Portfolio Optimization with PCC-GARCH-CVaR model

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Abstract

This thesis investigates the Conditional Value-at-Risk (CVaR) portfolio optimization approach combined with a univariate GARCH model and pair-copula constructions (PCC) to determine the optimal asset allocation for a portfolio.

The methodology focuses on minimizing CVaR as the risk measure in replacement of variance used in the traditional optimization framework of Markowitz. GARCH model provides a tool for predicting and analyzing the time-varying volatility financial assets are exposed to, while copulas allow us to model the non-linear dependence structure and margins separately.

We compare the performance of the CVaR optimized portfolio with other investment strategies such as Constant-Mix and Buy-and-Hold. Although the selection of strategy depends on the investor risk profile, it is empirically shown that the proposed CVaR optimized portfolio outperforms the other two investment strategies based on the accumulated wealth in the long run.

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Chapter 1

Introduction

In investment decision making portfolio optimization is an important process of diversification of which proportions of financial instruments held in a portfolio are determined to achieve a maximum expected return contingent on a desired level of risk. The classical approach, known as modern portfolio theory (MPT) started by Markowitz [1952] aim to optimize a portfolio's expected return with respect to variance as risk measure. This Mean-Variance framework is the foundation for later developments in portfolio optimization. As investors are most concerned about potential losses and its extent, the risk measure of interest is not necessarily the one that regards the overall market price dispersion on average like variance, but rather specifically the negative divergence. Hence alternative risk measures other than variance are introduced and researched subsequently.

Value-at-Risk (VaR) is a very popular risk measure first proposed by Baumol [1963]. It gives an indication of the worst scenario of loss for a given time horizon at a given confidence level. VaR has been widely used as the acceptable risk measure by many financial institutions and regulators, such as the Basel Committee on Banking Supervision. However, as financial return series are often not well presented by a elliptical or normal distribution, VaR is not a coherent risk measure according to Artzner et al. [1999] due to its undesirable mathematical properties such as non-subadditivity and non-convexity. R.McKay and T.E.Keefer [1996] also points out that VaR can be ill-behaved as a function of portfolio posistion as it can display multiple local extrema, creating difficulty for determining the optimal combination of risk factors. Taking such shortcomings into account, Rockafellar and Uryasev [2000] proposed a new approach to portfolio optimization using Conditional Value-at-Risk (CVaR) as the risk measure. CVaR quantifies the mean loss exceeding VaR and is proved to be a coherent risk measure in Pflug [2000]. Moreover, Rockafellar and Uryasev [2002] points out that it provides optimization short-cuts when dealing large-scale calculations through linear programming[1]

[1] Technical background of linear programming is outside the scope of thesis, for details see for example Kall

techniques, that otherwise could be non-feasible in practice.

This thesis employs the Mean-CVaR methodology introduced by Rockafellar and Uryasev [2000] to optimize a typical portfolio for a Norwegian life-insurance company. As the topic concerns investment choices and asset allocation we primarily aims at optimizing the portfolio's expected return by evaluating market risk with CVaR.

As the CVaR optimization procedure uses return scenarios as input, we carry out simulations generated from a model fitted to historical data. This requires therefore appropriate modeling of financial return series and their interdependence structure. It is commonly acknowledged that return series in financial markets are not independently identically normal distributed but leptokurtotic and subject to volatility clustering. These are important factors to be considered when calculating the risk a portfolio faces. It is a well known phenomenon that volatility varies over time and that it tends to come in clusters; appearing periods of low volatility and periods of high volatility. It is also shown that the volatility of financial time series are autocorrelated; that is, current volatility depends on past volatility. Since the volatility is not directly observable, it is essential to have an adequately good model for estimation and forecast. We will apply the most widely used model, GARCH(1,1) introduced by Bollerslev [1986] to model the volatility of univariate return series.

Additionally, in order to accurately estimate the CVaR of the portfolio, we need to effectively capture the dependence structure between the asset returns, especially in the tails where extreme events may occur concurrently. Copulas allow one to model the dependence structure and margins separately, and therefore provide more flexibility. This thesis will use the general theoretical framework of copula and apply the pair-copula construction (PCC) and vine structure introduced by Bedford and Cooke [2002] to model the interdependence between the residuals of the return series.

Since the introduction of CVaR as alternative risk measure, there has been extensive research comparing the CVaR portfolio optimization and Markowitz [1952]'s Mean-Variance approach such as Aas and Low [2012] and Deng et al. [2011]'s work. We, on the other hand, investigate the performance of the CVaR optimized portfolio in terms of rebalancing strategy by comparing other asset allocation strategies such as Constant-Mix and Buy-and-Hold.

This thesis is structured as follows. In Chapter 2 we present a summary of the CVaR optimization methodology. Chapter 3 describes the univariate GARCH model in the context of volatility clustering in the financial market. Chapter 4 introduces the general theoretical framework of copulas while Chapter 5 details the pair-copula composition (PCC) and canonical vine structure of a general multivariate distribution and its parameter estimation. In Chapter 6 we outline the portfolio optimization problem by combining the PCC-GARCH

and Mayer [2011]

model with CVaR optimization, while we carry out the empirical analysis in Chapter 7. Finally, the conclusion and suggestions for future work are presented in Chapter 8.

Chapter 2

Portfolio Optimization

The concepts of modern portfolio theory dates back to Markowitz [1952]'s introduction of risk management through portfolio diversification and selection based upon efficient frontier. The traditional framework of optimization is based on the use of variance as the measure of financial risk. The return of a portfolio is considered as a random variable R and modeled as a weighted sum of asset returns $r_1, ..., r_n$,

$$\mathrm{R} = \sum_{i=1}^{n} w_i r_i,$$

where w_i represents the weights of each asset in the portfolio subject to the constraint $\sum_{i=1}^{n} w_i = 1$. As the Markowitz framework uses the sample mean and variance as estimates for return and risk, minimizing the variance-covariance of the assets yields a set of feasible portfolios bounded by a curved line called the efficient frontier. The efficient frontier represents the set of optimal portfolios that generate the highest expected return for a given level of risk, or the lowest expected risk for a given return. As a result, one can construct at least one optimal portfolio from the available instruments with the expected return and risk corresponding to the point along the efficient frontier.

However, the Mean-Variance portfolio optimization has several drawbacks as the resulting efficient frontier relies on the sample estimates that are computed under the assumption of multivariate normality. Hence, unless the asset returns are normally distributed, which is highly improbable in practice, it is inappropriate to estimate risk by sample variance as it does not capture fat tails and skewness in the underlying distribution. Moreover, the variance accounts for both downside and upside movements in asset returns equally whilst the intention of risk management is to capture downside risk, that is, the risk of loss.

More recently, other risk measures such as Value-at-Risk (VaR) and conditional Value-at-risk (CVaR) have been used in portfolio optimization. This thesis employs the approach introduced by Rockafellar and Uryasev [2000] where the expected portfolio return is optimized by minimizing CVaR. The following sections give an overview of the basic concepts in risk management prior to illustrating the CVaR optimization problem.

The main references in this chapter are Markowitz [1952], Artzner et al. [1999], Rockafellar and Uryasev [2000], Krokhmal et al. [2002], McNeil et al. [2005],Morgan [1996], Würtz et al. [2009],G.Cornuejols and Tutuncu [2006] and Aas and Low [2012].

2.1 Risk Concepts

Morgan [1996] defines risk as *the degree of uncertainty of future net returns*. In the context of finance and insurance there are three main risk categories to be considered, namely, *credit risk*; *market risk* and *operational risk*. *Credit risk* is the risk of loss of financial capital such as loan repayments due to the failure of a borrower to meet contractual obligations, commonly known as the *default*. Credit risk arises from situations where a current debt or investment is expected to receive future cash payments. Consequently, investors or lenders require interest payment to compensate such risk of default of the borrower. The greater the credit risk, the higher the interest rate. Credit risk is therefore closely related to interest rate as a way of evaluating a borrower's ability to meet scheduled payment obligations. *Market risk* is the risk of losses in financial positions due to adverse movements in market prices affected by factors such as economic recession, political unrest and events that have substantial impact on the overall performance of the financial markets. This type of risk is omnipresent in various financial sectors and cannot be eliminated, it is however, possible to hedge against market risk by diversification. It is now a common practice to diversify a portfolio across different industries as opposed to financial markets since they have become increasingly correlated as a result of globalization. *Operational risk* refers to the risk of operational failures due to inadequate internal processes, systems and human errors.

Although insurance companies are certainly exposed to *insurance risk* that arises from a policyholder's mortality and morbidity status, the most critical risk associated with asset allocation and portfolio management is market risk. Notably, comprehensive market risk management involves techniques such as stress testing, worst scenario analysis combined with careful use of statistical risk measures. In the context of portfolio optimization, the scope of this thesis covers only the statistical approach to risk measurement.

2.2 Risk Measures

Selection of appropriate risk measures is central to optimization problem. A good risk measure should therefore have a list of desirable properties. This leads to the concept of so-called *coherent* risk measures presented by Artzner et al. [1999]. Let $\rho(\cdot)$ be the risk measure and let R_1 and R_2 be two random variables representing for example, two assets in a portfolio. A risk measure is *coherent* if it satisfies the following axioms:

Axiom T - Translation invariance

$$\rho(R_1 + l) = \rho(R_1) - l \tag{2.1}$$

Axiom S - Subaddivity

$$\rho(R_1 + R_2) \leq \rho(R_1) + \rho(R_2) \tag{2.2}$$

Axiom PH - Positive homogeneity

$$\rho(lR_1) = l\rho(R_1) \tag{2.3}$$

Axiom M - Monotonicity

$$\rho(R_1) \leq \rho(R_2), R_2 \leq R_1 \tag{2.4}$$

Axiom T states that adding a quantity l to an asset reduces the risk by the equivalent amount. Axiom S reflects the idea that risk can be reduced by diversification. The total risk faced by a portfolio is less than or equal to the sum of the risks of its individual assets. Axiom PH says that if one increases the amount invested in one asset one increases the risk with the same factor. Axiom M means that if the value of R_1 in general is larger than that of R_2, then the risk of R_1 is less than or equal to that of R_2. Note that Axiom S and Axiom PH together ensures the convexity of a risk measure, which implies that if there exists a local minimum, it is also the global minimum. *Convexity* and *subadditivity* are important properties of risk measures used for solving optimization problems. In particular, convex and continuously differentiable functions are easy to minimize numerically as addressed in Rockafellar and Uryasev [2000].

Banking laws and regulations issued by current financial regulators formulate risk measures as percentiles of the loss distribution, such as Value-at-Risk (VaR). Basel II for examples, considers Value-at-Risk (VaR) as the preferred risk measure. However, VaR has its shortcomings as it is only a coherent risk measure when the underlying assets follow normal or lognormal distributions. It is otherwise unstable and difficult to handle numerically when dealing with heavy tailed distributions as stressed by Rockafellar and Uryasev [2002]. The following section introduces the concept of *Conditional Value-at-Risk* (CVaR) as an alternative statistical measure of market risk.

2.2.1 Conditional Value-at-Risk

Often considered as an upper bound for Value-at-Risk (VaR), CVaR estimates the extent of loss on average given a loss occur. In order to understand the concept of CVaR, we first need to define VaR, a measure of the maximum potential change in portfolio value for a given probability over a fixed time horizon Δ. Generally, CVaR is the weighted average of loss exceeding VaR at some confidence level. In the case of a continuous distribution it is simply the expected loss that exceeds VaR. CVaR is thus also known as *expected shortfall*. Consider a portfolio of n instruments over a time period Δ. Then, the loss function is the negative of the return of the portfolio, where portfolio return is given as the sum of returns on individual instruments $\boldsymbol{r}$ scaled by weights $\boldsymbol{w}$

$$f_L(\boldsymbol{w}, \boldsymbol{r}) = -(w_1 r_1 + \cdots + w_n r_n) = -\boldsymbol{w}^T \boldsymbol{r} \tag{2.5}$$

We define then the probability of the loss $f_L(\boldsymbol{w}, \boldsymbol{r})$ not exceeding l as

$$\Phi(w, l) = \int_{f_L(\boldsymbol{w},\boldsymbol{r}) \leq l} p(\boldsymbol{r}) d\boldsymbol{r}, \tag{2.6}$$

where $p(\boldsymbol{r})$ is the joint density function of the random returns and $\Phi(\boldsymbol{w}, l)$ is the cumulative distribution loss function associated with $\boldsymbol{w}$ and is continuous and non-decreasing with respect to l.

The idea behind Value-at-Risk (VaR) is to consider the worst case scenario. Investors are interested in odds of large losses. VaR is an estimate of the maximum expected loss at a given confidence level over a fixed time period Δ. For instance, for an investment of 100$ daily, a 95%-VaR of -6% can be interpreted as we are 95% confident that the daily loss will not exceed 6$. Formally, VaR with respect to the portfolio weights $\boldsymbol{w}$ for a given confidence level $\alpha \in (0, 1)$ is given by the smallest l such that the probability of the loss $f_L(\boldsymbol{w}, \boldsymbol{r})$ exceeding l is at most $1 - \alpha$

$$\text{VaR}_\alpha(\boldsymbol{w}) = min\{l : \Phi(\boldsymbol{r}, l) \geq \alpha\}. \tag{2.7}$$

Although VaR is widely used for measuring downside risk, it does not account for scenarios exceeding VaR. In other words, VaR does not distinguish the extent of losses beyond the threshold. Furthermore, it does not satisfy all Axioms in (2.1)-(2.4), the criterion for coherence. It has undesirable properties such as *non-subadditivity* and *non-convexity* which fails to support the idea of diversification. With non-subadditivity we mean that the VaR of a portfolio is not necessarily bounded above by the sum of the VaR of the the individual assets. In other words, a diversified portfolio could be exposed to more risk and hence higher capital requirement than a less diversified one. Furthermore, the non-convex characteristic of VaR means that a local minimizer of VaR does not necessarily imply a global minimum. Consequently, VaR has its limitation as a risk measure for the portfolio optimization problem. An illustration of VaR and CVaR is given in figure 2.1.

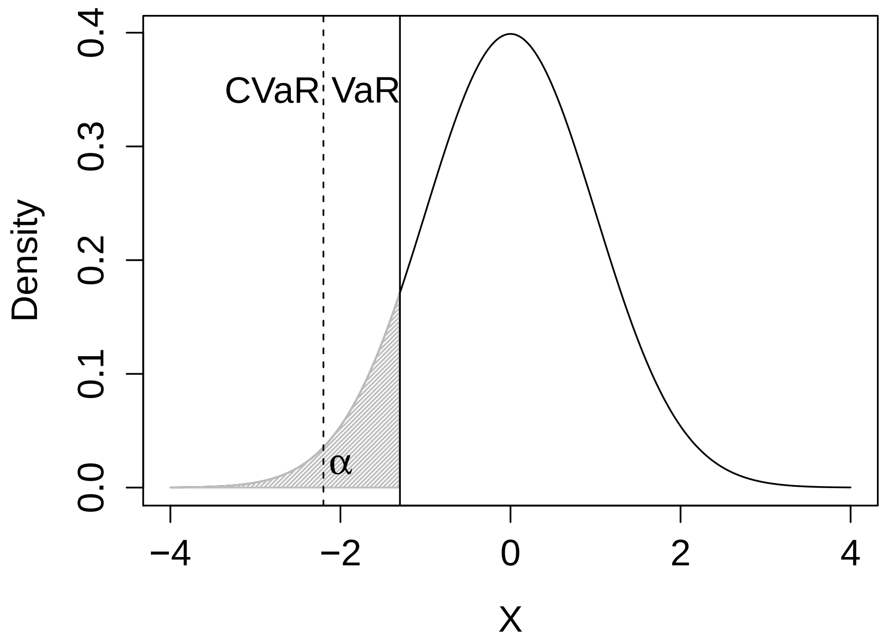

Figure 2.1: Graphical representation of VaR and CVaR at confidence level α

Formally, the CVaR associated with the portfolio weights $\boldsymbol{w}$ for a given confidence level $\alpha \in [0, 1]$ is defined as

$$\text{CVaR}_\alpha(\boldsymbol{w}) = \frac{1}{1-\alpha} \int_{f_L(\boldsymbol{w},\boldsymbol{r}) \geq VaR_\alpha(\boldsymbol{w})} f_L(\boldsymbol{w},\boldsymbol{r}) p(\boldsymbol{r}) d\boldsymbol{r} \tag{2.8}$$

Notably, the probability of $f_L(\boldsymbol{w},\boldsymbol{r})$ exceeding or equal to $VaR_\alpha(\boldsymbol{w})$ accumulates to $1-\alpha$. The definition of CVaR also ensures that $VaR \leq CVaR$, hence minimizing CVaR of a portfolio naturally implies a low VaR as well. Besides the properties mentioned above, the value $CVaR_\alpha(\boldsymbol{w})$ also behaves continuously with respect to $\alpha \in [0, 1]$. This is another feature of CVaR that makes it advantageous to VaR, its *stability* in the event of fat-tailed loss distributions. (for detailed proof see Rockafellar and Uryasev [2002]).

2.3 Efficient frontier

In the event of portfolio selection, we are faced with all possible combinations of assets in a risk-return space. The efficient frontier, in the shape of hyperbola, highlights a set of optimal portfolios with the greatest expected return given a risk level, or those with lowest risk level for a given expected return. In other words, the efficient frontier pinpoints the intersection between the portfolios with maximum return and those with minimum risk.

Figure 2.2 illustrates an example of the efficient frontier. The region on the right-hand side of the curve shows the set of all attainable portfolio combinations. The black dot is identified as portfolio with minimum risk. For the same level of risk, investors would always prefer the portfolio with higher expected return; given the different expected return investors would also prefer the portfolio with a higher return and a lower level of risk. Therefore, the efficient frontier is represented by the solid red curve starting from the portfolio with minimum risk and along the upward slope.

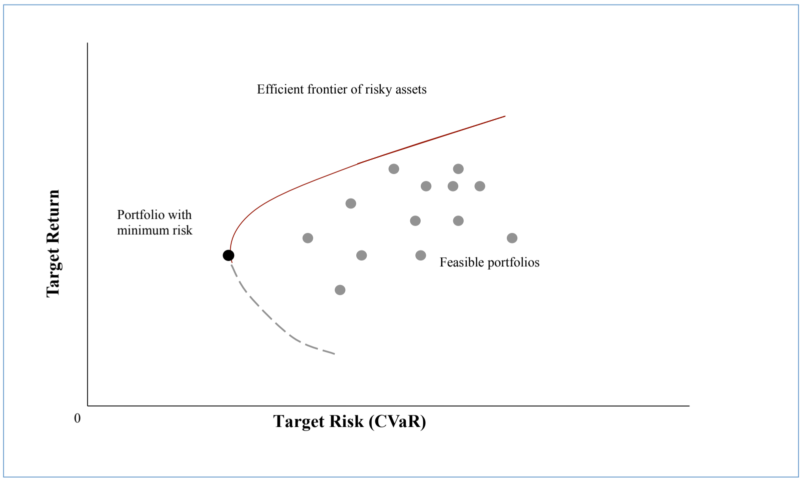

Figure 2.2: Efficient frontier of risky assets

2.4 Mean-CVaR Portfolio Optimization

Rockafellar and Uryasev [2000] introduces their approach to portfolio optimization that computes VaR and optimizes CVaR in terms of the following function:

$$\mathrm{F}_{\alpha}(\boldsymbol{w}, l) = l + \frac{1}{1-\alpha} \int_{f_L(\boldsymbol{w},\boldsymbol{r}) \geq l} (f_L(\boldsymbol{w}, \boldsymbol{r}) - l) p(\boldsymbol{r}) d\boldsymbol{r}. \tag{2.9}$$

Alternatively, it can also be formulated as following:

$$\mathrm{F}_{\alpha}(\boldsymbol{w}, l) = l + \frac{1}{1-\alpha} E\{[f_L(\boldsymbol{w}, \boldsymbol{r}) - l]^+\} \tag{2.10}$$

where $[f_L(\boldsymbol{w}, \boldsymbol{r}_s) - l]^+ = max[f_L(\boldsymbol{w}, \boldsymbol{r}_s) - l, 0]$. This function $F_{\alpha}(\boldsymbol{w}, l)$ of l is a convex function that has the following crucial properties:

$$\mathrm{VaR}_{\alpha}(\boldsymbol{w}) \in \arg\min_{l} F_{\alpha}(\boldsymbol{w}, l), \tag{2.11}$$

$$\mathrm{CVaR}_{\alpha}(\boldsymbol{w}) = \min_{l} F_{\alpha}(\boldsymbol{w}, l). \tag{2.12}$$

In other words, $CVaR_{\alpha}(\boldsymbol{w})$ is the minimum value of $F_{\alpha}(\boldsymbol{w}, l)$ and $VaR_{\alpha}(\boldsymbol{w})$, belonging to the non-empty, closed and bounded set of l for which the minimum is attained, is a minimizer of $F_{\alpha}(\boldsymbol{w}, l)$.

Note that the joint density $p(\boldsymbol{r})$ is often not known. Hence historical values of returns or simulated return scenarios $\boldsymbol{r}_1, ..., \boldsymbol{r}_s$ using the volatility and correlation estimates for the underlying portfolio assets are instead used. Therefore, Rockafellar and Uryasev [2000] approximate the integral in equation (2.10) by sampling the probability distribution of $\boldsymbol{r}$ according to density $p(\boldsymbol{r})$. As a result, the corresponding approximation to $F_{\alpha}(\boldsymbol{w}, l)$ based on scenarios $s = 1, ..., S$ is then

$$\tilde{F}_{\alpha}(\boldsymbol{w}, l) = l + \frac{1}{(1-\alpha)S} \sum_{s=1}^{S} (f_L(\boldsymbol{w}, \boldsymbol{r}_s) - l)^+. \tag{2.13}$$

The function $\tilde{F}_{\alpha}(\boldsymbol{w}, l)$ can then be transformed into a linear expression by replacing $(f_L(\boldsymbol{w}, \boldsymbol{r}_s) - l)^+$ with a dummy variable Z_s. We can hence solve the optimization problem by minimizing the following

$$\mathrm{l} + \frac{1}{(1-\alpha)S} \sum_{s=1}^{S} Z_s \tag{2.14}$$

subject to the following set of linear constraints:

$$\mathrm{Z}_s \geq f_L(\boldsymbol{w}, \boldsymbol{r}_s) - l. \tag{2.15}$$

$$\mathrm{Z}_s \geq 0, \tag{2.16}$$

$$\boldsymbol{w}^T E(\boldsymbol{r}) \geq R \tag{2.17}$$

$$\boldsymbol{w}^T \mathbf{1} = 1 \tag{2.18}$$

$$\boldsymbol{w} \geq 0. \tag{2.19}$$

Würtz et al. [2009] stress that constraints (2.15) and (2.16) alone cannot ensure $Z_s = (f_L(\boldsymbol{w}, \boldsymbol{r}_s) - l)^+$. However, it is justified that since the function to be minimized involves a positive multiplier of Z_s, an optimal solution can only be found when Z_s is the maximum of $f_L(\boldsymbol{w}, \boldsymbol{r}_s) - l$ and 0. Constraint (2.17) implies that the expected return on a portfolio should be bounded below by an arbitrary value R. Constraint (2.18) simply says that the sum of the weights of a portfolio should be 1. Constraint (2.19) ensures that all assets in the portfolio take long position, that is no short-selling. This constraint can modified according to portfolio strategy.

2.5 Compared to Mean-Variance Portfolio Optimization

Rockafellar and Uryasev [2000] has shown that the Mean-Variance (MV) and CVaR approaches to selecting optimal portfolio generates the same efficient frontier in case of a normally distributed loss function. However, differences arise when the underlying loss distribution is characterised by non-normality and asymmetry.

Consider the same portfolio of n instruments, the MV optimization problem aims to minimize the portfolio variance

$$\boldsymbol{w}^T \hat{\Sigma} \boldsymbol{w} \tag{2.20}$$

subject to the constraints

$$\boldsymbol{w}^T E(\boldsymbol{r}) = R \tag{2.21}$$

$$\boldsymbol{w}^T \mathbf{1} = 1. \tag{2.22}$$

Here, the vector $\boldsymbol{w}$ denotes portfolio weights of each instrument, matrix $\hat{\Sigma}$ represents an estimate of the covariance between returns of the instruments, and R is the target return.

For a given return, Krokhmal et al. [2002] has shown that the CVaR optimal portfolio has a higher standard deviation than that of the efficient MV portfolio. Whilst the MV optimal portfolio has higher CVaR than that of CVaR portfolio. As the confidence level increases the discrepancy between the two approaches also increases.

Chapter 3

GARCH

In finance, volatility is an important concept that measures the movements in market prices. It is therefore an important factor for investors to take into account when calculating the risks they face. It is a well known phenomenon that volatility varies over time, and that it tends cluster in periods of low volatility and periods of high volatility. It is also shown that the volatility of financial time series is autocorrelated, that is, current volatility is dependent on the past.

Since volatility is not directly observable, it is therefore essential to be able to estimate and predict volatility with a good model. Much literature has been written about the modeling of univariate volatility. This chapter presents the *generalized autoregressive conditionally heteroscedastic* (GARCH) model introduced by Bollerslev [1986] as a practical tool for forecasting *volatility* of a financial time series that varies over time. We concentrate on univariate GARCH models. Multivariate GARCH models are not considered in the context of this thesis as we will use copula instead for modeling time-varying dependencies.

The main references in this chapter are Aas and Dimakos [2004], Bollerslev [1986], McNeil et al. [2005], Zivot and Wang [2006], Posedel [2005], Bradley and Taqqu [2003] and Fernandez and Steel [1998].

3.1 Volatility

In times of stress market prices tend to fluctuate to a great extent and the magnitude of successive price movements are increasingly correlated over time. Note that when studying the magnitude of consecutive price movements we consider the logarithmic differences of market prices or indices, known as *log-returns*. The advantage of log-scaling is that compounded log-returns can be conveniently computed by summation. For example, annual log-returns are simply given as the sum of daily log-returns. Formally, a time series of daily log-returns

$\{X_t\}_{t\in\mathbf{Z}}$ is expressed as

$$X_t = \ln(P_t) - \ln(P_{t-1}) \tag{3.1}$$

where $\{P_t\}_{t\in\mathbf{Z}}$ are observed market prices. Unless otherwise specified, the use of term *returns* all refers to log-returns.

For a stochastic process $\{X_t\}_{t\in\mathbf{Z}}$ representing returns from financial market variables, its variance conditioned on the history $\{\mathcal{F}_{t-1}\}_{t\in\mathbf{Z}}$, [1]

$$\mathrm{Var}(X_t|\mathcal{F}_{t-1}),$$

is not constant over time due to the presence of *volatility clustering*. Formally, volatility is defined as

$$\sigma_t = \sqrt{Var(X_t|\mathcal{F}_{t-1})}. \tag{3.2}$$

Volatility clustering can be detected by computing the autocorrelation function of the absolute and/or squared values of returns defined as follows,

$$\rho_{abs}(h) = \frac{Cov(|X_t|, |X_{t-h}|)}{\sqrt{Var(|X_t|)Var(|X_{t-h}|)}} = \frac{\gamma_{abs}(h)}{\gamma_{abs}(0)}, \tag{3.3}$$

$$\rho_{squared}(h) = \frac{Cov(X_t^2, X_{t-h}^2)}{\sqrt{Var(X_t^2)Var(X_{t-h}^2)}} = \frac{\gamma_{squared}(h)}{\gamma_{squared}(0)}. \tag{3.4}$$

Here, $h \geq 0$ denotes the lag in time and $\rho(h) = \rho(-h)$, which means that autocorrelation is an even function. An autocorrelation function with positive values for a large amount of lags implies volatility clustering.

In general, characteristics of log-returns on indices, interest rates, commodity prices and other financial instruments are characterized by the following so-called *stylized facts*:

1. Return series are not independently identically distributed.
2. Return series are heavy, fat-tailed and asymmetric, hence, not Gaussian.
3. Absolute or squared values of return series display strong serial correlation.
4. Volatility is time varying and extreme values appear in clusters.
5. Volatility is mean-reverting, that is, in the long run it will settle down to a certain level.

The following sections focus on the technique used for modeling financial returns with time-varying volatility: GARCH-models, in particular, the widely used GARCH(1,1) model.

[1] Conditioning over $\mathcal{F}_{t-1}$ means conditioning over all past observations $X_{t-1}, X_{t-2}, ..., X_1$

3.2 GARCH process

The method of modeling time-varying conditional variance (squared volatility) was first introduced by Engle [1982] with the ARCH (Autoregressive Conditional Heteroskedastic) process. This model separates unconditional and conditional variance, where the latter is allowed to vary over time as a function of historical errors, also known as residuals. To illustrate, let $\{X_t\}_{t\in\mathbf{Z}}$ be a stochastic process of asset daily returns in terms of sample expected returns and residuals:

$$\mathrm{X}_t = \mu + \epsilon_t \tag{3.5}$$

An ARCH(p) process is given by

$$\begin{aligned} \epsilon_t &= \sigma_t z_t, \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 \end{aligned} \tag{3.6}$$

where z_t is iid $WN(0,1)$ [2] and $\alpha_0 > 0, \alpha_i \geq 0, i = 1, ..., p$. Given accruals of information $\mathcal{F}_{t-1}$, σ_t is known at time t and this ensures that

$$\mathrm{E}[\epsilon_t|\mathcal{F}_{t-1}] = E[\sigma_t z_t|\mathcal{F}_{t-1}] = \sigma_t E[z_t|\mathcal{F}_{t-1}] = \sigma_t E[z_t] = 0 \tag{3.7}$$

$$\mathrm{Var}[\epsilon_t|\mathcal{F}_{t-1}] = E[\sigma_t^2 z_t^2|\mathcal{F}_{t-1}] = \sigma_t^2 E[z_t|\mathcal{F}_{t-1}] = \sigma_t^2 E[z_t] = \sigma_t^2 \tag{3.8}$$

A few years later Bollerslev [1986] introduced the GARCH model as an extension to the ARCH process to "*allow for both a longer memory and a more flexible lag structure.*" Bollerslev [1986] Consider the same stochastic process $\{X_t\}_{t\in\mathbf{Z}}$ defined in Equation (3.5). A GARCH(p, q) process is formally defined as

$$\begin{aligned} \epsilon_t &= \sigma_t z_t, \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \end{aligned} \tag{3.9}$$

where z_t is iid $WN(0,1)$ and $\alpha_0 > 0, \alpha_i \geq 0, i = 1, ..., p, \beta_j \geq 0, j = 1, ..., q$. For $q = 0$ the process reduces to the ARCH(p) process. Note that the ARCH process expresses the conditional variance (squared volatility) as a linear function of the past p-period squared residuals ϵ_{t-i}^2 only, whilst the GARCH process allows the conditional variance σ_t^2 to be dependent on the past q-period conditional variances σ_{t-j}^2 in addition to the past p-period squared values of residuals.

[2] $\{z_t\}$ is a white noise process if $\{z_t\}$ is a sequence of uncorrelated random variables, each with mean 0 and variance σ^2. Here $\sigma^2 = 1$.

3.2.1 GARCH(1,1)

In practice, the most commonly used univariate GARCH model is the simplest GARCH(1,1) model given by

$$\begin{aligned} \epsilon_t &= \sigma_t z_t, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \end{aligned} \tag{3.10}$$

where z_t is iid $WN(0,1)$ and the three parameters satisfy $0 < \alpha_0, 0 \leq \alpha_1 \leq 1, 0 \leq \beta_1 \leq 1$ and $\alpha_1 + \beta_1 \leq 1$. The GARCH(1,1) model provides an adequate fit to financial time series and produces the *stylized facts* presented in previous section. Notably, the variance process σ_t^2 in GARCH(1,1) is *(weakly) stationary*[3] if $\alpha_1 + \beta_1 < 1$. To forecast future volatility, we are interested in predicting σ_{t+h} for $h \geq 1$. Formally, it translates into the following information $\mathcal{F}_t$ which is *known* at time t:

$$\begin{aligned} \mathrm{E}[\sigma_{t+1}^2 | \mathcal{F}_t] &= \alpha_0 + \alpha_1 E[\epsilon_t^2] + \beta_1 E[\sigma_t^2] \\ &= \alpha_0 + (\alpha_1 + \beta_1)\sigma_t^2. \end{aligned}$$

[3] A time series $\{X_t\}_{t \in \mathbf{Z}}$ is weakly stationary if it satisfies the following:

$$\begin{aligned} \mathrm{E}(X_t^2) &< \infty, \\ E(X_t) &= \mu, \\ cov(X_t, X_{t+h}) &= \gamma_x(h), \forall t \end{aligned}$$

These conditions mean that the first and second moments of a stationary time series are time invariant.

When looking $h > 1$ steps forward, we proceed to follow a recursive scheme step-wise and this yields a general formula as follows:

$$\begin{aligned}
\mathrm{E}[\sigma^2_{t+2}|\mathcal{F}_t] &= \alpha_0 + \alpha_1 E[\epsilon^2_{t+1}] + \beta_1 E[\sigma^2_{t+1}] \\
&= \alpha_0 + (\alpha_1 + \beta_1) E[\sigma^2_{t+1}] \\
&= \alpha_0 + (\alpha_1 + \beta_1)[\alpha_0 + (\alpha_1 + \beta_1)\sigma^2_t] \\
&= \alpha_0 [1 + (\alpha_1 + \beta_1)] + (\alpha_1 + \beta_1)^2 \sigma^2_t \\
E[\sigma^2_{t+3}|\mathcal{F}_t] &= \alpha_0 + \alpha_1 E[\epsilon^2_{t+2}] + \beta_1 E[\sigma^2_{t+2}] \\
&= \alpha_0 + (\alpha_1 + \beta_1) E[\sigma^2_{t+2}] \\
&= \alpha_0 + (\alpha_1 + \beta_1) \left[\alpha_0 [1 + (\alpha_1 + \beta_1)] + (\alpha_1 + \beta_1)^2 \sigma^2_t\right] \\
&= \alpha_0 [1 + (\alpha_1 + \beta_1) + (\alpha_1 + \beta_1)^2] + (\alpha_1 + \beta_1)^3 \sigma^2_t \\
&. \\
&. \\
&. \\
E[\sigma^2_{t+h}|\mathcal{F}_t] &= \alpha_0 [1 + (\alpha_1 + \beta_1) + (\alpha_1 + \beta_1)^2 + \cdots + (\alpha_1 + \beta_1)^{h-1}] + (\alpha_1 + \beta_1)^h \sigma^2_t \\
&= \alpha_0 \sum_{n=0}^{h-1} (\alpha_1 + \beta_1)^n + (\alpha_1 + \beta_1)^h \sigma^2_t \\
&= \alpha_0 \frac{1 - (\alpha_1 + \beta_1)^h}{1 - \alpha_1 - \beta_1} + (\alpha_1 + \beta_1)^h \sigma^2_t.
\end{aligned}$$

As $h \longrightarrow \infty$, we obtain the stationary variance $E[\sigma^2_{t+h}|\mathcal{F}_t] \longrightarrow \dfrac{\alpha_0}{1 - \alpha_1 - \beta_1}$. In other words, this is the long-run level where volatility will *mean-revert* to for a stationary GARCH(1,1) model. Furthermore, the sum of parameters α_1 and β_1 are often known as *persistence* that describes how quickly volatilities decay after a shock in financial markets. Since $\alpha_1 + \beta_1 < 1$, for h steps ahead in time, we have $(\alpha_1 + \beta_1)^h \longrightarrow 0$ as $h \longrightarrow \infty$. The pace of convergence is therefore dependent of $\alpha_1 + \beta_1$, the closer the size of persistence to 1, the longer it takes for a shock to be forgotten in the market.

3.3 Non-Gaussian Error Distributions

When modeling the volatility of return series with GARCH models one needs to consider both the marginal distribution which is the distribution of errors ϵ_t and the conditional distribution which is the distribution of ϵ_t/σ_t. Financial time series such as daily log-returns are known for displaying fat and heavy-tailed characteristics that are non-Gaussian. GARCH models with conditionally normally distributed errors are therefore sometimes unable to provide a good fit accounting for asymmetry and leptokurtic shape of marginal distribution. Consequently, alternative error distributions are considered depending on the class of asset returns. The following section give an overview of three alternative conditional distributions in univariate case: *Generalized Error Distribution*, *standard Student's t-distribution* and *skew Student's t-distribution*.

3.3.1 Generalized Error Distribution

The *generalized error distribution* (GED) first introduced by Nelson (1991) is an example of distribution to capture fat tails. Formally, the density of a GED variable with mean zero and variance one is given by

$$f_\nu(z) = \frac{\nu \exp\left[-(1/2)|z/\lambda|^\nu\right]}{\lambda 2^{\frac{\nu+1}{\nu}} \Gamma(1/\nu)}, \tag{3.11}$$

where

$$\lambda = \left[\frac{2^{-2/\nu}\Gamma(1/\nu)}{\Gamma(3/\nu)}\right]^{\frac{1}{2}} \tag{3.12}$$

and $\nu > 0$ is the shape parameter that controls the thickness of the tail. When $\nu = 2$, the GED density function becomes the standard normal; when $\nu > 2$, the distribution of x has thinner tails than that of normal distribution; and when $\nu < 2$, x has thicker tails than the normal. When $\nu \longrightarrow \infty$ the GED density function approaches to the uniform distribution.

3.3.2 Student's t-distribution

The Student's t distribution is a another commonly used fat-tailed distribution for modelling asset returns. The classical Student's t density is symmetric and have a single peak like the normal distribution. Formally, the density function of the *standard student's t-distribution* is given by

$$f_\nu(z) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi s \nu}} \left(1 + \frac{z^2}{s\nu}\right)^{-\frac{\nu+1}{2}}, \tag{3.13}$$

where ν is the degrees of freedom that controls the thickness of the tails and s the scale parameter. As the number of degrees of freedom increases, the t-distribution approaches the standard normal distribution. The mean and variance of the Student's t distribution are

$$\mathrm{E(Z)} = 0, \nu > 0$$
$$Var(Z) = \frac{s\nu}{\nu - 2}, \nu > 2$$

Since the error term z_t in a GARCH model following Student's t-distribution has variance 1 conditional on past events, the scale parameter s under the standard t-distribution must then be $\frac{(\nu-2)}{\nu}$.

Note that although both *Generalized Error Distribution* GED and *standard Student's t-distribution* account for fatter tails than that of the normal distribution, they are still symmetric. To account for any skewness and asymmetry, suitable alternative would be for example, the *skew Student's t-distribution.*

3.3.3 Skew Student's t-distribution

There are several different *skew Student's t-distributions.* We will use the Fernandez and Steel [1998] version in the context of this thesis. The probability density function is given by

$$f(z) = \frac{2}{\gamma + \frac{1}{\gamma}} \left[f_\nu(\frac{z}{\gamma}) I_{[0,\infty)}(z) + f_\nu(\gamma z) I_{(-\infty,0)}(z) \right]. \tag{3.14}$$

Here $I(\cdot)$ is the indicator function and $f_\nu(\cdot)$ is the PDF of standard Student's t-distribution. $\gamma > 0$ is the skewness parameter and when $\gamma = 1$, the skewness is 0 and f becomes the standard Student's t distribution with ν degrees of freedom.

3.4 Parameter Estimation

In practice, parameter estimation of GARCH models is performed on the basis of historical data. The most commonly used approach is maximum likelihood. We are interested in finding estimates that maximize the probability of which generates the set of known data. The likelihood is constructed and computed conditional on accruals of past information and volatility σ_t defined recursively in terms of σ_{t-1}.

As the parameter estimates are dependent of the historic data used for fitting GARCH models, the behaviour of estimates needs to be distinguished with respect to the underlying distribution. Consequently, this leads to the following two methods of GARCH model estimation: *maximum likelihood estimation (MLE)* and *quasi-maximum likelihood estimation (QMLE).* In MLE, the likelihood function uses the true distribution of data. This method generates estimates that contain more information. The challenge is, however, that it requires the distribution of the data set to be known, which in practice is difficult to obtain. In QMLE, on the other hand, we do not know the true distribution of the data set, but assume the error term is asymptotically i.i.d Gaussian distributed given all past information. This method is adequate to obtain good parameter estimates and is more practical and easier to implement.

To illustrate MLE, consider the GARCH(1,1) model specified in Equation(3.8). We assume now z_t is i.i.d standard normal, hence ϵ_t conditional on past information will follow a Gaussian distribution with mean 0 and variance σ_t^2. The conditional density function is then given by

$$f(\epsilon_t|\mathcal{F}_{t-1}) = \frac{1}{\sqrt{2\pi}\sigma_t} e^{-\frac{\epsilon_t^2}{2\sigma_t^2}} \tag{3.15}$$

Let $\Theta = (\alpha_0, \alpha_1, \beta_1)$ be the parameter vector. Then, the log-likelihood function of the GARCH(1,1) model conditional on historic values is

$$
\begin{aligned}
l(\Theta) &= \sum_{t=1}^{n} \ln \left(\frac{1}{\sqrt{2\pi}\sigma_t} e^{-\frac{\epsilon_t^2}{2\sigma_t^2}} \right) \\
&= \sum_{t=1}^{n} -\frac{1}{2} \ln 2\pi - \ln \sigma_t - \frac{\epsilon_t^2}{2\sigma_t^2}
\end{aligned}
\tag{3.16}
$$

where $\sigma_t = \sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2}$. Note that both methods require a starting value for σ_0 in order to carry the numerical maximazition of log-likelihood. The value can often be chosen as the sample variance.

3.5 Simulation

A GARCH(1,1) process with estimated parameters $\alpha_0, \alpha_1, \beta_1$ can be generated with the following simulation algorithm:

• Simulate z with respect to appropriate conditional distributions:

$$
\begin{aligned}
z &\sim ged_\nu(0,1), or \\
z &\sim t_\nu(0,1), or \\
z &\sim st_{\nu,\gamma}(0,1).
\end{aligned}
$$

• Generate sample of T observations using dynamics

$$
\begin{aligned}
\sigma_t^2 &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \\
\epsilon_t &= \sigma_t z_t
\end{aligned}
$$

Note that the algorithm requires an initial value for σ_0 and ϵ_0. They can for example, be chosen as the last estimated value of the GARCH(1,1) process.

Chapter 4

Copulas

While the GARCH model allows us to predict and analyze the volatility of financial time series when it varies over time, we need, in addition, a tool to model the dependence structure between the asset returns.

When studying dependence among random variables, linear correlation is by far the most commonly used measure. However, as it is a measure of *linear* dependence, linear correlation is limited to those of multivariate normal and elliptical distributions. Returns from financial markets such as bond indices, exchange rates and equity prices are known for non-normal behavior. Distribution of such returns are often skewed with heavier left tails than under normality, implying greater likelihood of large losses than gains. Hence, measures based on multivariate normal assumption cannot fully account for the dependence structure of financial assets. Alternatively, appropriate modeling of time-varying dependence between assets across financial markets can be performed by the use of a *copula*.

Copula is Latin for "*a link, tie, bond* " (Cassell's Latin Dictionary) and was first employed in a statistical sense by Abe Sklar (1959). He described copulas as "*functions that join or 'couple´ multivariate distribution functions to their one-dimensional marginal distribution functions*" Nelsen [2006]. The advantage of copula-based approach is it provides a way of modeling dependence structure between random variables independently of their margins. That is, a multivariate distribution can be decomposed into marginal distributions that are selected freely and then linked through suitable copulas. Therefore, it provides the opportunity to study the marginal distribution functions and the copula separately. Consequently, a given copula can result in various multivariate distributions by selecting different marginal distribution functions.

The following sections give the definition of a copula, examples of different copula families and dependence measures derived from copulas. Estimation of copula parameters and simulation from copulas will be presented in the next chapter as the approaches are different in the context of *pair-copula construction*.

Main references in this chapter are Nelsen [2006], Embrechts et al. [1986], Embrechts et al. [1999], Aas et al. [2007], Aas [2004] McNeil et al. [2005] and Brechmann [2010].

4.1 Definition

A copula can be understood as a function that links univariate marginal distributions to form multivariate distribution functions. A d-dimensional copula is a distribution function on $[0,1]^d$ with uniformly distributed margins on $[0,1]$. Its role for describing dependence among random variables was first introduced in Sklar's theorem (1959) which states the following:

Theorem 4.1. ***Sklar's theorem***
Let F be a d-dimensional distribution function with margins $F_1, ..., F_n$. Then there exists an unique copula C such that for all $\boldsymbol{x} = (x_1, ..., x_d)' \in R^d$,

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)). \tag{4.1}$$

If $F_1, ..., F_d$ are continuous, then C is unique; otherwise, C is uniquely determined on $RanF_1 \times ...F_d$. Conversely, if C is a copula and F_1, F_d are distribution functions, then the function F is a joint distribution function with margins $F_1, ..., F_d$.

Now denote $x_i = F_i^{-1}(u_i), i = 1, ..., d$ where $F_i^{-1}(u_i)'s$ are the inverse marginal distribution functions. It follows directly that $u_i = F_i(x_i)$. Inserting into (4.1) will give us the following corollary from Sklar's theorem:

$$\begin{aligned} \mathrm{F}(\mathrm{F}_1^{-1}(u_1), ..., F_d^{-1}(u_d)) &= C(F_1(F_1^{-1}(u_1), ..., F_d(F_d^{-1}(u_d)) \\ &= C(u_1, ..., u_d). \end{aligned} \tag{4.2}$$

Equations (4.1) and (4.2) show how the copula link marginal distribution functions to form the multivariate distribution and how the copula can be extracted from multivariate distribution functions, respectively. Note that $F_i^{-1}(u_i)'s$, the inverse marginal distribution functions are also known as the quantile functions of the marginals. Hence, Equation (4.2) displays $C(u_1, ..., u_d)$ as the joint probability of which $x_1, ..., x_d$ are below their respective $u_1, ..., u_d$-quantiles. The density function f follows also directly by differentiating (2.1) using chain rule

$$f(x_1, ..., x_d) = c(F_1(x_1), ..., F_d(x_d))f_1(x_1) \times ... \times f_d(x_d) \tag{4.3}$$

The next section presents examples of copulas from two main categories: *elliptical* copulas that do not possess a closed form and are *implied* by well-known multivariate distribution functions; *archimedean* copulas that have simple, closed, *explicit* forms and are not derived from well-known multivariate distributions.

4.2 Elliptical copulas

An elliptical distribution has the following density function:

$$f(\boldsymbol{x}) = c_d|\Sigma|^{-\frac{1}{2}}g(\boldsymbol{x}-\boldsymbol{\mu})\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}), \tag{4.4}$$

with constant c_d , mean vector $\boldsymbol{\mu}$ and covariance matrix Σ.

Elliptical copulas with an absolutely continuous joint distribution function F with continuous, strictly increasing marginals $F_1, ..., F_d$ can be constructed as follows. The copula density c is given by the density of corresponding multivariate distribution function divided by the product of its marginal densities. Formally, it can be expressed by rewriting (2.3) with $x_i = F_i^{-1}$ for $i = 1, ..., d$ defined in (2.2):

$$\mathrm{c}(\mathbf{u}) = \frac{f(F_1^{-1}(u_1), ..., F_d^{-1}(u_d))}{f_1(F_1^{-1}(u_1))...f_d(F_d^{-1}(u_d))}. \tag{4.5}$$

Since elliptical copulas are derived from known distributions, they are therefore also called *implicit* copulas. The following sections present the two most well-known examples of implicit copulas constructed using the method as shown above.

4.2.1 Gaussian copula

A d-dimensional multivariate Gaussian distributed random vector $\mathbf{X}$ has the representation:

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}}e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}, \tag{4.6}$$

with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ.

The Gaussian copula is constructed from the multivariate standard normal distribution. Hence the mean vector $\boldsymbol{\mu}$ becomes the null-vector and the covariance matrix Σ the symmetric positive definite correlation matrix $R \in [-1, 1]^d$. This yields the multivariate Gaussian copula density according to (2.5) as

$$c(\boldsymbol{u}) = \frac{\frac{1}{(2\pi)^{\frac{d}{2}}|\boldsymbol{R}|^{\frac{1}{2}}}e^{-1/2\boldsymbol{x}^T\boldsymbol{R}^{-1}\boldsymbol{x}}}{\prod_{j=1}^{d}\frac{1}{(2\pi)^{1/2}e^{-1/2x_j^2}}} = \frac{e^{-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{R}^{-1}\boldsymbol{x}}}{|\boldsymbol{R}|^{1/2}e^{-\frac{1}{2}\sum_{j=1}^{d}x_j^2}}, \tag{4.7}$$

where $\boldsymbol{x} = (\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d))$.

For simplicity we now consider the bivariate case. The copula density and its corresponding distribution function are given as

$$c_\rho(u_1, u_2) = \frac{1}{(1-\rho^2)^{\frac{1}{2}}}exp\left\{-\frac{\rho^2(x_1^2+x_2^2)-2\rho x_1x_2}{2(1-\rho^2)}\right\} \tag{4.8}$$

$$C_\rho(u_1,u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} exp\left\{-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right\} dx_1 dx_2 \tag{4.9}$$

where ρ is the parameter of the Gaussian copula.

4.2.2 Student's t-copula

A d-dimensional multivariate Student t distributed random vector $\mathbf{X}$ has the representation:

$$f(\boldsymbol{x}) = \frac{\Gamma[(\nu+d)/2]}{(\pi\nu)^{d/2}\Gamma(\nu/2)|\boldsymbol{S}|^{1/2}} \left[1 + \frac{(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{S}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}{\nu}\right]^{-\frac{\nu+d}{2}}, \tag{4.10}$$

with mean vector $\boldsymbol{\mu}$, $\nu > 0$ degrees of freedom and scale matrix $\boldsymbol{S}$.

Similar to the Gaussian copula, the t-copula is also based on standard t distribution, hence the mean vector $\boldsymbol{\mu}$ becomes the null-vector and the scale matrix S the correlation matrix $R \in [-1,1]^d$. This yields the t-copula density according to (4.5) as following

$$\begin{aligned} c(\boldsymbol{u}) &= \frac{\frac{\Gamma\left(\frac{\nu+d}{2}\right)}{(\pi\nu)^{d/2}\Gamma(\frac{\nu}{2})|\boldsymbol{R}|^{1/2}} \left(1 + \frac{\boldsymbol{x}'\boldsymbol{R}^{-1}\boldsymbol{x}}{\nu}\right)^{-\frac{\nu+d}{2}}}{\prod_{j=1}^{d} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{(\pi\nu)^{1/2}\Gamma(\frac{\nu}{2})} \left(1 + \frac{x_j^2}{\nu}\right)^{-\frac{\nu+1}{2}}} \\ &= \frac{\Gamma\left(\frac{\nu+d}{2}\right)\Gamma\left(\frac{\nu}{2}\right)^{d-1} \left(1 + \frac{\boldsymbol{x}'\boldsymbol{R}^{-1}\boldsymbol{x}}{\nu}\right)^{-\frac{\nu+d}{2}}}{|\boldsymbol{R}|^{1/2}\Gamma\left(\frac{\nu+1}{2}\right)^d \prod_{j=1}^{d} \left(1 + \frac{x_j^2}{\nu}\right)^{-\frac{\nu+1}{2}}}, \end{aligned} \tag{4.11}$$

where $\boldsymbol{x} = (t_\nu^{-1}(u_1), ..., t_\nu^{-1}(u_d))$. t_ν^{-1} is the inverse standard univariate student's t-distribution with expectation 0, variance $\frac{\nu}{\nu-2}$ and ν degrees of freedom.

Now consider the bivariate case again. The Student's t-copula density function is given by:

$$c_{\rho,\nu}(u_1,u_2) = \frac{\Gamma(\frac{\nu+2}{2})/\Gamma(\frac{\nu}{2})}{\nu\pi dt_\nu(x_1) dt_\nu(x_2)\sqrt{1-\rho^2}} \left\{1 + \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right\}^{-\frac{(\nu+2)}{2}}, \tag{4.12}$$

where $dt(\cdot)$ and $t_\nu^{-1}(\cdot)$ are the probability density and the quantile function of the univariate standard t distribution with linear correlation coefficient ρ and $\nu > 0$ degrees of freedom. The corresponding distribution function for bivariate t copula is

$$C_{\rho,\nu}(u_1,u_2) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \left\{1 + \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right\}^{-\frac{(\nu+2)}{2}} dx_1 dx_2. \tag{4.13}$$

The Student's t-copula has degrees of freedom ν in addition to ρ as parameters compared to the Gaussian copula that only has one parameter, ρ. The degrees of freedom ν has significant contribution to describing extreme co-movements among financial risk factors. The tendency to exhibit joint extreme movements decreases as the degrees of freedom ν increases. Consequently, one can study the dependence structure among assets regardless of their marginal behaviour. Notably, as the Student's t distribution approaches normality with increasing degrees of freedom; the Student's t copula also approaches the Gaussian copula as the degrees of freedom increase. Examples of simulations from the Gaussian and Student's t-copulas are shown in Figure 4.1. Note the similarity between the Gaussian copula in (a) and the t-copula in (c).

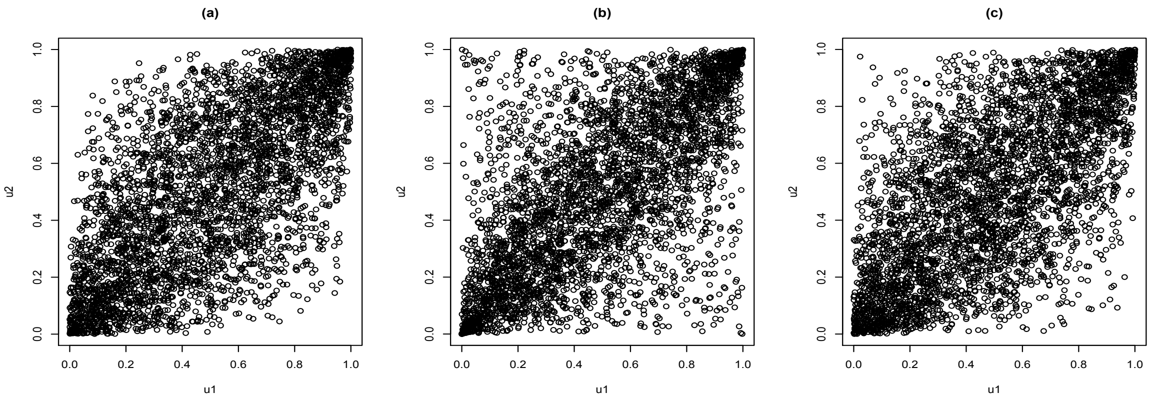

Figure 4.1: 4000 simulated points from (a) Gaussian copula with $\rho = 0.7$, (b) t-copula with $\rho = 0.7, \nu = 3$ and (c) t-copula with $\rho = 0.7, \nu = 30$ respectively.

4.3 Archimedean copulas

Archimedean copulas are another important class of copulas that are easily constructed with *explicit* forms. They are therefore also called *explicit* copulas. This family of copulas are especially useful for modeling the credit risk of a portfolio. Formally, a d-dimensional Archimedean copula is defined as follows.

Theorem 4.2. *Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(0) = \infty$ and $\varphi(1) = 0$. φ^{-1} denote the inverse of φ such that it is completely monotonic. Then*

$$C(\boldsymbol{u}) = \varphi^{-1}(\varphi(u_1) + ... + \varphi(u_d)) \tag{4.14}$$

is a copula.

φ is called the *generator* of a copula. We now present two of the most commonly used Archimedean copula families in bivariate case for the simplicity of expression.

4.3.1 Clayton copula

The Clayton copula is an asymmetric copula with larger dependence in the negative tail than in the positive. It has a generator $\varphi(t) = \frac{1}{\delta}(t^{-\delta} - 1)$ that gives $\varphi^{-1}(t) = (\delta\varphi(t) + 1)^{-\frac{1}{\delta}}$. The density for bivariate Clayton copula can be expressed as

$$C(u_1, u_2) = (u_1^{-\delta} + u_2^{-\delta} - 1)^{\frac{1}{\delta}}, \tag{4.15}$$

where $0 < \delta < \infty$. The copula density can be obtained by differentiating (2.14) with respect to u_1 and u_2:

$$\begin{aligned} c(u_1, u_2) &= \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} \\ &= \frac{\partial}{\partial u_1}\left[\frac{\partial C(u_1, u_2)}{\partial u_2}\right] \\ &= \frac{\partial}{\partial u_1}\left[-\frac{1}{\delta}(u_1^{-\delta} + u_2^{-\delta} - 1)^{\frac{1}{\delta}-1}(-\delta u_2^{-\delta-1})\right] \\ &= \frac{\partial}{\partial u_1}\left[(u_1^{-\delta} + u_2^{-\delta} - 1)^{\frac{1}{\delta}-1}(u_2^{-\delta-1})\right] \\ &= (-\frac{1}{\delta} - 1)(u_1^{-\delta} + u_2^{-\delta} - 1)^{\frac{1}{\delta}-2}(-\delta u_1^{-\delta-1})(u_2^{-\delta-1}) \\ &= -(\frac{1+\delta}{\delta})(u_1^{-\delta} + u_2^{-\delta} - 1)^{\frac{1}{\delta}-2}(-\delta)(u_1 u_2)^{-1-\delta} \\ &= (1+\delta)(u_1^{-\delta} + u_2^{-\delta} - 1)^{\frac{1}{\delta}-2}(u_1 u_2)^{-1-\delta}. \end{aligned} \tag{4.16}$$

δ is the parameter that contains information about dependence structure. Independence is implied by $\delta \longrightarrow 0$ whilst perfect dependence is obtained when $\delta \longrightarrow \infty$. Figure 4.2 shows simulated examples of such limiting cases.

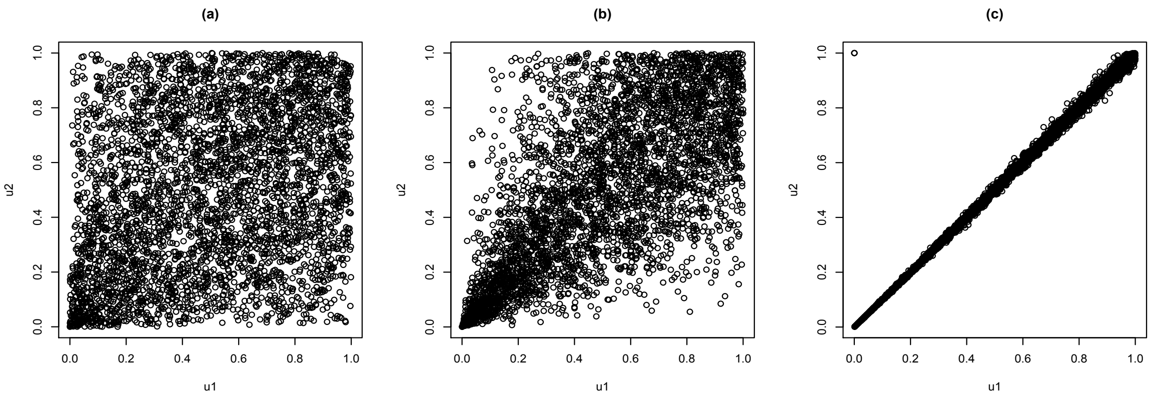

Figure 4.2: 4000 simulated points from Clayton copula with (a)$\delta = 0.5$, (b)$\delta = 2$,(c)$\delta = 100$.

4.3.2 Gumbel copula

The Gumbel copula is also an asymmetric copula. However as opposed to the Clayton copula it displays greater dependence in the positive tail than in the negative. Its generator is $\varphi(t) = (-\log t)^{\delta}$ which yields $\varphi^{-1}(t) = e^{-(\varphi(t))^{\frac{1}{\delta}}}$. Using the same method, the Gumbel copula distribution function is given by

$$C(u_1, u_2) = exp[-((-\log u_1)^{\delta} + (-\log u_2)^{\delta})^{\frac{1}{\delta}}], \tag{4.17}$$

where $0 < \delta \leq 1$ is the parameter for dependence measure. Independence is implied by $\delta = 1$ whilst perfect dependence is suggested when $\delta \longrightarrow 0$. Figure 4.3 shows simulations from a Gumbel copula with different values of δ. Similarly, the corresponding copula density can be obtained by differentiating (2.17) with respect to u_1 and u_2 and is given by

$$\begin{aligned}
c(u_1, u_2) &= \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} \\
&= \frac{\partial}{\partial u_1}\left[\frac{\partial C(u_1, u_2)}{\partial u_2}\right] \\
&= \frac{\partial}{\partial u_1}\left\{C(u_1, u_2)(-\frac{1}{\delta})\left[(-\log u_1)^{\delta} + (-\log u_2)^{\delta}\right]^{\frac{1}{\delta}-1}\delta(-\log u_2)^{\delta-1}(-\frac{1}{u_2})\right\} \\
&= \frac{1}{u_2}(-\log u_2)^{\delta-1}\frac{\partial}{\partial u_1}\left\{C(u_1, u_2)\left[(-\log u_1)^{\delta} + (-\log u_2)^{\delta}\right]^{\frac{1}{\delta}-1}\right\} \\
&= \frac{1}{u_2}(-\log u_2)^{\delta-1}\left\{C(u_1, u_2)\frac{1}{u_1}(-\log u_1)^{\delta-1}\left[(-\log u_1)^{\delta} + (-\log u_2)^{\delta}\right]^{(\frac{1}{\delta}-1)2}\right. \\
&\quad + C(u_1, u_2)(\frac{1}{\delta}-1)\left[(-\log u_1)^{\delta} + (-\log u_2)^{\delta}\right]^{\frac{1}{\delta}-2}\delta(-\log u_1)^{\delta-1}(-\frac{1}{u_1}) \\
&= \frac{1}{u_2}(-\log u_2)^{\delta-1}C(u_1, u_2)\frac{1}{u_1}(-\log u_1)^{\delta-1}\left\{\left[(-\log u_1)^{\delta} + (-\log u_2)^{\delta}\right]^{(\frac{1}{\delta}-1)2}\right. \\
&\quad + \left[(-\log u_1)^{\delta} + (-\log u_2)^{\delta}\right]^{\frac{1}{\delta}-2}(\delta - 1) \\
&= \frac{C(u_1, u_2)}{u_1 u_2}\frac{(\log u_1 \log u_2)^{\delta-1}}{((-\log u_1)^{\delta} + (-\log u_2)^{\delta})^{2-\frac{1}{\delta}}} \times \left[((-\log u_1)^{\delta} + (-\log u_2)^{\delta})^{\frac{1}{\delta}} + \delta - 1\right].
\end{aligned} \tag{4.18}$$

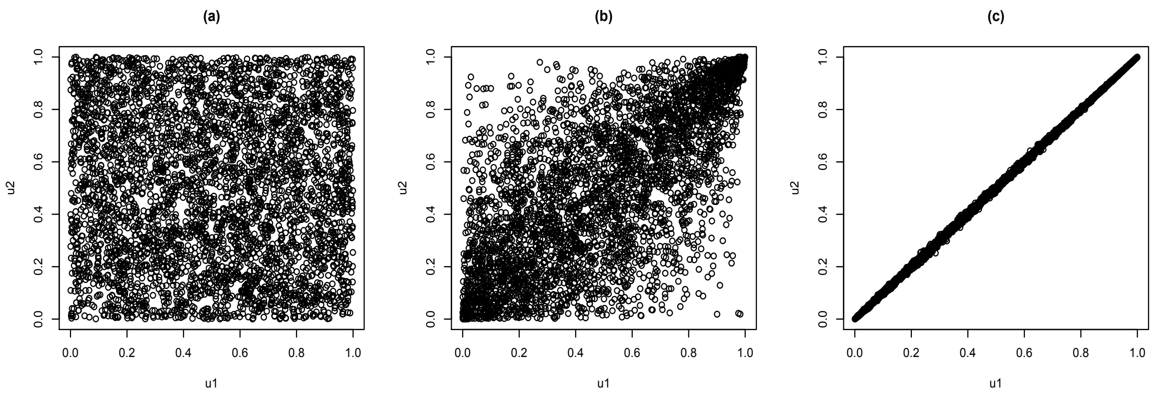

Figure 4.3: 4000 simulated points from Gumbel copula with (a)$\delta = 1$, (b)$\delta = 2$,(c)$\delta = 100$

4.4 Dependence measures

As addressed in Embrechts et al. [1999], unless financial returns are presented by multivariate normality or ellipticality the use of linear correlation as dependency measure becomes problematic. Following are some of the limitations of the usual Pearson linear correlation that ought to be repeated and amplified:

1. Correlation is only invariant under strictly increasing *linear* transformations, that is,X and Y do not yield the same correlation as $log(X)$ and $log(Y)$.

2. Not all values between 1 and -1 are necessarily attainable as correlation is dependent on the marginal distribution.

3. A correlation of 1 does not necessarily imply perfect positive dependence and a correlation of -1 does not necessarily imply perfect negative dependence.

4. Correlation is not defined for random variable with non-finite variances.

This means the use of the linear correlation coefficient as a dependence measure can be misleading when dealing with financial risk factors that are heavy-tailed distributed with non-finite second moments. On the contrary, copulas are invariant under *nonlinear* continuous and increasing transformation of the margins. This proves to be immensely useful for analysis of financial risk factors as it is common practice to use logarithmic returns on the indices and commodity prices.

This section focuses on two kinds of copula-based dependence measures as alternatives to linear correlation coefficient: *concordance* and *coefficients of tail dependence.*

4.4.1 Concordance

As opposed to *correlation*, *concordance* does not have the limitations of linear correlation mentioned previously and can be understood as a measure of ”association” that refers to any types of dependence structure between two random variables. According to Nelsen [2006] two pairs of random vectors (X_1, X_2) and $(\tilde{X}_1, \tilde{X}_2)$ are called *concordant* if $(X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) > 0$, that is $X_1 < \tilde{X}_1$ and $X_2 > \tilde{X}_2$. The pairs are *discordant* if $(X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) < 0$. The following section presents two important measures of concordance: *Kendall's tau* and *Spearman's rho*.

Kendall's tau

Kendall's τ for the random vector $(X_1, X_2)^T$ is defined as the probability of concordance minus the probability of discordance. Formally, it is defined as

$$\rho_\tau(X_1, X_2) = P((X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) > 0) - P((X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) < 0) \tag{4.19}$$

where $(\tilde{X}_1, \tilde{X}_2)$ is independent copy of (X_1, X_2). If X_1 and X_2 are continuous random variables with a unique copula C, Kendall's tau can be written as

$$\rho_\tau(X_1, X_2) = 4 \int \int_{[0,1]^2} C(u_1, u_2) dC(u_1, u_2) - 1. \tag{4.20}$$

Notably, for elliptical such as Gaussian and Student's t-copula Kendall's tau can be expressed in terms of the linear correlation coefficient ρ as follows:

$$\rho_\tau(X_1, X_2) = \frac{2}{\pi} arcsin \rho \tag{4.21}$$

For Archimedean copulas on the other hand, Kendall's tau can be expressed in terms of dependence parameter. For Clayton copula, it is

$$\rho_\tau(X_1, X_2) = \frac{\delta}{\delta + 2} \tag{4.22}$$

and for Gumbel copula it is

$$\rho_\tau(X_1, X_2) = 1 - \frac{1}{\delta} \tag{4.23}$$

Spearman's rho

Spearman's rho can also be expressed in terms of concordance and discordance. It is formally defined as

$$\rho_s(X_1, X_2) = 3(P((X_1 - \tilde{X}_1)(X_2 - \check{X}_2) > 0) - P((X_1 - \tilde{X}_1)(X_2 - \check{X}_2) < 0)), \tag{4.24}$$

where $\tilde{X}_1$ and $\check{X}_2$ are independent. However, it can be more intuitively understood as the linear correlation between probability-transformed random variables:

$$\rho_s(X_1, X_2) = \rho(F(X_1), F(X_2)). \tag{4.25}$$

If X_1 and X_2 are continuous random variables with a unique copula C, Spearman's rho is given by

$$\rho_s(X_1, X_2) = 12 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 3. \tag{4.26}$$

For elliptical copulas such as the Gaussian and Student's t-copula, Spearman's rho is given in terms of the linear correlation coefficient as follows:

$$\rho_s(X_1, X_2) = \frac{6}{\pi} arcsin \tfrac{1}{2} \rho. \tag{4.27}$$

Unlike the linear correlation coefficient, Kendall's tau and Spearman's rho are invariant under strictly increasing transformations. Although this is an indication that they are better alternatives as dependence measures, concordance also has its limitations as a dependence measure. For example, different copulas may have the same concordance. Hence, Embrechts et al. [1999] stresses that simple scalar measurement of dependence, both linear correlation and concordance ought to be used with caution. One should instead *choose a model for the dependence structure that reflects more detailed knowledge of the risk management in hand.* One such alternative is to examine the tail dependence, presented in the following section.

4.4.2 Tail dependence

Kendall's τ and Spearman's ρ discussed in previous section measures the amount of overall dependence over the whole unit space $[0, 1]^2$. In risk management, one is also interested in capturing the "association" between one large value of variable X_1 and one large value of another variable X_2. This leads to the concept of tail dependence. It refers to the degree of dependence in the corner of the lower-left quadrant or upper-right quadrant of a bivariate distribution. Tail dependence is therefore a measure of dependence between extreme events and proved to be useful for preventing concurrent large losses.

For two continuous random variables X_1 and X_2 with marginal distribution functions F_{X1} and F_{X2}, the coefficient of upper tail dependence is

$$\lambda_u(X_1, X_2) = \lim_{\alpha \to 1} P(X_2 > F_{X_2}^{-1}(\alpha) \mid X_1 > F_{X_1}^{-1}(\alpha)). \tag{4.28}$$

Provided that the limit $\lambda_u \in [0, 1]$ exists, the upper tail dependence coefficient describes the probability of large value of X_2 given large value of X_1. If $\lambda_u = 0$, it implies asymptotic independence between X_1 and X_2, which means that extreme values occur in isolation.

Analogously, the coefficient of lower tail dependence is

$$\lambda_l(X_1, X_2) = \lim_{\alpha \to 0} P(X_2 \leq F_{X_2}^{-1}(\alpha) \mid X_1 \leq F_{X_1}^{-1}(\alpha)). \tag{4.29}$$

Rewriting the conditional probability in Equation (4.28) the coefficient of lower tail dependence can be expressed in terms of a unique bivariate copula C:

$$\begin{aligned} \lambda_l(X_1, X_2) &= \lim_{\alpha \to 0} \frac{P(X_2 \leq F_{X_2}^{-1}(\alpha), X_1 \leq F_{X_1}^{-1}(\alpha))}{X_1 \leq F_{X_1}^{-1}(\alpha))} \\ &= \frac{C(\alpha, \alpha)}{\alpha}, \end{aligned} \tag{4.30}$$

and the upper tail dependence coefficient can be expressed in terms of a *joint survival function*[1]

$$\begin{aligned} \lambda_u(X_1, X_2) &= \lim_{\alpha \to 1} \frac{P(X_2 > F_{X_2}^{-1}(\alpha), X_1 > F_{X_1}^{-1}(\alpha))}{X_1 \leq F_{X_1}^{-1}(\alpha))} \\ &= \frac{\bar{C}(\alpha, \alpha)}{1 - \alpha}. \end{aligned} \tag{4.31}$$

Both coefficients have hence the invariance property and are independent of the asset returns margins.

In the case of elliptical distributions, the upper tail dependence coefficient is equal to the lower tail dependence coefficient. The Gaussian copula has probability 0 for joint extreme events regardless of the correlation coefficient ρ:

$$\lambda_u(X_1, X_2) = \lambda_l(X_1, X_2) = 2 \lim_{x \to -\infty} \Phi\left(x \frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right) = 0. \tag{4.32}$$

The Student's t-copula is asymptotically dependent in the tails regardless of the value of the linear correlation ρ. The tail dependence coefficients are given by:

$$\lambda_u(X_1, X_2) = \lambda_l(X_1, X_2) = 2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right). \tag{4.33}$$

Note that stronger tail dependence is implied by lower degrees of freedom ν and greater linear correlation ρ.

The tail dependence coefficients for Archimedean copulas have simple closed form. The upper tail coefficient for the Clayton copula is zero, $\lambda_u(X_1, X_2) = 0$ meaning that it is lower tail dependent. The lower tail dependence coefficient is

$$\lambda_l(X_1, X_2) = 2^{\frac{1}{\delta}}. \tag{4.34}$$

For the Gumbel copula, its lower tail dependence coefficient is zero whilst the upper coefficient is

$$\lambda_u(X_1, X_2) = 2 - 2^{\frac{1}{\delta}}. \tag{4.35}$$

[1] a joint survival copula is defined as $\bar{C}(u_1, u_2) = P(U_1 > u_1, U_2 > u_2)$

Chapter 5

Vine-Copula

In theory, construction of higher-dimensional copulas with more than two variables is fully feasible. In practice, however, when dealing with large financial data sets of which not all pairs of risk factors have the same dependence structure, and therefore inequivalent tail dependence; building higher-dimensional copulas is difficult and inflexible. This is due to the limited number of parameters of which most copulas contain, often only one. Consequently, the number of suitable and available parametric higher-dimensional copulas is rather scarce.

This chapter presents the method for modeling the dependence structure for multivariate data based on the *pair-copula construction* (PCC). A pair-wise approach is used since building higher dimensional copulas can be challenging and increasingly difficult with higher dimensions as aforementioned. Since there is a large number of bivariate copulas, the idea is to *decompose* a multivariate distribution into bivariate copulas as simple building blocks. The structure of the *pair-copula* decomposition is then illustrated by *vines*, a graphical model of storing the construction steps and dependence structure presented by Bedford and Cooke [2002]. This method of decomposing a multivariate distribution allows us to customize different pairs of risk factors with various bivariate copulas, while modeling the marginal distributions independent of the dependence structure.

The main references of this chapters are Aas et al. [2007], Brechmann [2010], Bedford and Cooke [2002], J.Dißmann et al. [2013] and Schirmacher and Schirmacher [2008].

5.1 Pair-Copula Constructions

The concept of pair-copula constructions is based on decomposing a n-dimensional joint density function f as

$$f(x_1, ..., x_n) = f(x_1)f(x_2|x_1)f(x_3|x_2, x_1)...f(x_n|x_{n-1}, ..., x_2, x_1). \tag{5.1}$$

Recall Equation (2.3) derived from Sklar's Theorem stating that the joint density function f can be expressed as a product of the marginal density functions and their corresponding copula densities.

$$f(x_1, ..., x_d) = c(F_1(x_1), ..., F_d(x_d))f_1(x_1) \times ... \times f_d(x_d).$$

In the bivariate case, the conditional density function for variable x_1 conditioned on variable x_2 can be expressed in terms of *pair-copula* density and a marginal density

$$\begin{aligned} f(x_1|x_2) &= \frac{f(x_1, x_2)}{f(x_1)} \\ &= c_{12}(F_1(x_1), F_2(x_2))f_2(x_2). \end{aligned} \tag{5.2}$$

Similarly for the third factor in equation (5.1) we can decompose the three-variable conditional density function as

$$\begin{aligned} f(x_3|x_2, x_1) &= c_{32|1}(F(x_3|x_1), F(x_2|x_1))f(x_3|x_1) \\ &= c_{32|1}(F(x_3|x_1), F(x_2|x_1))c_{31}(F_3(x_3), F_1(x_1))f_3(x_3), \end{aligned} \tag{5.3}$$

or

$$\begin{aligned} f(x_3|x_2, x_1) &= c_{31|2}(F(x_3|x_2), F(x_1|x_2))f(x_3|x_2) \\ &= c_{31|2}(F(x_3|x_2), F(x_1|x_2))c_{32}(F_3(x_3), F_2(x_2))f_3(x_3), \end{aligned} \tag{5.4}$$

where $c_{32|1}$ and $c_{31|2}$ are two different pair-copula densities. As shown above there are many possible decompositions of the same conditional density function. The number of decompositions increases as the dimension of random variables increases. Following the arguments above, the conditioned density of a variable x conditioned on a vector of variables $\boldsymbol{v}$ is given by

$$f(x|\boldsymbol{v}) = c_{xv_j|\boldsymbol{v}_{-j}}\left(F(x|\boldsymbol{v}_{-j}), F(v_j|\boldsymbol{v}_{-j})\right) f(x|\boldsymbol{v}_{-j}), \tag{5.5}$$

where $\boldsymbol{v}$ is a d-dimensional vector of variables . v_j denotes a randomly chosen component of $\boldsymbol{v}$ and v_{-j} is hence the $(d-1)$-dimensional vector without the jth component v_j. By iteration, the construction is straight forward. Moreover, when decomposing a joint density function into marginal densities and pair-copulas there exists numerous choices of conditional densities. As a result, we can have many different re-parametrisations given a unique factorisation which leads to a large number of possible pair-copula constructions.

In order to compute the density in Equation (5.5) we need an expression for the conditional distribution function $F(x|\boldsymbol{v})$. For every j, Joe [1996] has shown that

$$F(x|\boldsymbol{v}) = \frac{\partial C_{xv_j|\boldsymbol{v}_{-j}}\left(F(x|\boldsymbol{v}_{-j}), F(v_j|\boldsymbol{v}_{-j})\right)}{\partial F(v_j|\boldsymbol{v}_{-j})}, \tag{5.6}$$

where $C_{xv_j|\boldsymbol{v}_{-j}}$ is a bivariate copula distribution function. The calculation of these conditional distributions are recursive by nature. A special case where $\boldsymbol{v}$ only consists of one component simplifies Equation (5.6) to

$$F(x|v) = \frac{\partial C_{xv}(F(x), F(v))}{\partial F(v)}. \tag{5.7}$$

Recall from the definition of a copula that it is a function of uniformly distributed margins. If x and v are uniformly distributed, equation (5.7) then simplifies to

$$F(x|v) = \frac{\partial C_{xv}(x, v)}{\partial v}. \tag{5.8}$$

The representation (5.8) proves to be very useful for writing the simulation and likelihood evaluation algorithms. We hence define a function $h(\cdot)$ as

$$h(x, v, \Theta) = F(x|v) = \frac{\partial C_{xv}(x, v, \Theta)}{\partial v}, \tag{5.9}$$

where v denotes the conditioning variable and Θ the parameters in copula C that represents the joint distribution function of x and v. The inverse of the h-function with respect to the first variable is defined as h^{-1}, also known as the inverse conditional distribution function. See Appendix A for detailed representations of h-function and h^{-1} for the bivariate Gaussian, Student's t, Clayton and Gumbel copula.

5.2 Regular Vines

As aforementioned there is a vast range of possibilities for pair-copula constructions (PCC). For example, for a 5-dimensional density function there are 240 different ways of decomposition. Therefore, it is necessary to have a tool for organising the large number of pair-copula constructions for high dimensional distributions. *Regular vines* first introduced by Bedford and Cooke (2001) is a such graphical tool that conveniently depicts the dependence structure of which conditional specifications are made for the joint distribution.

Briefly speaking, a regular vine of d-dimensions is a nested set of $d-1$ trees and $\frac{d(d-1)}{2}$ edges such that the nodes of tree $i+1$ are the edges of tree i and two nodes of tree $i+1$ are connected by an edge only if they share a common node in tree i. A formal definition of regular vines is given as follows Brechmann [2010]:

Definition 5.1 (**Regular vine**). *$\mathcal{V}$ is a regular vine on d elements if*
(i) $\mathcal{V} = (T_1, ..., T_d)$.
(ii) $T_1 = (N_1, E_1)$ is a tree with nodes $N_1 = \{1, ..., d\}$. For $i = 2, ..., n-1, T_i = (N_i, E_i)$ is a tree with nodes $N_i = E_{i-1}$.
*(iii) (**proximity condition**) For $i = 2, ..., n-1$, if $\{a, b\} \in E_i$, where $a = \{a_1, a_2\}$ and $b = \{b_1, b_2\}$, then exactly one of the a_j's equals one of the b_j's.*

The *proximity condition* simply expresses that two nodes are joined by a common edge in tree T_i only if the corresponding edges in tree T_{i-1} share a common node.

We will now concentrate on one special kind of regular vines, namely the *canonical* vine (C-vine). Note that there are also other types of vines that illustrate specific ways of density decomposition, such as the *drawable* vine (D-vine). In the context of portfolio optimization, the scope of this thesis considers only the canonical vine.

5.2.1 C-vine

Intuitively, the canonical vine describes a scenario where one variable acts as the key component and is associated with all other variables in the group. Formally,

Definition 5.2 (**C-vine**). *A regular vine is called a C-vine if each tree $T_i, i = 1, ..., d-1$ has a unique node of degree $d - i$.*

By degree we mean the number of edges connected to the corresponding node. Each tree in a C-vine is a star with one unique node that connects to all other nodes. The key variable that is known or analyzed to govern the dependence structure among the variables in a data set, is located as the root node at level 1 of the nested set of trees. The key variable of tree i becomes the conditioning variable in tree $i+1$. There are d possible conditioning sets in T_2, $d-1$ possibilities in T_3 and so on until 1 different conditioning sets at the last level of tree T_{d-1}. As result, there are $d(d-1)(d-2)\cdots 3 = \frac{d!}{2}$ different canonical vines on d elements.

Figure 5.1 gives an example of a canonical vine for a 6-dimensional joint density function where variable 6 plays the key role in the dependence structure. Note that the edges of each tree identifies a common conditioning variable. It consists of 5 trees $T_i, i = 1, .., 5$ where the nodes in T_1 represents the marginal density functions $f_1, f_2, ..., f_6$ and each edge in T_i corresponds to a pair-copula density function with the subscript labelled. For a 6-dimensional distribution there are $6! = 720$ permutations of $x_1, x_2, ..., x_6$ but only $\dfrac{6!}{2} = 360$ of these are different.

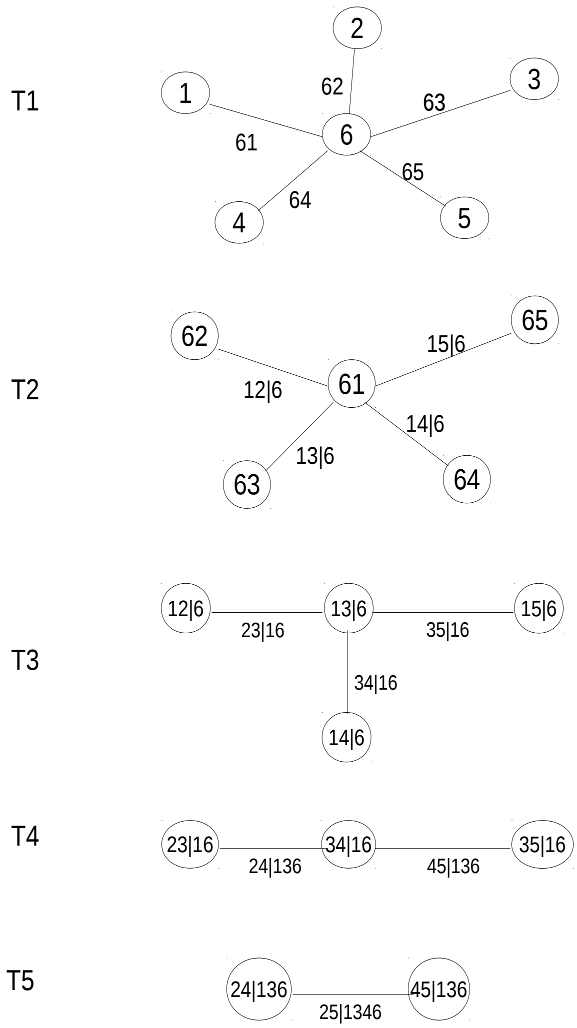

Figure 5.1: A C-vine representation with 6 variables, 5 trees and 15 edges.

The joint density function $f(x_1, ..., x_6)$ corresponding to the C-vine decomposition in Figure 5.1 can be decomposed as following:

$$
\begin{aligned}
f(x_1, x_2, x_3, x_4, x_5, x_6) = f_1(x_1)f_2(x_2)f_3(x_3)f_4(x_4)f_5(x_5)f_6(x_6) \\
\cdot c_{61}(F_6(x_6), F_1(x_1))c_{62}(F_6(x_6), F_2(x_2))c_{63}(F_6(x_6), F_3(x_3)) \\
\cdot c_{64}(F_6(x_6), F_4(x_4))c_{65}(F_6(x_6), F_5(x_5)) \\
\cdot c_{12|6}(F(x_1|x_6), F(x_2|x_6))c_{13|6}(F(x_1|x_6), F(x_3|x_6)) \\
\cdot c_{14|6}(F(x_1|x_6), F(x_4|x_6))c_{15|6}(F(x_1|x_6), F(x_5|x_6)) \\
\cdot c_{23|16}(F(x_2|x_1, x_6), F(x_3|x_1, x_6))c_{35|16}(F(x_3|x_1, x_6), F(x_5|x_1, x_6)) \\
\cdot c_{34|16}(F(x_3|x_1, x_6), F(x_4|x_1, x_6)) \\
\cdot c_{24|136}(F(x_2|x_1, x_3, x_6), F(x_4|x_1, x_3, x_6)) \\
\cdot c_{45|136}(F(x_4|x_1, x_3, x_6), F(x_5|x_1, x_3, x_6)) \\
\cdot c_{25|1346}(F(x_2|x_1, x_3, x_4, x_6), F(x_5|x_1, x_3, x_4, x_6)).
\end{aligned}
$$

In general, the joint density function $f(x_1, ..., x_d)$ with a C-vine decomposition can be written as

$$\prod_{k=1}^{d} f(x_k) \prod_{j=1}^{d-1} \prod_{i=1}^{d-j} c_{j,j+i|1,...,j-1} \left(F(x_j|x_1, ..., x_{j-1}), F(x_{j+1}|x_1, ..., x_{j-1}) \right), \tag{5.10}$$

where index i passes through the edges in each level of trees and j specifies the level of the tree.

5.3 Simulation

Simulation from the canonical vine is based on the following general algorithm for sampling n dependent uniform $[0, 1]$ variables. First, sample n independent uniform random numbers $u_i \in [0, 1]$ and then set

$$
\begin{aligned}
\mathrm{x}_1 &= u_1 \\
x_2 &= F^{-1}_{2|1}(u_2|x_1) \\
x_3 &= F^{-1}_{3|1,2}(u_3|x_1, x_2) \\
&. \\
&. \\
&. \\
x_{n-1} &= F^{-1}_{n-1|1,2,...,n-2}(u_{n-1}|x_1, x_2, .., x_{n-2}) \\
x_n &= F^{-1}_{n|1,2,...,n-1}(u_n|x_1, x_2, .., x_{n-1}).
\end{aligned}
$$

The implementation of this algorithm requires calculation of the marginal conditional distribution $F_{x|v}(x|v)$ and its inverse $F_{x|v}^{-1}(x|v)$. Equation (5.6) will allow us to recursively compute the conditional distribution and the inverse can then be obtained by using $h(\cdot)$ defined in equation (5.9). Then by choosing the variable v_j in (5.6) to be the *last* conditioning variable it will result in a canonical vine,

$$\mathrm{F}(\mathrm{x}_j|x_1,...,x_{j-1}) = \frac{\partial C_{j,j-1|1,2,...,j-2}\left(F(x_j|x_1,...,x_{j-2}), F(x_{j-1}|x_1,...,x_{j-2})\right)}{\partial F(x_{j-1}|x_1,...,x_{j-2})}. \tag{5.11}$$

We use Algorithm 1 from Aas et al. [2007] as follows to generate a sample $x_1, ..., x_n$ from a canonical vine:

Sample $w_1, ..., w_n$ independently on $U(0,1)$
$x_1 = v_{1,1} = w_1$
for $i \leftarrow 2, ..., n$
  $v_{i,1} = w_i$
  for $k \leftarrow i-1, i-2, ..., 1$
    $v_{i,1} = h^{-1}(v_{i,1}, v_{k,k}, \Theta_{k,i-k})$
  end for
  $x_1 = v_{1,1}$
  if $i == n$ **then**
    stop
  end if
  for $j \leftarrow 1, ..., i-1$
    $v_{i,j+1} = h(v_{i,j}, v_{j,j}, \Theta_{j,i-j})$
  end for
end for

5.4 Selection of Regular Vine Tree Structure

The construction of regular vines requires that we select pairs of variables that are to be linked with a copula. A sequential construction method described by Brechmann [2010] is used. That is, we construct one tree at a time starting from the top. The objective is to find a structure that captures as much dependence as possible in the first tree, T_1. The selection of variable pairs will also influence the selection of the corresponding copula family. In other words, modeling of the dependence structure between random variables with high dependencies in the first tree will have great impact on the model fit. We proceed to select the pairs of variables that display the strongest pairwise dependencies based on *Kendall's tau* as a dependence measure. We compute the empirical Kendall's tau $\hat{\tau}_{i,j}$ for each pair of variables $(i, j)_{1\leq i<j\leq d}$ and select the spanning tree that maximizes the sum of absolute value of empirical Kendall's taus:

$$\max \sum_{\text{edges} e_{ij} \text{in spanning tree}} |\hat{\tau}_{i,j}|.$$

For a C-vine tree structure, we select the node with strongest dependencies to all other nodes to be the root node. In practice, the root node can be identified by summing up the columns in the empirical Kendall's tau matrix and choosing the one with maximum sum.

5.5 Selection of Copula Family

Besides selecting the tree structure we also need to select a copula family for each pair of variables. In the context of this thesis we consider the following copula families:

- Gaussian (no tail dependence)
- Student's t (lower and upper tail dependence)
- Clayton (lower tail dependence)
- Gumbel (upper tail dependence).

The Student's t-copula will not be selected if the maximum likelihood estimation gives a degrees of freedom greater than 30, as the Student-t copula then can simply be replaced by the Gaussian.

To determine which copula family that gives the best fit we use the *Akaike informasjon criterion* AIC introduced by Akaike (1973) as the selection criteria. It is defined as

$$AIC := -2\sum_{i=1}^{d} \log f(x_i|\hat{\mathbf{\Theta}}) + 2k, \tag{5.12}$$

where $\hat{\mathbf{\Theta}}$ are the maximum likelihood estimates and k denotes the number of parameters in the model. Note that for AIC to be valid, the number of observations needs to be sufficiently large compared to the number of parameters, as the AIC penalizes the log-likehood as the number of parameter increases. For a given specific copula density c, the AIC of the chosen pairs of variable is given by

$$AIC := -2\sum_{i=1}^{d} \log c(u_{i1}, u_{i2}|\hat{\mathbf{\Theta}}) + 2k. \tag{5.13}$$

We compute the AIC's for each possible family for each pair of variables and then choose the copula family that generates the smallest AIC.

5.6 Parameter Estimation

Once the tree structure is determined using vines and an appropriate bivariate copula family is selected for each pair of variables, the next step is to estimate the copula parameters via maximum likelihood. Depending on the selected copula family there is at least one parameter to be estimated.

Suppose that we have $\mathbf{x}_{i,t}$ observations for variable $i = 1, ..., d$ at the t-th time point where $t = 1, ..., T$. The likelihood function for a canonical vine of d-dimension is,

$$\begin{aligned} L = & \prod_{t=1}^{T} \prod_{k=1}^{d-1} f(x_{k,t}) \\ & \cdot \prod_{j=1}^{d-1} \prod_{i=1}^{d-1} c_{j,j+i|1,...,j-1} \left(F(x_{j,t}|x_{1,t}, ..., x_{j-1,t}), F(x_{j+1,t}|x_{1,t}, ..., x_{j-1,t}) \right) \end{aligned} \tag{5.14}$$

and its corresponding log-likelihood function assuming uniform margins ($f(x_{t,k}) = 1$) is,

$$\begin{aligned} l = & \sum_{t=1}^{T} \sum_{k=1}^{d-1} log(f(x_{k,t})) \\ & + \sum_{j=1}^{d-1} \sum_{i=1}^{d-1} log \left[c_{j,j+i|1,...,j-1} \left(F(x_{j,t}|x_{1,t}, ..., x_{j-1,t}), F(x_{j+1,t}|x_{1,t}, ..., x_{j-1,t}) \right) \right] \\ & = \sum_{t=1}^{T} \sum_{j=1}^{d-1} \sum_{i=1}^{d-1} log \left[c_{j,j+i|1,...,j-1} \left(F(x_{j,t}|x_{1,t}, ..., x_{j-1,t}), F(x_{j+1,t}|x_{1,t}, ..., x_{j-1,t}) \right) \right]. \end{aligned} \tag{5.15}$$

In practice, the likelihood for a canonical vine can be evaluated according to Algorithm 3 from Aas et al. [2007] as follows:

log-likelihood = 0
for $i \leftarrow 1, ..., n$
  $\boldsymbol{v}_{0,1} = \boldsymbol{x}_i$
end for
for $j \leftarrow 1, ..., n-1$
  for $i \leftarrow 1, ..., n-j$
    log-likelihood = log-likelihood + $L(\boldsymbol{v}_{j-1,1}, \boldsymbol{v}_{j-1,i+1}, \Theta_{j,i})$
  end for
  if $j == n-1$ **then**
    stop
  end if
  for $i \leftarrow 1, ..., n-j$
    $\boldsymbol{v}_{j,i} = h(\boldsymbol{v}_{j-1,i+1}, \boldsymbol{v}_{j-1,1}, \Theta_{j,i})$
  end for

end for

In general, numerical optimization of the likelihood is needed as estimators on closed form cannot be obtained. One method to determine the starting values of the log-likelihood function is to sequentially estimate the parameters in the pair-copula construction tree by tree. Such estimations are straightforward and non-time consuming as we are only dealing with two dimensions at a time. We proceed according to the following algorithm:

1. Estimate the parameters of the copulas in tree 1 from the original data.

2. Compute observations in tree 2, that is the conditional distribution functions using the copula parameters estimated from tree 1 and h-functions.

3. Estimate the copulas parameters in tree 2 using the observations computed in step 2.

4. Compute observations in tree 3 using the copula parameters estimated from tree 3 and h-functions.

5. Estimate the copulas parameters in tree 3 using the observations computed in step 4.

6. Continue until the last tree is reached.

Chapter 6

PCC-GARCH-CVaR model

In this chapter we will show how to apply the mean-CVaR portfolio optimization by combining the use of the pair-copula construction (PCC) with univariate GARCH models. In order to calculate the portfolio weights we proceed as follows:

1. *estimate* the parameters of the PCC-GARCH,
2. *simulate* daily asset returns scenarios from this model,
3. use the simulated data as inputs when *optimizing* portfolio weights by minimizing CVaR for a given expected return.

6.1 Estimation

Consider the following model for daily log-returns $r_{i,t}$ for asset $i = 1, ..., d$ at day $t = 1, ..., n$:

$$r_{i,t} = \mu_i + \epsilon_{i,t}. \tag{6.1}$$

Here μ_i is the sample mean and $\epsilon_{i,t}$ represents the error term following a GARCH(1,1) process given as

$$\begin{aligned} \epsilon_{i,t} &= \sigma_{i,t} z_{i,t} \\ \sigma_{i,t}^2 &= \alpha_{i,0} + \alpha_{i,1}\epsilon_{i,t-1}^2 + \beta_{i,1}\sigma_{i,t-1}^2 \end{aligned} \tag{6.2}$$

where $z_{i,t}$ is iid white noise with mean 0 and variance 1. In the context of this thesis, $z_{i,t}$ is distributed according to Generalized Error Distribution (GED), Student's t distribution or skew Student's t-distribution depending on the dataset. We use the MLE discussed in Section 3.4 to obtain estimates of $\hat{\alpha}_{i,0}$, $\hat{\alpha}_{i,1}$ and $\hat{\beta}_{i,1}$ for all assets i. At this stage we also perform a Goodness-of-Fit(GoF) test to check how well the GARCH(1,1) model captures the volatility clustering of the process. This can be evaluated by graphical inspection of the autocorrelation function of the absolute (or squared) value of the standardized residuals given as

$$\hat{z}_{i,t} = \frac{\hat{\epsilon}_{i,t}}{\hat{\sigma}_{i,t}}. \tag{6.3}$$

Little or no autocorrelation implies an adequately good fit of the GARCH(1,1) model.

Prior to estimating the parameters of the *pair-copula construction* it is necessary to transform the margins onto copula scale so that they are uniformly distributed. In practice, the true distribution function of the margins F_i's are often unknown. Hence, it is common to use the *empirical distribution function* to transform the standardized residuals:

$$\hat{F}_i(\hat{z}) = \frac{1}{n} \sum_{t=1}^{n} \mathbf{1}_{\{\hat{z}_{i,t} \leq \hat{z}\}} \tag{6.4}$$

where $\mathbf{1}_{\{\hat{z}_{i,t} \leq \hat{z}\}}$ is the indicator function. The uniform margins $\boldsymbol{U_{i,t}}$ for $i = 1, ..., d$ and $t = 1, ..., n$ can then be constructed as follows, see Brechmann [2010]:

$$\mathrm{U}_{i,t} = \frac{rank(\hat{z}_{i,t})}{n+1} = \frac{n}{n+1} \hat{F}_i(\hat{z}_{i,t}). \tag{6.5}$$

We then fit the pair-copula-construction to the resulting uniform variables. In this thesis we assume that the dependence structure is modelled by a canonical vine (C-Vine). We first examine Kendall's tau discussed in Section 5.4 to select the root node that represent the asset of the strongest pairwise dependence in T1 of the C-vine.

Given the selected C-vine tree structure, we then proceed to choose the most appropriate bivariate copulas from different copula families using AIC as selection criterion (see Section 5.5). This thesis particularly focuses on the Gaussian, Student's t, Clayton and Gumbel copula due to their distinctive properties. While the two elliptical copulas are symmetric and the two Archimedean copulas are asymmetric, the Gaussian copula exhibits no tail dependence; the Student's t-copula captures both lower and upper tail dependence; the Clayton copula is able to parameterize lower tail dependence, whilst the Gumbel copula the upper tail dependence. The parameters in the bivariate copulas are estimated using the maximum likelihood estimation presented in Section 5.6.

6.2 Simulation

We use the following procedure to generate asset returns: For each simulation $n = 1, ..., 5000$:

- Generate samples of $u_i, i = 1, ..., d$ from the C-vine,
- For each i, convert u_i to GED/Student's t/Skew Student's t distributed samples z_i using the quantile function, that is, the inverse of the corresponding distribution functions.

- Compute the standard deviation $\sigma_{i,t}$ using the estimated GARCH-model,
- Finally, determine the daily log-returns as $r_{i,t}^{sim} = \mu_i + \sigma_{i,t} z_i$.

6.3 CVaR Optimization

When performing portfolio estimation, we consider two different scenarios of asset returns.

Scenario 1 It is common practice that the investor set the expected returns manually based on what they think about the asset return in the future. Hence, for each asset i, we adjust the simulated returns as follows: we subtract the historical mean and add an annual hypothetical expected return adjusted with respect to the number of trading days in a year, which is 250:

$$\mathrm{r}_{i,t}^{S_1} = r_{i,t}^{sim} - \mu_i + \frac{C_i}{250 \times 100} \tag{6.6}$$

Here μ_i is the mean historical return and C_i is the hypothetical expected return. We then proceed to CVaR optimization as presented in Section 2.3 where $\boldsymbol{r} = r_{i,t}^{S_1}$.

Scenario 2 Alternatively, we use the asset returns from the previous day as the expected returns. Intuitively, it means that we think yesterday's market price would be the best guess for today's market performance. We implement the CVaR optimization with $\boldsymbol{r} = r_{i,t}^{S_2}$ as inputs expressed as follows:

$$\mathrm{r}_{i,t}^{S_2} = r_{i,t}^{sim} - \mu_i + r_{i,t-1}. \tag{6.7}$$

Here $r_{i,t-1}$ is the previous day's log-return for asset i and its value is obtained directly from the data set.

Chapter 7

Empirical studies and analysis

The empirical studies and analysis employ a sample data extending from March 27, 2005 to March 26, 2008, in total 1108 trading days. It consists of daily log-returns from the following six indexes:

- BRIX: Norwegian bond index,
- WGBI: World citigroup bond index,
- ST2X: Government Bond Index, fix modified duration of 0.50 years,
- MSCI: Morgan Stanley World Index,
- OSEBX: Oslo Stock Exchange main index,
- OSE4040: Oslo Stock Exchange Real estate index.

We use a similar 'rolling window' approach as Aas and Low [2012]. The procedure is outlined as follows and illustrated in Figure 7.1:

- Optimization 1: Use day 1 to day 750 to estimate the PCC-GARCH model and determine the portfolio weights for day 751.
- Optimization 2: Use day 2 to day 751 to estimate the PCC-GARCH model and determine the portfolio weights for day 752.
- $\vdots$
- Optimization 358: Use day 358 to day 1107 to estimate the PCC-GARCH model and determine the portfolio weights for day 1108.

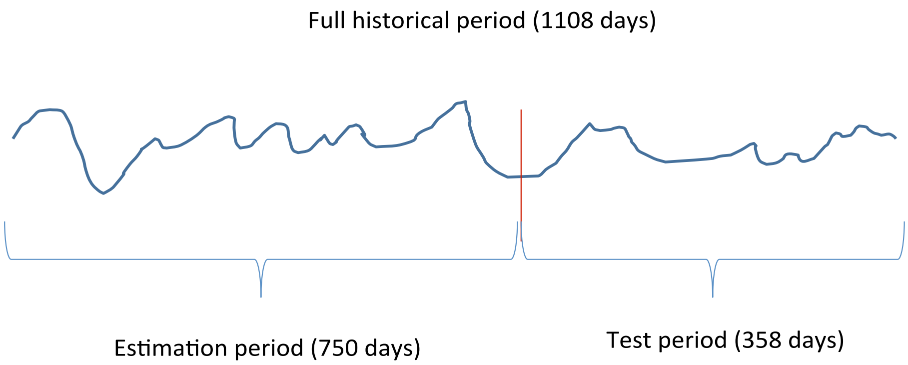

Figure 7.1: Optimization with respect to time horizon

For each of the days 751 to 1107 we simulate 5000 return scenarios from the estimated model as described in Section 6.2 and compute portfolio weights for 358 consecutive days using $\alpha = 0.99$ assuming no transaction costs and requiring an expected annual return of at least 6%.

The procedure of data analysis, simulation, estimation and optimization is carried out using R Programming language in addition to functions from packages `fGarch`, `VineCopula` and `fPortfolio`. See Appendix C for a summary of the R codes.

7.1 Data and descriptive statistics

We first perform a preliminary analysis of the whole data sample, illustrated in Figure 7.2.

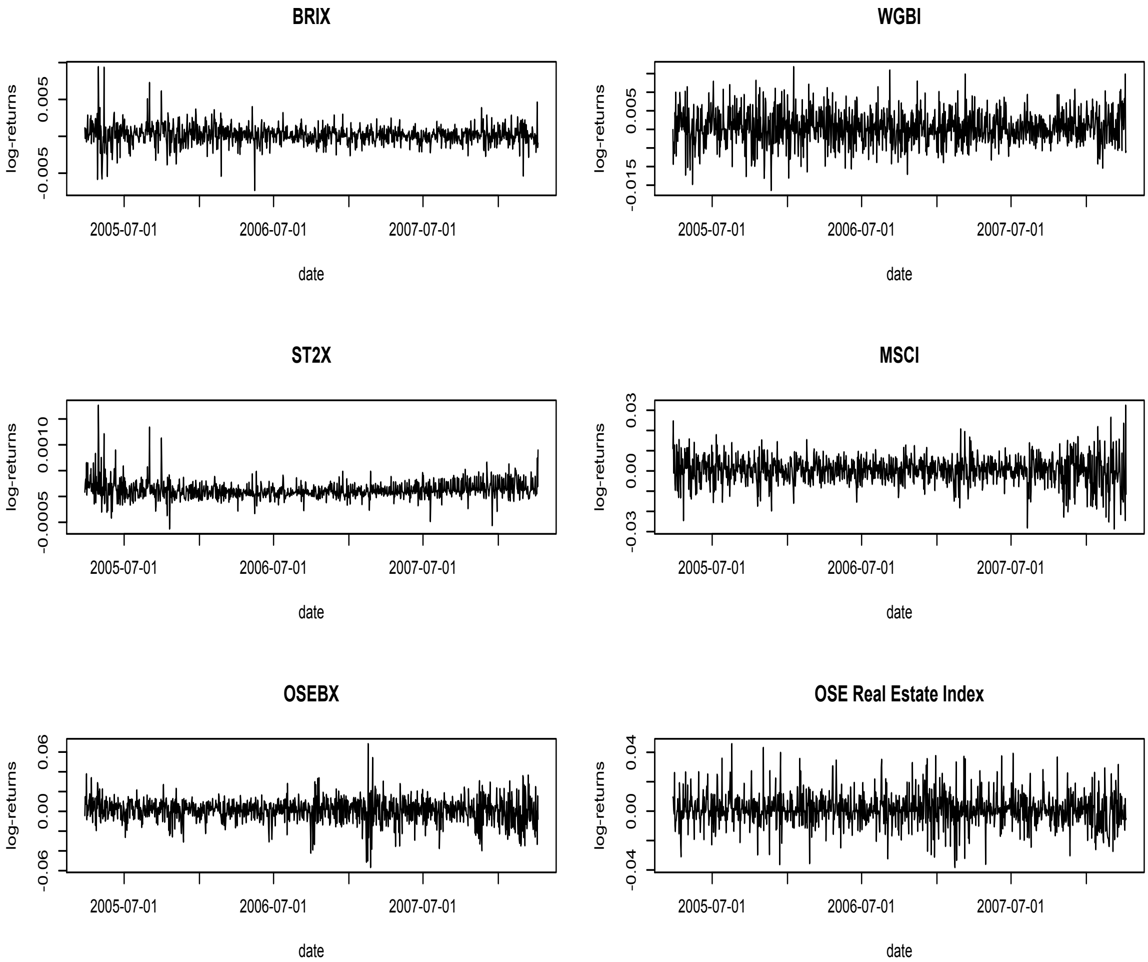

Figure 7.2: Daily log return series of the 6 indices during the period from from 27.03.2005 to 26.03.2008.

As shown in Table 7.1, both the kurtosis of each index, which is the excess value compared to the normal distribution, and the skewness of each index deviates from 0. In particular, ST2X with skewness 1.95 and kurtosis 13.69 departs significantly from normality. This indicates that all six return series are leptokurtic.

Furthermore, we carried out an informal test for normality by examining normal qq-plots fitted to the return series. As shown in Figure 7.3, there is significant deviations from the straight line in the tails for all six return series. This is another indication that the asset distributions do not follow the normal distribution but are more heavy-tailed.

Table 7.1: Preliminary descriptive statistics of the daily log-returns

Index	Minimum	Maximum	Mean	Std.dev	Skewness	Kurtosis
BRIX	-0.0073382667	0.009435948	0.0001852427	2.0780641	0.32508067	7.7023594
WGBI	-0.0164103320	0.016844566	0.0004136735	7.0616510	0.07951543	0.6456767
ST2X	-0.0006285101	0.001765514	0.0001217093	0.2732148	1.95201614	13.6884738
MSCI	-0.0286703647	0.032471356	0.0003167322	10.7862910	-0.27184690	1.8929980
OSEBX	-0.0567249498	0.068339285	0.0009464953	19.3306520	-0.34219546	2.4839024
OSE4040	-0.0382786963	0.045792428	0.0010606256	17.3779398	0.25757974	2.0241062

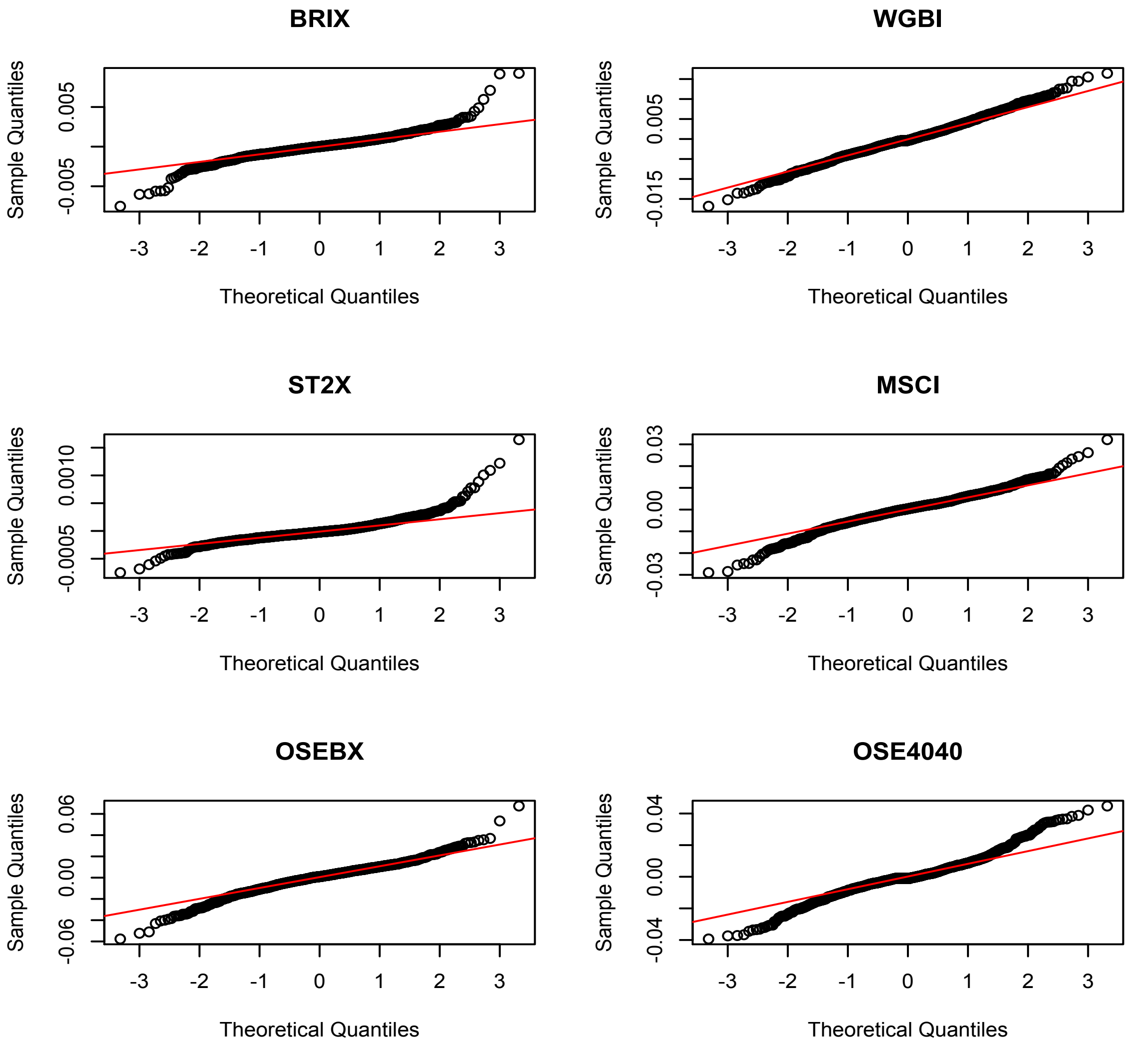

Figure 7.3: Normal QQ-plots of the daily log returns of all indices

Since the data set consists of financial time series, the returns series are expected to display volatility clustering, periods of high volatility and low volatility. Figure 7.3 shows the autocorrelation function of the absolute values of returns. Autocorrelation describes the correlation between observations at two different times. If the returns series are independent over time, the absolute values of the returns should be uncorrelated. That is, less than 5% of the values will exceed the dotted lines in the ACF-plots, which is 1.5 for a period of 30. The presence of volatility clustering can therefore be detected by a strong autocorrelation in the absolute values of the returns. We see that the autocorrelation functions in Figure 7.4 have positive values for a relatively large number of lags. This is an indication of volatility clustering and that a GARCH model might fit the data well.

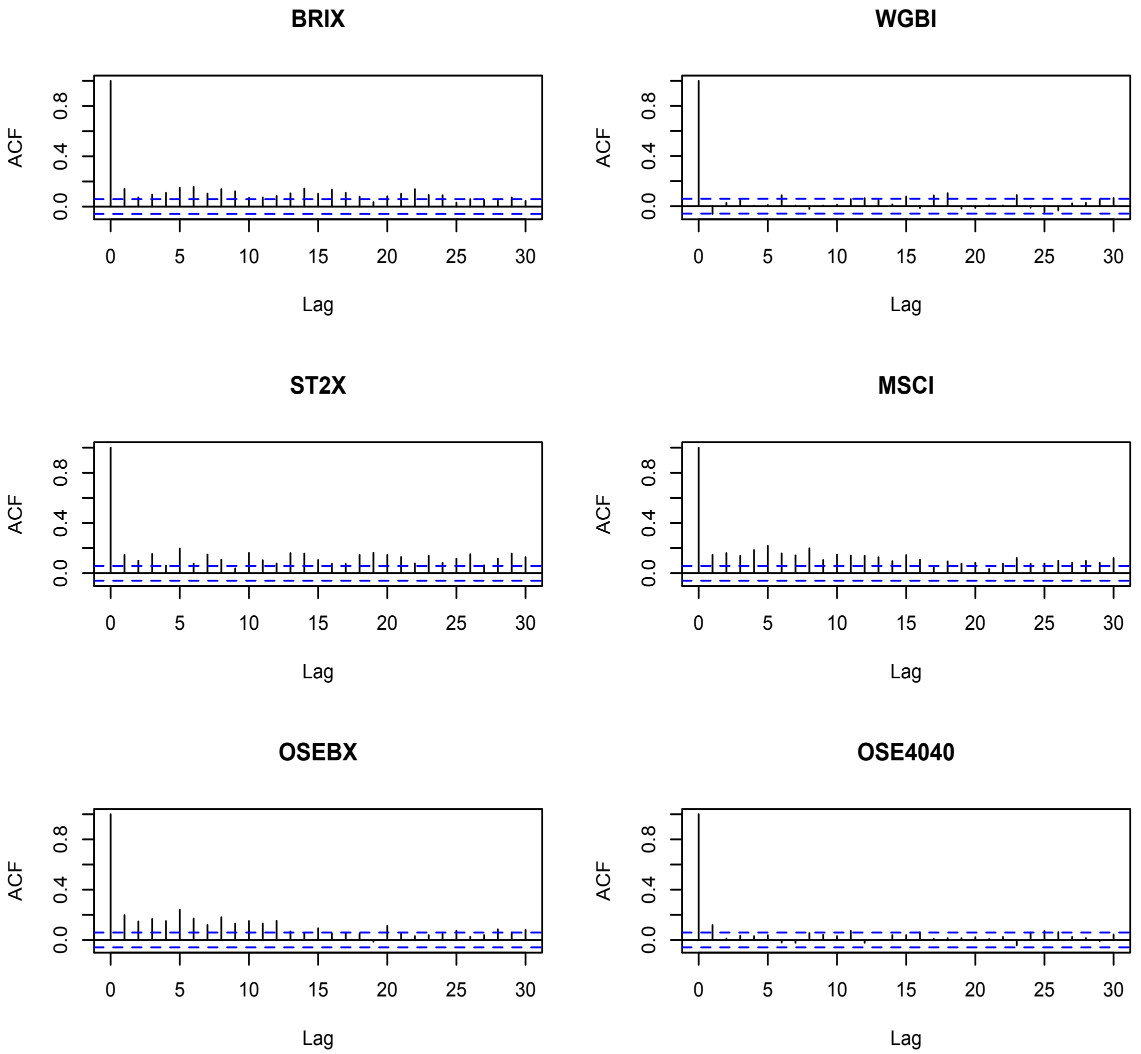

Figure 7.4: ACF of the absolute values of the daily log returns of all indices

7.2 GARCH application

GARCH(1,1) models are fitted to the six return series, each with non-Gaussian error distribution. Tables 7.2-7.7 present the results of parameter estimations from selected estimation periods. Naturally the mean return for each asset have a large variation, whereas the estimated parameters $\hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta}_1$, shape and skewness has little deviation in comparison. In general, the sum of the parameter estimates $\hat{\alpha}_1 + \hat{\beta}_1$ approaches to 1 towards the end of estimation period. Notably, this is when we enter the last quarter of 2007 and the first of 2008, where the financial crisis started to hit the global financial markets.

Table 7.2: Estimated parameters for the Norwegian bond index (BRIX) using the GED as error distribution in the GARCH (1,1) model

Estimation period	μ	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	shape
1 - 750	-5.788e-19	4.755e-06	0.054	0.919	1.164
50 - 799	-7.105e-19	3.487e-08	0.011	0.987	1.235
100 - 849	1.009e-18	5.438e-07	0.016	0.979	1.219
150 - 899	1.631e-18	2.772e-07	0.012	0.984	1.267
200 - 949	-2.341e-19	6.156e-07	0.017	0.976	1.359
250 - 999	3.936e-19	1.514e-06	0.0199	0.965	1.263
300 - 1049	-1.380e-18	2.148e-06	0.019	0.959	1.254
358 - 1107	-1.04E-18	1.77E-06	0.016	0.967	1.206

Table 7.3: Estimated parameters for the World citigroup bond index (WGBI) using the GED as error distribution in the GARCH (1,1) model

Estimation period	μ	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	shape
1 - 750	2.895e-18	0.002	1e-08	0.201	1.444
50 - 799	1.055e-18	0.002	1e-08	1e-08	1.496
100 - 849	8.829e-18	0.0006	1e-08	0.688	1.429
150 - 899	-1.628e-18	1.992e-09	0.0102	0.989	1.534
200 - 949	-5.8395e-18	1.894e-09	0.010	0.988	1.524
250 - 999	-1.091e-18	1.037e-06	0.0085	0.990	1.535
300 - 1049	-4.932e-19	0.001	1e-08	0.292	1.569
358 - 1107	-4.75E-18	1.22E-05	0.0216	0.972	1.666

Table 7.4: Estimated parameters for the Government Bond Index (ST2X) using the Skew Student's t as error distribution in the GARCH (1,1) model.

Estimation period	μ	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	skew	shape
1 - 750	5.297e-18	2.707e-06	0.021	0.967	1.310	3.635
50 - 799	1.253e-18	1.360e-06	0.017	0.974	1.207	4.179
100 - 849	2.546e-18	1.536e-06	0.011	0.979	1.193	4.057
150 - 899	6.732e-19	1.374e-06	0.009	0.982	1.198	3.801
200 - 949	3.218e-18	1.737e-05	0.066	0.828	1.182	4.053
250 - 999	-3.842e-18	1.388e-05	0.039	0.885	1.223	3.691
300 - 1047	-3.871e-18	1.688e-10	0.010	0.992	1.244	3.861
358 - 1107	1.53E-18	8.73E-06	0.036	0.934	1.319	3.179

Table 7.5: Estimated parameters for the Morgan Stanley World Index (MSCI) using the Student's t as error distribution in the GARCH (1,1) model

Estimation period	μ	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	shape
1 - 750	2.704e-19	1.365e-06	0.064	0.898	10
50 - 799	1.796e-19	2.069e-06	0.062	0.872	10
100 - 849	-2.566e-19	1.951e-06	0.059	0.874	10
150 - 899	3.150e-19	2.275e-06	0.067	0.855	9.802
200 - 949	5.619e-19	2.245e-06	0.066	0.858	9.901
250 - 999	1.954e-19	1.596e-06	0.076	0.875	10
300 - 1049	1.091e-18	1.231e-06	0.083	0.885	10
358 - 1107	-3.78E-19	6.86E-07	0.085	0.905	10

Table 7.6: Estimated parameters for the Oslo Stock Exchange main index (OSEBX) using the Student's t as error distribution in the GARCH (1,1) model

Estimation period	μ	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	shape
1 - 750	1.330e-18	7.445e-06	0.108	0.826	8.540
50 - 799	2.637e-19	7.878e-06	0.115	0.818	9.166
100 - 849	2.315e-18	8.076e-06	0.112	0.820	8.562
150 - 899	-8.343e-19	9.606e-06	0.124	0.799	7.797
200 - 949	-1.382e-19	9.259e-06	0.117	0.808	7.824
250 - 999	1.015e-18	9.345e-06	0.141	0.790	8.481
300 - 1049	4.659e-19	9.460e-06	0.141	0.796	9.301
358 - 1107	-5.26E-19	8.99E-06	0.145	0.808	10

Table 7.7: Estimated parameters for the Oslo Stock Excange Real estate index (OSE4040) using the Student's t as error distribution in the GARCH (1,1) model

Estimation period	μ	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	shape
1 - 750	1.394e-18	0.00017	0.298	1e-08	2.580
50 - 799	2.179e-18	0.00018	0.299	1e-08	2.505
100 - 849	-1.202e-18	0.00019	0.332	1e-08	2.456
150 - 899	-9.480e-19	0.00015	0.278	0.072	2.599
200 - 949	-8.902e-20	0.00014	0.274	0.177	2.549
250 - 999	-4.026e-19	0.00010	0.177	0.288	2.789
300 - 1049	-1.470e-18	0.00011	0.176	0.201	2.929
358 - 1107	-8.12E-19	7.05E-06	0.0554	0.912	3.198

For the estimation period that start at day 200 we also present a more detailed analysis. A graphical inspection of goodness-of-fit is carried out by examining the ACFs of the squared standardized residuals and as shown in Figures 7.5-10, there are no autocorrelation. Notably, ST2X and OSE4040 display a high level of volatility for the selected estimation period. This consequently has an impact on their proportion held in the optimal portfolio. (See Section 7.4)

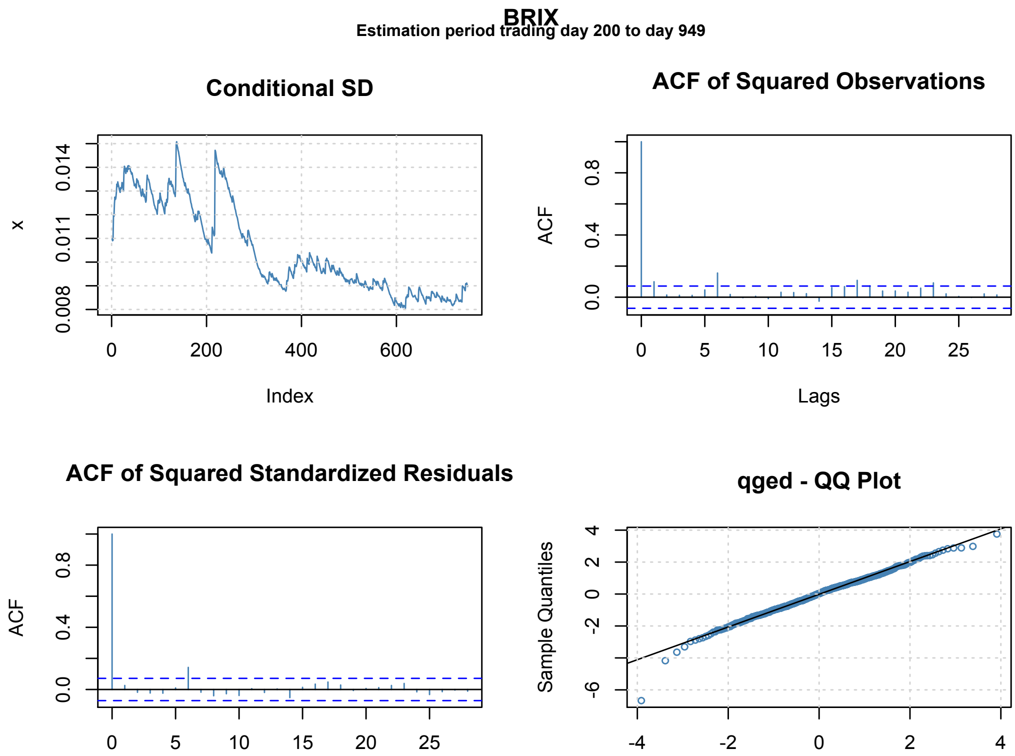

Figure 7.5: Selected plots from GARCH(1,1) model fitted for BRIX

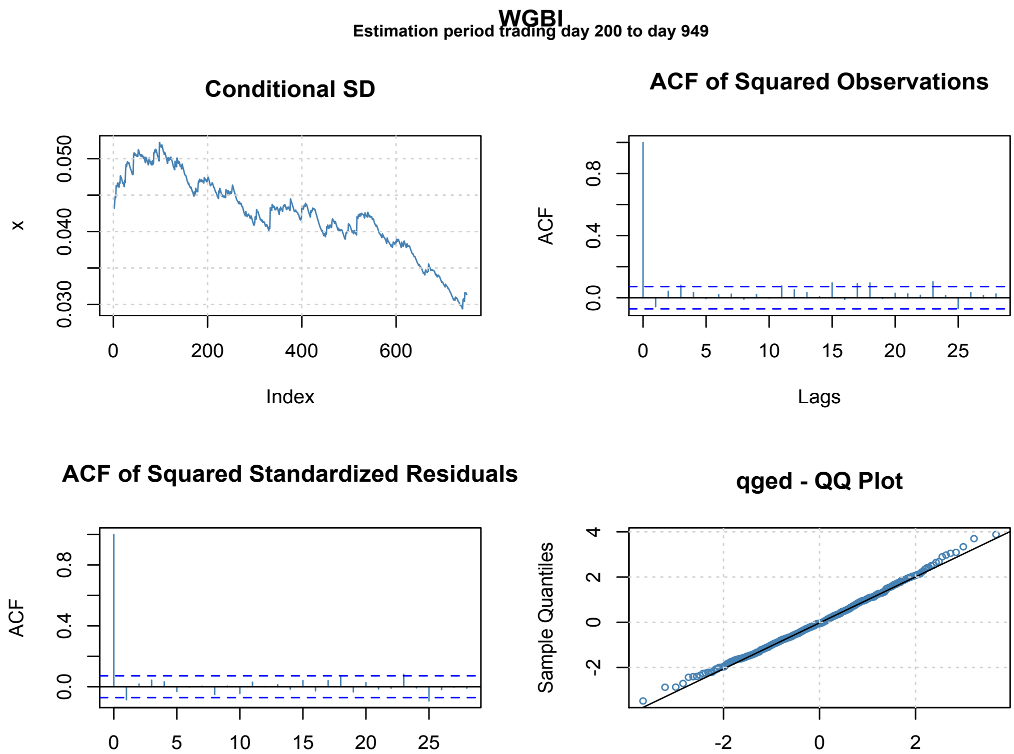

Figure 7.6: Selected plots from GARCH(1,1) model fitted for WGBI

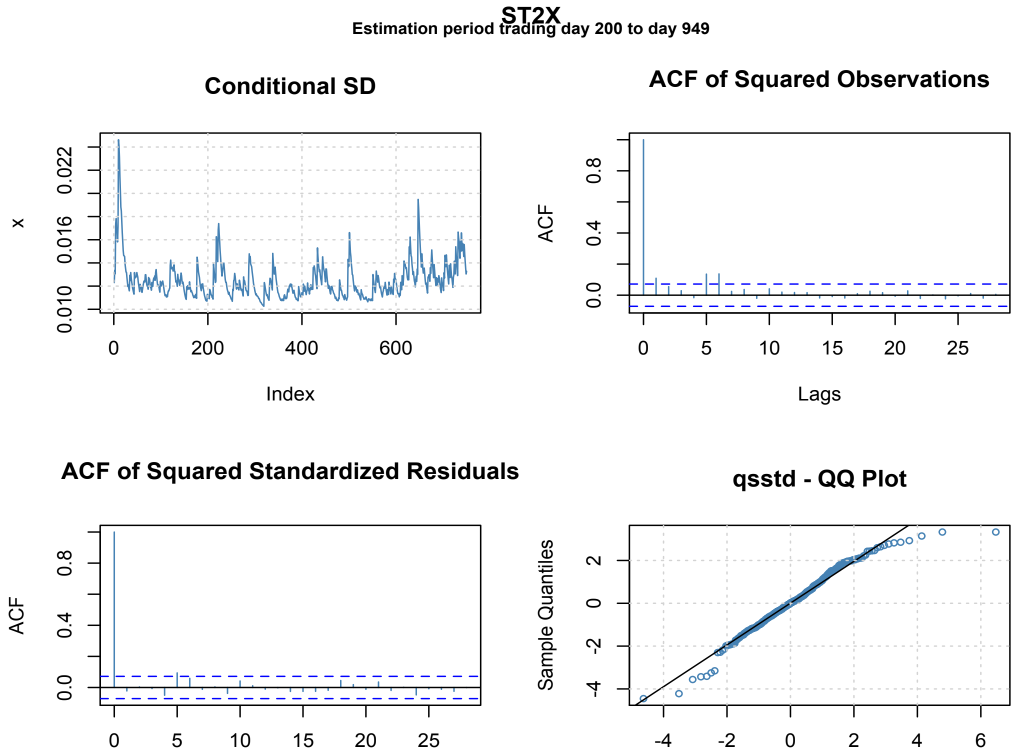

Figure 7.7: Selected plots from GARCH(1,1) model fitted for ST2X

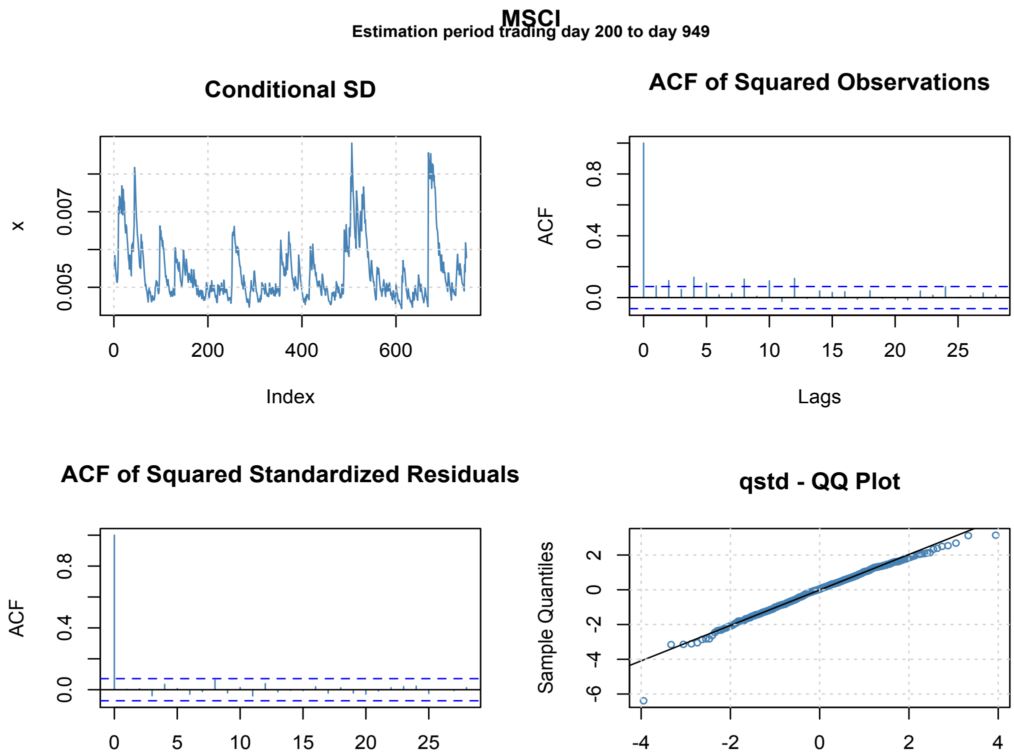

Figure 7.8: Selected plots from GARCH(1,1) model fitted for MSCI

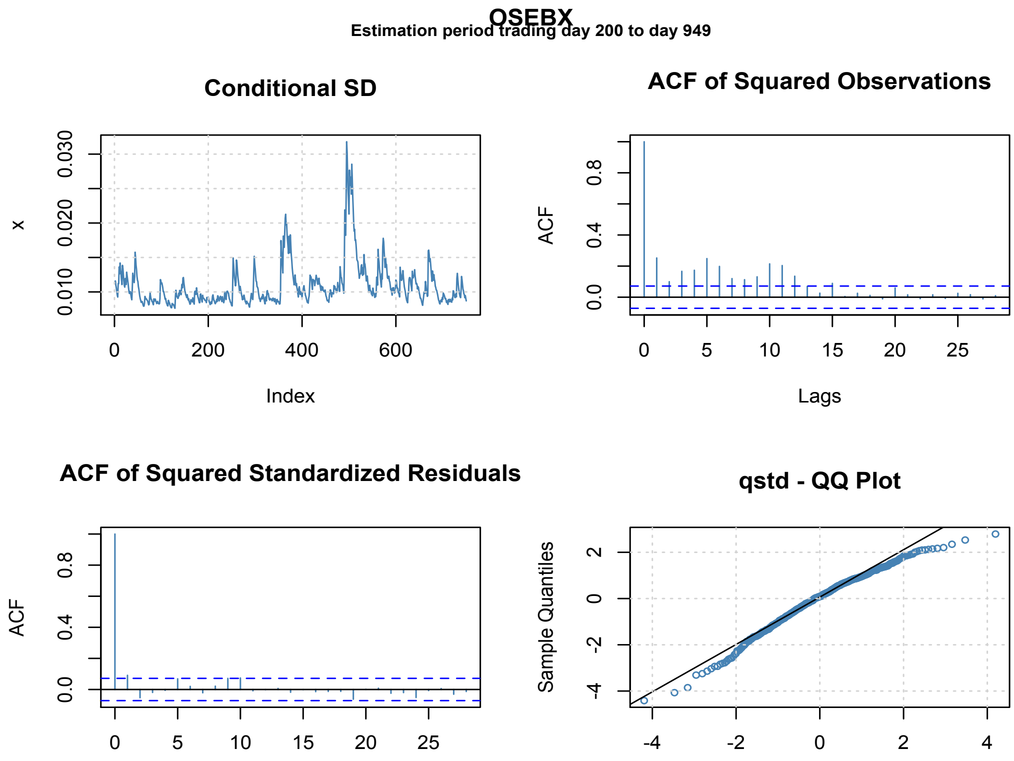

Figure 7.9: Selected plots from GARCH(1,1) model fitted for OSEBX

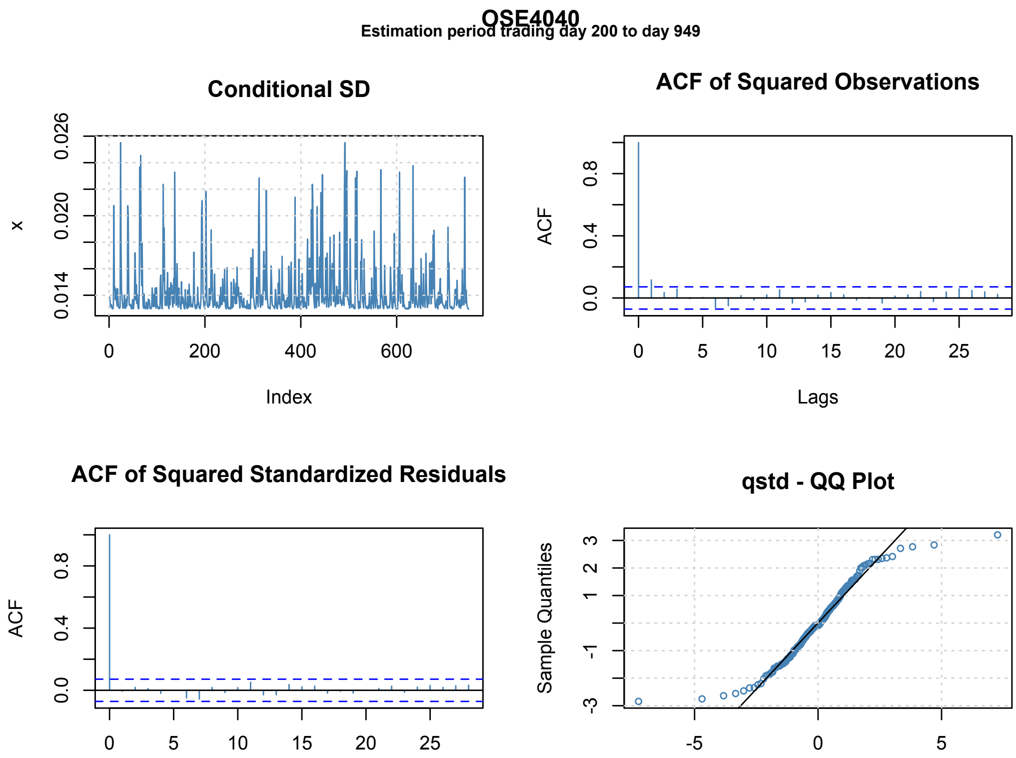

Figure 7.10: Selected plots from GARCH(1,1) model fitted for OSE4040

7.3 Pair-copula-construction

After fitting a univariate GARCH model to each return series, we transform the standardized residuals onto copula scale using the empirical cumulative distribution function according to Equations (6.4) and (6.5). Thereafter, as a first step of fitting a pair-copula-construction with a C-vine structure, we need to determine the root node in Tree 1. To do so we examine the empirical Kendall's tau discussed in Section 5.4 to select the asset that has the strongest pairwise dependence to all others. To illustrate the change in dependence structure between the asset returns subject to volatility clustering over time, we compare the empirical Kendall's taus from three different estimation periods: [1-750], [200-949] and [358-1107].

Table 7.8: Empirical Kendall's tau computed pairwise for estimation period [1-750]

Index	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
BRIX	1.000	0.225	0.306	-0.058	-0.087	-0.016
WGBI	0.225	1.000	0.121	-0.062	-0.072	-0.015
ST2X	0.306	0.121	1.000	-0.011	-0.037	-0.003
MSCI	-0.058	-0.062	-0.011	1.000	0.288	0.063
OSEBX	-0.087	-0.072	-0.037	0.288	1.000	0.148
OSE4040	-0.016	-0.015	-0.003	0.063	0.148	1.000

Table 7.9: Empirical Kendall's tau computed pairwise for estimation period [200-949]

Index	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
BRIX	1.000	0.221	0.255	-0.058	-0.073	-0.005
WGBI	0.221	1.000	0.102	-0.053	-0.039	0.0002
ST2X	0.255	0.102	1.000	-0.023	-0.017	-0.0101
MSCI	-0.058	-0.053	-0.023	1.000	0.322	0.084
OSEBX	-0.073	-0.039	-0.017	0.322	1.000	0.162
OSE4040	-0.005	0.0002	-0.0101	0.084	0.162	1.000

Table 7.10: Empirical Kendall's tau computed pairwise for estimation period [358-1107]

Index	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
BRIX	1.000	0.217	0.205	-0.104	-0.116	-0.021
WGBI	0.217	1.000	0.114	-0.119	-0.091	-0.043
ST2X	0.205	0.114	1.000	-0.044	-0.044	-0.014
MSCI	-0.104	-0.119	-0.044	1.000	0.372	0.098
OSEBX	-0.116	-0.091	-0.044	0.372	1.000	0.193
OSE4040	-0.021	-0.043	-0.014	0.098	0.193	1.000

We then sum up the absolute values of each column and subtract 1 to evaluate which risk factor that has the strongest association with all the others.

Table 7.11: Sums of the absolute values of empirical Kendall's taus for selected estimation periods

Index	[1-750]	[200-949]	[358-1107]
BRIX	0.6930523	0.6129595	0.6634944
WGBI	0.4951778	0.4146862	0.5844557
ST2X	0.4784157	0.4076297	0.4212657
MSCI	0.4834864	0.5401727	0.739101
OSEBX	0.6328851	0.6125323	0.8175131
OSE4040	0.245846	0.2623765	0.3692853

Interestingly, as the "rolling-window" moves along the sample timeline, there appears to be a shift in the dependence structure. More specifically, we stand between BRIX and OSEBX as the root node based on the sums in Table 7.11. The results from estimation period [1-750] indicates that BRIX has the strongest association with all other risk factors. Whilst estimation period [358-1107] selects OSEBX to be the governing risk factor of the data set. Notably, the two risk factors display an almost equivalently strong association to other risk factors for estimation period [200-749]. This is an indication of the possibility of having two key assets in our portfolio. In fact, until estimation period starting in day 202, BRIX is selected as the root node, whilst from estimation period starting in day 203 till the end as we approach the financial crisis 2007-2008 OSEBX is chosen to be the root node. Figure 7.11 and 7.12 illustrates the C-vine tree structure for for estimation period [200-749] and [358-1107] with BRIX and OSEBX as the respective root node.

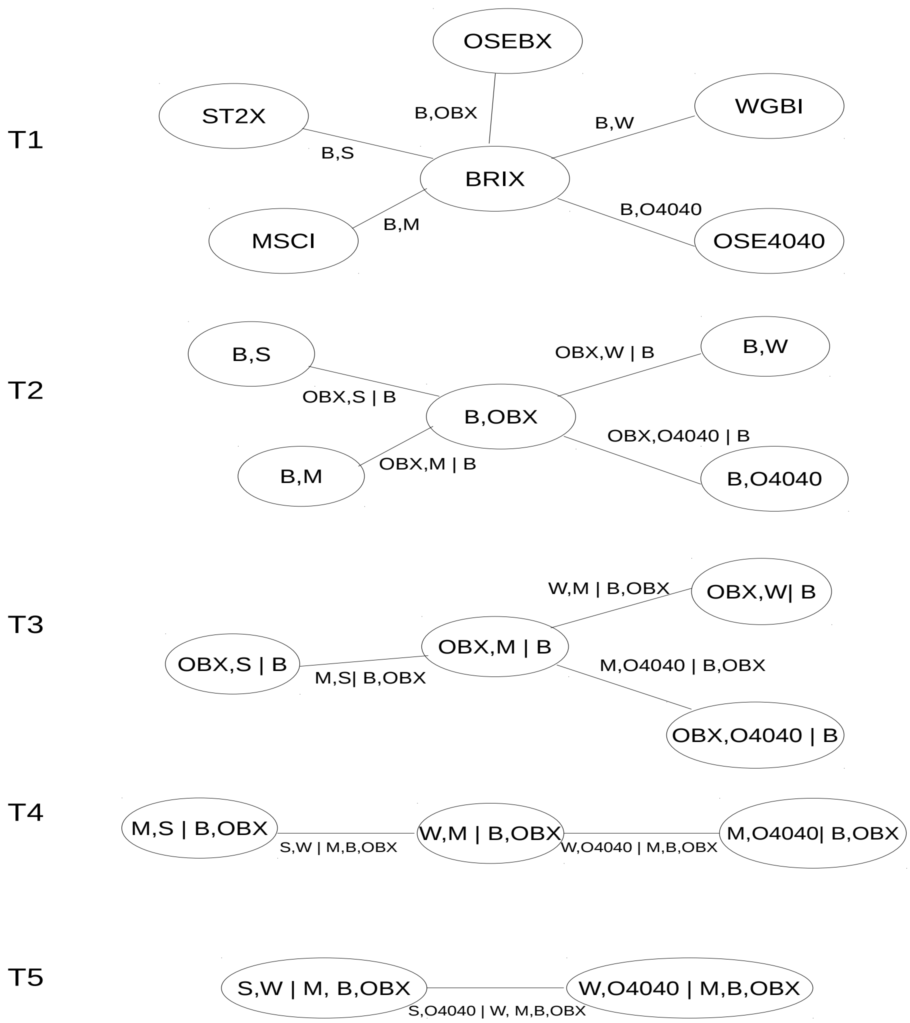

Figure 7.11: Canonical vine structure for the sample data for estimation period [200-949]

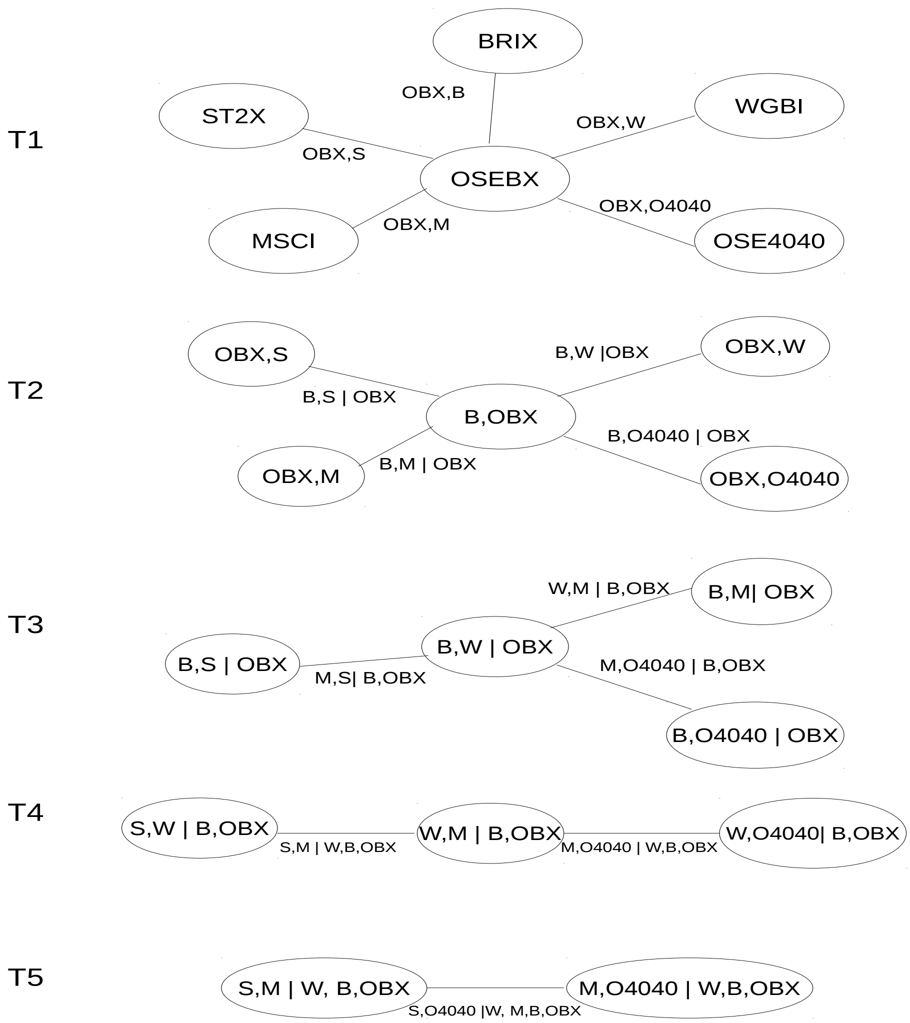

Figure 7.12: Canonical vine structure for the sample data for estimation period [358-1107]

Having determined the root node, we proceed to select the bivariate copula families for each pair of risk factors. For our analysis we consider the Gaussian, Student's t, Clayton and Gumbel copulas to maintain a variety of tail dependence characteristics. Notably, as the parameters estimated by the GARCH(1,1) model are slightly different for each estimation period, this eventually has an impact on the results of bivariate copula families selected for pairs of risk factors. For Tree 1 of our C-vine structure, the choice of copula families switches between the Gaussian and the Student's t copula. Recall that the Gaussian copula is essentially just the Student's t copula with a large degrees of freedom, that is, greater than 30 in our case.

To illustrate, we present the estimates of the C-vine structure for the estimation period that starts at day 200. Here only the Gaussian, Student's t and Gumbel copulas are selected.

Table 7.12: Estimated parameters for PCC C-vine structure for estimation period [200-949]

Pair	Copula type	Par1	Par2
B,S	Student's t	0.4	6.14
B,W	Gaussian	0.35	
B,M	Student's t	-0.1	13.58
B,OBX	Gaussian	-0.12	
B,O4040	Gaussian	-0.01	
OBX, W |B	Gaussian	-0.04	
OBX, S |B	Gumbel	1.01	
OBX, M |B	Student's t	0.49	9.46
OBX, O4040 |B	Gaussian	0.27	
M, S |B,OBX	Gaussian	-0.02	
W, M |B,OBX	Gaussian	-0.01	
M, O4040 |B,OBX	Gumbel	-0.03	
S, W |M,B,OBX	Gaussian	0.02	
W, O4040 |M,B,OBX	Gaussian	1.03	
S, O4040 |W,M,B,OBX	Gaussian	0.004	

7.4 CVaR portfolio optimization

For CVaR optimization we generate 5000 returns scenarios from the estimated PCC-GARCH model. Notably, to check whether this is a sufficient number of simulations, we performed the optimization also with 10 000 returns for a selected number of test periods. The resulted weights for the risk factors were not significantly different from that using 5000 simulated returns. In fact, they were the same. Hence we conclude that it is sufficient with 5000 simulations. We now proceed to present the results of CVaR optimization with respect to the two different scenarios described in Section 6.3.

Scenario 1 Recall that the simulated returns are adjusted by subtracting the historical mean and adding an annual hypothetical expected return adjusted with respect to the number of trading days in a year. According to common market opinions the hypothetical annual expected returns C_i are chosen to be 2.5%, 3.5%, 1.5%, 6.0%, 10% and 5% for BRIX, WGBI, ST2X, MSCI, OSEBX and OSE4040 respectively.

We use the PCC-GARCH model from the estimation period starting in day 200 to illustrate the CVaR optimization with confidence level $\alpha = 99\%$. We apply the Mean-CVaR optimization and obtain the efficient frontier of the portfolio under different expected returns, as shown in Figure 7.13. The graph shows the minimum variance locus (faded dotted curve from the red point representing the minimum risk portfolio) and the efficient frontier (solid dotted curve) for 50 equidistant return points. Note that the target return axis is scaled to *daily* basis, hence it should not be confused with expected annual rate of returns. In addition there are the risk-return points for the individual risk factors and the equal weights portfolio. We want our expected annual portfolio return to be at least 6%. Hence, to find the the optimal portfolio we simply search along the efficient frontier and as a result, obtain the following weights for our risk factors: 6.7% BRIX, 38.8% WGBI, 0% ST2X, 22% MSCI, 30.8% OSEBX and 1.7% OSE4040.

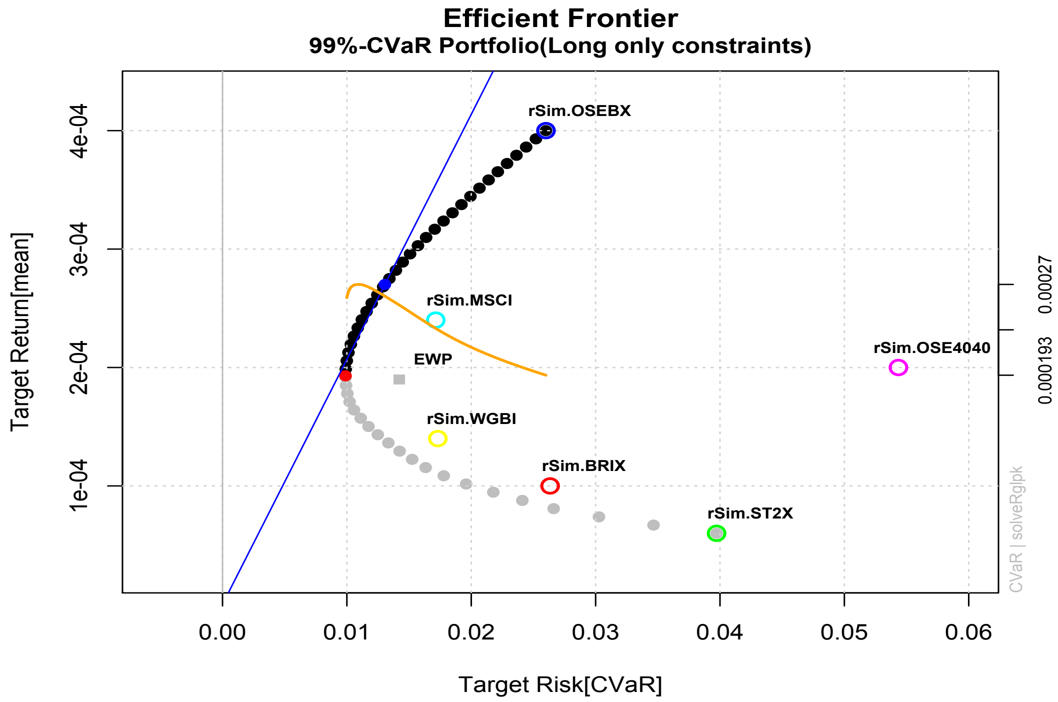

Figure 7.13: Efficient frontier Scenario 1

The colored circles named after indexes in Figure 7.13 represent the feasible portfolios containing only one risk factor. In the case of estimation period [200-949], both ST2X and OSE4040 display relatively high risk levels for low rate of returns; consequently this is reflected in the asset allocation choice of our portfolio: 0% ST2X and 1.7% OSE4040. Furthermore, OSEBX and BRIX display nearly identical level of risk but significant difference in return;

this is also directly reflected in the portfolio weights assigned: 30.8% OSEBX for a higher rate of return and a much smaller proportion of BRIX at 6.7%. At last, the weights allocation to MSCI and WGBI seem also reasonable since the discrepancy in returns is relatively small between the two risk factors for an almost identical level of risk .

The complete list of optimal portfolio weights is given in Appendix B.1. A graphical illustration of the change in portfolio weights for each risk factor is shown in Figure 7.14.

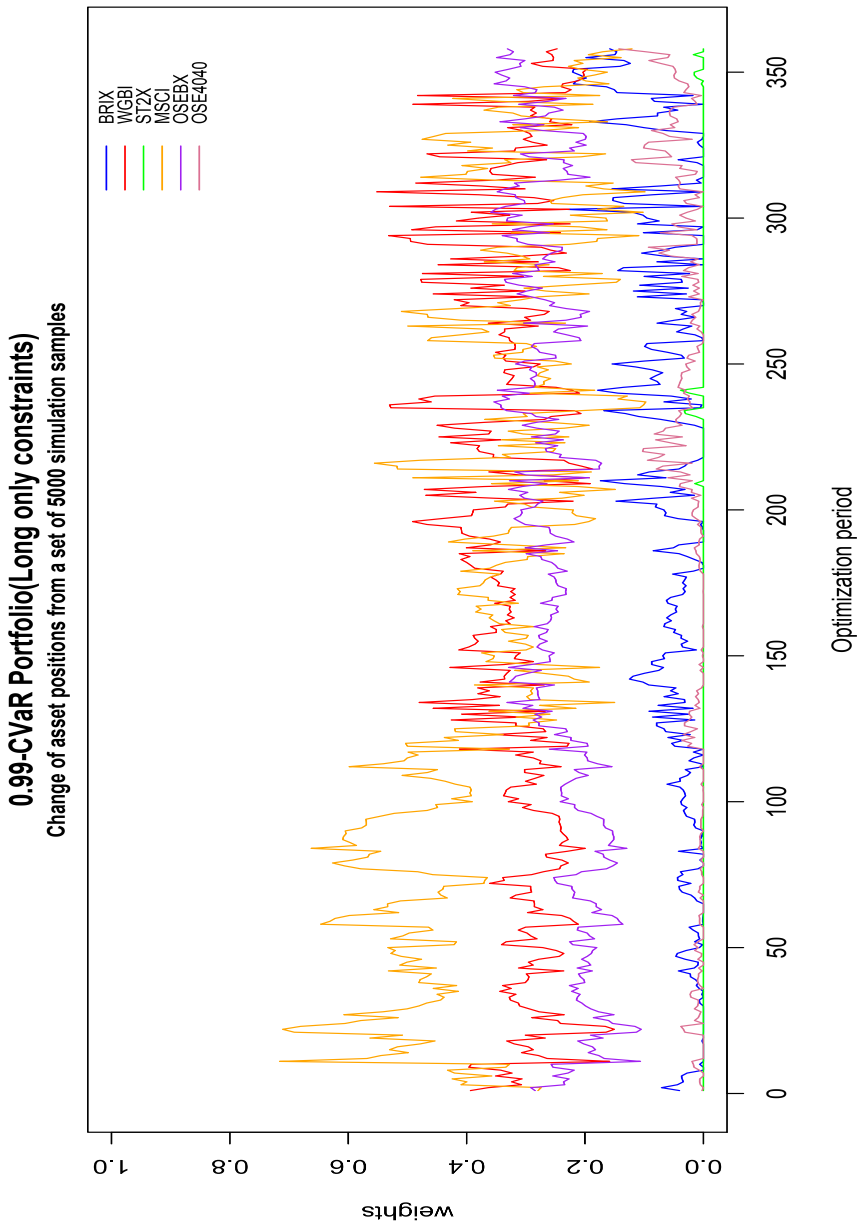

Figure 7.14: Portfolio weights of each assets

The optimization suggests a relative large proportion of MSCI in the beginning of the estimation period, as it is an asset subject to quite significant market risk in the beginning of the estimation period. However, in the first part of the test period the weight of this asset gradually decreases towards the first quarter of 2008. The second relatively large holding in the portfolio is WGBI, a less risky asset in comparison.

Scenario 2 All conditions held alike, we replaced the hypothetical expected return term with the log-return from previous day. Consequently, it has generated a significantly different efficient frontier shown in Figure 7.15 than that of Scenario 1 in order to achieve the expected annual portfolio return of at least 6%. We obtain the following weights for our risk factors: 29.0% BRIX, 39.2% WGBI, 3.2% ST2X, 11.4% MSCI, 0% OSEBX and 17.2% OSE4040.

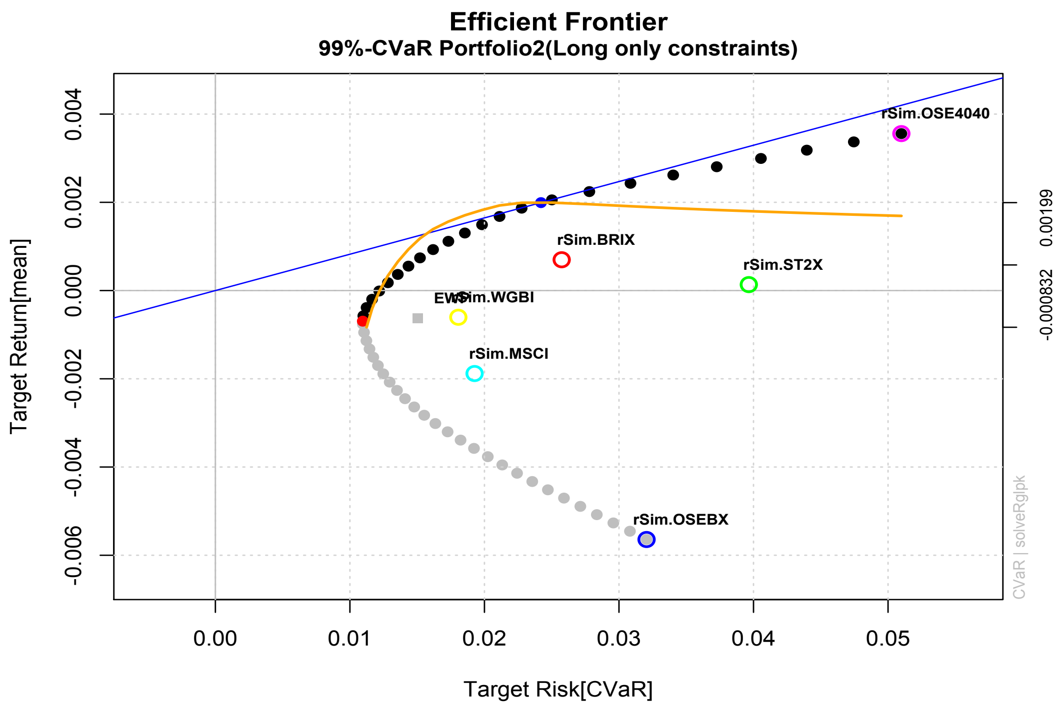

Figure 7.15: Efficient frontier Scenario 2

As illustrated above, both ST2X and OSE4040 display relatively high risk levels for low rate of returns; consequently small proportion of ST2X is allocated and OSE4040, positioned furthermost along the efficient frontier is assigned a fair proportion of 17.2%. The optimization suggests a significantly large holding of BRIX and WGBI in the portfolio as both risk factors display a relatively high rate of return for lower risk in comparison. MSCI and OSEBX, on the other hand, are assigned with rather small proportion as they are positioned in the unfavorable lower region of feasible, but low return portfolios. Recall that for estimation period starting in day 200, there is a presence of two key risk factors BRIX and OSEBX which shares the strongest association with other assets. While the portfolio still holds a proportion of BRIX, OSEBX has become undesirable for investment as it is increasingly subject to market risk towards the 2007-2008 financial crisis. For complete list of optimal portfolio weights generated from Scenario 2 see Appendix B.2.

As a final step, we compare the performance of the CVaR optimized portfolio where dependence structure is modeled by the PCC-GARCH model to portfolios managed based on the *Constant-Mix* and *Buy-and-Hold* rebalancing strategies. A Buy-and-Hold strategy is simply a "do-nothing" strategy where portfolio assets are bought and held according to an initial combination of weights. The Constant-Mix strategy requires investors to rebalance the portfolio to maintain a constant proportion of the assets held based on a standard initial mix of weights, see for example Perold and Sharpe [1988]. For our analysis we apply the same standard initial weights as in Aas et al. [2014] for the Constant-Mix and Buy-and-Hold strategies. That is, 10% BRIX, 11% WGBI, 40% ST2X, 12% MSCI, 4% OSEBX and 21% OSE4040. The CVaR strategy on the other hand rebalances the portfolio with the weights obtained from optimization procedure (See Appendix B for the complete list of portfolio weights).

We hypothetically invest 100 NOK at April 3, 2007 and then compute the accumulated wealth obtained till March 26, 2008 using the three different strategies, assuming long-only positions and no transaction costs. That is, the assets in our portfolio can only be held with positive weights and the portfolio is rebalanced on a daily basis.

For CVaR optimization and the Constant-Mix strategy, the accumulated wealth W_t at day t is computed as:

$$W_t = W_{t-1}(1 + R_t) \tag{7.1}$$

where $R_t = \sum_{i=1}^{6} w_{i,t} r_{i,t}^{arit}$ is the arithmetic portfolio return at day t. Further, $r_{i,t}^{arit} = \exp(r_{i,t}) - 1$ is the arithmetic return transformed from the log-return described in Section 6.3, and $w_{i,t}$ is the portfolio weights for asset i.

For the Buy-and-Hold strategy where rebalancing is not required, the accumulated portfolio wealth W_t at day t is computed as:

$$W_t = \sum_{i=1}^{6} W_{i,t}. \tag{7.2}$$

Here $W_{i,t}$ for $i = 1, 2, ..., 6$ is the accumulated wealth of asset i and is given as

$$W_{i,t} = W_{i,0} \prod_{t=1}^{358} (1 + r_{i,t}^{arit}), \tag{7.3}$$

where $W_{i,0} = 100 \cdot w_{i,t}$ represents the initial wealth of asset i.

Figure 7.16 and 7.17 illustrate the performance of the three portfolio strategies with respect to the two different scenarios. For Scenario 1, the CVaR optimized portfolio outperforms the other two by a slight margin until we enter the financial crisis in 2008 where drastic declines took place in the global financial markets. The Contant-Mix strategy displays a better performance in the last period. Not surprisingly, the Buy-and-Hold portfolio underperforms throughout the whole test period as it is a "do-nothing" strategy and the growth of portfolio wealth is under full exposure of market risk. In comparison, the CVaR optimized portfolio from Scenario 2 overall has accumulated greater wealth. Particularly it outperforms the other two strategies significantly towards the end of 2007 throughout the first quarter of 2008. We can therefore conclude that the CVaR optimization methodology is more advantageous in the event of large market price movements and high volatility.

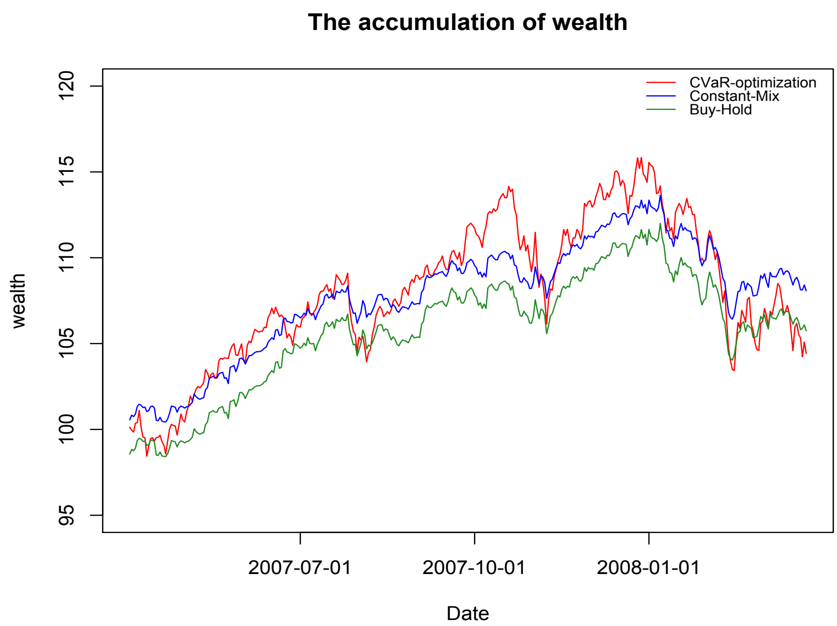

Figure 7.16: The accumulation of wealth if 100 NOK is invested using the asset positions resulted from CVaR optimization Scenario 1, Constant-Mix strategy or Buy-and-Hold strategy.

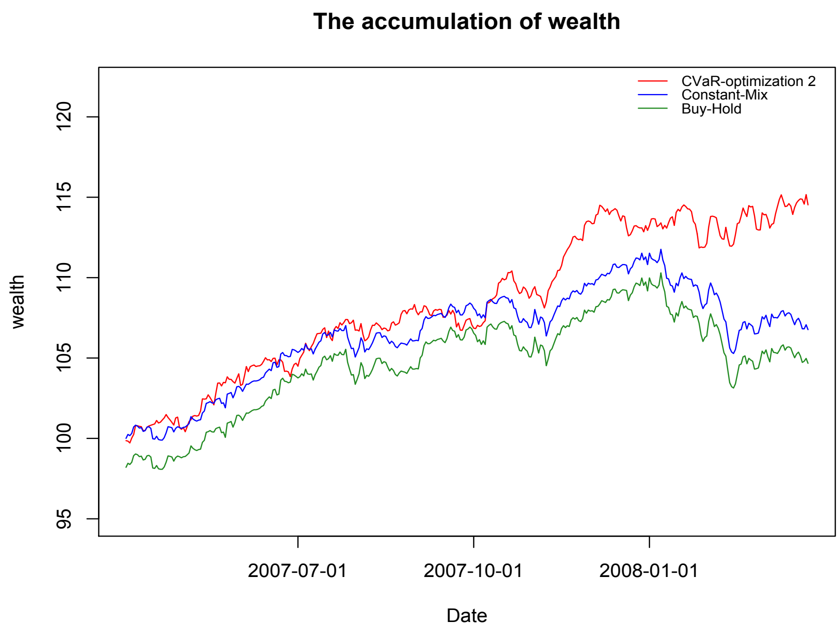

Figure 7.17: The accumulation of wealth if 100 NOK is invested using the asset positions resulted from CVaR optimization Scenario 2, Constant-Mix strategy or Buy-and-Hold strategy.

Chapter 8

Conclusion and future work

This thesis incorporates a non-linear dependence structure and time-varying volatility in the asset returns using the PCC-GARCH model and applies Mean-CVaR portfolio optimization to analyse the feasibility of the strategy. Compared to those using the Constant-Mix and Buy-and-Hold strategy our analysis of the evolution of wealth using CVaR optimized portfolios generated from Scenario 1 and 2 provide a positive validation of our method. In particular, the wealth accumulation based on CVaR optimized portfolio generated from Scenario 2 indicate a very promising outlook, even in the event of 2007-2008 financial crisis.

However, we need to bear in mind that the CVaR optimization has been carried out in the absence of transaction costs. The portfolio is also assumed to be rebalanced according to the optimal weights on a daily basis. The great upside benefits from our CVaR optimization strategy may lose its advantage if we incorporate transaction costs. The CVaR strategy may produce large changes in asset positions, hence in reality, a rebalancing frequency of daily basis could incur large transaction cost. There are, however, solutions to such problems. We may for example,

1. incorporate transaction cost as an additional constraint in the CVaR optimization problem, Krokhmal et al. [2002]

2. reduce the rebalancing frequency, for example, based on weekly or monthly monitoring only rebalances if the asset returns differ significantly from the previous after rebalancing. (For more detailed research on rebalancing strategy see Mendes and Marques [2012])

Furthermore, we have in the thesis only considered canonical vines. Our empirical result of the empirical studies indicate that there are in fact, two risk factors (BRIX and OSEBX) governing the dependence structure in our portfolio. To improve the results we can for example consider using a *regular* vine or a D-vine instead. As opposed to canonical vine, D-vines are symmetric in structure and each tree is a path. Notably, the solely use of canonical vine and D-vine can sometimes be limited due to their specific graphical structures, hence the more general representation *regular* vine would be far more advantageous as it allows the combination of C-vines and D-vines. This adds to the flexibility when modeling the dependence structure of the portfolio assets.

Finally, another modification that might improve our results is to relax the assumption of no short selling as part of the optimization constraints. In comparison to long-only portfolio, it may reduce the downside risk if investors are allowed to consider both long and short position. In particular, when market prices started to fall as we enter the first quarter of 2008.

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Appendix A: h-function

A.1 The bivariate Gaussian copula

We derive the h-function from the distribution function for the bivariate Gaussian copula given in Equation (4.9):

$$
\begin{aligned}
C_\rho(u_1, u_2) &= \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} exp\left\{-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right\} dx_1 dx_2 \\
&= \int_{-\infty}^{a_1} \int_{-\infty}^{a_2} g(x_1, x_2) dx_2 dx_1
\end{aligned}
$$

Recall the density and distribution function of the standard normal distribution as following:

$$
\begin{aligned}
f(z) &= \frac{1}{\sqrt{2\pi}} exp(-\frac{z^2}{2}) \\
\Phi(h) &= \int_{-\infty}^{h} f(z) dz
\end{aligned}
$$

We then apply (5.9) to obtain $h(u_1, u_1)$:

$$
\begin{aligned}
h(u_1, u_2) &= F_{12}(u_1|u_2) \\
&= \frac{\partial C_\rho(u_1, u_2)}{\partial u_2} \\
&= \frac{\partial}{\partial u_2} \int_{-\infty}^{a_1} \int_{-\infty}^{a_2} g(x_1, x_2) dx_2 dx_1 \\
&= \frac{\partial a_2}{\partial u_2} \frac{\partial}{\partial a_2} \int_{-\infty}^{a_1} \int_{-\infty}^{a_2} g(x_1, x_2) dx_2 dx_1,
\end{aligned}
$$

where

$$\frac{\partial a_2}{\partial u_2} = \frac{\partial}{\partial u_2}\Phi^{-1}(u_2)$$
$$= \frac{1}{f(a_2)},$$
$$f(a_2) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{{a_2}^2}{2}).$$

$$\begin{aligned}
h(u_1,u_2) &= \frac{1}{f(a_2)}\frac{\partial}{\partial a_2}\int_{-\infty}^{a_1}\int_{-\infty}^{a_2} g(x_1,x_2)dx_2dx_1 \\
&= \frac{1}{f(a_2)}\int_{-\infty}^{a_1}\left(\frac{\partial}{\partial a_2}\int_{-\infty}^{a_2} g(x_1,x_2)dx_2\right)dx_1 \\
&= \frac{1}{f(a_2)}\int_{-\infty}^{a_1} g(x_1,a_2)dx_1 \\
&= \frac{1}{f(a_2)}\int_{-\infty}^{a_1}\frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}}exp\left\{-\frac{x_1^2-2\rho x_1 a_2+a_2^2}{2(1-\rho^2)}\right\}dx_1 \\
&= \frac{1}{f(a_2)}\int_{-\infty}^{a_1}\frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}}exp\left\{-\frac{(x_1^2-\rho a_2)^2+a_2^2-\rho^2a_2^2}{2(1-\rho^2)}\right\}dx_1 \\
&= \frac{1}{f(a_2)}\frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}}\int_{-\infty}^{a_1} exp\left\{-\frac{(x_1^2-\rho a_2)^2}{2(1-\rho^2)}\right\}exp\left\{-\frac{a_2^2}{2}\right\}dx_1 \\
&= \frac{1}{f(a_2)}\frac{1}{\sqrt{2\pi}}exp\left\{-\frac{a_2^2}{2}\right\}\int_{-\infty}^{a_1}\frac{1}{\sqrt{2\pi(1-\rho^2)}}exp\left\{-\frac{(x_1^2-\rho a_2)^2}{2(1-\rho^2)}\right\}dx_1 \\
&= \frac{f(a_2)}{f(a_2)}\int_{-\infty}^{a_1}\frac{1}{\sqrt{2\pi(1-\rho^2)}}exp\left\{-\frac{(x_1^2-\rho a_2)^2}{2(1-\rho^2)}\right\}dx_1 \\
&= \int_{-\infty}^{a_1}\frac{1}{\sqrt{2\pi(1-\rho^2)}}exp\left\{-\frac{(x_1^2-\rho a_2)^2}{2(1-\rho^2)}\right\}dx_1
\end{aligned}$$

Denote then

$$\sigma = \sqrt{1-\rho^2}$$
$$\mu = \rho a_2$$

The bivariate Gaussian copula has the following h-function:

$$\begin{aligned} h(u_1,u_2) &= \int_{-\infty}^{a_1} \frac{1}{\sqrt{2\pi\sigma}} exp\left\{-\frac{(x_1^2-\mu)^2}{2\sigma)}\right\} dx_1 \\ &= \int_{-\infty}^{\frac{a_1-\mu}{\sigma}} f(z)dz \\ &= \Phi\left(\frac{a_1-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{\Phi^{-1}(u_1)-\rho\Phi^{-1}(u_2)}{\sqrt{1-\rho^2}}\right) \end{aligned} \tag{A.1.1}$$

The inverse of the h-function can then be derived straightforwardly from A(1.1) and is given by

$$h^{-1}(u_1,u_2) = \Phi\left(\Phi^{-1}(u_1)\sqrt{1-\rho^2} - \rho\Phi^{-1}(u_2)\right) \tag{A.1.2}$$

A.2 The bivariate Student's t-copula

The h-function of bivariate Student's t copula can be derived using the same method as previously used for the Gaussian copula. Due to the similarity detailed stepwise derivation is omitted. For bivariate Student's t-copula the h-function is given by

$$h(u_1,u_2) = t_{\nu+1}\left\{\frac{t_\nu^{-1}(u_1)-\rho t_\nu^{-1}(u_2)}{\sqrt{\frac{(\nu+(t_\nu^{-1}(u_2))^2)(1-\rho^2)}{\nu+1}}}\right\} \tag{A.2.1}$$

and the corresponding inverse h-function is given as

$$h^{-1}(u_1,u_2) = t_\nu\left\{t_\nu^{-1}(u_1)\sqrt{\frac{(\nu+(t_\nu^{-1}(u_2))^2)(1-\rho^2)}{\nu+1}} + \rho t_\nu^{-1}(u_2)\right\} \tag{A.2.2}$$

A.3 The bivariate Clayton copula

We derive the h-function from the distribution function for the bivariate Clayton copula given in Equation (4.15):

$$\begin{aligned} h(u_1, u_2) &= F_{12}(u_1|u_2) \\ &= \frac{\partial C(u_1, u_2)}{\partial u_2} \\ &= \frac{\partial}{\partial u_2}(u_1^{-\delta} + u_2^{-\delta} - 1)^{\frac{1}{\delta}} \\ &= -\frac{1}{\delta}(u_1^{-\delta} + u_2^{-\delta} - 1)^{\frac{1}{\delta}-1}(-\delta)u_2^{-\delta-1} \\ &= u_2^{-\delta-1}(u_1^{-\delta} + u_2^{-\delta} - 1)^{\frac{1}{\delta}-1} \end{aligned} \tag{A.3.1}$$

The inverse of the h-function can then be derived straightforwardly from A(3.1) and is given by

$$h^{-1}(u_1, u_2) = \left(\left(u_1 u_2^{\delta+1}\right)^{-\frac{\delta}{\delta+1}} + 1 - \nu^{-\delta}\right)^{-\frac{1}{\delta}} \tag{A.3.2}$$

A.4 The bivariate Gumbel copula

We derive the h-function from the distribution function for the bivariate Gumbel copula given in Equation (4.17):

$$\begin{aligned} h(u_1, u_2) &= F_{12}(u_1|u_2) \\ &= \frac{\partial C(u_1, u_2)}{\partial u_2} \\ &= \frac{\partial}{\partial u_2} exp[-((-\log u_1)^{\delta} + (-\log u_2)^{\delta})^{\frac{1}{\delta}}] \\ &= C(u_1, u_2)(-\frac{1}{\delta})[-((-\log u_1)^{\delta} + (-\log u_2)^{\delta})^{\frac{1}{\delta}-1}]\delta(-\log u_2)^{\delta-1}(-\frac{1}{u_2}) \\ &= C(u_1, u_2)(-\frac{1}{u_2})(-\log u_2)^{\delta-1}[-((-\log u_1)^{\delta} + (-\log u_2)^{\delta})^{\frac{1}{\delta}-1}] \end{aligned} \tag{A.4.1}$$

Note that in the case of the bivariate Gumbel copula, the inverse of h-function does not have an explicit form and must be derived numerically using methods such as Newton-Raphson. Aas et al. [2007]

Appendix B: Portfolio weights

B.1 CVaR optimized portfolio weights for Scenario 1

Table 1: Complete list of Portfolio weights for Scenario 1

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
1	750	0.0404	0.393564	0	0.279317	0.284446	0.002273
2	751	0.070587	0.363632	0	0.274196	0.291585	0
3	752	0.043885	0.307347	0	0.409358	0.234316	0.005093
4	753	0.033844	0.324038	0	0.399673	0.23624	0.006205
5	754	0.035943	0.306541	0	0.42435	0.227105	0.006061
6	755	0.03695	0.351126	0	0.357589	0.254336	0
7	756	0.027472	0.32344	0	0.420349	0.228739	0
8	757	0	0.34247	0	0.432342	0.218314	0.006875
9	758	0	0.396209	0	0.333915	0.254121	0.015755
10	759	0.006108	0.392096	0	0.327719	0.25718	0.016898
11	760	0	0.158432	0	0.715967	0.106377	0.019224
12	761	0	0.264098	0	0.56829	0.167612	0
13	762	0	0.278461	0	0.54495	0.176589	0
14	763	0	0.307077	0	0.498449	0.194474	0
15	764	0	0.287509	0	0.530247	0.182244	0
16	765	0	0.295783	0	0.516801	0.187416	0
17	766	0	0.319573	0	0.478143	0.202284	0
18	767	0.00227	0.33208	0	0.453563	0.212087	0
19	768	0	0.267273	0	0.563131	0.169596	0
20	769	0.000181	0.300776	0	0.508347	0.190695	0
21	770	0	0.16435	0	0.691257	0.113094	0.031298

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Table 1 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
22	771	0	0.150439	0	0.710873	0.104998	0.03369
23	772	0	0.165778	0	0.680551	0.115664	0.038007
24	773	0	0.241413	0	0.605152	0.153434	0
25	774	0	0.246229	0	0.578761	0.160157	0.014853
26	775	0	0.289849	0	0.515675	0.18586	0.008616
27	776	0	0.235275	0	0.607269	0.15117	0.006286
28	777	0	0.277448	0	0.541339	0.177008	0.004206
29	778	0	0.294068	0	0.500689	0.190123	0.01512
30	779	0.000101	0.310503	0	0.480583	0.199125	0.009688
31	780	0.006104	0.312603	0	0.467406	0.205392	0.008496
32	781	0.00702	0.325887	0	0.444059	0.214505	0.008528
33	782	0	0.327978	0	0.438158	0.212802	0.021061
34	783	0.002517	0.31685	0	0.44981	0.208392	0.022431
35	784	0	0.343897	0	0.413777	0.222454	0.019872
36	785	0.007066	0.324828	0	0.456356	0.211751	0
37	786	0.011875	0.339089	0	0.416898	0.226325	0.005813
38	787	0.008618	0.300944	0	0.48045	0.200543	0.009445
39	788	0.013255	0.296329	0	0.47871	0.201825	0.009881
40	789	0.021738	0.294688	0	0.467516	0.207813	0.008245
41	790	0.022118	0.299153	0	0.459694	0.210907	0.008128
42	791	0.042148	0.236051	0	0.532082	0.187514	0.002204
43	792	0.015393	0.312245	0	0.451283	0.213154	0.007925
44	793	0.009188	0.287609	0	0.512857	0.190346	0
45	794	0.008707	0.295627	0	0.50073	0.194936	0
46	795	0.030621	0.278348	0	0.476135	0.205629	0.009267
47	796	0.046432	0.243431	0	0.510589	0.196169	0.003379
48	797	0.043987	0.235853	0	0.527826	0.189225	0.003109
49	798	0.02912	0.248644	0	0.520739	0.187046	0.014451
50	799	0.006721	0.271328	0	0.532934	0.180213	0.008803
51	800	0.001301	0.341051	0	0.421084	0.22079	0.015774
52	801	0.014629	0.333217	0	0.417131	0.225894	0.00913
53	802	0.006331	0.281232	0	0.528577	0.18386	0

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Table 1 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
54	803	0.008408	0.291552	0	0.507913	0.192127	0
55	804	0.008646	0.298096	0	0.49683	0.196427	0
56	805	0.017247	0.312662	0	0.457036	0.213055	0
57	806	0.024008	0.300421	0	0.46425	0.211321	0
58	807	0	0.2112	0	0.646654	0.13607	0.006076
59	808	0.001047	0.219324	0	0.630953	0.142171	0.006505
60	809	0.000786	0.225202	0	0.622206	0.145553	0.006253
61	810	0.00041	0.239481	0	0.598346	0.154421	0.007343
62	811	0	0.296997	0	0.514829	0.188174	0
63	812	0	0.271181	0	0.556781	0.172039	0
64	813	0	0.285573	0	0.533394	0.181034	0
65	814	0	0.285389	0	0.533692	0.180919	0
66	815	0.010989	0.286268	0	0.511658	0.191084	0
67	816	0.014671	0.32632	0	0.432789	0.220714	0.005506
68	817	0.020964	0.316659	0	0.443571	0.218806	0
69	818	0.035734	0.290232	0	0.448181	0.217341	0.008511
70	819	0.040328	0.291776	0	0.440207	0.221696	0.005993
71	820	0.041123	0.291894	0	0.438541	0.222463	0.005979
72	821	0.021194	0.361099	0	0.370925	0.246782	0
73	822	0.043104	0.336755	0	0.369403	0.250739	0
74	823	0.04383	0.338627	0	0.364998	0.252544	0
75	824	0.034768	0.290226	0	0.457299	0.215033	0.002674
76	825	0.03859	0.270074	0	0.482638	0.20583	0.002867
77	826	0.009092	0.242933	0	0.578837	0.163699	0.005439
78	827	0	0.240775	0	0.60619	0.153035	0
79	828	0	0.228231	0	0.626574	0.145195	0
80	829	0.005784	0.239118	0	0.59238	0.158192	0.004525
81	830	0	0.264897	0	0.566992	0.168111	0
82	831	0	0.266575	0	0.564265	0.16916	0
83	832	0.041936	0.229958	0	0.545137	0.182969	0
84	833	0	0.199573	0	0.662482	0.129416	0.008528
85	834	0.005225	0.242983	0	0.586232	0.160302	0.005258

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Table 1 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
86	835	0.003481	0.238401	0	0.596914	0.155919	0.005285
87	836	0.005355	0.228638	0	0.609229	0.151464	0.005314
88	837	0.002965	0.231595	0	0.608044	0.151393	0.006003
89	838	0.00163	0.237075	0	0.603818	0.153215	0.004263
90	839	0	0.241566	0	0.604904	0.15353	0
91	840	0.012896	0.240358	0	0.577348	0.165126	0.004272
92	841	0.016156	0.241727	0	0.569325	0.168772	0.004019
93	842	0.014266	0.243813	0	0.569469	0.168424	0.004028
94	843	0.01499	0.242648	0	0.569896	0.168351	0.004115
95	844	0.028443	0.243877	0	0.546129	0.1802	0.001351
96	845	0.031281	0.255479	0	0.52239	0.189847	0.001003
97	846	0.033594	0.28716	0	0.466581	0.21167	0.000995
98	847	0.027884	0.305301	0	0.449052	0.217763	0
99	848	0.036727	0.291293	0	0.451986	0.217395	0.002599
100	849	0.038017	0.329402	0	0.39089	0.241692	0
101	850	0.034666	0.312199	0	0.425126	0.228009	0
102	851	0.030205	0.337519	0	0.392346	0.23993	0
103	852	0.031526	0.334792	0	0.3943	0.239381	0
104	853	0.034122	0.332808	0	0.392117	0.240521	0.000433
105	854	0.039735	0.325226	0	0.394419	0.240592	0.000028
106	855	0.06132	0.282131	0	0.419156	0.233509	0.003884
107	856	0.038919	0.300487	0	0.436185	0.224409	0
108	857	0.039795	0.284559	0	0.460426	0.215221	0
109	858	0.02693	0.269589	0	0.508874	0.194608	0
110	859	0.021754	0.301907	0	0.466062	0.210277	0
111	860	0.030497	0.302169	0	0.449243	0.218091	0
112	861	0	0.240514	0	0.59834	0.154527	0.006619
113	862	0.003975	0.276564	0	0.540578	0.178882	0
114	863	0.024201	0.274601	0	0.505846	0.195352	0
115	864	0.016117	0.287452	0	0.500119	0.196311	0
116	865	0.000069	0.316997	0	0.482199	0.200735	0
117	866	0.016813	0.287512	0	0.498718	0.196957	0

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Table 1 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
118	867	0	0.412125	0	0.327747	0.260129	0
119	868	0.04823	0.23012	0	0.49996	0.1952	0.026491
120	869	0.053468	0.227194	0	0.503111	0.196311	0.019916
121	870	0.034191	0.291639	0	0.414009	0.223826	0.036336
122	871	0.059754	0.26431	0	0.437635	0.223684	0.014616
123	872	0.022389	0.353141	0	0.338501	0.251477	0.034492
124	873	0.063528	0.266425	0	0.418883	0.229956	0.021209
125	874	0.066686	0.270646	0	0.420059	0.232566	0.010042
126	875	0.078926	0.316868	0	0.294753	0.277613	0.03184
127	876	0.078926	0.316868	0	0.294753	0.277613	0.03184
128	877	0.014723	0.425825	0	0.248136	0.287523	0.023794
129	878	0.085726	0.305286	0	0.316091	0.273272	0.019625
130	879	0.023529	0.408376	0	0.259402	0.284438	0.024255
131	880	0.091764	0.267208	0	0.362105	0.255664	0.023258
132	881	0.017549	0.456455	0	0.215483	0.304655	0.005858
133	882	0.076417	0.344593	0	0.276575	0.288312	0.014102
134	883	0.031279	0.480084	0	0.150019	0.331702	0.006916
135	884	0.031921	0.389846	0	0.293085	0.276338	0.008811
136	885	0.066373	0.343595	0	0.298264	0.278653	0.013115
137	886	0.042572	0.380788	0	0.288631	0.279837	0.00817
138	887	0.055339	0.361647	0	0.291863	0.279832	0.011319
139	888	0.046542	0.381156	0	0.287039	0.282251	0.003013
140	889	0.092333	0.269296	0	0.386719	0.251652	0
141	890	0.102328	0.377031	0	0.19291	0.327732	0
142	891	0.124707	0.335722	0	0.218075	0.321496	0
143	892	0.118729	0.33649	0	0.228035	0.316745	0
144	893	0.099674	0.332103	0	0.270892	0.297331	0
145	894	0.07527	0.326614	0	0.31874	0.273912	0.005464
146	895	0.067436	0.427993	0	0.175518	0.329053	0
147	896	0.087055	0.336848	0	0.277781	0.291066	0.007249
148	897	0.087316	0.287169	0	0.367083	0.258433	0
149	898	0.065704	0.319291	0	0.355407	0.259599	0

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Table 1 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
150	899	0.068484	0.316299	0	0.353154	0.260542	0.001522
151	900	0.064152	0.309285	0	0.374577	0.251987	0
152	901	0.011792	0.413016	0	0.299524	0.271937	0.003731
153	902	0.035049	0.396513	0	0.287399	0.281039	0
154	903	0.03738	0.385726	0	0.300558	0.276337	0
155	904	0.026666	0.380546	0	0.329064	0.263725	0
156	905	0.037665	0.386121	0	0.299381	0.276833	0
157	906	0.037005	0.388924	0	0.296063	0.278008	0
158	907	0.050058	0.350009	0	0.334826	0.265107	0
159	908	0.047202	0.349698	0	0.340686	0.262414	0
160	909	0.066577	0.359526	0	0.286567	0.285874	0.001457
161	910	0.06506	0.326832	0	0.344361	0.263748	0
162	911	0.05405	0.333778	0	0.353705	0.258458	0.000008
163	912	0.061873	0.319767	0	0.361816	0.256544	0
164	913	0.055201	0.331134	0	0.355856	0.25781	0
165	914	0.045253	0.324924	0	0.384599	0.245225	0
166	915	0.047421	0.327911	0	0.37568	0.248988	0
167	916	0.044416	0.32637	0	0.383818	0.245396	0
168	917	0.059846	0.352367	0	0.312641	0.275146	0
169	918	0.062417	0.318095	0	0.363512	0.255975	0
170	919	0.055251	0.326527	0	0.363246	0.254975	0
171	920	0.035178	0.318058	0	0.414646	0.232118	0
172	921	0.030683	0.3253	0	0.410892	0.232794	0.000331
173	922	0.034256	0.318416	0	0.415793	0.231535	0
174	923	0.029298	0.347623	0	0.377629	0.245451	0
175	924	0.035001	0.344669	0	0.371735	0.248595	0
176	925	0.031593	0.350551	0	0.368566	0.24929	0
177	926	0.028665	0.359296	0	0.359848	0.252192	0
178	927	0.05184	0.342549	0	0.343607	0.262004	0
179	928	0.016984	0.338348	0	0.409636	0.23011	0.004923
180	929	0	0.386256	0	0.360907	0.245736	0.007102
181	930	0	0.392878	0	0.349911	0.249922	0.007289

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Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
182	931	0.002629	0.40005	0	0.334556	0.256459	0.006305
183	932	0.014926	0.409409	0	0.297665	0.272794	0.005206
184	933	0.02716	0.394514	0	0.29682	0.274611	0.006895
185	934	0.041244	0.412385	0	0.235305	0.299317	0.011748
186	935	0.084734	0.266615	0	0.389482	0.246496	0.012673
187	936	0.050692	0.400393	0	0.232668	0.300971	0.015276
188	937	0.041612	0.360025	0	0.325608	0.265732	0.007023
189	938	0	0.342887	0	0.430647	0.218777	0.007689
190	939	0	0.361199	0	0.40269	0.229862	0.006249
191	940	0	0.395194	0	0.355006	0.249598	0.000201
192	941	0	0.408061	0	0.329606	0.258538	0.003795
193	942	0	0.408661	0	0.328215	0.258996	0.004127
194	943	0.001615	0.409695	0	0.324247	0.260907	0.003535
195	944	0.004616	0.467149	0	0.217468	0.301	0.009767
196	945	0	0.491389	0	0.193454	0.310767	0.00439
197	946	0.0428	0.442868	0	0.182303	0.31984	0.012189
198	947	0.065425	0.391761	0	0.214612	0.309359	0.018843
199	948	0.063251	0.389783	0	0.221719	0.306257	0.01899
200	949	0.066899	0.388404	0	0.219968	0.308018	0.016712
201	950	0.081027	0.334758	0	0.290575	0.284867	0.008773
202	951	0.078746	0.3303	0	0.302177	0.280069	0.008707
203	952	0.147507	0.220332	0	0.352889	0.271316	0.007956
204	953	0.083795	0.354407	0	0.239347	0.302391	0.020059
205	954	0.015251	0.46882	0	0.201601	0.309992	0.004336
206	955	0.055846	0.384408	0	0.242762	0.296734	0.02025
207	956	0.031238	0.471427	0	0.148818	0.329325	0.019192
208	957	0.070519	0.37078	0	0.212118	0.306111	0.040472
209	958	0.133222	0.191123	0.014446	0.357861	0.264529	0.038818
210	959	0.173928	0.268668	0	0.211417	0.327322	0.018665
211	960	0.063339	0.192696	0	0.490749	0.193369	0.059848
212	961	0.110851	0.312105	0	0.253994	0.300299	0.02275
213	962	0.091272	0.36217	0	0.189391	0.318449	0.038718

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Table 1 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
214	963	0.041577	0.186737	0	0.516609	0.175529	0.079548
215	964	0.031591	0.20615	0	0.517826	0.176116	0.068316
216	965	0.025956	0.22704	0	0.555654	0.172	0.01935
217	966	0.009691	0.235602	0	0.478447	0.181877	0.094383
218	967	0	0.35505	0	0.326151	0.243326	0.075474
219	968	0	0.354405	0	0.339521	0.240458	0.065615
220	969	0	0.378445	0	0.254368	0.264701	0.102486
221	970	0	0.382364	0	0.249583	0.266833	0.101219
222	971	0	0.378173	0	0.346373	0.246218	0.029236
223	972	0	0.426808	0	0.234079	0.283267	0.055846
224	973	0	0.342097	0	0.340814	0.236507	0.080582
225	974	0	0.446531	0	0.227459	0.290508	0.035502
226	975	0	0.36268	0	0.304324	0.24998	0.083017
227	976	0	0.358778	0	0.329608	0.243753	0.067862
228	977	0	0.420247	0	0.234763	0.281162	0.063827
229	978	0.014971	0.449861	0	0.193141	0.305857	0.03617
230	979	0.04261	0.369208	0	0.260309	0.282046	0.045827
231	980	0.071726	0.2682	0	0.369325	0.244499	0.04625
232	981	0.10936	0.257088	0.012383	0.297966	0.282922	0.040281
233	982	0.131371	0.206981	0.030325	0.304231	0.290202	0.03689
234	983	0.168047	0.217805	0.032267	0.207868	0.332419	0.041595
235	984	0.005059	0.52795	0	0.107291	0.341497	0.018202
236	985	0.002833	0.530292	0	0.100897	0.342366	0.023613
237	986	0.066436	0.459902	0	0.097303	0.353769	0.02259
238	987	0.020535	0.480438	0	0.144122	0.327615	0.02729
239	988	0.054531	0.453555	0	0.128585	0.339656	0.023673
240	989	0.144332	0.209293	0.023509	0.29487	0.294477	0.03352
241	990	0.17855	0.221722	0.03849	0.18377	0.348021	0.029447
242	991	0.135075	0.248999	0	0.286086	0.287061	0.042779
243	992	0.073296	0.332189	0	0.269294	0.284486	0.040735
244	993	0.078086	0.32784	0	0.269608	0.285514	0.038953
245	994	0.089884	0.323264	0	0.258121	0.292338	0.036394

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Table 1 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
246	995	0.080018	0.319986	0	0.280595	0.281927	0.037474
247	996	0.068787	0.336727	0	0.275784	0.282296	0.036406
248	997	0.0864	0.335531	0	0.261805	0.293539	0.022725
249	998	0.11489	0.287213	0	0.280362	0.289577	0.027956
250	999	0.15413	0.275875	0	0.223174	0.317233	0.029588
251	1000	0.099864	0.333593	0	0.250754	0.3019	0.013889
252	1001	0.032283	0.338532	0	0.352317	0.249279	0.02759
253	1002	0.03709	0.333868	0	0.354444	0.249857	0.02474
254	1003	0.060605	0.350955	0	0.305342	0.276562	0.006536
255	1004	0.087761	0.3184	0	0.304473	0.280547	0.00882
256	1005	0.099142	0.317695	0	0.281943	0.290532	0.010688
257	1006	0.095155	0.318911	0	0.299049	0.285482	0.001403
258	1007	0	0.327201	0	0.465747	0.207052	0
259	1008	0.002608	0.328164	0	0.459292	0.209936	0
260	1009	0.014331	0.345172	0	0.409673	0.230823	0
261	1010	0.045011	0.335909	0	0.362596	0.2528	0.003684
262	1011	0.033894	0.335773	0	0.374766	0.244767	0.010801
263	1012	0.019664	0.279217	0	0.499225	0.195793	0.0061
264	1013	0.078156	0.363286	0	0.233084	0.303488	0.021985
265	1014	0.048548	0.291933	0	0.408793	0.232136	0.018591
266	1015	0.013804	0.301775	0	0.465098	0.206456	0.012867
267	1016	0.037418	0.268646	0	0.466559	0.208032	0.019346
268	1017	0.028266	0.260811	0	0.510127	0.192392	0.008404
269	1018	0.046953	0.299297	0	0.415603	0.232186	0.005961
270	1019	0	0.409629	0	0.326163	0.259697	0.004512
271	1020	0	0.396166	0	0.347273	0.251436	0.005125
272	1021	0.002088	0.42089	0	0.308572	0.267637	0.000812
273	1022	0.111488	0.294482	0	0.279126	0.290304	0.0246
274	1023	0.028696	0.456347	0	0.193414	0.31461	0.006933
275	1024	0.118081	0.301156	0	0.266123	0.298204	0.016435
276	1025	0.025668	0.392613	0	0.302855	0.272088	0.006776
277	1026	0.107309	0.336341	0	0.227557	0.311086	0.017708

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Table 1 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
278	1027	0.035835	0.475841	0	0.147098	0.333291	0.007936
279	1028	0.037438	0.477169	0	0.140189	0.335873	0.009332
280	1029	0.100035	0.304298	0	0.269989	0.28935	0.036328
281	1030	0.023519	0.474656	0	0.170915	0.322014	0.008896
282	1031	0.144329	0.22488	0	0.319564	0.277756	0.033471
283	1032	0.13496	0.245096	0	0.314807	0.280088	0.025049
284	1033	0	0.44806	0	0.26092	0.284275	0.006745
285	1034	0.078338	0.279018	0	0.370137	0.250888	0.02162
286	1035	0	0.426775	0	0.293248	0.271424	0.008553
287	1036	0.067769	0.304219	0	0.307158	0.265759	0.055095
288	1037	0.104151	0.231442	0	0.332351	0.257078	0.074978
289	1038	0.069705	0.257724	0	0.376356	0.238939	0.057276
290	1039	0.047239	0.273838	0	0.347862	0.23824	0.092821
291	1040	0	0.465966	0	0.226517	0.296527	0.01099
292	1041	0	0.494573	0	0.180295	0.314354	0.010778
293	1042	0	0.495188	0	0.178565	0.314884	0.011363
294	1043	0	0.531852	0	0.10921	0.339754	0.019184
295	1044	0.101167	0.241598	0	0.324218	0.26026	0.072757
296	1045	0.010426	0.492801	0	0.163079	0.322478	0.011216
297	1046	0	0.452105	0	0.208986	0.295875	0.043034
298	1047	0.111111	0.225331	0	0.356644	0.253867	0.053047
299	1048	0.074874	0.417285	0	0.162176	0.332228	0.013437
300	1049	0.095367	0.379743	0	0.164888	0.33067	0.029332
301	1050	0.102051	0.327226	0	0.224556	0.306323	0.039844
302	1051	0.119886	0.39202	0	0.101788	0.359232	0.027074
303	1052	0.233272	0.226828	0	0.178794	0.350967	0.01014
304	1053	0	0.53018	0	0.134657	0.334163	0.000999
305	1054	0.133651	0.279957	0	0.256161	0.301621	0.02861
306	1055	0.152756	0.252933	0	0.258999	0.302499	0.032813
307	1056	0.15231	0.286379	0	0.238041	0.316501	0.006768
308	1057	0.085144	0.379708	0	0.203252	0.317875	0.014021
309	1058	0.000882	0.551737	0	0.098738	0.348256	0.000387

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Table 1 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
310	1059	0.154422	0.301522	0	0.168187	0.336071	0.039797
311	1060	0.072409	0.375546	0	0.198275	0.311254	0.042516
312	1061	0.003802	0.485663	0	0.152657	0.319109	0.03877
313	1062	0.011624	0.351858	0	0.351925	0.243025	0.041567
314	1063	0	0.283984	0	0.477453	0.191745	0.046817
315	1064	0	0.356546	0	0.401153	0.228774	0.013526
316	1065	0	0.360561	0	0.398557	0.230498	0.010383
317	1066	0	0.358908	0	0.364544	0.236804	0.039744
318	1067	0	0.335072	0	0.407701	0.221022	0.036204
319	1068	0.010671	0.313349	0	0.325379	0.236305	0.114296
320	1069	0.042524	0.264968	0	0.335391	0.235715	0.121401
321	1070	0	0.445397	0	0.206486	0.294363	0.053755
322	1071	0	0.466794	0	0.165336	0.309012	0.058858
323	1072	0.000052	0.319095	0	0.398128	0.21817	0.064555
324	1073	0	0.317086	0	0.35628	0.225911	0.100723
325	1074	0.006548	0.264164	0	0.431757	0.198212	0.099319
326	1075	0.010102	0.289032	0	0.393181	0.215165	0.09252
327	1076	0.002394	0.304169	0	0.475078	0.199473	0.018885
328	1077	0	0.289679	0	0.455029	0.197939	0.057354
329	1078	0	0.293488	0	0.435325	0.203022	0.068165
330	1079	0.063402	0.30598	0	0.27199	0.271138	0.08749
331	1080	0.100615	0.330656	0	0.210944	0.309356	0.048429
332	1081	0.130502	0.222368	0	0.321682	0.269666	0.055783
333	1082	0.192051	0.259624	0	0.162111	0.343531	0.042682
334	1083	0.145877	0.25607	0	0.251506	0.301499	0.045048
335	1084	0.077573	0.292087	0	0.307583	0.266937	0.05582
336	1085	0.077446	0.282216	0	0.322087	0.261011	0.057239
337	1086	0.043355	0.292061	0	0.370493	0.237238	0.056853
338	1087	0.066858	0.284935	0	0.3463	0.251691	0.050216
339	1088	0	0.490691	0	0.186787	0.311891	0.010632
340	1089	0.0826	0.295425	0	0.344963	0.262976	0.014036
341	1090	0.087838	0.236637	0	0.423401	0.232271	0.019853

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Table 1 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
342	1091	0.017698	0.482953	0	0.175289	0.320718	0.003342
343	1092	0.145817	0.23312	0	0.304107	0.284064	0.032892
344	1093	0.150519	0.223634	0	0.315057	0.281379	0.029412
345	1094	0.163864	0.240049	0	0.269333	0.302121	0.024633
346	1095	0.21941	0.237446	0.003408	0.159769	0.353411	0.026556
347	1096	0.21467	0.212291	0	0.198643	0.333335	0.041061
348	1097	0.197801	0.203305	0.010878	0.211875	0.327172	0.048971
349	1098	0.208524	0.201451	0.015416	0.184287	0.340672	0.04965
350	1099	0.220794	0.203427	0.015327	0.162289	0.351737	0.046427
351	1100	0.219214	0.196406	0	0.207571	0.329055	0.047754
352	1101	0.140934	0.27144	0	0.227769	0.308386	0.051471
353	1102	0.12339	0.290503	0	0.203679	0.310151	0.072277
354	1103	0.134179	0.254078	0	0.218145	0.301725	0.091873
355	1104	0.198286	0.253178	0	0.139157	0.349306	0.060073
356	1105	0.149352	0.256097	0.016933	0.194243	0.326551	0.056823
357	1106	0.146476	0.265233	0	0.169895	0.32087	0.097527
358	1107	0.158017	0.24744	0	0.120904	0.331101	0.142538

Concluded

B.2 CVaR optimized portfolio weights for Scenario 2

Table 2: Complete list of Portfolio weights for Scenario 2

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
1	750	0.1089638	0.3725078	0.0000000	0.4875070	0.0000000	0.0310215
2	751	0.1032596	0.5387238	0.0182285	0.2029181	0.1087576	0.0281124
3	752	0.0821158	0.4216474	0.0000000	0.4261221	0.0313629	0.0387518
4	753	0.0821158	0.4216474	0.0000000	0.4261221	0.0313629	0.0387518
5	754	0.2001530	0.0000000	0.7028376	0.0000000	0.0000000	0.0970094
6	755	0.1083116	0.5150143	0.0038127	0.2588263	0.1140351	0.0000000
7	756	0.5116511	0.0000000	0.0880577	0.1645471	0.0000000	0.2357442
8	757	0.2127916	0.4097043	0.0905909	0.0000000	0.0000000	0.2869133
9	758	0.1371352	0.4519903	0.0470338	0.2074150	0.0000000	0.1564257
10	759	0.0614788	0.4942763	0.0034766	0.4148301	0.0000000	0.0259382
11	760	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
12	761	0.0000000	0.7735992	0.0229387	0.1170421	0.0261894	0.0602306
13	762	0.0551807	0.1993743	0.7454450	0.0000000	0.0000000	0.0000000
14	763	0.0826237	0.1541263	0.0757705	0.6484859	0.0000000	0.0389935
15	764	0.0752211	0.2175573	0.0372359	0.6623634	0.0000000	0.0076223
16	765	0.0313493	0.4256073	0.0380059	0.4676216	0.0374159	0.0000000
17	766	0.0135794	0.4741861	0.0322633	0.4601162	0.0000000	0.0198550
18	767	0.0826876	0.5882143	0.0314415	0.1415066	0.1561500	0.0000000
19	768	0.0547926	0.4334518	0.0260187	0.4574490	0.0092063	0.0190816
20	769	0.0169176	0.4371230	0.0377660	0.4866511	0.0000000	0.0215423
21	770	0.0074036	0.5087448	0.0345443	0.4143949	0.0021276	0.0327847
22	771	0.0000000	0.5978581	0.0290662	0.3351669	0.0000000	0.0379087
23	772	0.0000000	0.7746860	0.0317306	0.1935834	0.0000000	0.0000000
24	773	0.0000000	0.4764312	0.0401965	0.4466789	0.0000000	0.0366934
25	774	0.2180258	0.0000000	0.1262262	0.3607787	0.2949694	0.0000000
26	775	0.0376586	0.2714996	0.0537033	0.5714237	0.0000000	0.0657148
27	776	0.0188293	0.2714996	0.0537033	0.5714237	0.0000000	0.0657148
28	777	0.0000000	0.8071725	0.0287252	0.1278562	0.0000000	0.0362461
29	778	0.0000000	0.7778817	0.0328994	0.1173642	0.0000000	0.0718547

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Table 2 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
30	779	0.2427289	0.0000000	0.1355866	0.3548735	0.0000000	0.2668109
31	780	0.1213645	0.3832312	0.0902501	0.2295659	0.0000000	0.1755883
32	781	0.0000000	0.7664624	0.0449136	0.1042583	0.0000000	0.0843657
33	782	0.0338092	0.4563916	0.0101961	0.4267521	0.0388666	0.0339844
34	783	0.3076246	0.2538461	0.1717444	0.0000000	0.0000000	0.2667849
35	784	0.0107953	0.5905980	0.0240396	0.2984803	0.0101017	0.0659850
36	785	0.0327859	0.5245812	0.0256979	0.4142348	0.0000000	0.0027002
37	786	0.0825959	0.2739472	0.0542179	0.5045039	0.0000000	0.0847351
38	787	0.0223023	0.4823111	0.0259553	0.4019883	0.0000000	0.0674430
39	788	0.0670390	0.4035953	0.0033872	0.4775115	0.0000000	0.0484669
40	789	0.0446733	0.4539613	0.0000000	0.4607095	0.0002543	0.0404016
41	790	0.2258853	0.1966659	0.0638654	0.1846304	0.2844652	0.0444878
42	791	0.5511153	0.0000000	0.3980778	0.0508070	0.0000000	0.0000000
43	792	0.0562734	0.4332932	0.0000000	0.4509202	0.0161310	0.0433822
44	793	0.0305591	0.4836484	0.0334374	0.3995469	0.0000000	0.0528081
45	794	0.1710830	0.0872464	0.0284234	0.6595829	0.0000000	0.0536642
46	795	0.0261432	0.4788464	0.0088113	0.4031912	0.0461234	0.0368845
47	796	0.0261432	0.4788464	0.0088113	0.4031912	0.0461234	0.0368845
48	797	0.1233202	0.2489010	0.0062175	0.4622612	0.1361280	0.0231722
49	798	0.0050274	0.7419677	0.0251754	0.1688925	0.0000000	0.0589369
50	799	0.1634092	0.1550140	0.0450229	0.3928966	0.2076690	0.0359883
51	800	0.1610994	0.0794214	0.0553966	0.4878534	0.1316935	0.0845357
52	801	0.0523469	0.5106459	0.0103241	0.3327612	0.0939219	0.0000000
53	802	0.1534553	0.2384886	0.0185394	0.4885356	0.0959511	0.0050300
54	803	0.1693614	0.1290163	0.0090293	0.6877346	0.0048584	0.0000000
55	804	0.0000000	0.7849869	0.1395380	0.0754752	0.0000000	0.0000000
56	805	0.0230890	0.5131288	0.0331449	0.4062539	0.0000000	0.0243835
57	806	0.0707217	0.3960462	0.0201465	0.4562638	0.0443061	0.0125157
58	807	0.0242749	0.5103208	0.0156084	0.4477377	0.0000000	0.0020581
59	808	0.0230075	0.4742801	0.0112082	0.3779574	0.0590753	0.0544714
60	809	0.3089889	0.3214179	0.3695932	0.0000000	0.0000000	0.0000000
61	810	0.1594312	0.0000000	0.1014125	0.4924321	0.1477205	0.0990037

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Table 2 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
62	811	0.0000000	0.7626283	0.1298891	0.0000000	0.0000000	0.1074826
63	812	0.0266155	0.4273828	0.0425738	0.4564002	0.0138290	0.0331987
64	813	0.0000000	0.7089373	0.1126094	0.1130048	0.0000000	0.0654485
65	814	0.1820070	0.0202847	0.0433469	0.7461619	0.0000000	0.0081995
66	815	0.0468446	0.3818739	0.0495588	0.4829608	0.0000000	0.0387618
67	816	0.0000000	0.7317046	0.0326734	0.1619173	0.0203047	0.0533999
68	817	0.0180404	0.6555064	0.1082572	0.1302630	0.0000000	0.0879330
69	818	0.0685169	0.4184454	0.0068878	0.4443482	0.0166195	0.0451823
70	819	0.1627639	0.0191446	0.0214395	0.7173880	0.0000000	0.0792640
71	820	0.1297607	0.2509859	0.0059515	0.4436340	0.1453284	0.0243395
72	821	0.1414750	0.2389141	0.0153948	0.5742735	0.0000000	0.0299427
73	822	0.4599310	0.0000000	0.0000000	0.5400690	0.0000000	0.0000000
74	823	0.0726111	0.4370904	0.0179054	0.4022752	0.0537388	0.0163790
75	824	0.0848140	0.5342714	0.0030975	0.3635115	0.0000000	0.0143056
76	825	0.0929707	0.3403523	0.0074551	0.4548281	0.0742298	0.0301640
77	826	0.0923603	0.3687052	0.0117399	0.4143008	0.0851191	0.0277748
78	827	0.0100719	0.4942332	0.0332067	0.4365526	0.0000000	0.0259356
79	828	0.0000000	0.0000000	1.0000000	0.0000000	0.0000000	0.0000000
80	829	0.1402411	0.5400770	0.0723799	0.0328303	0.0000000	0.2144717
81	830	0.1416163	0.3241885	0.1293838	0.0000000	0.4048114	0.0000000
82	831	0.0645165	0.3928398	0.0247289	0.3975407	0.1072345	0.0131396
83	832	0.0145641	0.5017952	0.0128022	0.4375551	0.0000000	0.0332834
84	833	0.1713429	0.1069967	0.0000000	0.6298561	0.0000000	0.0918043
85	834	0.0993361	0.2824269	0.0041766	0.5438993	0.0116427	0.0585184
86	835	0.0273293	0.4578570	0.0083533	0.4579425	0.0232853	0.0252326
87	836	0.0598206	0.3077939	0.0080566	0.5926720	0.0000000	0.0316569
88	837	0.1894593	0.1778572	0.2031286	0.1781938	0.0000000	0.2513611
89	838	0.0000000	0.6054250	0.0380500	0.3565250	0.0000000	0.0000000
90	839	0.0000000	0.6975687	0.1114786	0.0491105	0.0406112	0.1012309
91	840	0.0229284	0.4525283	0.0126816	0.4789113	0.0000000	0.0329504
92	841	0.1005721	0.2858594	0.0616139	0.3140721	0.2378825	0.0000000
93	842	0.0618453	0.7098540	0.1016802	0.0000000	0.0000000	0.1266204

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Table 2 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
94	843	0.0288033	0.4665359	0.0076624	0.4280477	0.0337213	0.0352293
95	844	0.1875194	0.2080410	0.0240485	0.3566278	0.1232407	0.1005228
96	845	0.0408278	0.6739271	0.0222717	0.2629734	0.0000000	0.0000000
97	846	0.1244048	0.4044137	0.0346040	0.2556122	0.1682553	0.0127099
98	847	0.2079819	0.1349003	0.0469364	0.2482510	0.3365106	0.0254199
99	848	0.0848096	0.3819921	0.0022039	0.4463930	0.0454722	0.0391291
100	849	0.1264470	0.4590751	0.0262153	0.1978945	0.1458280	0.0445402
101	850	0.5429689	0.0000000	0.1358322	0.3211989	0.0000000	0.0000000
102	851	0.3314202	0.0000000	0.6685798	0.0000000	0.0000000	0.0000000
103	852	0.0665180	0.4525387	0.0000000	0.4212417	0.0265027	0.0331990
104	853	0.0677594	0.4612326	0.0000000	0.3890461	0.0469589	0.0350030
105	854	0.5061632	0.3340367	0.1598001	0.0000000	0.0000000	0.0000000
106	855	0.1787801	0.0000000	0.2460647	0.0000000	0.5751552	0.0000000
107	856	0.1446311	0.1578252	0.1230324	0.2066598	0.3619653	0.0058864
108	857	0.1104820	0.3156504	0.0000000	0.4133195	0.1487753	0.0117727
109	858	0.2002054	0.4964339	0.0162570	0.1235420	0.0000000	0.1635617
110	859	0.1081138	0.5752524	0.0222017	0.1970032	0.0000000	0.0974291
111	860	0.7062429	0.0000000	0.2063713	0.0873858	0.0000000	0.0000000
112	861	0.0857666	0.3630832	0.0033480	0.4130618	0.1115476	0.0231928
113	862	0.0000000	0.3443833	0.2708431	0.0000000	0.0000000	0.3847736
114	863	0.1649235	0.3131438	0.0304817	0.3157389	0.0002592	0.1754529
115	864	0.0960443	0.3039718	0.0142447	0.5802470	0.0000000	0.0054922
116	865	0.1509449	0.1135920	0.0333607	0.6533128	0.0000000	0.0487896
117	866	0.0000000	0.0000000	1.0000000	0.0000000	0.0000000	0.0000000
118	867	0.0375429	0.7135339	0.0296146	0.1987123	0.0000000	0.0205963
119	868	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
120	869	0.0000000	0.7645995	0.2194607	0.0159398	0.0000000	0.0000000
121	870	0.0000000	0.4792889	0.1004667	0.2392994	0.1010199	0.0799251
122	871	0.0256274	0.7510290	0.1700291	0.0533146	0.0000000	0.0000000
123	872	0.0316879	0.6846106	0.1498810	0.0000000	0.0134011	0.1204195
124	873	0.0682809	0.3646172	0.0407318	0.4495561	0.0071687	0.0696453
125	874	0.2344517	0.0103807	0.1112153	0.3581946	0.2857577	0.0000000

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Table 2 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
126	875	0.0679338	0.5501190	0.0828266	0.0396637	0.1969286	0.0625282
127	876	0.0339669	0.6134032	0.0655889	0.1223695	0.0984643	0.0662072
128	877	0.0000000	0.6766873	0.0483512	0.2050753	0.0000000	0.0698862
129	878	0.1191953	0.3753838	0.0550444	0.3288813	0.0655867	0.0559084
130	879	0.0398127	0.6357456	0.0373684	0.2012081	0.0000000	0.0858651
131	880	0.2483307	0.0000000	0.7516693	0.0000000	0.0000000	0.0000000
132	881	0.0252252	0.8228026	0.0530282	0.0711974	0.0000000	0.0277466
133	882	0.0000000	0.2465104	0.7534896	0.0000000	0.0000000	0.0000000
134	883	0.1040567	0.4162042	0.0165349	0.2837039	0.1512202	0.0282801
135	884	0.0000000	0.7008539	0.0264155	0.2097290	0.0000000	0.0630017
136	885	0.0328848	0.5932107	0.0182587	0.2693429	0.0339671	0.0523357
137	886	0.0657697	0.4855676	0.0101020	0.3289569	0.0679341	0.0416698
138	887	0.0804209	0.4145969	0.0087356	0.4065022	0.0467970	0.0429474
139	888	0.6667639	0.0000000	0.0378161	0.2954200	0.0000000	0.0000000
140	889	0.1394312	0.3085473	0.0000000	0.4872792	0.0000000	0.0647423
141	890	0.1492222	0.5297994	0.0000000	0.1500460	0.1277114	0.0432210
142	891	0.1852377	0.3876908	0.0110059	0.1308510	0.2120657	0.0731489
143	892	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
144	893	0.1341774	0.3896831	0.0000000	0.3675304	0.0655171	0.0430919
145	894	0.1156689	0.4376685	0.0000000	0.4180523	0.0000000	0.0286103
146	895	0.2142583	0.6301391	0.0000000	0.0109295	0.0000000	0.1446732
147	896	0.1898315	0.1650012	0.0361829	0.4998296	0.0298880	0.0792668
148	897	0.1521506	0.3531335	0.0150689	0.3551222	0.0985863	0.0259385
149	898	0.1312478	0.4966364	0.0159449	0.2679011	0.0492931	0.0389766
150	899	0.1103451	0.6401393	0.0168208	0.1806800	0.0000000	0.0520148
151	900	0.0842702	0.4109155	0.0314588	0.4047525	0.0214937	0.0471093
152	901	0.0688593	0.5096940	0.0035174	0.3801335	0.0075185	0.0302774
153	902	0.0400415	0.7595239	0.0333680	0.0469793	0.0000000	0.1200873
154	903	0.1534748	0.2082087	0.0132121	0.4531865	0.1379679	0.0339499
155	904	0.0259516	0.6051441	0.0240849	0.3047556	0.0000000	0.0400638
156	905	0.0703649	0.5294262	0.0101206	0.3429577	0.0391693	0.0079614
157	906	0.0693876	0.4780858	0.0083049	0.4081003	0.0099815	0.0261399

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Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
158	907	0.7807100	0.1622413	0.0570487	0.0000000	0.0000000	0.0000000
159	908	0.2489476	0.5343973	0.0049232	0.0000000	0.2117318	0.0000000
160	909	0.5685120	0.0000000	0.0195519	0.0000000	0.2660073	0.1459289
161	910	0.1567095	0.4494670	0.0291526	0.2047385	0.1461650	0.0137673
162	911	0.1093501	0.2965732	0.0683992	0.2357526	0.2279512	0.0619738
163	912	0.0808447	0.4586565	0.0054224	0.4187866	0.0034872	0.0328025
164	913	0.0708612	0.5361411	0.0133016	0.3134171	0.0244232	0.0418559
165	914	0.0973011	0.4262642	0.0211102	0.3675517	0.0658368	0.0219360
166	915	0.3617869	0.4500403	0.0001777	0.1319259	0.0000000	0.0560692
167	916	0.0583767	0.5067785	0.0028074	0.4033796	0.0000000	0.0286578
168	917	0.0522206	0.4735810	0.0178390	0.4381677	0.0000000	0.0181917
169	918	0.0460646	0.4403835	0.0328706	0.4729557	0.0000000	0.0077256
170	919	0.0901877	0.4799338	0.0152180	0.3337145	0.0722593	0.0086868
171	920	0.1518137	0.7100335	0.0000000	0.1381528	0.0000000	0.0000000
172	921	0.0313217	0.5472547	0.0167915	0.3851863	0.0000000	0.0194458
173	922	0.0999171	0.2762597	0.0353449	0.4150535	0.0983670	0.0750578
174	923	0.0000000	0.4176886	0.2083200	0.0000000	0.0000000	0.3739913
175	924	0.0688901	0.2854944	0.0398637	0.6057518	0.0000000	0.0000000
176	925	0.0211583	0.7368809	0.0307559	0.1754173	0.0000000	0.0357875
177	926	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
178	927	0.0605685	0.5865913	0.0156485	0.2404408	0.0574333	0.0393176
179	928	0.0859162	0.6648617	0.0181514	0.1684960	0.0000000	0.0625746
180	929	0.4579340	0.1511514	0.3909146	0.0000000	0.0000000	0.0000000
181	930	0.0980833	0.2332758	0.0455535	0.5835664	0.0000000	0.0395210
182	931	0.0905308	0.2535636	0.0402931	0.6029051	0.0000000	0.0127074
183	932	0.1050213	0.5785452	0.0106253	0.0322466	0.2445432	0.0290184
184	933	0.0781708	0.7015073	0.0143612	0.0161233	0.1483285	0.0415088
185	934	0.0513204	0.8244695	0.0180972	0.0000000	0.0521138	0.0539992
186	935	0.0882976	0.4692741	0.0258525	0.4041759	0.0105167	0.0018833
187	936	0.0643429	0.5380134	0.0102201	0.3192494	0.0016459	0.0665282
188	937	0.1269383	0.4087767	0.0173161	0.4089764	0.0171460	0.0208466
189	938	0.0412655	0.7428607	0.0130752	0.1704581	0.0000000	0.0323404

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Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
190	939	0.0310713	0.5265515	0.0162852	0.3253255	0.0000000	0.1007665
191	940	0.0000000	0.0000000	1.0000000	0.0000000	0.0000000	0.0000000
192	941	0.0516822	0.5063912	0.0201874	0.3978874	0.0000000	0.0238518
193	942	0.0641873	0.5079827	0.0204222	0.3265910	0.0631253	0.0176915
194	943	0.0834164	0.4778093	0.0161414	0.3446495	0.0000000	0.0779834
195	944	0.0728015	0.4988636	0.0197440	0.3603773	0.0217905	0.0264230
196	945	0.1673995	0.1659176	0.0469684	0.4484201	0.0939463	0.0773481
197	946	0.0917358	0.6713239	0.0174551	0.0987102	0.0865051	0.0342699
198	947	0.1522729	0.6277291	0.0924631	0.0000000	0.0456215	0.0819134
199	948	0.1378275	0.2649687	0.0344444	0.5198785	0.0000000	0.0428809
200	949	0.2896276	0.3921797	0.0317556	0.1145120	0.0000000	0.1719250
201	950	0.0812958	0.4673297	0.0273480	0.2617902	0.1414986	0.0207376
202	951	0.1322839	0.4711093	0.0821915	0.2050289	0.0000000	0.1093864
203	952	0.0728817	0.5518008	0.0591947	0.2852262	0.0096939	0.0212027
204	953	0.0000000	0.5996226	0.0440943	0.1222104	0.2340727	0.0000000
205	954	0.1045575	0.8061292	0.0179507	0.0429153	0.0000000	0.0284473
206	955	0.4817340	0.0000000	0.1760460	0.0590544	0.0000000	0.2831657
207	956	0.0394825	0.7141352	0.0654054	0.1208935	0.0016024	0.0584810
208	957	0.5348990	0.3491006	0.1160004	0.0000000	0.0000000	0.0000000
209	958	0.1104575	0.3231068	0.1350791	0.1358765	0.1028356	0.1926445
210	959	0.5946502	0.1224989	0.0000000	0.2418298	0.0018360	0.0391852
211	960	0.0638695	0.6120749	0.1400577	0.0472608	0.0000000	0.1367372
212	961	0.5002921	0.1826805	0.0054610	0.2355817	0.0374461	0.0385387
213	962	0.5310281	0.3213363	0.0000000	0.1005399	0.0079786	0.0391171
214	963	0.0877361	0.3109915	0.1220544	0.3278855	0.0818721	0.0694604
215	964	0.0000000	0.5906576	0.1323720	0.2680402	0.0000000	0.0089301
216	965	0.0000000	0.6286351	0.1470251	0.2243399	0.0000000	0.0000000
217	966	0.2039679	0.5340327	0.1136106	0.1277930	0.0000000	0.0205958
218	967	0.4079359	0.4394302	0.0801961	0.0312461	0.0000000	0.0411916
219	968	0.0000000	0.5238962	0.1642371	0.2848369	0.0000000	0.0270299
220	969	0.0246610	0.5006459	0.1576685	0.0278168	0.1087409	0.1804670
221	970	0.2249322	0.5500122	0.1896560	0.0000000	0.0000000	0.0353995

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Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
222	971	0.0118731	0.5778524	0.1623746	0.2470738	0.0000000	0.0008260
223	972	0.0997964	0.0000000	0.7919777	0.0000000	0.0000000	0.1082259
224	973	0.0000000	0.5558365	0.1458048	0.2983587	0.0000000	0.0000000
225	974	0.0000000	0.7545972	0.0386686	0.0185281	0.1768840	0.0113220
226	975	0.0000000	0.6500289	0.0877949	0.1644385	0.0000000	0.0977377
227	976	0.0798494	0.6011268	0.1057732	0.2132506	0.0000000	0.0000000
228	977	0.0000000	0.0000000	1.0000000	0.0000000	0.0000000	0.0000000
229	978	0.0272410	0.5973152	0.0487156	0.1146768	0.0825696	0.1294817
230	979	0.0859488	0.4682928	0.0385860	0.4032532	0.0000000	0.0039192
231	980	0.0872253	0.5065099	0.1220065	0.1213173	0.0000000	0.1629410
232	981	0.0401089	0.4424267	0.1259785	0.2016058	0.0837615	0.1061186
233	982	0.3483929	0.0000000	0.5064686	0.0000000	0.0000000	0.1451385
234	983	0.1077031	0.4202875	0.0753764	0.2818623	0.0120651	0.1027055
235	984	0.0444063	0.6089060	0.0163885	0.2582133	0.0720859	0.0000000
236	985	0.0100521	0.7631822	0.0137104	0.1570393	0.0000000	0.0560160
237	986	0.3351323	0.4563154	0.0000000	0.2085523	0.0000000	0.0000000
238	987	0.0800494	0.5170306	0.0186747	0.2731248	0.0697887	0.0413319
239	988	0.8003463	0.1996537	0.0000000	0.0000000	0.0000000	0.0000000
240	989	0.1570881	0.4877035	0.0911162	0.0510165	0.0000000	0.2130757
241	990	0.1656271	0.3780676	0.0821482	0.0635141	0.2327622	0.0778809
242	991	0.2895250	0.5055694	0.1077611	0.0000000	0.0000000	0.0971445
243	992	0.0470842	0.4593141	0.0657472	0.3314154	0.0303380	0.0661010
244	993	0.0989481	0.4183857	0.0434512	0.3659859	0.0053101	0.0679190
245	994	0.1151818	0.3768682	0.0400848	0.2641678	0.1601920	0.0435055
246	995	0.1229554	0.3684492	0.0124175	0.3923299	0.0275503	0.0762976
247	996	0.1166568	0.3676798	0.0107939	0.4108772	0.0332160	0.0607762
248	997	0.7559366	0.0000000	0.2440634	0.0000000	0.0000000	0.0000000
249	998	0.2260937	0.1717094	0.0034683	0.5060543	0.0000000	0.0926743
250	999	0.1608392	0.6035557	0.0198924	0.0000000	0.0352753	0.1804375
251	1000	0.1486810	0.3779094	0.0000000	0.3931713	0.0190697	0.0611686
252	1001	0.9067249	0.0000000	0.0000000	0.0366963	0.0527905	0.0037883
253	1002	0.2816350	0.4168770	0.0000000	0.3014881	0.0000000	0.0000000

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Table 2 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
254	1003	0.0728907	0.6415264	0.0152020	0.1421484	0.0788175	0.0494151
255	1004	0.3796394	0.0000000	0.0482169	0.0534933	0.3887924	0.1298580
256	1005	0.8583954	0.0000000	0.1186093	0.0000000	0.0229953	0.0000000
257	1006	0.1692114	0.2860445	0.0097907	0.5349535	0.0000000	0.0000000
258	1007	0.1227021	0.3783710	0.0000000	0.4465597	0.0081576	0.0442096
259	1008	0.1386430	0.2548002	0.0113825	0.5126998	0.0824744	0.0000000
260	1009	0.0413706	0.5770999	0.0096920	0.1808760	0.1552947	0.0356669
261	1010	0.1417485	0.5251015	0.0000000	0.1773622	0.0000000	0.1557878
262	1011	0.2694552	0.0000000	0.1702323	0.1312193	0.0000000	0.4290932
263	1012	0.1534642	0.2835587	0.0034421	0.4467624	0.0830359	0.0297367
264	1013	0.0812947	0.5686200	0.0000000	0.1819832	0.1068268	0.0612753
265	1014	0.4374691	0.5616027	0.0009282	0.0000000	0.0000000	0.0000000
266	1015	0.4950139	0.4669752	0.0380109	0.0000000	0.0000000	0.0000000
267	1016	0.0631481	0.4117604	0.0000000	0.4523306	0.0078608	0.0649000
268	1017	0.5174093	0.3650555	0.1175351	0.0000000	0.0000000	0.0000000
269	1018	0.4091499	0.3933707	0.0831809	0.0000000	0.0000000	0.1142984
270	1019	0.0419887	0.7395399	0.0000000	0.1085611	0.0000000	0.1099103
271	1020	0.0000000	0.8339506	0.0261715	0.0000000	0.0000000	0.1398779
272	1021	0.0000000	0.7980231	0.0071031	0.1948737	0.0000000	0.0000000
273	1022	0.1989663	0.6693078	0.0440213	0.0000000	0.0518278	0.0358769
274	1023	0.1017579	0.4850110	0.0125942	0.3288859	0.0717509	0.0000000
275	1024	0.0000000	0.9999999	0.0000001	0.0000000	0.0000000	0.0000000
276	1025	0.5982829	0.3048106	0.0969064	0.0000000	0.0000000	0.0000000
277	1026	0.1038645	0.5208904	0.0000000	0.2249248	0.1107283	0.0395919
278	1027	0.0644882	0.6040130	0.0000000	0.2693664	0.0260239	0.0361084
279	1028	0.0673586	0.6484866	0.0000000	0.2152981	0.0348286	0.0340281
280	1029	0.3493156	0.2712792	0.0671998	0.0000000	0.0000000	0.3122054
281	1030	0.0226338	0.7440757	0.0084086	0.0957392	0.0871531	0.0419895
282	1031	0.0722463	0.5415374	0.0650742	0.3211421	0.0000000	0.0000000
283	1032	0.0000000	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000
284	1033	0.0345010	0.6959455	0.0024233	0.2449596	0.0000000	0.0221706
285	1034	0.0785550	0.6836020	0.0815723	0.1562707	0.0000000	0.0000000

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Table 2 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
286	1035	0.0559878	0.6958811	0.0134386	0.0555915	0.0000000	0.1791010
287	1036	0.0180919	0.6334282	0.0740296	0.1418782	0.0473373	0.0852348
288	1037	0.1190630	0.3289163	0.0507754	0.2363935	0.1495933	0.1152585
289	1038	0.1114315	0.3533747	0.0303328	0.4420118	0.0000000	0.0628492
290	1039	0.0622200	0.5672050	0.0579685	0.2039360	0.0000000	0.1086704
291	1040	0.0151432	0.7056387	0.0179555	0.0905993	0.0000000	0.1706632
292	1041	0.0536876	0.5499065	0.0127726	0.2725348	0.0146232	0.0964753
293	1042	0.0578379	0.6177125	0.0118462	0.2504881	0.0621153	0.0000000
294	1043	0.0256679	0.6278186	0.0069369	0.2398588	0.0302474	0.0694703
295	1044	0.0748509	0.4419015	0.0264767	0.2928222	0.0000000	0.1639487
296	1045	0.0681255	0.5754262	0.0145650	0.2859595	0.0559237	0.0000000
297	1046	0.0274931	0.6947274	0.0000000	0.2163351	0.0007750	0.0606695
298	1047	0.0996573	0.3761415	0.0152604	0.3619391	0.0331822	0.1138195
299	1048	0.0501016	0.6339927	0.0211215	0.0994052	0.1847197	0.0106594
300	1049	0.1412812	0.2944822	0.0201105	0.3332742	0.0977903	0.1130616
301	1050	0.1951180	0.4493925	0.0139509	0.0634631	0.0000000	0.2780756
302	1051	0.0000000	0.0000000	1.0000000	0.0000000	0.0000000	0.0000000
303	1052	0.9771384	0.0000000	0.0000000	0.0111575	0.0054433	0.0062608
304	1053	0.6414546	0.2621097	0.0000000	0.0707422	0.0256935	0.0000000
305	1054	0.1159419	0.4169761	0.0015548	0.3410325	0.0317028	0.0927919
306	1055	0.2004828	0.3388059	0.0220592	0.0036552	0.3418146	0.0931822
307	1056	0.1900867	0.5753923	0.0023421	0.2321789	0.0000000	0.0000000
308	1057	0.3018879	0.0837599	0.0320803	0.5822718	0.0000000	0.0000000
309	1058	0.0118792	0.7293506	0.0008980	0.2462892	0.0000000	0.0115830
310	1059	0.1694769	0.5771203	0.0067693	0.1283560	0.0000000	0.1182775
311	1060	0.1196298	0.3896934	0.0091540	0.2486778	0.0000000	0.2328449
312	1061	0.0890218	0.5552183	0.0151422	0.1193334	0.1822314	0.0390529
313	1062	0.0786816	0.4493352	0.0070020	0.3701496	0.0000000	0.0948316
314	1063	0.0000000	0.6913523	0.0221710	0.2864767	0.0000000	0.0000000
315	1064	0.0000000	0.7415053	0.0133656	0.2451290	0.0000000	0.0000000
316	1065	0.0532965	0.5649843	0.0257764	0.1323465	0.0000000	0.2235962
317	1066	0.0336849	0.7952407	0.0449099	0.1261645	0.0000000	0.0000000

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Table 2 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
318	1067	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
319	1068	0.6383786	0.1883179	0.0172486	0.0000000	0.0000000	0.1560549
320	1069	0.2767573	0.3766358	0.0344973	0.0000000	0.0000000	0.3121097
321	1070	0.0379778	0.6859992	0.0214611	0.1038952	0.1506668	0.0000000
322	1071	0.0000000	0.8118222	0.0147567	0.1017753	0.0000000	0.0716457
323	1072	0.0594936	0.4522003	0.0257530	0.3969396	0.0000000	0.0656134
324	1073	0.0670317	0.4770486	0.0351010	0.2261769	0.1119926	0.0826492
325	1074	0.0000000	0.6200222	0.0566428	0.3233350	0.0000000	0.0000000
326	1075	0.0497500	0.4289505	0.0289550	0.2120451	0.0000000	0.2802993
327	1076	0.0569928	0.4635365	0.0381357	0.2479726	0.0468869	0.1464756
328	1077	0.9358597	0.0000000	0.0000000	0.0421584	0.0156584	0.0063236
329	1078	0.9068440	0.0000000	0.0127289	0.0000000	0.0804271	0.0000000
330	1079	0.0794409	0.4556516	0.0872570	0.0336954	0.0721946	0.2717605
331	1080	0.1045278	0.5202390	0.0751337	0.0442367	0.2558628	0.0000000
332	1081	0.1509899	0.1460767	0.0681010	0.4161072	0.0000000	0.2187252
333	1082	0.9453478	0.0000000	0.0000000	0.0329341	0.0217181	0.0000000
334	1083	0.1318784	0.3039683	0.0462001	0.3720764	0.1077177	0.0381592
335	1084	0.1472541	0.5989327	0.0993746	0.1544387	0.0000000	0.0000000
336	1085	0.0848361	0.3388235	0.0498526	0.3908485	0.0026773	0.1329621
337	1086	0.0909505	0.3442355	0.0408619	0.3845701	0.0166664	0.1227155
338	1087	0.0185423	0.5622219	0.0357891	0.3834467	0.0000000	0.0000000
339	1088	0.0853908	0.3922508	0.0493403	0.4730181	0.0000000	0.0000000
340	1089	0.1522393	0.2222798	0.0628914	0.5625895	0.0000000	0.0000000
341	1090	0.0898570	0.3658086	0.0326976	0.4670468	0.0000000	0.0445900
342	1091	0.0370376	0.6680932	0.0040095	0.2651665	0.0000000	0.0256932
343	1092	0.1851046	0.3490841	0.0366606	0.1990566	0.0000000	0.2300941
344	1093	0.4138009	0.0000000	0.1217561	0.1540813	0.0000000	0.3103617
345	1094	0.1265048	0.3675958	0.0323375	0.3352656	0.0664079	0.0718884
346	1095	0.1508234	0.4401406	0.0181047	0.1810411	0.0971895	0.1127007
347	1096	0.2107099	0.5996114	0.0552068	0.1344719	0.0000000	0.0000000
348	1097	0.1510668	0.3092082	0.0388376	0.2433353	0.1362516	0.1213005
349	1098	0.1530675	0.3793157	0.0299724	0.2749395	0.0491685	0.1135364

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Table 2 Continued from previous page

Est.Start	Est.End	BRIX	WGBI	ST2X	MSCI	OSEBX	OSE4040
350	1099	0.1396085	0.4302802	0.0543036	0.1209661	0.1214965	0.1333450
351	1100	0.2747856	0.6066658	0.0924827	0.0000000	0.0000000	0.0260658
352	1101	0.0614941	0.5540234	0.0868297	0.1351738	0.0000000	0.1624791
353	1102	0.1273215	0.3914245	0.0898771	0.0898778	0.0000000	0.3014991
354	1103	0.2717630	0.3462730	0.0845564	0.2974076	0.0000000	0.0000000
355	1104	0.8494198	0.1505802	0.0000000	0.0000000	0.0000000	0.0000000
356	1105	0.1197855	0.4041098	0.1194895	0.0111951	0.2902574	0.0551627
357	1106	0.1125110	0.3465735	0.0577541	0.2645745	0.0000000	0.2185868
358	1107	0.1127334	0.3466997	0.0541752	0.1234840	0.0616363	0.3012715

Concluded

Appendix C: R code

C.1 R code

```
#load sample data and packages
dataset<-read.table("dataSet.txt")
attach(dataset)
colnames(dataset)<-c("BRIX","WGBI","ST2X","MSCI","OSEBX","OSE4040")
TS <- ts(dataset)
dato <- timeSequence(from = "2005-03-27", length.out = 1108, by = "day")
no.ts <- timeSeries(dataset[,1], dato)
io.ts <- timeSeries(dataset[,2], dato)
p.ts <- timeSeries(dataset[,3], dato)
ia.ts <- timeSeries(dataset[,4], dato)
na.ts <- timeSeries(dataset[,5], dato)
e.ts <- timeSeries(dataset[,6], dato)

library(fGarch)
library(VineCopula)
library(fPortfolio)


#descriptive statistics of the sample data
skewness<-skewness(dataset)
kurtosis<-kurtosis(dataset)
mean<-c(mean(dataset[,1]),mean(dataset[,2]),mean(dataset[,3]),mean(dataset
    [,4]),mean(dataset[,5]),mean(dataset[,6]))
std.dev<-c(sd(dataset[,1])*sqrt(250)*100, sd(dataset[,2])*sqrt(250)*100, sd(
    dataset[,3])*sqrt(250)*100, sd(dataset[,4])*sqrt(250)*100, sd(dataset[,5])
    *sqrt(250)*100, sd(dataset[,6])*sqrt(250)*100)
minimum<-c(min(dataset[,1]),min(dataset[,2]),min(dataset[,3]),min(dataset[,4])
    ,min(dataset[,5]),min(dataset[,6]))
maximum<-c(max(dataset[,1]),max(dataset[,2]),max(dataset[,3]),max(dataset[,4])
    ,max(dataset[,5]),max(dataset[,6]))
summary<-data.frame(cbind(minimum, maximum,mean,std.dev,skewness,kurtosis))
rownames(summary)<-c("BRIX","WGBI","ST2X","MSCI","OSEBX","OSE4040")
```

```
print(summary)

#Empirical cumulative distribution function used to obtain uniformly
    distributed margins
empDist <- function(data)
{
        n <- length(data)
        cumdist <- seq(1/(n+1), n/(n+1), 1/(n+1))
        unif <- cumdist[rank(data)]
        unif
}


#plot the 6 returns series
par(mfrow = c(3, 2))
#Norwegian bond index
plot(no.ts, main="BRIX", ylab="log-returns",xlab="date")
#World citigroup bond index
plot(io.ts, main="WGBI",ylab="log-returns",xlab="date")
#Government Bond Index, fix modified duration of 0.50 years
plot(p.ts, main="ST2X",ylab="log-returns",xlab="date")
#Morgan Stanley World Index
plot(ia.ts, main="MSCI",ylab="log-returns",xlab="date")
#Oslo Stock Exchange main index
plot(na.ts, main="OSEBX", ylab="log-returns",xlab="date")
#Oslo Stock Exchange Real estate index
plot(e.ts, main="OSE Real Estate Index",ylab="log-returns",xlab="date")
par(mfrow = c(1, 1))

#Testing for normality
par(mfrow = c(3, 2))
qqnorm(no.ts-mean(no.ts), main="BRIX")
qqline(no.ts-mean(no.ts), col = 2)
qqnorm(io.ts-mean(io.ts),main="WGBI")
qqline(io.ts-mean(io.ts), col = 2)
qqnorm(p.ts-mean(p.ts),main="ST2X")
qqline(p.ts-mean(p.ts), col = 2)
qqnorm(ia.ts-mean(ia.ts),main="MSCI")
qqline(ia.ts-mean(ia.ts), col = 2)
qqnorm(na.ts-mean(na.ts),main="OSEBX")
qqline(na.ts-mean(na.ts), col = 2)
qqnorm(e.ts-mean(e.ts),main="OSE4040")
qqline(e.ts-mean(e.ts), col = 2)
par(mfrow = c(1, 1))
```

```
#Detecting volatility clustering by inspecting the ACF of the absolute values
    of the log-returns
par(mfrow = c(3, 2))
acf(abs(no.ts),main="BRIX")
acf(abs(io.ts), main="WGBI")
acf(abs(p.ts), main="ST2X")
acf(abs(ia.ts), main="MSCI")
acf(abs(na.ts), main="OSEBX")
acf(abs(e.ts), main="OSE4040")
par(mfrow = c(1, 1))

#CVaR optimization with 5000 simulated returns

#Scenario 1

vektmat5<-matrix(0,ncol=6,nrow=358)
optFunk5<-function(no.ts,io.ts,p.ts,ia.ts,na.ts,e.ts,dag)
{
        print(dag)
        no.fit<-garchFit(~ garch(1,1),cond.dist="ged", data = 10*(no.ts[dag:(
            dag+749)]-mean(no.ts[dag:(dag+749)])), trace = FALSE)
        io.fit<-garchFit(~ garch(1,1),cond.dist="ged",data = 10*(io.ts[dag:(
            dag+749)]-mean(io.ts[dag:(dag+749)])), trace = FALSE)
        p.fit<-garchFit(~ garch(1,1),cond.dist="sstd", data = 100*(p.ts[dag:(
            dag+749)]-mean(p.ts[dag:(dag+749)])), trace = FALSE)
        ia.fit<-garchFit(~ garch(1,1), cond.dist="std",data = ia.ts[dag:(dag
            +749)]-mean(ia.ts[dag:(dag+749)]), trace = FALSE)
        na.fit<-garchFit(~ garch(1,1), cond.dist="std", data = na.ts[dag:(dag
            +749)]-mean(na.ts[dag:(dag+749)]), trace = FALSE)
        e.fit<-garchFit(~ garch(1,1),cond.dist="std", data = e.ts[dag:(dag
            +749)]-mean(e.ts[dag:(dag+749)]), trace = FALSE)

        no.unifVaR<-empDist((residuals(no.fit)/10)/(no.fit@sigma.t/10))
        io.unifVaR<-empDist((residuals(io.fit)/10)/(io.fit@sigma.t/10))
        p.unifVaR<-empDist((residuals(p.fit)/100)/(p.fit@sigma.t/100))
        ia.unifVaR<-empDist(residuals(ia.fit)/ia.fit@sigma.t)
        na.unifVaR<-empDist(residuals(na.fit)/na.fit@sigma.t)
        e.unifVaR<-empDist(residuals(e.fit)/e.fit@sigma.t)

        unifVaR<-data.frame(cbind(no.unifVaR, io.unifVaR, p.unifVaR, ia.
            unifVaR, na.unifVaR, e.unifVaR))

        TauMatrix(unifVaR)
```

```
        sum(abs(TauMatrix(unifVaR)[,1]))-1
        sum(abs(TauMatrix(unifVaR)[,2]))-1
        sum(abs(TauMatrix(unifVaR)[,3]))-1
        sum(abs(TauMatrix(unifVaR)[,4]))-1
        sum(abs(TauMatrix(unifVaR)[,5]))-1
        sum(abs(TauMatrix(unifVaR)[,6]))-1

cvm <- RVineStructureSelect(unifVaR, type=1, c(1:4))

set.seed(2)
uSim<-RVineSim(5000,cvm)

zSim.no<-qged(uSim[,1], nu=matrix(coef(no.fit))[5,])/sd(qged(uSim[,1],
     nu=matrix(coef(no.fit))[5,]))
zSim.io<-qged(uSim[,2], nu=matrix(coef(io.fit))[5,])/sd(qged(uSim[,2],
     nu=matrix(coef(io.fit))[5,]))
zSim.p<-qstd(uSim[,3],sd=matrix(coef(p.fit))[5,], nu=matrix(coef(p.fit
    ))[6,])/sd(qstd(uSim[,3],sd=matrix(coef(p.fit))[5,], nu=matrix(
    coef(p.fit))[6,]))
zSim.ia<-qt(uSim[,4],df=matrix(coef(ia.fit))[5,])/sd(qt(uSim[,4],df=
    matrix(coef(ia.fit))[5,]))
zSim.na<-qt(uSim[,5],df=matrix(coef(na.fit))[5,])/sd(qt(uSim[,5],df=
    matrix(coef(na.fit))[5,]))
zSim.e<-qt(uSim[,6],df=matrix(coef(e.fit))[5,])/sd(qt(uSim[,6],df=
    matrix(coef(e.fit))[5,]))

x <-cbind(coef(no.fit)) #estimates from no.fit
alpha0.no<-x[2,1]/100
alpha1.no<-x[3,1]
beta1.no<-x[4,1]
z.no<-zSim.no
a.no<-array(0,5000)
s.no<-array(0,5000)
s.no.start<-matrix(volatility(na.fit, type = "sigma"))[750,]
a.no.start<- matrix(no.ts[dag:(dag+749)]-mean(no.ts[dag:(dag+749)]))
    [750,]
for(i in 1:5000){s.no[i]<-sqrt(alpha0.no+alpha1.no*(a.no.start^2) +
    beta1.no*s.no.start^2)
a.no[i]<-z.no[i]*s.no[i]}
rSim.no<-mean(no.ts[dag:(dag+749)])+a.no

y <-cbind(coef(io.fit)) #estimates from io.fit
alpha0.io<-y[2,1]/100
alpha1.io<-y[3,1]
```

```
beta1.io<-y[4,1]
z.io<-zSim.io
a.io<-array(0,5000)
s.io<-array(0,5000)
s.io.start<-matrix(volatility(ia.fit, type = "sigma"))[750,]
a.io.start<- matrix(io.ts[dag:(dag+749)]-mean(io.ts[dag:(dag+749)]))
    [750,]
for(i in 1:5000){s.io[i]<-sqrt(alpha0.io+alpha1.io*(a.io.start^2) +
    beta1.io*s.io.start^2)
a.io[i]<-z.io[i]*s.io[i]}
rSim.io<-mean(io.ts[dag:(dag+749)])+a.io

w <-cbind(coef(p.fit)) #estimates from p.fit
alpha0.p<-w[2,1]/10000
alpha1.p<-w[3,1]
beta1.p<-w[4,1]
z.p<-zSim.p
a.p<-array(0,5000)
s.p<-array(0,5000)
s.p.start<-matrix(volatility(p.fit, type = "sigma"))[750,]
a.p.start<- matrix(p.ts[dag:(dag+749)]-mean(p.ts[dag:(dag+749)]))
    [750,]
for(i in 1:5000){s.p[i]<-sqrt(alpha0.p+alpha1.p*(a.p.start^2) + beta1.
    p*s.p.start^2)
a.p[i]<-z.p[i]*s.p[i]}
rSim.p<-mean(p.ts[dag:(dag+749)])+a.p

z <-cbind(coef(ia.fit)) #estimates from ia.fit
alpha0.ia<-z[2,1]
alpha1.ia<-z[3,1]
beta1.ia<-z[4,1]
z.ia<-zSim.ia
a.ia<-array(0,5000)
s.ia<-array(0,5000)
s.ia.start<-matrix(volatility(ia.fit, type = "sigma"))[750,]
a.ia.start<- matrix(ia.ts[dag:(dag+749)]-mean(ia.ts[dag:(dag+749)]))
    [750,]
for(i in 1:5000){s.ia[i]<-sqrt(alpha0.ia+alpha1.ia*(a.ia.start^2) +
    beta1.ia*s.ia.start^2)
a.ia[i]<-z.ia[i]*s.ia[i]}
rSim.ia<-mean(ia.ts[dag:(dag+749)])+a.ia

u <-cbind(coef(na.fit)) #estimates from na.fit
alpha0.na<-u[2,1]
```

```
alpha1.na<-u[3,1]
beta1.na<-u[4,1]
z.na<-zSim.na
a.na<-array(0,5000)
s.na<-array(0,5000)
s.na.start<-matrix(volatility(na.fit, type = "sigma"))[750,]
a.na.start<- matrix(na.ts[dag:(dag+749)]-mean(na.ts[dag:(dag+749)]))
    [750,]
for(i in 1:5000){s.na[i]<-sqrt(alpha0.na+alpha1.na*(a.na.start^2) +
    beta1.na*s.na.start^2)
a.na[i]<-z.na[i]*s.na[i]}
rSim.na<-mean(na.ts[dag:(dag+749)])+a.na

v <-cbind(coef(e.fit)) #estimates from e.fit
alpha0.e<-v[2,1]
alpha1.e<-v[3,1]
beta1.e<-v[4,1]
z.e<-zSim.e
a.e<-array(0,5000)
s.e<-array(0,5000)
s.e.start<-matrix(volatility(e.fit, type = "sigma"))[750,]
a.e.start<- matrix(e.ts[dag:(dag+749)]-mean(e.ts[dag:(dag+749)]))
    [750,]
for(i in 1:5000){s.e[i]<-sqrt(alpha0.e+alpha1.e*(a.e.start^2) + beta1.
    e*s.e.start^2)
a.e[i]<-z.e[i]*s.e[i]}
rSim.e<-mean(e.ts[dag:(dag+749)])+a.e

rSim.BRIX<-rSim.no-mean(rSim.no) + 2.5/(250*100)
rSim.WGBI<- rSim.io-mean(rSim.io) + 3.5/(250*100)
rSim.ST2X<-rSim.p-mean(rSim.p) + 1.5/(250*100)
rSim.MSCI<-rSim.ia-mean(rSim.ia) + 6.0/(250*100)
rSim.OSEBX<-rSim.na-mean(rSim.na) + 10.0/(250*100)
rSim.OSE4040<-rSim.e-mean(rSim.e) + 5.0/(250*100)

data<-cbind(rSim.BRIX,rSim.WGBI,rSim.ST2X,rSim.MSCI,rSim.OSEBX,rSim.
    OSE4040)
tsData <- timeSeries(data)
frontierSpec<-portfolioSpec()
setType(frontierSpec)<-"CVaR"
setSolver(frontierSpec)<-"solveRglpk"

setAlpha(frontierSpec)<-0.01
setNFrontierPoints(frontierSpec)<-50
```

```
        frontier1e<-portfolioFrontier(data=tsData,spec=frontierSpec,
            constraints="LongOnly")
        VaRtab1e <- getTargetRisk(frontier1e@portfolio)
        muTab1e<-getTargetReturn(frontier1e@portfolio)
        weightTab1e<- getWeights(frontier1e@portfolio)

        ind<-1
        while(muTab1e[ind,1]<0.06/250)
        ind<-ind+1
        optWeights1e <- weightTab1e[ind,]
        optWeights1e
}

for (dag in 1:358)
vektmat5[dag,]=optFunk5(no.ts,io.ts,p.ts,ia.ts,na.ts,e.ts,dag)
weights5<-data.frame(vektmat5)
write.csv(weights5, "weights5.csv")


#Scenario 2

vektmat5b<-matrix(0,ncol=6,nrow=358)
optFunk5b<-function(no.ts,io.ts,p.ts,ia.ts,na.ts,e.ts,dag)
{
        print(dag)
        no.fit<-garchFit(~ garch(1,1),cond.dist="ged", data = 10*(no.ts[dag:(
            dag+749)]-mean(no.ts[dag:(dag+749)])), trace = FALSE)
        io.fit<-garchFit(~ garch(1,1),cond.dist="ged",data = 10*(io.ts[dag:(
            dag+749)]-mean(io.ts[dag:(dag+749)])), trace = FALSE)
        p.fit<-garchFit(~ garch(1,1),cond.dist="sstd", data = 100*(p.ts[dag:(
            dag+749)]-mean(p.ts[dag:(dag+749)])), trace = FALSE)
        ia.fit<-garchFit(~ garch(1,1), cond.dist="std",data = ia.ts[dag:(dag
            +749)]-mean(ia.ts[dag:(dag+749)]), trace = FALSE)
        na.fit<-garchFit(~ garch(1,1), cond.dist="std", data = na.ts[dag:(dag
            +749)]-mean(na.ts[dag:(dag+749)]), trace = FALSE)
        e.fit<-garchFit(~ garch(1,1),cond.dist="std", data = e.ts[dag:(dag
            +749)]-mean(e.ts[dag:(dag+749)]), trace = FALSE)

        no.unifVaR<-empDist((residuals(no.fit)/10)/(no.fit@sigma.t/10))
        io.unifVaR<-empDist((residuals(io.fit)/10)/(io.fit@sigma.t/10))
        p.unifVaR<-empDist((residuals(p.fit)/100)/(p.fit@sigma.t/100))
        ia.unifVaR<-empDist(residuals(ia.fit)/ia.fit@sigma.t)
        na.unifVaR<-empDist(residuals(na.fit)/na.fit@sigma.t)
```

```
e.unifVaR<-empDist(residuals(e.fit)/e.fit@sigma.t)

unifVaR<-data.frame(cbind(no.unifVaR, io.unifVaR, p.unifVaR, ia.
    unifVaR, na.unifVaR, e.unifVaR))
cvm <- RVineStructureSelect(unifVaR, type=1, c(1:4))

set.seed(2)
uSim<-RVineSim(5000,cvm)

zSim.no<-qged(uSim[,1], nu=matrix(coef(no.fit))[5,])/sd(qged(uSim[,1],
     nu=matrix(coef(no.fit))[5,]))
zSim.io<-qged(uSim[,2], nu=matrix(coef(io.fit))[5,])/sd(qged(uSim[,2],
     nu=matrix(coef(io.fit))[5,]))
zSim.p<-qstd(uSim[,3],sd=matrix(coef(p.fit))[5,], nu=matrix(coef(p.fit
    ))[6,])/sd(qstd(uSim[,3],sd=matrix(coef(p.fit))[5,], nu=matrix(
    coef(p.fit))[6,]))
zSim.ia<-qt(uSim[,4],df=matrix(coef(ia.fit))[5,])/sd(qt(uSim[,4],df=
    matrix(coef(ia.fit))[5,]))
zSim.na<-qt(uSim[,5],df=matrix(coef(na.fit))[5,])/sd(qt(uSim[,5],df=
    matrix(coef(na.fit))[5,]))
zSim.e<-qt(uSim[,6],df=matrix(coef(e.fit))[5,])/sd(qt(uSim[,6],df=
    matrix(coef(e.fit))[5,]))

x <-cbind(coef(no.fit)) #estimates from no.fit
alpha0.no<-x[2,1]/100
alpha1.no<-x[3,1]
beta1.no<-x[4,1]
z.no<-zSim.no
a.no<-array(0,5000)
s.no<-array(0,5000)
s.no.start<-matrix(volatility(na.fit, type = "sigma"))[750,]
a.no.start<- matrix(no.ts[dag:(dag+749)]-mean(no.ts[dag:(dag+749)]))
    [750,]
for(i in 1:5000){s.no[i]<-sqrt(alpha0.no+alpha1.no*(a.no.start^2) +
    beta1.no*s.no.start^2)
a.no[i]<-z.no[i]*s.no[i]}
rSim.no<-mean(no.ts[dag:(dag+749)])+a.no

y <-cbind(coef(io.fit)) #estimates from io.fit
alpha0.io<-y[2,1]/100
alpha1.io<-y[3,1]
beta1.io<-y[4,1]
z.io<-zSim.io
a.io<-array(0,5000)
```

```
s.io<-array(0,5000)
s.io.start<-matrix(volatility(ia.fit, type = "sigma"))[750,]
a.io.start<- matrix(io.ts[dag:(dag+749)]-mean(io.ts[dag:(dag+749)]))
    [750,]
for(i in 1:5000){s.io[i]<-sqrt(alpha0.io+alpha1.io*(a.io.start^2) +
    beta1.io*s.io.start^2)
a.io[i]<-z.io[i]*s.io[i]}
rSim.io<-mean(io.ts[dag:(dag+749)])+a.io

w <-cbind(coef(p.fit)) #estimates from p.fit
alpha0.p<-w[2,1]/10000
alpha1.p<-w[3,1]
beta1.p<-w[4,1]
z.p<-zSim.p
a.p<-array(0,5000)
s.p<-array(0,5000)
s.p.start<-matrix(volatility(p.fit, type = "sigma"))[750,]
a.p.start<- matrix(p.ts[dag:(dag+749)]-mean(p.ts[dag:(dag+749)]))
    [750,]
for(i in 1:5000){s.p[i]<-sqrt(alpha0.p+alpha1.p*(a.p.start^2) + beta1.
    p*s.p.start^2)
a.p[i]<-z.p[i]*s.p[i]}
rSim.p<-mean(p.ts[dag:(dag+749)])+a.p

z <-cbind(coef(ia.fit)) #estimates from ia.fit
alpha0.ia<-z[2,1]
alpha1.ia<-z[3,1]
beta1.ia<-z[4,1]
z.ia<-zSim.ia
a.ia<-array(0,5000)
s.ia<-array(0,5000)
s.ia.start<-matrix(volatility(ia.fit, type = "sigma"))[750,]
a.ia.start<- matrix(ia.ts[dag:(dag+749)]-mean(ia.ts[dag:(dag+749)]))
    [750,]
for(i in 1:5000){s.ia[i]<-sqrt(alpha0.ia+alpha1.ia*(a.ia.start^2) +
    beta1.ia*s.ia.start^2)
a.ia[i]<-z.ia[i]*s.ia[i]}
rSim.ia<-mean(ia.ts[dag:(dag+749)])+a.ia

u <-cbind(coef(na.fit)) #estimates from na.fit
alpha0.na<-u[2,1]
alpha1.na<-u[3,1]
beta1.na<-u[4,1]
z.na<-zSim.na
```

```
a.na<-array(0,5000)
s.na<-array(0,5000)
s.na.start<-matrix(volatility(na.fit, type = "sigma"))[750,]
a.na.start<- matrix(na.ts[dag:(dag+749)]-mean(na.ts[dag:(dag+749)]))
    [750,]
for(i in 1:5000){s.na[i]<-sqrt(alpha0.na+alpha1.na*(a.na.start^2) +
    beta1.na*s.na.start^2)
a.na[i]<-z.na[i]*s.na[i]}
rSim.na<-mean(na.ts[dag:(dag+749)])+a.na

v <-cbind(coef(e.fit)) #estimates from e.fit
alpha0.e<-v[2,1]
alpha1.e<-v[3,1]
beta1.e<-v[4,1]
z.e<-zSim.e
a.e<-array(0,5000)
s.e<-array(0,5000)
s.e.start<-matrix(volatility(e.fit, type = "sigma"))[750,]
a.e.start<- matrix(e.ts[dag:(dag+749)]-mean(e.ts[dag:(dag+749)]))
    [750,]
for(i in 1:5000){s.e[i]<-sqrt(alpha0.e+alpha1.e*(a.e.start^2) + beta1.
    e*s.e.start^2)
a.e[i]<-z.e[i]*s.e[i]}
rSim.e<-mean(e.ts[dag:(dag+749)])+a.e

rSim.BRIX<-array(0,5000)
for(i in 1:5000){rSim.BRIX[i]<-rSim.no[i]-mean(rSim.no)+no.ts[dag
    +749]}

rSim.WGBI<-array(0,5000)
for(i in 1:5000){rSim.WGBI[i]<-rSim.io[i]-mean(rSim.io)+io.ts[dag
    +749]}

rSim.ST2X<-array(0,5000)
for(i in 1:5000){rSim.ST2X<-rSim.p[i]-mean(rSim.p)+p.ts[dag+749]}

rSim.MSCI<-array(0,5000)
for(i in 1:5000){rSim.MSCI[i]<-rSim.ia[i]-mean(rSim.ia)+ia.ts[dag
    +749]}

rSim.OSEBX<-array(0,5000)
for(i in 1:5000){rSim.OSEBX[i]<-rSim.na[i]-mean(rSim.na)+na.ts[dag
    +749]}
```

```
        rSim.OSE4040<-array(0,5000)
        for(i in 1:5000){rSim.OSE4040[i]<-rSim.e[i]-mean(rSim.e)+e.ts[dag
            +749]}

        data<-cbind(rSim.BRIX,rSim.WGBI,rSim.ST2X,rSim.MSCI,rSim.OSEBX,rSim.
            OSE4040)
        tsData <- timeSeries(data)
        frontierSpec<-portfolioSpec()
        setType(frontierSpec)<-"CVaR"
        setSolver(frontierSpec)<-"solveRglpk"

        setAlpha(frontierSpec)<-0.01
        setNFrontierPoints(frontierSpec)<-50

        frontier1g<-portfolioFrontier(data=tsData,spec=frontierSpec,
            constraints="LongOnly")
        VaRtab1g <- getTargetRisk(frontier1g@portfolio)
        muTab1g<-getTargetReturn(frontier1g@portfolio)
        weightTab1g<- getWeights(frontier1g@portfolio)

        ind<-1
        while(muTab1g[ind,1]<0.06/250)
        ind<-ind+1
        optWeights1g <- weightTab1g[ind,]
        optWeights1g
}

for (dag in 1:358)
vektmat5b[dag,]=optFunk5b(no.ts,io.ts,p.ts,ia.ts,na.ts,e.ts,dag)
weights5b<-data.frame(vektmat5b)
write.csv(weights5b, "weights5b.csv")


#Plot efficient frontiers
tailoredFrontierPlot(object=frontier1e,mText="99%-CVaR Portfolio(Long only 
    constraints)",risk="CVaR")
tailoredFrontierPlot(object=frontier1g,mText="99%-CVaR Portfolio2(Long only 
    constraints)",risk="CVaR")


#Plot the change in portfolio weights
vekt<-read.csv("weights5.csv")
vekt<-vekt[,-1]
head(vekt)
```

```
colnames(vekt)<-c("BRIX","WGBI","ST2X","MSCI","OSEBX","OSE4040")

plot.ts(vekt[,1],ylab="weights",ylim=c(0,1), main="0.99-CVaR␣Portfolio(Long␣
    only␣constraints)", col="blue")
text<-"Comparison␣of␣daily␣weights␣of␣5000␣simulated␣points"
mtext(text,side=3,line=0.5,font=2, cex=0.9)
par(new=T)
plot.ts(vekt[,2],ylim=c(0,1), ylab="",axes=F,col="red")
par(new=T)
plot.ts(vekt[,3],ylim=c(0,1),ylab="", axes=F,col="green")
par(new=T)
plot.ts(vekt[,4],ylim=c(0,1),ylab="", axes=F,col="orange")
par(new=T)
plot.ts(vekt[,5],ylim=c(0,1),ylab="", axes=F,col="purple")
par(new=T)
plot.ts(vekt[,6],ylim=c(0,1),ylab="", axes=F,col="palevioletred")
par(new=T)
legend("topright", c("BRIX","WGBI","ST2X","MSCI","OSEBX","OSE4040"), lty=1,
    col=c("blue","red","green","orange","purple","palevioletred"), bty="n",
    cex=0.75)
par(new=F)

#wealth computation with different strategies
#CVaR optimization
r.no<-exp(no.ts[751:1108,])-1
r.io<-exp(io.ts[751:1108,])-1
r.p<-exp(p.ts[751:1108,])-1
r.ia<-exp(ia.ts[751:1108,])-1
r.na<-exp(na.ts[751:1108,])-1
r.e<-exp(e.ts[751:1108,])-1

#Scenario 1
R1<-vekt[,1]*r.no+vekt[,2]*r.io+vekt[,3]*r.p+vekt[,4]*r.ia+vekt[,5]*r.na+vekt
    [,6]*r.e
W1<-array(0,358)
W0<-100#startverdi
W1[1]=W0*(1+R1[1])
for(i in 2:358){W1[i]=W1[i-1]*(1+R1[i])}

#Scenario 2
R4<-vektmat5b[,1]*r.no+vektmat5b[,2]*r.io+vektmat5b[,3]*r.p+vektmat5b[,4]*r.ia
    +vektmat5b[,5]*r.na+vektmat5b[,6]*r.e
W4<-array(0,358)
W0<-100#startverdi
```

```
W4[1]=W0*(1+R4[1])
for(i in 2:358){W4[i]=W4[i-1]*(1+R4[i])}


#Constant-Mix
vekt.new<-as.vector(c(0.10,0.11,0.4,0.12,0.04,0.21))
vekt.const<-cbind(matrix(vekt.new,ncol=6,nrow=358,byrow="TRUE"))
R2<-vekt.const[,1]*r.no+vekt.const[,2]*r.io+vekt.const[,3]*r.p+vekt.const[,4]*
   r.ia+vekt.const[,5]*r.na+vekt.const[,6]*r.e
W2<-array(0,358)
W0<-100
W2[1]=W0*(1+R2[1])
for(i in 2:358){W2[i]=W2[i-1]*(1+R2[i])}


#Buy-and-Hold
w.no.initial<-100*vekt.new[1]
w.io.initial<-100*vekt.new[2]
w.p.initial<-100*vekt.new[3]
w.ia.initial<-100*vekt.new[4]
w.na.initial<-100*vekt.new[5]
w.e.initial<-100*vekt.new[6]


w.no<-array(0,358)
w.no[1]<-w.no.initial*(1+r.no[1])
for(i in 2:358){w.no[i]<-w.no[i-1]*(1+r.no[i])}

w.io<-array(0,358)
w.io[1]<-w.io.initial*(1+r.io[1])
for(i in 2:358){w.io[i]<-w.io[i-1]*(1+r.io[i])}

w.p<-array(0,358)
w.p[1]<-w.p.initial*(1+r.p[1])
for(i in 2:358){w.p[i]<-w.p[i-1]*(1+r.p[i])}

w.ia<-array(0,358)
w.ia[1]<-w.ia.initial*(1+r.ia[1])
for(i in 2:358){w.ia[i]<-w.ia[i-1]*(1+r.ia[i])}

w.na<-array(0,358)
w.na[1]<-w.na.initial*(1+r.na[1])
for(i in 2:358){w.na[i]<-w.na[i-1]*(1+r.na[i])}
```

```
w.e<-array(0,358)
w.e[1]<-w.e.initial*(1+r.e[1])
for(i in 2:358){w.e[i]<-w.e[i-1]*(1+r.e[i])}

W3<-w.no+w.io+w.p+w.ia+w.na+w.e
w.no/W3

#Comparison of the three strategies
plot.ts(W1,ylab="wealth", main="change␣in␣wealth", ylim=c(95,125), col="red")
text<-"Comparison␣of␣wealth"
mtext(text,side=3,line=0.5,font=2, cex=0.9)
par(new=T)
plot.ts(W2,ylab="", ylim=c(95,125),axes=F,col="blue")
par(new=T)
plot.ts(W3,ylab="",ylim=c(95,125), axes=F,col="forestgreen")
par(new=T)
legend("topright", c("CVaR-optimization","Constant-Mix","Buy-Hold"), lty=1,
    col=c("red","blue","forestgreen"), bty="n", cex=0.75)
par(new=F)

plot.ts(W4,ylab="wealth", main="The␣accumulation␣of␣wealth", ylim=c(95,125),
    col="red")
par(new=T)
plot.ts(W2,ylab="", ylim=c(95,120),axes=F,col="blue")
par(new=T)
plot.ts(W3,ylab="",ylim=c(95,120), axes=F,col="forestgreen")
par(new=T)
legend("topright", c("CVaR-optimization␣2","Constant-Mix","Buy-Hold"), lty=1,
    col=c("red","blue","forestgreen"), bty="n", cex=0.75)
par(new=F)
```