the application of upscaling, we observe that we can obtain much more accurate coarse spaces by applying a set of solution-based probing vectors. A more detailed discussion is given in Section 4. To demonstrate the flexibility of the proposed methodology, we show in Section 5 a two-step preconditioner, where the first step is an upscaling of the fine-scale system, and the second step is a preconditioner for the upscaled system.

2 MCDD

2.1 Fine-scale system

We consider linear systems arising from fine-scale discretisation on cell centred grids, consisting of finite volumes ω_i. Here, the permeability tensors K are assumed to be constant on each volume ω_i, but may be discontinuous at the interfaces γ_{ij}, between two neighbouring volumes ω_i and ω_j. By integrating (1) over ω_i, and applying Green's theorem we obtain the integral equation for conservation of incompressible fluids,

$$\int_{\partial\omega_i} \mathbf{F}\cdot\nu_{\mathbf{i}} = \int_{\omega_i} q. \tag{2}$$

Here, $\mathbf{F} = -\mathbf{K}(\mathbf{x})\nabla u(\mathbf{x})$ represents the Darcy flux and ν_i is the outward normal vector to $\partial\omega_i$. A discretisation of (2) yields local mass conservation within ω_i, and the global discrete system of fine-scale equations takes the form

$$Au = b. \tag{3}$$

The system matrix A is in general non-symmetric.

2.2 Grids and scales

A primal coarse grid $\Omega = \bigcup \Omega_i$ is defined, such that each primal coarse cell Ω_i is a set of finite volumes ω_j on the fine scale and the interfaces of Ω_i align with the interfaces on the fine scale. The centre-most volume on the fine scale within Ω_i is defined as the coarse node V. By repeating the process, we can form a hierarchy of cell centred coarse grids Ω^l. As a preprocessing step we require mass conservation between all cells Ω_i^l on each level l, on which we will compute the solution. This gives us the possibility to construct a mass-conservative flow field on level l, from the approximate solution $\hat{u}$ at the same level. Let $A_i^l = R_i^l A$ be the restriction of the system matrix A to Ω_i^l. Acting on each primal coarse cell Ω_i^l, the integration matrix M_i^l sums all the rows of A_i^l into the row of the coarse node V. More precisely,

$$M_i^l = I + e_{iV}^l \left(\underline{1} - e_{iV}^l\right)^T, \tag{4}$$

where I is the identity matrix, e_{iV}^l is the unit vector identifying the row of the coarse node and $\underline{1}$ is the vector entirely filled with ones. We apply (4) on the linear system (3), which gives us the MCDD system

$$Cu = p, \tag{5}$$

where

$$C = Q^{(l)} A; \quad p = Q^{(l)} b \quad \text{and} \quad Q^{(l)} = \sum_l \sum_i \left(R_i^l\right)^T M_i^l R_i^l.$$

A dual coarse grid Ω' is also introduced, s.t. all the coarse nodes defined on the primal coarse grid Ω represent vertex nodes on the dual grid. A continuous path of connecting cells on the finer level, connecting two neighbouring primal coarse nodes, further define the interfaces on the dual grid. The boundary of Ω_i' consist of boundary nodes, sub-divided into edge and vertex nodes, as shown in Fig. 1(b). While the MCDD system of equations is defined on the primal grid, all local operations will be carried out on the dual grid.

3 SCHUR COMPLEMENT SYSTEM

We consider non-overlapping sub-domains on Ω', where the sub-domains only share common sub-interfaces. We will denote the boundary nodes and internal nodes on each sub-domain by subscript B and I, respectively. The boundary nodes B are further sub-divided into edge nodes E and vertex nodes V. The vertex nodes will here be of special

importance, since they define our coarse-scale degrees of freedom. By grouping the unknowns corresponding to the internal nodes I in u_I, and the unknowns located on the local interfaces of Ω_i' in u_B, we reorder the fine-scale problem (5), writing

$$\begin{bmatrix} C_{II} & C_{IB} \\ C_{BI} & C_{BB} \end{bmatrix} \begin{bmatrix} u_I \\ u_B \end{bmatrix} = \begin{bmatrix} p_I \\ p_B \end{bmatrix}. \tag{6}$$

All internal unknowns are now decoupled on each local sub-domain, and can formally be eliminated by a block Gaussian elimination of (6). This gives us the Schur complement system $Su_B = p_B - C_{BI}C_{II}^{-1}p_I$ on the interface, where $S = C_{BB} - C_{BI}C_{II}^{-1}C_{IB}$. The Schur complement system can be shown to be better conditioned[1], however the system is still quite expensive to solve. The multiplication of S with a vector $\mathbf{x}$ will require solving a Dirichlet problem on each local sub-domain Ω_i'.

By a similar grouping of unknowns u_E and u_V on the edge nodes E and vertex nodes V, respectively, we can write

$$\begin{bmatrix} S_{EE} & S_{EV} \\ S_{VE} & S_{VV} \end{bmatrix} \begin{bmatrix} u_E \\ u_V \end{bmatrix} = \begin{bmatrix} g_E \\ g_V \end{bmatrix}. \tag{7}$$

The reordered Schur complement matrix now has a sparse block structure, however each block is in general dense. We want to construct a simple approximation to $S_{EE}^{-1}S_{EV}$, and a reduced system-matrix A_c on a coarser scale, where A_c can be solved directly or applied as a coarse space for an additive Schwarz preconditioner. Note that this only modifies the matrices belonging to the equations for the edge unknowns. Thus, our solution still have the property of conserving mass on the coarse scale. Another observation is that the coarse-scale operator A_c will have the same general structure as the fine-scale operator A. This means that the same operations may be applied for A_c, and we can recursively construct mass-conservative operators on a hierarchy of levels.

For the construction of 2-level additive Schwarz preconditioners, numerical experiments indicate that the property of mass conservation may result in better conditioned problems for the fine-scale, see Figure 1(a), however this does not necessarily apply for multi-level Schwarz preconditioners. The approximation induced on the edges for the highest level, may destroy the property of mass conservation on all intermediate levels, meaning mass conservation can only be guaranteed on two scales simultaneously, that being the coarsest and finest scale. Consequently, the MCDD operators may be better suited as input parameters for a multigrid-type preconditioner, where the restricted residual is applied directly on each mass-conservative level.

4 INTERFACE PROBING APPROXIMATION

For the interface approximation on the local edge nodes, we consider the probing technique introduced in Chan and Mathew[2] and references therein. The aim is to construct an approximation of the Schur complement matrix on the edge, such that

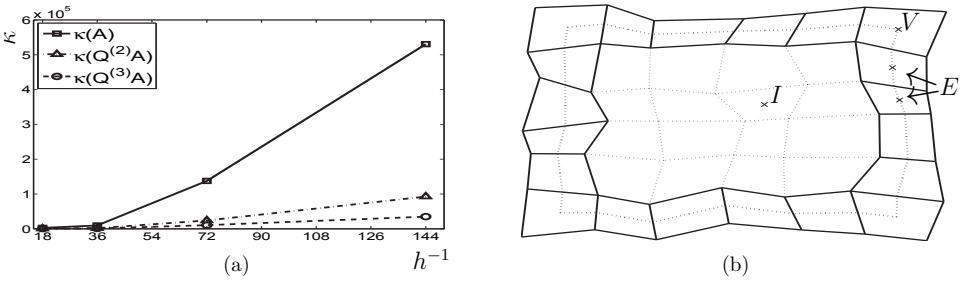

Figure 1: Figure (a) shows the condition number of the fine-scale solution with mass conservation on multiple levels, as we refine the grid. Figure (b) shows a single sub-domain Ω_i', where the bold lines indicate the boundary. The boundary cells are shared between two sub-domains, where each sub-domain only compute half-fluxes along the boundary.

$$\hat{S}_{EB}\mathbf{v}^i = S_{EB}\mathbf{v}^i = \mathbf{w}^i, \tag{8}$$

for some carefully chosen linearly independent probing vectors $\mathbf{v}^i$. Originally, the probing technique was applied on the square matrix S_{EE}, where the choice of probing vectors $\mathbf{v}^i = \sum_{j=i \bmod(n)} \mathbf{e}_j$ would lead to a low-band approximation of the Schur complement, which is fast to invert. The method was motivated by the observation of Golub and Mayers[4], that the coefficients of the Schur complement often had a rapid decay away from the diagonal, following the relation $|S_{ij}| = O\left(|i-j|^{-2}\right)$. In the case of anisotropic coefficients K_{ij} in the elliptic problem (1), we may have large off-diagonal elements, and the relation for the coefficients does not apply. If we instead probe S_{EB} on each Ω_i', the probing vectors $\mathbf{v}^i$ can be interpreted as boundary values for a Dirichlet problem on Ω_i'.

4.1 Solution-based probing vectors

We introduce solution-based probing vectors, to mimic upscaling techniques and therefore provide better approximation properties for heterogeneous and anisotropic problems. Let S_{EB} be the Schur complement on the edge, restricted to a single domain Ω_i', where B denotes the boundary of Ω_i' (see Figure 1(b)). The application of a probing vector $\mathbf{v}^i$ with the Schur complement matrix S_{EB} in (8), requires solving a Dirichlet problem on Ω_i', with boundary values $\mathbf{v}^i$. Thus, we may construct accurate upscaling methods by choosing probing vectors $\mathbf{v}^i$ that capture the important physical features of the local fine-scale solution. In fact, if the probing vectors could be chosen as the exact fine-scale solution restricted to the local boundary of Ω_i', the local approximation is exact, and the solution converges in one iteration. For the construction of solution-based probing vectors, we solve flow problems on a local domain covering the local support of S_{EB} (see Figure 1(b)). The restriction of the local solution to the boundary B will then be used as a probing vector, which will belong to the null space of the S_{EB}. We construct the

interface approximation $\hat{S}_{EB}$, from (8), filling only the diagonal elements and the element neighbours, corresponding to the largest couplings in C_{EB}. If the number of probing vectors are larger then the number of non-zero couplings in C_{EB}, we represent the additional neighbour connection(s) by an average of the remaining boundary elements. For each local problem, the calculation of a solution-based probing vector requires solving a local fine-scale problem on $\Sigma \supset \Omega_i'$. However, as the resulting probing vector $\mathbf{v}^i \in \text{null}(S_{EB})$ we do not need to solve the local fine-scale problem relating to the multiplication with S_{EB}. It follows from relation (8), that the preconditioner will be exact for those fine-scale problems captured by the local solutions. We will denote the preconditioner, MCDD-N, since its construction only depends on Neighbour relations.

4.2 Oscillating probing vectors

The interface probing preconditioners discussed in Chan and Mathew[2], are based on oscillating vectors, $\mathbf{v}^i = \sum_{j=i \bmod(n)} \mathbf{e}_j$. These preconditioners may give more robust approximations to the Schur complement, however they lack the physical interpretation provided by the solution-based probing vectors. Similar to standard upscaling techniques, the quality of the solution-based probing vectors will be case dependent. For robustness, all the probing vectors should not belong to the null space of S_{EB}. In fact, experience shows that by applying a few oscillating vectors, we get a more robust method. Thus, a combination of oscillating and solution-based probing vectors seems to be favourable. It is important that the chosen probing vectors are linearly independent. If two of the vectors $\mathbf{v}^i$ are close to being linearly dependent, the system (8) for calculating the approximation $\hat{S}_{EB}$ will be ill-conditioned.

5 NUMERICAL EXPERIMENTS

We demonstrate the Multilevel MCDD-N preconditioner for a heterogeneous flow problem generated by a random Log-Normal permeability field, with standard deviation 1 and a correlation length of 3 fine-grid cells in both the x- and y-direction. All calculations are performed on a (50x50) uniform mesh, with a coarsening factor of 5. For the global boundary conditions we consider $u = 1$ at the left boundary, $u = -1$ on the right boundary and no-flow conditions on the top and bottom boundary. We consider a two-step 3-level coarsening strategy, consisting of upscaling from the fine level, and preconditioning the intermediate level. For upscaling, we construct two solution-based probing vectors belonging to the null space of S_{EB}. We solve one problem with unit pressure drop in the horizontal direction and no-flow conditions on the vertical boundaries. Similarly, we solve a second problem with unit pressure drop in the vertical direction and no-flow conditions on the horizontal boundaries. All the local solutions are solved on a region with an overlap of 1, 2 and 3 sub-domains. We refer to MCDD-N(n) as the preconditioner with n sub-domains overlap. Additionally, two oscillating probing vectors are used. As preconditioners for the intermediate level we consider the interface probing preconditioner of

Chan and Mathew[4], using 3 and 5 oscillating vectors. We denote them MCDD-3P, and MCDD-5P, respectively. Both strategies are compared with an MCDD preconditioner using reduced boundary conditions. This is the same interface approximation applied with e.g. the multiscale finite-volume method of Jenny[7] for problems on regular Cartesian fine grid. It is also equivalent with the tangential component approximation $\hat{S}_{EB} = C_{EB}^T$, (see e.g. Smith et al.[9]). We denote the preconditioner MCDD-TC.

	MCDD-TC	MCDD-3P	MCDD-N(1)	MCDD-N(2)	MCDD-N(3)
mean	$4.2 \cdot 10^{-1}$	$9.5 \cdot 10^{-2}$	$4.8 \cdot 10^{-2}$	$1.2 \cdot 10^{-2}$	$2.6 \cdot 10^{-2}$
mean (92%)	$1.5 \cdot 10^{-1}$	$9.2 \cdot 10^{-2}$	$3.1 \cdot 10^{-2}$	$9.9 \cdot 10^{-3}$	$4.5 \cdot 10^{-3}$
mean (80%)	$1.4 \cdot 10^{-1}$	$9.1 \cdot 10^{-2}$	$2.1 \cdot 10^{-2}$	$8.1 \cdot 10^{-3}$	$3.7 \cdot 10^{-3}$

Table 1: Upscaling; We analyse the error in L^2-norm after one fine-scale iteration. The results for each method are the mean of 50 realisations of random Log-normal permeability fields. We also show the truncated means, where 2 and 5 realisations of both the low and high end of the results are discarded.

	MCDD-TC	MCDD-3P	MCDD-N(1)	MCDD-N(2)	MCDD-N(3)
unprec.	78	71	74	74	74
MCDD-TC	31	30	28	28	35
MCDD-3P	23	22	22	22	22
MCDD-5P	17	16	17	17	17

Table 2: Preconditioning; The table shows the number of iterations on the intermediate level, to meet a tolerance of 10^{-8}. Here each column represents different upscaling procedures, while the rows represent different preconditioners. All results are means of 50 realisations of random Log-normal permeability fields.

6 DISCUSSION

The results in Table 1 show that considerably more accurate coarse spaces can be achieved by applying only a few solution-based probing vectors, capturing the most important features of the fine-scale solution. While the upscaling method resulting from the tangential component approximation (MCDD-TC), fails to capture the correct flow field for many problems involving heterogeneous permeability, the probing technique (MCDD-3P) represents a more robust framework for approximating the flow on the boundary. Harmonic probing vectors (MCDD-N) can be applied to give better approximation properties for the interface probing technique. As for standard upscaling methods, the overall accuracy of the solution-based vectors relies on the localisation assumptions for the local problems. In general, the overall accuracy will increase with the size of the overlapping

region; The results show an improved accuracy of about a factor 2.5, per sub-domain. For robustness of the preconditioner, we need an independent set of probing vectors. For large overlapping regions, the local solution within the target region is less influenced by the boundary conditions and we may get similar flow behaviour for different boundary set up. This may cause inaccurate approximations to the local Schur complement. The residual on the local boundaries can be used to build local error estimates and adaptive strategies for constructing accurate operators on the coarse scale or efficient smoothers for the fine-scale. Table 2 shows that algebraic preconditioners may be constructed and applied to coarser levels, independently of the choice of upscaling procedure. The oscillating probing vectors applied with the (MCDD-3P and MCDD-5P) seem to be efficient to capture the oscillating nature of the residual. However, a systematic investigation of the quality of the MCDD preconditioners is beyond the scope of this paper, and a more systematic study of the localisation approximation and the properties of $\hat{S}_{EB}$ is needed.

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